1. Let $A_0 = \begin{bmatrix} 10 & 8 & 12 & 15 & 15 \\ 8 & 14 & 8 & 7 & 9 \\ 12 & 8 & 10 & 13 & 9 \\ 15 & 7 & 13 & 4 & 10 \\ 15 & 9 & 9 & 10 & 4 \end{bmatrix}$, $A_1 = \begin{bmatrix} 12 & 11 & 14 & 10 & 3 \\ 11 & 14 & 10 & 14 & 6 \\ 14 & 10 & 16 & 18 & 4 \\ 10 & 14 & 18 & 18 & 8 \\ 3 & 6 & 4 & 8 & 8 \end{bmatrix}$, $A_2 = \begin{bmatrix} 4 & 13 & 12 & 16 & 6 \\ 13 & 4 & 14 & 9 & 15 \\ 12 & 14 & 6 & 5 & 5 \\ 16 & 9 & 5 & 2 & 6 \\ 6 & 15 & 5 & 6 & 8 \end{bmatrix}$.

Suppose $A : \mathbb{R}^2 \to S^5$ is defined by

$$A(x) = A_0 + x_1 A_1 + x_2 A_2.$$

Let $\lambda_1(x) \geq \lambda_2(x) \geq \lambda_3(x) \geq \lambda_4(x) \geq \lambda_5(x)$ denote the eigenvalues of $A(x)$.

(a) Formulate the problem of minimizing the spread of the eigenvaules $\lambda_1(x) - \lambda_5(x)$ as an SDP. (10%)

(b) Solve (a) by using MATLAB with the CVX tool. What are the optimal point and optimal value? (10%)

(a)

$$\begin{aligned} \text{minimize} \quad & \lambda_1(x) - \lambda_5(x) \\ \text{subject to} \quad & A(x) \preceq \lambda_1(x) I \\ & A(x) \succeq \lambda_5(x) I \end{aligned}$$

(b)

```
clear all; close all; clc;
A0 = [10 8 12 15 15; 8 14 8 7 9; 12 8 10 13 9; 15 7 13 4 10; 15 9 9 10 4];
A1 = [12 11 14 10 3; 11 14 10 14 6; 14 10 16 18 4; 10 14 18 18 8; 3 6 4 8 8];
A2 = [4 13 12 16 6; 13 4 14 9 15; 12 14 6 5 5; 16 9 5 2 6; 6 15 5 6 8];

cvx_begin sdp
    variables x1 x2 lamda1 lamda2
    minimize( lamda1-lamda2 )
    subject to
        A = A0 + x1*A1 + x2*A2;
        A <= lamda1*eye(5);
        A >= lamda2*eye(5);
cvx_end

fprintf("x1 = %f, x2 = %f\n",x1,x2);
```

```
Status: Solved
Optimal value (cvx_optval): +28.1544

x1 = -0.596605, x2 = -0.335843
```

optimal points

2. Prove that $x^\star = (1, 1/2, -1)$ is optimal for the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) = (1/2)x^T P x + q^T x + r \\ \text{subject to} & -1 \le x_i \le 1, \quad i = 1,2,3 \end{array}$$

where

$$P = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}, \quad q = \begin{bmatrix} -22 \\ -14.5 \\ 13 \end{bmatrix}, \quad r = 1.$$

(Hint: you may verify the optimality condition.) (15%)

$$\nabla f_0(x) = Px + q \Rightarrow \nabla f_0(x^*) = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \\ -1 \end{bmatrix} + \begin{bmatrix} -22 \\ -14.5 \\ 13 \end{bmatrix}$$

$$= \begin{bmatrix} 21 \\ 14.5 \\ -11 \end{bmatrix} + \begin{bmatrix} -22 \\ -14.5 \\ 13 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

By the optimal condition, we know that if x^* is optimal point

$$\nabla f_0(x^*)^T (y - x^*) \ge 0 \; ; \; \forall y \in \text{feasible set}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 - 1 \\ y_2 - \frac{1}{2} \\ y_3 + 1 \end{bmatrix} \ge 0$$

$$\Rightarrow -(y_1 - 1) + 2(y_3 + 1) \ge 0$$

$$\Rightarrow -y_1 + 2y_3 + 3 \ge 0$$

for all y satisfying $-1 \le y_i \le 1$; $i = 1, 2, 3$ is true #

3. Consider a problem of the form

$$\begin{array}{ll} \text{minimize} & f_0(x)/(c^T x + d) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

where $f_0, f_1, \ldots, f_m$ are convex, and the domain of the objective function is defined as $\{x \in \operatorname{dom} f_0 : c^T x + d > 0\}$.

