ENG2009

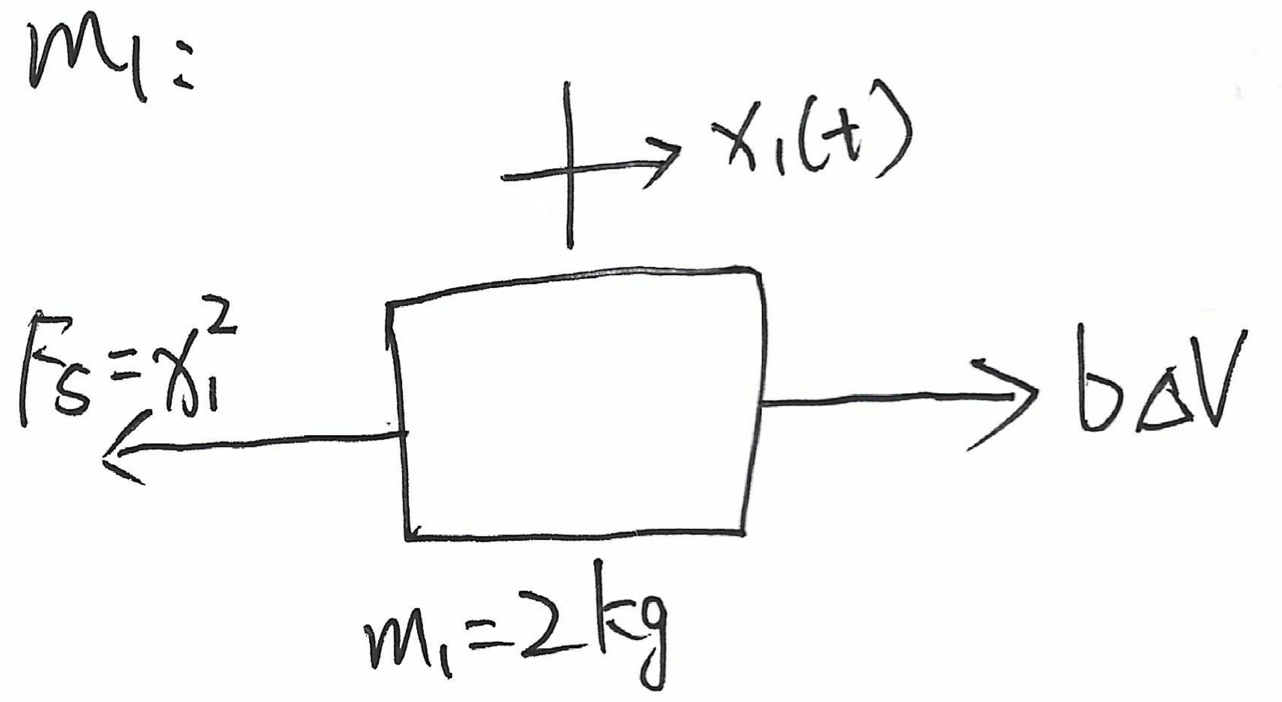
Part A (Modelling)

Question 1:

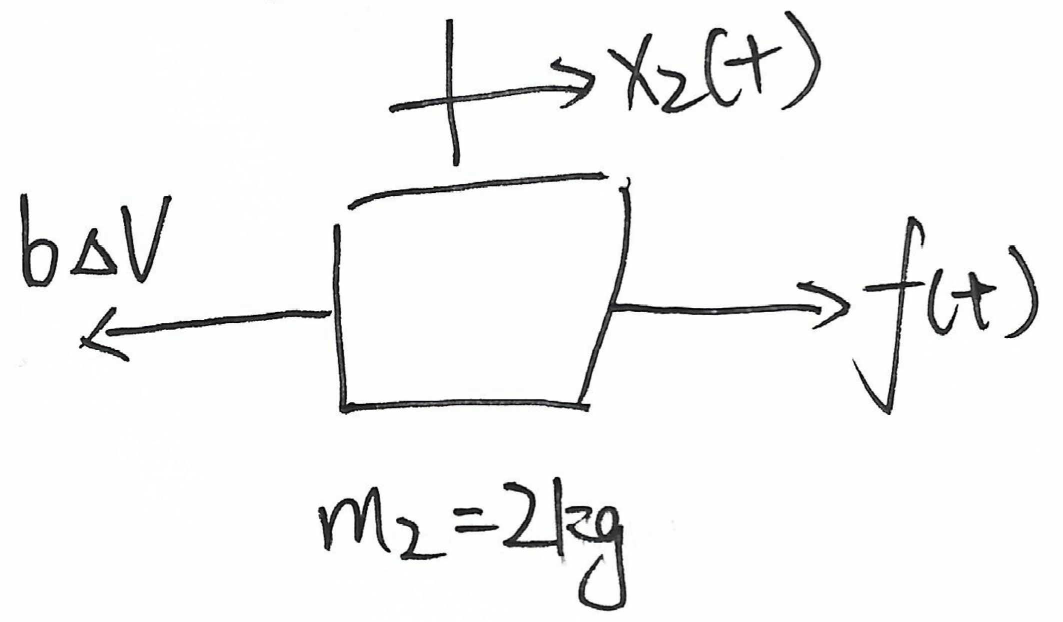
Real-life example: Car suspension system

Question 2: free-body-diagram

m_1:



m_2:



notice: ΔV is the difference between V_1 and V_2.

$\Delta V = \cancel{b} V_2 - V_1$ $\qquad F_s = x_1^2(t)$ eq(1)

use math model:

$\Sigma F_1 = m_1 a \qquad \Sigma F_2 = m_2 a$

$b\Delta V - x_1^2 = m_1 a \qquad f(t) - b\Delta V = m_2 a$

$bV_2 - bV_1 - x_1^2 = m_1 a \qquad f(t) - bV_2 + bV_1 = m_2 a$

for $V = \frac{dx(t)}{dt}$, $a = \frac{d^2x(t)}{dt^2}$, $b = 2$, $m_1 = m_2 = 2$

We have

$$m_1: 2\frac{dx_2(t)}{dt} - 2\frac{dx_1(t)}{dt} - x_1^2(t) = 2\frac{d^2x_1(t)}{dt^2}$$

$$m_2: f(t) - 2\frac{dx_2(t)}{dt} + 2\frac{dx_1(t)}{dt} = 2\frac{d^2x_2(t)}{dt^2}.$$

rearranged, we have

$$\begin{cases} 2\frac{d^2x_1(t)}{dt^2} + 2\frac{dx_1(t)}{dt} + x_1^2(t) - 2\frac{dx_2(t)}{dt} = 0 & eq(2) \\ 2\frac{d^2x_2(t)}{dt^2} + 2\frac{dx_2(t)}{dt} - 2\frac{dx_1(t)}{dt} = f(t) & eq(3) \end{cases}$$

Question 3:

Let ($t = ts + \delta t$) ($ts = 0$) and because $x_1(0)=1$ $x_2(0)=0$, $f(0)=0$.

for eq(2)

$$2\frac{d^2x_1(ts+\delta t)}{dt^2} + 2\frac{dx_1(ts+\delta t)}{dt} + x_1^2(ts+\delta t) - 2\frac{d^2x_2(ts+\delta t)}{dt^2} = 0.$$

since ts is a constant

$$\frac{d^2x_1(ts+\delta t)}{dt^2} = \frac{d^2x_1(\delta t)}{dt^2} \qquad \frac{dx_1(ts+\delta t)}{dt} = \frac{dx_1(\delta t)}{dt} \qquad f(ts+\delta t) = f(\delta t)$$

For $x_1^2(ts+\delta t)$, Let $f(t) = x_1^2(t)$.

$$f(ts+\delta t) = f(ts) + c\delta t \qquad c = \frac{df(t)}{dt}\Big|_{t=ts}.$$

$$= x_1^2(ts) + 2x_1(\delta t)$$

becaus $ts = 0$, $x_1(0) = 1$

so, $f(ts+\delta t) = 1 + 2\delta x_1(t)$

Thus, the linearise function of eq(2) and eq(3) is.

