Computational Physics Homework 4

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Summary

This assigment presents two numerical studies solved with adaptive integrators. First, the Brusselator chemical oscillator (Exercise 8.18) was integrated for parameters $a = 1$, $b = 3$ from $t = 0$ to $t = 20$ with a stringent per-unit-time accuracy target; the adaptive algorithm concentrates steps where the solution varies rapidly and yields the expected oscillatory behavior. Second, the orbital evolution of one component of a symmetric supermassive black hole binary was studied under a central potential and a simplified dynamical friction (DF) model. Using an adaptive fourth-order Runge–Kutta integrator I (a) validated conservative dynamics to choose an accuracy tolerance, (b) integrated an $A = B = 1$ DF run stopping when the radius reached the Schwarzschild scale $r_s = 10^{-7}$ (code units), (c) scanned inspiral time versus the ratio B/A across $A, B \in [0.5, 10]$, and (d) tested sensitivity to initial tangential speed factors 0.4, 0.8 and 1.5 of the circular speed. Main conclusions: adaptive methods effectively allocate effort to stiff regions; inspiral time first decreases with B/A then increases; the time is not sensitive to the initial speed.

Methods

- Brusselator: integrate

$$\dot{x} = 1 - (b+1)x + ax^2y, \qquad \dot{y} = bx - ax^2y$$

for $a = 1$, $b = 3$, $x(0) = y(0) = 0$, $t \in [0, 20]$, enforcing a strict per-unit-time accuracy.

- Black hole binary: integrate the equation of motion

$$\ddot{\mathbf{r}} = -\frac{1}{4r^3}\mathbf{r} - \frac{A}{v^3 + B}\mathbf{v},$$

where $r = \|\mathbf{r}\|$ and $v = \|\mathbf{v}\|$. Tasks included conservative validation, a sample DF run $A = B = 1$, a parameter scan of inspiral time over $(A, B) \in [0.5, 10]^2$, and tests of initial tangential speed sensitivity.

Units and diagnostics - Nondimensional choices: $G = M = 1$, distance unit $L_0 = 100$ pc; in these code units $r_s = 10^{-7}$. - State ordering: Brusselator state $[x, y]$; BH integrator state $[x, y, v_x, v_y]$. - Diagnostics: for the conservative BH test monitor energy $E = \frac{1}{2}v^2 - 1/(4r)$ and angular momentum $L = xv_y - yv_x$; for DF runs record time of first crossing $r \le r_s$, and log per-run statistics.

Results

Brusselator - The adaptive Bulirsch–Stoer strategy produced the expected oscillatory evolution for $x(t)$ and $y(t)$. Recorded interval boundary markers cluster where the solution gradients are largest, illustrating effective adaptivity and efficient allocation of computational effort. See Fig. 1.

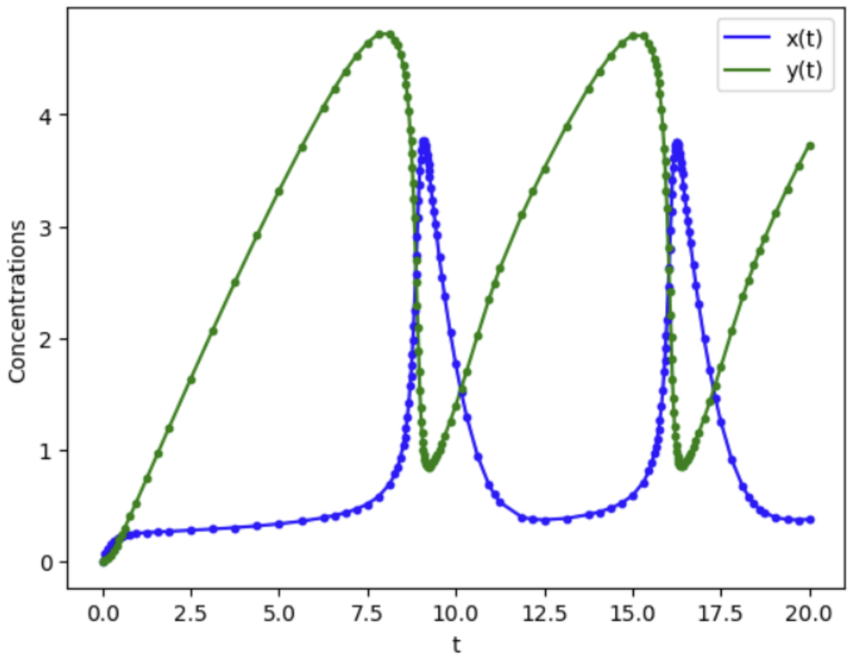

Figure 1: adaptive Bulirsch–Stoer strategy

Black hole binary with dynamical friction
- Conservative validation: using the analytic initial tangential speed tuned to yield periapsis $r_{\text{peri}} = 10^{-7}$, the adaptive RK4 conserved energy and angular momentum to small relative variations over ten orbital periods at the validated tolerance ($d = 10^{-8}$). See Fig. 2
- Representative DF run (A=B=1): the BH exhibits inward spiraling with shrinking periapsides; the $\log_{10} r(t)$ oscillates until the stopping radius r_s is reached. See Fig. 3 4
- Dependence on B/A: Scanning $A, B \in [0.5, 10]$ shows inspiral time first decreases with B/A then increases. See Fig. 5 6
- Initial velocity sensitivity: changing the initial tangential factor from 0.8 to 0.4 and 1.5 produces slight shifts in inspiral time. The inspiral time is insensitive to the initial velocity. See Fig. 7 8

