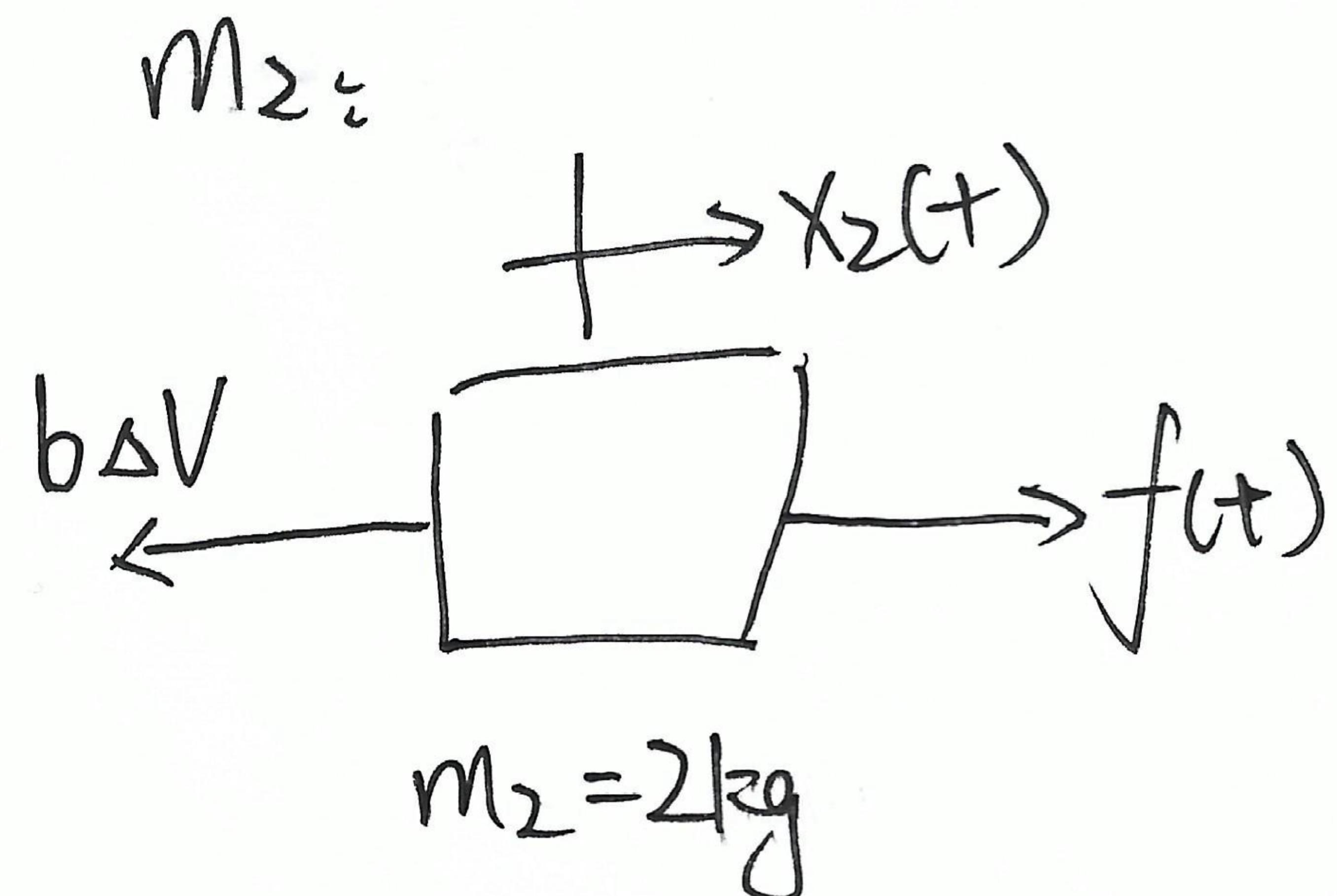
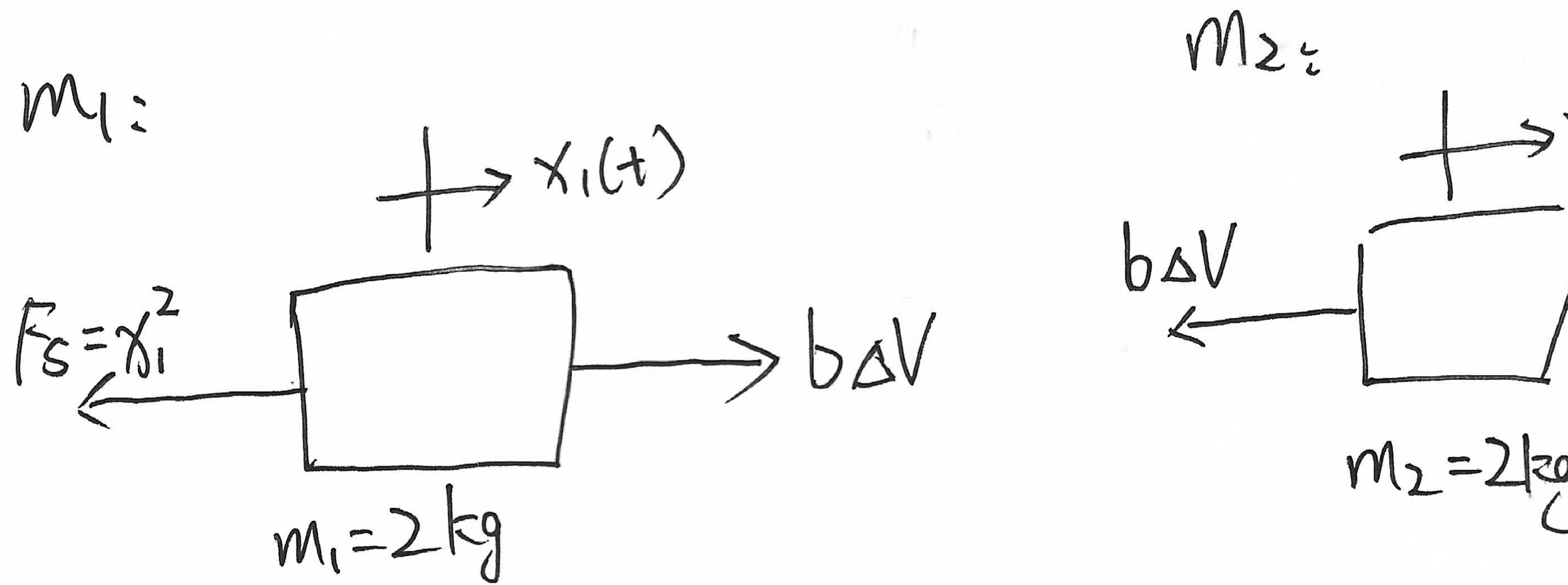


## Part A (Modelling)

Question 1:

Real-life example: Car suspension system

Question 2: free-body-diagram



notice:  $\Delta V$  is the difference between  $V_1$  and  $V_2$ .

