Korea Advanced Institute of Science and Technology

Probability Theory Qualifying Exam

Problems and Solutions

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1. Let $M_X(t) = \mathbb{E}[e^{tX}]$ be a moment generating function of X. Suppose that $M_X(t)$ is finite in some neighborhood of $t = 0$. Show that there exist constants $a, b > 0$ such that

$$\mathbb{P}(|X| \geq t) \leq ae^{-bt}, \quad \forall t > 0.$$

Sol. Let U be a neighborhood of $t = 0$ such that $M_X(t) < \infty$. Let $b \in U$ be a positive number such that $M_X(\pm b) < \infty$. For such b and any $t > 0$, by Markov's inequality,

$$\mathbb{P}(|X| \geq t) = \mathbb{P}(X \geq t) + \mathbb{P}(-X \geq t) = \mathbb{P}(e^{bX} \geq e^{bt}) + \mathbb{P}(e^{-bX} \geq e^{bt}) \leq e^{-bt}\mathbb{E}[e^{bX}] + e^{-bt}\mathbb{E}[e^{-bX}].$$

By letting $a := \mathbb{E}[e^{bX}] + \mathbb{E}[e^{-bX}]$, the desired inequality is shown.

2. Let X_1, X_2,...be a sequence of independent random variables such that $\mathbb{P}(X_n = 1) = p_n$ and $\mathbb{P}(X_n = 0) = 1 - p_n$.

   (1) Show that $X_n \to 0$ in probability if and only if $p_n \to 0$.

   (2) Show that $X_n \to 0$ almost surely if and only if $\sum_{n=1}^{\infty} p_n < \infty$.

Sol. (1) For $\varepsilon \in (0, 1)$, $\mathbb{P}(|X_n| \geq \varepsilon) = p_n \to 0$.

(2) For $\varepsilon \in (0, 1)$, by Borel-Cantelli lemmas, $\mathbb{P}(|X_n| \geq \varepsilon \text{ i.o.}) = 0$ if and only if $\sum_{n=1}^{\infty} \mathbb{P}(|X_n| \geq \varepsilon) = \sum_{n=1}^{\infty} p_n < \infty$. Hence $|X_n| < \varepsilon$ eventually with probability 1—X_n converges to 0 almost surely— if and only if sum of all p_n is finite.

3. Suppose that $\{X_n\}_{n\geq 1}$ and X are (real-valued) random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded and continuous function. Prove or provide a counterexample in each case:

   (1) If $X_n \to X$ in probability, then $f(X_n) \to f(X)$ in probability.

   (2) If $X_n \to X$ in distribution, then $f(X_n) \to f(X)$ in distribution.

Sol. (1) This is same with problem 2, 2023 Feb.
From bounded continuity, it is sufficient to define B_m for $m \leq \log_2(1/2M)$, where $M > 0$ is a global bound of the function f.

(2) A sequence of random variables $\{Y_n\}_{n\geq 1}$ converges to Y in distribution if and only if $\mathbb{E}[g(Y_n)] \to \mathbb{E}[g(Y)]$ for any bounded, continuous g.
Since X_n converges to X in distribution, for any continuous, bounded g, we have

$$\mathbb{E}[g(f(X_n))] \to \mathbb{E}[g(f(X))],$$

because $g \circ f$ is a bounded continuous function. Since g was arbitrary, we may conclude that $f(X_n) \to f(X)$ in distribution. (Set $Y_n = f(X_n)$ and $Y = f(X)$.)

4. Let X_1, X_2,...be a sequence of i.i.d. random variables such that $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}(X_i) < \infty$. Show that

$$\frac{2}{n^2} \sum_{1\leq i<j\leq n} X_i X_j \to \mu^2$$

in probability.

Sol. We will observe its L^2 convergence. First,

$$\mathbb{E}\left|\frac{2}{n^2}\sum_{1\leq i<j\leq n} X_iX_j - \mu^2\right|^2 = \mathbb{E}\left[\frac{4}{n^4}\left(\sum_{1\leq i<j\leq n} X_iX_j\right)^2 - \frac{4\mu^2}{n^2}\sum_{1\leq i<j\leq n} X_iX_j + \mu^4\right]$$

and

$$\mathbb{E}\left[\frac{4\mu^2}{n^2}\sum_{1\le i<j\le n} X_iX_j\right] = \frac{4\mu^2}{n^2}\binom{n}{2}\mu^2 = \frac{2(n-1)\mu^4}{n}.$$

On the other hand,

$$\mathbb{E}\left[\left(\sum_{1\le i<j\le n} X_iX_j\right)^2\right] = \mathbb{E}\left[\sum_{1\le i<j\le n} X_i^2X_j^2 + \sum_{\text{all indices are different}} X_i^2X_jX_k + \sum_{\substack{1\le i<j\le n,\, i\neq k,l \\ 1\le k<l\le n,\, j\neq k,l}} X_iX_jX_kX_l\right]$$
$$= \binom{n}{2}\operatorname{Var}(X_1)^2 + 3\binom{n}{3}\mu^2\operatorname{Var}(X_1) + \binom{n}{2}\binom{n-2}{2}\mu^4 = \frac{\mu^4}{4}n^4 + \mathrm{O}(n^3).$$

Hence

$$\mathbb{E}\left|\frac{2}{n^2}\sum_{1\le i<j\le n} X_iX_j - \mu^2\right|^2 = \mu^4 - 2\mu^4 + \mu^4 + \mathrm{O}(1/n).$$

That is, $\frac{2}{n^2}\sum_{1\le i<j\le n} X_iX_j$ converges to μ^2 in L^2. Which immediately implies convergence in probability by Markov inequality; $\mathbb{P}(|Y_n - Y| \ge \varepsilon) = \mathbb{P}(|Y_n - Y|^2 \ge \varepsilon^2) \le \varepsilon^{-2}\mathbb{E}|Y_n - Y|^2 \to 0$ if $Y_n \to Y$ in L^2.