$$\begin{cases} \frac{2d^2\delta x_1(t)}{dt^2} + \frac{2d\delta x_1(t)}{dt} + (1+2\delta x_1(t)) - 2\frac{d^2\delta x_2(t)}{dt} = 0 & eq(4) \\ 2\frac{d^2\delta x_2(t)}{dt^2} + 2\frac{d\delta x_2(t)}{dt} - 2\frac{d\delta x_1(t)}{dt} = \delta f(t) & eq(5) \end{cases}$$

Part B (Analytical and numerical methods)

$$\begin{cases} 2\frac{d^2 x_1(t)}{dt^2} + 2\frac{dx_1(t)}{dt} + (1+2x_1(t)) - 2\frac{dx_2(t)}{dt} = 0 & eq(6) \\ 2\frac{d^2 x_2(t)}{dt^2} + 2\frac{dx_2(t)}{dt} - 2\frac{dx_1(t)}{dt} = f(t) & eq(7). \end{cases}$$

Question 4.

Laplace transform:

$$L\left(\frac{d^2 x_1(t)}{dt^2}\right) = L(\ddot{x}_1(t)) = s^2 X_1(s) - sx_1(0) - \dot{x}(0)$$

$$L\left(\frac{dx_1(t)}{dt}\right) = L(\dot{x}_1(t)) = sX_1(s) - x(0)$$

$$L(f(t)) = F(s)$$

$$L(x_1(t)) = X_1(s)$$

Because ignoring the linearisation point (l is the linearisation point)

so l could be ignored.

Hence,

$$eq(6) \Rightarrow 2\left[s^2 X_1(s) - sX_1(0) - \dot{X}_1(0)\right] + 2\left[sX_1(s) - X_1(0)\right] + 2X_1(s)$$
$$-2\left[sX_2(s) - X_2(0)\right] = 0.$$

$$eq(7) \Rightarrow 2\left[s^2 X_2(s) - sX_2(0) - \dot{X}_2(0)\right] + 2\left[sX_2(s) - X_2(0)\right] - 2\left[sX_1(s) - X_1(0)\right] = F(s)$$

use $X_1(0) = 0,\ X_2(0) = 0,\ \dot{X}_1(0) = 0,\ \dot{X}_2(0) = 0.$

so,

$$eq(6) \Rightarrow 2s^2 X_1(s) + 2sX_1(s) + 2X_1(s) - 2sX_2(s) = 0 \Rightarrow (s^2 + s + 1)X_1(s) = sX_2(s)$$

$$eq(7) \Rightarrow 2s^2 X_2(s) + 2sX_2(s) - 2sX_1(s) = F(s) \Rightarrow (2s^2 + 2s)X_2(s) - 2sX_1(s) = F(s)$$

the solution of question 4 is

$$\begin{cases} (s^2+s+1)X_1(s) - sX_2(s) = 0 \\ (2s^2+2s)X_2(s) - 2sX_1(s) = F(s) \end{cases}$$

Question 5. a)

the definition of transfer function is

$$G(s) = \frac{\text{output}}{\text{input}} = \frac{X_2(s)}{F(s)}$$

Solve the equation:

$$\begin{cases} (s^2+s+1)X_1(s) - sX_2(s) = 0 \\ (2s^2+2s)X_2(s) - 2sX_1(s) = F(s) \end{cases}$$

$$X_1(s) = \frac{s}{s^2+s+1} X_2(s)$$

$$(2s^2+2s)X_2(s) - 2s \cdot \frac{s}{s^2+s+1} X_2(s) = F(s).$$

$$\frac{2s^4+2s^3+2s^2+2s^3+2s^2+2s-2s^2}{s^2+s+1} X_2(s) = F(s).$$

$$\frac{2s^4+4s^3+2s^2+2s}{s^2+s+1} X_2(s) = F(s)$$

So,

$$G(s) = \frac{X_2(s)}{F(s)} = \frac{s^2+s+1}{2s^4+4s^3+2s^2+2s}$$

b) 4^{th} order system (the highest power of denominator).

