
KAIST ANALYSIS QUALIFYING EXAM
PROBLEMS AND SOLUTIONS

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1 Real Analysis

1.1 2024 Feb Real

1. Prove that the set of $x \in \mathbb{R}$ such that there exist infinitely many fractions p/q , with relatively prime integers p and q such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^3}$$

is a set of (Lebesgue) measure zero.

Sol. See [5], p. 46 Problem 1.

2. Suppose that f and g are measurable functions on \mathbb{R}^d . Prove the following statements:
 - (a) If f is integrable and g is bounded, then $f * g$ is uniformly continuous.
 - (b) If f and g are integrable, and g is bounded, then $(f * g)(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

Sol. (a) See [Problem 2 in 2022 August](#).
 (b) Use [part \(b\), problem 2 in 2023 February](#) and previous result.

3. Prove the following statements:

- (a) If $1 \leq p < q < \infty$, then $L^p(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^q(\mathbb{R})$.
- (b) If $f \in L^r(\mathbb{R})$ for some $r < \infty$, then $\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty$.

Sol. See [Problem 4 in 2021 August](#).

4. Prove that $L^p(\mathbb{R}^d)$ ($p \in [1, \infty)$) with the Lebesgue measure is a Hilbert space if and only if $p = 2$.

Sol. See [Problem 4 in 2022 August](#).

5. For a signed measure ν , prove that its total variation $|\nu|$ is a (positive) measure that satisfies $\nu \leq |\nu|$.
6. Let μ and ν be σ -finite measures on the Borel sets of the positive real line $[0, \infty)$. Suppose that $\phi(t) := \nu([0, t])$ is finite for every $t > 0$. Prove that for any μ -measurable function $f : [0, \infty) \rightarrow [0, \infty)$,

$$\int_0^\infty \phi(f(x)) d\mu(x) = \int_0^\infty \mu(\{x : f(x) > t\}) d\nu(t).$$

1.2 2023 Aug Real

1. Let $A \subset \mathbb{R}$ be a Lebesgue measurable set whose Lebesgue measure is strictly positive. Prove that there exists $B \subset A$ such that B is not Lebesgue measurable.

Sol.

2. Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a Lebesgue integrable function. Prove the following:

- (a) $\lim_{y \rightarrow 0} \int f(x+y)dx = \int f(x)dx$
 (b) $\lim_{k \rightarrow \infty} \int f(x)e^{-x^2/k}dx = \int f(x)dx$

- Sol. (a) Let $\varepsilon > 0$ be given. By approximating f to compactly supported continuous function g with L^1 error less than $\varepsilon > 0$, we have

$$\begin{aligned} \left| \int f(x+y) - f(x)dx \right| &\leq \int |f(x+y) - f(x)|dx \\ &\leq \int |f(x+y) - g(x+y)| + |g(x+y) - g(x)| + |g(x) - f(x)|dx \\ &\leq 2\varepsilon + \int |g(x+y) - g(x)|dx. \end{aligned}$$

By uniform continuity of g , $|g(x+y) - g(x)| < \varepsilon$ if $|y| < \delta$ for some δ . Hence, if $|y| < \delta$,

$$\int |g(x+y) - g(x)|dx \leq \int_E \varepsilon dx = \mu(E) \cdot \varepsilon$$

where $E = \text{supp } g \cup (\delta + \text{supp } g)$, which has finite measure. Therefore

$$\left| \int f(x+y) - f(x)dx \right| \leq \varepsilon(2 + \mu(E))$$

gives that the limit is valid.

- (b) From $|f(x)(1 - \exp(-x^2/k))| \leq |f(x)|$, by using DCT,

$$\lim_{k \rightarrow \infty} \int f(x)(1 - \exp(-x^2/k))dx = \int \lim_{k \rightarrow \infty} f(x)(1 - \exp(-x^2/k))dx = 0.$$

3. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying

$$F(x) = \int_a^x f(y) dy$$

for a Lebesgue integrable function f . Prove that F is absolutely continuous (with respect to the Lebesgue measure).

4. Let \mathcal{H} be a separable Hilbert space and T be a non-zero linear bounded operator on \mathcal{H} . Suppose that T is compact and symmetric. Prove that $\|T\|$ or $-\|T\|$ is an eigenvalue of T .

5. Suppose that \mathcal{M} is a σ -algebra in a set X and μ a finite measure on (X, \mathcal{M}) . We say that a sequence of measurable functions $\{f_n\} \rightarrow f$ in measure if for every $\varepsilon > 0$

$$\mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) \rightarrow 0$$

as $n \rightarrow \infty$.

(a) Prove that if $f_n \rightarrow f$ almost everywhere (with respect to μ) then $f_n \rightarrow f$ in measure.

(b) Prove that if $f_n \rightarrow f$ in measure then $\{f_n\}$ has a subsequence that converges to f almost everywhere (with respect to μ).

Sol. (a) Fix $\varepsilon > 0$. Let $E_n := \{x : |f_n(x) - f(x)| > \varepsilon\}$. Since the sequence $\{f_n\}$ converges to f almost everywhere, the set

$$\bigcap_{N \geq 1} \bigcup_{n \geq N} E_n$$

has zero measure. As μ is finite, we deduce that

$$\mu\left(\bigcap_{N \geq 1} \bigcup_{n \geq N} E_n\right) = \lim_{N \rightarrow \infty} \mu\left(\bigcup_{n \geq N} E_n\right) \leq \lim_{N \rightarrow \infty} \sum_{n \geq N} \mu(E_n) = 0$$

and hence $\mu(E_n) \rightarrow 0$, i.e., f_n converges to f in measure.

(b) Let $E_{\varepsilon, n} := \{x : |f_n(x) - f(x)| > \varepsilon\}$. Then there exists an increasing subsequence n_k such that $\mu(E_{2^{-k}, n_k}) \leq 2^{-k}$. Then

$$\mu\left(\bigcup_{k=1}^{\infty} E_{2^{-k}, n_k}\right) \leq \sum_{k=1}^{\infty} \mu(E_{2^{-k}, n_k}) \leq \sum_{k=1}^{\infty} 2^{-k} < \infty$$

and

$$\mu \left(\bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} E_{2^{-k}, n_k} \right) = \lim_{N \rightarrow \infty} \mu \left(\bigcup_{k=N}^{\infty} E_{2^{-k}, n_k} \right) \leq \lim_{N \rightarrow \infty} \sum_{k=N}^{\infty} \mu(E_{2^{-k}, n_k}) = 0.$$

For $x \notin \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} E_{2^{-k}, n_k}$, there exists some $N \geq 1$ such that $|f_{n_k}(x) - f(x)| \leq 2^{-k}$ for all $k \geq N$. That is, $f_{n_k}(x) \rightarrow f(x)$. Hence if $f_{n_k} \rightarrow f$ fails, then $x \in \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} E_{2^{-k}, n_k}$, which has zero measure, i.e., $f_{n_k} \rightarrow f$ almost everywhere.

6. Assume that μ is a σ -finite measure on S . Suppose that $1 \leq p \leq q \leq \infty$ and $1/p + 1/q = 1$. Prove that, for every $f \in L^p(S, \mu)$,

$$\|f\|_p = \sup \left\{ \left| \int_S f g d\mu \right| : g \in L^q(S, \mu), \|g\|_q = 1 \right\}.$$

Sol. By Hölder's inequality, if $\|g\|_q = 1$, then

$$\left| \int_S f g d\mu \right| \leq \left(\int_S |f|^p d\mu \right)^{1/p} \left(\int_S |g|^q d\mu \right)^{1/q} = \|f\|_p$$

and the inequality " \geq " is shown.

If $\|f\|_p = 0$, there is nothing to show. Assume $\|f\|_p \neq 0$. To show the reversed inequality, let

$$g(x) = \frac{|f|^p}{\|f\|_p^{p/q} f}.$$

Then

$$\int |g|^q d\mu = \frac{1}{\|f\|_p^p} \int |f|^{pq-q} = \frac{\|f\|_p^p}{\|f\|_p^p} = 1$$

and hence $g \in L^q(S, \mu)$ with $\|g\|_q = 1$. Its evaluation is

$$\int f g d\mu = \frac{1}{\|f\|_p^{p/q}} \int |f|^p d\mu = \|f\|_p^{p-p/q} = \|f\|_p.$$

Hence $\|f\|_p$ is the supremum of the given set. Note that $p < \infty$.

1.3 2023 Feb Real

1. Let $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a continuous mapping. Prove that, if A is a Borel subset of \mathbb{R}^n , then $f^{-1}(A)$ is a Borel subset of \mathbb{R}^m .

Sol. Since f is continuous, f^{-1} preserves openness and closedness. Hence if A is either open or closed in \mathbb{R}^n , so is $f^{-1}(A)$ in \mathbb{R}^m .

Furthermore, the inverse image preserves unions and intersections. As a Borel set is generated by countable unions and intersections of open and closed sets, if A is a Borel subset, so is $f^{-1}(A)$.

2. Prove the following:

- (a) There exists a positive continuous function f on \mathbb{R} so that f is integrable on \mathbb{R} , but $\limsup_{x \rightarrow \infty} f(x) = \infty$.
 (b) If f is uniformly continuous on \mathbb{R} and integrable, then $\lim_{|x| \rightarrow \infty} f(x) = 0$.

Sol. (a) Let $g(x) = \exp(-x^2)$, which is continuous, integrable on \mathbb{R} , and positive. Let

$$h(x) = (-|x| + 1)\mathbf{1}_{|x| \leq 1}.$$

For each $k \in \mathbb{Z}$, let

$$h_k(x) = 2^{|k|} h(4^{|k|}(x - k)).$$

Finally, define

$$f(x) = g(x) + \sum_{k \in \mathbb{Z} \setminus \{0\}} h_k(x),$$

where the series is well defined; for each $l \in \mathbb{Z} \setminus \{0\}$ and $x \in (l - 4^{|l|}, l + 4^{|l|})$, we have

$$\sum_{k \in \mathbb{Z}} h_k(x) = h_l(x).$$

Then

$$\int f = \int g + \int \sum_k h_k = \int g + \sum_k \int h_k = \int g + \sum_k 2^{-|k|+1} < \infty.$$

However, for each $n \in \mathbb{N}$, $f(n) > h_n(n) = 2^n$ and hence $\limsup_{x \rightarrow \infty} f(x) = \infty$.

- (b) Suppose f does not vanish at infinity. Then there exist $\varepsilon > 0$ and a sequence $\{x_n\}$ with $x_n + 1 < x_{n+1}$ such that $|f(x_n)| > \varepsilon$. For such ε , there exists $1 > \delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon/2$. Which means that $|x - x_n| < \delta$ implies $|f(x)| > \varepsilon/2$. By continuity, on $(x_n - \delta, x_n + \delta)$, $f(x)$ is either positive or negative in whole interval. But then,

$$\left| \int_{x_n - \delta}^{x_n + \delta} f(x) dx \right| > \varepsilon \delta$$

and

$$\int_{\mathbb{R}} |f(x)| dx \geq \sum_{n=1}^{\infty} \int_{x_n - \delta}^{x_n + \delta} |f(x)| dx \geq \sum_{n=1}^{\infty} \varepsilon \delta = \infty,$$

contradicts that f is integrable.

3. Suppose that $a, b > 0$. Let

$$f(x) = \begin{cases} x^a \sin(x^{-b}), & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that f is of bounded variation in $[0, 1]$ if and only if $a > b$.

4. For a bounded linear operator T on a Hilbert space \mathcal{H} , we say that T is an isometry if $\|Tf\| = \|f\|$ for all $f \in \mathcal{H}$.

(a) Prove that $T^*T = I$ if T is an isometry.

(b) Prove that if an isometry T is surjective then it is unitary and $TT^* = I$.