5. Let X_1, X_2,...be a sequence of i.i.d. random variables such that $\mathbb{P}(X_n = 0) = \mathbb{P}(X_n = 2) = 1/2$. Let $\{\mathcal{F}_n\}_{n\ge1}$ be the canonical filtration associated to $X_1, X_2, \ldots$. Define $Y_n := \prod_{k=1}^n X_k$.
   (1) Show that Y_n is a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n\ge1}$.
   (2) Show that it is NOT possible to find a random variable Z with $\mathbb{E}|Z| < \infty$ such that $Y_n = \mathbb{E}[Z|\mathcal{F}_n]$.

Sol. (1) First, $Y_n \ge 0$ almost surely, and by independence,

$$\mathbb{E}Y_n = \prod_{k=1}^n \mathbb{E}X_k = 1 < \infty.$$

Clearly Y_n is adapted to $\mathcal{F}_n$, and

$$\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = \mathbb{E}[X_{n+1}Y_n|\mathcal{F}_n] = Y_n\mathbb{E}[X_{n+1}|\mathcal{F}_n] = Y_n\mathbb{E}[X_{n+1}] = Y_n.$$

Hence it is a martingale.

(2) If such random variable exists, then Y_n is uniformly integrable. A uniformly integrable martingale converges in L^1 and almost surely sense simultaneously.
By the way, to use Borel-Cantelli lemmas, we will use $\mathbb{E}[\sqrt{X_i}] = 1/\sqrt{2}$. Since $\mathbb{E}[\sqrt{Y_n}] = \prod_{k=1}^n \mathbb{E}[\sqrt{X_k}] = (1/\sqrt{2})^n$, we have

$$\mathbb{P}(|Y_n| \ge \varepsilon) = \mathbb{P}(Y_n \ge \varepsilon) = \mathbb{P}(\sqrt{Y_n} \ge \sqrt{\varepsilon}) \le \frac{1}{\sqrt{\varepsilon}}\mathbb{E}[\sqrt{Y_n}] = \frac{1}{\sqrt{\varepsilon}2^{n/2}} \to 0$$

for any $\varepsilon > 0$. Hence Y_n converges to 0 in probability. Because almost sure convergence implies convergence in probability, and the limit in probability is unique, we have $Y_n \to 0$ almost surely, and it holds in L^1 sense.
However,

$$\mathbb{E}[|Y_n - 0|] = \mathbb{E}[Y_n] = 1 \not\to 0$$

has a contradiciton. Therefore such random variable Z cannot exist.

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1. State and prove the central limit theorem. (You can use the fact "$c_n \to c \in \mathbb{C} \Rightarrow (1+\frac{c_n}{n})^n \to e^c$" without a proof.)

Sol. Let X_1, X_2,...be a sequence of i.i.d. random variables such that $\mathbb{E}X_i = \mu$ and $\mathrm{Var}(X_i) = \sigma^2 \in (0,\infty)$. For $S_n = X_1 + \cdots + X_n$,

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \Rightarrow Z \sim N(0,1).$$

Proof. By shifting, without loss of generality, we may assume $\mu = 0$. The characteristic function $\varphi_n(t)$ of S_n is given by

$$\varphi_n(t) = \mathbb{E}\left[\exp\left(it\sum_{k=1}^n \frac{X_k}{n^{1/2}\sigma}\right)\right] = \prod_{k=1}^n \mathbb{E}\left[\exp\left(i\frac{t}{n^{1/2}\sigma}X_k\right)\right] = \left(\mathbb{E}\left[\exp\left(i\frac{t}{n^{1/2}\sigma}X_1\right)\right]\right)^n.$$

From the Taylor expansion of e^{ix}, we have an error estimate $|e^{ix} - \sum_{k=0}^n (ix)^k/k!| \le \min(|x|^3, 2|x|^2)$(details are omitted; see Durret's textbook), and hence

$$\begin{aligned}\left|\mathbb{E}\left[\exp\left(i\frac{t}{n^{1/2}\sigma}X_1\right)\right] - 1 + \frac{t^2}{2n}\right| &\le \mathbb{E}\left[\min\left(\left|\frac{t}{n^{1/2}\sigma}\right|^3 |X_1|^3, 2\left|\frac{t}{n^{1/2}\sigma}\right|^2 |X_1|^2\right)\right] \\ &= \frac{t^2}{n\sigma^2}\mathbb{E}[\min(|t|/n^{1/2}\sigma)|X_1|^3, 2|X_1|^2)]\end{aligned}$$

with $\mathbb{E}[\min(|t|/n^{1/2}\sigma)|X_1|^3, 2|X_1|^2)] \to 0$ by DCT. That is, the error term is in $o(1/n)$, and hence

$$\left(\mathbb{E}\left[\exp\left(i\frac{t}{n^{1/2}\sigma}X_1\right)\right]\right)^n = \left(1 - \frac{t^2}{2n} + o(1/n)\right)^n \to e^{-t^2/2}.$$

Since it is the characteristic function for standard normal distribution and continuous at $t = 0$, we may conclude that $\frac{S_n}{n^{1/2}\sigma} \Rightarrow N(0,1)$, the desired result. □

2. Let $\{X_1\}_{n=1,2,\cdots}$ and X be (real-valued) random variables. Suppose that X_n converges to X in probability. Prove that for any continuous function $f : \mathbb{R} \to \mathbb{R}$, $f(X_n)$ also converges to $f(X)$ in probability.

Sol. (Note: This is a proof using the property of continuity. You can find another proof in the Durret's book, Theorem 2.3.4.)

Fix $\varepsilon > 0$. For integer m, define the set B_m as

$$B_m := \{x \in \mathbb{R} : \exists y : |x-y| < 2^{-m},\ |f(x) - f(y)| \ge \varepsilon\}.$$

(Equivalently, B_m^c is the set of all $x \in \mathbb{R}$ such that its corresponding δ of given ε is larger than 2^{-m}.) By its definition, $B_m \subset B_k$ if $m > k$, and from continuity, we have $\bigcap_{m\in\mathbb{Z}} B_m = \varnothing$.

