

IND-GEOMETRIC STACKS

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ABSTRACT. We develop the theory of ind-geometric stacks, in particular their coherent and ind-coherent sheaf theory. This provides a convenient framework for working with equivariant sheaves on ind-schemes, especially in derived settings. Motivating examples include the double affine Hecke category and its relatives from the theory of Coulomb branches.

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1. INTRODUCTION

Equivariant sheaves are central objects in geometric representation theory. If G is an algebraic group acting on a variety X , then G -equivariant sheaves on X can be reformulated as sheaves on the quotient X/G . In general X/G will be an algebraic stack, and it will contain strictly less information than X together with its G -action. But it is useful to discuss X/G independently of X and G , for the same reasons it is useful to discuss smooth manifolds independently of an atlas: many results are stated and proved most clearly in coordinate-free terms, and choosing coordinates can obscure important symmetries.

In applications one often considers more general geometric settings. Our immediate motivation is the work of Braverman-Finkelberg-Nakajima [BFN18], who associate a space $\mathcal{R}_{G,N}$ to a reductive group G and a representation N . This is acted on by $G_{\mathcal{O}}$, the group of maps $\mathrm{Spec} \mathcal{O} \rightarrow G$ (where $\mathcal{O} = \mathbb{C}[[t]]$). The Coulomb branches of certain gauge theories associated to G and N can be interpreted as the spectra of the $G_{\mathcal{O}}$ -equivariant Borel-Moore and K-homology of $\mathcal{R}_{G,N}$. In [CW23a] we construct a canonical basis in the latter, induced by

a nonstandard t-structure on the category $\mathrm{Coh}^{G_{\mathcal{O}}}(\mathcal{R}_{G,N})$ of $G_{\mathcal{O}}$ -equivariant coherent sheaves on $\mathcal{R}_{G,N}$.

This setting differs from that of the first paragraph in three main ways. First, $\mathcal{R}_{G,N}$ is an ind-scheme rather than a variety. Second, $G_{\mathcal{O}}$ is an infinite-type group scheme rather than an algebraic group. And third, $\mathcal{R}_{G,N}$ is an object of derived rather than classical algebraic geometry. The considerations in the first paragraph are greatly amplified by this last point: already $\mathcal{R}_{G,N}$ is an object of higher-categorical mathematics, whereas in the classical setting one can avoid such complications by avoiding the language of stacks.

On the other hand, the first and third points imply that $\mathcal{R}_{G,N}/G_{\mathcal{O}}$ is not an algebraic (i.e. Artin) stack, even in the derived sense [TV08]. Instead, $\mathcal{R}_{G,N}/G_{\mathcal{O}}$ is an example of what we call an *ind-geometric stack*. The purpose of this paper is to develop the basic theory of such objects, in particular to provide a foundation for our study of $\mathrm{Coh}^{G_{\mathcal{O}}}(\mathcal{R}_{G,N})$ in [CW23a].

1.1. Summary of definitions and results. We refer to Section 2 for detailed conventions. For now the reader may take k to be a field of characteristic zero and CAlg_k the (enhanced homotopy) category of nonpositively graded commutative dg k -algebras.

In Section 3 we review the basic theory of geometric stacks. Following [Lur18, Sec. 9] a geometric stack will mean a functor $X : \mathrm{CAlg}_k \rightarrow \mathcal{S}$ which satisfies flat descent, has affine diagonal, and admits a flat cover $\mathrm{Spec} A \rightarrow X$ (here \mathcal{S} is the category of spaces). We caution that this corresponds to the notion of zero-geometric stack in the terminology of [TV08, Sec. 1.3.3]. Two key results established in this section are that geometric stacks are convergent (Proposition 3.13) and that they are compact in the category of convergent 1-stacks (Proposition 3.16).

Section 4 contains the definition and basic theory of ind-geometric stacks. An ind-geometric stack is a filtered colimit $X \cong \mathrm{colim} X_{\alpha}$ of truncated geometric stacks along closed immersions in the category of convergent stacks. Here truncated means the structure sheaves of the X_{α} are bounded. This definition naturally extends the derived notion of ind-scheme considered in [GR14]. An ind-geometric stack is reasonable if the maps among the X_{α} are almost finitely presented, and this in turn naturally extends the notion of reasonableness considered in [BD96, Ras19].

In Sections 5 and 6 we develop the theory of coherent and ind-coherent sheaves on ind-geometric stacks. In each case the category of sheaves on an ind-geometric stack $X \cong \mathrm{colim} X_{\alpha}$ as the colimit under pushforward of the categories of sheaves on the X_{α} , as in the case of ind-schemes. We describe the functoriality of such sheaves under pushforward, finite Tor-dimension pullback, and proper !-pullback, and establish various compatibilities between these. Note that in this generality we use the term coherent sheaf for what would be called a bounded almost perfect sheaf in [Lur18], or a bounded pseudocoherent complex in [Ill71].

The most subtle point here is the definition of ind-coherent sheaves on a general geometric stack X . We define $\mathrm{IndCoh}(X)$ as the left anticompletion of $\mathrm{QCoh}(X)$ [Lur18, Thm. C.5.8.8],

which characterizes $\mathrm{IndCoh}(X)$ by a universal property with respect to bounded, colimit-preserving functors. This is the natural extension of Krause's category of injective complexes beyond classical derived categories [Kra05]. The discrepancy between quasi-coherent and ind-coherent sheaves is minor in the setting of geometric stacks, but substantial in the setting of ind-geometric stacks (or ind-schemes). In particular, even if we are mainly interested in coherent sheaves on ind-geometric stacks, it is necessary to introduce ind-coherent sheaves in order to make sense of various adjoints, in particular $!$ -pullback and sheaf Hom .

We caution that, despite the notation, $\mathrm{IndCoh}(X)$ is not compactly generated in general. However, this does hold for the class of coherent ind-geometric stacks, studied in Section 7. A geometric stack X is coherent if it admits a flat cover $\mathrm{Spec} A \rightarrow X$ with A a coherent ring, and if the abelian category $\mathrm{QCoh}(X)^\heartsuit$ is compactly generated. The second condition is automatic if A is Noetherian, and beyond this it can be managed using the notion of tamely presented morphism studied in [CW23b]. An ind-geometric stack is coherent if every reasonable geometric substack is coherent.

In Section 8 we study sheaf Hom in the setting of ind-coherent sheaves on ind-geometric stacks. Although ind-coherent sheaves do not have a tensor product in general, they do admit external products. This is sufficient to have a good notion of sheaf Hom via the formula $\mathcal{H}om(\mathcal{F}, -) = (\mathcal{F} \boxtimes -)^R \Delta_{X*}$, where $\mathcal{F} \in \mathrm{Coh}(X)$ and $(\mathcal{F} \boxtimes -)^R : \mathrm{IndCoh}(X \times X) \rightarrow \mathrm{IndCoh}(X)$ is the right adjoint of the external product. In the geometric case this is compatible with the usual quasi-coherent sheaf Hom . Moreover, it is compatible with pushforward, hence is determined from the geometric case by the formula $\mathcal{H}om(i_*(\mathcal{F}'), -) \cong i_* \mathcal{H}om(\mathcal{F}', i^!(-))$ for any reasonable geometric substack $i : X' \rightarrow X$ and any $\mathcal{F}' \in \mathrm{Coh}(X')$. This treatment of sheaf Hom will play an important role in establishing the rigidity of the monoidal category $\mathrm{Coh}^{G_O}(\mathcal{R}_{G,N})$ in [CW23a].

