SMC Math Integration Club
Alternative Method for Irrational Integrals

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Irrational integrals are not fun to evaluate. I mean, take a look at this:

$$\int \frac{dx}{(x-1)\sqrt{x^2-1}}$$

Yes, you CAN do a trigonometric substitution and completely waste a day doing it, but what if there was an alternative method? A method more suited for integrals like the one we are seeing.
Well, lucky for us, there is a substitution that makes integrals like above more doable.

Suppose, $I = \displaystyle\int \frac{dx}{(x-\alpha)^k\sqrt{ax^2+bx+c}}$ and let, $t = \dfrac{1}{x-\alpha}$

$$x-\alpha = \frac{1}{t} \implies dx = -t^{-2}dt$$

$$x = \frac{1}{t}+\alpha = \frac{1+\alpha t}{t} \implies x^2 = \frac{1+2\alpha t+\alpha^2t^2}{t^2}$$

$$ax^2+bx+c = \frac{a+2a\alpha t+a\alpha^2t^2+bt+b\alpha t^2+ct^2}{t^2} = \frac{(a\alpha^2+b\alpha+c)t^2+(2a\alpha+b)t+a}{t^2}$$

With the correct substitution, the integral should now look like:

$$I = \int \frac{dx}{(x-\alpha)^k\sqrt{ax^2+bx+c}} = -\int \frac{t^k\cdot t^{-2}dt}{\sqrt{\frac{(a\alpha^2+b\alpha+c)t^2+(2a\alpha+b)t+a}{t^2}}}$$

$$I = -\int \frac{t^k\cdot t^{-2}\cdot tdt}{\sqrt{(a\alpha^2+b\alpha+c)t^2+(2a\alpha+b)t+a}} = -\int \frac{t^{k-1}dt}{\sqrt{(a\alpha^2+b\alpha+c)t^2+(2a\alpha+b)t+a}}$$

To sum it up, $I = \displaystyle\int \frac{dx}{(x-\alpha)^k\sqrt{ax^2+bx+c}} = -\int \frac{t^{k-1}dt}{\sqrt{(a\alpha^2+b\alpha+c)t^2+(2a\alpha+b)t+a}}$

And alothough the integral looks more complicated, it actually makes integrals of that form doable!

1) Evaluate $\displaystyle\int \frac{dx}{(x-1)\sqrt{x^2-1}}$ using the above substitution.

2) $\displaystyle\int \frac{dx}{(x-3)\sqrt{4x^2-20x+25}}$

3) $\displaystyle\int \frac{du}{(u-7)^3\sqrt{25u^2-370u+1369}}$

4) $\displaystyle\int \frac{dx}{(x-1)\sqrt{x^2+2x-8}}$

5) $\displaystyle\int \frac{\cos(\theta)d\theta}{\sin(\theta)\sqrt{-\cos^2(\theta)+2\sin(\theta)-7}-\sqrt{\sin^2(\theta)+2(\sin(\theta)-4)}}$