KAIST Analysis Qualifying Exam

Problems and Solutions

Contents

1 Real Analysis

In many context, μ will denote Lebesgue measure on appropriate dimensional Euclidean space, if it does not mentioned.

1.1 2024 Feb Real

1. Prove that the set of $x \in \mathbb{R}$ such that there exist infinitely many fractions p/q, with relatively prime integers p and q such that

$$\left| x - \frac{p}{q} \right| \le \frac{1}{q^3}$$

is a set of (Lebesgue) measure zero.

Sol. See [5], p. 46 Problem 1. Because the Lebesgue measure μ is shift invariant, we only consider the case in the interval $[0, 1]$. We will denote $p \perp q$ for relatively prime integers p and $q > 0$, or $p = 0$.

Give an enumeration on the set of all rationals in $[0, 1]$. Denote the n-th element as p_n/q_n, where $p_n \perp q_n$. Let $E_n = (p_n/q_n - 1/q_n^3, p_n/q_n + 1/q_n^3)$. Then

$$\mu\left(\bigcup_n E_n\right) \le \sum_n \mu(E_n) = \sum_k \sum_{n:q_n=k} \mu(E_n) \le \sum_k \frac{2(k+1)}{k^3} < \infty,$$

where the second inequality is from that $0 \le p_n \le q_n$. Then we have

$$\mu\left(\bigcap_{N\ge1} \bigcup_{n\ge N} E_n\right) = \lim_{N\to\infty} \mu\left(\bigcup_{n\ge N} E_n\right) \le \lim_{N\to\infty} \sum_{n\ge N} \mu(E_n) = 0.$$

If $x \in [0, 1]$ has infinitely many such fractions, then $x \in \bigcap_{N\ge1} \bigcup_{n\ge N} E_n$, since for each q, at most one p with $p \perp q$ can satisfy $|x - p/q| \le 1/q^3$. By countable additivity, the set of such x has zero measure.

2. Suppose that f and g are measurable functions on $\mathbb{R}^d$. Prove the following statements:

   (a) If f is integrable and g is bounded, then $f * g$ is uniformly continuous.

   (b) If f and g are integrable, and g is bounded, then $(f * g)(x) \to 0$ as $|x| \to \infty$.

Sol. (a) See Problem 2 in 2022 February.

(b) By Fubini-Tonelli's theorem,

$$\begin{aligned}\int |f * g(x)|dx &= \int \left| \int f(x-y)g(y)dy \right| dx \\ &\le \int \int |f(x-y)||g(y)|dydx \\ &= \int \int |f(x-y)||g(y)|dxdy = \|f\|_1 \|g\|_1 < \infty\end{aligned}$$

and hence $f * g \in L^1$. Use part (b), problem 2 in 2023 February and previous result.

3. Prove the following statements:

   (a) If $1 \le p < q < \infty$, then $L^p(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^q(\mathbb{R})$.

   (b) If $f \in L^r(\mathbb{R})$ for some $r < \infty$, then $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

Sol. See Problem 4 in 2021 August.

4. Prove that $L^p(\mathbb{R}^d)$ ($p \in [1, \infty)$) with the Lebesgue measure is a Hilbert space if and only if $p = 2$.

Sol. See Problem 4 in 2022 August.

5. For a signed measure ν, prove that its total variation $|\nu|$ is a (positive) measure that satisfies $\nu \le |\nu|$.

Sol. See [4] p. 117 Theorem 6.2 or [5] p. 286 Proposition 4.1. If you want to use Hahn decomposition, see [1] p. 87 Theorem 3.4.

6. Let μ and ν be σ-finite measures on the Borel sets of the positive real line $[0, \infty)$. Suppose that $\phi(t) := \nu([0, t))$ is finite for every $t > 0$. Prove that for any μ-measurable function $f : [0, \infty) \to [0, \infty)$,

$$\int_0^\infty \phi(f(x))d\mu(x) = \int_0^\infty \mu(\{x : f(x) > t\})d\nu(t).$$

Sol. It is almost same with Problem 3 in 2022 February.

$$\begin{aligned}\int_0^\infty \phi(f(x))d\mu(x) &= \int_0^\infty \int_0^\infty \mathbf{1}_{[0,f(x))}(t)d\nu(t)d\mu(x)\\ &= \int_0^\infty \int_0^\infty \mathbf{1}_{[0,f(x))}(t)d\mu(x)d\nu(t)\\ &= \int_0^\infty \mu(\{x : f(x) > t\})d\nu(t).\end{aligned}$$

1.2 2023 Aug Real

1. Let $A \subset \mathbb{R}$ be a Lebesgue measurable set whose Lebesgue measure is strictly positive. Prove that there exists $B \subset A$ such that B is not Lebesgue measurable.

Sol. See [5], p. 44 Exercise 32(b). It is just imitation to construct Vitali's set.

Without loss of generality, assume A has finite measure. Since A has nonzero measure, there is an interval I such that $A \cap I$ has nonzero measure. It is known that the set of all representives of the quotient group $\mathbb{R}/\mathbb{Q}$ is nonmeasurable, where all representives are in I. Let N be such set.

Then $A \cap N \subset A \cap I$ is nonmeasurable; let $\{q_k\}$ be an enumeration of all rationals in I. Clearly $A \cap I \subseteq \bigsqcup_k (A \cap (N + q_k))$. By the way, there is an interval J such that

$$J \supseteq \bigsqcup_k N + q_k \supseteq I.$$

If $A \cap N$ were measurable,

$$\begin{aligned} 0 < \mu(A \cap I) &\leq \mu\left(\bigsqcup_k (A \cap (N + q_k))\right) \\ &= \sum_k \mu(A \cap (N + q_k)) \\ &= \sum_k \mu(A \cap N) \leq \mu(A \cap J) < \infty \end{aligned}$$

and $A \cap N$ would have zero measure. But then $\mu(A \cap I) = 0$, a contradiction.

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue integrable function. Prove the following:

   (a) $\displaystyle\lim_{y \to 0} \int f(x + y)dx = \int f(x)dx$

   (b) $\displaystyle\lim_{k \to \infty} \int f(x)e^{-x^2/k}dx = \int f(x)dx$

Sol. (a) Let $\varepsilon > 0$ be given. By approximating f to compactly supported continuous function g with L^1 error less than $\varepsilon > 0$, we have

$$\left|\int f(x+y) - f(x)dx\right| \leq \int |f(x+y) - f(x)|dx$$
$$\leq \int |f(x+y) - g(x+y)| + |g(x+y) - g(x)| + |g(x) - f(x)|dx$$
$$\leq 2\varepsilon + \int |g(x+y) - g(x)|dx.$$

By uniform continuity of g, $|g(x+y)-g(x)| < \varepsilon$ if $|y| < \delta$ for some δ. Hence, if $|y| < \delta$,

$$\int |g(x+y) - g(x)|dx \leq \int_E \varepsilon dx = \mu(E) \cdot \varepsilon$$

where $E = \operatorname{supp} g \cup (\delta + \operatorname{supp} g)$, which has finite measure. Therefore

$$\left|\int f(x+y) - f(x)dx\right| \leq \varepsilon(2 + \mu(E))$$

gives that the limit is valid.

(b) From $|f(x)(1 - \exp(-x^2/k))| \leq 2|f(x)|$, by using DCT,

$$\lim_{k\to\infty} \int f(x)(1 - \exp(-x^2/k))dx = \int \lim_{k\to\infty} f(x)(1 - \exp(-x^2/k))dx = 0.$$

3. Let $F : \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$F(x) = \int_a^x f(y)dy$$

for a Lebesgue integrable function f. Prove that F is absolutely continuous (with respect to the Lebesgue measure).

Sol. For $M > 0$, let $E_M := \{x : |f(x)| > M\}$. Let $\varepsilon > 0$. If $E \subset \mathbb{R}$ is measurable with finite measure,

$$\int_E |f(y)|dy = \int_{E\cap E_M} |f(y)|dy + \int_{E\cap E_M^{\complement}} |f(y)|dy$$
$$\leq \int_{E\cap E_M} |f(y)|dy + M\mu(E \cap E_M)$$
$$\leq \int_{E_M} |f(y)|dy + M\mu(E).$$

By DCT, the first summand goes to zero as $M \to \infty$, by dominating $|f\mathbf{1}_{E_M}| \le |f|$. Then choose $M > 0$ so that the first summand is less than $\varepsilon/2$. If $\mu(E) < \varepsilon/2M$, $|\int_E f| < \varepsilon$.
Hence, for given ε, any collection of disjoint intervals $\{(a_k, b_k)\}$ with $\sum_k (b_k - a_k) < \delta$, we have

$$\sum_k |F(b_k) - F(a_k)| \le \sum_k \int_{(a_k,b_k)} |f(y)|dy = \int_{\bigsqcup_k (a_k,b_k)} |f(y)|dy < \varepsilon.$$

Hence the signed measure induced by F is absolutely abs

4. Let $\mathcal{H}$ be a separable Hilbert space and T be a non-zero linear bounded operator on $\mathcal{H}$. Suppose that T is compact and symmetric. Prove that $\|T\|$ or $-\|T\|$ is an eigenvalue of T.

Sol. See [5] p. 192, Lemma 6.5.

5. Suppose that $\mathcal{M}$ is a σ-algebra in a set X and μ a finite measure on $(X, \mathcal{M})$. We say that a sequence of measurable functions $\{f_n\} \to f$ in measure if for every $\varepsilon > 0$

$$\mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) \to 0$$

as $n \to \infty$.

(a) Prove that if $f_n \to f$ almost everywhere (with respect to μ) then $f_n \to f$ in measure.

(b) Prove that if $f_n \to f$ in measure then $\{f_n\}$ has a subsequence that converges to f almost everywhere (with respect to μ).

Sol. (a) Fix $\varepsilon > 0$. Let $E_n := \{x : |f_n(x) - f(x)| > \varepsilon\}$. Since the sequence $\{f_n\}$ converges to f almost everywhere, the set

$$\bigcap_{N\ge 1} \bigcup_{n \ge N} E_n$$

has zero measure; x is in the set if and only if for any $N \ge 1$, $|f_n(x) - f(x)| > \varepsilon$ for some $n \ge N$, i.e., $f_n(x)$ does not converge to $f(x)$.
As μ is finite, we can deduce that

$$\mu\left(\bigcap_{N\ge 1} \bigcup_{n\ge N} E_n\right) = \lim_{N\to\infty} \mu\left(\bigcup_{n\ge N} E_n\right) \le \lim_{N\to\infty} \sum_{n\ge N} \mu(E_n) = 0$$

and hence $\mu(E_n) \to 0$, i.e., f_n converges to f in measure.

(b) Let $E_{\varepsilon,n} := \{x : |f_n(x) - f(x)| > \varepsilon\}$. Then there exists an increasing subsequence n_k such that $\mu(E_{2^{-k},n_k}) \le 2^{-k}$. Then

$$\mu\left(\bigcup_{k=1}^{\infty} E_{2^{-k},n_k}\right) \le \sum_{k=1}^{\infty} \mu(E_{2^{-k},n_k}) \le \sum_{k=1}^{\infty} 2^{-k} < \infty$$

and

$$\mu\left(\bigcap_{N=1}^{\infty}\bigcup_{k=N}^{\infty} E_{2^{-k},n_k}\right) = \lim_{N\to\infty} \mu\left(\bigcup_{k=N}^{\infty} E_{2^{-k},n_k}\right) \le \lim_{N\to\infty} \sum_{k=N}^{\infty} \mu(E_{2^{-k},n_k}) = 0.$$

For $x \notin \bigcap_{N=1}^{\infty}\bigcup_{k=N}^{\infty} E_{2^{-k},n_k}$, there exists some $N \ge 1$ such that $|f_{n_k}(x) - f(x)| \le 2^{-k}$ for all $k \ge N$. That is, $f_{n_k}(x) \to f(x)$. Hence if $f_{n_k} \to f$ fails, then $x \in \bigcap_{N=1}^{\infty}\bigcup_{k=N}^{\infty} E_{2^{-k},n_k}$, which has zero measure, i.e., $f_{n_k} \to f$ almost everywhere.

6. Assume that μ is a σ-finite measure on S. Suppose that $1 \le p \le q \le \infty$ and $1/p + 1/q = 1$. Prove that, for every $f \in L^p(S,\mu)$,

$$\|f\|_p = \sup\left\{\left|\int_S fg\,d\mu\right| : g \in L^q(S,\mu), \|g\|_q = 1\right\}.$$

Sol. By Hölder's inequality, if $\|g\|_q = 1$, then

$$\left|\int_S fg\,d\mu\right| \le \left(\int_S |f|^p d\mu\right)^{1/p} \left(\int_S |g|^q d\mu\right)^{1/q} = \|f\|_p$$

and the inequality "$\ge$" is shown.

If $\|f\|_p = 0$, there is nothing to show. Assume $\|f\|_p \ne 0$. To show the reversed inequality, let

$$g(x) = \frac{|f|^p}{\|f\|_p^{p/q} f}.$$

Then

$$\int |g|^q d\mu = \frac{1}{\|f\|_p^p} \int |f|^{pq-q} = \frac{\|f\|_p^p}{\|f\|_p^p} = 1$$

and hence $g \in L^q(S,\mu)$ with $\|g\|_q = 1$. Its evaluation is

$$\int fg\,d\mu = \frac{1}{\|f\|_p^{p/q}} \int |f|^p d\mu = \|f\|_p^{p-p/q} = \|f\|_p.$$

Hence $\|f\|_p$ is the supremum of the given set. Note that $p < \infty$.