Let us close by noting that another (ultimately equivalent) definition of the category $\mathrm{IndCoh}^{G_O}(\mathcal{R}_{G,N})$ is provided by the formalism of renormalizable prestacks developed in [Ras19]. The main difference between this formalism and the one developed here is whether $\mathcal{R}_{G,N}/G_O$ is viewed as a quotient of an inductive limit of schemes or as an inductive limit of quotients of schemes. The latter view turns out to be more efficient for our purposes.

On one hand, this lets results about various classes of morphisms be reduced to the affine case in fewer steps, non-affine schemes being bypassed entirely. This will be a decisive advantage in our study of morphisms with coherent pullback in [CW23b]. On the other hand, compared to quotients, inductive limits have a more dramatic effect on coherent sheaf theory, insofar as they break the close relationship between quasi-coherent and ind-coherent sheaves. The result is that certain technical complications (e.g. the appearance of multiple anticompletion/renormalization steps) are avoided if inductive limits are delayed until after all quotients have been taken.

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2. CONVENTIONS AND NOTATION

Our notation generally follows [Lur18] unless otherwise specified. We write Cat_∞ (resp. $\widehat{\text{Cat}}_\infty$) for the ∞ -category of small (resp. not necessarily small) ∞ -categories, $\mathcal{S} \subset \text{Cat}_\infty$ for the ∞ -category of spaces, Sp for the ∞ -category of spectra, and $\mathcal{P}\text{r}^{\text{L}} \subset \widehat{\text{Cat}}_\infty$ for the category of presentable ∞ -categories and functors which preserve small colimits. We use the terms category and ∞ -category interchangeably, and say ordinary category when we specifically mean a category in the traditional sense. Given morphisms $f : X \rightarrow Y$, $g : Y \rightarrow Z$ in an ∞ -category \mathcal{C} , we often refer by abuse to the composition $g \circ f$, with the understanding that this is only well-defined up to homotopy.

Our most significant departure from [Lur18] is that we use cohomological indexing and notation for t-structures. Thus if \mathcal{C} has a t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ with heart \mathcal{C}^\heartsuit we write $\tau^{\leq n} : \mathcal{C} \rightarrow \mathcal{C}^{\leq n}$, $H^n : \mathcal{C} \rightarrow \mathcal{C}^\heartsuit$, etc., for the associated functors. This is of course more consistent with the general conventions in algebraic geometry, but it does create some awkwardness in that it remains most convenient to write $\tau_{\leq n}$ for the subcategory of n -truncated objects in an ∞ -category \mathcal{D} . Thus, for example, $\tau_{\leq n}(\mathcal{C}^{\leq 0})$ and $\mathcal{C}^{[-n, 0]}$ refer to the same subcategory of \mathcal{C} . The reader can remain oriented by distinguishing subscripts from superscripts, which should respectively be read homologically and cohomologically.

If $\mathcal{C} \in \widehat{\text{Cat}}_\infty$ is monoidal we write $\text{CAlg}(\mathcal{C})$ for the category of commutative algebra objects of \mathcal{C} . When $\mathcal{C} \cong \text{Sp}^{\leq 0}$ is the category of connective spectra we omit it from the notation, so that CAlg denotes the category of connective \mathbb{E}_∞ -rings (we omit the superscript used in [Lur18] since we never consider nonconnective \mathbb{E}_∞ -rings). Its full subcategory $\tau_{\leq 0}\text{CAlg}$ is equivalently the ordinary category of ordinary commutative rings.

Given $A \in \text{CAlg}$, we write $\text{CAlg}_A := \text{CAlg}_{A/}$ for the category of commutative A -algebras, $\tau_{\leq n}\text{CAlg}_A$ for the subcategory of n -truncated algebras, and $\tau_{<\infty}\text{CAlg}_A := \bigcup_n \tau_{\leq n}\text{CAlg}_A$ for its subcategory of truncated algebras (i.e. n -truncated for some n). If A is an ordinary ring containing \mathbb{Q} then CAlg_A is equivalently the (enhanced homotopy) category of nonpositively graded commutative dg algebras. We write Mod_A for the category of A -modules (i.e. A -module objects in the category of spectra). If A is an ordinary ring this is the (enhanced) unbounded derived category of ordinary A -modules (i.e. of Mod_A^\heartsuit).

Throughout the text we fix a Noetherian base $k \in \text{CAlg}$. That is, $H^0(k)$ is an ordinary Noetherian ring and $H^n(k)$ is finitely generated over $H^0(k)$ for all $n < 0$.

We write PStk_k for the category of prestacks over $\text{Spec } k$, i.e. functors $\text{CAlg}_k \rightarrow \mathcal{S}$. Similarly we write $\widehat{\text{PStk}}_k$, $\text{PStk}_{k, \leq n}$ for the categories of functors from $\tau_{<\infty}\text{CAlg}_k$, $\tau_{\leq n}\text{CAlg}_k$ to \mathcal{S} . A stack will mean a prestack which is a sheaf for the fpqc topology on CAlg_k [Lur18, Prop. B.6.1.3], and we denote the category of stacks by $\text{Stk}_k \subset \text{PStk}_k$. Similarly we write $\widehat{\text{Stk}}_k \subset \widehat{\text{PStk}}_k$, $\text{Stk}_{k, \leq n} \subset \text{PStk}_{k, \leq n}$ for the subcategories of fpqc sheaves on $\tau_{<\infty}\text{CAlg}_k$, $\tau_{\leq n}\text{CAlg}_k$ (note that $\tau_{<\infty}\text{CAlg}_k$ does not admit arbitrary pushouts, but for the application of

[Lur18, Prop. A.3.2.1] in defining the fpqc topology closure under flat pushouts is sufficient). We write Spec for the Yoneda embedding $\mathrm{CAlg}_k \rightarrow \mathrm{Stk}_k$.

Given $k' \in \mathrm{CAlg}_k$, the natural functor $\mathrm{PStk}_{k'} \rightarrow (\mathrm{PStk}_k)_{/\mathrm{Spec} k'}$ is an equivalence by [Lur09, Cor. 5.1.6.12], and we record the following analogue for stacks.

Lemma 2.1. *The equivalence $\mathrm{PStk}_{k'} \rightarrow (\mathrm{PStk}_k)_{/\mathrm{Spec} k'}$ restricts to an equivalence $\mathrm{Stk}_{k'} \rightarrow (\mathrm{Stk}_k)_{/\mathrm{Spec} k'}$.*

We will make use several times of the following standard fact (c.f. [Gai, Sec. 1.3.4]).