5. Suppose that \mathcal{M} is a σ -algebra in a set X and μ a (positive) measure on (X, \mathcal{M}) . For $f \in L^1(\mu)$, define a signed measure λ on (X, \mathcal{M}) by $\lambda(E) = \int_E f d\mu$ for $E \in \mathcal{M}$. Prove that

$$|\lambda|(E) = \int_E |f| d\mu.$$

6. Let F be an increasing function on $[0, 1]$ with $F(0) = 0$ and $F(1) = 1$. Let μ be a Borel measure defined by $\mu((a, b)) = F(b-) - F(a+)$ and $\mu(0) = \mu(1) = 0$. Suppose that the function F satisfies a Lipschitz condition $|F(x) - F(y)| \leq A|x - y|$ for some $A > 0$. Prove that $\mu \ll m$, where m is the Lebesgue measure on $[0, 1]$.

1.4 2022 Aug Real

1. Suppose that $A \subset E \subset B \subset \mathbb{R}$, where A and B are Lebesgue measurable sets of finite measure. Prove that if $m(A) = m(B)$, then E is Lebesgue measurable.

Sol. The set $E \setminus A$ has zero measure;

$$m_*(E \setminus A) \leq m_*(B \setminus A) = m(B \setminus A) = m(B) - m(A) = 0.$$

Since A and B are both finite measurable sets,

$$m(B \setminus A) = m(B) - m(A).$$

Therefore, E is measurable because E is the union of two measurable sets A and $E \setminus A$.

2. Prove the following generalization of Lebesgue's dominated convergence theorem: Suppose that f_1, f_2, \dots are measurable functions on \mathbb{R}^d and $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for a.e. $x \in \mathbb{R}^d$. Suppose also that g_1, g_2, \dots are nonnegative, integrable functions such that $|f_k(x)| \leq g_k(x)$ and $\lim_{n \rightarrow \infty} g_n(x) = g(x)$ exists for a.e. $x \in \mathbb{R}^d$. Prove that if g is integrable with $\int g = \lim_{n \rightarrow \infty} \int g_n$ then $\int f = \lim_{n \rightarrow \infty} \int f_n$.

Sol. [1] p.59 Exercise 20.

Imitate the proof of Lebesgue's dominated convergence theorem;

Since f is measurable and $|f| \leq g$ almost everywhere, $f \in L^1$. By taking real and imaginary parts it suffices to assume that f_n and f are real-valued, in which case we have By Fatou's lemma,

$$\begin{aligned} \int 2g &= \int \liminf_{n \rightarrow \infty} (g_n + g - |f_n - f|) \leq \liminf_{n \rightarrow \infty} \int (g_n + g - |f_n - f|) \\ &= 2 \int g + \liminf_{n \rightarrow \infty} \left(- \int |f_n - f| \right) \\ &= 2 \int g - \limsup_{n \rightarrow \infty} \int |f_n - f| \end{aligned}$$

and hence $\limsup_{n \rightarrow \infty} \int |f_n - f| \leq 0$. Which implies that

$$\lim_{n \rightarrow \infty} \left| \int (f_n - f) \right| = 0$$

and hence $\lim_{n \rightarrow \infty} \int f_n = \int f$.

3. Suppose that $F : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous and increasing. Let $A = F(a)$, $B = F(b)$. Prove the following:

- (a) If $E \subset [A, B]$ is measurable, then $F^{-1}(E) \cap \{F'(x) > 0\}$ is measurable.
- (b) There exists such an F that is strictly increasing, $F'(x) = 0$ on a set of positive measure, and there is a measurable subset $E \subset [A, B]$ so that $m(E) = 0$ but $F^{-1}(E)$ is not measurable.

Sol. [5] p. 149 Exercise 20.

- (a) First, we will prove the statement

$$m(\mathcal{O}) = \int_{F^{-1}(\mathcal{O})} F'(x) dx$$

where \mathcal{O} is open in $[A, B]$.

Because every open set in \mathbb{R} is a union of disjoint open intervals and inverse image preserves the union, it is sufficient to show that the statement holds for open intervals.

Let I be an open interval in $[A, B]$. Even though it contains an endpoint of $[A, B]$, because the measure of singleton is zero, its measure is same with removing the endpoint. Hence further assume that I has no endpoint. Let $I = (F(u), F(v))$. If $F'(u) = 0$, then replace $F(u)$ to $F(u')$, where $u' = \sup\{x : F(x) = F(u)\}$, and similarly replace $F(v)$ to $F(v')$ where $v' = \inf\{x : F(x) = F(v)\}$ if $F'(v) = 0$. Then

$$\begin{aligned} m(I) &= F(v') - F(u') \\ &= \int_{u'}^{v'} F'(x) dx \\ &= \int_{(u', v')} F'(x) dx \\ &= \int_{F^{-1}(I)} F'(x) dx \end{aligned}$$

where the second equality is from absolute continuity. Therefore the statement in the hint is shown.

Let $E \subset [A, B]$ be a measurable set. The set $P := \{x : F'(x) > 0\} = (F')^{-1}((0, \infty))$ is measurable set because F' is measurable. Then both have

G_δ sets G and G' such that $m(G \setminus E) = m(G' \setminus P) = 0$. The claim is that $F^{-1}(G) \cap G'$ is a G_δ set where the difference with $F^{-1}(E) \cap P$ has zero measure.

By elementary set operations,

$$\begin{aligned} & (F^{-1}(G) \cap G') \setminus (F^{-1}(E) \cap P) \\ &= (F^{-1}(G \setminus E) \cap G') \cup (F^{-1}(G) \cap (G' \setminus P)) \\ &= (F^{-1}(G \setminus E) \cap (P \cup (G' \setminus P))) \cup (F^{-1}(G) \cap (G' \setminus P)) \\ &= (F^{-1}(G \setminus E) \cap P) \cup (F^{-1}(G) \cap (G' \setminus P)). \end{aligned}$$

To verify our claim, it is sufficient to show that $F^{-1}(G \setminus E) \cap P$ has zero measure, as $m(F^{-1}(G) \cap (G' \setminus P))$ is bounded by $m(G' \setminus P) = 0$.

Since $G \setminus E$ has zero measure, there exists open O_n such that $(G \setminus E) \subset O_n$ and $m(O_n \setminus (G \setminus E)) = m(O_n) \leq 1/n$. Then

$$\begin{aligned} \frac{1}{n} &\geq m(O_n) = \int_{F^{-1}(O_n)} F'(x) dx \\ &\geq \int_{F^{-1}(\bigcap_i O_i)} F'(x) dx \\ &\geq \int_{F^{-1}(\bigcap_i O_i) \cap P} F'(x) dx \end{aligned}$$

for all n , and as $F'(x) > 0$ on $F^{-1}(\bigcap_i O_i) \cap P$, the set $F^{-1}(\bigcap_i O_i) \cap P$ has zero measure. As $G \setminus E \subset \bigcap_i O_i$, $F^{-1}(G \setminus E) \cap P$ also has zero measure.

- (b) Construct Cantor-like set C by removing the middle $1/4^n$ from each 2^{n-1} subintervals. Then $m(C) = 1 - 1/4 - 2 \times 1/4^2 - 2^2 \times 1/4^3 - \dots = 1/2 > 0$. As C is measurable, its complement K on $[0, 1]$ is also measurable. Hence $\mathbf{1}_K$ is measurable function, and the integral from 0 to x is measurable function. The claim is that $F(x) := \int_0^x \mathbf{1}_K(t) dt$ satisfies strictly increasing and absolute continuity, and $F'(x) = 0$ on nonzero measure set.

- Let $x, y \in [0, 1]$ with $x < y$. Then

$$F(y) - F(x) = \int_x^y \mathbf{1}_K(u) du \geq 0$$

and it is monotonically increasing. If either x or y , without loss of generality x , is in K , then as K is open, some open ball $B_x(r) \subset K$ exists with $r < y - x$. Then the integral is bigger than the measure of $B_x(r) \cap K$,

and it is positive. If both x and y are in C , as C has empty interior, there exists some nonempty open $U \subset K \cap (x, y)$. Then the integral becomes the measure of $U \cap K \cap (x, y)$, which is positive. This shows that F is strictly increasing.

- Since F is defined as the integral of integrable function, by proposition 1.12 in chapter 2, it immediately satisfies absolute continuity.
- By Lebesgue differentiation theorem, $F'(x) = \mathbf{1}_K(x)$ for a.e. $x \in [0, 1]$. Hence $F'(x) = 0$ a.e. on C .

As K is open in \mathbb{R} , K can be expressed as the disjoint union of open intervals. Indeed, such open intervals are removed intervals in constructing Cantor-like set C . Let $\{D_i\}$ be the collection of such intervals. Then by injectivity of F ,

$$F(K) = F\left(\bigsqcup_i D_i\right) = \bigsqcup_i F(D_i),$$

and if a_i is the left endpoint of the interval D_i , then

$$F(D_i) = \left\{ \int_0^x \mathbf{1}_K : x \in D_i \right\} = \left\{ F(a_i) + \int_{a_i}^x \mathbf{1}_K : x \in D_i \right\}$$

gives that

$$\begin{aligned} m(F(D_i)) &= m\left(\left\{ F(a_i) + \int_{a_i}^x \mathbf{1}_K : x \in D_i \right\}\right) \\ &= m\left(\left\{ \int_{a_i}^x \mathbf{1}_K : x \in D_i \right\}\right) \\ &= m(\{x - a_i : x \in D_i\}) = m(D_i). \end{aligned}$$

Therefore

$$m(F(K)) = \sum_{i=1}^{\infty} m(F(D_i)) = \sum_{i=1}^n \frac{2^{i-1}}{4^i} = \frac{1}{2} = m([F(1) - F(0)]).$$

As $m(F(K)) + m(F(C)) = m([F(1) - F(0)])$, $F(C)$ has zero measure.

Let U be a subset of C , which is nonmeasurable. Such U exists since C has positive measure. Then choose $E = F(U)$ so that $m(E) \leq m(F(C)) = 0$, whereas $F^{-1}(F(U)) = U$ is nonmeasurable.

4. Let \mathcal{B} be a Banach space.

(a) Prove that \mathcal{B} is a Hilbert space if and only if

$$\|f + g\|^2 + \|f - g\|^2 = 2(\|f\|^2 + \|g\|^2)$$

for any $f, g \in \mathcal{B}$.

(b) Prove that $L^p(\mathbb{R}^d)$ ($p \in [1, \infty)$) with the Lebesgue measure is a Hilbert space if and only if $p = 2$.

Sol. (a) A Hilbert space is always a Banach space, where it satisfies described parallelogram law.

Conversely, suppose that \mathcal{B} satisfies the parallelogram law. Define the inner product $\langle \cdot, \cdot \rangle$ on \mathcal{B} as *polarization*:

$$\langle f, g \rangle := \frac{1}{4} \sum_{k=1}^4 i^k \|f + i^k g\|^2.$$

Then it satisfies the axioms of inner product:

- For $f \in \mathcal{B}$,

$$\langle f, f \rangle = \frac{1}{4} \sum_{k=1}^4 i^k \|f + i^k f\|^2 = \frac{1}{4} \cdot 4 \|f\|^2 \geq 0$$

and the equality holds if and only if $f = 0$. Thus it satisfies positive definiteness.

- Let $f, g \in \mathcal{B}$. Then

$$\begin{aligned} \langle f, g \rangle &= \frac{1}{4} \sum_{k=1}^4 i^k \|f + i^k g\|^2 \\ &= \frac{1}{4} (i \|f + ig\|^2 - \|f - g\|^2 - i \|f - ig\|^2 + \|f + g\|^2) \\ &= \frac{1}{4} (i \| -if + g \|^2 - \| -f + g \|^2 - i \| if + g \|^2 + \| f + g \|^2) \\ &= \frac{1}{4} \sum_{k=1}^4 i^{-k} \|g + i^k f\|^2 = \overline{\langle g, f \rangle}. \end{aligned}$$

That is, it satisfies conjugate symmetry.