Question 6. a)

zeros: roots of numerator:

set $s^2+s+1=0$

$$S=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-1\pm\sqrt{1-4}}{2}=-0.5\pm0.86603i$$

zero: $S_1=-0.5+0.86603i$ $S_2=-0.5-0.86603i$

poles: roots of denominator

set $2s^4+4s^3+2s^2+2s=0$

$$s(s^3+2s^2+s+1)=0.$$

We know the three poles are:

$S_1=-0.12256+0.74486i$

$S_2=-0.12256-0.74486i$

$S_3=0$

and $S_4\in(-2,0)$

use the Bisection method to find the other real pole.

set $F(s)=s^3+2s^2+s+1$

in order to use the bisection method,

we need to determine the monotonicity of $F(s)$.

$F'(s) = 3s^2+4s+1$, the minimum value is $\frac{4ac-b^2}{4a} = 1-\frac{16}{12} = -\frac{1}{3}$

the roots are. $\frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-4\pm\sqrt{4}}{6}$, which are $-\frac{1}{3}$ and -1

so, when $s\in(-\infty,-\frac{1}{3})$, $F'(s)>0$. $F(s)$ monotonically increasing.

$s\in(-\frac{1}{3},-1)$, $F'(s)<0$, $F(s)$ monotonically decreasing.

$s\in(-1,+\infty)$, $F'(s)>0$, $F(s)$ monotonically increasing.

Hence, there is a local minimum value in $s=-\frac{1}{3}$, local maximum value $s=-1$

$F(-1) = 1 > 0$. $F(-\frac{1}{3}) = 0.8519$

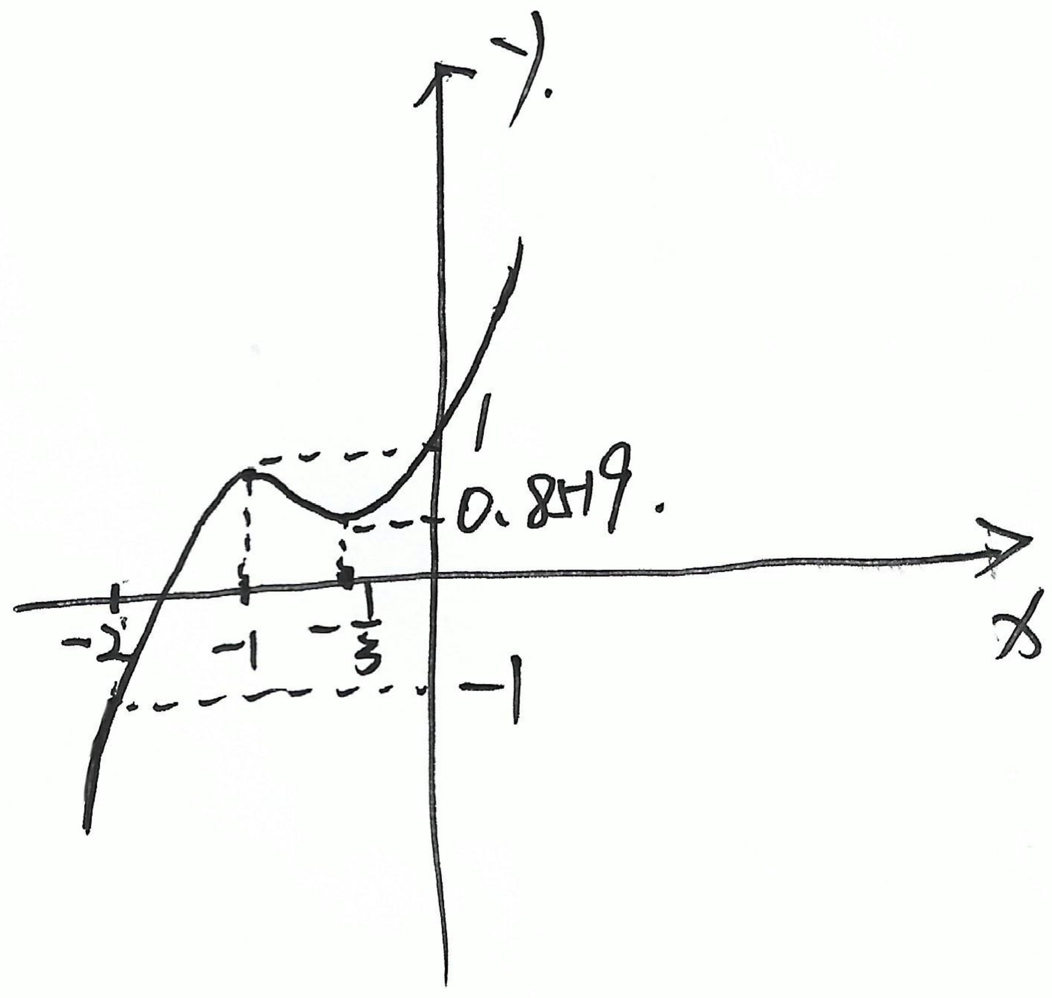
because the other real roots of $F(s)$ is between -2 and 0.