Then

$$\begin{aligned}&\mathbb{P}(|f(X_n) - f(X)| \ge \varepsilon) \\ =&\mathbb{P}(X \in B_m,\ |f(X_n) - f(X)| \ge \varepsilon) + \mathbb{P}(X \notin B_m,\ |f(X_n) - f(X)| \ge \varepsilon) \\ \le&\mathbb{P}(X \in B_m) + \mathbb{P}(|X_n - X| \ge 2^{-m})\end{aligned}$$

and

$$\limsup_{n\to\infty} \mathbb{P}(|f(X_n) - f(X)| \ge \varepsilon) \le \mathbb{P}(X \in B_m) + \limsup_{n\to\infty} \mathbb{P}(|X_n - X| \ge 2^{-m}) = \mathbb{P}(X \in B_m)$$

for any m. Since $\mathbb{P}(X \in B_m)$ converges to 0 as m varies to ∞, the left side of the inequality must be 0. Therefore $f(X_n)$ converges to $f(X)$ in probability.

3. Let $\{X_1\}_{n=1,2,\dots}$ be independent random variables such that

$$X_n = \begin{cases} 1 & \text{with probability } \frac{1}{2n}, \\ 0 & \text{with probability } 1-\frac{1}{n}, \\ -1 & \text{with probability } \frac{1}{2n}. \end{cases}$$

Let $Y_1 := X_1$ and

$$Y_n = \begin{cases} X_n & \text{if } Y_{n-1}=0, \\ nY_{n-1}|X_n| & \text{if } Y_{n-1}\neq 0. \end{cases}$$

Show that $\{Y_n\}_{n\geq 1}$ is a martingale with respect to $\mathcal{F}_n = \sigma(Y_1,\cdots,Y_n)$.

Sol. $\{Y_n\}_{n\geq 1}$ is a collection of L^1 random variables: Y_1 is clearly L^1. Suppose all Y_k with $k<n$ is L^1. Then

$$\mathbb{E}[|Y_n|] = \mathbb{E}[|X_n|\mathbf{1}_{Y_{n-1}=0}] + \mathbb{E}[n|Y_{n-1}||X_n|\mathbf{1}_{Y_{n-1}\neq 0}] \leq \mathbb{E}[|X_n|] + n\mathbb{E}[|Y_{n-1}|] < \infty$$

and hence Y_n is also L^1. Adaptedness is obvious. Finally, as X_i is independent to $\mathcal{F}_j$ if $j<i$, we have

$$\begin{aligned} \mathbb{E}[Y_{n+1}|\mathcal{F}_n] &= \mathbb{E}[X_{n+1}\mathbf{1}_{Y_n=0}|\mathcal{F}_n] + \mathbb{E}[(n+1)Y_n|X_{n+1}|\mathbf{1}_{Y_n\neq 0}|\mathcal{F}_n] \\ &= \mathbf{1}_{Y_n=0}\mathbb{E}[X_{n+1}|\mathcal{F}_n] + (n+1)Y_n\mathbf{1}_{Y_n\neq 0}\mathbb{E}[|X_{n+1}||\mathcal{F}_n] \\ &= 0 + (n+1)Y_n\frac{1}{n+1} = Y_n. \end{aligned}$$

Hence $\{Y_n\}_{n\geq 1}$ is a martingale.

4. Let $\{X_1\}_{n=1,2,\dots}$ be i.i.d random variables with $\mathbb{P}(X_n=-1)=\mathbb{P}(X_n=1)=1/2$. Set $S_0:=0$ and $S_n := X_1+\cdots+X_n$ for $n\geq 1$. For positive integers a, b, define

$$\tau := \inf\{n\geq 1 : S_n = -a \text{ or } S_n = b\}.$$

Compute $\mathbb{E}\tau$. (Hint: Consider a sequence $S_n^2 - n$.)

Sol. The given $\tau \geq 0$ is a stopping time. The sequences S_n and S_n^2-n are martingales(the proof will be omitted). Hence $S_{n\wedge\tau}$ and $S_{n\wedge\tau}^2 - (n\wedge\tau)$ are also martingales.

Clearly,

$$\mathbb{E}[S_\tau] = -a\mathbb{P}(S_\tau=-a) + b\mathbb{P}(S_\tau=b) = -a\mathbb{P}(S_\tau=-a) + b(1-\mathbb{P}(S_\tau=a)).$$

By martingale property, $\mathbb{E}[S_{n\wedge\tau}]=0$, with $|S_{n\wedge\tau}| \leq a\wedge b$ by the definition of τ, and $S_{n\wedge\tau}\to S_\tau$ almost surely. By DCT,

$$\mathbb{E}[S_\tau] = \lim_{n\to\infty}\mathbb{E}[S_{n\wedge\tau}] = 0.$$

Hence

$$\frac{b}{b+a} = \mathbb{P}(S_\tau=-a), \qquad \frac{a}{b+a} = \mathbb{P}(S_\tau=b).$$

Likewise, from $\mathbb{E}[S_{n\wedge\tau}^2] = \mathbb{E}[n\wedge\tau]$ and $n\wedge\tau \nearrow \tau$, by MCT and DCT, we have

$$\mathbb{E}[\tau] = \lim_{n\to\infty}\mathbb{E}[n\wedge\tau] = \lim_{n\to\infty}\mathbb{E}[S_{n\wedge\tau}^2] = \mathbb{E}[S_\tau^2] \leq (a\wedge b)^2 < \infty.$$

The precise calculation of $\mathbb{E}[S_\tau^2]$ is as following:

$$\begin{aligned} \mathbb{E}[S_\tau^2] &= \mathbb{E}[S_\tau^2; S_\tau=-a] + \mathbb{E}[S_\tau^2; S_\tau=b] = a^2\mathbb{P}(S_\tau=-a) + b^2\mathbb{P}(S_\tau=b) \\ &= a^2\frac{b}{b+a} + b^2\frac{a}{b+a} = \frac{a^2b+ab^2}{b+a} = ab. \end{aligned}$$

5. Let $\{X_1\}_{n=1,2,\dots}$ be i.i.d random variables with $\mathbb{E}[X_1]=0$. Let $\alpha>0$ be a constant. Show that the following two statements are equivalent.

(a) $\lim_{n\to\infty} \frac{X_n}{n^{1/\alpha}} = 0$ almost surely.

(b) $\mathbb{E}[|X_1|^\alpha] < \infty$.