- First, for $f, g \in \mathcal{B}$,

$$\begin{aligned}\langle f, -g \rangle &= \frac{1}{4} \sum_{k=1}^4 i^k \|f - i^k g\|^2 \\ &= -\frac{1}{4} \sum_{k=1}^4 i^{k+2} \|f + i^{k+2} g\|^2 \\ &= -\langle f, g \rangle\end{aligned}$$

and

$$\begin{aligned}\langle f, ig \rangle &= \frac{1}{4} \sum_{k=1}^4 i^k \|f + i^{k+1} g\|^2 \\ &= -\frac{i}{4} \sum_{k=1}^4 i^{k+1} \|f + i^{k+1} g\|^2 \\ &= -i\langle f, g \rangle.\end{aligned}$$

By conjugate symmetry, $\langle if, g \rangle = i\langle f, g \rangle$.

Let $f_1, f_2 \in \mathcal{B}$. Then

$$\begin{aligned}\langle f_1 + f_2, g \rangle &= \frac{1}{4} \sum_{k=1}^4 i^k \|f_1 + f_2 + i^k g\|^2 \\ &= \frac{1}{4} \sum_{k=1}^4 i^k (2\|f_1\|^2 + 2\|f_2 + i^k g\|^2 - \|f_1 - f_2 - i^k g\|^2) \\ &= \frac{1}{4} \sum_{k=1}^4 i^k (2\|f_1\|^2 + 2\|f_2 + i^k g\|^2 \\ &\quad - (2\|f_1 - i^k g\|^2 + 2\|f_2\|^2 - \|f_1 + f_2 - i^k g\|^2)) \\ &= \frac{1}{2} \sum_{k=1}^4 i^k (\|f_1\|^2 + \|f_2\|^2 + \|f_2 + i^k g\|^2 - \|f_1 - i^k g\|^2) \\ &\quad + \frac{1}{4} \sum_{k=1}^4 i^k \|f_1 + f_2 - i^k g\|^2 \\ &= 2(\langle f_2, g \rangle - \langle f_1, -g \rangle) + \langle f_1 + f_2, -g \rangle \\ &= 2(\langle f_2, g \rangle + \langle f_1, g \rangle) - \langle f_1 + f_2, g \rangle\end{aligned}$$

so that $\langle f_1 + f_2, g \rangle = \langle f_1, g \rangle + \langle f_2, g \rangle$.

By these properties, for $n \in \mathbb{Z}$, $\langle (n+1)f, g \rangle = \langle nf, g \rangle + \langle f, g \rangle = (n+1)\langle f, g \rangle$ is valid.

For a nonzero integer n ,

$$\langle f, g \rangle = \left\langle \frac{n}{n}f, g \right\rangle = n \left\langle \frac{1}{n}f, g \right\rangle$$

so that $\frac{1}{n}\langle f, g \rangle = \langle \frac{1}{n}f, g \rangle$. Hence $\langle qf, g \rangle = q\langle f, g \rangle$ for $q \in \mathbb{Q} + i\mathbb{Q}$. As $\mathbb{Q} + i\mathbb{Q}$ is dense in \mathbb{C} and since \mathcal{B} is complete, $\langle zf, g \rangle = z\langle f, g \rangle$ for all $z \in \mathbb{C}$. Hence it is linear in first component.

This inner product induces same norm given in \mathcal{B} , by definition. Therefore it becomes a Hilbert space automatically.

- (b) If $p = 2$, then $L^2(\mathbb{R}^d)$ is a Hilbert space with inner product $\langle f, g \rangle := \int f \bar{g} dm$. Conversely, let $f = \mathbf{1}_{(0,1)^d}$ and $g = \mathbf{1}_{(1,2)^d}$. Then

$$\|f + g\|_p^2 + \|f - g\|_p^2 = 2 \left(\int \mathbf{1}_{(0,1)^d \cup (1,2)^d} dm \right)^{2/p} = 2 \cdot (2d)^{2/p}$$

and

$$2(\|f\|_p^2 + \|g\|_p^2) = 2 \left\{ \left(\int \mathbf{1}_{(0,1)^d} dm \right)^{2/p} + \left(\int \mathbf{1}_{(1,2)^d} dm \right)^{2/p} \right\} = 4d^{2/p}.$$

so that $2 \cdot (2d)^{2/p} = 4d^{2/p}$ if and only if $p = 2$. Hence if $p \neq 2$, then parallelogram law fails, and thus it cannot be a Hilbert space.

5. Let μ be a σ -finite measure on a measure space X . Prove that every measurable set of infinite measure in X contains measurable sets of arbitrary large finite measure.

Sol. Let $X = \bigcup_{n \in \mathbb{N}} E_n$, where E_n has finite measure. Let $E'_n = \bigcup_{i=1}^n E_i$. Then each E'_n has finite measure, and $X = \bigcup_{n \in \mathbb{N}} E'_n$.

Let S be a subset of infinite measure. Then

$$S = S \cap X = S \cap \left(\bigcup_{n \in \mathbb{N}} E'_n \right) = \bigcup_{n \in \mathbb{N}} (S \cap E'_n).$$

As the sequence $S \cap E'_n$ is increasing,

$$\mu(S) = \mu \left(\bigcup_{n \in \mathbb{N}} (S \cap E'_n) \right) = \lim_{n \rightarrow \infty} \mu(S \cap E'_n) = \infty.$$

Hence for any $M > 0$, there exists some $N \in \mathbb{N}$ such that $\mu(S \cap E'_n) > M$ if $n \geq N$, where $S \cap E'_n \subset S$.

6. Let S be a set of all complex, measurable, simple functions on a measure space X with a positive measure μ , satisfying that, for any $f \in S$,

$$\mu(\text{supp}(f)) < \infty.$$

Prove that S is dense in $L^p(X, \mu)$ for any $1 \leq p < \infty$.

Sol. [4] p.69 Theorem 3.13.

It is clear that $S \subset L^p(\mu)$. Suppose $f \geq 0$, $f \in L^p(\mu)$, and define $\{s_n(x)\}$ as

$$s_n(x) = \begin{cases} \lfloor 2^n f(x) \rfloor 2^{-n} & \text{if } 0 \leq f(x) < n, \\ n & \text{if } n \leq f(x) \leq \infty. \end{cases}$$

Then s_n converges to f pointwisely. The support of s_n is $\{x : 2^{-n} \leq f(x)\}$ ¹.

This set has finite measure since

$$\begin{aligned} \mu(\{f(x) \geq 2^{-n}\}) &= \int_{\{f(x) \geq 2^{-n}\}} d\mu \\ &= 2^{np} \int_{\{f(x) \geq 2^{-n}\}} 2^{-np} d\mu \\ &\leq 2^{np} \int_{\{f(x) \geq 2^{-n}\}} f^p d\mu \\ &\leq 2^{np} \|f\|_p^p < \infty. \end{aligned}$$

Hence $\{s_n\}$ is a sequence in S .

Since $|f - s_n|^p \leq (|f| + |s_n|)^p \leq 2^p |f|^p$, DCT shows that $\|f - s_n\|_p \rightarrow 0$ as $n \rightarrow \infty$. Thus f is in \bar{S} , the topological closure of S . The general case follows immediately, by decomposing $f = (\text{Re } f)^+ - (\text{Re } f)^- + i(\text{Im } f)^+ - i(\text{Im } f)^-$.

¹There are several issues in defining the terminology *support*; [5] p. 53 defines the support of a function as the set of all points where the function does not vanishes, whereas [4] p. 38 definition 2.9 says that the support of a function is the closure of the set defined in [5]. In this problem, we will follow the former definition.

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1. For a given set $E \in \mathbb{R}^d$, define $\mathcal{O}_n = \{x \in \mathbb{R}^d : d(x, E) < 1/n\}$.

- (a) Show that $m(E) = \lim_{n \rightarrow \infty} m(\mathcal{O}_n)$ if E is compact, where m is the Lebesgue measure.
- (b) Show that the conclusion in (a) may be false for E closed and unbounded; or E open and bounded.

Sol. (a) First, the set \mathcal{O}_n is open; let $x \in \mathcal{O}_n$, and let $\delta = d(x, E) = \inf\{d(x, w) : w \in E\}$.

If $d(x, y) < 1/n - \delta$, then

$$\begin{aligned} d(y, E) &= \inf_{z \in E} d(y, z) \\ &\leq \inf_{z \in E} (d(y, x) + d(x, z)) \\ &= d(y, x) + \inf_{z \in E} d(x, z) \\ &< \frac{1}{n} - \delta + \delta = \frac{1}{n}, \end{aligned}$$

that is, $y \in \mathcal{O}_n$, and hence \mathcal{O}_n is open, and hence it is measurable.

The set \mathcal{O}_1 has finite measure; since E is bounded, E is a subset of $B_N(0)$, which has finite measure. Then if $x \notin B_{N+1}(0)$, then

$$d(x, E) = \inf_{z \in E} d(x, z) \geq \inf_{z \in B_N(0)} d(x, z) \geq 1$$

and thus $x \notin \mathcal{O}_1$. That is, $\mathcal{O}_1 \subset B_{N+1}(0)$. By monotonicity of measure, \mathcal{O}_1 has finite measure.

If $x \in \mathcal{O}_n$ for all $n \in \mathbb{N}$, then $d(x, E) < \inf 1/n = 0$, i.e., x is a limit point of E . Since E is closed, $x \in E$. That is, $\bigcap_n \mathcal{O}_n \subset E$. Conversely, the reversed inclusion is trivial.

Hence, $\{\mathcal{O}_n\}_{n=1}^\infty$ is a decreasing sequence of open sets, whose intersection is E . Therefore

$$m(E) = m\left(\bigcap_n \mathcal{O}_n\right) = \lim_{n \rightarrow \infty} m(\mathcal{O}_n).$$

- (b) If the bounded condition is omitted, there is a counterexample; For $d = 1$, choose $E = \mathbb{N}$. Then $\mathcal{O}_n = \bigcup_{k \in \mathbb{N}} (k - 1/n, k + 1/n)$ and $m(\mathcal{O}_n) = \infty$ for all n , but $m(E) = 0$.

If the closed condition is omitted, there is a counterexample; Let C be the standard Cantor set. For given $r > 0$, let $n \in \mathbb{N}$ be sufficiently large so that $r > 2^{-n}$. For $x \in C$, x lies in a subinterval in n -th construction, whose length is 2^{-n} . Then $(x - r, x + r)$ contains an element in $[0, 1] \setminus C$. That is, $C \subset \overline{[0, 1] \setminus C}$. Hence $[0, 1]$ is the closure of $[0, 1] \setminus C$. By letting $E = [0, 1] \setminus C$, E is open and bounded with $m(E) = 1/2$.

As $[0, 1] = \overline{E}$, for any $p \in [0, 1]$, $(p - 1/n, p + 1/n) \cap E \neq \emptyset$ for all $n \in \mathbb{N}$. Hence $d(p, E) = 0 < 1/n$, and $[0, 1] \subset \mathcal{O}_n$ for all n . Clearly \mathcal{O}_1 is bounded by boundedness of E , and therefore

$$m\left(\bigcap_{n=1}^{\infty} \mathcal{O}_n\right) = \lim_{n \rightarrow \infty} m(\mathcal{O}_n) \geq \lim_{n \rightarrow \infty} m([0, 1]) = 1 \neq 0 = m(E).$$

2. Show that $f * g$ is uniformly continuous when f is integrable and g is bounded.

Sol. Let $\varepsilon > 0$. Let h be a compactly supported continuous function which approximates f with error less than $\varepsilon/2$ in L^1 norm, i.e., $\|f - h\|_{L^1} < \varepsilon/2$.

