Computational Physics Homework 6

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Summary

This report details the computational study of three distinct physical systems involving stochastic processes. First, we simulated the 2D Ising model to investigate ferromagnetic phase transitions. Using the Metropolis-Hastings algorithm, we observed that the system spontaneously breaks symmetry and develops a non-zero magnetization below the critical temperature. Second, we addressed the "Dimer Covering" optimization problem using Simulated Annealing. Our results demonstrate that the density of the final packing is critically dependent on the cooling schedule; a slower exponential decay allows the system to escape local "jammed" minima. Finally, we analyzed the spectral properties of stochastic signals. By generating a random walk from Gaussian white noise, we confirmed via Fourier analysis that the power spectrum transitions from a flat profile (k^0) for uncorrelated noise to a Brownian profile (k^{-2}) for the integrated walk.

Methods

Problem 1: The 2D Ising Model

We modeled a ferromagnetic material using a square lattice of size $N = 20 \times 20$, where each site i hosts a spin $s_i = \pm 1$. The total energy is determined by the nearest-neighbor Hamiltonian $E = -J \sum_{\langle i,j \rangle} s_i s_j$, where $J = 1.0$. We simulated the system using the Metropolis-Hastings algorithm. At each step, a single spin flip was proposed and accepted with probability $P = \min(1, e^{-\Delta E / k_B T})$. We performed simulations at dimensionless temperatures $T = 1.0$, $T = 2.0$, and $T = 3.0$ for 10^6 steps.

Problem 2: Dimer Covering via Simulated Annealing

The objective was to maximize the number of non-overlapping dimers on a 50×50 grid ($E = -N_{\text{dimers}}$). The system was evolved using Simulated Annealing with an exponential cooling schedule $T(t) = T_0 e^{-t/\tau}$. We tested different cooling time constants. The Markov chain utilized both "addition" moves (filling empty spots) and "removal" moves (deleting existing dimers with probability $e^{-1/T}$) to prevent the system from becoming trapped in local minima.

Problem 3: Spectral Analysis of Random Walks

We implemented a Linear Congruential Generator (LCG) with parameters $a = 1664525$, $c = 1013904223$, and $m = 2^{32}$ to produce uniform random numbers. These were transformed into Gaussian variables via the Box-Muller transform. A 1D random walk $x(t)$ was constructed by cumulatively summing these steps: $x(t) = \sum x(t-1) + \delta_t$. We computed the power spectrum $P(k) = |\tilde{x}(k)|^2$ using the Fast Fourier Transform (FFT).

Results and Discussion

1. Phase Transition in the Ising Model

At $T = 1.0$, the system rapidly evolved from a random configuration ($M \approx 0$) to a highly ordered state ($M \to \pm 400$). As shown in Figure 1, the magnetization stabilizes at a non-zero value, confirming spontaneous symmetry breaking. At $T = 3.0$, thermal fluctuations dominated, keeping the net magnetization near zero.