Sol. Let $\varepsilon >$ be arbitrary. Since the function $x \mapsto \mathbb{P}(|X_1|^\alpha \geq x)$ is decreasing on $[0,\infty)$, we have

$$\mathbb{E}[|X_1|^\alpha] = \int_0^\infty \mathbb{P}(|X_1|^\alpha \geq x)dx \sim \sum_{n=0}^\infty \mathbb{P}(|X_1|^\alpha \geq n\varepsilon^\alpha) = \sum_{n=0}^\infty \mathbb{P}(|X_n| \geq n^{1/\alpha}\varepsilon).$$

By Borel-Cantelli lemmas, the sum is finite if and only if $\mathbb{P}(|X_n| < n^{1/\alpha}\varepsilon \text{ eventually}) = 1$, i.e., $\frac{X_n}{n^{1/\alpha}} \to 0$ almost surely. Hence both are equivalent.

6. Let $\{X_1\}_{n=1,2,\cdots}$ be an i.i.d sequence of standard normal random variables. Show that almost surely,

$$\lim_{n\to\infty} \frac{\max\{X_1,\cdots,X_n\}}{\sqrt{\log n}} = \sqrt{2}.$$

Hint: If X is a standard normal random variable, then for any $t > 1$,

$$\frac{1}{2\sqrt{2\pi}}\frac{1}{t}e^{-t^2/2} \leq \mathbb{P}(X > t) \leq \frac{1}{\sqrt{2\pi}}\frac{1}{t}e^{-t^2/2}.$$

Sol. For $k = 1, 1/2$, by differentiating $t \mapsto \mathbb{P}(X > t) - k\frac{1}{t\sqrt{2\pi}}e^{-t^2/2}$, for sufficiently large $t > 0$, we have the inequality in the hint.

For $0 \leq \varepsilon \ll 1$, we have $\mathbb{P}(X_n \geq (\sqrt{2}+\varepsilon)\sqrt{\log n}) \sim \frac{1}{n^{1+\varepsilon'}\sqrt{\log n}}$ by hint, where $\varepsilon' = \sqrt{2}\varepsilon + \varepsilon^2/2$. By integral test, the series of them converges if and only if $\varepsilon = 0$. By Borel-Cantelli lemma, therefore, we have $\mathbb{P}(X_n \geq \sqrt{2\log n} \text{ i.o}) = 1$ and $\mathbb{P}(X_n \geq (\sqrt{2}+\varepsilon)\sqrt{\log n} \text{ i.o.}) = 0$. As ε was arbitrary, we have

$$\limsup_{n\to\infty} \frac{X_n}{\sqrt{\log n}} = \sqrt{2} \text{ almost surely.}$$

Note following observation: Let $\{a_n\}$ and $\{b_n\}$ be real sequences such that $0 < b_n \nearrow \infty$ and

$$\limsup a_n/b_n = \alpha \in (-\infty,\infty).$$

For $\varepsilon > 0$, there exists N such that $n \geq N$ implies $a_n \leq (\alpha+\varepsilon)b_n$. Since $b_n \nearrow \infty$, we can further assume that $a_i < (\alpha+\varepsilon)b_n$ for all $i \leq N$, if $n \geq N$. Then if $n \geq N$, $\max a_i < (\alpha+\varepsilon)b_n$, i.e., $\max a_i/b_n < \alpha+\varepsilon$. Hence $\limsup \max a_i/b_n \leq \alpha$.

Applying this result on the sequence of *real numbers* $a_n = X_n(\omega)$ and $b_n = \sqrt{\log n}$, where ω satisfies $\limsup_{n\to\infty} X_n(\omega)/\sqrt{\log n} = \sqrt{2}$, we get

$$\limsup_{n\to\infty} \frac{\max\{X_i\}}{\sqrt{\log n}} \leq \sqrt{2} \text{ almost surely.}$$

By the way, we have

$$\mathbb{P}(\max\{X_i\} \leq (\sqrt{2}-\varepsilon)\sqrt{\log n}) = \left(\mathbb{P}(X_1 \leq (\sqrt{2}-\varepsilon)\sqrt{\log n})\right)^n = \left(1 - \frac{C_\varepsilon}{n^{1-\varepsilon''}\sqrt{\log n}}\right)^n,$$

where $\varepsilon'' = \sqrt{2}\varepsilon - \varepsilon^2/2$ and $C_\varepsilon = \frac{1}{2\sqrt{2\pi}(\sqrt{2}+\varepsilon)} > 0$. Let $s_n = n^{1-\varepsilon''}\sqrt{\log n}$. Then as $(1-t/s_n)^{s_n} \to e^{-t}$, for sufficiently large n, we have $(1-C_\varepsilon/s_n)^{s_n} < e^{-C_\varepsilon} + \delta =: k < 1$, and for such n,

$$\left(1-\frac{C_\varepsilon}{s_n}\right)^n = \left(\left(1-\frac{C_\varepsilon}{s_n}\right)^{s_n}\right)^{n/s_n} \leq k^{n/s_n} \leq k^n.$$

Hence, by using Borel-Cantelli lemma again, $\mathbb{P}(\max\{X_i\} \le (\sqrt{2}-\varepsilon)\sqrt{\log n} \text{ i.o.}) = 0$. That is,

$$\liminf_{n\to\infty} \frac{\max\{X_i\}}{\sqrt{\log n}} \ge \sqrt{2} \text{ almost surely.}$$

Therefore the limit converges to $\sqrt{2}$ almost surely.

cf. The inequality given by hint is valid: for upper tail,

$$\mathbb{P}(X > t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-x^2/2} dx \le \frac{1}{\sqrt{2\pi}} \int_t^\infty \frac{x}{t} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \frac{1}{t} e^{-t^2/2}.$$

For lower tail, if we let $g(t) = \mathbb{P}(X > t) - \frac{1}{\sqrt{2\pi}} \frac{t}{t^2+1} e^{-t^2/2}$,

$$\begin{aligned} g'(t) &= -\frac{1}{\sqrt{2\pi}} e^{-t^2/2} - \frac{1}{\sqrt{2\pi}} \frac{t}{t^2+1} e^{-t^2/2} \left(-t + \frac{1}{t} - \frac{2t}{t^2+1}\right) \\ &= -\frac{1}{\sqrt{2\pi}} e^{-t^2/2} \left(1 + \frac{t}{t^2+1} \left(-t + \frac{1}{t} - \frac{2t}{t^2+1}\right)\right) \\ &= -\frac{1}{\sqrt{2\pi}} \frac{1}{(t^2+1)^2} e^{-t^2/2} < 0 \end{aligned}$$

with $g(0+) = 1/2 > 0$ implies g is decreasing function on $(0, \infty)$. Hence

$$\frac{t}{t^2+1} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \le \mathbb{P}(X > t) \le \frac{1}{t} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}.$$

This type inequality is called *Mills' ratio*.