Let $|g| \leq M$ with $M > 0$. Then

$$\begin{aligned} |f * g(x + t) - f * g(x)| &= \left| \int_{\mathbb{R}^d} (f(x + t - y) - f(x - y))g(y)dy \right| \\ &\leq M \int_{\mathbb{R}^d} |f(x + t - y) - f(x - y)|dy \\ &= M \int_{\mathbb{R}^d} |f(t + u) - f(u)|du \end{aligned}$$

and from

$$|f(t + u) - f(u)| \leq |f(t + u) - h(t + u)| + |h(t + u) - h(u)| + |h(u) - f(u)|,$$

we get

$$\begin{aligned} &\int_{\mathbb{R}^d} |f(t + u) - f(u)|du \\ &\leq \int_{\mathbb{R}^d} |f(t + u) - h(t + u)| + |h(t + u) - h(u)| + |h(u) - f(u)|du \\ &= 2\|f - h\|_{L^1} + \int_{\mathbb{R}^d} |h(t + u) - h(u)|du. \end{aligned}$$

From uniform continuity on compact set, if $\|t\|$ is sufficiently small, the last term can be bounded by $\varepsilon|\operatorname{supp} h|$, where $|\cdot|$ denotes the Lebesgue measure. Hence $|f * g(x+t) - f * g(x)| < M\varepsilon(1 + |\operatorname{supp} h|)$, and the conclusion holds.

The construction of such h is as following: Let $R > 0$ be sufficiently large so that $\|f - f\mathbf{1}_{\{x: \|x\| \leq R\}}\|_{L^1} < \varepsilon/2$. On the compact set $K_R := \{x : \|x\| \leq R\}$, by Lusin's theorem, there exists a continuous function h on K_R with compact support, such that $\|f\mathbf{1}_{K_R} - h\|_{L^1} < \varepsilon/2$.

There exists $\delta > 0$ satisfying $|E| < \delta$ implies $\int_E |f| < \varepsilon$. Let $\eta > 0$ be sufficiently small so that $|K_{R+\eta} \setminus K_R| < \delta$ and $|K_{R+\eta} \setminus K_R| \max |h(x)| < \varepsilon$. Finally, on $K_{R+\eta} \setminus K_R$, for each unit vector v , define by piecewisely linear between $(Rv, h(Rv))$ and $((R+\eta)v, 0)$. Then h is continuous, compactly supported, and

$$\begin{aligned} \|f - h\|_{L^1} &= \int_{\mathbb{R}^d} |f(x) - h(x)| dx \\ &= \int_{K_R} |f(x) - h(x)| dx + \int_{K_{R+\eta} \setminus K_R} |f(x) - h(x)| dx \\ &\quad + \int_{K_{R+\eta}^c} |f(x) - h(x)| dx \\ &\leq \varepsilon/2 + \int_{K_{R+\eta} \setminus K_R} |f(x)| + |h(x)| dx + \varepsilon/2 \\ &\leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon. \end{aligned}$$

By replacing ε to $\varepsilon/3$, we get the desired result.

3. Suppose that f is integrable on \mathbb{R}^k . For each $\alpha > 0$, define $E_\alpha = \{x \in \mathbb{R}^k : |f(x)| > \alpha\}$.

Prove that

$$\int_{\mathbb{R}^k} |f(x)| dx = \int_0^\infty m(E_\alpha) d\alpha.$$

(Here, m is the Lebesgue measure.)

Sol. By applying the Fubini-Tonelli theorem,

$$\begin{aligned} \int_0^\infty m(E_\alpha) d\alpha &= \int_0^\infty \int_{\mathbb{R}^k} \mathbf{1}_{|f(x)| > \alpha} dx d\alpha \\ &= \int_{\mathbb{R}^k} \int_0^\infty \mathbf{1}_{|f(x)| > \alpha} d\alpha dx \\ &= \int_{\mathbb{R}^k} |f(x)| dx. \end{aligned}$$

4. Let \mathcal{H} be a Hilbert space and $T : \mathcal{H} \rightarrow \mathcal{H}$ a bounded linear operator.

If T is self-adjoint, prove that

$$\|T\| = \sup_{x \in \mathcal{H}} \{|\langle Tx, x \rangle| : \|x\| \leq 1\}.$$

Sol. See [5] p. 184.

Let $M = \sup\{|\langle Tf, f \rangle| : \|f\| = 1\}$. As $\|T\| = \sup\{|\langle Tf, g \rangle| : \|f\| \leq 1, \|g\| \leq 1\}$, clearly $M \leq \|T\|$. Conversely, let $f, g \in \mathcal{H}$ whose norm is at most 1. Then

$$\langle Tf, g \rangle = \frac{1}{4} \sum_{k=1}^4 i^k \langle T(f + i^k g), f + i^k g \rangle$$

and by self-adjoint property,

$$\operatorname{Re} \langle Tf, g \rangle = \frac{1}{4} (\langle T(f + g), f + g \rangle - \langle T(f - g), f - g \rangle).$$

From $|\langle Th, h \rangle| \leq M\|h\|^2$ and parallelogram law,

$$|\operatorname{Re} \langle Tf, g \rangle| \leq \frac{M}{2} (\|f\|^2 + \|g\|^2) \leq M.$$

By replacing g by $e^{i\theta}g$, we may conclude that $|\langle Tf, g \rangle| \leq M$. By taking supremum over f and g , $\|T\| \leq M$.

5. Suppose that (X, μ) is a measure space such that $\mu(A) > 0 \Rightarrow \mu(A) \geq 1$.

Prove that, if $1 \leq p \leq q \leq \infty$, then

$$\|f\|_{L^\infty(X, \mu)} \leq \|f\|_{L^q(X, \mu)} \leq \|f\|_{L^p(X, \mu)} \leq \|f\|_{L^1(X, \mu)}.$$

Sol. It suffices to show the inequality only for nonnegative functions.

It holds for integrable simple functions; Let $\varphi = \sum_{k=1}^n c_k \mathbf{1}_{E_k}$ be the canonical form of a simple function. Then

$$\begin{aligned} \|\varphi\|_p^q &= \left(\sum_{k=1}^n |c_k|^p \mu(E_k) \right)^{q/p} \\ &\geq \sum_{k=1}^n |c_k|^q (\mu(E_k))^{q/p} \\ &\geq \sum_{k=1}^n |c_k|^q (\mu(E_k)) = \|\varphi\|_q^q, \end{aligned}$$

where the first inequality is from $(1+x)^p \geq 1+x^p$ and mathematical induction, and the property $\mu(A) > 0$ implies $\mu(A) \geq 1$ is used for the second inequality. Therefore $\|\varphi\|_{L^q(X,\mu)} \leq \|\varphi\|_{L^p(X,\mu)} \leq \|\varphi\|_{L^1(X,\mu)}$ is valid. By the way,

$$\|\varphi\|_{\infty}^q = \max_{\mu(E_k) \neq 0} |c_k|^q \leq \sum_{k=1}^n |c_k|^q \mu(E_k),$$

hence $\|\varphi\|_{L^{\infty}(X,\mu)} \leq \|\varphi\|_{L^q(X,\mu)}$ is valid.

Let $\{\varphi_n\}$ and $\{\psi_n\}$ be sequences of positive simple functions such that $\{\varphi_n(x)\}$ and $\{\psi_n(x)\}$ are increasing sequences for almost every x , and $\varphi_n(x) \rightarrow f_+(x) := \max(f(x), 0)$ and $\psi_n(x) \rightarrow f_-(x) := \max(-f(x), 0)$. Then for $r \in \{1, p, q\}$,

$$\begin{aligned} \|f\|_{L^r(X,\mu)}^r &= \int_X |f|^r d\mu = \int_X |f_+|^r + |f_-|^r d\mu = \int_X \left| \lim_{n \rightarrow \infty} \varphi_n \right|^r + \left| \lim_{n \rightarrow \infty} \psi_n \right|^r d\mu \\ &= \int_X \lim_{n \rightarrow \infty} |\varphi_n|^r + \lim_{n \rightarrow \infty} |\psi_n|^r d\mu = \lim_{n \rightarrow \infty} \int_X |\varphi_n|^r + |\psi_n|^r d\mu \\ &= \lim_{n \rightarrow \infty} \int_X |\varphi_n + \psi_n|^r d\mu, \end{aligned}$$

where $\varphi_n + \psi_n$ is a simple function. Because the integration by approximating simple functions is well defined, the inequalities are valid except the first one.

To simplify, let $\|f\| := \|f\|_{L^{\infty}(X,\mu)}$. For simple functions $\sigma_n = \varphi_n + \psi_n$, let $\sigma_n(x) = \sum_{m=1}^{N_n} s_{m,n} \mathbf{1}_{E_{m,n}}$. Then $|s_{m,n}| \leq \|f\|$ for all possible pairs (m, n) , and $\|\sigma_n\|_{L^{\infty}(X,\mu)} \leq \|f\|$. Conversely, because $\|\sigma_n\|_{L^{\infty}(X,\mu)}$ increases by its construction, if $\|\sigma_n\|_{L^{\infty}(X,\mu)}$ does not converge to $\|f\|$, then for some $k > 0$, $\|\sigma_n\|_{L^{\infty}(X,\mu)} < \|f\| - k$ holds for every n . Then on the set $E = \{x \in X : |f(x)| > \|f\| - k\}$, $\sigma_n(x)$ cannot converge to $f(x)$, where $\mu(E) > 0$. It has a contradiction, and thus $\|f\| = \lim_{n \rightarrow \infty} \|\sigma_n\|_{L^{\infty}(X,\mu)}$. This argument guarantees the first inequality.

6. Let $C([a, b])$ be the vector space of continuous functions on the closed and bounded interval $[a, b]$. Prove the following:

- (a) For a given Borel measure μ on this interval with $\mu([a, b]) < \infty$,

$$f \mapsto \ell(f) = \int_a^b f(x) d\mu(x)$$

is a linear functional on $C([a, b])$, which is positive in the sense that $\ell(f) \geq 0$ if $f \geq 0$.

- (b) For any positive linear functional ℓ on $C([a, b])$, there exists a unique finite Borel measure μ such that

$$\ell(f) = \int_a^b f(x) d\mu(x)$$

for all $f \in C([a, b])$.

Sol. [4] p. 40, theorem 2.14. (Riesz representation theorem for Borel measures)

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1. Prove the following statements in \mathbb{R}^n :

- (a) A countable union of (Lebesgue) measurable sets is (Lebesgue) measurable.
- (b) Closed sets are (Lebesgue) measurable.

Sol. [5] p 17, p 18.

- (a) Let $\{E_i\}_{i=1}^{\infty}$ be a countable collection of measurable subsets of \mathbb{R}^n . Let $\varepsilon > 0$ be given. Then by definition, for each i , there exists open V_i , containing E_i such that $m_*(V_i \setminus E_i) < \varepsilon 2^{-i}$, where m_* denotes exterior measure. Then,

$$\bigcup_{i=1}^{\infty} (V_i \setminus E_i) \supset \bigcup_{i=1}^{\infty} V_i \setminus \bigcup_{i=1}^{\infty} E_i$$

and by monotonicity and σ -subadditivity of exterior measure,

$$m_* \left(\bigcup_{i=1}^{\infty} (V_i \setminus E_i) \right) \leq \sum_{i=1}^{\infty} m_*(V_i \setminus E_i) \leq \sum_{i=1}^{\infty} \varepsilon 2^{-i} = \varepsilon.$$

On the other hands, we found an open set $\bigcup V_i$ containing $\bigcup E_i$, where its difference has exterior measure less than given ε . By the definition of Lebesgue measurable set, it is measurable.

- (b) First, every closed set can be expressed as the union of compact sets; for closed $F \subset \mathbb{R}^n$,

$$F = \bigcup_{r=1}^{\infty} (F \cap \overline{B_r(0)})$$

where $\overline{B_r(0)}$ is a closed ball of center the origin and radius r . By (a), it is sufficient to show that every compact set is Lebesgue measurable.