Lemma 2.2. *Let $\mathcal{C} \cong \mathrm{colim} \mathcal{C}_\alpha$ be a small colimit in $\mathcal{P}\mathbf{r}^{\mathrm{L}}$, with $F_\alpha : \mathcal{C}_\alpha \rightarrow \mathcal{C}$ the canonical functors and $G_\alpha : \mathcal{C} \rightarrow \mathcal{C}_\alpha$ their right adjoints. Then any $X \in \mathcal{C}$ can be written as $X \cong \mathrm{colim} F_\alpha G_\alpha(X)$.*

3. GEOMETRIC STACKS

In this section we recall the basic theory of geometric stacks, and establish some results we will need in later sections. The most important of these are that geometric stacks are convergent (Proposition 3.13) and are compact in the category of convergent 1-stacks (Proposition 3.16).

3.1. Definitions. Recall our convention that a stack means a functor $\mathrm{CAlg}_k \rightarrow \mathcal{S}$ satisfying fpqc descent (here k is our fixed Noetherian base), and that the category of stacks is denoted by Stk_k . Our usage of the term geometric stack follows [Lur18, Ch. 9] (up to the presence of the base k), though we caution again that the terminology varies in the literature.

Definition 3.1. A stack X is geometric if its diagonal $X \rightarrow X \times X$ is affine and there exists faithfully flat morphism $\mathrm{Spec} B \rightarrow X$ in Stk_k . A morphism $X \rightarrow Y$ in Stk_k is geometric if for any morphism $\mathrm{Spec} A \rightarrow Y$, the fiber product $X \times_Y \mathrm{Spec} A$ is geometric. We write $\mathrm{GStk}_k \subset \mathrm{Stk}_k$ for the full subcategory of geometric stacks.

Note here that products are taken in Stk_k , hence are implicitly over $\mathrm{Spec} k$. Also note that affineness of $X \rightarrow X \times X$ implies that any morphism $\mathrm{Spec} B \rightarrow X$ is affine. In particular, (faithful) flatness of such a morphism is defined by asking that its base change to any affine scheme is such. More generally, a morphism $X \rightarrow Y$ in GStk_k is (faithfully) flat if its composition with any faithfully flat $\mathrm{Spec} A \rightarrow Y$ is (faithfully) flat. A faithfully flat morphism of geometric stacks will also be called a flat cover.

Definition 3.2. A geometric stack X is n -truncated if it admits a flat cover $\mathrm{Spec} A \rightarrow X$ such that A is n -truncated. We say $X \in \mathrm{GStk}_k$ is truncated if it is n -truncated for some n , and denote by $\mathrm{GStk}_k^+ \subset \mathrm{GStk}_k$ the full subcategory of truncated geometric stacks.

Alternatively, note that the restriction functor $(-)_{\leq n} : \mathrm{PStk}_k \rightarrow \mathrm{PStk}_{k, \leq n}$ takes Stk_k to $\mathrm{Stk}_{k, \leq n}$ [Lur18, Prop. A.3.3.1], and write $i_{\leq n} : \mathrm{Stk}_{k, \leq n} \rightarrow \mathrm{Stk}_k$ for the left adjoint of this

restriction and $\tau_{\leq n} : \mathrm{Stk}_k \rightarrow \mathrm{Stk}_k$ for their composition. Then if X is geometric $\tau_{\leq n}X$ is an n -truncated geometric stack called the n -truncation of X , and X is n -truncated if and only if the natural map $\tau_{\leq n}X \rightarrow X$ is an isomorphism [Lur18, Cor. 9.1.6.8, Prop. 9.1.6.9].

Our terminology follows [Lur18, Def. 9.1.6.2], but we caution that what we call n -truncatedness is called n -coconnectedness in [GR17a]. We also note that this use of the symbol $\tau_{\leq n}$ and of the term truncation are different from their usual meaning in terms of truncatedness of mapping spaces, but in practice no ambiguity will arise (and this abuse has the pleasant feature that $\tau_{\leq n}\mathrm{Spec} A \cong \mathrm{Spec} \tau_{\leq n}A$).

Proposition 3.3. *Geometric morphisms are stable under composition and base change in Stk_k . If $f : X \rightarrow Y$ is a morphism in Stk_k , then f is geometric if X and Y are, and X is geometric if f and Y are. In particular, GStk_k is closed under fiber products in Stk_k .*

Proof. When $k = S$ is the sphere spectrum this is [Lur18, Prop. 9.3.1.2, Ex. 9.3.1.10]. In general it suffices to show GStk_k is the preimage of $\mathrm{GStk} := \mathrm{GStk}_S$ under $\mathrm{Stk}_k \cong \mathrm{Stk}/_{\mathrm{Spec} k} \rightarrow \mathrm{Stk}$ (see Lemma 2.1). Clearly $X \in \mathrm{Stk}_k$ has a flat cover $\mathrm{Spec} A \rightarrow X$ over $\mathrm{Spec} k$ if and only if it does over $\mathrm{Spec} S$. Now note that if $f : Y \rightarrow Z$, $g : Z \rightarrow W$ are morphisms in Stk and g is affine, f is affine if and only if $g \circ f$ is (since any $\mathrm{Spec} B \rightarrow Y$ factors through $Y \times_Z \mathrm{Spec} B \rightarrow Y$). The morphism $X \times_{\mathrm{Spec} k} X \rightarrow X \times_{\mathrm{Spec} S} X$ is affine since it is a base change of $\mathrm{Spec} k \rightarrow \mathrm{Spec}(k \otimes_S k)$, hence $X \rightarrow X \times_{\mathrm{Spec} k} X$ is affine if and only if $X \rightarrow X \times_{\mathrm{Spec} S} X$ is. \square

Next recall that any stack X has an associated category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves, defined as the limit of the categories Mod_A over all maps $\mathrm{Spec} A \rightarrow X$. If X is geometric this is equivalent to the corresponding limit over the Čech nerve of any flat cover [Lur18, Prop. 9.1.3.1]. This assignment extends to a functor $\mathrm{Stk}_k^{\mathrm{op}} \rightarrow \mathcal{P}\mathrm{r}^{\mathrm{L}}$ taking $f : X \rightarrow Y$ to the functor $f^* : \mathrm{QCoh}(Y) \rightarrow \mathrm{QCoh}(X)$. In the truncated geometric case we define coherent sheaves as follows.

Definition 3.4. If $X \in \mathrm{GStk}_k^+$, then $\mathcal{F} \in \mathrm{QCoh}(X)$ is coherent if $f^*(\mathcal{F})$ is a coherent A -module for some (equivalently, any) flat cover $\mathrm{Spec} A \rightarrow X$. We write $\mathrm{Coh}(X) \subset \mathrm{QCoh}(X)$ for the full subcategory of coherent sheaves.

While the above definition makes sense when X is not truncated, without additional hypotheses the resulting category $\mathrm{Coh}(X)$ may be degenerate (for example, it may contain no nonzero objects). It will be convenient to exclude such degenerate cases from our discussion, though our treatment of coherent sheaves on ind-geometric stacks will include well-behaved non-truncated geometric stacks within its scope. We also caution that the above notion of coherence differs from that of [Lur18, Def. 6.4.3.1].