For $t > 1$, we have $t/(t^2+1) = 1/(t + t^{-1}) < 1/2t$ by arithmetic-geometric mean.

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1. State and prove Kolmogorov's 0-1 Law.

Sol. For independent random variables $\{X_n\}$, let $\mathcal{F}'_n := \sigma(X_n, X_{n+1}, \cdots)$, and let $\mathcal{T} := \bigcap_n \mathcal{F}'_n$. If an event A is $\mathcal{T}$-measurable, then $\mathbb{P}(A) \in \{0, 1\}$.

Proof. Two σ-fields $\sigma(X_1, \cdots, X_n)$ and $\mathcal{F}'_{n+1}$ are independent; $\sigma(X_1, \cdots, X_n)$ and $\sigma(X_{n+1}, \cdots, X_{n+j})$ are clearly independent. As $\bigcup_j \sigma(X_{n+1}, \cdots, X_{n+j})$ is a π-system containing whole probability space, the generated σ-fields by each one are independent.

Similarly, $\sigma(X_1, \cdots)$ and $\mathcal{T}$ are independent; From above argument, $\sigma(X_1, \cdots, X_n)$ and $\mathcal{T}$ are independent. As $\bigcup_n \sigma(X_1, \cdots, X_n)$ is a π-system containing whole probability space, the generated σ-fields by each one are independent.

Finally, as $\mathcal{T} \subset \sigma(\bigcup_n \sigma(X_1, \cdots, X_n)) = \sigma(X_1, \cdots)$, if $A \in \mathcal{T}$, A is independent with itself. Therefore $\mathbb{P}(A) = \mathbb{P}(A \cap A) = \mathbb{P}(A)^2$, or equivalently $\mathbb{P}(A) \in \{0, 1\}$. $\square$

2. Let X_1, X_2, ..., X_n be i.i.d. random variables with $\mathbb{E}[|X_i|] < \infty$. Define $S_n := X_1 + \cdots + X_n$. Compute $\mathbb{E}[X_1|S_n]$.

Sol. By the linearity of conditional expectation, we have

$$S_n = \mathbb{E}[S_n|S_n] = \sum_{k=1}^{n} \mathbb{E}[X_k|S_n] \overset{\text{i.i.d.}}{=} n\mathbb{E}[X_1|S_n]$$

and thus $\mathbb{E}[X_1|S_n] = S_n/n$.

3. X is a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}[X^2] < \infty$. Let $\mathcal{G} \subseteq \mathcal{F}$ be a σ-algebra. Show that $\mathbb{E}[X|\mathcal{G}]$ is a minimizer of $\mathbb{E}[(X - Y)^2]$ over all $\mathcal{G}$-measurable random variables Y.

Sol.

$$\begin{aligned}
\mathbb{E}[(X - Y)^2] &= \mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}] + \mathbb{E}[X|\mathcal{G}] - Y)^2] \\
&= \mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])^2] + 2\mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])(\mathbb{E}[X|\mathcal{G}] - Y)] + \mathbb{E}[(\mathbb{E}[X|\mathcal{G}] - Y)^2]
\end{aligned}$$

and by letting $Z = \mathbb{E}[X|\mathcal{G}]$ as $\mathcal{G}$-measurable random variable, we have

$$\begin{aligned}
\mathbb{E}[(X - Z)(Z - Y)] &= \mathbb{E}[XZ - Z^2 - XY + YZ] \\
&= \mathbb{E}[XZ] - \mathbb{E}[Z\mathbb{E}[X|\mathcal{G}]] - \mathbb{E}[XY] + \mathbb{E}[Y\mathbb{E}[X|\mathcal{G}]] \\
&= \mathbb{E}[XZ] - \mathbb{E}[\mathbb{E}[XZ|\mathcal{G}]] - \mathbb{E}[XY] + \mathbb{E}[\mathbb{E}[XY|\mathcal{G}]] = 0.
\end{aligned}$$

Hence

$$\mathbb{E}[(X - Y)^2] = \mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])^2] + \mathbb{E}[(X|\mathcal{G}] - Y)^2]$$

is minimized when $Y = \mathbb{E}[X|\mathcal{G}]$.

4. Suppose that events $\{A_n\}_{n=1,2,\cdots}$ are independent and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$. Show that

$$\mathbb{P}(A_n \text{ infinitely often}) = 1.$$

(Hint: Use the formulation $\{A_n \text{infinitely often}\} = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n$ and the inequality $1 - x \leq e^{-x}$ for $x \geq 0$.)

Sol. We will show that $\mathbb{P}(\{A_n \text{ infinitely often}\}^c) = 0$.

Fix arbitrary $N \in \mathbb{N}$. The equality in the hint is quite obvious, and $\{\bigcap_{n=N}^{M} A_n^c\}_{M=N,N+1,N+2,\cdots}$ is a decreasing sequence of events. Hence for $N \in \mathbb{N}$, we have

$$\mathbb{P}\left(\bigcap_{n=N}^{M} A_n^c\right) = \prod_{n=N}^{M} (1 - \mathbb{P}(A_n)) \leq \prod_{n=N}^{M} \exp(-\mathbb{P}(A_n)) = \exp\left(-\sum_{n=N}^{M} \mathbb{P}(A_n)\right) \to 0$$

as $M \to \infty$, and

$$\mathbb{P}\left(\bigcap_{n=N}^{\infty} A_n^c\right) = \mathbb{P}\left(\bigcap_{M \geq N} \bigcap_{n=N}^{M} A_n^c\right) = \lim_{M\to\infty} \mathbb{P}\left(\bigcap_{n=N}^{M} A_n^c\right) = 0.$$

Finally, as $\{\bigcap_{n=N}^{\infty} A_n^c\}_{N=1,2,\dots}$ is an increasing sequence of events, we have

$$\mathbb{P}\left(\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} A_n^c\right) = \lim_{N\to\infty} \mathbb{P}\left(\bigcap_{n=N}^{\infty} A_n^c\right) = \lim_{N\to\infty} 0 = 0.$$