1.3 2023 Feb Real

1. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous mapping. Prove that, if A is a Borel subset of $\mathbb{R}^n$, then $f^{-1}(A)$ is a Borel subset of $\mathbb{R}^m$.

Sol. Since f is continuous, f^{-1} preserves openness and closedness. Hence if A is either open or closed in $\mathbb{R}^n$, so is $f^{-1}(A)$ in $\mathbb{R}^m$.

Furthermore, the inverse image preserves unions and intersections. As a Borel set is generated by countable unions and intersections of open and closed sets, if A is a Borel subset, so is $f^{-1}(A)$.

2. Prove the following:
   (a) There exists a positive continuous function f on $\mathbb{R}$ so that f is integrable on $\mathbb{R}$, but $\limsup_{x\to\infty} f(x) = \infty$.
   (b) If f is uniformly continuous on $\mathbb{R}$ and integrable, then $\lim_{|x|\to\infty} f(x) = 0$.

Sol. (a) Let $g(x) = \exp(-x^2)$, which is continuous, integrable on $\mathbb{R}$, and positive. Let

$$h(x) = (-|x| + 1)\mathbf{1}_{|x|\le 1}.$$

For each $k \in \mathbb{Z}$, let

$$h_k(x) = 2^{|k|}h(4^{|k|}(x - k)).$$

Finally, define

$$f(x) = g(x) + \sum_{k\in\mathbb{Z}\setminus\{0\}} h_k(x),$$

where the series is well defined; for each $l \in \mathbb{Z}\setminus\{0\}$ and $x \in (l - 4^{|l|}, l + 4^{|l|})$, we have

$$\sum_{k\in\mathbb{Z}} h_k(x) = h_l(x).$$

Then

$$\int f = \int g + \int \sum_k h_k = \int g + \sum_k \int h_k = \int g + \sum_k 2^{-|k|+1} < \infty.$$

However, for each $n \in \mathbb{N}$, $f(n) > h_n(n) = 2^n$ and hence $\limsup_{x\to\infty} f(x) = \infty$.

(b) Suppose f does not vanish at infinity. Then there exist $\varepsilon > 0$ and a sequence $\{x_n\}$ with $x_n + 1 < x_{n+1}$ such that $|f(x_n)| > \varepsilon$. For such ε, there exists $1 > \delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon/2$. Which means that $|x - x_n| < \delta$ implies $|f(x)| > \varepsilon/2$. By continuity, on $(x_n - \delta, x_n + \delta)$, $f(x)$ is either positive or negative in whole interval. But then,

$$\left| \int_{x_n-\delta}^{x_n+\delta} f(x)dx \right| > \varepsilon\delta$$

and

$$\int_{\mathbb{R}} |f(x)|dx \geq \sum_{n=1}^{\infty} \int_{x_n-\delta}^{x_n+\delta} |f(x)|dx \geq \sum_{n=1}^{\infty} \varepsilon\delta = \infty,$$

contradicts that f is integrable.

3. Suppose that $a, b > 0$. Let

$$f(x) = \begin{cases} x^a \sin(x^{-b}), & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that f is of bounded variation in $[0, 1]$ if and only if $a > b$.

Sol. Let $x_0 = 0 < x_1 < \cdots < x_N = 1$ be a partition of $[0, 1]$. Then

$$\sum_{i=0}^{N-1} |f(x_{i+1}) - f(x_i)| = |f(x_1)| + \sum_{i=1}^{N-1} |f(x_{i+1}) - f(x_i)|$$

and by mean value theorem, $f(x_{i+1}) - f(x_i)| = (x_{i+1} - x_i)f'(\tilde{x}_i)$ for some $\tilde{x}_i \in (x_i, x_{i+1})$. Hence

$$|f(x_1)| + \sum_{i=1}^{N-1} |f(x_{i+1}) - f(x_i)| = |f(x_1)| + \sum_{i=1}^{N-1} |f'(\tilde{x}_i)|(x_{i+1} - x_i)$$

As the partition becomes finer, the sum of differences increases. Moreover, from the definition of Darboux's integral, which is equivalent with Riemann integral,

$$\sum |f'(\tilde{x}_i)(x_{i+1} - x_i) \to \int_0^1 |f'(x)|dx.$$

The derivative of f is given by $f'(x) = x^{a-1}(a\sin(x^{-b}) - bx^{-b}\cos(x^{-b})$ on $(0,1]$. If $a > b$, then

$$\int_0^1 |f'(x)|dx \le \int_0^1 x^{a-1}dx + b\int_0^1 x^{a-b-1}dx < \infty.$$

Conversely, if $a \le b$, let $t_0 = 1$ and $t_n = (\pi/2 + n\pi)^{-1/b}$. By taking summation over a partition $P_n = \{t_0 > t_1 > \cdots > t_n > 0\}$, we have

$$\begin{aligned}\sum |f(t_k) - f(t_{k+1})| &= \sum \left(\frac{1}{\pi/2 + k\pi}\right)^{a/b} + \left(\frac{1}{\pi/2 + (k+1)\pi}\right)^{a/b} \\ &\ge C\sum k^{-a/b}.\end{aligned}$$

As $n \to \infty$, $\sum k^{-a/b} \to \infty$. Hence it cannot be a bounded variation.

4. For a bounded linear operator T on a Hilbert space $\mathcal{H}$, we say that T is an isometry if $\|Tf\| = \|f\|$ for all $f \in \mathcal{H}$.

   (a) Prove that $T^*T = I$ if T is an isometry.

   (b) Prove that if an isometry T is surjective then it is unitary and $TT^* = I$.

Sol. (a) Let T be an isometry on $\mathcal{H}$. First, for $f, g \in \mathcal{H}$,

$$\|f-g\|^2 = \|f\|^2 - 2\,\mathrm{Re}\langle f, g\rangle + \|g\|^2 = \|Tf\|^2 - 2\,\mathrm{Re}\langle Tf, Tg\rangle + \|Tg\|^2$$

and hence $\mathrm{Re}\langle f, g\rangle = \mathrm{Re}\langle Tf, Tg\rangle$. By substituting f into $-if$, $\langle f, g\rangle = \langle Tf, Tg\rangle$.
Then

$$\langle f, T^*Tg\rangle = \langle Tf, Tg\rangle = \langle f, g\rangle$$

for all $f, g \in \mathcal{H}$ implies that $T^*T = I$.

(b) By surjectivity, for each $f \in \mathcal{H}$, $f = Tg$ for some $g \in \mathcal{H}$. Then $\|T^*f\| = \|T^*Tg\| = \|g\|$. Since T is an isometry, $\|f\| = \|Tg\| = \|g\|$. Hence T^* is also an isometry, and hence $(T^*)^*T^* = TT^* = I$. Injectivity directly follows.

5. Suppose that $\mathcal{M}$ is a σ-algebra in a set X and μ a (positive) measure on $(X, \mathcal{M})$. For $f \in L^1(\mu)$, define a signed measure λ on $(X, \mathcal{M})$ by $\lambda(E) = \int_E f d\mu$ for $E \in \mathcal{M}$. Prove that

$$|\lambda|(E) = \int_E |f| d\mu.$$

Sol. By Hahn decomposition and Jordan decomposition, $\lambda = \lambda^+ - \lambda^-$ and $|\lambda| = \lambda^+ + \lambda^-$ for two mutually singular positive measures $\lambda^\pm$, where

$$\lambda^+(E) = \lambda(E \cap X^+), \quad \lambda^-(E) = -\lambda(E \cap X^-),$$

where $X^\pm$ are disjoint subsets of X, such that $\lambda(A) \geq 0$ (resp. ≤ 0) for any measurable A with $A \subset X^+$ (resp. $\subset X^-$). By the uniqueness of Hahn decomposition up to symmetric difference, $f \geq 0$ on X^+ a.e., and $f < 0$ on X^- a.e.

Then we have

$$\begin{aligned} |\lambda|(E) &= \lambda^+(E) + \lambda^-(E) \\ &= \int_{E\cap X^+} f d\mu - \int_{E \cap X^-} f d\mu \\ &= \int_{E\cap X^+} |f| d\mu + \int_{E\cap X^-} |f| d\mu = \int_E |f| d\mu. \end{aligned}$$

It can be shown without using Hahn decomposition. For a partition $\{E_j\}$ of E,

$$\sum_j \left| \int_{E_j} f d\mu \right| \leq \sum_j \int_{E_j} |f| d\mu = \int_E |f| d\mu$$

guarantees that $|\lambda|(E) \leq \int_E |f| d\mu$. Conversely, $E^+ := \{x \in E : |f(x)| \geq 0\}$ and $E^- := \{x \in E : |f(x)| < 0\}$ form a measurable partition of E, with

$$\left| \int_{E^+} f d\mu \right| + \left| \int_{E^-} f d\mu \right| = \int_{E^+} |f| d\mu + \int_{E^-} |f| d\mu = \int_E |f| d\mu.$$

Hence the supremum of $\sum_j |\lambda(E_j)|$ over all measurable partitions of E is $\int_E |f| d\mu$.

6. Let F be an increasing function on $[0, 1]$ with $F(0) = 0$ and $F(1) = 1$. Let μ be a Borel measure defined by $\mu((a, b)) = F(b-) - F(a+)$ and $\mu(0) = \mu(1) = 0$. Suppose that the function F satisfies a Lipschitz condition $|F(x) - F(y)| \leq A|x - y|$ for some $A > 0$. Prove that $\mu \ll m$, where m is the Lebesgue measure on $[0, 1]$.

Sol. By Lipschitz condition, F is continuous, and hence $\mu((a, b)) = F(b) - F(a)$. Then immediately

$$\mu\left(\bigsqcup_{n=1}^\infty (a_n, b_n) \right) = \sum_{n=1}^\infty \mu((a_n, b_n)) \leq A \sum_{n=1}^\infty (b_n - a_n) = A \cdot m \left(\bigsqcup_{n=1}^\infty (a_n, b_n) \right).$$

That is, for open $U \subset [0,1]$, $\mu(U) \le A \cdot m(U)$.

Let E be a measurable set with $m(E) = 0$. Then there exists a decreasing sequence of open sets V_n in $[0,1]$, such that $E \subset V_n$ and $m(V_n) < m(E) + 1/n$, and

$$\mu(E) \le \mu(V_n) \le A \cdot m(V_n) < A \cdot m(E) + A/n$$

for all n. Hence $\mu(E) \le A \cdot m(E) = 0$. Therefore $\mu \ll m$ is shown.

1.4 2022 Aug Real

1. Suppose that $A \subset E \subset B \subset \mathbb{R}$, where A and B are Lebesgue measurable sets of finite measure. Prove that if $m(A) = m(B)$, then E is Lebesgue measurable.

Sol. The set $E \setminus A$ has zero measure;

$$m_*(E \setminus A) \leq m_*(B \setminus A) = m(B \setminus A) = m(B) - m(A) = 0.$$

Since A and B are both finite measurable sets,

$$m(B \setminus A) = m(B) - m(A).$$

Therefore, E is measurable because E is the union of two measurable sets A and $E \setminus A$.

2. Prove the following generalization of Lebesgue's dominated convergence theorem: Suppose that f_1, f_2,...are measurable functions on $\mathbb{R}^d$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. $x \in \mathbb{R}^d$. Suppose also that g_1, g_2,...are nonnegative, integrable functions such that $|f_k(x)| \leq g_k(x)$ and $\lim_{n\to\infty} g_n(x) = g(x)$ exists for a.e. $x \in \mathbb{R}^d$. Prove that if g is integrable with $\int g = \lim_{n\to\infty} \int g_n$ then $\int f = \lim_{n\to\infty} \int f_n$.

Sol. [1] p.59 Exercise 20.

Imitate the proof of Lebesgue's dominated convergence theorem;

Since f is measurable and $|f| \leq g$ almost everywhere, $f \in L^1$. By taking real and imiginary parts it suffices to assume that f_n and f are real-valued, in which case we have By Fatou's lemma,

$$\begin{aligned}\int 2g = \int \liminf_{n\to\infty}(g_n + g - |f_n - f|) &\leq \liminf_{n\to\infty} \int (g_n + g - |f_n - f|) \\ &= 2\int g + \liminf_{n\to\infty}\left(-\int |f_n - f|\right) \\ &= 2\int g - \limsup_{n\to\infty} \int |f_n - f|\end{aligned}$$

and hence $\limsup_{n\to\infty} \int |f_n - f| \leq 0$. Which implies that

$$\lim_{n\to\infty} \left| \int (f_n - f) \right| = 0$$

and hence $\lim_{n\to\infty} \int f_n = \int f$.

3. Supppose that $F : [a, b] \to \mathbb{R}$ is absolutely continuous and increasing. Let $A = F(a)$, $B = F(b)$. Prove the following:

   (a) If $E \subset [A, B]$ is measurable, then $F^{-1}(E) \cap \{F'(x) > 0\}$ is measurable.