$$\Delta V = V_2 - V_1 \quad F_s = x_1^2 \text{ eq(1)}$$

use math model:

$$\sum F_1 = m_1 a$$

$$\sum F_2 = m_2 a$$

$$b\Delta V - x_1^2 = m_1 a \quad f(t) - b\Delta V = m_2 a$$

$$bV_2 - bV_1 - x_1^2 = m_1 a \quad f(t) - bV_2 + bV_1 = m_2 a$$

$$\text{for } V = \frac{dx(t)}{dt}, \quad a = \frac{d^2x(t)}{dt^2}, \quad b = 2, \quad m_1 = m_2 = 2$$

We have

$$m_1: 2 \frac{dV_2}{dt} - 2 \frac{dV_1}{dt} - x_1^2(t) = 2 \frac{d^2x_1(t)}{dt^2}$$

$$m_2: f(t) - 2 \frac{dV_2}{dt} + 2 \frac{dV_1}{dt} = 2 \frac{d^2x_2(t)}{dt^2}$$

rearranged, we have

$$\left\{ \begin{array}{l} 2 \frac{d^2x_1(t)}{dt^2} + 2 \frac{dx_1(t)}{dt} + x_1^2(t) - 2 \frac{dx_2(t)}{dt} = 0 \end{array} \right. \quad \text{eq(2)}$$

$$\left\{ \begin{array}{l} 2 \frac{d^2x_2(t)}{dt^2} + 2 \frac{dx_2(t)}{dt} - 2 \frac{dx_1(t)}{dt} = f(t) \end{array} \right. \quad \text{eq(3)}$$

Question 3:  
Let ( $t = t_s + \delta t$ ) ( $t_s = 0$ ) and because  $x_1(0) = 1$ ,  $x_2(0) = 0$ ,  $f(0) = 0$ .

for eq(2)

$$2 \frac{d^2x_1(t_s + \delta t)}{dt^2} + 2 \frac{dx_1(t_s + \delta t)}{dt} + x_1^2(t_s + \delta t) - 2 \frac{dx_2(t_s + \delta t)}{dt} = 0.$$

since  $t_s$  is a constant

$$\frac{d^2x_1(t_s + \delta t)}{dt^2} = \frac{d^2x_1(\delta t)}{dt^2} \quad \frac{dx_1(t_s + \delta t)}{dt} = \frac{dx_1(\delta t)}{dt} \quad f(t_s + \delta t) = f(\delta t)$$

For  $x_1^2(t_s + \delta t)$ , let  $f(t) = x_1^2(t)$ .

$$f(t_s + \delta t) = f(t_s) + c\delta t \quad C = \frac{df(t)}{dt} \Big|_{t=t_s}$$

$$= x_1^2(t_s) + 2x_1(\delta t)$$

because  $t_s = 0$ ,  $x_1(0) = 1$

$$\text{so, } f(t_s + \delta t) = 1 + 2\delta x_1(t)$$

Thus, the linearise function of eq(2) and eq(3) is.

$$\left\{ \begin{array}{l} \frac{2d^2\delta x_1(t)}{dt^2} + \frac{2d\delta x_1(t)}{dt} + (1+2\delta x_1(t)) - 2\frac{d^2\delta x_2(t)}{dt^2} = 0 \quad \text{eq(4)} \\ 2\frac{d^2\delta x_2(t)}{dt^2} + 2\frac{d\delta x_2(t)}{dt} - 2\frac{d\delta x_1(t)}{dt} = \delta f(t) \end{array} \right. \quad \text{eq(5)}$$

Part B (Analytical and numerical methods)

$$\left\{ \begin{array}{l} 2\frac{d^2x_1(t)}{dt^2} + 2\frac{dx_1(t)}{dt} + (1+2x_1(t)) - 2\frac{dx_2(t)}{dt} = 0 \quad \text{eq(6)} \\ 2\frac{d^2x_2(t)}{dt^2} + 2\frac{dx_2(t)}{dt} - 2\frac{dx_1(t)}{dt} = f(t) \end{array} \right. \quad \text{eq(7)}$$

Question 4.

Laplace transform:

$$L\left(\frac{d^2x_1(t)}{dt^2}\right) = L(\ddot{x}_1(t)) = s^2 X_1(s) - s\dot{x}_1(0) - \ddot{x}(0)$$

$$L\left(\frac{dx_1(t)}{dt}\right) = L(\dot{x}_1(t)) = sX_1(s) - x_1(0)$$

$$L(f(t)) = F(s)$$

$$L(x_1(t)) = X_1(s)$$

Because ignoring the linearisation point (1 is the linearisation point)  
so 1 could be ignored.

Hence,

$$\text{eq(6)} \Rightarrow 2[s^2\dot{X}_1(s) - sX_1(0) - \dot{X}_1(0)] + 2[sX_1(s) - X_1(0)] + 2X_1(s) - 2[sX_2(s) - X_2(0)] = 0.$$

$$\text{eq(7)} \Rightarrow 2[s^2\dot{X}_2(s) - sX_2(0) - \dot{X}_2(0)] + 2[sX_2(s) - X_2(0)] - 2[sX_1(s) - X_1(0)] = F(s)$$

use  $X_1(0) = 0, X_2(0) = 0, \dot{X}_1(0) = 0, \dot{X}_2(0) = 0$ .

so,

$$\text{eq(6)} \Rightarrow 2s^2\dot{X}_1(s) + 2sX_1(s) + 2X_1(s) - 2sX_2(s) = 0 \Rightarrow (s^2 + s + 1)X_1(s) = sX_2(s)$$

$$\text{eq(7)} \Rightarrow 2s^2\dot{X}_2(s) + 2sX_2(s) - 2sX_1(s) = F(s) \Rightarrow (2s^2 + 2s)X_2(s) - 2sX_1(s) = F(s)$$

the solution of question 4 is

$$\begin{cases} (s^2 + s + 1)X_1(s) - sX_2(s) = 0 \\ (2s^2 + 2s)X_2(s) - 2sX_1(s) = F(s) \end{cases}$$

Question 5. a)

the definition of transfer function is

$$G(s) = \frac{\text{output}}{\text{Input}} = \frac{X_2(s)}{F(s)}$$

Solve the equation:

$$\begin{cases} (s^2 + s + 1)X_1(s) - sX_2(s) = 0 \\ (2s^2 + 2s)X_2(s) - 2sX_1(s) = F(s) \end{cases}$$

$X_1(s) = \frac{s}{s^2 + s + 1} X_2(s)$

$$(2s^2 + 2s)X_2(s) - 2s \cdot \frac{s}{s^2 + s + 1} X_2(s) = F(s).$$

$$\frac{2s^4 + 2s^3 + 2s^2 + 2s^3 + 2s^2 + 2s - 2s^2}{s^2 + s + 1} X_2(s) = F(s).$$

$$\frac{2s^4 + 4s^3 + 2s^2 + 2s}{s^2 + s + 1} X_2(s) = F(s)$$

So,

$$G(s) = \frac{X_2(s)}{F(s)} = \frac{s^2 + s + 1}{2s^4 + 4s^3 + 2s^2 + 2s}$$

b) 4<sup>th</sup> order system (the highest power of denominator).