3.2. Properties of morphisms. We now recall some key properties of morphisms in the setting of geometric stacks. Given a morphism $f : X \rightarrow Y$ in GStk_k , the pushforward

$f_* : \mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(Y)$ is defined as the right adjoint of f^* . For f_* to be well-behaved one needs additional hypotheses on f . A morphism $f : X \rightarrow Y$ in GStk_k is of cohomological dimension $\leq n$ if $f_*(\mathrm{QCoh}(X)^{\leq 0}) \subset \mathrm{QCoh}(Y)^{\leq n}$, and is of finite cohomological dimension if it is of cohomological dimension $\leq n$ for some n .

Proposition 3.5. *Morphisms of finite cohomological dimension are stable under composition and base change in GStk_k . A morphism $f : X \rightarrow Y$ of geometric stacks is of finite cohomological dimension if and only if its base change along any given flat cover $\mathrm{Spec} A \rightarrow Y$ is. In this case $f_* : \mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(Y)$ is continuous, and for any Cartesian square*

$$(3.6) \quad \begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ h' \downarrow & & \downarrow h \\ X & \xrightarrow{f} & Y \end{array}$$

in GStk_k the Beck-Chevalley transformation $h^* f_*(\mathcal{F}) \rightarrow f'_* h'^*(\mathcal{F})$ is an isomorphism for all $\mathcal{F} \in \mathrm{QCoh}(X)$.

Proof. Stability under composition is immediate, while stability under base change and flat locality on the target follow from [HLP14, Prop. A.1.9] (whose proof applies to geometric stacks, not just algebraic stacks with affine diagonal). The remaining properties then follow from [Lur18, Prop. 9.1.5.3, Prop. 9.1.5.7] or [HLP14, Prop. A.1.5]. \square

A key case is that of proper morphisms. We say $f : X \rightarrow Y$ is proper if it is representable and for any $\mathrm{Spec} A \rightarrow Y$, the fiber product $X \times_Y \mathrm{Spec} A$ is proper over $\mathrm{Spec} A$ in the sense of [Lur18, Def. 5.1.2.1]. In particular, this requires that $X \times_Y \mathrm{Spec} A$ be a quasi-compact separated algebraic space.

Proposition 3.7. *Proper morphisms are stable under composition and base change in Stk_k . If f and g are morphisms in Stk_k such that $g \circ f$ and g are proper, then so is f . Proper morphisms of geometric stacks are of finite cohomological dimension. If f is a proper, almost finitely presented morphism of truncated geometric stacks, then f_* takes $\mathrm{Coh}(X)$ to $\mathrm{Coh}(Y)$.*

Proof. Stability under base change is immediate, and the claims about composition follow from [Lur18, Prop. 5.1.4.1, Prop. 6.3.2.2]. By Proposition 3.5 finiteness of cohomological dimension can be checked after base change along a flat cover $h : \mathrm{Spec} A \rightarrow Y$, where it follows from [Lur18, Prop. 2.5.4.4, Prop. 3.2.3.1]. If f is almost of finite presentation and f' is its base change along h , then f'_* preserves coherence by [Lur18, Thm. 5.6.0.2]. Then so does $h^* f_*$ by Proposition 3.5, hence so does f_* by definition. \square

We mostly consider properness together with the following finiteness condition. Recall following [Lur18, Def. 17.4.1.1] that a morphism $f : X \rightarrow Y$ of stacks is (locally) almost

of finite presentation if, for any n and any filtered colimit $A \cong \operatorname{colim} A_\alpha$ in $\tau_{\leq n} \mathbf{CAlg}_k$, the canonical map

$$(3.8) \quad \operatorname{colim} X(A_\alpha) \rightarrow X(A) \times_{Y(A)} \operatorname{colim} Y(A_\alpha)$$

is an isomorphism (we omit the word locally by default, as all morphisms we consider will be quasi-compact or effectively so).

Proposition 3.9 ([Lur18, Rem. 17.4.1.3, Rem. 17.4.1.5]). *Almost finitely presented morphisms are stable under composition and base change in \mathbf{Stk}_k . If f and g are composable morphisms in \mathbf{Stk}_k such that $g \circ f$ and g are almost of finite presentation, then so is f .*

A morphism $f : X \rightarrow Y$ in \mathbf{GStk}_k is of Tor-dimension $\leq n$ if $f^*(\mathbf{QCoh}(Y)^{\geq 0}) \subset \mathbf{QCoh}(Y)^{\geq n}$, and is of finite Tor-dimension if it is of Tor-dimension $\leq n$ for some n . We have the following variant of standard results.

Proposition 3.10. *Morphisms of Tor-dimension $\leq n$ are stable under base change in \mathbf{GStk}_k , and morphisms of finite Tor-dimension are also stable under composition. A morphism $f : X \rightarrow Y$ of geometric stacks is of Tor-dimension $\leq n$ if and only if its base change along any given flat cover $h : \operatorname{Spec} A \rightarrow Y$ is.*

Proof. Stability under composition is immediate. Flat locality on the target follows since if h' and f' are defined by base change, then h'^* is conservative, t-exact, and satisfies $h'^* f'^* \cong f'^* h'^*$. If $h : \operatorname{Spec} A \rightarrow Y$ is arbitrary, then h' is affine, hence h'_* is conservative, t-exact, and satisfies $f'^* h'_* \cong h'_* f'^*$ [Lur18, Prop. 9.1.5.7]. Stability under base change along affine morphisms follows, and arbitrary base change now follows by composing an arbitrary $Y' \rightarrow Y$ with a flat cover $\operatorname{Spec} B \rightarrow Y'$. \square

By extension, say a geometric morphism in \mathbf{Stk}_k is of Tor-dimension $\leq n$ if its base change to any geometric stack is so, and is of finite Tor-dimension if it is of Tor-dimension $\leq n$ for some n .

Next we discuss external products. Given $X, Y \in \mathbf{GStk}_k$, the external product of $\mathcal{F} \in \mathbf{QCoh}(X)$ and $\mathcal{G} \in \mathbf{QCoh}(Y)$ is defined by

$$\mathcal{F} \boxtimes \mathcal{G} := \pi_X^*(\mathcal{F}) \otimes \pi_Y^*(\mathcal{G}) \in \mathbf{QCoh}(X \times Y).$$

In this paper we only consider external products under the hypothesis that k is an ordinary (Noetherian) ring of finite global dimension, in which case the following result holds.