   (b) There exists such an F that is strictly increasing, $F'(x) = 0$ on a set of positive measure, and there is a measurable subset $E \subset [A, B]$ so that $m(E) = 0$ but $F^{-1}(E)$ is not measurable.

Sol. [5] p. 149 Exercise 20.

(a) First, we will prove the statement

$$m(\mathcal{O}) = \int_{F^{-1}(\mathcal{O})} F'(x)dx$$

where $\mathcal{O}$ is open in $[A, B]$.

Because every open set in $\mathbb{R}$ is a union of disjoint open intervals and inverse image preserves the union, it is sufficient to show that the statement holds for open intervals.

Let I be an open interval in $[A, B]$. Even though it contains an endpoint of $[A, B]$, because the measure of singleton is zero, its measure is same with removing the endpoint. Hence further assume that I has no endpoint. Let $I = (F(u), F(v))$. If $F'(u) = 0$, then replace $F(u)$ to $F(u')$, where $u' = \sup\{x : F(x) = F(u)\}$, and similarly replace $F(v)$ to $F(v')$ where $v' = \inf\{x : F(x) = F(v)\}$ if $F'(v) = 0$. Then

$$\begin{aligned} m(I) &= F(v') - F(u') \\ &= \int_{u'}^{v'} F'(x)dx \\ &= \int_{(u',v')} F'(x)dx \\ &= \int_{F^{-1}(I)} F'(x)dx \end{aligned}$$

where the second equality is from absolute continuity. Therefore the statement in the hint is shown.

Let $E \subset [A, B]$ be a measurable set. The set $P := \{x : F'(x) > 0\} = (F')^{-1}((0, \infty))$ is measurable set because F' is measurable. Then both have

G_δ sets G and G' such that $m(G \setminus E) = m(G' \setminus P) = 0$. The claim is that $F^{-1}(G) \cap G'$ is a G_δ set where the differnece with $F^{-1}(E) \cap P$ has zero measure.

By elementary set operations,

$$\begin{aligned} &(F^{-1}(G) \cap G') \setminus (F^{-1}(E) \cap P) \\ =&(F^{-1}(G \setminus E) \cap G') \cup (F^{-1}(G) \cap (G' \setminus P)) \\ =&(F^{-1}(G \setminus E) \cap (P \cup (G' \setminus P)) \cup (F^{-1}(G) \cap (G' \setminus P)) \\ =&(F^{-1}(G \setminus E) \cap P) \cup (F^{-1}(G) \cap (G' \setminus P)). \end{aligned}$$

To verify our claim, it is sufficient to show that $F^{-1}(G \setminus E) \cap P$ has zero measure, as $m(F^{-1}(G) \cap (G' \setminus P))$ is bounded by $m(G' \setminus P) = 0$.

Since $G \setminus E$ has zero measure, there exists open O_n such that $(G \setminus E) \subset O_n$ and $m(O_n \setminus (G \setminus E)) = m(O_n) \leq 1/n$. Then

$$\begin{aligned} \frac{1}{n} \geq m(O_n) &= \int_{F^{-1}(O_n)} F'(x)dx \\ &\geq \int_{F^{-1}(\bigcap_i O_i)} F'(x)dx \\ &\geq \int_{F^{-1}(\bigcap_i O_i) \cap P} F'(x)dx \end{aligned}$$

for all n, and as $F'(x) > 0$ on $F^{-1}(\bigcap_i O_i) \cap P$, the set $F^{-1}(\bigcap_i O_i) \cap P$ has zero measure. As $G \setminus E \subset \bigcap_i O_i$, $F^{-1}(G \setminus E) \cap P$ also has zero measure.

(b) Construct Cantor-like set C by removing the middle $1/4^n$ from each 2^{n-1} subintervals. Then $m(C) = 1 - 1/4 - 2 \times 1/4^2 - 2^2 \times 1/4^3 - \cdots = 1/2 > 0$. As C is measurable, its complement K on $[0, 1]$ is also measurable. Hence $\mathbf{1}_K$ is measurable function, and the integral from 0 to x is measurable function. The claim is that $F(x) := \int_0^x \mathbf{1}_K(t)dt$ satisfies strictly increasing and absolute continuity, and $F'(x) = 0$ on nonzero measure set.

- Let $x, y \in [0, 1]$ with $x < y$. Then

  $$F(y) - F(x) = \int_x^y \mathbf{1}_K(u)du \geq 0$$

  and it is monotonically increasing. If either x or y, without loss of generality x, is in K, then as K is open, some open ball $B_x(r) \subset K$ exists with $r < y - x$. Then the integral is bigger than the measure of $B_x(r) \cap K$, and it is positive. If both x and y are in C, as C has empty interior, there exists some nonempty open $U \subset K \cap (x, y)$. Then the integral becomes the measure of $U \cap K \cap (x, y)$, which is positive. This shows that F is strictly increasing.

- Since F is defined as the integral of integrable function, by proposition 1.12 in chapter 2, it immediately satisfies absolute continuity.
- By Lebesgue differentiation theorem, $F'(x) = \mathbf{1}_K(x)$ for a.e. $x \in [0, 1]$. Hence $F'(x) = 0$ a.e. on C.

As K is open in $\mathbb{R}$, K can be expressed as the disjoint union of open intervals. Indeed, such open intervals are removed intervals in constructing Cantor-like set C. Let $\{D_i\}$ be the collection of such intervals. Then by injectivity of F,

$$F(K) = F\left(\bigsqcup_i D_i\right) = \bigsqcup_i F(D_i),$$

and if a_i is the left endpoint of the interval D_i, then

$$F(D_i) = \left\{\int_0^x \mathbf{1}_K : x \in D_i\right\} = \left\{F(a_i) + \int_{a_i}^x \mathbf{1}_K : x \in D_i\right\}$$

gives that

$$\begin{aligned} m(F(D_i)) &= m\left(\left\{F(a_i) + \int_{a_i}^x \mathbf{1}_K : x \in D_i\right\}\right) \\ &= m\left(\left\{\int_{a_i}^x \mathbf{1}_K : x \in D_i\right\}\right) \\ &= m(\{x - a_i : x \in D_i\}) = m(D_i). \end{aligned}$$

Therefore

$$m(F(K)) = \sum_{i=1}^{\infty} m(F(D_i)) = \sum_{i=1}^{n} \frac{2^{i-1}}{4^i} = \frac{1}{2} = m([F(1) - F(0)]).$$

As $m(F(K)) + m(F(C)) = m([F(1) - F(0)])$, $F(C)$ has zero measure. Let U be a subset of C, which is nonmeasurable. Such U exists since C has positive measure. Then choose $E = F(U)$ so that $m(E) \le m(F(C)) = 0$, whereas $F^{-1}(F(U)) = U$ is nonmeasurable.

4. Let $\mathcal{B}$ be a Banach space.

(a) Prove that $\mathcal{B}$ is a Hilbert space if and only if

$$\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2)$$

for any $f, g \in \mathcal{B}$.

(b) Prove that $L^p(\mathbb{R}^d)$ ($p \in [1, \infty)$) with the Lebesgue measure is a Hilbert space if and only if $p = 2$.

Sol. (a) A Hilbert space is always a Banach space, where it satsifies described paralellogram law.

Conversely, suppose that $\mathcal{B}$ satisfies the paralellogram law. Define the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{B}$ as *polarization*:

$$\langle f, g \rangle := \frac{1}{4} \sum_{k=1}^{4} i^k \|f + i^k g\|^2.$$

Then it satisfies the axioms of inner product:

- For $f \in \mathcal{B}$,

$$\langle f, f \rangle = \frac{1}{4} \sum_{k=1}^{4} i^k \|f + i^k f\|^2 = \frac{1}{4} \cdot 4\|f\|^2 \geq 0$$

  and the equality holds if and only if $f = 0$. Thus it satisfies positive definiteness.

- Let $f, g \in \mathcal{B}$. Then

$$\begin{aligned}
\langle f, g \rangle &= \frac{1}{4} \sum_{k=1}^{4} i^k \|f + i^k g\|^2 \\
&= \frac{1}{4}(i\|f + ig\|^2 - \|f - g\|^2 - i\|f - ig\|^2 + \|f + g\|^2) \\
&= \frac{1}{4}(i\| - if + g\|^2 - \| - f + g\|^2 - i\|if + g\|^2 + \|f + g\|^2) \\
&= \frac{1}{4} \sum_{k=1}^{4} i^{-k} \|g + i^k f\|^2 = \overline{\langle g, f \rangle}.
\end{aligned}$$

  That is, it satisfies conjugate symmetry.

- First, for $f, g \in \mathcal{B}$,

$$\begin{aligned}\langle f, -g\rangle &= \frac{1}{4}\sum_{k=1}^{4} i^k \|f - i^k g\|^2 \\ &= -\frac{1}{4}\sum_{k=1}^{4} i^{k+2} \|f + i^{k+2} g\|^2 \\ &= -\langle f, g\rangle\end{aligned}$$

  and

$$\begin{aligned}\langle f, ig\rangle &= \frac{1}{4}\sum_{k=1}^{4} i^k \|f + i^{k+1} g\|^2 \\ &= -\frac{i}{4}\sum_{k=1}^{4} i^{k+1} \|f + i^{k+1} g\|^2 \\ &= -i\langle f, g\rangle.\end{aligned}$$

  By conjugate symmetry, $\langle if, g\rangle = i\langle f, g\rangle$.
  Let $f_1, f_2 \in \mathcal{B}$. Then

$$\begin{aligned}\langle f_1 + f_2, g\rangle &= \frac{1}{4}\sum_{k=1}^{4} i^k \|f_1 + f_2 + i^k g\|^2 \\ =&\frac{1}{4}\sum_{k=1}^{4} i^k (2\|f_1\|^2 + 2\|f_2 + i^k g\|^2 - \|f_1 - f_2 - i^k g\|^2) \\ =&\frac{1}{4}\sum_{k=1}^{4} i^k (2\|f_1\|^2 + 2\|f_2 + i^k g\|^2 \\ &\qquad - (2\|f_1 - i^k g\|^2 + 2\|f_2\|^2 - \|f_1 + f_2 - i^k g\|)) \\ =&\frac{1}{2}\sum_{k=1}^{4} i^k (\|f_1\|^2 + \|f_2\|^2 + \|f_2 + i^k g\|^2 - \|f_1 - i^k g\|^2) \\ &+ \frac{1}{4}\sum_{k=1}^{4} i^k \|f_1 + f_2 - i^k g\|^2 \\ =&2(\langle f_2, g\rangle - \langle f_1, -g\rangle) + \langle f_1 + f_2, -g\rangle \\ =&2(\langle f_2, g\rangle + \langle f_1, g\rangle) - \langle f_1 + f_2, g\rangle\end{aligned}$$

so that $\langle f_1 + f_2, g\rangle = \langle f_1, g\rangle + \langle f_2, g\rangle$.
By these properties, for $n \in \mathbb{Z}$, $\langle (n+1)f, g\rangle = \langle nf, g\rangle + \langle f, g\rangle = (n+1)\langle f, g\rangle$ is valid.
For a nonzero integer n,

$$\langle f, g\rangle = \left\langle \frac{n}{n} f, g \right\rangle = n \left\langle \frac{1}{n} f, g \right\rangle$$

so that $\frac{1}{n}\langle f, g\rangle = \langle \frac{1}{n} f, g\rangle$. Hence $\langle qf, g\rangle = q\langle f, g\rangle$ for $q \in \mathbb{Q} + i\mathbb{Q}$. As $\mathbb{Q} + i\mathbb{Q}$ is dense in $\mathbb{C}$ and since $\mathcal{B}$ is complete, $\langle zf, g\rangle = z\langle f, g\rangle$ for all $z \in \mathbb{C}$. Hence it is linear in first component.

This inner product induces same norm given in $\mathcal{B}$, by definition. Therefore it becomes a Hilbert space automatically.

(b) If $p = 2$, then $L^2(\mathbb{R}^d)$ is a Hilbert space with inner product $\langle f, g\rangle := \int f\overline{g}dm$. Conversely, let $f = \mathbf{1}_{(0,1)^d}$ and $g = \mathbf{1}_{(1,2)^d}$. Then

$$\|f+g\|_p^2 + \|f-g\|_p^2 = 2\left(\int \mathbf{1}_{(0,1)^d \cup (1,2)^d} dm\right)^{2/p} = 2 \cdot (2d)^{2/p}$$

and

$$2(\|f\|_p^2 + \|g\|_p^2) = 2\left\{\left(\int \mathbf{1}_{(0,1)^d} dm\right)^{2/p} + \left(\int \mathbf{1}_{(1,2)^d} dm\right)^{2/p}\right\} = 4d^{2/p}.$$

so that $2 \cdot (2d)^{2/p} = 4d^{2/p}$ if and only if $p = 2$. Hence if $p \neq 2$, then paralellogram law fails, and thus it cannot be a Hilbert space.

5. Let μ be a σ-finite measure on a measure space X. Prove that every measurable set of infinite measure in X contains measurable sets of arbitrary large finite measure.