and $F(s)$ monotonically incresing between $(-\infty,-\frac{1}{3})$

So, the real roots must between $(-2,-1)$.

$F(-2)=-1$

the draft of $F(s)$ is on the left side.



use the bisection method:

$F(\frac{-2-1}{2}) = F(-1.5) = 0.625 > 0$. roots must in $(-2,-1.5)$

$F(\frac{-2-1.5}{2}) = F(-1.75) = 0.0156 > 0$. roots must in $(-2,-1.75)$

$F(\frac{-2-1.75}{2}) = F(-1.875) = -0.4355 < 0$. roots in $(-1.875,-1.75)$

$F(\frac{-1.875-1.75}{2}) = F(-1.8125) = -0.1965 < 0$ roots in $(-1.8125,-1.75)$

$F(\frac{-1.8125-1.75}{2}) = F(-1.7813) = -0.0872 < 0$ roots in $(-1.7813,-1.75)$.

$F\left(\frac{-1.7813-1.75}{2}\right) = F(-1.7657) = -0.0351 < 0$ roots in $(-1.7657, -1.75)$.

$F\left(\frac{-1.7657-1.75}{2}\right) = F(-1.7579) = -0.0096 < 0$ roots in $(-1.7579, -1.75)$

$F\left(\frac{-1.7579-1.75}{2}\right) = F(-1.7540) = 0.0028 > 0$.

notice that $(0.0028-0) \times 100\% = 0.28\% < 1\%$

So we can assume that -1.7540 approximately equal to the real roots.

So, the zeros of the transfer function is.

$$\begin{cases} S_1 = -0.5 + 0.86603i \\ S_2 = -0.5 - 0.86603i \end{cases}$$

the poles ~~of~~ is

$$\begin{cases} S_1 = -0.12256 + 0.74486i \\ S_2 = -0.12256 - 0.74486i \\ S_3 = 0 \\ S_4 = -1.754 \end{cases}$$

Question 6 b).

the system is stable.