Proposition 3.11. *Suppose k is an ordinary ring of finite global dimension. If $X, Y \in \mathbf{GStk}_k$, $\mathcal{F} \in \mathbf{QCoh}(X)^+$, and $\mathcal{G} \in \mathbf{QCoh}(Y)^+$, then $\mathcal{F} \boxtimes \mathcal{G} \in \mathbf{QCoh}(X \times Y)^+$. In particular, $X \times Y$ is truncated if X and Y are, and $\mathcal{F} \boxtimes \mathcal{G}$ is coherent if \mathcal{F} and \mathcal{G} are.*

Proof. It suffices to check the claim for the pullbacks of \mathcal{F}, \mathcal{G} to flat covers of X and Y , hence we may assume X and Y are affine. If $\mathcal{F} \in \mathbf{QCoh}(X)^{\geq m}$ and $\mathcal{G} \in \mathbf{QCoh}(Y)^{\geq n}$, then since the

underlying k -module of $\mathcal{F} \boxtimes \mathcal{G}$ is $\mathcal{F} \otimes_k \mathcal{G}$ we have $\mathcal{F} \boxtimes \mathcal{G} \in \mathrm{QCoh}(X \times Y)^{\geq \ell}$, where ℓ is $m + n$ minus the global dimension of k . The last claims follow by taking $\mathcal{F} \cong \mathcal{O}_X$ and $\mathcal{G} \cong \mathcal{O}_Y$, and by recalling that almost perfect sheaves are closed under pullbacks and tensor products. \square

3.3. Convergence. Recall that $\widehat{\mathrm{PStk}}_k$ denotes the category of functors $\tau_{<\infty} \mathrm{CAlg}_k \rightarrow \mathcal{S}$, and $\widehat{\mathrm{Stk}}_k \subset \widehat{\mathrm{PStk}}_k$ the subcategory of functors satisfying fpqc descent. The restriction functor $(-)<\infty : \mathrm{PStk}_k \rightarrow \widehat{\mathrm{PStk}}_k$ has a fully faithful right adjoint, by which we generally regard $\widehat{\mathrm{PStk}}_k$ as a subcategory of PStk_k . One says a prestack is convergent (or nilcomplete) if it is contained in this subcategory. Explicitly, $X \in \mathrm{PStk}_k$ is convergent if for all $A \in \mathrm{CAlg}_k$ the natural morphism $X(A) \rightarrow \lim X(\tau_{\leq n} A)$ is an isomorphism. We have an induced notion of convergent stack, which is unambiguous in the following sense.

Lemma 3.12. *The inclusion $\widehat{\mathrm{PStk}}_k \subset \mathrm{PStk}_k$ identifies $\widehat{\mathrm{Stk}}_k$ with $\mathrm{Stk}_k \cap \widehat{\mathrm{PStk}}_k$.*

Proof. That $\mathrm{Stk}_k \cap \widehat{\mathrm{PStk}}_k \subset \widehat{\mathrm{Stk}}_k$ follows from the definition of the fpqc topology and from [Lur18, Prop. A.3.3.1] (closure of $\tau_{<\infty} \mathrm{CAlg}_k$ under pushouts along flat morphisms is sufficient to apply this to $\widehat{\mathrm{PStk}}_k$). Now suppose $X \in \widehat{\mathrm{Stk}}_k \subset \widehat{\mathrm{PStk}}_k \subset \mathrm{PStk}_k$. Note that $\tau_{\leq n} : \mathrm{Sp}^{\leq 0} \rightarrow \tau_{\leq n} \mathrm{Sp}^{\leq 0}$ preserves finite products since it preserves colimits and $\mathrm{Sp}^{\leq 0}$ is additive, hence $\tau_{\leq n} : \mathrm{CAlg}_k \rightarrow \tau_{\leq n} \mathrm{CAlg}_k$ preserves finite products by [Lur17, Cor. 3.2.2.5]. Then if $A \cong \prod_{i=1}^n A_i$ is a finite product in CAlg_k , we have $X(A) \cong \lim_n X(\tau_{\leq n} A) \cong \lim_{n,i} X(\tau_{\leq n} A_i) \cong \lim_i X(A_i)$. Similarly, if $A \rightarrow A^0$ in CAlg_k is faithfully flat and A^\bullet its Čech nerve, then $X(A) \cong \lim_n X(\tau_{\leq n} A) \cong \lim_{n,i} X(\tau_{\leq n} A^i) \cong \lim_i X(A^i)$. Thus $X \in \mathrm{Stk}_k$ by [Lur18, Prop. A.3.3.1]. \square

We now have the following result in the geometric case.

Proposition 3.13. *Geometric stacks are convergent.*

Proof. Suppose $X \in \mathrm{GStk}_k$ and $A \in \mathrm{CAlg}_k$. It suffices to treat the case where $k = S$ is the sphere spectrum. In general, $\mathrm{Map}_{\mathrm{Stk}_k}(\mathrm{Spec} A, X)$ is the fiber of the map $\mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec} A, X) \rightarrow \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec} A, \mathrm{Spec} k)$ induced by composition with $X \rightarrow \mathrm{Spec} k$ over the point corresponding to $\mathrm{Spec} A \rightarrow \mathrm{Spec} k$ [Lur09, Lem. 5.5.5.12] (recall $\mathrm{Stk}_k \cong \mathrm{Stk}/_{\mathrm{Spec} k}$ per Lemma 2.1). Since the same holds for each $\tau_{\leq n} A$, it follows that X is convergent over $\mathrm{Spec} k$ if it is convergent over $\mathrm{Spec} S$ (since $\mathrm{Spec} k$ is).

Consider the natural diagram

$$(3.14) \quad \begin{array}{ccc} \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec} A, X) & \longrightarrow & \mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)}(\mathrm{QCoh}(X)^{\leq 0}, \mathrm{Mod}_A^{\leq 0}) \\ \downarrow & & \downarrow \\ \lim \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec} \tau_{\leq n} A, X) & \longrightarrow & \lim \mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)}(\mathrm{QCoh}(X)^{\leq 0}, \mathrm{Mod}_{\tau_{\leq n} A}^{\leq 0}). \end{array}$$

By [Lur, Lem. 5.4.6] the top map is a monomorphism if the corresponding map from $\mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec} A, X)$ to $\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)}(\mathrm{QCoh}(X), \mathrm{Mod}_A)$ is a monomorphism, which in turn follows from Tannaka duality [Lur18, Prop. 9.3.0.3]. Since this also holds replacing A with

$\tau_{\leq n}A$, and since monomorphisms are stable under limits, it follows similarly that the bottom map is a monomorphism. The right map is an isomorphism, since we have

$$\mathrm{Mod}_A^{\leq 0} \cong \lim_m \mathrm{Mod}_A^{[m,0]} \cong \lim_{m,n} \mathrm{Mod}_{\tau_{\leq n}A}^{[m,0]} \cong \lim_n \mathrm{Mod}_{\tau_{\leq n}A}^{\leq 0},$$

in $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)$. Here the first and third equivalences follow from the relevant t-structures being left complete and $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty) \rightarrow \widehat{\mathrm{Cat}}_\infty$ preserving limits [Lur17, Cor. 3.2.2.5], the second from $\mathrm{Mod}_A^{[m,0]} \rightarrow \mathrm{Mod}_{\tau_{\leq n}A}^{[m,0]}$ being an equivalence for $m \leq -n$. It follows that the left map in (3.14) is a monomorphism, so we must show it is essentially surjective.