Sol. Let $X = \bigcup_{n\in\mathbb{N}} E_n$, where E_n has finite measure. Let $E_n' = \bigcup_{i=1}^n E_i$. Then each E_n' has finite measure, and $X = \bigcup_{n\in\mathbb{N}} E_n'$.

Let S be a subset of infinite measure. Then

$$S = S \cap X = S \cap \left(\bigcup_{n\in\mathbb{N}} E_n'\right) = \bigcup_{n\in\mathbb{N}} (S \cap E_n').$$

As the sequence $S \cap E_n'$ is increasing,

$$\mu(S) = \mu\left(\bigcup_{n\in\mathbb{N}} (S \cap E_n')\right) = \lim_{n\to\infty} \mu(S \cap E_n') = \infty.$$

Hence for any $M > 0$, there exists some $N \in \mathbb{N}$ such that $\mu(S \cap E_n') > M$ if $n \geq N$, where $S \cap E_n' \subset S$.

6. Let S be a set of all complex, measurable, simple functions on a measure space X with a positive measure μ, satisfying that, for any $f \in S$,

$$\mu(\text{supp}(f)) < \infty.$$

Prove that S is dense in $L^p(X, \mu)$ for any $1 \leq p < \infty$.

Sol. [4] p.69 Theorem 3.13.

It is clear that $S \subset L^p(\mu)$. Suppose $f \geq 0$, $f \in L^p(\mu)$, and define $\{s_n(x)\}$ as

$$s_n(x) = \begin{cases} \lfloor 2^n f(x) \rfloor 2^{-n} & \text{if } 0 \leq f(x) < n, \\ n & \text{if } n \leq f(x) \leq \infty. \end{cases}$$

Then s_n converges to f pointwisely. The support of s_n is $\{x : 2^{-n} \leq f(x)\}$[1].

This set has finite measure since

$$\begin{aligned} \mu(\{f(x) \geq 2^{-n}\}) &= \int_{\{f(x) \geq 2^{-n}\}} d\mu \\ &= 2^{np} \int_{\{f(x) \geq 2^{-n}\}} 2^{-np} d\mu \\ &\leq 2^{np} \int_{\{f(x) \geq 2^{-n}\}} f^p d\mu \\ &\leq 2^{np} \|f\|_p^p < \infty. \end{aligned}$$

Hence $\{s_n\}$ is a sequence in S.

Since $|f - s_n|^p \leq (|f| + |s_n|)^p \leq 2^p |f|^p$, DCT shows that $\|f - s_n\|_p \to 0$ as $n \to \infty$. Thus f is in $\overline{S}$, the topological closure of S. The general case follows immediately, by decomposing $f = (\text{Re } f)^+ - (\text{Re } f)^- + i(\text{Im } f)^+ - i(\text{Im } f)^-$.

[1] There are several issues in defining the terminology *support*; [5] p. 53 defines the support of a function as the set of all points where the function does not vanishes, whereas [4] p. 38 definition 2.9 says that the support of a function is the closure of the set defined in [5]. In this problem, we will follow the former definition.

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1. For a given set $E \in \mathbb{R}^d$, define $\mathcal{O}_n = \{x \in \mathbb{R}^d : d(x, E) < 1/n\}$.

   (a) Show that $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$ if E is compact, where m is the Lebesgue measure.

   (b) Show that the conclusion in (a) may be false for E closed and unbounded; or E open and bounded.

Sol. (a) First, the set $\mathcal{O}_n$ is open; let $x \in \mathcal{O}_n$, and let $\delta = d(x, E) = \inf\{d(x, w) : w \in E\}$.
If $d(x, y) < 1/n - \delta$, then

$$\begin{aligned} d(y, E) &= \inf_{z\in E} d(y, z) \\ &\le \inf_{z\in E}(d(y, x) + d(x, z)) \\ &= d(y, x) + \inf_{z\in E} d(x, z) \\ &< \frac{1}{n} - \delta + \delta = \frac{1}{n}, \end{aligned}$$

that is, $y \in \mathcal{O}_n$, and hence $\mathcal{O}_n$ is open, and hence it is measurable.
The set $\mathcal{O}_1$ has finite measure; since E is bounded, E is a subset of $B_N(0)$, which has finite measure. Then if $x \notin B_{N+1}(0)$, then

$$d(x, E) = \inf_{z\in E} d(x, z) \ge \inf_{z\in B_N(0)} d(x, z) \ge 1$$

and thus $x \notin \mathcal{O}_1$. That is, $\mathcal{O}_1 \subset B_{N+1}(0)$. By monotonicity of measure, $\mathcal{O}_1$ has finite measure.
If $x \in \mathcal{O}_n$ for all $n \in \mathbb{N}$, then $d(x, E) < \inf 1/n = 0$, i.e., x is a limit point of E. Since E is closed, $x \in E$. That is, $\bigcap_n \mathcal{O}_n \subset E$. Conversely, the reversed inclusion is trivial.
Hence, $\{\mathcal{O}_n\}_{n=1}^{\infty}$ is a decreasing sequence of open sets, whose intersection is E. Therefore

$$m(E) = m\left(\bigcap_n \mathcal{O}_n\right) = \lim_{n\to\infty} m(\mathcal{O}_n).$$

(b) If the bounded condition is omitted, there is a counterexample; For $d = 1$, choose $E = \mathbb{N}$. Then $\mathcal{O}_n = \bigcup_{k\in\mathbb{N}}(k - 1/n, k + 1/n)$ and $m(\mathcal{O}_n) = \infty$ for all n, but $m(E) = 0$.

If the closed condition is omitted, there is a counterexample; Let C be the standdard Cantor set. For given $r > 0$, let $n \in \mathbb{N}$ be sufficiently large so that $r > 2^{-n}$. For $x \in C$, x lies in a subinterval in n-th construction, whose length is 2^{-n}. Then $(x-r, x+r)$ contains an element in $[0,1] \setminus C$. That is, $C \subset \overline{[0,1] \setminus C}$. Hence $[0,1]$ is the closure of $[0,1] \setminus C$. By letting $E = [0,1] \setminus C$, E is open and bounded with $m(E) = 1/2$.

As $[0,1] = \overline{E}$, for any $p \in [0,1]$, $(p - 1/n, p + 1/n) \cap E \neq \varnothing$ for all $n \in \mathbb{N}$. Hence $d(p, E) = 0 < 1/n$, and $[0,1] \subset \mathcal{O}_n$ for all n. Clearly $\mathcal{O}_1$ is bounded by boundedness of E, and therefore

$$m\left(\bigcap_{n=1}^{\infty} \mathcal{O}_n\right) = \lim_{n\to\infty} m(\mathcal{O}_n) \geq \lim_{n\to\infty} m([0,1]) = 1 \neq 0 = m(E).$$

2. Show that $f * g$ is uniformly continuous when f is integrable and g is bounded.

Sol. Let $\varepsilon > 0$. Let h be a compactly supported continuous function which approximates f with error less than $\varepsilon/2$ in L^1 norm, i.e., $\|f - h\|_{L^1} < \varepsilon/2$.

Let $|g| \leq M$ with $M > 0$. Then

$$\begin{aligned} |f * g(x+t) - f * g(x)| &= \left| \int_{\mathbb{R}^d} (f(x+t-y) - f(x-y)) g(y) dy \right| \\ &\leq M \int_{\mathbb{R}^d} |f(x+t-y) - f(x-y)| dy \\ &= M \int_{\mathbb{R}^d} |f(t+u) - f(u)| du \end{aligned}$$

and from

$$|f(t+u) - f(u)| \leq |f(t+u) - h(t+u)| + |h(t+u) - h(u)| + |h(u) - f(u)|,$$

we get

$$\begin{aligned} &\int_{\mathbb{R}^d} |f(t+u) - f(u)| dy \\ \leq &\int_{\mathbb{R}^d} |f(t+u) - h(t+u)| + |h(t+u) - h(u)| + |h(u) - f(u)| du \\ =& 2\|f - h\|_{L^1} + \int_{\mathbb{R}^d} |h(t+u) - h(u)| du. \end{aligned}$$

From uniform continuity on compact set, if $\|t\|$ is sufficiently small, the last term can be bounded by $\varepsilon|\operatorname{supp} h|$, where $|\cdot|$ denotes the Lebesgue measure. Hence $|f * g(x+t) - f * g(x)| < M\varepsilon(1 + |\operatorname{supp} h|)$, and the conclsion holds.

The construction of such h is as following: Let $R > 0$ be sufficiently large so that $\|f - f\mathbf{1}_{\{x:\|x\|\le R\}}\|_{L^1} < \varepsilon/2$. On the compact set $K_R := \{x : \|x\| \le R\}$, by Lusin's theorem, there exists a continuous function h on K_R with compact support, such that $\|f\mathbf{1}_{K_R} - h\|_{L^1} < \varepsilon/2$.

There exists $\delta > 0$ satisfying $|E| < \delta$ implies $\int_E |f| < \varepsilon$. Let $\eta > 0$ be sufficiently small so that $|K_{R+\eta} \setminus K_R| < \delta$ and $|K_{R+\eta} \setminus K_R| \max |h(x)| < \varepsilon$. Finally, on $K_{R+\eta} \setminus K_R$, for each unit vector v, define by piecewisely linear between $(Rv, h(Rv))$ and $((R+\eta)v, 0)$. Then h is continuous, compactly supported, and

$$\begin{aligned}
\|f - h\|_{L^1} &= \int_{\mathbb{R}^d} |f(x) - h(x)|dx \\
&= \int_{K_R} |f(x) - h(x)|dx + \int_{K_{R+\eta}\setminus K_R} |f(x) - h(x)|dx \\
&\qquad + \int_{K_{R+\eta}^{\complement}} |f(x) - h(x)|dx \\
&\le \varepsilon/2 + \int_{K_{R+\eta}\setminus K_R} |f(x)| + |h(x)|dx + \varepsilon/2 \\
&\le \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.
\end{aligned}$$

By replacing ε to $\varepsilon/3$, we get the desired result.

3. Suppose that f is integrable on $\mathbb{R}^k$. For each $\alpha > 0$, define $E_\alpha = \{x \in \mathbb{R}^k : |f(x)| > \alpha\}$.

   Prove that
$$\int_{\mathbb{R}^k} |f(x)|dx = \int_0^\infty m(E_\alpha)d\alpha.$$
   (Here, m is the Lebesgue measure.)

Sol. By applying the Fubini-Tonelli theorem,

$$\begin{aligned}
\int_0^\infty m(E_\alpha)d\alpha &= \int_0^\infty \int_{\mathbb{R}^k} \mathbf{1}_{|f(x)|>\alpha} dx d\alpha \\
&= \int_{\mathbb{R}^k} \int_0^\infty \mathbf{1}_{|f(x)|>\alpha} d\alpha dx \\
&= \int_{\mathbb{R}^k} |f(x)|dx.
\end{aligned}$$

4. Let $\mathcal{H}$ be a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ a bounded linear operator.

   If T is self-adjoint, prove that

$$\|T\| = \sup_{x \in \mathcal{H}} \{|\langle Tx, x\rangle| : \|x\| \le 1\}.$$

Sol. See [5] p. 184.

Let $M = \sup\{|\langle Tf, f\rangle| : \|f\| = 1\}$. As $\|T\| = \sup\{|\langle Tf, g\rangle| : \|f\| \le 1, \|g\| \le 1\}$, clearly $M \le \|T\|$. Conversely, let $f, g \in \mathcal{H}$ whose norm is at most 1. Then

$$\langle Tf, g\rangle = \frac{1}{4}\sum_{k=1}^{4} i^k \langle T(f + i^k g), f + i^k g\rangle$$

and by self-adjoint property,

$$\mathrm{Re}\langle Tf, g\rangle = \frac{1}{4}(\langle T(f+g), f+g\rangle - \langle T(f-g), f-g\rangle).$$

From $|\langle Th, h\rangle| \le M\|h\|^2$ and paralellogram law,

$$|\,\mathrm{Re}\langle Tf, g\rangle| \le \frac{M}{2}(\|f\|^2 + \|g\|^2) \le M.$$

By replacing g by $e^{i\theta}g$, we may conclude that $|\langle Tf, g\rangle| \le M$. By taking supremum over f and g, $\|T\| \le M$.

5. Suppose that (X, μ) is a measure space such that $\mu(A) > 0 \Rightarrow \mu(A) \ge 1$.

   Prove that, if $1 \le p \le q \le \infty$, then

$$\|f\|_{L^\infty(X,\mu)} \le \|f\|_{L^q(X,\mu)} \le \|f\|_{L^p(X,\mu)} \le \|f\|_{L^1(X,\mu)}.$$

Sol. It suffices to show the inequality only for nonnegative functions.

