4. Constructing Kernels

⇒ Choosing a Kernel $K(x, z) = \phi(x)^T \phi(z)$, we can implicitly map data to a high dimentional Space.

↳ One way to generate Kernel is to explicitly define the mapping ϕ to a higher dimensional space, and then work with the corresponding K.

⇒ Here we are intrested in directly constructing Kernels.

That we think gives an appropriate Similarity measure for our learning problem.

*Let:

→ K_1, K_2 are Kernels over $R^n \times R^n$

→ $a \in R^+$

→ $f: R^n \mapsto R$

→ $\phi: R^n \to R^d$

→ K_3 be a Kernel over $R^d \times R^d$

→ $P(x)$ → Polynomial over x with positive coefficients.

Q) For each K, state whether it is necessarily a Kernel.

(a) $K(x,z) = K_1(x,z) + K_2(x,z)$

$$\phi_1^T(x)\,\phi_1(z) + \phi_2^T(x)\,\phi_2(z)$$

$K(x,z) = K(z,x)$

$K(x,x) = K_1(x,x) + K_2(x,x) \geq 0$

$\Rightarrow$ Let K^i be Square matrix for Kernal K_i.

$K = K^1 + K^2$ {at is Symmetric)

$z^T K z = z^T K^1 z + z^T K^2 z \geq 0$ {Positive Semi difinite}

$\Rightarrow$ hense K is a Valid Kernel.

(b) $K(x,z) = K_1(x,z) - K_2(x,z)$

$K(z,x) = K^1(z,x) - K^2(z,x)$

$= K^1(x,z) - K^2(z,x)$

$= K(x,z)$ {hense Symmetric}

$z^T K z = z^T K_1 z - z^T K^2 z$

{Not Positive Semi definite}

(c) $K(x,z) = a K_1(x,z)$

$K(z,x) = a K^1(z,x) = a K^1(x,z) = K(x,z)$ {Symmetric}

$z^T K z = z^T a K^1 z = a z^T K^1 z$ {as $a \in R^+$, it is Positive Semi difinit}

Valid Kernel

d $K(x,z) = -a K_1(x,z)$

Similar to c, but as $a \in R^{+}$, $-a z^T K' z$ is not positive semidefinite.

$\Rightarrow$ Hence K is Not a Valid Kernel.

e $K(x,z) = K_1(x,z) K_2(x,z)$

$$K(z,x) = K^1(z,x) K^2(z,x)$$

$$= K^1(x,z) K^2(x,z)$$

$$= K(x,z) \quad \{\text{Symmetric}\}$$

$$z^T K z = z^T K^1 K^2 z$$

$$= \sum_{i,j} z_i \phi^1(x^{(i)})^T \phi^1(x^{(j)}) \phi^2(x^{(i)})^T \phi^2(x^{(j)}) z_j$$

$$= \sum_{i,j} z_i \sum_{k=1}^{n} \phi^1(x^{(i)})_k \phi^1(x^{(j)})_k \sum_{p=1}^{m} \phi^2(x^{(i)})_p \phi^2(x^{(j)})_p z_j$$

$$= \sum_{k=1}^{n} \sum_{p=1}^{m} \left(\sum_{i=1}^{m} z_i \phi^1(x^{(i)})_k \phi^2(x^{(i)})_p \right) \left(\sum_{j=1}^{m} z_i \phi^1(x^{(j)})_k \phi^2(x^{(j)})_p \right)$$

$$= \sum_{k=1}^{n} \sum_{p=1}^{m} \left(\sum_{i=1}^{m} z_i \phi^1(x^{(i)})_k \phi^2(x^{(i)})_p \right)^2 \geq 0$$

{It is positive semidefinite}

Valid Kernel

(f) $K(x,z) = f(x) f(z)$

$$K(z,x) = f(z) f(x)$$
$$= f(x) f(z)$$
$$= K(x,z) \quad \{\text{Symmetric}\}$$

$$z^T K z = \sum_{j,i} z_i f(x^{(i)}) f(x^{(j)}) z_j = (z^T x)^2$$

valid Kernel $\quad \{\text{If } x_i = f(x^{(i)})\}$

(g) $K(x,z) = K_3(\phi(x), \phi(z))$

= Valid Kernel

(h) $K(x,z) = P(K_1(x,z))$

$$P(K_1) = \underline{b_m} K_1^m + \underline{b_{m-1}} K_1^{m-1} + \cdots + \underline{b_1} K$$

* Power of Kernel is valid Kernel
* aK is valid Kernel
* $K_1 + K_2$ is valid Kernel

So $P(K_1)$ is Valid Kernel