Since the horizontal maps are monomorphisms, it suffices to show the right isomorphism restricts to an essential surjection between their essential images. By Tannaka duality [Lur18, Prop. 9.3.0.3] the essential image of the bottom map consists of systems $\{G_n : \mathrm{QCoh}(X)^{\leq 0} \rightarrow \mathrm{Mod}_{\tau_{\leq n}A}^{\leq 0}\}$ of symmetric monoidal functors which preserve small colimits and flat objects, similarly for the top map. Let $G : \mathrm{QCoh}(X)^{\leq 0} \rightarrow \mathrm{Mod}_A^{\leq 0}$ be the functor associated to some such system $\{G_n\}$ under the right isomorphism. Then G preserves small colimits since $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ is closed under limits in $\widehat{\mathrm{Cat}}_\infty$ [Lur09, Prop. 5.5.3.13]. Moreover, if $G_n(\mathcal{F}) \cong \tau_{\leq n}A \otimes_A G(\mathcal{F})$ is flat over $\tau_{\leq n}A$ for all n , then since $\tau_{\leq n}G(\mathcal{F}) \cong \tau_{\leq n}(\tau_{\leq n}A \otimes_A G(\mathcal{F}))$ by [Lur17, Prop. 7.1.3.15] it follows from the definition of flatness that $G(\mathcal{F})$ is flat over A . Thus G is in the image of the top map in (3.14), establishing the claim. \square

Since $\tau_{\leq n}\mathrm{CAlg}_k$ is closed under products and targets of flat morphisms in $\tau_{<\infty}\mathrm{CAlg}_k$, the restriction functor $(-)_{\leq n} : \widehat{\mathrm{PStk}}_k \rightarrow \mathrm{PStk}_{k,\leq n}$ takes $\widehat{\mathrm{Stk}}_k$ to $\mathrm{Stk}_{k,\leq n}$ [Lur18, Prop. A.3.3.1]. We write $\widehat{i}_{\leq n} : \mathrm{Stk}_{k,\leq n} \rightarrow \widehat{\mathrm{Stk}}_k$ for the resulting left adjoint and $\widehat{\tau}_{\leq n} : \widehat{\mathrm{Stk}}_k \rightarrow \widehat{\mathrm{Stk}}_k$ for their composition.

The functors $(-)_{\leq n}$ induce an equivalence $\widehat{\mathrm{Stk}}_k \cong \lim \mathrm{Stk}_{k,\leq n}$ in $\widehat{\mathrm{Cat}}_\infty$ [Lur18, Ex. A.7.1.6], hence for any $X \in \widehat{\mathrm{Stk}}_k$ the natural map $\mathrm{colim} \widehat{\tau}_{\leq n}X \rightarrow X$ is an isomorphism by Lemma 2.2. More explicitly, if $\widehat{\tau}_{\leq n}^{\mathrm{pre}} : \widehat{\mathrm{PStk}}_k \rightarrow \widehat{\mathrm{PStk}}_k$ denotes the composition of $(-)_{\leq n} : \widehat{\mathrm{PStk}}_k \rightarrow \mathrm{PStk}_{k,\leq n}$ and its left adjoint, then $\mathrm{colim} \widehat{\tau}_{\leq n}^{\mathrm{pre}}X \rightarrow X$ is an isomorphism since $\tau_{<\infty}\mathrm{CAlg}_k = \bigcup_n \tau_{\leq n}\mathrm{CAlg}_k$ and since colimits in $\widehat{\mathrm{PStk}}_k$ are computed objectwise. But $\widehat{\tau}_{\leq n}$ is the sheafification of $\widehat{\tau}_{\leq n}^{\mathrm{pre}}$, so $\mathrm{colim} \widehat{\tau}_{\leq n}X \rightarrow X$ is an isomorphism since sheafification is continuous. In particular, if X is a geometric stack Proposition 3.13 implies $\tau_{\leq n}X \cong \widehat{\tau}_{\leq n}X$, hence we obtain the following corollary.

Proposition 3.15. *For any geometric stack X we have $X \cong \mathrm{colim} \tau_{\leq n}X$ in $\widehat{\mathrm{Stk}}_k$.*

Now let $1\text{-}\widehat{\mathrm{Stk}}_k \subset \widehat{\mathrm{Stk}}_k$ denote the full subcategory consisting of X such that $X(A)$ is an $(n+1)$ -truncated space for all $A \in \tau_{\leq n}\mathrm{CAlg}_k$. Then Proposition 3.13 is refined by the following result, the first half of which is a variant of [GR17a, Cor. I.2.4.3.4], the second of [GR14, Lem. 1.3.6].

Proposition 3.16. *Geometric stacks are objects of $1\text{-}\widehat{\mathrm{Stk}}_k$. Moreover, $1\text{-}\widehat{\mathrm{Stk}}_k$ is closed under filtered colimits in $\widehat{\mathrm{PStk}}_k$, and truncated geometric stacks are compact as objects of $1\text{-}\widehat{\mathrm{Stk}}_k$.*

Proof. Let X be a geometric stack. To show $X \in 1\text{-}\widehat{\text{Stk}}_k$, it suffices to show that $X_{\leq n}$ belongs to $\tau_{\leq n+1}\text{Stk}_{k,\leq n}$, the category of $(n+1)$ -truncated objects of $\text{Stk}_{k,\leq n}$, since for $A \in \tau_{\leq n}\text{CAlg}_k$ we have $X(A) \cong \text{Map}_{\text{Stk}_{k,\leq n}}((\text{Spec } A)_{\leq n}, X_{\leq n})$.

Let $\text{Spec } B \rightarrow X$ be a flat cover, so that $(\text{Spec } B)_{\leq n} \rightarrow X_{\leq n}$ is a flat cover in $\text{Stk}_{k,\leq n}$. If $(\text{Spec } B)_{\leq n}^\bullet$ is its Čech nerve, we have $X_{\leq n} \cong \text{colim}(\text{Spec } B)_{\leq n}^m$ in $\text{Stk}_{k,\leq n}$. Equivalently, $X_{\leq n}$ is the sheafification of the same colimit taken in $\text{PStk}_{k,\leq n}$. Since sheafification is left exact it preserves $(n+1)$ -truncated objects [Lur09, Prop. 5.5.6.16], hence it suffices to show the colimit taken in $\text{PStk}_{k,\leq n}$ is $(n+1)$ -truncated. For any m an object of $\text{PStk}_{k,\leq n}$ is m -truncated if its values on $\tau_{\leq n}\text{CAlg}_k$ are m -truncated spaces [Lur09, Rem. 5.5.8.26]. Each $(\text{Spec } B)_{\leq n}^m$ is affine since X is geometric, hence is then n -truncated in $\text{PStk}_{k,\leq n}$ since $\tau_{\leq n}\text{CAlg}_k$ is an $(n+1)$ -category. The claim then follows since values of colimits in $\text{PStk}_{k,\leq n}$ are computed objectwise, and since the geometric realization of a groupoid of n -truncated spaces is $(n+1)$ -truncated.