It holds for integrable simple functions; Let $\varphi = \sum_{k=1}^n c_k \mathbf{1}_{E_k}$ be the canonical form of a simple function. Then

$$\begin{aligned}
\|\varphi\|_p^q &= \left(\sum_{k=1}^n |c_k|^p \mu(E_k)\right)^{q/p} \\
&\ge \sum |c_k|^q (\mu(E_k))^{q/p} \\
&\ge \sum |c_k|^q (\mu(E_k)) = \|\varphi\|_q^q,
\end{aligned}$$

where the first inequality is from $(1+x)^p \geq 1 + x^p$ and mathematical induction, and the property $\mu(A) > 0$ implies $\mu(A) \geq 1$ is used for the second inequality. Therefore $\|\varphi\|_{L^q(X,\mu)} \leq \|\varphi\|_{L^p(X,\mu)} \leq \|\varphi\|_{L^1(X,\mu)}$ is valid. By the way,

$$\|\varphi\|_\infty^q = \max_{\mu(E_k)\neq 0} |c_k|^q \leq \sum_{k=1}^n |c_k|^q \mu(E_k),$$

hence $\|\varphi\|_{L^\infty(X,\mu)} \leq \|\varphi\|_{L^q(X,\mu)}$ is valid.

Let $\{\varphi_n\}$ and $\{\psi_n\}$ be sequences of positive simple functions such that $\{\varphi_n(x)\}$ and $\{\psi_n(x)\}$ are increasing sequences for almost every x, and $\varphi_n(x) \to f_+(x) := \max(f(x), 0)$ and $\psi_n(x) \to f_-(x) := \max(-f(x), 0)$. Then for $r \in \{1, p, q\}$,

$$\begin{aligned}
\|f\|_{L^r(X,\mu)}^r &= \int_X |f|^r d\mu = \int_X |f_+|^r + |f_-|^r d\mu = \int_X \left|\lim_{n\to\infty} \varphi_n\right|^r + \left|\lim_{n\to\infty} \psi_n\right|^r d\mu \\
&= \int_X \lim_{n\to\infty} |\varphi_n|^r + \lim_{n\to\infty} |\psi_n|^r d\mu = \lim_{n\to\infty} \int_X |\varphi_n|^r + |\psi_n|^r d\mu \\
&= \lim_{n\to\infty} \int_X |\varphi_n + \psi_n|^r d\mu,
\end{aligned}$$

where $\varphi_n + \psi_n$ is a simple function. Because the integration by approximating simple functions is well defined, the inequalities are valid except the first one.

To simplify, let $\|f\| := \|f\|_{L^\infty(X,\mu)}$. For simple functions $\sigma_n = \varphi_n + \psi_n$, let $\sigma_n(x) = \sum_{m=1}^{N_n} s_{m,n} \mathbf{1}_{E_{m,n}}$. Then $|s_{m,n}| \leq \|f\|$ for all possible pairs (m, n), and $\|s_n\|_{L^\infty(X,\mu)} \leq \|f\|$. Conversely, because $\|s_n\|_{L^\infty(X,\mu)}$ increases by its construction, if $\|s_n\|_{L^\infty(X,\mu)}$ does not converge to $\|f\|$, then for some $k > 0$, $\|s_n\|_{L^\infty(X,\mu)} < \|f\| - k$ holds for every n. Then on the set $E = \{x \in X : |f(x)| > \|f\| - k\}$, $s_n(x)$ cannot not converge to $f(x)$, where $\mu(E) > 0$. It has a contradiction, and thus $\|f\| = \lim_{n\to\infty} \|s_n\|_{L^\infty(X,\mu)}$. This argument guarantees the first inequality.

6. Let $C([a,b])$ be the vector space of continuous functions on the closed and bounded interval $[a,b]$. Prove the following:

   (a) For a given Borel measure μ on this interval with $\mu([a,b]) < \infty$,

   $$f \mapsto \ell(f) = \int_a^b f(x) d\mu(x)$$

   is a linear functional on $C([a,b])$, which is positive in the sense that $\ell(f) \geq 0$ if $f \geq 0$.

(b) For any positive linear functional ℓ on $C([a,b])$, there exists a unique finite Borel measure μ such that

$$\ell(f) = \int_a^b f(x) d\mu(x)$$

for all $f \in C([a,b])$.

Sol. [4] p. 40, theorem 2.14. (Riesz representation theorem for Borel measures)

1.6 2021 Aug Real

1. Prove the following statements in $\mathbb{R}^n$:

   (a) A countable union of (Lebesgue) measurable sets is (Lebesgue) measurable.

   (b) Closed sets are (Lebesgue) measurable.

Sol. [5] p 17, p 18.

(a) Let $\{E_i\}_{i=1}^{\infty}$ be a countable collection of measurable subsets of $\mathbb{R}^n$. Let $\varepsilon > 0$ be given. Then by definition, for each i, there exists open V_i, containing E_i such that $m_*(V_i \setminus E_i) < \varepsilon 2^{-i}$, where m_* denotes exterior measure. Then,

$$\bigcup_{i=1}^{\infty}(V_i \setminus E_i) \supset \bigcup_{i=1}^{\infty} V_i \setminus \bigcup_{i=1}^{\infty} E_i$$

and by monotonicity and σ-subadditivity of exterior measure,

$$m_*\left(\bigcup_{i=1}^{\infty}(V_i \setminus E_i)\right) \leq \sum_{i=1}^{\infty} m_*(V_i \setminus E_i) \leq \sum_{i=1}^{\infty} \varepsilon 2^{-i} = \varepsilon.$$

On the other hands, we found an open set $\bigcup V_i$ containing $\bigcup E_i$, where its difference has exterior measure less than given ε. By the definition of Lebesgue measurable set, it is measurable.

(b) First, every closed set can be expressed as the union of compact sets; for closed $F \subset \mathbb{R}^n$,

$$F = \bigcup_{r=1}^{\infty}(F \cap \overline{B_r(0)})$$

where $\overline{B_r(0)}$ is a closed ball of center the origin and radius r. By (a), it is sufficient to show that every compact set is Lebesgue measurable.

Suppose F is compact, and let $\varepsilon > 0$ be given. By the definition of exterior measure, there exists an open set V such that $F \subset V$ and $m_*(V) \leq m_*(F)+\varepsilon$. Then $V \setminus F$ is open, and it can be expressed as almost disjoint closed cubes, i.e.,

$$V \setminus F = \bigcup_{k=1}^{\infty} Q_k.$$

For a fixed N, the finite union $K = \bigcup_{j=1}^{N} Q_j$ is compact. Therefore $d(K, F) > 0$. Since $(K \cup F) \subset V$,

$$m_*(V) \geq m_*(F) + m_*(K) = m_*(F) + \sum_{j=1}^{N} m_*(Q_j).$$

Hence, $\sum_{j=1}^{N} m_*(Q_j) \leq m_*(V) - m_*(F) \leq \varepsilon$, and this also holds in the limit as N tends to infinity. Hence

$$m_*(V \setminus F) = m_*\left(\bigcup_{k=1}^{\infty} Q_k\right) \leq \sum_{k=1}^{\infty} m_*(Q_k) \leq \varepsilon,$$

and hence F is measurable.

2. Suppose that $f : [0, b] \to \mathbb{R}$ is (Lebesgue) integrable. Let

$$g(x) = \int_x^b \frac{f(t)}{t} dt$$

for $x \in (0, b]$. Prove that

$$\int_0^b g(x)dx = \int_0^b f(t)dt.$$

Sol.

$$\begin{aligned}
\int_0^b g(x)dx &= \int_0^b \int_x^b \frac{f(t)}{t} dtdx \\
&= \int_0^b \int_0^t \frac{f(t)}{t} dxdt \\
&= \int_0^b \frac{f(t)}{t} \int_0^t dxdt \\
&= \int_0^b f(t)dt
\end{aligned}$$

and the statement is shown. The second equality is valid due to Fubini-Tonelli theorem.

3. Construct an increasing function on $\mathbb{R}$ whose set of discontinuities is $\mathbb{Q}$.

Sol. [3] p. 97 Remark 4.31.

Let $\{q_i\}_{i=1}^{\infty}$ be an enumeration of $\mathbb{Q}$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{i:q_i \le x} 2^{-i}.$$

As $2^{-i} > 0$ for all $i \in \mathbb{N}$ and $\sum 2^{-i}$ converges, its partial sums converge. Hence $f(x)$ is well-defined.

If $x < y$, then

$$f(y) - f(x) = \sum_{i:x<q_i \le y} 2^{-i}$$

and since there must exist a rational q_i between x and y, $f(y) - f(x) > 0$. Hence f is (strictly) increasing.

Let x_0 be j-th rational. If we set $\varepsilon = 2^{-j-1}$, then whatever $\delta > 0$ is, if $t < x_0$, then

$$f(x_0) - f(t) = \sum_{i:t<q_i \le x_0} 2^{-i} \ge 2^{-j} > 2^{-j-1} = \varepsilon$$

so that f is not continous at x_0.

Let x_1 be irrational. Let $\varepsilon > 0$ be given. Let N be the smallest integer such that $2^{-N} < \varepsilon/2$. Pick

$$\delta = \min\{|x_1 - q_i| : i < N\}.$$

Then if $x_1 < t < x_1 + \delta$, then

$$f(t) - f(x_1) = \sum_{i:x_1<q_i \le t} 2^{-i} \le \sum_{i:x_1<q_i \le x_1+\delta} 2^{-i} \le \sum_{i \ge N} 2^{-i} = 2^{-N+1} < \varepsilon.$$

Similarly, if $x_1 - \delta < t < x_1$, then

$$f(x_1) - f(t) = \sum_{i:t<q_i \le x_1} 2^{-i} \le \sum_{i:x_1-\delta<q_i \le x_1} 2^{-i} \le \sum_{i \ge N} 2^{-i} = 2^{-N+1} < \varepsilon.$$

Hence if $|t - x_1| < \delta$, then $|f(t) - f(x_1)| < \varepsilon$. That is, f is continuous at x_1.

4. Prove the following statements:

   (a) If $1 \le p < q < \infty$, then $L^p(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^q(\mathbb{R})$.

   (b) If $f \in L^r(\mathbb{R})$ for some $r < \infty$, then $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

Sol. (a) Let $f \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then $\mu(\{x : |f(x)| > \|f\|_\infty\}) = 0$. Let $E = \{x : |f(x)| > \|f\|_\infty\}$. Then

$$\begin{aligned}\int |f|^q d\mu &= \int |f|^p |f|^{q-p} d\mu \\ &= \int_E |f|^p |f|^{q-p} d\mu + \int_{E^\complement} |f|^p |f|^{q-p} d\mu \\ &= \int_{E^\complement} |f|^p |f|^{q-p} d\mu \\ &\le \int_{E^\complement} |f|^p \|f\|_\infty^{q-p} d\mu \\ &= \|f\|_\infty^{q-p} \int_{E^\complement} |f|^p d\mu \le \|f\|_\infty^{q-p} \|f\|_p^p < \infty\end{aligned}$$

and thus $f \in L^q(\mathbb{R})$.

(b) First, assume that $\|f\|_\infty < \infty$. Then $f \in L^p$ for all $p \ge r$, by part (a).
For sufficiently small $\varepsilon > 0$, consider $E_\varepsilon := \{x : |f(x)| > \|f\|_\infty - \varepsilon\}$, whose measure is not zero. Then for $p \ge r$,

$$\begin{aligned}\|f\|_p^p = \int_X |f|^p d\mu &\ge \int_{E_\varepsilon} |f|^p d\mu \\ &= \int_{E_\varepsilon} (\|f\|_\infty - \varepsilon)^p d\mu \\ &= (\|f\|_\infty - \varepsilon)^p \mu(E_\varepsilon)\end{aligned}$$

and hence $\|f\|_p \ge (\|f\|_\infty - \varepsilon)(\mu(E_\varepsilon))^{1/p}$. By taking lower limit over $p \to \infty$, we get

$$\liminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty - \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, it turns out that $\liminf_{p\to\infty} \|f\|_p \ge \|f\|_\infty$.
Conversely, as $|f(x)| \le \|f\|_\infty$ almost everywhere, for $p \ge r$,

$$\begin{aligned}\|f\|_p^p = \int_X |f|^p d\mu &= \int_X |f|^{p-r} |f|^r d\mu \\ &\le \int_X \|f\|_\infty^{p-r} |f|^r d\mu \\ &= \|f\|_\infty^{p-r} \|f\|_r^r\end{aligned}$$

and hence $\|f\|_p \le \|f\|_\infty^{1-r/p} \|f\|_r^{r/p}$. By taking upper limit over $p \to \infty$, we get

$$\limsup_{p\to\infty} \|f\|_p \le \|f\|_\infty.$$

Therefore $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$, for $p \ge r$.

The case for $f \notin L^\infty$ is analogous. Let $S_M = \{x : |f(x)| > M\}$ for $M > 0$. Then $\mu(S_M) \neq 0$. Hence

$$\|f\|_p^p = \int_X |f|^p d\mu \ge \int_{S_M} |f|^p d\mu = \int_{S_M} M^p d\mu = M^p \mu(S_M)$$

and thus $\liminf_{p\to\infty} \|f\|_p \ge M$ for any positive M. This implies that

$$\liminf_{p\to\infty} \|f\|_p = \infty.$$

5. Let X be a Banach space, and let A and B be linear operators on X. Assume that A is invertible and $\|B - A\| \cdot \|A^{-1}\| < 1$. Prove that B is invertible.

Sol. First assume that $A = I$. Let $\|I - B\| = c < 1$. For each $y \in X$, let $T_y(x) = y + (I - B)x$. Then

$$\|T_y(x) - T_y(x')\| = \|(I - B)(x - x')\| < c\|x - x'\|$$

and by Banach fixed point theorem, T_y has a unique fixed point f_y. That is, $y + (I - B)f_y = f_y$, and $Bf_y = y$. Then the map $L : y \mapsto f_y$ satisfies $BL = I$.