5. Let $\{X_n\}_{n=1,2,\dots}$ be an i.i.d. sequence of exponential random variables (i.e., the probability density function is given by $f(x) = e^{-x}$ for $x \geq 0$). Show that almost surely,

$$\limsup_{n\to\infty} \frac{X_n}{\log n} = 1.$$

Sol. If X is an exponential random variable (with parameter $\lambda = 1$), for $t > 0$, we have

$$\mathbb{P}(X \geq t) = \int_t^{\infty} e^{-x} dx = e^{-t}.$$

Then

$$\sum \mathbb{P}(X_n \geq \log n) = \sum \mathbb{P}(X_1 \geq \log n) = \sum \frac{1}{n} = \infty,$$

and as X_i are independent random variables, by the second Borel-Cantelli lemma,

$$\mathbb{P}(X_n \geq \log n \text{ i.o.}) = 1.$$

In other words, $\limsup X_n / \log n \geq 1$ almost surely.

On the other hand, for any positive ε,

$$\sum \mathbb{P}(X_n \geq (1+\varepsilon)\log n) = \sum \mathbb{P}(X_1 \geq (1+\varepsilon)\log n) = \sum \frac{1}{n^{1+\varepsilon}} < \infty,$$

and thus by first Borel-Cantelli lemma,

$$\mathbb{P}(X_n \geq (1+\varepsilon)\log n \text{ i.o.}) = 0$$

for any positive ε. As $\varepsilon > 0$ was arbitrary, we have $\limsup X_n / \log n \leq 1$ and we may deduce that the upper limit of the random variable $X_n / \log n$ is 1.

6. Let $\{X_n\}_{n=0,1,\dots}$ be a martingale with $X_0 = 0$ such that $\mathbb{E}[(X_n - X_{n-1})^2] = 1$ for all $n \geq 1$. Show that almost surely,

$$\frac{X_n}{n} \to 0.$$

(Hint: First show that $\frac{X_{a_n}}{a_n} \to 0$ a.s. along a suitable subsequence $\{a_n\}_{n=0,1,\dots}$ using Borel-Cantelli lemma. Then, extend it to the full sequence.)

Sol. By the property of conditional expectation,

$$\mathbb{E}[X_{n+1}X_n] = \mathbb{E}[\mathbb{E}[X_{n+1}X_n|\mathcal{F}_n]] = \mathbb{E}[X_n\mathbb{E}[X_{n+1}|\mathcal{F}_n]] = \mathbb{E}[X_n^2].$$

Then, $\mathbb{E}[(X_n - X_{n-1})^2] = \mathbb{E}[X_n^2 - X_{n-1}^2] = 1$ with $\mathbb{E}[X_0^2] = 0$ implies $\mathbb{E}[X_n^2] = n$.

By Borel-Cantelli lemma,

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_{n^2}| \geq \varepsilon n^2) \leq \sum_{n=1}^{\infty} \frac{\mathbb{E}X_{n^2}^2}{\varepsilon^2 n^4} = \sum_{n=1}^{\infty} \frac{1}{\varepsilon^2 n^2} < \infty$$

implies $|X_{n^2}| < \varepsilon n^2$ eventually. Since ε was arbitrary, we have shown that the convergence given in the hint. The above argument can be generalized. Let $i > j$. Then

$$\begin{aligned}\mathbb{E}[X_i X_j] &= \mathbb{E}[\mathbb{E}[X_i X_j|\mathcal{F}_j]] = \mathbb{E}[X_j \mathbb{E}[X_i|\mathcal{F}_j]] \\ &= \mathbb{E}[X_j \mathbb{E}[\mathbb{E}[X_i|\mathcal{F}_{i-1}]|\mathcal{F}_j]] \\ &= \mathbb{E}[X_j \mathbb{E}[X_{i-1}|\mathcal{F}_j]] = \cdots = \mathbb{E}[X_j \mathbb{E}[X_j|\mathcal{F}_j]] = \mathbb{E}[X_j^2].\end{aligned}$$

Therefore, we have $\mathbb{E}[(X_i - X_j)^2] = 1$ for all $i \neq j$. By using this, for $m \in \{\lfloor\sqrt{n}\rfloor, \lceil\sqrt{n}\rceil\}$, we have

$$\mathbb{P}(|X_n - X_{m^2}| \geq \varepsilon m^2) \leq \frac{\mathbb{E}[(X_n - X_{m^2})^2]}{\varepsilon^2 m^4} \leq \frac{1}{\varepsilon^2 m^4} = \mathrm{O}\left(\frac{1}{n^2}\right).$$

Hence, Borel-Cantelli lemma says $|X_n - X_{m^2}| < \varepsilon m^2$ eventually, i.e., $|X_n - X_{m^2}|/m^2 \to 0$ almost surely. Finally, the inequality

$$\frac{X_n}{\lceil\sqrt{n}\rceil^2} = \frac{X_n - X_{\lceil\sqrt{n}\rceil^2}}{\lceil\sqrt{n}\rceil^2} + \frac{X_{\lceil\sqrt{n}\rceil^2}}{\lceil\sqrt{n}\rceil^2} \leq \frac{X_n}{n} \leq \frac{X_n}{\lfloor\sqrt{n}\rfloor^2} = \frac{X_n - X_{\lfloor\sqrt{n}\rfloor^2}}{\lfloor\sqrt{n}\rfloor^2} + \frac{X_{\lfloor\sqrt{n}\rfloor^2}}{\lfloor\sqrt{n}\rfloor^2}$$

guarantees that X_n/n vanishes almost surely.

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1. Let $A_1, A_2, \ldots$be a sequence of events on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that if $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(A_n \text{ infinitely often}) = 0$.

Sol. This is just first Borel-Cantelli lemma.