Question 6. a)

zeros: roots of numerator:

set  $s^2 + s + 1 = 0$

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{-4}}{2} = -0.5 \pm 0.86603i$$

$$\text{zero: } S_1 = -0.5 + 0.86603i; \quad S_2 = -0.5 - 0.86603i$$

poles: roots of denominator

set  $2s^4 + 4s^3 + 2s^2 + 2s = 0$

$$s(s^3 + 2s^2 + s + 1) = 0$$

We know the three poles are:

$$S_1 = -0.12256 + 0.74486i$$

$$S_2 = -0.12256 - 0.74486i$$

$$S_3 = 0$$

and  $S_4 \in (-2, 0)$

use the Bisection method to find the other real pole.

set  $F(s) = s^3 + 2s^2 + s + 1$

in order to use the bisection method,

We need to determine the monotonicity of  $F(s)$ .

$$F'(s) = 3s^2 + 4s + 1, \text{ the minimum value is } \frac{4ac-b^2}{4a} = 1 - \frac{16}{12} = -\frac{1}{3}$$

$$\text{the roots are. } \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{4}}{6}, \text{ which are}$$

so, when  $s \in (-\infty, -\frac{1}{3})$ ,  $F'(s) > 0$ ,  $F(s)$  monotonically increasing.

$s \in (-\frac{1}{3}, -1)$ ,  $F'(s) < 0$ ,  $F(s)$  monotonically decreasing.

$s \in (-1, +\infty)$ ,  $F'(s) > 0$ ,  $F(s)$  monotonically increasing.

Hence, there is a local minimum value in  $s = -\frac{1}{3}$ , local maximum value

$$F(-1) = 1 > 0, F(-\frac{1}{3}) = 0.8519$$

$$s = -$$

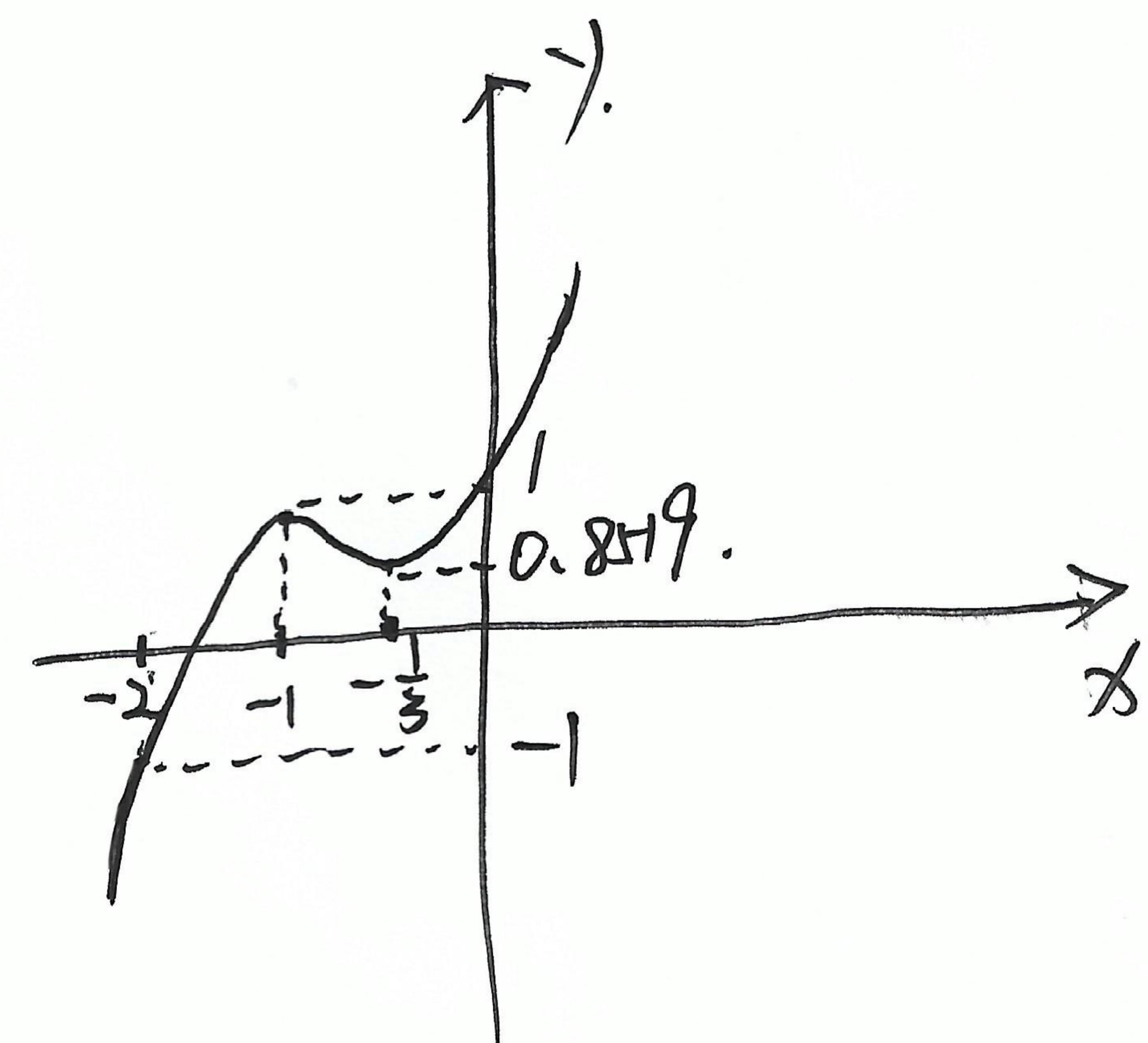
because the other real roots of  $F(s)$  is between  $-2$  and  $0$ .

and  $F(s)$  monotonically increasing between  $(-\infty, -\frac{1}{3})$

So, the real roots must between  $(-2, -1)$ .

$$F(-2) = -1$$

the draft of  $F(s)$  is on the left side.



use the bisection method:

$$F\left(\frac{-2-1}{2}\right) = F(-1.5) = 0.625 > 0. \quad \text{roots must in } (-2, -1.5)$$

$$F\left(\frac{-2-1.5}{2}\right) = F(-1.75) = 0.0156 > 0. \quad \text{roots must in } (-2, -1.75)$$

$$F\left(\frac{-2-1.75}{2}\right) = F(-1.875) = -0.4375 < 0. \quad \text{roots in } (-1.875, -1.75)$$

$$F\left(\frac{-1.875-1.75}{2}\right) = F(-1.8125) = -0.1965 < 0 \quad \text{roots in } (-1.8125, -1.75)$$

$$F\left(\frac{-1.8125-1.75}{2}\right) = F(-1.7813) = -0.0872 < 0 \quad \text{roots in } (-1.7813, -1.75)$$

$$F\left(\frac{-1.7813 - 1.75}{2}\right) = F(-1.7657) = -0.035 < 0 \text{ roots in } (-1.7657, -1.75).$$

$$F\left(\frac{-1.7657 - 1.75}{2}\right) = F(-1.7579) = -0.0096 < 0 \text{ roots in } (-1.7579, -1.75)$$

$$F\left(\frac{-1.7579 + 1.75}{2}\right) = F(-1.7540) = 0.0028 > 0.$$

notice that  $(0.0028 - 0) \times 100\% = 0.28\% < 1\%$

So we can assume that  $-1.7540$  approximately equal to the real roots.