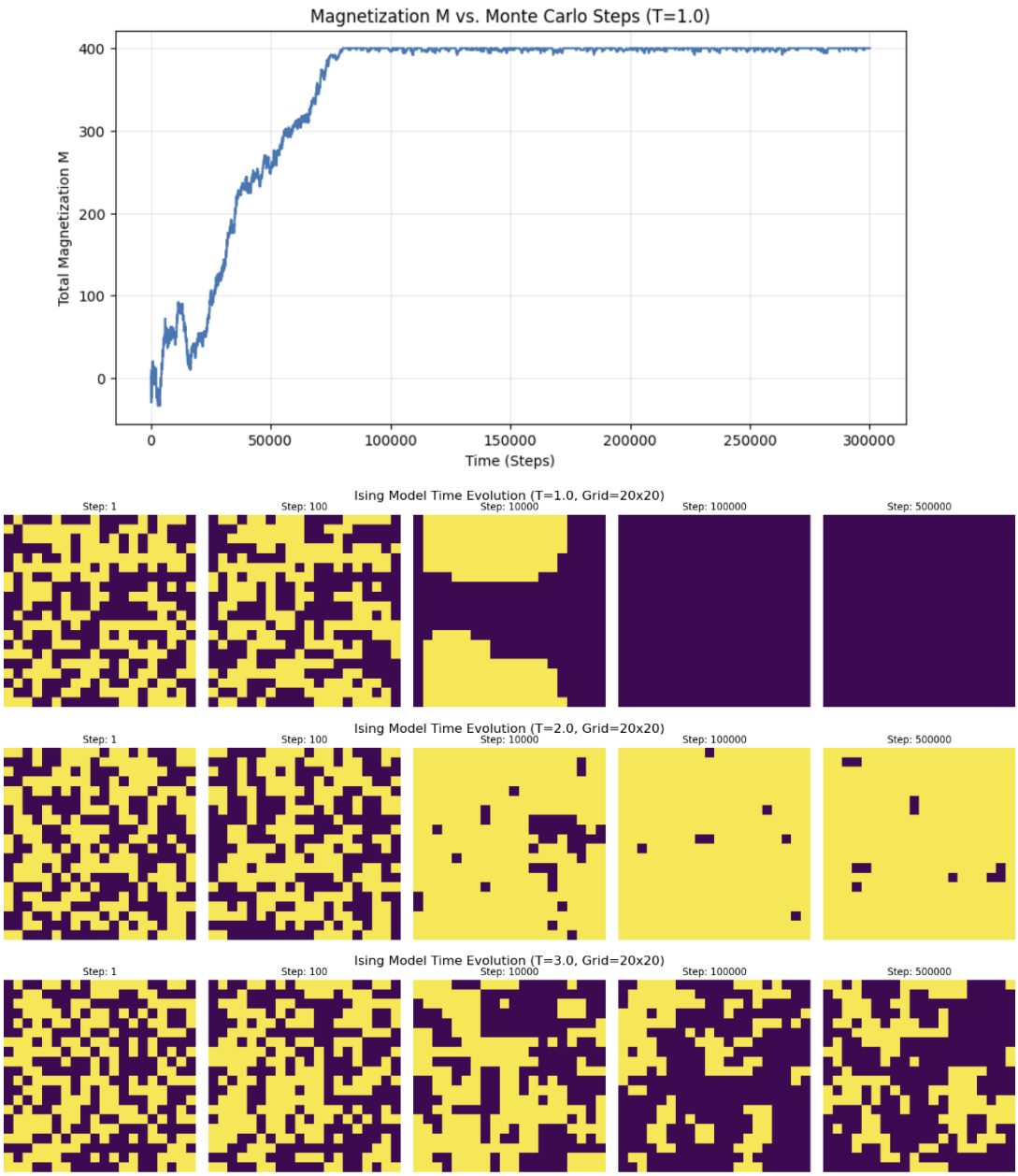

Figure 1: Time evolution of the total magnetization M for the 2D Ising model. At $T = 1.0$, the system spontaneously magnetizes, while at $T = 3.0$, it remains in a disordered paramagnetic state.

2. Dimer Covering and Cooling Schedules

Figure 2 illustrates the evolution of the lattice from a sparse early state to a dense final covering. The efficiency of this packing was strictly determined by the cooling rate τ. As summarized in Table 1, fast cooling ($\tau = 100$) resulted in a "jammed" state with significantly lower coverage. Slow cooling ($\tau = 50,000$) allowed the system to reorganize, achieving a near-perfect filling fraction.

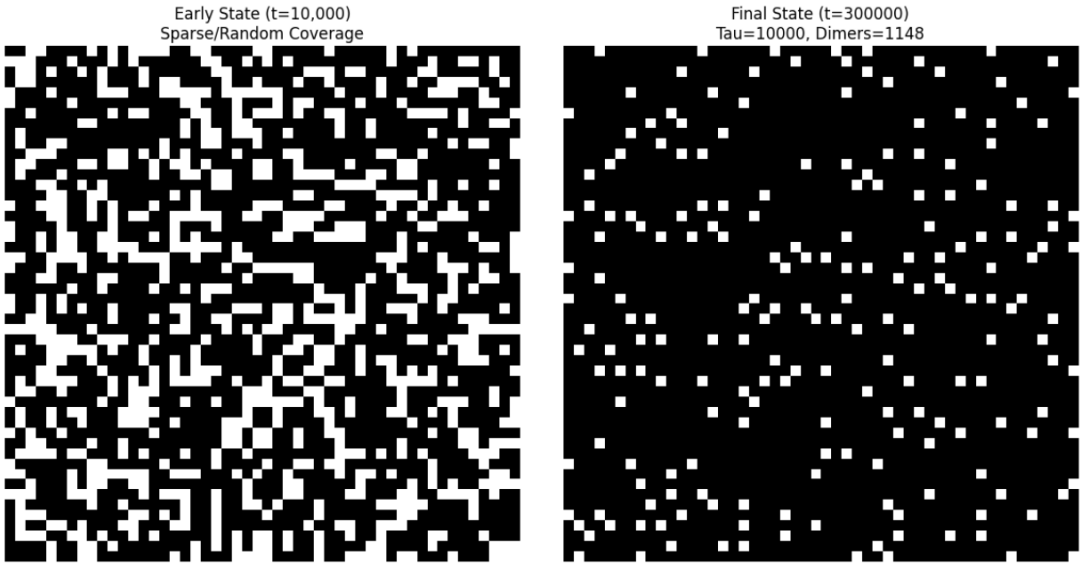

Figure 2: Snapshots of the Dimer Covering system. Left: Early state (random/sparse). Right: Final state (dense packing) after simulated annealing.