Suppose F is compact, and let $\varepsilon > 0$ be given. By the definition of exterior measure, there exists an open set V such that $F \subset V$ and $m_*(V) \leq m_*(F) + \varepsilon$. Then $V \setminus F$ is open, and it can be expressed as almost disjoint closed cubes, i.e.,

$$V \setminus F = \bigcup_{k=1}^{\infty} Q_k.$$

For a fixed N , the finite union $K = \bigcup_{j=1}^N Q_j$ is compact. Therefore $d(K, F) > 0$. Since $(K \cup F) \subset V$,

$$m_*(V) \geq m_*(F) + m_*(K) = m_*(F) + \sum_{j=1}^N m_*(Q_j).$$

Hence, $\sum_{j=1}^N m_*(Q_j) \leq m_*(V) - m_*(F) \leq \varepsilon$, and this also holds in the limit as N tends to infinity. Hence

$$m_*(V \setminus F) = m_*\left(\bigcup_{k=1}^{\infty} Q_k\right) \leq \sum_{k=1}^{\infty} m_*(Q_k) \leq \varepsilon,$$

and hence F is measurable.

2. Suppose that $f : [0, b] \rightarrow \mathbb{R}$ is (Lebesgue) integrable. Let

$$g(x) = \int_x^b \frac{f(t)}{t} dt$$

for $x \in (0, b]$. Prove that

$$\int_0^b g(x) dx = \int_0^b f(t) dt.$$

Sol.

$$\begin{aligned} \int_0^b g(x) dx &= \int_0^b \int_x^b \frac{f(t)}{t} dt dx \\ &= \int_0^b \int_0^t \frac{f(t)}{t} dx dt \\ &= \int_0^b \frac{f(t)}{t} \int_0^t dx dt \\ &= \int_0^b f(t) dt \end{aligned}$$

and the statement is shown. The second equality is valid due to Fubini-Tonelli theorem.

3. Construct an increasing function on \mathbb{R} whose set of discontinuities is \mathbb{Q} .

Sol. [3] p. 97 Remark 4.31.

Let $\{q_i\}_{i=1}^{\infty}$ be an enumeration of \mathbb{Q} . Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \sum_{i: q_i \leq x} 2^{-i}.$$

As $2^{-i} > 0$ for all $i \in \mathbb{N}$ and $\sum 2^{-i}$ converges, its partial sums converge. Hence $f(x)$ is well-defined.

If $x < y$, then

$$f(y) - f(x) = \sum_{i: x < q_i \leq y} 2^{-i}$$

and since there must exist a rational q_i between x and y , $f(y) - f(x) > 0$. Hence f is (strictly) increasing.

Let x_0 be j -th rational. If we set $\varepsilon = 2^{-j-1}$, then whatever $\delta > 0$ is, if $t < x_0$, then

$$f(x_0) - f(t) = \sum_{i: t < q_i \leq x_0} 2^{-i} \geq 2^{-j} > 2^{-j-1} = \varepsilon$$

so that f is not continuous at x_0 .

Let x_1 be irrational. Let $\varepsilon > 0$ be given. Let N be the smallest integer such that $2^{-N} < \varepsilon/2$. Pick

$$\delta = \min\{|x_1 - q_i| : i < N\}.$$

Then if $x_1 < t < x_1 + \delta$, then

$$f(t) - f(x_1) = \sum_{i: x_1 < q_i \leq t} 2^{-i} \leq \sum_{i: x_1 < q_i \leq x_1 + \delta} 2^{-i} \leq \sum_{i \geq N} 2^{-i} = 2^{-N+1} < \varepsilon.$$

Similarly, if $x_1 - \delta < t < x_1$, then

$$f(x_1) - f(t) = \sum_{i: t < q_i \leq x_1} 2^{-i} \leq \sum_{i: x_1 - \delta < q_i \leq x_1} 2^{-i} \leq \sum_{i \geq N} 2^{-i} = 2^{-N+1} < \varepsilon.$$

Hence if $|t - x_1| < \delta$, then $|f(t) - f(x_1)| < \varepsilon$. That is, f is continuous at x_1 .

4. Prove the following statements:

(a) If $1 \leq p < q < \infty$, then $L^p(\mathbb{R}) \cap L^\infty(\mathbb{R}) \subset L^q(\mathbb{R})$.

(b) If $f \in L^r(\mathbb{R})$ for some $r < \infty$, then $\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty$.

Sol. (a) Let $f \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then $\mu(\{x : |f(x)| > \|f\|_\infty\}) = 0$. Let $E = \{x : |f(x)| > \|f\|_\infty\}$. Then

$$\begin{aligned} \int |f|^q d\mu &= \int |f|^p |f|^{q-p} d\mu \\ &= \int_E |f|^p |f|^{q-p} d\mu + \int_{E^c} |f|^p |f|^{q-p} d\mu \\ &= \int_{E^c} |f|^p |f|^{q-p} d\mu \\ &\leq \int_{E^c} |f|^p \|f\|_\infty^{q-p} d\mu \\ &= \|f\|_\infty^{q-p} \int_{E^c} |f|^p d\mu \leq \|f\|_\infty^{q-p} \|f\|_p^p < \infty \end{aligned}$$

and thus $f \in L^q(\mathbb{R})$.

- (b) First, assume that $\|f\|_\infty < \infty$. Then $f \in L^p$ for all $p \geq r$, by part (a). For sufficiently small $\varepsilon > 0$, consider $E_\varepsilon := \{x : |f(x)| > \|f\|_\infty - \varepsilon\}$, whose measure is not zero. Then for $p \geq r$,

$$\begin{aligned} \|f\|_p^p &= \int_X |f|^p d\mu \geq \int_{E_\varepsilon} |f|^p d\mu \\ &= \int_{E_\varepsilon} (\|f\|_\infty - \varepsilon)^p d\mu \\ &= (\|f\|_\infty - \varepsilon)^p \mu(E_\varepsilon) \end{aligned}$$

and hence $\|f\|_p \geq (\|f\|_\infty - \varepsilon)(\mu(E_\varepsilon))^{1/p}$. By taking lower limit over $p \rightarrow \infty$, we get

$$\liminf_{p \rightarrow \infty} \|f\|_p \geq \|f\|_\infty - \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, it turns out that $\liminf_{p \rightarrow \infty} \|f\|_p \geq \|f\|_\infty$.

Conversely, as $|f(x)| \leq \|f\|_\infty$ almost everywhere, for $p \geq r$,

$$\begin{aligned} \|f\|_p^p &= \int_X |f|^p d\mu = \int_X |f|^{p-r} |f|^r d\mu \\ &\leq \int_X \|f\|_\infty^{p-r} |f|^r d\mu \\ &= \|f\|_\infty^{p-r} \|f\|_r^r \end{aligned}$$

and hence $\|f\|_p \leq \|f\|_\infty^{1-r/p} \|f\|_r^{r/p}$. By taking upper limit over $p \rightarrow \infty$, we get

$$\limsup_{p \rightarrow \infty} \|f\|_p \leq \|f\|_\infty.$$

Therefore $\lim_{p \rightarrow \infty} \|f\|_p = \|f\|_\infty$, for $p \geq r$.

The case for $f \notin L^\infty$ is analogous. Let $S_M = \{x : |f(x)| > M\}$ for $M > 0$. Then $\mu(S_M) \neq 0$. Hence

$$\|f\|_p^p = \int_X |f|^p d\mu \geq \int_{S_M} |f|^p d\mu = \int_{S_M} M^p d\mu = M^p \mu(S_M)$$

and thus $\liminf_{p \rightarrow \infty} \|f\|_p \geq M$ for any positive M . This implies that

$$\liminf_{p \rightarrow \infty} \|f\|_p = \infty.$$

5. Let X be a Banach space, and let A and B be linear operators on X . Assume that A is invertible and $\|B - A\| \cdot \|A^{-1}\| < 1$. Prove that B is invertible.

Sol. First assume that $A = I$. Let $\|I - B\| = c < 1$. For each $y \in X$, let $T_y(x) = y + (I - B)x$. Then

$$\|T_y(x) - T_y(x')\| = \|(I - B)(x - x')\| < c\|x - x'\|$$

and by Banach fixed point theorem, T_y has a unique fixed point f_y . That is, $y + (I - B)f_y = f_y$, and $Bf_y = y$. Then the map $L : y \mapsto f_y$ satisfies $BL = I$.

Consider the map T_{By} , which has a fixed point LB_y . But then, $T_{By}(y) = By + y - By = y$ implies y is the fixed point of T_{By} . By the uniqueness of fixed point, we have $LB_y = y$. That is, $LB = I$. Therefore $LB = BL = I$, i.e., B has the inverse $B^{-1} = L$.

For general invertible A with $\|B - A\| \cdot \|A^{-1}\| < 1$, since $\|BA^{-1} - I\| \leq \|B - A\| \|A^{-1}\| < 1$, we get that BA^{-1} has the inverse. Hence B also has the inverse.

6. Assume that (X, \mathcal{M}, μ) and (Y, \mathcal{N}, ν) are σ -finite complete measure spaces. Prove that, for any $\mathcal{M} \times \mathcal{N}$ -measurable function f on $X \times Y$, if $1 \leq q \leq p < \infty$, then²

$$\left[\int_X \left(\int_Y |f(x, y)|^q d\nu(y) \right)^{p/q} d\mu(x) \right]^{1/p} \leq \left[\int_Y \left(\int_X |f(x, y)|^p d\mu(x) \right)^{q/p} d\nu(y) \right]^{1/q}.$$

²Due to marginal issue, it is typesetted as `\textstyle`, which makes it smaller than usual size.

Sol. The given inequality is equivalent to

$$\left[\int_X \left(\int_Y |f(x, y)|^q d\nu(y) \right)^{p/q} d\mu(x) \right]^{q/p} \leq \int_Y \left(\int_X |f(x, y)|^p d\mu(x) \right)^{q/p} d\nu(y).$$

Let $r = p/q \geq 1$. Then by standard Minkowski's inequality,

$$\left[\int \left(\int |f(x, y)|^q d\nu(y) \right)^r d\mu(x) \right]^{1/r} \leq \int \left[\int (|f(x, y)|^q)^r d\mu(x) \right]^{1/r} d\nu(y)$$

and

$$\left[\int \left(\int |f(x, y)|^q d\nu(y) \right)^{p/q} d\mu(x) \right]^{q/p} \leq \int \left[\int |f(x, y)|^p d\mu(x) \right]^{q/p} d\nu(y)$$

is valid, which is the equivalent inequality.

1.7 2021 Feb Real

1. Let $f : [0, 1] \rightarrow [0, M]$ be a bounded (Lebesgue) measurable function. Show that f is Riemann integrable if and only if f is continuous almost everywhere.

Sol. [1], p. 57, Theorem 2.28 (b).

Let \int^R denote Riemann integration and \int^L denote Lebesgue integration. Before proving main statement, we will prove that Riemann integrability implies Lebesgue integrability.

Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded Riemann integrable function. As f is Riemann integrable, there is a sequence of partitions $\{P_n = \{a = t_0^{(n)} < \dots < t_{k_n}^{(n)} = b\}\}$, satisfying:

- $P_n \subset P_{n+1}$ for all $n \in \mathbb{N}$,
- $|P_n| \rightarrow 0$, where $|P_n| = \max |t_j^{(n)} - t_{j-1}^{(n)}|$,
- both upper and lower Riemann sums converge to $\int^R f$.

Then by the settings, the two simple functions $G_n(x) := \sum_{j=1}^{k_n} M_j^{(n)} \mathbf{1}_{(t_j^{(n)}, t_{j+1}^{(n)}]}(x)$ and $g_n(x) := \sum_{j=1}^{k_n} m_j^{(n)} \mathbf{1}_{(t_j^{(n)}, t_{j+1}^{(n)}]}(x)$, where

$$M_j^{(n)} = \sup_{t_j^{(n)} < x \leq t_{j+1}^{(n)}} f(x), \quad m_j^{(n)} = \inf_{t_j^{(n)} < x \leq t_{j+1}^{(n)}} f(x),$$

satisfy $\int^R G_n = \int^L G_n \rightarrow \int^R f$, $\int^R g_n = \int^L g_n \rightarrow \int^R f$.