So, the zeros of the transfer function is.

$$\begin{cases} S_1 = -0.5 + 0.86603i \\ S_2 = -0.5 - 0.86603i \end{cases}$$

the poles ~~off~~ is

$$\begin{cases} S_1 = -0.12256 + 0.74486i \\ S_2 = -0.12256 - 0.74486i \\ S_3 = 0 \\ S_4 = -1.754 \end{cases}$$

Question 6 b).

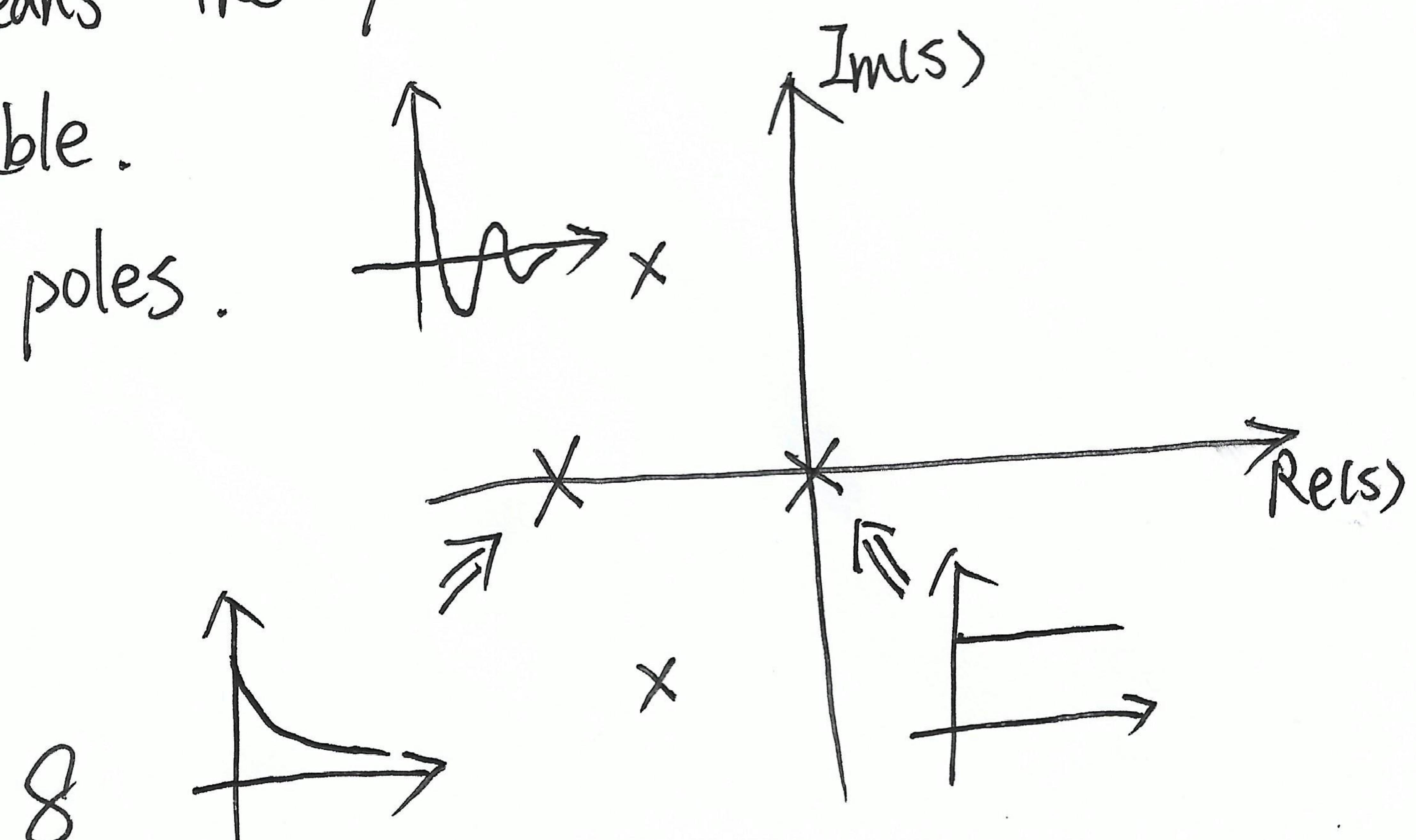
the system is stable.

we have 4 poles.  
3 poles have negative real roots which are  $-0.12256$ ,  $-0.12256$  and  $-1.754$