The event A_n infinitely often can be written as $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$, where $\{\bigcup_{k=n}^{\infty} A_k\}$ is a decreasing sequence of events. Hence

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right) = \lim_{n\to\infty} \mathbb{P}\left(\bigcup_{k=n}^{\infty} A_k\right) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} \mathbb{P}(A_k) = 0.$$

2. Let X be a random variable with mean 0 and variance σ^2. Show that for any $\lambda > 0$,

$$\mathbb{P}(X \geq \lambda) \leq \frac{\sigma^2}{\sigma^2 + \lambda^2}.$$

(Hint: Consider the function $\phi(x) = (x + \frac{\sigma^2}{\lambda})^2$.)

Sol. The function ϕ given in the hint has (global) minimum 0 at $x = -\frac{\sigma^2}{\lambda}$, and increases on $(-\frac{\sigma^2}{\lambda}, \infty)$. Hence by Markov's inequality, we have

$$\begin{aligned}
\mathbb{P}(X \geq \lambda) \leq \mathbb{P}(\phi(X) \geq \phi(\lambda)) &\leq \frac{\mathbb{E}[\varphi(X)]}{\varphi(\lambda)} \\
&= \frac{\mathbb{E}[X^2 + \frac{2\sigma^2}{\lambda}X + \frac{\sigma^4}{\lambda^2}]}{(\lambda\frac{\sigma^2}{\lambda})^2} \\
&= \frac{\sigma^2 + \frac{\sigma^4}{\lambda^2}}{(\lambda\frac{\sigma^2}{\lambda})^2} = \frac{\sigma^2(\lambda^2 + \sigma^2)}{(\lambda\frac{\sigma^2}{\lambda})^2} = \frac{\sigma^2}{\lambda^2 + \sigma^2}.
\end{aligned}$$

3. Suppose that X and Y are independent random variables with the same exponential density

$$f(x) = \theta e^{-\theta x}, \quad x > 0.$$

Show that the sum $X + Y$ and the ratio X/Y are independent.

Sol. First, we will calculate density of $X + Y$ and X/Y. For notational convenience, let $f_W(x)$ be the density of random variable W. By independence, the joint density of X and Y is same with $f_X(x)f_Y(y)$.

It is well-known fact that the density of sum of two independent random variables is the convolution of densities of each other. Hence

$$\begin{aligned}
f_{X+Y}(x) = (f_X * f_Y)(x) &= \int f_X(y) f_Y(x-y)dy \\
&= \int_0^x \theta e^{-\theta y} \theta e^{-\theta(x-y)} dy = \theta^2 x e^{-\theta x}.
\end{aligned}$$

On the other hand, since X and Y are positive almost surely, we have

$$\begin{aligned}
\mathbb{P}(X/Y \leq t) &= \mathbb{P}(X \leq tY) \\
&= \int_0^{\infty} \int_0^{ty} \theta e^{-\theta x} \theta e^{-\theta y} dx dy \\
&= \int_0^{\infty} \theta e^{-\theta y}(1 - \theta e^{-\theta t y}) dy \\
&= 1 - \frac{1}{1+t} = \frac{t}{1+t}.
\end{aligned}$$

Therefore

$$\mathbb{P}(X+Y \le s)\mathbb{P}(X/Y \le t) = \int_0^s \theta^2 x e^{-\theta x} dx \frac{t}{1+t} = \frac{t}{1+t}(1-(1+\theta s)e^{-\theta s}).$$

By calculating integral of joint density directly, we have

$$\begin{aligned}
\mathbb{P}(X+Y \le s, X/Y \le t) &= \mathbb{P}(X/t \le Y \le s-X, 0 \le X \le st/(1+t)) \\
&= \int_0^{st/(1+t)} \int_{x/t}^{s-x} \theta e^{-\theta x} \theta e^{-\theta y} dy dx \\
&= \int_0^{st/(1+t)} \theta e^{-\theta x}(-e^{-\theta(s-x)} + e^{-\theta x/t}) dx \\
&= \int_0^{st/(1+t)} -\theta e^{-\theta s} + \theta e^{-\theta(1+t)x/t} dx \\
&= -\theta \frac{st}{1+t} e^{-\theta s} + \frac{t}{1+t} - \frac{t}{1+t} e^{-\theta s} \\
&= \frac{t}{1+t}(1 - e^{-\theta s} - \theta s e^{-\theta s}).
\end{aligned}$$

As they are same, we may conclude that these two random variables $X+Y$ and X/Y are independent.

4. Let $X_1, X_2, \ldots$ be an i.i.d. sequence of random variables with $\mathbb{E}[X_i] = 0$ and $\operatorname{Var} X_i = 1$. Show that

$$\limsup_n \frac{X_1 + \cdots + X_n}{\sqrt{n}} = \infty \text{ almost surely.}$$

Sol. Fix M. The event $A_M = \{\limsup_n \frac{X_1+\cdots+X_n}{\sqrt{n}} \le M\}$ is in tail σ-field, and by Kolmogorov's zero-one law, its probability is either 0 or 1. By central limit theorem and reversed Fatou's lemma, we have

$$\mathbb{P}\left(\limsup_n \frac{X_1+\cdots+X_n}{\sqrt{n}} \le M\right) \ge \limsup_n \mathbb{P}\left(\frac{X_1+\cdots+X_n}{\sqrt{n}} \le M\right) = \Phi(M) > 0$$

where $\Phi(x)$ is the cumulative distribution of standard normal distribution. Hence whatever M is, $\mathbb{P}(A_M) = 1$. In other words, for any M, $\limsup_n \frac{X_1+\cdots+X_n}{\sqrt{n}} \ge M$ almost surely, and thus the upper limit must be ∞ almost surely.

5. Let $X_1, X_2, \ldots$ be an i.i.d. sequence of random variables with $\mathbb{E}[X_i] = 0$ and $\operatorname{Var} X_i = 1$. Let T be a stopping time with respect to the natural filtration such that $\mathbb{E}[T] < \infty$. Define $S_n = X_1 + \cdots + X_n$.

   (a) Show that both S_n and $S_n^2 - n$ are a martingale with respect to the natural filtration.