Consider the map T_{By}, which has a fixed point LBy. But then, $T_{By}(y) = By + y - By = y$ implies y is the fixed point of T_{By}. By the uniqueness of fixed point, we have $LBy = y$. That is, $LB = I$. Therefore $LB = BL = I$, i.e., B has the inverse $B^{-1} = L$.

For general invertible A with $\|B - A\| \cdot \|A^{-1}\| < 1$, since $\|BA^{-1} - I\| \le \|B - A\|\|A^{-1}\| < 1$, we get that BA^{-1} has the inverse. Hence B also has the inverse.

6. Assume that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite complete measure spaces. Prove that, for any $\mathcal{M} \times \mathcal{N}$-measurable function f on $X \times Y$, if $1 \le q \le p < \infty$, then[2]

$$\left[\textstyle\int_X \left(\int_Y |f(x,y)|^q d\nu(y)\right)^{p/q} d\mu(x)\right]^{1/p} \le \left[\textstyle\int_Y \left(\int_X |f(x,y)|^p d\mu(x)\right)^{q/p} d\nu(y)\right]^{1/q}.$$

[2]Due to marginal issue, it is typesetted as `\textstyle`, which makes it smaller than usual size.

Sol. The given inequality is equivalent to

$$\left[\int_X \left(\int_Y |f(x,y)|^q d\nu(y)\right)^{p/q} d\mu(x)\right]^{q/p} \leq \int_Y \left(\int_X |f(x,y)|^p d\mu(x)\right)^{q/p} d\nu(y).$$

Let $r = p/q \geq 1$. Then by standard Minkowski's inequality,

$$\left[\int \left(\int |f(x,y)|^q d\nu(y)\right)^r d\mu(x)\right]^{1/r} \leq \int \left[\int (|f(x,y)|^q)^r d\mu(x)\right]^{1/r} d\nu(y)$$

and

$$\left[\int \left(\int |f(x,y)|^q d\nu(y)\right)^{p/q} d\mu(x)\right]^{q/p} \leq \int \left[\int |f(x,y)|^p d\mu(x)\right]^{q/p} d\nu(y)$$

is valid, which is the equivalent inequality.

1.7 2021 Feb Real

1. Let $f : [0,1] \to [0, M]$ be a bounded (Lebesgue) measurable function. Show that f is Riemann integrable if and only if f is continuous almost everywhere.

Sol. [1], p. 57, Theorem 2.28 (b).

Let $\int^R$ denote Riemann integration and $\int^L$ denote Lebesgue integration. Before proving main statement, we will prove that Riemann integrablity implies Lebesgue integrablity.

Let $f : [a, b] \to \mathbb{R}$ be a bounded Riemann integrable function. As f is Riemann integrable, there is a sequence of partitions $\{P_n = \{a = t_0^{(n)} < \cdots < t_{k_n}^{(n)} = b\}\}$, satisfying:

- $P_n \subset P_{n+1}$ for all $n \in \mathbb{N}$,
- $|P_n| \to 0$, where $|P_n| = \max |t_j^{(n)} - t_{j-1}^{(n)}|$,
- both upper and lower Riemann sums converge to $\int^R f$.

Then by the settings, the two simple functions $G_n(x) := \sum_{j=1}^{k_n} M_j^{(n)} \mathbf{1}_{(t_j^{(n)}, t_{j+1}^{(n)}]}(x)$ and $g_n(x) := \sum_{j=1}^{k_n} m_j^{(n)} \mathbf{1}_{(t_j^{(n)}, t_{j+1}^{(n)}]}(x)$, where

$$M_j^{(n)} = \sup_{t_j^{(n)} < x \le t_{j+1}^{(n)}} f(x), \quad m_j^{(n)} = \inf_{t_j^{(n)} < x \le t_{j+1}^{(n)}} f(x),$$

satisfy $\int^R G_n = \int^L G_n \to \int^R f$, $\int^R g_n = \int^L g_n \to \int^R f$.

Moreover, since both $G_n(x)$ and $g_n(x)$ are bounded on $[a, b]$, and $G_n(x) \ge G_{n+1}(x)$ and $g_n(x) \le g_{n+1}(x)$. Hence they converge to $G(x)$ and $g(x)$, respectively, where $g_n \le g \le f \le G \le G_n$ for all n. By MCT (or DCT), we have

$$\lim_{n\to\infty} \int^L G_n = \int^L G, \quad \lim_{n\to\infty} \int^L g_n = \int^L g.$$

Therefore $\int^L G = \int^R G = \int^R f$ and $\int^L g = \int^R g = \int^R f$. This gives that $\int^L (G - g) = 0$. The inequality $G \ge g$ gives that $G = g$ a.e., and hence $f = G = g$ a.e. Hence f is measurable. Since it is bounded measurable function on a bounded interval, it is Lebesgue integrable, with $\int^L f = \int^R f$.

Let $f:[a,b]\to\mathbb{R}$ be a bounded function. Suppose f is Riemann integrable. Use same settings from above proof. Let

$$H(x)=\limsup_{y\to x} f(y):=\lim_{\delta\to 0}\sup_{|y-x|<\delta} f(y),$$
$$h(x)=\liminf_{y\to x} f(y):=\lim_{\delta\to 0}\inf_{|y-x|<\delta} f(y).$$

Assume $x\notin\bigcup_k P_k$. Then for any n, there is $\delta_n>0$ such that $(x-\delta_n,x+\delta_n)\subset(t_j^{(n)},t_{j+1}^{(n)}]$. Then for sufficiently large l, x belongs to $(t_{j'}^{(n+l)},t_{j'+1}^{(n+l)}]$ with

$$(t_{j'}^{(n+l)},t_{j'+1}^{(n+l)}]\subset(x-\delta_n,x+\delta_n)\subset(t_j^{(n)},t_{j+1}^{(n)}].$$

This is because of the second setting. Hence

$$M_{j'}^{(n+l)}\le\sup_{|y-x|<\delta_n} f(y)\le M_j^{(n)}$$

and by letting $n\to\infty$, $\delta_n\to 0$ and hence

$$\lim_{n\to\infty} M_{j'}^{(n+l)}=G(x)\le\lim_{n\to\infty}\sup_{|y-x|<\delta} f(y)=H(x)\le\lim_{n\to\infty}M_j^{(n)}=G(x).$$

That is, $G(x)=H(x)$. Similarly $g(x)=h(x)$.

Let $N=\{x:g(x)=G(x)\}$. Then on $N\setminus\bigcup_k P_k$, $H(x)=G(x)=g(x)=h(x)$, i.e., upper limit and lower limit of f at x is same, and hence f is continuous at x. Since the measure of $N\setminus\bigcup_k P_k$ is same with the measure of $[a,b]$, f is continuous a.e.

Conversely, if f is not Riemann integrable, then the measure of $[a,b]\setminus N$ is nonzero, and thus the set of discontinuity has nonzero measure.

2. Let $\{u_n:\mathbb{R}\to\mathbb{R}\}$ be a sequence of continuous functions on $\mathbb{R}$ that are equicontinuous and satisfy $|u_n(x)|\le\frac{1}{1+|x|^2}$ for all n. Show that there is a convergence subsequence in L^1-norm. (Hint. You may use Arzelà-Ascoli theorem)

Sol. For $k\in\mathbb{N}$, let $E_k:=\{x\in\mathbb{R}:|x|\le k\}$. Since $\frac{1}{1+|x|^2}\le 1$, by Arzelà-Ascoli theorem, $\{u_n\}$ has a uniformly convergent subsequence $\{u_{1,n}\}$ on E_1. On E_2, the subsequence $\{u_{1,n}\}$ has a uniformly convergent subsequence $\{u_{2,n}\}$. By repeating this process, for the subsequence $\{u_{m,n}\}$ which converges uniformly on E_m, choose a subsequence $\{u_{m+1,n}\}$ which converges uniformly on E_{m+1}.

Then $\{u_{n,n}\}$ is a desired subsequence; Let $\varepsilon > 0$ be given. Choose N such that $\int_{E_N^\complement} \frac{1}{1+|x|^2} dx < \frac{\varepsilon}{4}$. From the construction of $\{u_{n,n}\}$, it converges uniformly on E_N. Hence, if m, n are sufficiently large, then

$$\int_{E_N} |u_{n,n}(x) - u_{m,m}(x)| dx \le \int_{E_N} \frac{\varepsilon}{4N} dx = \frac{\varepsilon}{2}.$$

On $E_N^\complement$, for the chosen indices m and n,

$$\begin{aligned}\int_{E_N^\complement} |u_{n,n}(x) - u_{m,m}(x)| dx &\le \int_{E_N^\complement} |u_{n,n}(x)| dx + \int_{E_N^\complement} |u_{m,m}(x)| dx \\ &\le \int_{E_N^\complement} \frac{2}{1+x^2} dx < \frac{\varepsilon}{2}.\end{aligned}$$

Hence

$$\int_{\mathbb{R}} |u_{n,n}(x) - u_{m,m}(x)| dx < \varepsilon.$$

Therefore $\{u_{n,n}\}$ is a Cauchy sequence in L^1, which is complete.

3. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. For given $\varepsilon > 0$, there exists a continuous function $g(x)$ such that $g'(x)$ exists and equals 0 almost everywhere and

$$\sup_{x\in[0,1]} |f(x) - g(x)| \le \varepsilon.$$

(Hint. Mimic Cantor function.)

Sol. Without loss of generality, let $f(0) = 0$. For given ε, define a sequence $\{a_n\}$ as following: $a_0 = 0$, and

$$a_{n+1} := \begin{cases} \inf\{x > a_n : |f(x) - f(a_n)| = \varepsilon\} & \text{if it exists,} \\ 1 & \text{otherwise.} \end{cases}$$

Then $a_N = 1$ for some N whatever ε is; If it does not happen, $\{f(a_n)\}$ diverges or oscilating. More precisely, $a_n \nearrow \alpha \in (0,1]$. By the definition of a_n and the continuity of f, we have $f(a_n) = m_n \varepsilon$ for some $m_n \in \mathbb{Z}$.

If $\{m_n\}$ is bounded, there exists a subsequence $\{a_{n_k}\}$ such that $f(a_{n_k}) = i\varepsilon$ for odd k and $j\varepsilon$ for even k, where $i \ne j$. Then

$$\lim_{k\to\infty} f(a_{n_{2k}}) \ne \lim_{k\to\infty} f(a_{n_{2k+1}}),$$

which contradicts to continuity at α.

Similarly, if $\{m_n\}$ is unbounded, some subsequence $\{a_{n_k}\}$ satisfies that $|f(a_{n_k})| \to \infty$, and thus continuity at α fails.

For such chosen a_n, let $E_n = [a_n, a_{n+1}]$, and let $\delta = \min(a_{n+1} - a_n)/3$. Define the continuous function g as following: on $[0, \delta]$, $g(x) = f(0)$, on $[1-\delta, 1]$, $g(x) = f(1)$, and

$$g(x) := \begin{cases} f(a_n) & x \in (a_n + \delta, a_{n+1} - \delta), \\ C_n(x) & x \in [a_n - \delta, a_n + \delta], \end{cases}$$

where $C_n(x)$ is a Cantor function with appropriate translation and scaling. Then from the construction of a_n, $|f(x) - g(x)| \le \varepsilon$ for all $x \in [0, 1]$, and $g'(x) = 0$ for almost every $x \in [0, 1]$.

4. We define the 1d Fourier transform by $\widehat{f} = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx$.

   (a) Assume that for each integer N, we have a decay $|\widehat{f}(\xi)| \le C_N \frac{1}{1+|\xi|^N}$. Show that $f \in C^\infty \cap L^2$.

   (b) Show that if we further assume $|\widehat{f}(\xi)| \le Ce^{-\alpha|\xi|}$ for some $\alpha > 0$, then $f(x)$ is real-analytic.

5.

1.8 2020 Aug Real

1. Find a sequence of functions $\{\varphi_n\}_{n=1}^{\infty}$ on $[0,1]$ such that $\{\varphi_n\}$ is a dense subset of $L^p(\Omega)$ for any $p \in [1,\infty)$.

Sol. It will be discussed only for $\Omega = \mathbb{R}$ with standard Lebesgue measure.

2. Prove that for any $f \in L^1(\mathbb{R})$, its Fourier transform $\widehat{f}$ is continuous and $\lim_{|x|\to\infty} \widehat{f}(x) = 0$, that is, $\widehat{f} \in C_0(\mathbb{R})$.