Code

See https://github.com/hw3926/Computational-Physics/tree/main/HW4

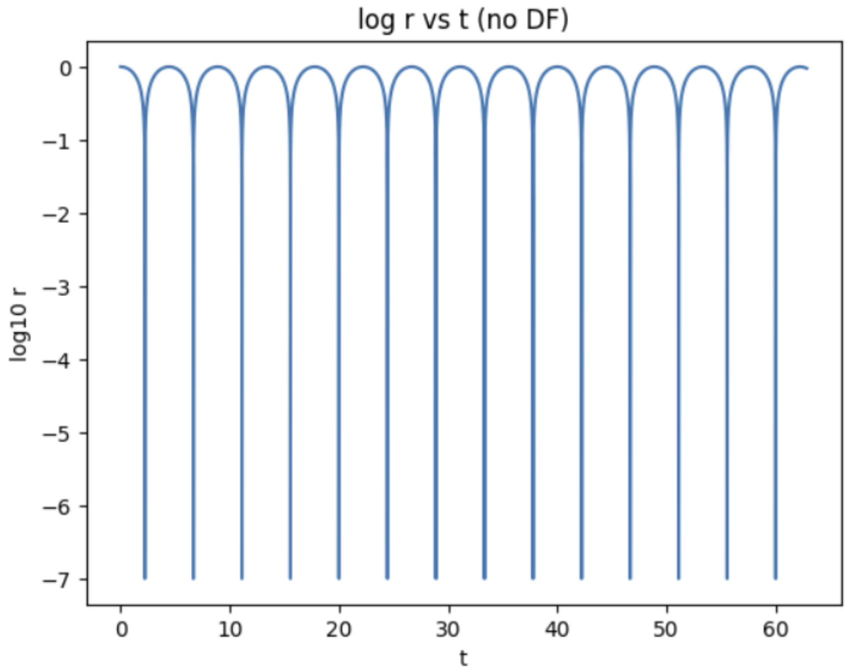

Figure 2: validated tolerance at $d = 10^{-8}$

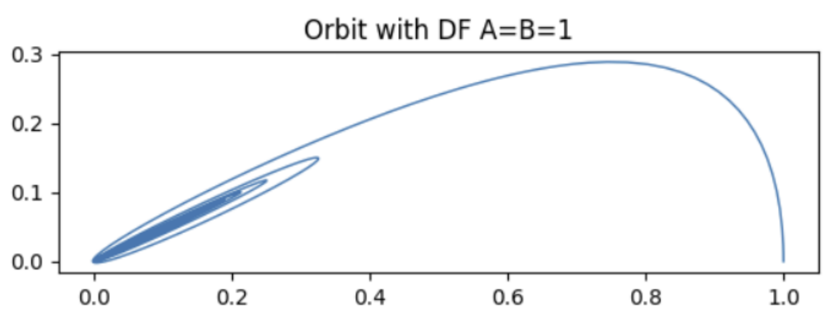

Figure 3: orbit

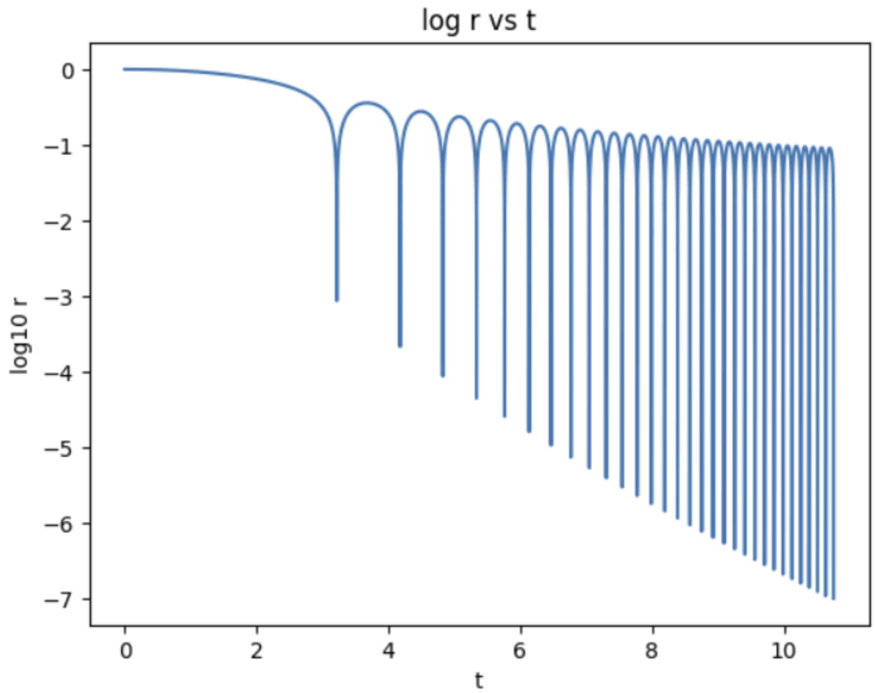

Figure 4: $\log\, r - t$

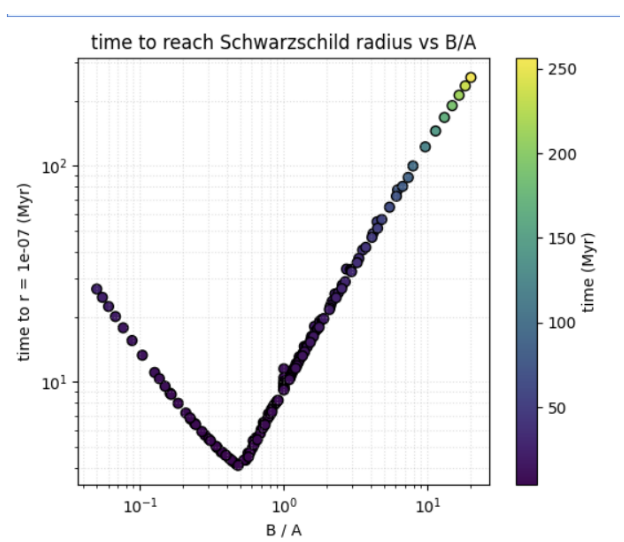

Figure 5: $t-\ B/A$ at $v_{\text{inital}} = 0.8\ v_{\text{circular orbit}}$