Moreover, since both $G_n(x)$ and $g_n(x)$ are bounded on $[a, b]$, and $G_n(x) \geq G_{n+1}(x)$ and $g_n(x) \leq g_{n+1}(x)$. Hence they converge to $G(x)$ and $g(x)$, respectively, where $g_n \leq g \leq f \leq G \leq G_n$ for all n . By MCT (or DCT), we have

$$\lim_{n \rightarrow \infty} \int^L G_n = \int^L G, \quad \lim_{n \rightarrow \infty} \int^L g_n = \int^L g.$$

Therefore $\int^L G = \int^R G = \int^R f$ and $\int^L g = \int^R g = \int^R f$. This gives that $\int^L (G - g) = 0$. The inequality $G \geq g$ gives that $G = g$ a.e., and hence $f = G = g$ a.e. Hence f is measurable. Since it is bounded measurable function on a bounded interval, it is Lebesgue integrable, with $\int^L f = \int^R f$.

Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function. Suppose f is Riemann integrable. Use same settings from above proof. Let

$$H(x) = \limsup_{y \rightarrow x} f(y) := \lim_{\delta \rightarrow 0} \sup_{|y-x| < \delta} f(y),$$

$$h(x) = \liminf_{y \rightarrow x} f(y) := \lim_{\delta \rightarrow 0} \inf_{|y-x| < \delta} f(y).$$

Assume $x \notin \bigcup_k P_k$. Then for any n , there is $\delta_n > 0$ such that $(x - \delta_n, x + \delta_n) \subset (t_j^{(n)}, t_{j+1}^{(n)})$. Then for sufficiently large l , x belongs to $(t_{j'}^{(n+l)}, t_{j'+1}^{(n+l)})$ with

$$(t_{j'}^{(n+l)}, t_{j'+1}^{(n+l)}) \subset (x - \delta_n, x + \delta_n) \subset (t_j^{(n)}, t_{j+1}^{(n)}).$$

This is because of the second setting. Hence

$$M_{j'}^{(n+l)} \leq \sup_{|y-x| < \delta_n} f(y) \leq M_j^{(n)}$$

and by letting $n \rightarrow \infty$, $\delta_n \rightarrow 0$ and hence

$$\lim_{n \rightarrow \infty} M_{j'}^{(n+l)} = G(x) \leq \lim_{n \rightarrow \infty} \sup_{|y-x| < \delta} f(y) = H(x) \leq \lim_{n \rightarrow \infty} M_j^{(n)} = G(x).$$

That is, $G(x) = H(x)$. Similarly $g(x) = h(x)$.

Let $N = \{x : g(x) = G(x)\}$. Then on $N \setminus \bigcup_k P_k$, $H(x) = G(x) = g(x) = h(x)$, i.e., upper limit and lower limit of f at x is same, and hence f is continuous at x . Since the measure of $N \setminus \bigcup_k P_k$ is same with the measure of $[a, b]$, f is continuous a.e.

Conversely, if f is not Riemann integrable, then the measure of $[a, b] \setminus N$ is nonzero, and thus the set of discontinuity has nonzero measure.

2. Let $\{u_n : \mathbb{R} \rightarrow \mathbb{R}\}$ be a sequence of continuous functions on \mathbb{R} that are equicontinuous and satisfy $|u_n(x)| \leq \frac{1}{1+|x|^2}$ for all n . Show that there is a convergence subsequence in L^1 -norm. (Hint. You may use Arzelà-Ascoli theorem)

Sol. For $k \in \mathbb{N}$, let $E_k := \{x \in \mathbb{R} : |x| \leq k\}$. Since $\frac{1}{1+|x|^2} \leq 1$, by Arzelà-Ascoli theorem, $\{u_n\}$ has a uniformly convergent subsequence $\{u_{1,n}\}$ on E_1 . On E_2 , the subsequence $\{u_{1,n}\}$ has a uniformly convergent subsequence $\{u_{2,n}\}$. By repeating this process, for the subsequence $\{u_{m,n}\}$ which converges uniformly on E_m , choose a subsequence $\{u_{m+1,n}\}$ which converges uniformly on E_{m+1} .

Then $\{u_{n,n}\}$ is a desired subsequence; Let $\varepsilon > 0$ be given. Choose N such that $\int_{E_N^c} \frac{1}{1+|x|^2} dx < \frac{\varepsilon}{4}$. From the construction of $\{u_{n,n}\}$, it converges uniformly on E_N . Hence, if m, n are sufficiently large, then

$$\int_{E_N} |u_{n,n}(x) - u_{m,m}(x)| dx \leq \int_{E_N} \frac{\varepsilon}{4N} dx = \frac{\varepsilon}{2}.$$

On E_N^c , for the chosen indices m and n ,

$$\begin{aligned} \int_{E_N^c} |u_{n,n}(x) - u_{m,m}(x)| dx &\leq \int_{E_N^c} |u_{n,n}(x)| dx + \int_{E_N^c} |u_{m,m}(x)| dx \\ &\leq \int_{E_N^c} \frac{2}{1+x^2} dx < \frac{\varepsilon}{2}. \end{aligned}$$

Hence

$$\int_{\mathbb{R}} |u_{n,n}(x) - u_{m,m}(x)| dx < \varepsilon.$$

Therefore $\{u_{n,n}\}$ is a Cauchy sequence in L^1 , which is complete.

3. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function. For given $\varepsilon > 0$, there exists a continuous function $g(x)$ such that $g'(x)$ exists and equals 0 almost everywhere and

$$\sup_{x \in [0, 1]} |f(x) - g(x)| \leq \varepsilon.$$

(Hint. Mimic Cantor function.)

Sol. Without loss of generality, let $f(0) = 0$. For given ε , define a sequence $\{a_n\}$ as following: $a_0 = 0$, and

$$a_{n+1} := \begin{cases} \inf\{x > a_n : |f(x) - f(a_n)| = \varepsilon\} & \text{if it exists,} \\ 1 & \text{otherwise.} \end{cases}$$

Then $a_N = 1$ for some N whatever ε is; If it does not happen, $\{f(a_n)\}$ diverges or oscillating. More precisely, $a_n \nearrow \alpha \in (0, 1]$. By the definition of a_n and the continuity of f , we have $f(a_n) = m_n \varepsilon$ for some $m_n \in \mathbb{Z}$.

If $\{m_n\}$ is bounded, there exists a subsequence $\{a_{n_k}\}$ such that $f(a_{n_k}) = i\varepsilon$ for odd k and $j\varepsilon$ for even k , where $i \neq j$. Then

$$\lim_{k \rightarrow \infty} f(a_{n_{2k}}) \neq \lim_{k \rightarrow \infty} f(a_{n_{2k+1}}),$$

which contradicts to continuity at α .

Similarly, if $\{m_n\}$ is unbounded, some subsequence $\{a_{n_k}\}$ satisfies that $|f(a_{n_k})| \rightarrow \infty$, and thus continuity at α fails.

For such chosen a_n , let $E_n = [a_n, a_{n+1}]$, and let $\delta = \min(a_{n+1} - a_n)/3$. Define the continuous function g as following: on $[0, \delta]$, $g(x) = f(0)$, on $[1 - \delta, 1]$, $g(x) = f(1)$, and

$$g(x) := \begin{cases} f(a_n) & x \in (a_n + \delta, a_{n+1} - \delta), \\ C_n(x) & x \in [a_n - \delta, a_n + \delta], \end{cases}$$

where $C_n(x)$ is a Cantor function with appropriate translation and scaling. Then from the construction of a_n , $|f(x) - g(x)| \leq \varepsilon$ for all $x \in [0, 1]$, and $g'(x) = 0$ for almost every $x \in [0, 1]$.

4. We define the 1d Fourier transform by $\widehat{f} = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx$.

- (a) Assume that for each integer N , we have a decay $|\widehat{f}(\xi)| \leq C_N \frac{1}{1+|\xi|^N}$. Show that $f \in C^\infty \cap L^2$.
- (b) Show that if we further assume $|\widehat{f}(\xi)| \leq C e^{-\alpha|\xi|}$ for some $\alpha > 0$, then $f(x)$ is real-analytic.

5.

1.8 2020 Aug Real

1. Find a sequence of functions $\{\varphi_n\}_{n=1}^\infty$ on $[0, 1]$ such that $\{\varphi_n\}$ is a dense subset of $L^p(\Omega)$ for any $p \in [1, \infty)$.

Sol. It will be discussed only for $\Omega = \mathbb{R}$ with standard Lebesgue measure.

2. Prove that for any $f \in L^1(\mathbb{R})$, its Fourier transform \widehat{f} is continuous and $\lim_{|x| \rightarrow \infty} \widehat{f}(x) = 0$, that is, $\widehat{f} \in C_0(\mathbb{R})$.

Sol. The Fourier transform of $f \in L^1(\mathbb{R})$ is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx.$$

Hence

$$\begin{aligned} |\widehat{f}(\xi + h) - \widehat{f}(\xi)| &= \left| \int_{\mathbb{R}} f(x) (e^{-2\pi i x (\xi + h)} - e^{-2\pi i x \xi}) dx \right| \\ &= \left| \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} (e^{-2\pi i x h} - 1) dx \right| \\ &\leq \int_{\mathbb{R}} |f(x)| |e^{-2\pi i x h} - 1| dx \leq C \int_{\mathbb{R}} |f(x)| dx = C \|f\|_{L^1} \end{aligned}$$

for some $C > 0$, if $|h|$ is sufficiently small. By DCT, we have

$$\begin{aligned} \lim_{h \rightarrow 0} (\widehat{f}(\xi + h) - \widehat{f}(\xi)) &= \lim_{h \rightarrow 0} \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} (e^{-2\pi i x h} - 1) dx \\ &= \int_{\mathbb{R}} \lim_{h \rightarrow 0} f(x) e^{-2\pi i x \xi} (e^{-2\pi i x h} - 1) dx = 0, \end{aligned}$$

that is, \widehat{f} is continuous.

The second part is the lemma called *Riemann-Lebesgue Lemma*. Let g be a compactly supported continuous function. By substituting x into $x + 1/2\xi$ in the definition of Fourier transform, we have

$$\widehat{g}(\xi) = \int_{\mathbb{R}} g\left(x + \frac{1}{2\xi}\right) e^{-2\pi i x \xi - \pi i} dx = - \int_{\mathbb{R}} g\left(x + \frac{1}{2\xi}\right) e^{-2\pi i x \xi} dx.$$

Since g is continuous and has compact support, $g(x) - g(x + 1/2\xi) \rightarrow 0$ for any $x \in \mathbb{R}$ as $|\xi| \rightarrow \infty$. By DCT, we have

$$\widehat{g}(\xi) \leq \frac{1}{2} \int_{\mathbb{R}} \left| g(x) - g\left(x + \frac{1}{2\xi}\right) \right| dx \rightarrow 0$$

as $|\xi| \rightarrow 0$. Finally, for $f \in L^1$, let g be a continuous function with compact support such that $\|f - g\|_{L^1} < \varepsilon$. Then

$$|\widehat{f}(\xi)| \leq |\widehat{f}(\xi) - \widehat{g}(\xi)| + |\widehat{g}(\xi)| \leq \|f - g\|_{L^1} + |\widehat{g}(\xi)| \leq \varepsilon + |\widehat{g}(\xi)|$$

and

$$\limsup_{|\xi| \rightarrow \infty} |\widehat{f}(\xi)| \leq \varepsilon$$

whatever ε is. That is, \widehat{f} vanishes at infinity.

3. Let $\{f_n\}_{n=1}^\infty$ be a sequence in $L^p([0, 1])$ for $p \in (1, \infty)$. Suppose that there exists a $f \in L^p([0, 1])$ satisfying $\lim_{n \rightarrow \infty} \int_0^1 f_n(x)g(x)dx = \int_0^1 f(x)g(x)dx$ for any $g \in L^q([0, 1])$ with $\frac{1}{p} + \frac{1}{q} = 1$. Prove that $\lim_{n \rightarrow \infty} \|f_n - f\|_p = 0$ if $\lim_{n \rightarrow \infty} \|f_n\|_p = \|f\|_p$.

Sol.

