SMC Math Integration Club
Gamma Functions, or Generalized Factorial Functions

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What comes up in your mind when you think of factorials?

$1! = 1$, $2! = 2 \cdot 1$, $3! = 3 \cdot 2 \cdot 1$, $4! = 4 \cdot 3 \cdot 2 \cdot 1$, ... , $n! = n \cdot (n-1) \ ... \ \cdot 3 \cdot 2 \cdot 1$

Matter of fact, what *are* factorials? Why do we care about recursive multiplication?

Take a look at these sets of letters. $\{a\}, \{a, b\}, \{a, b, c\}$

How many ways can we arrange the order of these letters within the sets?

The first is easy: one. There is one element, so there is only one way to arrange the letter.

Let's try for all of the sets

$\{a, b\}, \{b, a\}$ There are two ways.

$\{a, b, c\}, \{a, c, b\}, \{b, a, c\}, \{b, c, a\}, \{c, a, b\}, \{c, b, a\}$ There are six ways!

See the pattern?

How does this correlate with integrals and integration?

Four Mathematicians: John Wallis, Daniel Bernoulli, Leonhard Euler, and Adrien-Marie Legendre communicated back and forth regarding factorials, and we know this because the letters they sent each other are actually a public record!

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\, dx$$

This function, the generalized factorial function, is widely known as the Gamma function, named after Legendre who perfected it after much iterations. The idea is that if you set t to any number, not just the positive integers, the function will give you the factorial of that number.

The limitations are that t cannot be any negative integers and zero, but think about it. How does one compute the factorial of 1/2?

That's why it is called the generalized factorial function because you can now compute not just positive integers' factorials, but factorials of numbers extending to other domains as well.

Jason's to do:

Prove that: $\alpha\Gamma(\alpha) = \Gamma(\alpha + 1)$

1) Compute $\Gamma(1) = \int_0^\infty e^{-x} x^{1-1}\, dx$

2) Compute $\frac{1}{2}!$ using the Gamma function.

That is, $\int_0^\infty e^{-x} x^{\frac{1}{2}-1}\, dx$

3) $\int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx$ Hint: This is an even function.

4) $\int_0^\infty x^{2019} e^{-2020x}\, dx$

5) $\int_0^\infty x^5 e^{-x^4}\,dx$