   (b) Prove that
   $$\mathbb{E}[S_T] = 0.$$

   (c) Prove that
   $$\operatorname{Var}(S_T) = \mathbb{E}[T].$$

Sol. (a) Suppose X_i are L^1 random variables.
First,

$$\mathbb{E}[|S_n|] \le \sum_{i=1}^n \mathbb{E}[|X_i|] = n < \infty$$

and

$$\mathbb{E}[|S_n^2 - n|] \le \mathbb{E}[S_n^2 + n] = \operatorname{Var}(S_n) + n = \sum_{i=1}^n \operatorname{Var} X_i + n = 2n < \infty$$

implies both are in L^1. Adaptedness is obvious. Finally,

$$\mathbb{E}[S_{n+1}|\mathcal{F}_n] = \mathbb{E}[S_n + X_{n+1}|\mathcal{F}_n] = \mathbb{E}[S_n|\mathcal{F}_n] + \mathbb{E}[X_{n+1}|\mathcal{F}_n] = S_n + \mathbb{E}[X_{n+1}] = S_n$$

and

$$\begin{aligned}\mathbb{E}[S_{n+1}^2 - (n+1)|\mathcal{F}_n] &= \mathbb{E}[S_n^2 + 2S_nX_{n+1} + X_{n+1}^2 - (n+1)|\mathcal{F}_n] \\ &= \mathbb{E}[S_n^2|\mathcal{F}_n] + \mathbb{E}[2S_nX_{n+1}|\mathcal{F}_n] + \mathbb{E}[X_{n+1}^2|\mathcal{F}_n] - (n+1) \\ &= S_n^2 + 2S_n\mathbb{E}[X_n] + \mathbb{E}[X_{n+1}^2] - (n+1) = S_n^2 - n.\end{aligned}$$

Hence they are martingales.

(b) The collection of random variables $\{S_{n\wedge T}\}_{n\geq 1}$ is dominated for some integrable random variable(see Durret's textbook 4.8.5); first observe that

$$\mathbb{E}[|S_{n+1} - S_n||\mathcal{F}_n] = \mathbb{E}[|X_{n+1}||\mathcal{F}_n] = \mathbb{E}|X_{n+1}| = \mathbb{E}|X_1| =: M < \infty.$$

By letting $S_0 = 0$, we can write

$$S_{n\wedge T} = \sum_{m=0}^{\infty}(S_{m+1} - S_m)\mathbf{1}_{T>m} = \sum_{m=0}^{\infty} X_{m+1}\mathbf{1}_{T>m}.$$

Then $|S_{n\wedge T}|$ is dominated by $\sum_{m=0}^{\infty}|X_{m+1}|\mathbf{1}_{T>m}$, with finite expectation: for each m,

$$\mathbb{E}[\mathbf{1}_{T>m}|X_{m+1}|] = \mathbb{E}[\mathbb{E}[\mathbf{1}_{T>m}|X_{m+1}||\mathcal{F}_m] = \mathbb{E}[\mathbf{1}_{T>m}\mathbb{E}[|X_{m+1}||\mathcal{F}_m] = \mathbb{E}[M\mathbf{1}_{T>m}],$$

and by MCT,

$$\mathbb{E}\left[\sum_{m=0}^{\infty}|X_{m+1}|\mathbf{1}_{T>m}\right] = \sum_{m=0}^{\infty}\mathbb{E}[M\mathbf{1}_{T>m}] = M\sum_{m=0}^{\infty}\mathbb{P}(T > m) = M\mathbb{E}T < \infty.$$

Therefore, $\mathbb{E}[S_{n\wedge T}] = \mathbb{E}[S_{1\wedge T}] = 0 \to \mathbb{E}[S_T] = 0$ holds.

(c) We have $\mathbb{E}[S_{n\wedge T}^2 - (n\wedge T)] = 0$ as it is a martingale. By monotone convergence,

$$\mathbb{E}[S_{n\wedge T}^2] = \mathbb{E}[n\wedge T] \nearrow \mathbb{E}T < \infty.$$

Thus $\mathbb{E}[S_{n\wedge T}^2] \leq \mathbb{E}T$ for all n. Hence, L^p convergence for martingale implies $S_{n\wedge T} \to S_T$ in L^2(and a.s.). Therefore $\mathbb{E}[S_{n\wedge T}^2] \to \mathbb{E}[S_T^2] = \mathbb{E}T$. From previous part, we have $\mathbb{E}[S_T] = 0$, and thus $\mathbb{E}[S_T^2] = \mathrm{Var}(S_T)$.

6. Let $X_1, X_2, \ldots$be an i.i.d. sequence of random variables with $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = -1) = 1 - p$, where $\frac{1}{2} < p < 1$. Let $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$.

(a) Let $\phi(x) = (\frac{1-p}{p})^x$. Prove that $\phi(S_n)$ is a martingale with respect to the natural filtration.

(b) Let $T_x = \inf\{n \geq 1 : S_n = x\}$. Prove that for any positive integer k,

$$\mathbb{P}(T_{-k} < T_k) = \frac{1}{1 + \phi(-k)}.$$

Sol. (a) The random variable $\phi(S_n)$ is positive for any n, and by independence,

$$\begin{aligned}\mathbb{E}[\phi(S_n)] &= \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{X_1+\cdots+X_n}\right] \\ &= \prod_{i=1}^{n}\mathbb{E}\left[\left(\frac{1-p}{p}\right)^{X_i}\right] = \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{X_1}\right]^n = (1-p) + p = 1 < \infty\end{aligned}$$

and it is L^1 random variable. Adaptedness is obvious. Finally,

$$\mathbb{E}[\phi(S_{n+1})|\mathcal{F}_n] = \mathbb{E}[\phi(S_n)\phi(X_{n+1})|\mathcal{F}_n] = \phi(S_n)\mathbb{E}[\phi(X_{n+1})|\mathcal{F}_n] = \phi(S_n).$$

Hence it is a martingale.

(b) Let $T = T_{-k} \wedge T_k$ be a new stopping time. Because $\phi(S_{n\wedge T})$ is a bounded martingale, by DCT,

$$1 = \mathbb{E}[\phi(S_{n\wedge T})] \to \mathbb{E}[\phi(S_T)] = 1$$

and

$$1 = \mathbb{E}[\phi(S_T)] = \mathbb{P}(T_{-k} < T_k)\phi(-k) + (1 - \mathbb{P}(T_{-k} < T_k))\phi(k).$$

Hence

$$\mathbb{P}(T_{-k} < T_k) = \frac{\phi(k)}{1+\phi(k)} = \frac{1}{1+\phi(-k)}.$$