Time Constant (τ)	**Final Dimer Count**	**Fill Percentage**
100	790	63.20%
1000	1,138	91.04%
10000	1,148	91.84%
50000	1,169	93.52%

Table 1: Dependence of the final dimer coverage density on the exponential cooling time constant τ.

3. Power Laws in Spectral Analysis

We verified our generator by ploting the histogram (Figure 3) and calculating the spectrum (Figure 4). Then we studied the random walk process (Figure 5). The spectral analysis in Figure 6 confirmed the theoretical scaling for Brownian motion. The power spectrum follows a linear trend on the log-log scale with a slope of -2, consistent with the $P(k) \propto k^{-2}$ power law predicted for integrated white noise.

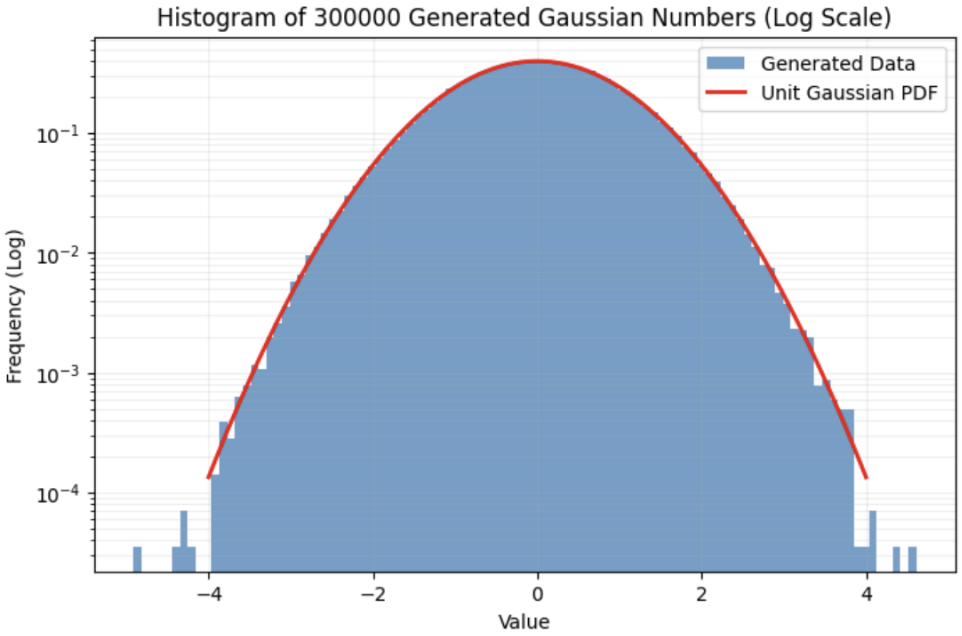

Figure 3: Gaussian histogram

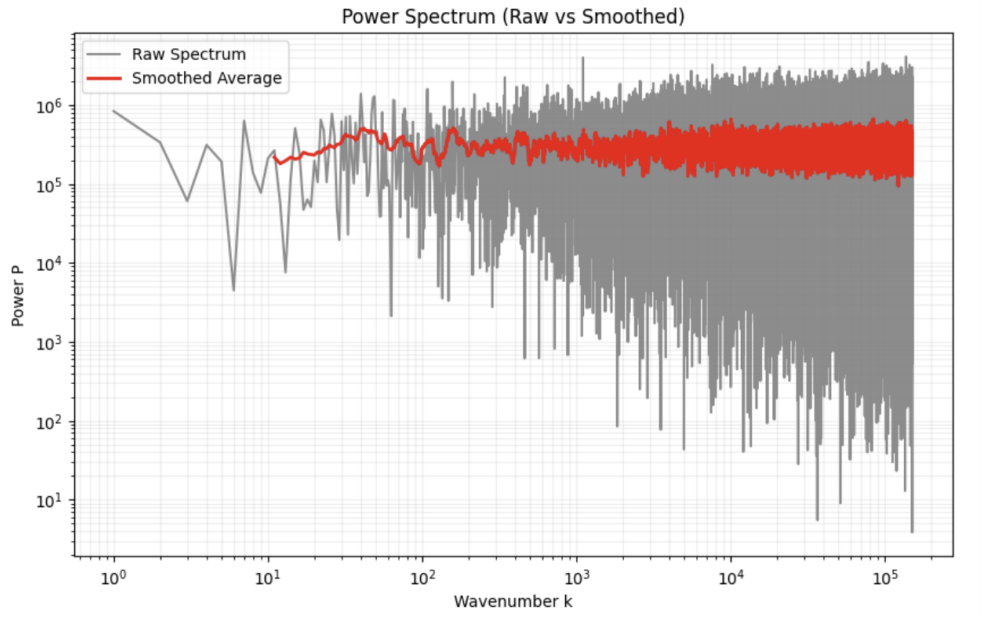

Figure 4: Power spectrum of Gaussian distribution

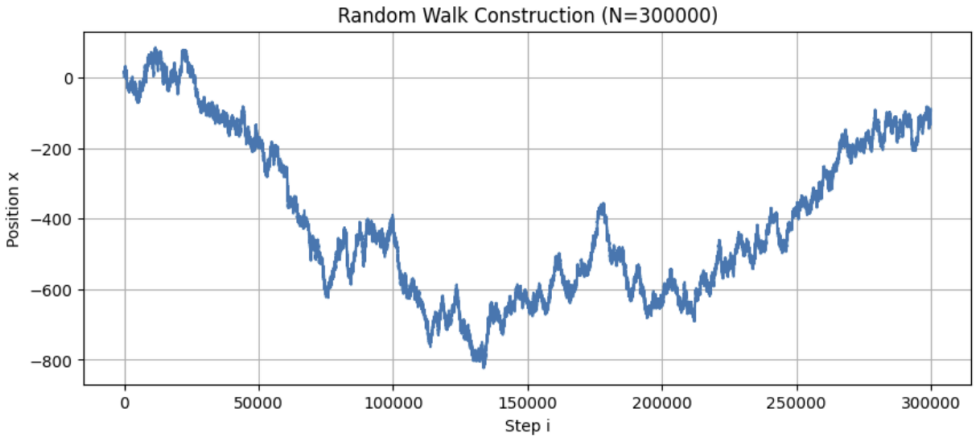

Figure 5: Trajectory of the constructed 1D Random Walk over 10,000 steps.

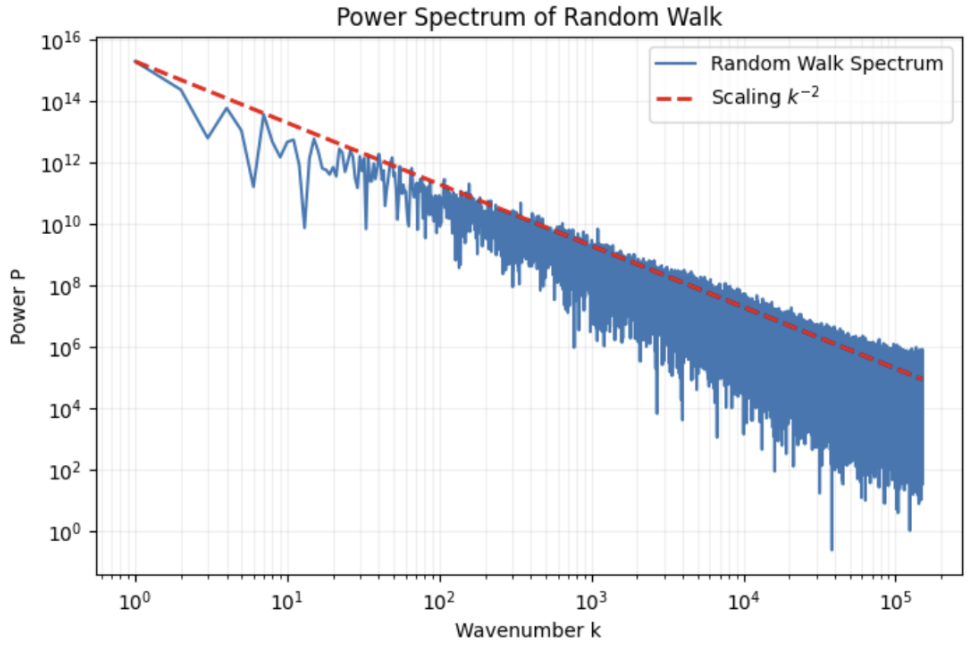

Figure 6: Power spectrum of the Random Walk. The red dashed line indicates the theoretical k^{-2} scaling.

Code Availability

All code used to generate these results and figures is available at the following repository: `https://github.com/[your-username]/computational-physics-hw`