Sol. The Fourier transform of $f \in L^1(\mathbb{R})$ is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x\xi} dx.$$

Hence

$$\begin{aligned}
|\widehat{f}(\xi+h) - \widehat{f}(\xi)| &= \left| \int_{\mathbb{R}} f(x)(e^{-2\pi i x(\xi+h)} - e^{2\pi i x\xi}) dx \right| \\
&= \left| \int_{\mathbb{R}} f(x) e^{-2\pi i x\xi}(e^{-2\pi i x h} - 1) dx \right| \\
&\le \int_{\mathbb{R}} |f(x)||(e^{-2\pi i x h} - 1)| dx \le C \int_{\mathbb{R}} |f(x)| dx = C\|f\|_{L^1}
\end{aligned}$$

for some $C > 0$, if $|h|$ is sufficiently small. By DCT, we have

$$\begin{aligned}
\lim_{h\to 0}(\widehat{f}(\xi+h) - \widehat{f}(\xi)) &= \lim_{h\to 0} \int_{\mathbb{R}} f(x) e^{-2\pi i x\xi}(e^{-2\pi i x h} - 1) dx \\
&= \int_{\mathbb{R}} \lim_{h\to 0} f(x) e^{-2\pi i x\xi}(e^{-2\pi i x h} - 1) dx = 0,
\end{aligned}$$

that is, $\widehat{f}$ is continous.

The second part is the lemma called *Riemann-Lebesgue Lemma*. Let g be a compactly supported continuous function. By substituting x into $x + 1/2\xi$ in the definition of Fourier transform, we have

$$\widehat{g}(\xi) = \int_{\mathbb{R}} g\left(x + \frac{1}{2\xi}\right) e^{-2\pi i x\xi - \pi i} dx = -\int_{\mathbb{R}} g\left(x + \frac{1}{2\xi}\right) e^{-2\pi i x\xi} dx.$$

Since g is continuous and has compact support, $g(x) - g(x + 1/2\xi) \to 0$ for any $x \in \mathbb{R}$ as $|\xi| \to \infty$. By DCT, we have

$$\widehat{g}(\xi) \le \frac{1}{2} \int_{\mathbb{R}} \left| g(x) - g\left(x + \frac{1}{2\xi}\right) \right| \to 0$$

as $|\xi| \to 0$. Finally, for $f \in L^1$, let g be a continuous function with compact support such that $\|f - g\|_{L^1} < \varepsilon$. Then

$$|\widehat{f}(\xi)| \leq |\widehat{f}(\xi) - \widehat{g}(\xi)| + |\widehat{g}(\xi)| \leq \|f - g\|_{L^1} + |\widehat{g}(\xi)| \leq \varepsilon + |\widehat{g}(\xi)|$$

and

$$\limsup_{|\xi| \to \infty} |\widehat{f}(\xi)| \leq \varepsilon$$

whatever ε is. That is, $\widehat{f}$ vanishes at infinity.

3. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $L^p([0,1])$ for $p \in (1, \infty)$. Suppose that there exists a $f \in L^p([0,1])$ satisfying $\lim_{n\to\infty} \int_0^1 f_n(x)g(x)dx = \int_0^1 f(x)g(x)dx$ for any $g \in L^q([0,1])$ with $\frac{1}{p} + \frac{1}{q} = 1$. Prove that $\lim_{n\to\infty} \|f_n - f\|_p = 0$ if $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$.

Sol.

1.9 2020 Feb Real

2 Complex Analysis

2.1 2024 Feb Complex

1. Prove that $\sum_{n=1}^{\infty} e^{-n^2} z^n$ is an entire function.

2. Find all entire functions f such that $f(n\pi) = 0$ for any $n \in \mathbb{Z}$ and $|f(x+iy)| \le Ce^{|y|} < \infty$, $x, y \in \mathbb{R}$ for some $C > 0$.

3. Find all entire functions f which satisfies the property that for some $R, C > 0$, $|f(z)| \ge C$ when $|z| \ge R$.

Sol. Let f be an entire function satisfying given properties. As f is continuous on compact set $\{z : |z| \le R\}$, it is bounded on the compact set by $M > C/2$. Then the modulus of $g(z) = f(z) + 2M$ is bounded below by some $M' > 0$; $|g(z)| > M$ on $\{z : |z| \le R\}$ and $|g(z)| \ge 2M - C$ on $\{z : |z| \ge R\}$. Then $1/g(z)$ is bounded entire function, and by Liouville's theorem, $1/g(z)$ is constant. That is, $f(z)$ is constant function. Hence $f(z) \equiv k$ for some $|k| \ge C$.

4. Let $f : \mathbb{C} \to \mathbb{C}$ be a function. Prove that f is entire if f^2 is entire and f is continuous.

Sol. Because f^2 is entire and f and f^2 share their zeros, the zeros of f should be isolated or $f \equiv 0$. If $f \equiv 0$, it is obvious. If $f(z) \not\equiv 0$, for $f(z) \neq 0$,

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{f^2(z+h) - f^2(z)}{h} \frac{1}{f(z+h) + f(z)} = \frac{(f^2)'(z)}{2f(z)}.$$

This gives that f is differentiable on whole complex plane except its zeros, and is holomorphic.

For a zero z_0, let D be a domain of z_0 with $f(z) \neq 0$ except $z = z_0$. Denote $D_0 = D \setminus \{z_0\}$. By Riemann's theorem, a holomorphic function $f|_{D_0} : D_0 \to \mathbb{C}$ can be holomorphically extended to $f|_D : D \to \mathbb{C}$ with $f(z_0) = 0$, since f is continuous at $z = z_0$. Hence f is also entire.

5. Evaluate $\int_0^{2\pi} \frac{\cos^2\theta}{5+3\cos\theta} d\theta$.

Sol. By substituting $z = e^{i\theta}$, the given integral becomes

$$\begin{aligned}&\int_{|z|=1} \frac{\frac{1}{4}(z+\frac{1}{z})^2}{5+\frac{3}{2}(z+\frac{1}{z})}\frac{dz}{iz} \\ =&-i\int_{|z|=1}\frac{z^2+2+\frac{1}{z^2}}{6z^2+20z+6}dz \\ =&-i\int_{|z|=1}\frac{z^4+2z^2+1}{z^2(6z^2+20z+6)}dz \\ =&-i\int_{|z|=1}\frac{z^4+2z^2+1}{2z^2(z+3)(3z+1)}dz.\end{aligned}$$

In the region $\{|z|<1\}$, the function $\frac{z^4+2z^2+1}{2z^2(z+3)(3z+1)}$ has singularities only at $z=0$ and $z=-1/3$, and both are poles. By residue theorem,

$$\begin{aligned}&\int_{|z|=1}\frac{z^4+2z^2+1}{2z^2(z+3)(3z+1)}dz \\ =&2\pi i\left(\frac{d}{dz}\bigg|_{z=0}\frac{z^4+2z^2+1}{(z+3)(3z+1)}+\frac{z^4+2z^2+1}{z^2(z+3)}\bigg|_{z=-1/3}\right) \\ =&2\pi i\left(10+\frac{25}{6}\right)=\frac{85\pi i}{3}.\end{aligned}$$

Therefore the given integral becomes $\frac{85\pi}{3}$.

6. Show that a polynomial $f(z) = z^5 + 2z^3 + 1$ has no zero in $D(0;2/3)$, three zeros in $D(0,1)\setminus D(0,2/3)$ and two zeros in $D(0,2)\setminus\overline{D(0,1)}$.

Sol. Use Rouché's theorem.

On $\{|z|=2/3\}$, $|z^5+1|\geq 1-|z|^5 = 211/243 > 16/27 = |2z^3|$. Hence the disk $D(0,2/3)$ has no zeros of z^5+2^3+1, because $z^5+1=0$ has no roots in the disk.

On $\{|z|=1\}$, for any $a>0$,

$$|z^5+(2+a)z^3| = |z^2+2+a| \geq 1+a, \quad |1-az^3|\leq 1+a.$$

In the first inequality, $|z^5+(2+a)z^3| = 1+a$ if and only if $z=\pm i$, and $|1-a(\pm i)^3| < 1+a$. In the second inequality, $|1-az^3| = 1+a$ if and only if $z^3=-1$. If $z=-1$, $|z^2+2+a| = 3+a$. If $z^2-z+1=0$, then $|z^2+2+a| = a+1+a^2$.

Hence $|z^5 + (2+a)z^3| > |1 - az^3|$ for all z with $|z| = 1$. Thus the disk $D(0,1)$ has three zeros of $z^5 + 2^3 + 1$, because $z^5 + (2+a)z^3 = z^3(z^2 + 2 + a) = 0$ has only three roots in the disk.

Finally, on $\{|z| = 2\}$, $|z^5| = 32 \geq 17 = 1 + 2|z^3|$. Hence the disk $D(0,2)$ has all five zeros of $z^5 + 2^3 + 1$. Since there are three zeros in $D(0,1)$, $D(0,2) \setminus \overline{D(0,1)}$ contains two zeros of $z^5 + 2z^3 + 1$.

2.2 2023 Aug Complex

1. Let $f(z)$ be entire function such that $|e^{f(z)}| \leq |z|$ for $|z| \geq 1$. What can you say about $f(z)$?

2. Find a branch of $\sqrt{z(1-z)}$ so that it becomes a holomorphic (single-valued) function on $\mathbb{C} \setminus [0,1]$.

3. Evaluate the following improper integral

$$\int_0^\infty \frac{\log x}{(1+x^2)(x^2+4)} dx.$$

4. Find a partial fraction decomposition of

$$\frac{\pi}{\cos(\pi z)}.$$

5. Find a conformal map of the vertical strip $\{-1 < \operatorname{Re} z < 1\}$ onto the unit disc $\{|z| < 1\}$.

6. Supoose that $D \neq \mathbb{C}$ is a simply connected domain. Construct an injective conformal map $f : D \longrightarrow \{|z| < 1\}$. (Do not quote Riemann mapping theorem. This problem asks a part of its proof.)

7. Let $D \neq \mathbb{C}$ be a simply connected domain. Suppose that $f : D \to D$ a holomorphic function having a fixed point $f(a) = a$. Show that $|f'(a)| \leq 1$. Moreover if $|f'(a)| = 1$, then f is a homeomorphism of D.

2.3 2023 Feb Complex

1. Let $f(z)$ is holomorphic in a connected domain D. Assume that $f(z)$ is constant on a curve $C \subset D$. Show that $f(z)$ is constant in D.

Sol.

2. Evaluate the following improper integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^2} dx.$$

Sol.

3. Prove that the following infinite product converges and evaluate it

$$\prod_{n=1}^{\infty} \left(1 + \frac{(-1)^{n+1}}{n}\right).$$

4. Denote the upper half plane by $\mathbb{H} = \{\operatorname{Im} z > 0\}$. Find most general form of linear fractional transforms that maps $\mathbb{H}$ onto $\mathbb{H}$. Show that any conformal self-map of $\mathbb{H}$ is of that form.

5. Find poles and their principal parts of $\frac{1}{\sin^2 z}$. Verify the partial fraction formula

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{k=-\infty}^{\infty} \frac{1}{(z-k)^2}.$$

From this deduce that

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{z \neq 0} \left(\frac{1}{z-k} + \frac{1}{k}\right).$$

6. Construct an entire function that has simple zeros at the points n^2, for each $n \in \mathbb{N}$ and no other zeros.

2.4 2022 Aug Complex

1. Let $\mathbb{C}_\infty$ be the Riemann sphere. Show that if $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is meromorphic, then f is rational.

Sol. Let S be a subset of $\mathbb{C}_\infty$ where f has a pole at each $z \in S$. If S had a limit point p, then f cannot be neither analytic at p nor have an isolated singularity at p. Hence S cannot have a limit point. Since $\mathbb{C}_\infty$ is compact, S must be finite. Let $S \cap \mathbb{C} = \{P_1, \cdots, P_k\}$. So, $f(z)(z-P_1)^{n_1} \cdots (z-P_k)^{n_k} =: F(z)$ is entire function on $\mathbb{C}$, where n_i is order of pole P_i. Then either $\infty \in S$ or not.

If $\infty \in S$, $f(1/z)$ has a pole at $z = 0$. then $F(1/z)$ has a pole at $z = 0$, that is,

$$F(1/z) = \sum_{n=-n_0}^{\infty} a_n z^n$$

and

$$F(z) = \sum_{n=-n_0}^{\infty} a_n z^{-n}.$$

Since F does not have essential singularity at $z = 0$, $a_n \equiv 0$ if $n \geq N$. Hence

$$f(z) = \frac{F(z)}{(z-P_1)^{n_1} \cdots (z-P_k)^{n_k}} = \frac{\sum_{n=-n_0}^{N} a_n z^{-n}}{(z-P_1)^{n_1} \cdots (z-P_k)^{n_k}}$$

is a rational function.

If $\infty \notin S$, then $f(1/z)$ has removable singularity at $z = 0$. That is, $\lim_{z \to 0} f(1/z)$ is well-defined, and hence

$$\begin{aligned} F(1/z) &= f(1/z)(1/z - P_1)^{n_1} \cdots (1/z - P_k)^{n_k} \\ &= \frac{f(1/z)(1 - zP_1)^{n_1} \cdots (1 - zP_k)^{n_k}}{z^{n_1 + \cdots + n_k}} \end{aligned}$$

has either a pole at $z = 0$ with order at most $n_1 + \cdots + n_k$, or a removable singularity.

If it has a removable singularity, then $F(z)$ has removable singularity at $z = \infty$, and hence $F|_{\mathbb{C}}(z)$ is bounded on $\{z : |z| \geq R\}$ for some R. Then $F|_{\mathbb{C}}(z)$ is bounded on whole $\mathbb{C}$, and by Liouville's theorem, $F(z)$ is a constant function. Hence

$$f(z) = \frac{C}{(z-P_1)^{n_1} \cdots (z-P_k)^{n_k}}$$

is a rational function.