1.9 2020 Feb Real

2 Complex Analysis

2.1 2024 Feb Complex

1. Prove that $\sum_{n=1}^{\infty} e^{-n^2} z^n$ is an entire function.
2. Find all entire functions f such that $f(n\pi) = 0$ for any $n \in \mathbb{Z}$ and $|f(x + iy)| \leq Ce^{|y|} < \infty$, $x, y \in \mathbb{R}$ for some $C > 0$.
3. Find all entire functions f which satisfies the property that for some $R, C > 0$, $|f(z)| \geq C$ when $|z| \geq R$.

Sol. Let f be an entire function satisfying given properties. As f is continuous on compact set $\{z : |z| \leq R\}$, it is bounded on the compact set by $M > C/2$. Then the modulus of $g(z) = f(z) + 2M$ is bounded below by some $M' > 0$; $|g(z)| > M$ on $\{z : |z| \leq R\}$ and $|g(z)| \geq 2M - C$ on $\{z : |z| \geq R\}$. Then $1/g(z)$ is bounded entire function, and by Liouville's theorem, $1/g(z)$ is constant. That is, $f(z)$ is constant function. Hence $f(z) \equiv k$ for some $|k| \geq C$.

4. Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a function. Prove that f is entire if f^2 is entire and f is continuous.

Sol. Because f^2 is entire and f and f^2 share their zeros, the zeros of f should be isolated or $f \equiv 0$. If $f \equiv 0$, it is obvious. If $f(z) \not\equiv 0$, for $f(z) \neq 0$,

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \rightarrow 0} \frac{f^2(z+h) - f^2(z)}{h} \frac{1}{f(z+h) + f(z)} = \frac{(f^2)'(z)}{2f(z)}.$$

This gives that f is differentiable on whole complex plane except its zeros, and is holomorphic.

For a zero z_0 , let D be a domain of z_0 with $f(z) \neq 0$ except $z = z_0$. Denote $D_0 = D \setminus \{z_0\}$. By Riemann's theorem, a holomorphic function $f|_{D_0} : D_0 \rightarrow \mathbb{C}$ can be holomorphically extended to $f|_D : D \rightarrow \mathbb{C}$ with $f(z_0) = 0$, since f is continuous at $z = z_0$. Hence f is also entire.

5. Evaluate $\int_0^{2\pi} \frac{\cos^2 \theta}{5+3\cos \theta} d\theta$.

Sol. By substituting $z = e^{i\theta}$, the given integral becomes

$$\begin{aligned} & \int_{|z|=1} \frac{\frac{1}{4}(z + \frac{1}{z})^2}{5 + \frac{3}{2}(z + \frac{1}{z})} \frac{dz}{iz} \\ &= -i \int_{|z|=1} \frac{z^2 + 2 + \frac{1}{z^2}}{6z^2 + 20z + 6} dz \\ &= -i \int_{|z|=1} \frac{z^4 + 2z^2 + 1}{z^2(6z^2 + 20z + 6)} dz \\ &= -i \int_{|z|=1} \frac{z^4 + 2z^2 + 1}{2z^2(z+3)(3z+1)} dz. \end{aligned}$$

In the region $\{|z| < 1\}$, the function $\frac{z^4+2z^2+1}{2z^2(z+3)(3z+1)}$ has singularities only at $z = 0$ and $z = -1/3$, and both are poles. By residue theorem,

$$\begin{aligned} & \int_{|z|=1} \frac{z^4 + 2z^2 + 1}{2z^2(z+3)(3z+1)} dz \\ &= 2\pi i \left(\left. \frac{d}{dz} \right|_{z=0} \frac{z^4 + 2z^2 + 1}{(z+3)(3z+1)} + \left. \frac{z^4 + 2z^2 + 1}{z^2(z+3)} \right|_{z=-1/3} \right) \\ &= 2\pi i \left(10 + \frac{25}{6} \right) = \frac{85\pi i}{3}. \end{aligned}$$

Therefore the given integral becomes $\frac{85\pi}{3}$.

6. Show that a polynomial $f(z) = z^5 + 2z^3 + 1$ has no zero in $D(0; 2/3)$, three zeros in $D(0, 1) \setminus D(0, 2/3)$ and two zeros in $D(0, 2) \setminus \overline{D(0, 1)}$.

Sol. Use Rouché's theorem.

On $\{|z| = 2/3\}$, $|z^5 + 1| \geq 1 - |z|^5 = 211/243 > 16/27 = |2z^3|$. Hence the disk $D(0, 2/3)$ has no zeros of $z^5 + 2z^3 + 1$, because $z^5 + 1 = 0$ has no roots in the disk.

On $\{|z| = 1\}$, for any $a > 0$,

$$|z^5 + (2+a)z^3| = |z^2 + 2 + a| \geq 1 + a, \quad |1 - az^3| \leq 1 + a.$$

In the first inequality, $|z^5 + (2+a)z^3| = 1 + a$ if and only if $z = \pm i$, and $|1 - a(\pm i)^3| < 1 + a$. In the second inequality, $|1 - az^3| = 1 + a$ if and only if $z^3 = -1$. If $z = -1$, $|z^2 + 2 + a| = 3 + a$. If $z^2 - z + 1 = 0$, then $|z^2 + 2 + a| = a + 1 + a^2$.

Hence $|z^5 + (2+a)z^3| > |1 - az^3|$ for all z with $|z| = 1$. Thus the disk $D(0, 1)$ has three zeros of $z^5 + 2^3 + 1$, because $z^5 + (2+a)z^3 = z^3(z^2 + 2 + a) = 0$ has only three roots in the disk.

Finally, on $\{|z| = 2\}$, $|z^5| = 32 \geq 17 = 1 + 2|z^3|$. Hence the disk $D(0, 2)$ has all five zeros of $z^5 + 2^3 + 1$. Since there are three zeros in $D(0, 1)$, $D(0, 2) \setminus \overline{D(0, 1)}$ contains two zeros of $z^5 + 2z^3 + 1$.

2.2 2023 Aug Complex

1. Let $f(z)$ be entire function such that $|e^{f(z)}| \leq |z|$ for $|z| \geq 1$. What can you say about $f(z)$?
2. Find a branch of $\sqrt{z(1-z)}$ so that it becomes a holomorphic (single-valued) function on $\mathbb{C} \setminus [0, 1]$.
3. Evaluate the following improper integral

$$\int_0^\infty \frac{\log x}{(1+x^2)(x^2+4)} dx.$$

4. Find a partial fraction decomposition of

$$\frac{\pi}{\cos(\pi z)}.$$

5. Find a conformal map of the vertical strip $\{-1 < \operatorname{Re} z < 1\}$ onto the unit disc $\{|z| < 1\}$.
6. Suppose that $D \neq \mathbb{C}$ is a simply connected domain. Construct an injective conformal map $f : D \rightarrow \{|z| < 1\}$. (Do not quote Riemann mapping theorem. This problem asks a part of its proof.)
7. Let $D \neq \mathbb{C}$ be a simply connected domain. Suppose that $f : D \rightarrow D$ a holomorphic function having a fixed point $f(a) = a$. Show that $|f'(a)| \leq 1$. Moreover if $|f'(a)| = 1$, then f is a homeomorphism of D .

2.3 2023 Feb Complex

1. Let $f(z)$ is holomorphic in a connected domain D . Assume that $f(z)$ is constant on a curve $C \subset D$. Show that $f(z)$ is constant in D .

Sol.

2. Evaluate the following improper integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{(1+x^2)^2} dx.$$

Sol.

3. Prove that the following infinite product converges and evaluate it

$$\prod_{n=1}^{\infty} \left(1 + \frac{(-1)^{n+1}}{n} \right).$$

4. Denote the upper half plane by $\mathcal{H} = \{\operatorname{Im} z > 0\}$. Find most general form of linear fractional transforms that maps \mathcal{H} onto \mathcal{H} . Show that any conformal self-map of \mathcal{H} is of that form.
5. Find poles and their principal parts of $\frac{1}{\sin^2 z}$. Verify the partial fraction formula

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{k=-\infty}^{\infty} \frac{1}{(z-k)^2}.$$

From this deduce that

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{z \neq 0} \left(\frac{1}{z-k} + \frac{1}{k} \right).$$

6. Construct an entire function that has simple zeros at the points n^2 , for each $n \in \mathbb{N}$ and no other zeros.

2.4 2022 Aug Complex

1. Let \mathbb{C}_∞ be the Riemann sphere. Show that if $f : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ is meromorphic, then f is rational.

Sol. Let S be a subset of \mathbb{C}_∞ where f has a pole at each $z \in S$. If S had a limit point p , then f cannot be neither analytic at p nor have an isolated singularity at p . Hence S cannot have a limit point. Since \mathbb{C}_∞ is compact, S must be finite. Let $S \cap \mathbb{C} = \{P_1, \dots, P_k\}$. So, $f(z)(z - P_1)^{n_1} \dots (z - P_k)^{n_k} =: F(z)$ is entire function on \mathbb{C} , where n_i is order of pole P_i . Then either $\infty \in S$ or not.

If $\infty \in S$, $f(1/z)$ has a pole at $z = 0$. then $F(1/z)$ has a pole at $z = 0$, that is,

$$F(1/z) = \sum_{n=-n_0}^{\infty} a_n z^n$$

and

$$F(z) = \sum_{n=-n_0}^{\infty} a_n z^{-n}.$$

Since F does not have essential singularity at $z = 0$, $a_n \equiv 0$ if $n \geq N$. Hence

$$f(z) = \frac{F(z)}{(z - P_1)^{n_1} \dots (z - P_k)^{n_k}} = \frac{\sum_{n=-n_0}^N a_n z^{-n}}{(z - P_1)^{n_1} \dots (z - P_k)^{n_k}}$$

is a rational function.

If $\infty \notin S$, then $f(1/z)$ has removable singularity at $z = 0$. That is, $\lim_{z \rightarrow 0} f(1/z)$ is well-defined, and hence

$$\begin{aligned} F(1/z) &= f(1/z)(1/z - P_1)^{n_1} \dots (1/z - P_k)^{n_k} \\ &= \frac{f(1/z)(1 - zP_1)^{n_1} \dots (1 - zP_k)^{n_k}}{z^{n_1 + \dots + n_k}} \end{aligned}$$

has either a pole at $z = 0$ with order at most $n_1 + \dots + n_k$, or a removable singularity.

If it has a removable singularity, then $F(z)$ has removable singularity at $z = \infty$, and hence $F|_{\mathbb{C}}(z)$ is bounded on $\{z : |z| \geq R\}$ for some R . Then $F|_{\mathbb{C}}(z)$ is bounded on whole \mathbb{C} , and by Liouville's theorem, $F(z)$ is a constant function. Hence

$$f(z) = \frac{C}{(z - P_1)^{n_1} \dots (z - P_k)^{n_k}}$$

is a rational function.

If it is a pole of order d , then $F(z)z^d = z^d f(z)(z - P_1)^{n_1} \cdots (z - P_k)^{n_k}$ has removable singularity at $z = \infty$, and by same argument, $F(z)z^d$ is a constant function. Hence

$$f(z) = \frac{C'}{z^d(z - P_1)^{n_1} \cdots (z - P_k)^{n_k}}$$

is a rational function.

2. (a) Evaluate

$$\int_{-1}^1 \frac{\sqrt{1-x^2}}{1+x^2} dx$$

- (b) Check if the integral is integrable. If so, evaluate it.

$$\int_0^\infty \frac{\log x}{x^b - 1} dx, \quad b > 1$$

Sol. (a)

(b)

3. Denote $\mathbb{D} = \{z : |z| < 1\}$. Show if $f : \mathbb{D} \rightarrow \mathbb{D}$ is analytic, then

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

Moreover, if $f(z)$ is a conformal self-map of \mathbb{D} , then the equality holds. (Hint: Use the conformal self-map of \mathbb{D} sending 0 to z_0 and its inverse.)

Sol. This is called *Schwartz-Pick Lemma*.