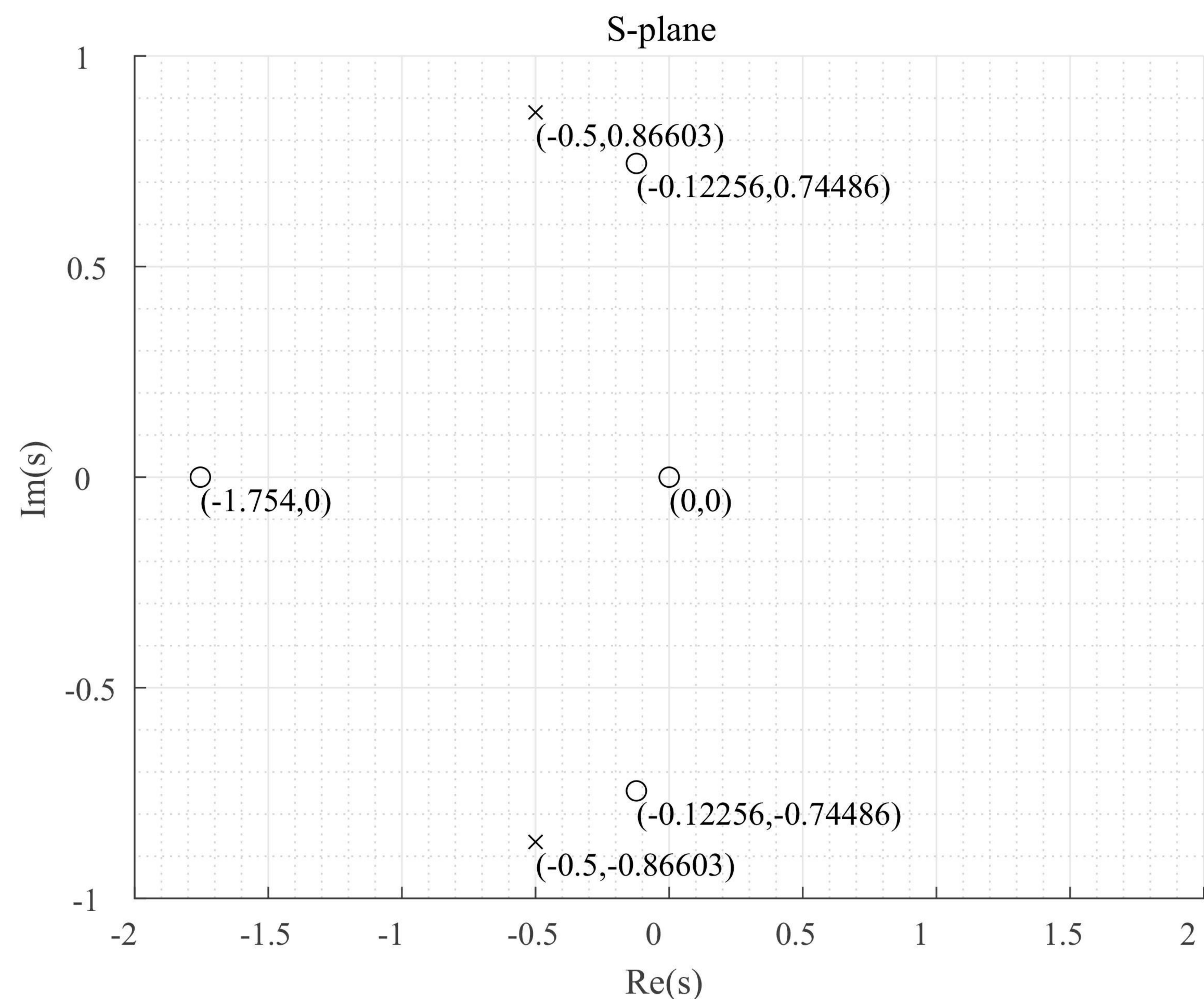
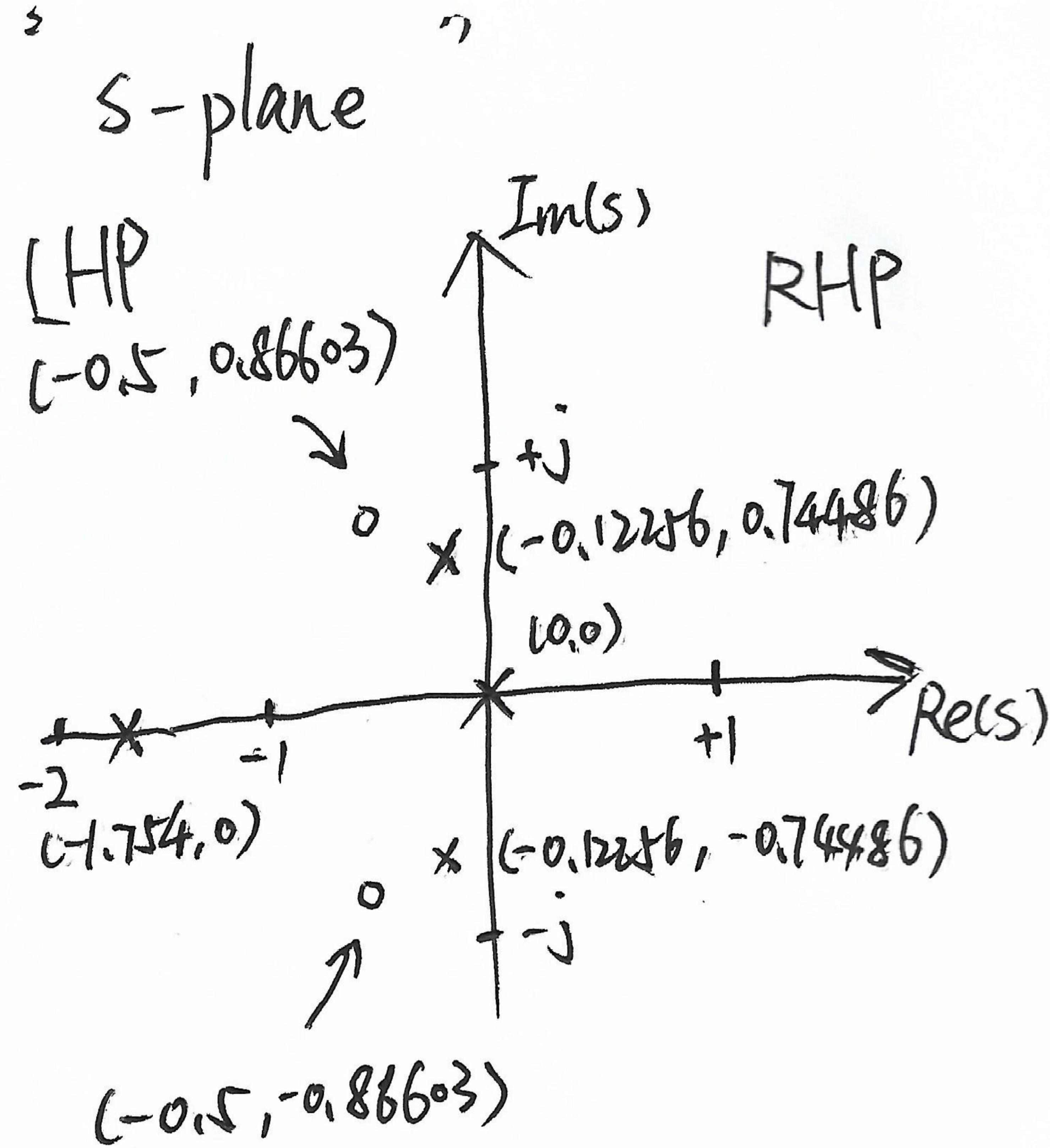
1 poles is zero, which means the system will do nothing.

it doesn't effect stable or unstable.

There is a draft of this poles.



Question 6 c)



Question 7

We have the transfer function

$$G(s) = \frac{X_2(s)}{F(s)} = \frac{s^2 + s + 1}{2s^4 + 4s^3 + 2s^2 + 2s}$$

$$\Rightarrow X_2(s) = F(s) \frac{s^2 + s + 1}{2s^4 + 4s^3 + 2s^2 + 2s}$$

when input is a unit impulse.

$$f(t) = \delta(t) \Rightarrow L(f(t)) = L(\delta(t)) = 1$$

$$\text{Hence, } F(s) = 1$$

$$X_2(s) = \frac{s^2 + s + 1}{2s^4 + 4s^3 + 2s^2 + 2s}$$

Final Value Theorem is

$$\lim_{t \rightarrow \infty} f_2(t) = \lim_{s \rightarrow 0} s X_2(s) = s \cdot \left. \frac{s^2 + s + 1}{2s^4 + 4s^3 + 2s^2 + 2s} \right|_{s=0} = \left. \frac{1}{2} \frac{s^2 + s + 1}{s^3 + 2s^2 + s + 1} \right|_{s=0} = \frac{1}{2}$$

## Question 8.

$$2\frac{d^2x_1(t)}{dt^2} + 2\frac{dx_1(t)}{dt} + (1 + 2x_1(t)) - 2\frac{dx_2(t)}{dt} = 0 \quad \text{eq. 6.}$$

$$2\frac{d^2x_2(t)}{dt^2} + 2\frac{dx_2(t)}{dt} - 2\frac{dx_1(t)}{dt} = f(t) \quad \text{eq. 7}$$

use  $\ddot{x}_1 = \frac{d^2x_1(t)}{dt^2}$     $\dot{x}_1 = \frac{dx_1(t)}{dt}$     $\ddot{x}_2 = \frac{d^2x_2(t)}{dt^2}$     $\dot{x}_2 = \frac{dx_2(t)}{dt}$

$$\left. \begin{array}{l} 2\ddot{x}_1 + 2\dot{x}_1 + 1 + 2x_1 - 2\dot{x}_2 = 0 \quad (1) \\ 2\ddot{x}_2 + 2\dot{x}_2 - 2\dot{x}_1 = f(t) \end{array} \right. \quad \begin{array}{l} 2\dot{z}_2 = -2z_1 - 2z_2 + 2z_4 - 0.5 \\ z_3 = z_4 \\ 2\dot{z}_4 = 2z_2 - 2z_4 + f(t) \end{array}$$

use the state space matrix.

$$\text{Let } z_1 = x_1, z_2 = \dot{x}_1, z_3 = x_2, z_4 = \dot{x}_2$$

we have

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = -z_1 - z_2 + z_4 - 0.5 \quad \text{— derived from (1)}$$

$$\dot{z}_3 = z_4$$

$$\dot{z}_4 = +z_2 - z_4 + \frac{1}{2}f(t) \quad \text{— derived from (2)}$$

use the matrix.