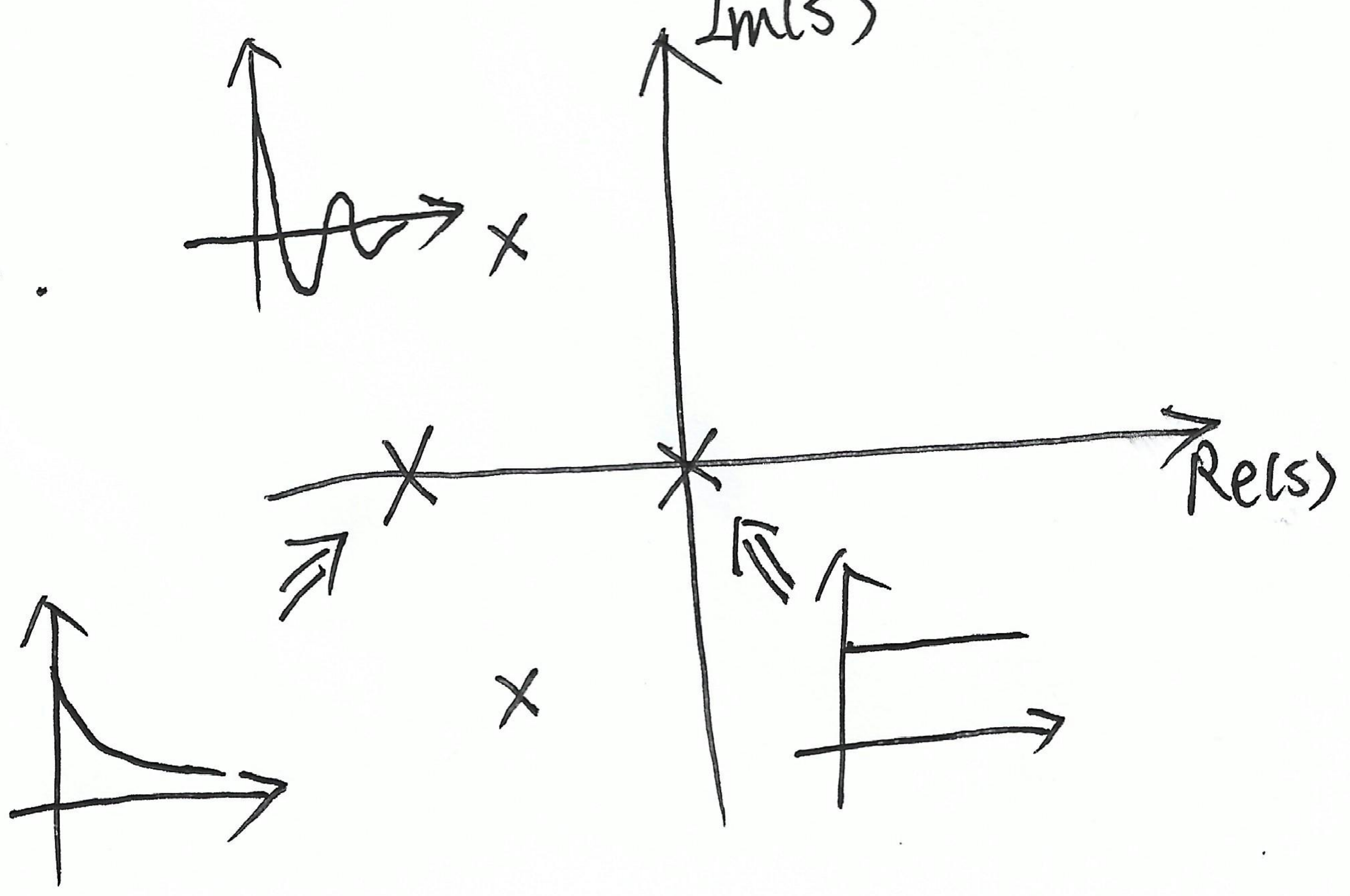
we have 4 poles.