We now claim $X_{\leq n}$ is compact in $\tau_{\leq n+1}\text{Stk}_{k,\leq n}$. By the argument of [GR14, Lem. 1.3.3] we have that $\tau_{\leq n+1}\text{Stk}_{k,\leq n}$ is closed under filtered colimits in $\text{PStk}_{k,\leq n}$ (noting that by [Lur09, Prop. 5.5.6.16] we have $\tau_{\leq n+1}\text{Stk}_{k,\leq n} = \text{Stk}_{k,\leq n} \cap \tau_{\leq n+1}\text{PStk}_{k,\leq n}$ since $\text{Stk}_{k,\leq n}$ is closed under limits in $\text{PStk}_{k,\leq n}$). It follows that each $(\text{Spec } B)_{\leq n}^m$ is compact in $\tau_{\leq n+1}\text{Stk}_{k,\leq n}$ since it is so in $\text{PStk}_{k,\leq n}$. Moreover, $X_{\leq n}$ is their colimit in $\tau_{\leq n+1}\text{Stk}_{k,\leq n}$ since it is their colimit in $\text{Stk}_{k,\leq n}$. But $\tau_{\leq n+1}\text{Stk}_{k,\leq n}$ is an $(n+2)$ -category, so by the proof of [Lur17, Lem. 1.3.3.10] $X_{\leq n}$ is also the colimit in $\tau_{\leq n+1}\text{Stk}_{k,\leq n}$ of the $(\text{Spec } B)_{\leq n}^m$ over the finite subdiagram $\Delta_{s,\leq n+2}^{op} \subset \Delta_s^{op}$. It follows that $X_{\leq n}$ is compact in $\tau_{\leq n+1}\text{Stk}_{k,\leq n}$ [Lur09, Cor. 5.3.4.15].

Next note that [Lur18, Prop. A.3.3.1] also implies $\widehat{\text{Stk}}_k$ is the full subcategory of $X \in \widehat{\text{PStk}}_k$ such that $X_{\leq n} \in \text{Stk}_{k,\leq n}$ for all n . But then $\tau_{\leq n+1}\text{Stk}_{k,\leq n} = \text{Stk}_{k,\leq n} \cap \tau_{\leq n+1}\text{PStk}_{k,\leq n}$ implies $1\text{-}\widehat{\text{Stk}}_k$ is the full subcategory of $X \in \widehat{\text{PStk}}_k$ such that $X_{\leq n} \in \tau_{\leq n+1}\text{Stk}_{k,\leq n}$ for all n . Closure of $1\text{-}\widehat{\text{Stk}}_k$ under filtered colimits then follows since $(-)_\leq n : \widehat{\text{PStk}}_k \rightarrow \text{PStk}_{k,\leq n}$ is continuous and since as recalled above $\tau_{\leq n+1}\text{Stk}_{k,\leq n}$ is closed under filtered colimits in $\text{PStk}_{k,\leq n}$.

Now suppose X is an n -truncated geometric stack, let $Y \cong \text{colim } Y_\alpha$ be a filtered colimit in $1\text{-}\widehat{\text{Stk}}_k$, and consider the following diagram.

$$\begin{array}{ccc}
\text{colim Map}_{1\text{-}\widehat{\text{Stk}}_k}(X, Y_\alpha) & \longrightarrow & \text{colim Map}_{\tau_{\leq n+1}\text{Stk}_{k,\leq n}}(X_{\leq n}, (Y_\alpha)_{\leq n}) \\
\downarrow & & \downarrow \\
& & \text{Map}_{\tau_{\leq n+1}\text{Stk}_{k,\leq n}}(X_{\leq n}, \text{colim}(Y_\alpha)_{\leq n}) \\
\downarrow & & \downarrow \\
\text{Map}_{1\text{-}\widehat{\text{Stk}}_k}(X, Y) & \longrightarrow & \text{Map}_{\tau_{\leq n+1}\text{Stk}_{k,\leq n}}(X_{\leq n}, Y_{\leq n})
\end{array}$$

The bottom right map is an isomorphism since by the preceding paragraph $(-)_\leq n$ restricts to a continuous functor $1\text{-}\widehat{\text{Stk}}_k \rightarrow \tau_{\leq n+1}\text{Stk}_{k,\leq n}$. Since X is n -truncated it is the image of $X_{\leq n}$ under the left adjoint $\widehat{i}_{\leq n} : \text{Stk}_{k,\leq n} \rightarrow \widehat{\text{Stk}}_k$ of $(-)_\leq n$, hence the horizontal maps are

isomorphisms. The top right map is an isomorphism since $X_{\leq n} \in \tau_{\leq n+1}\widehat{\text{Stk}}_{k,\leq n}$ is compact, hence the left map is an isomorphism and X is compact in $1\text{-}\widehat{\text{Stk}}_k$. \square

3.4. Closed immersions. We conclude this section with some minor results on closed immersions we will use in treating ind-geometric stacks. Following [GR14, Sec. 1.6] we say a map $f : X \rightarrow Y$ in Stk_k is a closed immersion if for any $\text{Spec } A \rightarrow Y$, the map $\tau_{\leq 0}(X \times_Y \text{Spec } A) \rightarrow \tau_{\leq 0}\text{Spec } A$ is a closed immersion of ordinary affine schemes. The following statement follows immediately from its classical counterpart.

Proposition 3.17. *Closed immersions are stable under composition and base change in Stk_k . If f and g are composable morphisms in Stk_k such that $g \circ f$ and g are closed immersions, then so is f .*

A closed immersion need not be affine (for example, the inclusion of a closed subscheme of a classical ind-scheme is typically not affine in the derived sense), but in GStk_k we have the following.

Proposition 3.18. *Closed immersions between geometric stacks are affine.*

Proof. Let $f : X \rightarrow Y$ be a closed immersion between geometric stacks, and let $\text{Spec } A \rightarrow Y$ be arbitrary. The base change $f' : X' \rightarrow \text{Spec } A$ of f factors through a map $f'' : X' \rightarrow \text{Spec } B$, where $B := f'_*(\mathcal{O}_{X'}) \in \text{CAlg}_A$. We claim f'' is an isomorphism. By Tannaka duality [Lur18, Thm. 9.2.0.2] it suffices to show $f'_* : \text{QCoh}(X') \rightarrow \text{Mod}_B$ is an equivalence. By Barr-Beck-Lurie [Lur17, Thm. 4.7.4.5] it further suffices to show $f'_* : \text{QCoh}(X') \rightarrow \text{Mod}_A$ is conservative and preserves small colimits.

Since f is a closed immersion $\tau_{\leq 0}f'' : \tau_{\leq 0}X' \rightarrow \tau_{\leq 0}\text{Spec } B$ is an isomorphism, hence f'' restricts to an equivalence $\text{QCoh}(X')^\heartsuit \cong \text{Mod}_B^\heartsuit$. It follows that the restriction of f'_* to $\text{QCoh}(X')^\heartsuit$ is conservative, continuous, and factors through Mod_A^\heartsuit . It follows as in [HLP14, Lem. A.1.6] that f'_* is t-exact.