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} f(t) + \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 0 \end{pmatrix}$$

put it in python code.

i) see .py file.

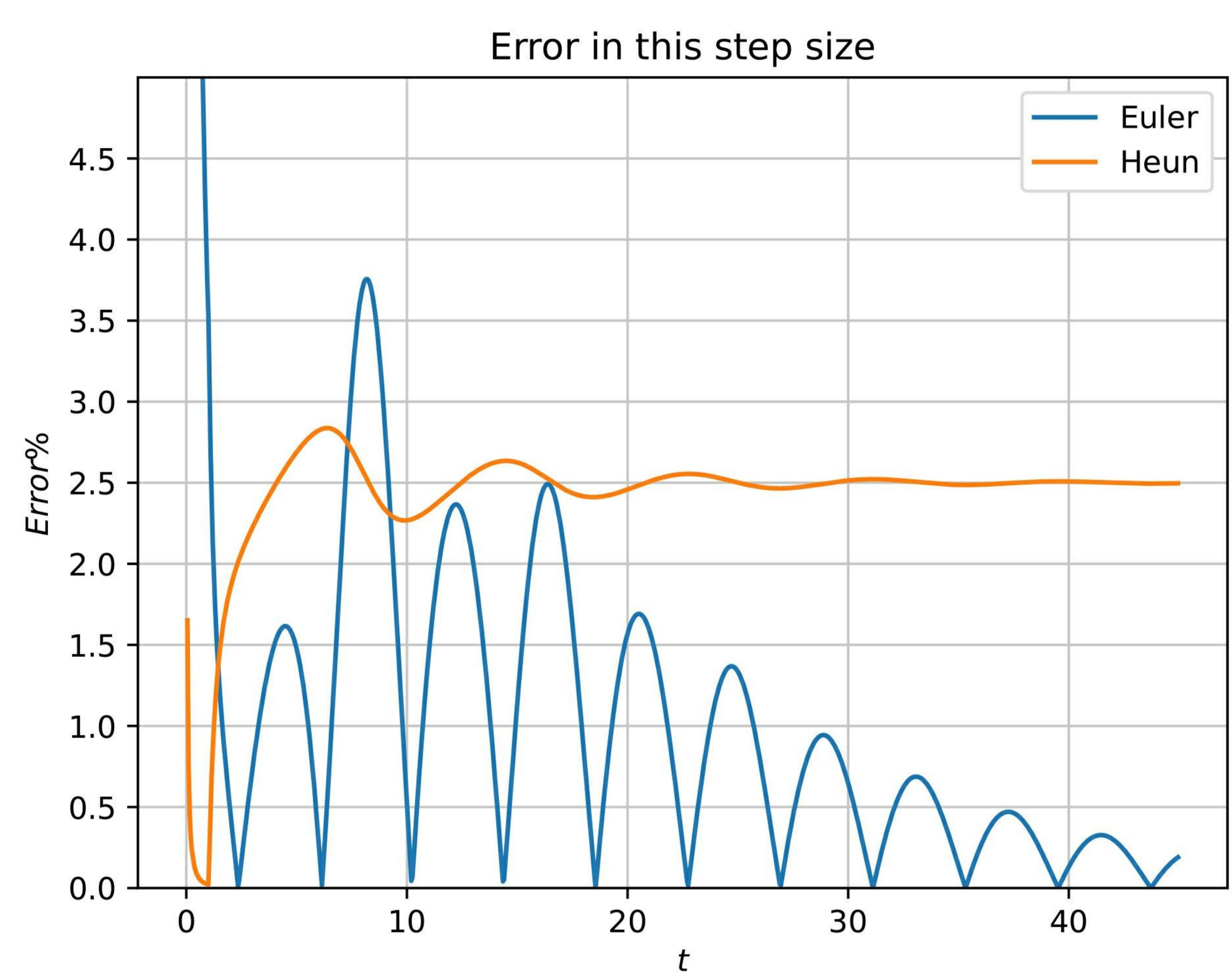
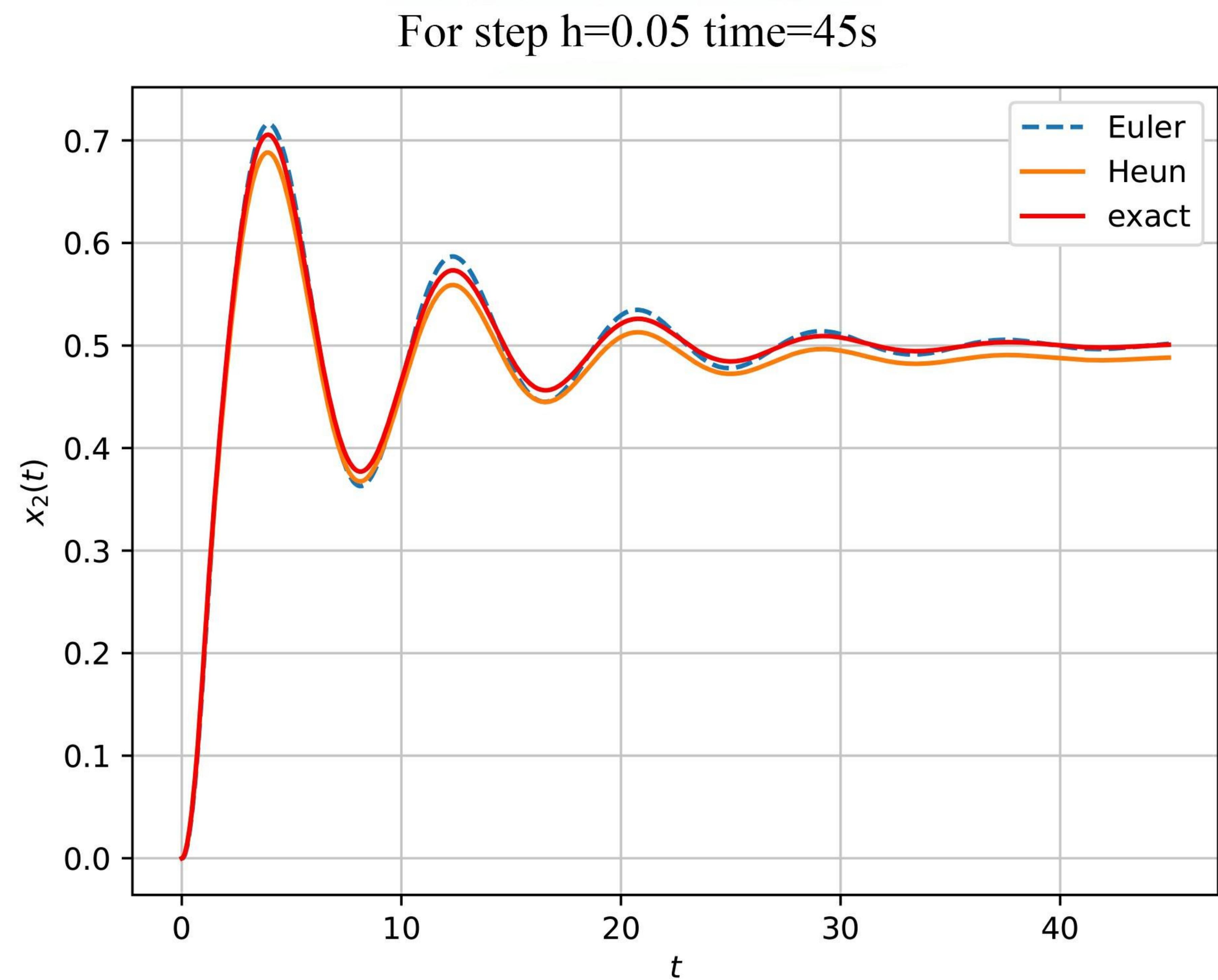
ii) Euler: set  $h=0.05$ , time = 45s.  
in the second plot (error in step size)

