1. Backtracking line search. Suppose that f is strongly convex with $mI \preceq \nabla^2 f(x) \preceq MI$. Let Δx be a descent direction at x. Show that the backtracking stopping condition holds for

$$0 < t \leq -\frac{\nabla f(x)^T \Delta x}{M\|\Delta x\|_2^2}.$$

Use this to give an upper bound on the number of backtracking iterations. (20%)

$$f(y) = f(x) + \nabla f(x)^T (y-x) + \frac{1}{2} \nabla^2 f(x) (y-x)^2$$

$$\leq f(x) + \nabla f(x)^T (y-x) + \frac{M}{2} (y-x)^2 \qquad (\text{use } \nabla^2 f(x) \leq MI)$$

use $y = x + t\Delta x$

then $f(x+t\Delta x) \leq f(x) + t \nabla f(x)^T \Delta x + \left(\frac{M}{2}\right) t^2 \Delta x^T \Delta x$

hence $f(x+t\Delta x) \leq f(x) + \alpha t \nabla f(x)^T \Delta x$

if $\quad (1-\alpha) t \nabla f(x)^T \Delta x + \left(\frac{M}{2}\right) t^2 \Delta x^T \Delta x \leq 0$

i.e the exit condition certainly holds if $0 \leq t \leq t_0$

$$(1-\alpha) \nabla f(x)^T \Delta x + \left(\frac{M}{2}\right) t \Delta x^T \Delta x \leq 0$$

$$\Rightarrow \left(\frac{M}{2}\right) t \Delta x^T \Delta x \leq -(1-\alpha) \nabla f(x)^T \Delta x$$

$$\Rightarrow t \leq -2(1-\alpha) \frac{\nabla f(x)^T \Delta x}{M \Delta x^T \Delta x}$$

So $\quad t_0 = -2(1-\alpha) \dfrac{\nabla f(x)^T \Delta x}{M \Delta x^T \Delta x} \geq -\dfrac{\nabla f(x)^T \Delta x}{M \Delta x^T \Delta x}$

then $\quad 0 \leq t \leq -\dfrac{\nabla f(x)^T \Delta x}{M \|\Delta x\|_2^2}$

2. Suppose $Q \succeq 0$. The following modified problem

$$\begin{array}{ll} \text{minimize} & f(x) + (Ax-b)^T Q (Ax-b) \\ \text{subject to} & Ax = b. \end{array}$$

is equivalent to the original problem (10.1). Is the Newton step for this problem the same as the Newton step for the original problem? (20%)

The Newton step of the new problem satisfies

$$\begin{bmatrix} H + A^TQA & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ w \end{bmatrix} = \begin{bmatrix} -g - 2A^TQAx + 2A^TQb \\ 0 \end{bmatrix}$$

From the second equation $A\Delta x = 0$. Then

$$\begin{bmatrix} H & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ w \end{bmatrix} = \begin{bmatrix} -g - 2A^TQAx + 2A^TQb \\ 0 \end{bmatrix}$$

and
$$\begin{bmatrix} H & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} -g \\ 0 \end{bmatrix}$$

where $\tilde{w} = w + 2QAx - 2Qb$

We conclude that the Newton Steps are equal. #

3. Adding a norm bound to ensure strong convexity of the centering problem. Suppose we add the constraint $x^T x \le R^2$ to the problem (11.1):

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \ i = 1, \ldots, m \\ & Ax = b \\ & x^T x \le R^2. \end{array}$$

Let $\tilde{\phi}$ denote the logarithmic barrier function for this modified problem. Find $m > 0$ for which $\nabla^2(tf_0(x) + \tilde{\phi}(x)) \succeq mI$ holds, for all feasible x. (20%)

Let ϕ denote the logarithmic barrier of the original problem.
The constraint $x^T x \le R^2$ adds the term $-\log(R^2 - x^T x)$ to the logarithmic barrier, so we have

$$tf_0(x) + \tilde{\phi}(x) = tf_0(x) + \phi(x) - \log(R^2 - x^T x)$$

$$\nabla^2(tf_0(x) + \tilde{\phi}(x)) = \nabla^2(tf_0(x) + \phi(x)) + \frac{2}{(R^2 - x^T x)} I + \frac{4xx^T}{(R^2 - x^T x)^2}$$

$$\succeq \nabla^2(tf_0(x) + \phi(x)) + \left(2/R^2\right) I$$

$$\succeq \left(2/R^2\right) I$$

So we can take $m = 2/R^2$ #

4. (Equality constrained entropy maximization.) Consider the equality constrained entropy maximization problem

$$\begin{array}{ll} \text{minimize} & f(x) = \sum_{i=1}^{5} x_i \log x_i \\ \text{subject to} & Ax = b \end{array}$$

where $A = \begin{bmatrix} 4 & 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 20 \\ 15 \end{bmatrix}$ with $\operatorname{dom} f = \mathbb{R}^n_{++}$.

Compute the solution of the problem using the following methods.

(a) Standard Newton method (Algorithm 10.1) with initial point $x^{(0)} = [1\ 2\ 3\ 4\ 5]^T$. (30%)

(b) Infeasible start Newton method (Algorithm 10.2) with initial point $\nu^{(0)} = \mathbf{0}$, $x^{(0)} = [1\ 2\ 3\ 4\ 5]^T$, and also $x^{(0)} = [5\ 2\ 3\ 4\ 5]^T$. (30%)

Verify that the two methods compute the same optimal point.
Note that $\operatorname{dom} f$ is not $\mathbb{R}^5$ and thus in the update step of x, you have to check that $x + t\Delta x \in \operatorname{dom} f$.

(a)

```
The initial point is : [1 2 3 4 5]
The optimal point is : [1.3193,1.8738,2.6611,3.7791,5.3667]
The optimal value is : 1.8188e+01
fx >>
```

(b)

```
(1) The initial point is : [1 2 3 4 5]
    The optimal point is : [1.3194,1.8738,2.661,3.779,5.3668]
    The optimal value is : 1.8188e+01
(2) The initial point is : [5 2 3 4 5]
    The optimal point is : [1.3194,1.8738,2.661,3.779,5.3668]
    The optimal value is : 1.8188e+01
fx >>
```

5. You were asked to prove that $x^\star = (1, 1/2, -1)$ is optimal for the following optimization problem in HW#4:

$$\begin{array}{ll} \text{minimize} & f_0(x) = (1/2)x^T Px + q^T x + r \\ \text{subject to} & -1 \le x_i \le 1, \quad i = 1, 2, 3 \end{array}$$

where

$$P = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}, \quad q = \begin{bmatrix} -22 \\ -14.5 \\ 13 \end{bmatrix}, \quad r = 1.$$

Implement a barrier method for solving this QP. Assume that the initial point is $x^\star(0) = (0, 0, 0)$. Plot the duality gap versus Newton steps (such as Fig. 11.4). Verify that the barrier method computes the optimal point.

($t = 1$, tolerance 10^{-6}, $\alpha = 0.01$, $\beta = 0.5$)

(40%)

```
The optimal point with u = 2
    1.0000
    0.5000
   -1.0000

The optimal point with u = 10
    1.0000
    0.5000
   -1.0000

The optimal point with u = 50
    1.0000
    0.5000
   -1.0000

The optimal point with u = 150
    1.0000
    0.5000
   -1.0000
```