3 poles have negative real roots which are -0.12256, -0.12256 and -1.754

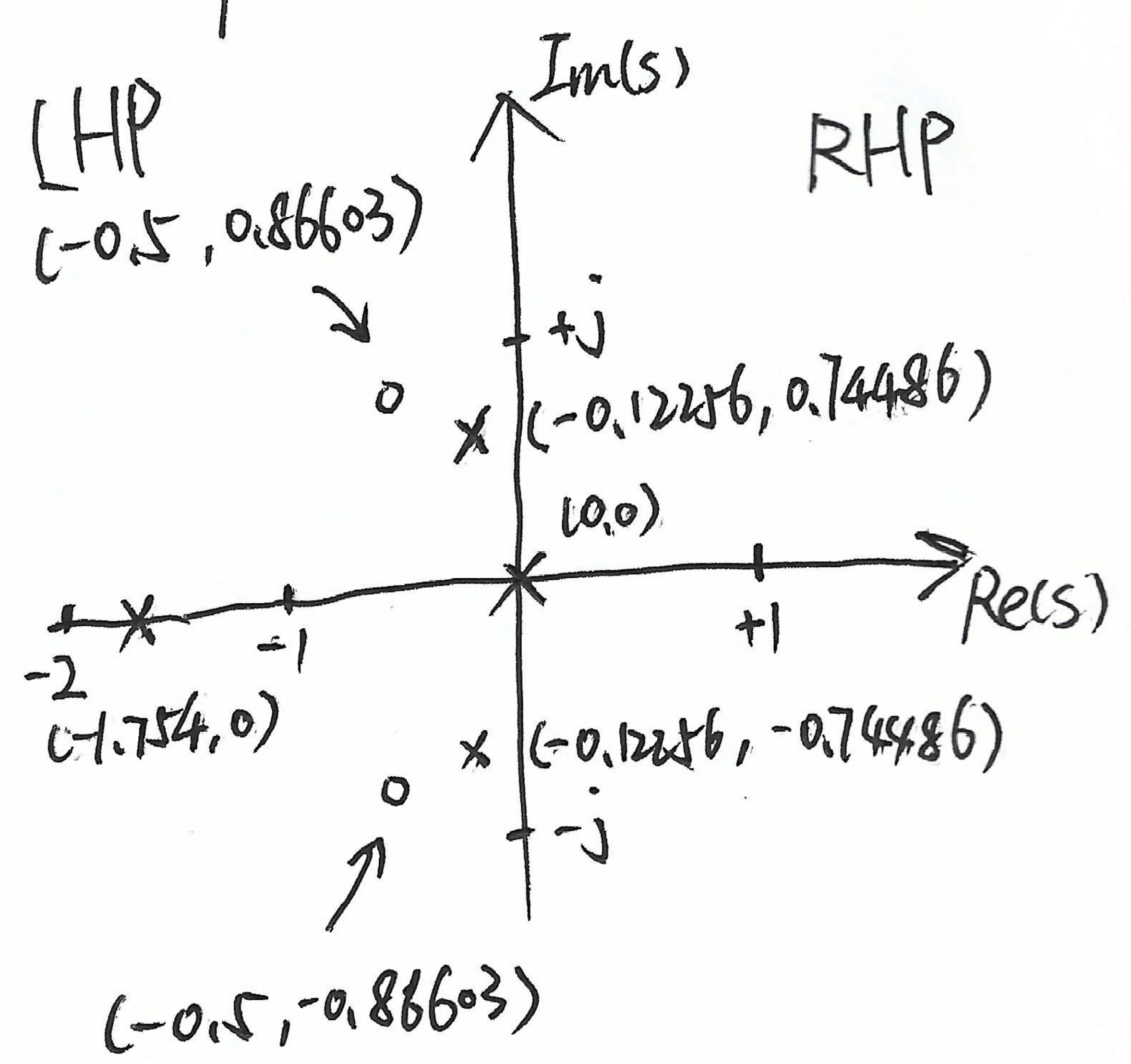
1 poles is zero, which means the system will do nothing.