If it is a pole of order d, then $F(z)z^d = z^d f(z)(z-P_1)^{n_1}\cdots(z-P_k)^{n_k}$ has removable singularity at $z=\infty$, and by same argument, $F(z)z^d$ is a constant function. Hence

$$f(z) = \frac{C'}{z^d(z-P_1)^{n_1}\cdots(z-P_k)^{n_k}}$$

is a rational function.

2. (a) Evaluate

$$\int_{-1}^{1}\frac{\sqrt{1-x^2}}{1+x^2}dx$$

(b) Check if the integral is integrable. If so, evaluate it.

$$\int_0^\infty \frac{\log x}{x^b-1}dx,\ b>1$$

Sol. (a)

(b)

3. Denote $\mathbb{D}=\{z:|z|<1\}$. Show if $f:\mathbb{D}\to\mathbb{D}$ is analytic, then

$$|f'(z)| \le \frac{1-|f(z)|^2}{1-|z|^2}.$$

Moreover, if $f(z)$ is a conformal self-map of $\mathbb{D}$, then the equality holds. (Hint: Use the conformal self-map of $\mathbb{D}$ sending 0 to z_0 and its inverse.)

Sol. This is called *Schwartz-Pick Lemma.*

If $w\in\mathbb{D}$, then set

$$\varphi_w(z) := \frac{z-w}{1-z\overline{w}}$$

Then φ is a conformal self-map of $\mathbb{D}$ which maps w to 0. Elementary algebra shows that φ_w is invertible and that its inverse is φ_{-w}. Now, for the function f given in the problem, we consider

$$g = \varphi_{f(z_0)}\circ f\circ \varphi_{z_0}^{-1}:\mathbb{D}\to\mathbb{D}.$$

Then

$$g(0) = \varphi_{f(z_0)}(f(\varphi_{z_0}^{-1}(0))) = \varphi_{f(z_0)}(f(z_0)) = 0$$

and hence Schwarz's lemma can be applied, i.e., $|g'(0)| \le 1$, where

$$\begin{aligned} g'(0) &= \varphi'_{f(z_0)}(f(z_0)) \cdot f'(z_0) \cdot \frac{1}{\varphi'_{z_0}(z_0)} \\ &= \frac{1}{1-|f(z_0)|^2} \cdot f'(z_0) \cdot 1 - |z_0|^2 \\ &= \frac{1-|z_0|^2}{1-|f(z_0)|^2} f'(z_0) \end{aligned}$$

so that $|f'(z_0)| \le (1-|f(z_0)|^2)/(1-|z_0|^2)$. As the choice of z_0 is arbitrary, the given inequality holds.

From Schwarz's lemma, the equality holds if and only if $g(z) = e^{i\lambda}z$ for some $\lambda \in \mathbb{R}$. This is a conformal self-map of $\mathbb{D}$, and $f = \varphi^{-1}_{f(z_0)} \circ g \circ \varphi_{z_0}$ is a composition of conformal self-maps, which is also a conformal self-map.

4. Let $f(z)$ be the Riemann map of a simply connected domain D onto the unit disk $\mathbb{D}$. Suppose $f(z_0) = 0$ and $f'(z_0) > 0$. Show that if $g(z)$ is an analytic function on D such that $|g(z)| \le 1$ for $z \in D$ and $g(z_0) = 0$, then Re $g'(z_0) \le f'(z_0)$.

Sol. As f is a Riemann map, it has the inverse $f^{-1} : \mathbb{D} \to D$, which is analytic. Then $h := g \circ f^{-1} : \mathbb{D} \to \mathbb{D}$ satisfies the conditions for Schwarz's lemma. Hence $|h'(0)| \le 1$, where

$$h'(0) = g'(f^{-1}(0)) \cdot \frac{1}{f'(z_0)} = \frac{g'(z_0)}{f'(z_0)}$$

and $f'(z_0) > 0$ so that $|g'(z_0)| \le f'(z_0)$. As Re $g'(z_0) \le |\text{Re } g'(z_0)| \le |g'(z_0)|$ is obvious, the given inequality is valid.

5. (a) Let $\{a_n\} \subset \mathbb{C} \setminus \{0\}$ be a sequence[3]. Show that $\prod_{n=1}^{\infty}(1-\frac{z}{a_n})$ is entire if and only if $\sum_{n=1}^{\infty} \frac{1}{z-a_n}$ is meromorphic.

   (b) Find a meromorphic function $f(z)$ which has poles only at $z = n$ for each positive integer n with order n.

Sol. (a) Suppose $f(z) = \prod_{n=1}^{\infty}(1-\frac{z}{a_n})$ is entire. Then the infinite product converges uniformly, and logarithmic derivative is valid. Hence

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \frac{-1/a_n}{1-z/a_n} = \sum_{n=1}^{\infty} \frac{1}{z-a_n}$$

[3] The condition that the set has no limit points would have to be added.

is analytic except the points where $f(z) = 0$. Such points form a set $S = \{a_1, a_2, \cdot\}$, and at $z_0 \in S$, it has a pole. $\sum_{n=1}^{\infty} \frac{1}{z-a_n}$ has no singularities except poles, i.e., it is meromorphic.

Conversely,

2.5 2022 Feb Complex

1. Let $\{a_n\}_{n=1}^\infty \subset \mathbb{C}$ be a sequence such that $\sum_{n=1}^\infty \frac{1}{|a_n|}$ diverges but $\sum_{n=1}^\infty \frac{1}{|a_n|^2}$ converges. Find an entire function that has zeros only at $\{a_n\}_{n=1}^\infty$. (You need to verify that your example is entire.)

Sol. This is an example of Weierstrass' product theorem.

Clearly $a_n \neq 0$ for all n. Since $\sum_{n=1}^\infty \frac{1}{|a_n|^2}$ converges absolutely, without loss of generality, assume that $|a_n|$ is increasing sequence. Consider the infinite product

$$\prod_{n=1}^\infty \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{z}{a_n}\right)^n\right).$$

It converges if and only if the series

$$\sum_{n=1}^\infty \left[\operatorname{Log}\left(1 - \frac{z}{a_n}\right) + \left(\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{z}{a_n}\right)^n\right)\right]$$

converges. Suppose $|z| < R$. By Taylor expansion, if n is sufficiently large so that $|z/a_n| \leq R/|a_n| < 1/2 < 1$, then

$$\operatorname{Log}\left(1 - \frac{z}{a_n}\right) + \left(\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{z}{a_n}\right)^n\right) = -\sum_{k=n+1}^\infty \frac{1}{k}\left(\frac{z}{a_n}\right)^k$$

and

$$\left| -\sum_{k=n+1}^\infty \frac{1}{k}\left(\frac{z}{a_n}\right)^k \right| \leq \frac{1}{n+1}\left|\frac{R}{a_n}\right|^n \sum_{j=1}^\infty \left(\frac{1}{2}\right)^j < \frac{1}{2^n}$$

so that

$$\begin{aligned}
&\left|\sum \left[\operatorname{Log}\left(1 - \frac{z}{a_n}\right) + \left(\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{z}{a_n}\right)^n\right)\right]\right| \\
\leq &\sum \left|\left[\operatorname{Log}\left(1 - \frac{z}{a_n}\right) + \left(\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{z}{a_n}\right)^n\right)\right]\right| \\
\leq &\sum \frac{1}{2^n} < \infty
\end{aligned}$$

for sufficiently large n's, and hence it converges uniformly on $|z| \leq R$. Hence this product is analytic on $\{z : |z| < R\}$. As the choice of R is arbitrary, it may be concluded that this infinite product is entire.

2. Let $f : D \to D$ be analytic in a simply connected domain $D \subsetneq \mathbb{C}$ having a fixed point in D. Show that $|f'(a)| \leq 1$ for all $a \in D$. Show if $|f'(a)| = 1$ for some $a \in D$, then f is bijective on D.

Sol. Indeed, by choosing $f(z) = z^2$ and D as the unit disk, it satisfies all given condition but does not satisfy the conclusion. However, by lettig a as *the unique fixed point*, it has no problem. See [2] p. 403 Example 11.29.

Let $\mathbb{D}$ be the unit disk, and consider the Riemann map $\varphi : D \to \mathbb{D}$ with $\varphi(a) = 0$. Let $g = \varphi \circ f \circ \varphi^{-1}$. Then $g : \mathbb{D} \to \mathbb{D}$ and $g(0) = 0$.

Since φ is conformal, it is guaranteed that $\varphi'(a) \neq 0$. By Schwarz's lemma,

$$g'(0) = \varphi'(a) \cdot f'(a) \cdot \frac{1}{\varphi'(a)} = f'(a),$$

and thus $|g'(0)| = |f'(a)| \leq 1$. Moreover, the equality holds if and only if $g(z) = \lambda z$ with $|\lambda| = 1$. In this condition, $f(z) = \varphi^{-1}(\lambda\varphi(z))$ and this is a composition of bijections. Hence f must be a bijection.

3. Let D be a domain and $f : D \to \mathbb{C}$ be an analytic function with $f'(a) \neq 0$ for some $a \in D$. Show that the derivative $df(a)$ is a composition of rotation and dilation in $\mathbb{C}$. (Here, $df(a)$ is the gradient of f, when one understand $f : D \subset \mathbb{R}^2 \to \mathbb{R}^2$)

Sol. Let $z = x + iy$, and let $f(x + iy) = u(x, y) + iv(x, y)$. Let $c = |f'(a)| \neq 0$. Then by Cauchy-Riemann equation,

$$\begin{aligned} df(a) &= \begin{pmatrix} u_x(a) & u_y(a) \\ v_x(a) & v_y(a) \end{pmatrix} = \begin{pmatrix} u_x(a) & -v_x(a) \\ v_x(a) & u_x(a) \end{pmatrix} \\ &= \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} u_x(a)/c & -v_x(a)/c \\ v_x(a)/c & u_x(a)/c \end{pmatrix} \end{aligned}$$

where

$$\left(\frac{u_x(a)}{c}\right)^2 + \left(\frac{v_x(a)}{c}\right)^2 = \frac{u_x(a)^2 + v_x(a)^2}{c^2} = \frac{|f'(a)|^2}{|f'(a)|^2} = 1.$$

That is, there exists $\theta \in \mathbb{R}$ such that

$$\cos\theta = \frac{u_x(a)}{c}, \ \sin\theta = \frac{v_x(a)}{c}.$$

Therefore $df(a)$ is a composition of dilation matrix

$$\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$$

and rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

4. Let D be a connected domain and $\{f_n\}$ a sequence of injective analytic functions on D. Assume that $\{f_n\}$ converges uniformly on each compact subset of D. Show that the limit function f is either injective or constant.

Sol. Assume that f is neither injective nor constant. Then there is a complex number w such that $f(z) = w$ has at least two solutions in D. Let K be a connected compact subset of D where the equation $f(z) = w$ has more than two solutions, and no solutions on ∂K. As $f_n(z) - w$ converges to $f(z) - w$ uniformly on K, by Hurwitz's theorem, the number of zeros of $f(z)-w$ is equal to the number of zeros of $f_n(z) - w$ for sufficiently large n. But it contradicts that $f_n(z) - w$ is injective for all n. Hence the assumption fails.

5. Let f be analytic and satisfy $|f(z)| \leq M$ on $|z - z_0| < R$ for some $M, R > 0$. Show that if $f(z)$ has a zero of order m at z_0, then

$$|f(z)| \leq \frac{M}{R^m}|z - z_0|^m, \quad |z - z_0| < R.$$

Show that if the equality holds at some point, then $f(z) = C(z - z_0)^m$ for some C.

Sol. Since f has a zero of order m at z_0, $g(z) = f(z)/(z-z_0)^m$ has removable singularity at z_0, and $\lim_{z\to z_0} g(z) \neq 0$. Then by maximum modulus theorem, for any $0 < r < R$,

$$\max_{|z-z_0|=r} |g(z)| \leq \frac{M}{r^m}$$

and by letting $r \to R$, $|g(z)| \leq M/R^m$. Hence $|f(z)| \leq M|z - z_0|^m/R^m$.

From maximum modulus, the equality holds if and only if g is constant function. Thus $f(z) = C(z - z_0)^m$ for some C.

6. Let D be a domain and $f : D \to \mathbb{C}$ be an analytic function. Assume that $f(a_n) = 0$ for all n, where $\{a_n\}_{n=1}^{\infty} \subset D$ is a convergent sequence in $\mathbb{C}$. Prove or disprove that $f \equiv 0$.

Sol. Let $D = \{z : \text{Re}(z) > 0\}$, $a_n = 1/n$ for all n and $f(z) = \sin(\pi/z)$. Then clearly a_n converges to $0 \in \mathbb{C}$, $f(z) \not\equiv 0$, but $f(a_n) = \sin(n\pi) = 0$.

It is because the limit point of a_n is not in D. If it is a point of D, then by uniqueness theorem, f should be zero function.

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2.7 2021 Feb Complex

2.8 2020 Aug Complex

2.9 2020 Feb Complex

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