Prove that the above problem is equivalent to the following problem:

$$\begin{array}{ll} \text{minimize} & g_0(y,t) \\ \text{subject to} & g_i(y,t) \le 0, \quad i = 1, \ldots, m \\ & Ay = bt \\ & c^T y + dt = 1 \end{array}$$

where g_i is the perspective of f_i (see §3.2.6). The variables are $y \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

Further show that this problem is convex. (20%)

① Suppose x is feasible in original problem. Define

$t = \dfrac{1}{c^T x + d}$, where $t > 0$; $y = \dfrac{x}{c^T x + d}$

$\Rightarrow g_0(y,t) = \frac{1}{t} f_0\left(\frac{y}{t}\right) = \frac{1}{t} \cdot f_0(x) = \dfrac{f_0(x)}{c^T x + d}$

$\Rightarrow g_i(y,t) \le 0 \quad i = 1, \ldots, m \qquad \because f_i(x) \le 0 \quad i = 1, \ldots, m$

$\Rightarrow Ay = bt \Rightarrow \dfrac{Ax}{c^T x + d} = \dfrac{b}{c^T x + d} \Rightarrow Ax = b$

$\Rightarrow c^T y + dt = 1 \Rightarrow \dfrac{c^T x}{c^T x + d} + \dfrac{d}{c^T x + d} = 1$

Then $t \cdot y$ are feasible in the equivalent problem

② Suppose $y \cdot t$ are feasible for the transformed problem.

We must have $t > 0$. Define $x = y/t$.

We have $x \in \operatorname{dom} f_i$ for $i = 0, \ldots, m$

$\Rightarrow \dfrac{f_0(x)}{c^T x + d} = \dfrac{f_0(y/t)}{c^T (y/t) + d} = \dfrac{t f_0(y/t)}{c^T y + dt} = t f_0(y/t) = g_0(y,t)$

$\Rightarrow f_i(x) = f_i\left(\frac{y}{t}\right) = t g_i(y,t) \le 0$

$\Rightarrow Ax = b \Rightarrow A\left(\frac{y}{t}\right) = b \Rightarrow Ay = bt$

Then x is feasible in the original problem, with the objective value $g_0(y, t)$

③ So the transformed problem is equivalent to the original problem. #

And this problem is convex #

4. Formulate the following problem as a linear program.

$$\text{minimize} \qquad ||Ax - b||_\infty$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, and $b \in \mathbb{R}^m$. Show that your linear program is equivalent to the original problem. (15%)

Equivalent to the linear program

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & Ax - b \le t\mathbf{1} \\ & Ax - b \ge -t\mathbf{1} \end{aligned}$$

in the variable x, t. To see the equivalence, assume x is fixed in this problem, and we optimize only over t.

The constraints say that

$$-t \le a_k^T x - b_k \le t$$

for each k, i.e., $t \ge |a_k^T x - b_k|$, i.e.,

$$t \ge \max_k |a_k^T x - b_k| = \|Ax - b\|_\infty$$

Clearly, if x is fixed, the optimal value of the LP is $p^*(x) = \|Ax-b\|_\infty$. Therefore optimizing over t and x simultaneously is equivalent to the original problem.

5. (Dual of general LP). Find the dual function of the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \le h \\ & Ax = b. \end{array}$$

Give the dual problem, and make the implicit equality constraints explicit. (15%)

The Lagrangian is

$$\begin{aligned} L(x,\lambda,v) &= c^T x + \lambda^T (Gx-h) + v^T (Ax-b) \\ &= (c^T + \lambda^T G + v^T A)x - \lambda^T h - v^T b \end{aligned}$$

which is an affine function of x.

① The dual function is given by

$$g(\lambda,v) = \inf_x \{L(x,\lambda,v)\} = \begin{cases} -\lambda^T h - v^T b & ; \text{ if } c^T + \lambda^T G + v^T A = 0 \\ -\infty & ; \text{ otherwise} \end{cases}$$

② The dual problem is

$$\begin{array}{ll} \text{maximize} & g(\lambda,v) \\ \text{subject to} & \lambda \ge 0 \end{array}$$

③ After making the implicit constraint explicit, we have

$$\begin{array}{ll} \text{maximize} & -h^T\lambda - b^T v \\ \text{subject to} & c + G^T\lambda + A^T v = 0 \\ & \lambda \ge 0 \end{array}$$

6. Derive a dual problem for

$$\text{minimize} \quad -\sum_{i=1}^{m} \log(b_i - a_i^T x)$$

with domain $\{x : a_i^T x < b_i, i = 1, \ldots, m\}$. First introduce new variables y_i and equality constraints $y_i = b_i - a_i^T x$. (15%)

We derive the dual of the problem

$$\begin{aligned} &\text{minimize} && -\sum_{i=1}^{m} \log y_i \\ &\text{subject to} && y = b - Ax \end{aligned}$$

where $A \in \mathbb{R}^{m \times n}$ has a_i^T as its ith row. The Lagrangian is

$$L(x, y, v) = -\sum_{i=1}^{m} \log y_i + v^T(y - b + Ax)$$

and the dual function is

$$g(v) = \inf_{x,y} \{L(x, y, v)\} = \inf_{x,y} \{-\sum_{i=1}^{m} \log y_i + v^T(y - b + Ax)\}$$

The term $v^T A x$ is unbounded unless $A^T v = 0$ and the term in y are unbounded unless $v > 0$. And achieve their minimum for $y_i = 1/v_i$. The dual function

① $$g(v) = \begin{cases} \sum_{i=1}^{m} \log v_i + m - b^T v & ; A^T v = 0, \ v > 0 \\ -\infty & ; \text{otherwise} \end{cases}$$

And the dual problem

② $$\begin{aligned} &\text{maximize} && \sum_{i=1}^{m} \log v_i - b^T v + m \\ &\text{subject to} && A^T v = 0 \end{aligned}$$