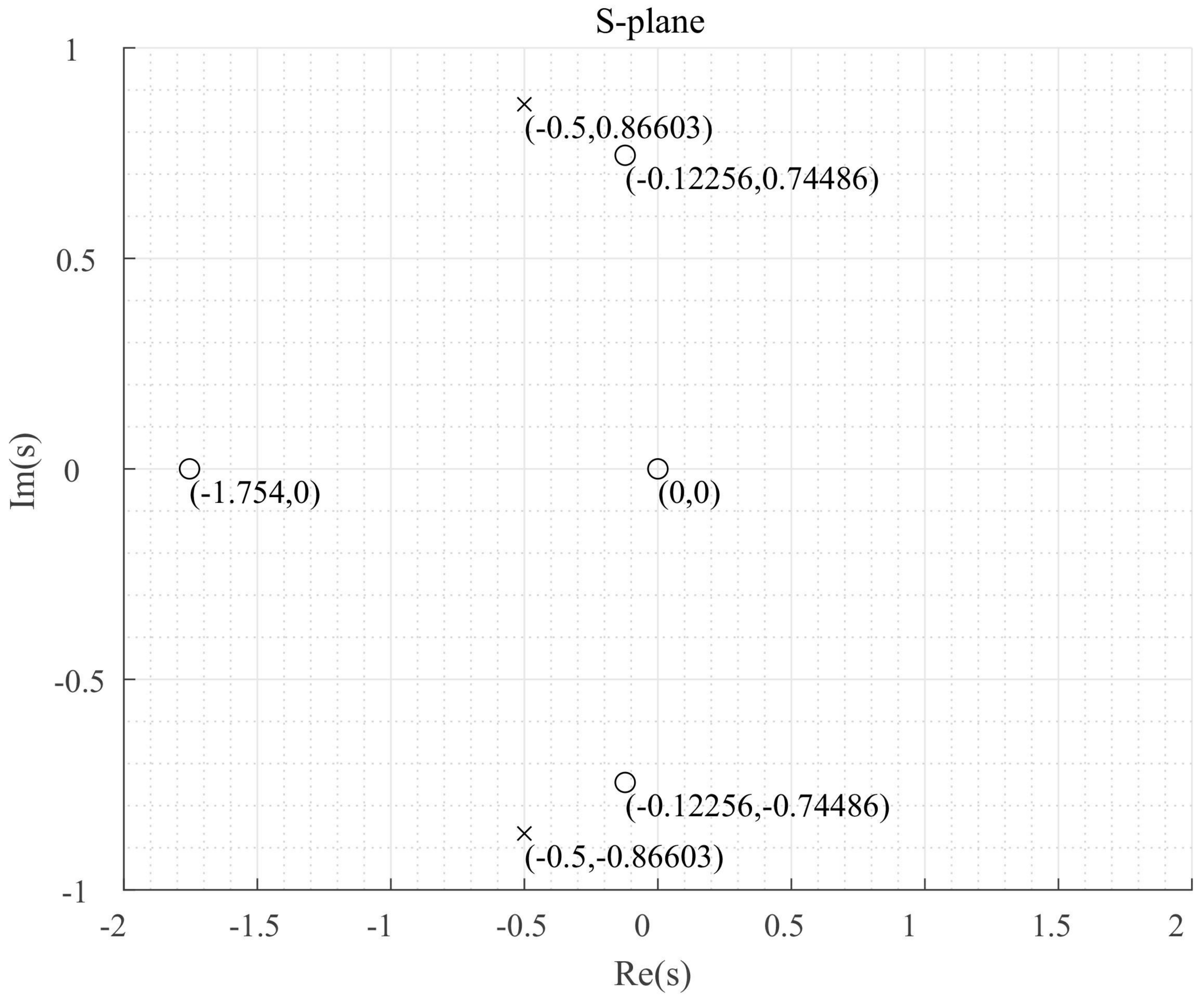
it doesn't effect stable or unstable.

There is a draft of this poles.



Question 6 c)





Question 7

We have the transfer function

$$G(s) = \frac{X_2(s)}{F(s)} = \frac{s^2+s+1}{2s^4+4s^3+2s^2+2s}$$

$$\Rightarrow X_2(s) = F(s)\frac{s^2+s+1}{2s^4+4s^3+2s^2+2s}$$

when input is a unit impulse.

$$f(t) = \delta(t) \Rightarrow \mathcal{L}(f(t)) = \mathcal{L}(\delta(t)) = 1$$

Hence, $F(s) = 1$

$$X_2(s) = \frac{s^2+s+1}{2s^4+4s^3+2s^2+2s}$$

Final Value Theorem is

$$\lim_{t\to\infty} x_2(t) = \lim_{s\to 0} sX_2(s) = s\cdot\frac{s^2+s+1}{2s^4+4s^3+2s^2+2s}\bigg|_{s=0} = \frac{1}{2}\,\frac{s^2+s+1}{s^3+2s^2+s+1}\bigg|_{s=0} = \frac{1}{2}$$