most of the error is larger than 1%

set  $h=0.01$ . time = 45s.

Error larger than 1% at the beginning only,  
later all errors below 1%.

So, the appropriate step size  $h=0.01$ .



Heun:

Set  $h=0.01$ , time = 45s.

All of the error are less than 1%

iii)

set  $h=0.01$ , time = 45s.

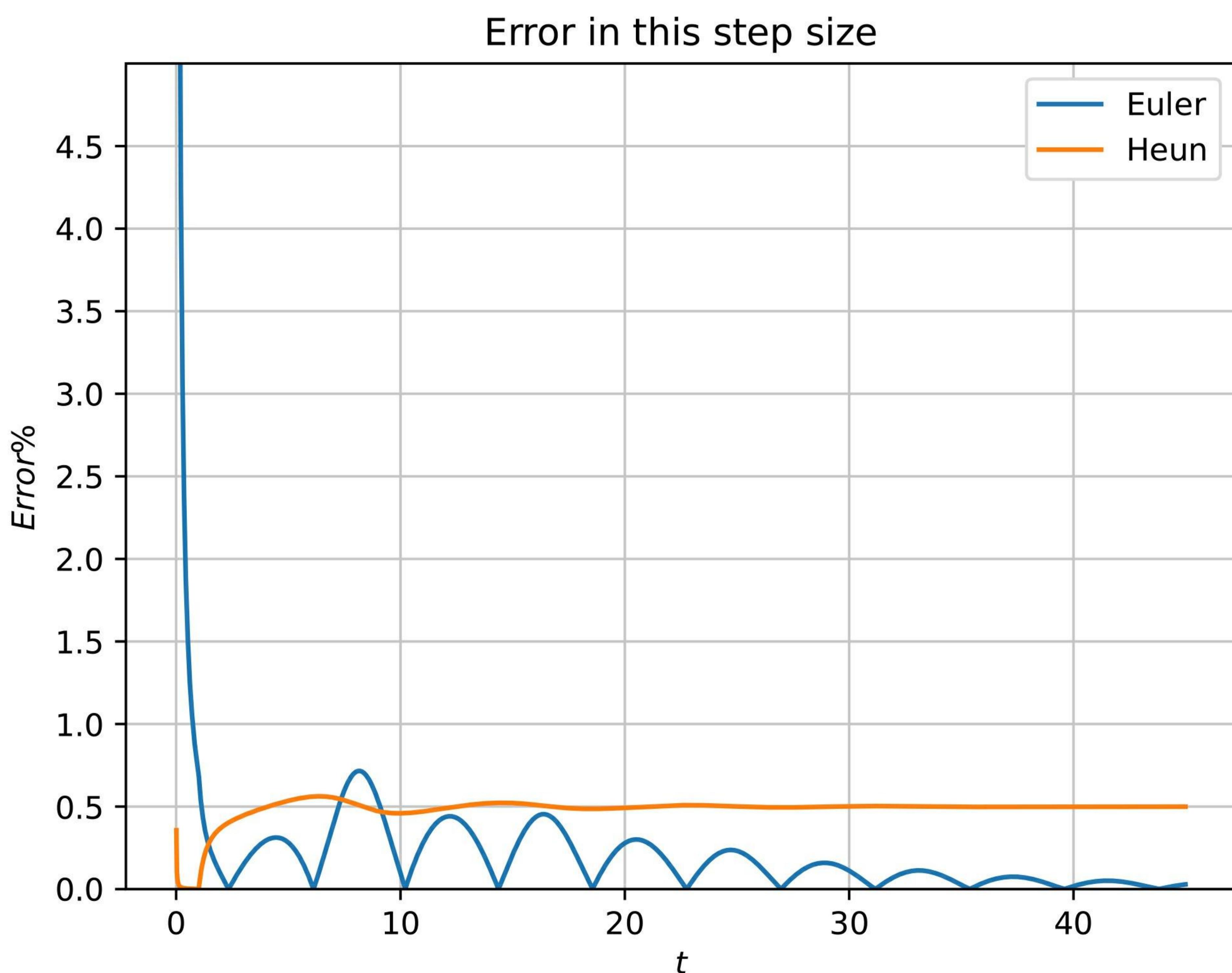
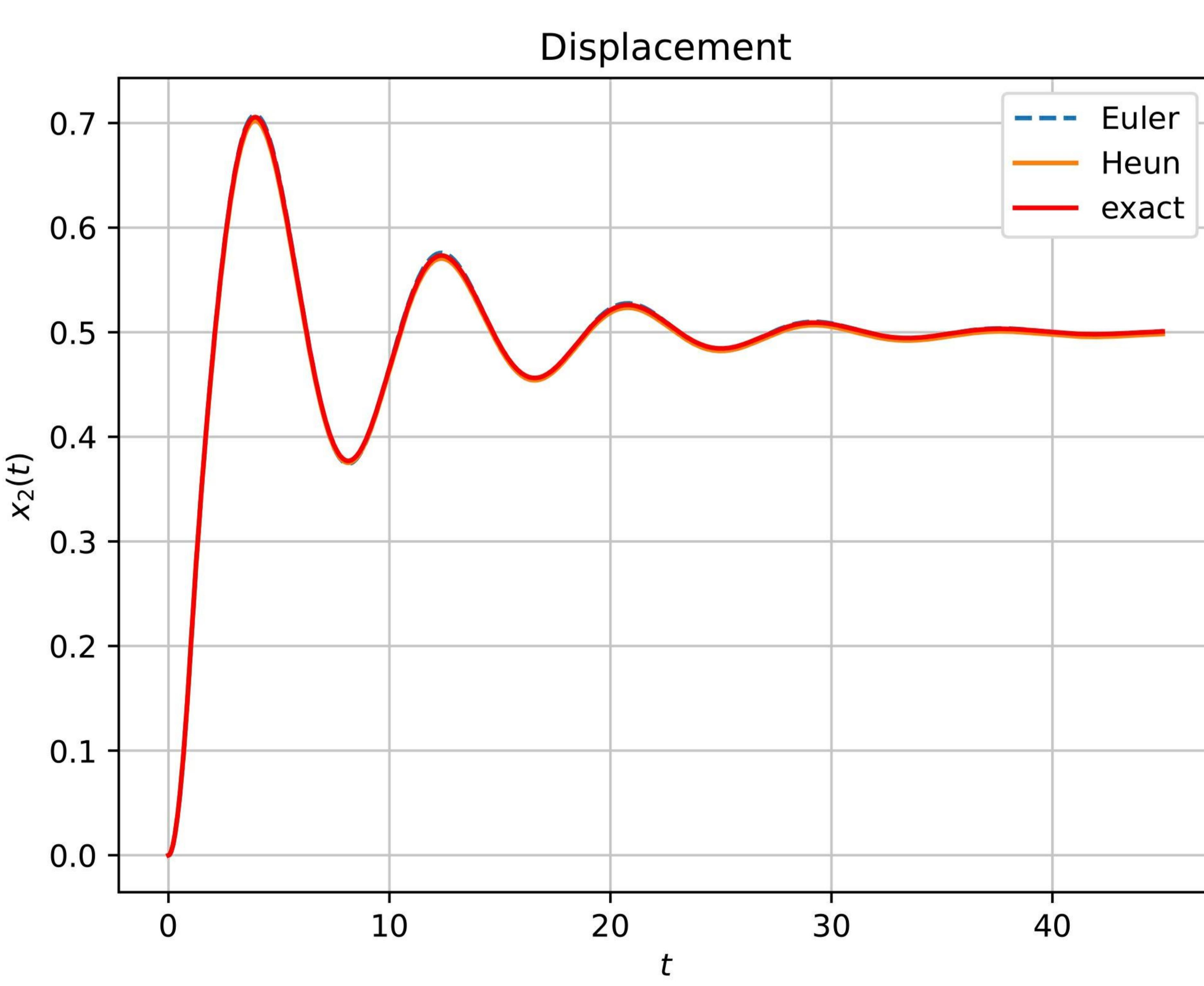
as we can see, when time = 30s,  
the Euler's error is less than 0.25%  
the Heun's error is approximately equal to 0.5%