If $w \in \mathbb{D}$, then set

$$\varphi_w(z) := \frac{z - w}{1 - \overline{w}z}$$

Then φ is a conformal self-map of \mathbb{D} which maps w to 0. Elementary algebra shows that φ_w is invertible and that its inverse is φ_{-w} . Now, for the function f given in the problem, we consider

$$g = \varphi_{f(z_0)} \circ f \circ \varphi_{z_0}^{-1} : \mathbb{D} \rightarrow \mathbb{D}.$$

Then

$$g(0) = \varphi_{f(z_0)}(f(\varphi_{z_0}^{-1}(0))) = \varphi_{f(z_0)}(f(z_0)) = 0$$

and hence Schwarz's lemma can be applied, i.e., $|g'(0)| \leq 1$, where

$$\begin{aligned} g'(0) &= \varphi'_{f(z_0)}(f(z_0)) \cdot f'(z_0) \cdot \frac{1}{\varphi'_{z_0}(z_0)} \\ &= \frac{1}{1 - |f(z_0)|^2} \cdot f'(z_0) \cdot 1 - |z_0|^2 \\ &= \frac{1 - |z_0|^2}{1 - |f(z_0)|^2} f'(z_0) \end{aligned}$$

so that $|f'(z_0)| \leq (1 - |f(z_0)|^2)/(1 - |z_0|^2)$. As the choice of z_0 is arbitrary, the given inequality holds.

From Schwarz's lemma, the equality holds if and only if $g(z) = e^{i\lambda}z$ for some $\lambda \in \mathbb{R}$. This is a conformal self-map of \mathbb{D} , and $f = \varphi_{f(z_0)}^{-1} \circ g \circ \varphi_{z_0}$ is a composition of conformal self-maps, which is also a conformal self-map.

4. Let $f(z)$ be the Riemann map of a simply connected domain D onto the unit disk \mathbb{D} . Suppose $f(z_0) = 0$ and $f'(z_0) > 0$. Show that if $g(z)$ is an analytic function on D such that $|g(z)| \leq 1$ for $z \in D$ and $g(z_0) = 0$, then $\operatorname{Re} g'(z_0) \leq f'(z_0)$.

Sol. As f is a Riemann map, it has the inverse $f^{-1} : \mathbb{D} \rightarrow D$, which is analytic. Then $h := g \circ f^{-1} : \mathbb{D} \rightarrow \mathbb{D}$ satisfies the conditions for Schwarz's lemma. Hence $|h'(0)| \leq 1$, where

$$h'(0) = g'(f^{-1}(0)) \cdot \frac{1}{f'(z_0)} = \frac{g'(z_0)}{f'(z_0)}$$

and $f'(z_0) > 0$ so that $|g'(z_0)| \leq f'(z_0)$. As $\operatorname{Re} g'(z_0) \leq |\operatorname{Re} g'(z_0)| \leq |g'(z_0)|$ is obvious, the given inequality is valid.

5. (a) Let $\{a_n\} \subset \mathbb{C} \setminus \{0\}$ be a sequence³. Show that $\prod_{n=1}^{\infty} (1 - \frac{z}{a_n})$ is entire if and only if $\sum_{n=1}^{\infty} \frac{1}{z - a_n}$ is meromorphic.
 (b) Find a meromorphic function $f(z)$ which has poles only at $z = n$ for each positive integer n with order n .

Sol. (a) Suppose $f(z) = \prod_{n=1}^{\infty} (1 - \frac{z}{a_n})$ is entire. Then the infinite product converges uniformly, and logarithmic derivative is valid. Hence

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty} \frac{-1/a_n}{1 - z/a_n} = \sum_{n=1}^{\infty} \frac{1}{z - a_n}$$

³The condition that the set has no limit points would have to be added.

is analytic except the points where $f(z) = 0$. Such points form a set $S = \{a_1, a_2, \cdot\}$, and at $z_0 \in S$, it has a pole. $\sum_{n=1}^{\infty} \frac{1}{z-a_n}$ has no singularities except poles, i.e., it is meromorphic.

Conversely,

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1. Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be a sequence such that $\sum_{n=1}^{\infty} \frac{1}{|a_n|}$ diverges but $\sum_{n=1}^{\infty} \frac{1}{|a_n|^2}$ converges. Find an entire function that has zeros only at $\{a_n\}_{n=1}^{\infty}$. (You need to verify that your example is entire.)

Sol. This is an example of Weierstrass' product theorem.

Clearly $a_n \neq 0$ for all n . Since $\sum_{n=1}^{\infty} \frac{1}{|a_n|^2}$ converges absolutely, without loss of generality, assume that $|a_n|$ is increasing sequence. Consider the infinite product

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp \left(\frac{z}{a_n} + \frac{1}{2} \left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{n} \left(\frac{z}{a_n}\right)^n \right).$$

It converges if and only if the series

$$\sum_{n=1}^{\infty} \left[\operatorname{Log} \left(1 - \frac{z}{a_n}\right) + \left(\frac{z}{a_n} + \frac{1}{2} \left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{n} \left(\frac{z}{a_n}\right)^n \right) \right]$$

converges. Suppose $|z| < R$. By Taylor expansion, if n is sufficiently large so that $|z/a_n| \leq R/|a_n| < 1/2 < 1$, then

$$\operatorname{Log} \left(1 - \frac{z}{a_n}\right) + \left(\frac{z}{a_n} + \frac{1}{2} \left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{n} \left(\frac{z}{a_n}\right)^n \right) = - \sum_{k=n+1}^{\infty} \frac{1}{k} \left(\frac{z}{a_n}\right)^k$$

and

$$\left| - \sum_{k=n+1}^{\infty} \frac{1}{k} \left(\frac{z}{a_n}\right)^k \right| \leq \frac{1}{n+1} \left| \frac{R}{a_n} \right|^n \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j < \frac{1}{2^n}$$

so that

$$\begin{aligned} & \left| \sum_{n=1}^{\infty} \left[\operatorname{Log} \left(1 - \frac{z}{a_n}\right) + \left(\frac{z}{a_n} + \frac{1}{2} \left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{n} \left(\frac{z}{a_n}\right)^n \right) \right] \right| \\ & \leq \sum_{n=1}^{\infty} \left| \left[\operatorname{Log} \left(1 - \frac{z}{a_n}\right) + \left(\frac{z}{a_n} + \frac{1}{2} \left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{n} \left(\frac{z}{a_n}\right)^n \right) \right] \right| \\ & \leq \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty \end{aligned}$$

for sufficiently large n 's, and hence it converges uniformly on $|z| \leq R$. Hence this product is analytic on $\{z : |z| < R\}$. As the choice of R is arbitrary, it may be concluded that this infinite product is entire.

2. Let $f : D \rightarrow D$ be analytic in a simply connected domain $D \subsetneq \mathbb{C}$ having a fixed point in D . Show that $|f'(a)| \leq 1$ for all $a \in D$. Show if $|f'(a)| = 1$ for some $a \in D$, then f is bijective on D .

Sol. Indeed, by choosing $f(z) = z^2$ and D as the unit disk, it satisfies all given condition but does not satisfy the conclusion. However, by letting a as *the unique fixed point*, it has no problem. See [2] p. 403 Example 11.29.

Let \mathbb{D} be the unit disk, and consider the Riemann map $\varphi : D \rightarrow \mathbb{D}$ with $\varphi(a) = 0$. Let $g = \varphi \circ f \circ \varphi^{-1}$. Then $g : \mathbb{D} \rightarrow \mathbb{D}$ and $g(0) = 0$.

Since φ is conformal, it is guaranteed that $\varphi'(a) \neq 0$. By Schwarz's lemma,

$$g'(0) = \varphi'(a) \cdot f'(a) \cdot \frac{1}{\varphi'(a)} = f'(a),$$

and thus $|g'(0)| = |f'(a)| \leq 1$. Moreover, the equality holds if and only if $g(z) = \lambda z$ with $|\lambda| = 1$. In this condition, $f(z) = \varphi^{-1}(\lambda \varphi(z))$ and this is a composition of bijections. Hence f must be a bijection.

3. Let D be a domain and $f : D \rightarrow \mathbb{C}$ be an analytic function with $f'(a) \neq 0$ for some $a \in D$. Show that the derivative $df(a)$ is a composition of rotation and dilation in \mathbb{C} . (Here, $df(a)$ is the gradient of f , when one understand $f : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$)

Sol. Let $z = x + iy$, and let $f(x + iy) = u(x, y) + iv(x, y)$. Let $c = |f'(a)| \neq 0$. Then by Cauchy-Riemann equation,

$$\begin{aligned} df(a) &= \begin{pmatrix} u_x(a) & u_y(a) \\ v_x(a) & v_y(a) \end{pmatrix} = \begin{pmatrix} u_x(a) & -v_x(a) \\ v_x(a) & u_x(a) \end{pmatrix} \\ &= \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} u_x(a)/c & -v_x(a)/c \\ v_x(a)/c & u_x(a)/c \end{pmatrix} \end{aligned}$$

where

$$\left(\frac{u_x(a)}{c}\right)^2 + \left(\frac{v_x(a)}{c}\right)^2 = \frac{u_x(a)^2 + v_x(a)^2}{c^2} = \frac{|f'(a)|^2}{|f'(a)|^2} = 1.$$

That is, there exists $\theta \in \mathbb{R}$ such that

$$\cos \theta = \frac{u_x(a)}{c}, \quad \sin \theta = \frac{v_x(a)}{c}.$$

Therefore $df(a)$ is a composition of dilation matrix

$$\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$$

and rotation matrix

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

4. Let D be a connected domain and $\{f_n\}$ a sequence of injective analytic functions on D . Assume that $\{f_n\}$ converges uniformly on each compact subset of D . Show that the limit function f is either injective or constant.

Sol. Assume that f is neither injective nor constant. Then there is a complex number w such that $f(z) = w$ has at least two solutions in D . Let K be a connected compact subset of D where the equation $f(z) = w$ has more than two solutions, and no solutions on ∂K . As $f_n(z) - w$ converges to $f(z) - w$ uniformly on K , by Hurwitz's theorem, the number of zeros of $f(z) - w$ is equal to the number of zeros of $f_n(z) - w$ for sufficiently large n . But it contradicts that $f_n(z) - w$ is injective for all n . Hence the assumption fails.

5. Let f be analytic and satisfy $|f(z)| \leq M$ on $|z - z_0| < R$ for some $M, R > 0$. Show that if $f(z)$ has a zero of order m at z_0 , then

$$|f(z)| \leq \frac{M}{R^m} |z - z_0|^m, \quad |z - z_0| < R.$$

Show that if the equality holds at some point, then $f(z) = C(z - z_0)^m$ for some C .

Sol. Since f has a zero of order m at z_0 , $g(z) = f(z)/(z - z_0)^m$ has removable singularity at z_0 , and $\lim_{z \rightarrow z_0} g(z) \neq 0$. Then by maximum modulus theorem, for any $0 < r < R$,

$$\max_{|z - z_0| = r} |g(z)| \leq \frac{M}{r^m}$$

and by letting $r \rightarrow R$, $|g(z)| \leq M/R^m$. Hence $|f(z)| \leq M|z - z_0|^m/R^m$.

From maximum modulus, the equality holds if and only if g is constant function. Thus $f(z) = C(z - z_0)^m$ for some C .

6. Let D be a domain and $f : D \rightarrow \mathbb{C}$ be an analytic function. Assume that $f(a_n) = 0$ for all n , where $\{a_n\}_{n=1}^\infty \subset D$ is a convergent sequence in \mathbb{C} . Prove or disprove that $f \equiv 0$.

Sol. Let $D = \{z : \operatorname{Re}(z) > 0\}$, $a_n = 1/n$ for all n and $f(z) = \sin(\pi/z)$. Then clearly a_n converges to $0 \in \mathbb{C}$, $f(z) \not\equiv 0$, but $f(a_n) = \sin(n\pi) = 0$.

It is because the limit point of a_n is not in D . If it is a point of D , then by uniqueness theorem, f should be zero function.

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