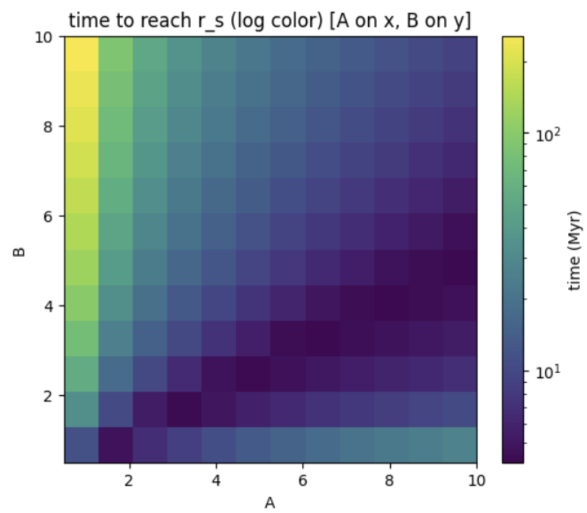

Figure 6: $t-\ A, B$ at $v_{\text{inital}} = 0.8\ v_{\text{circular orbit}}$

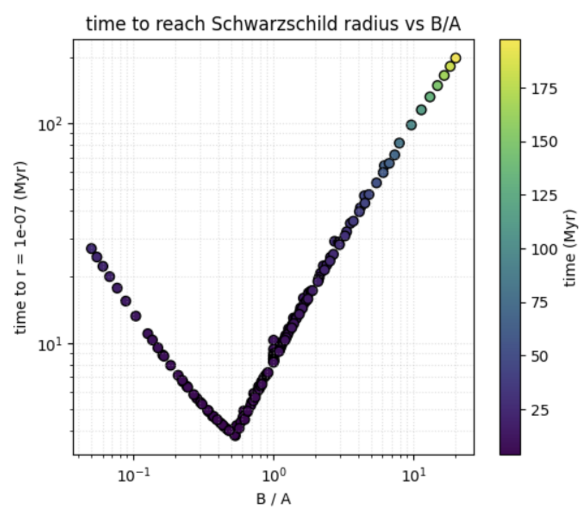

Figure 7: $t-\ B/A$ at $v_{\text{inital}} = 0.4\ v_{\text{circular orbit}}$

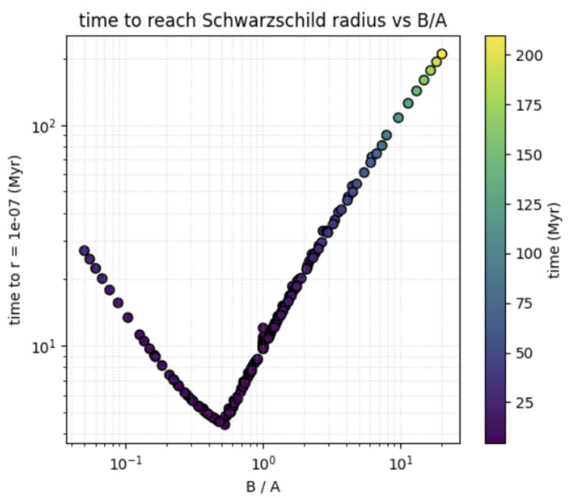

Figure 8: $t-$ B/A at $v_{\text{inital}} = 1.5\ v_{\text{circular orbit}}$