So, choose the end time which

is 30s + 15s, in 45s.

in time = 45s

the Euler's error is approximately equal to 0.1%  
the Heun's error is approximately equal to 0.5%



(iv) Euler

$$E_t = 8.926 \times 10^{-4}$$

$$|E_t| \% = 0.1785\%$$

Heun

$$E_t = 1.751 \times 10^{-3}$$

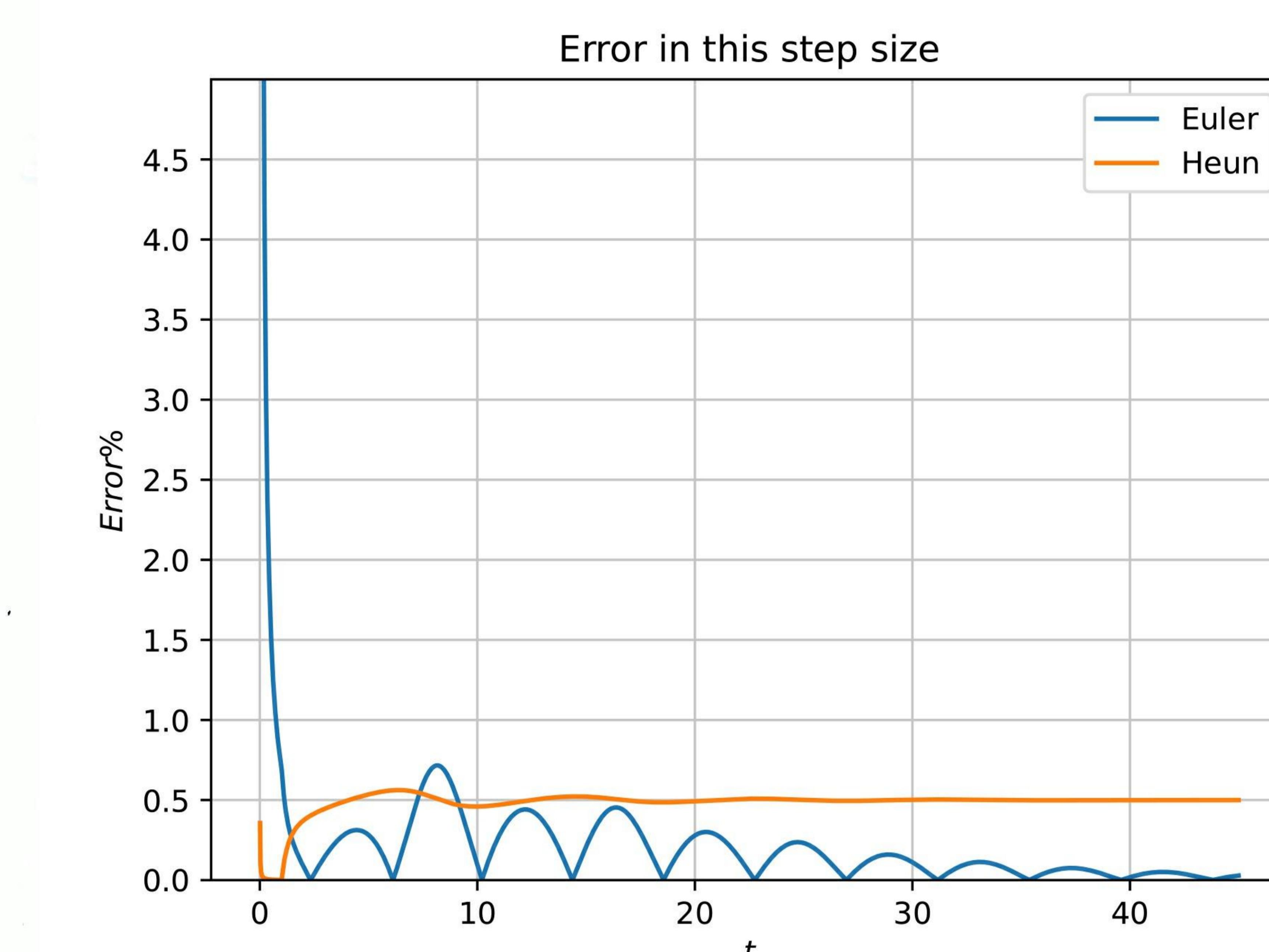
$$|E_t| \% = 0.35\%$$

Discuss

in step  $h=0.01$ , time = 45s,

Euler's method are more accurate than Heun's method.

and the ~~error~~ difference between Euler, Heun and exact is less than 1%




Terminal output

```
the final Euler method error is 0.0008925771159923324
the final Euler method error percentage is 0.17851542319846647
the final Heun method error is 0.0017513999471015507
the final Heun method error percentage is 0.35027998942031013
```

(V.)

Displacement



continue V.

discuss :

The system is an underdamped system.

reason 1: It cross its stable final value (0.5) many times.

reason 2: It has two symmetric poles in LHP, which is

$(-0.12256, \pm 0.74486)$   
and the other poles is on the coordinate axis,  
which will effect nothing with the system.

reason 3: The figure show the system is periodically damped. Additionally,  
the damp amplitude is reduce and it's final value is 0.5