Suppose now that $f'_*(\mathcal{F}) \cong 0$ for some $\mathcal{F} \in \text{QCoh}(X')$. It follows from the preceding paragraph that $\mathcal{H}^n(\mathcal{F}) \cong 0$ for all $n \in \mathbb{Z}$. Since $\text{QCoh}(X')$ is t-complete we then have $\mathcal{F} \cong 0$, hence f'_* is conservative. Similarly, suppose $\mathcal{F} \cong \text{colim } \mathcal{F}_\alpha$ is a filtered colimit in $\text{QCoh}(X')$, and let $\phi : \text{colim } f'_*(\mathcal{F}_\alpha) \rightarrow f'_*(\mathcal{F})$ denote the natural map. Since the t-structures on $\text{QCoh}(X')$ and Mod_A are compatible with filtered colimits, it follows from the preceding paragraph that $\mathcal{H}^n(\phi)$ is an isomorphism for all $n \in \mathbb{Z}$. That ϕ is an isomorphism, hence that f'_* is continuous, now follows from the t-completeness of Mod_A . But then f'_* preserves all small colimits since it is exact [Lur09, Prop. 5.5.1.9]. \square

Proposition 3.19. *Let $f : X \rightarrow Y_\alpha$, $h : Y' \rightarrow Y_\alpha$, and $i : Y_\alpha \rightarrow Y_\beta$ be morphisms of geometric stacks. If i is a closed immersion, the map $\tau_{\leq 0}(X \times_{Y_\alpha} Y') \rightarrow \tau_{\leq 0}(X \times_{Y_\beta} Y')$ is an isomorphism.*

Proof. Let $X'_\alpha := X \times_{Y_\alpha} Y'$, $X'_\beta := X \times_{Y_\beta} Y'$, and suppose first that X and Y' are affine. Fixing a flat cover $\mathrm{Spec} B_\beta \rightarrow Y_\beta$ we obtain flat covers $\mathrm{Spec} A'_\alpha \rightarrow X'_\alpha$, $\mathrm{Spec} A'_\beta \rightarrow X'_\beta$ by base change. It suffices to show the induced morphism $\tau_{\leq 0} \mathrm{Spec} A'_\alpha \rightarrow \tau_{\leq 0} \mathrm{Spec} A'_\beta$ is an isomorphism [Lur09, Lem. 6.2.3.16]. If $\mathrm{Spec} A \rightarrow X$, $\mathrm{Spec} B' \rightarrow Y'$, and $\mathrm{Spec} B_\alpha \rightarrow Y_\alpha$ are also obtained by base change from $\mathrm{Spec} B_\beta \rightarrow Y_\beta$, then $\tau_{\leq 0} \mathrm{Spec} B_\alpha \rightarrow \tau_{\leq 0} \mathrm{Spec} B_\beta$ is a closed immersion since i is. But then $\tau_{\leq 0} \mathrm{Spec} A'_\alpha \rightarrow \tau_{\leq 0} \mathrm{Spec} A'_\beta$ is an isomorphism since $A'_\alpha \cong A \otimes_{B_\alpha} B'$ and $A'_\beta \cong A \otimes_{B_\beta} B'$.

In the general case, fix flat covers $\mathrm{Spec} C \rightarrow X$ and $\mathrm{Spec} D' \rightarrow Y'$, and let $\mathrm{Spec} C'_\alpha := \mathrm{Spec} C \times_{Y_\alpha} \mathrm{Spec} D'$ and $\mathrm{Spec} C'_\beta := \mathrm{Spec} C \times_{Y_\beta} \mathrm{Spec} D'$. By the preceding paragraph $\tau_{\leq 0} \mathrm{Spec} C'_\alpha \rightarrow \tau_{\leq 0} \mathrm{Spec} C'_\beta$ is an isomorphism. But since it is the base change of $\tau_{\leq 0} X'_\alpha \rightarrow \tau_{\leq 0} X'_\beta$ along the flat cover $\tau_{\leq 0} \mathrm{Spec} C'_\beta \rightarrow \tau_{\leq 0} X'_\beta$, the claim follows. \square

4. IND-GEOMETRIC STACKS

We now define our main objects of study and establish their basic properties. Key technical results include the consistency of the notions of truncated and reasonable geometric substack (Proposition 4.5), and the closure of ind-geometric stacks under fiber products (Proposition 4.18) and ind-closed filtered colimits (Proposition 4.13).

4.1. Definitions. Recall from Proposition 3.13 that GStk_k is contained in the full subcategory $\widehat{\mathrm{Stk}}_k \subset \mathrm{Stk}_k$ of convergent stacks. This subcategory plays a central role in our discussion because colimits in $\widehat{\mathrm{Stk}}_k$ are typically more natural than colimits in Stk_k . For example, any $\mathrm{Spec} A$ is the colimit of its truncations $\mathrm{Spec} \tau_{\leq n} A$ in $\widehat{\mathrm{Stk}}_k$, but not in Stk_k unless A is itself truncated. In particular, the inclusion $\widehat{\mathrm{Stk}}_k \subset \mathrm{Stk}_k$ does not preserve colimits in general.

Definition 4.1. An *ind-geometric stack* is a convergent stack X which admits an expression $X \cong \mathrm{colim}_\alpha X_\alpha$ as a filtered colimit in $\widehat{\mathrm{Stk}}_k$ of truncated geometric stacks along closed immersions. We call such an expression an *ind-geometric presentation* of X . We call it a *reasonable presentation* if the structure maps are almost finitely presented, and say X is *reasonable* if it admits a reasonable presentation.

We write $\mathrm{indGStk}_k \subset \widehat{\mathrm{Stk}}_k$ (resp. $\mathrm{indGStk}_k^{\mathrm{reas}} \subset \widehat{\mathrm{Stk}}_k$) for the full subcategory of ind-geometric (resp. reasonable ind-geometric) stacks. This is the natural extension of the derived notion of ind-scheme introduced in [GR14] (see Proposition 4.21) and the derived notion of reasonableness introduced in [Ras19].

Any geometric stack X is ind-geometric since $X \cong \mathrm{colim} \tau_{\leq n} X$ in $\widehat{\mathrm{Stk}}_k$ by Proposition 3.15. On the other hand, not every geometric stack is reasonable, a basic example being the self-intersection of the origin in \mathbb{A}^∞ . However, we have the following result, where we say a geometric stack is locally coherent if it admits a flat cover $\mathrm{Spec} A \rightarrow X$ such that A is coherent.

Proposition 4.2. *If a geometric stack X is locally coherent, then it is reasonable.*

Proof. Let $\mathrm{Spec} A \rightarrow X$ be a flat cover such that A is coherent. Then $\mathrm{Spec} \tau_{\leq n} A \rightarrow \mathrm{Spec} \tau_{\leq n+1} A$ is almost of finite presentation for all n [Lur18, Cor. 5.2.2.2]. It follows that $\tau_{\leq n} X \rightarrow \tau_{n+1} X$ is as well [Lur18, Prop. 4.1.4.3], noting that $\mathrm{Spec} \tau_{\leq n} A$ is the base change of $\mathrm{Spec} A$ to $\tau_{\leq n} X$. Thus X is reasonable by Proposition 3.15. \square

Truncatedness plays an essential role in our discussion due to the following variant of [GR14, Lem. 1.3.6] (though see Remark 4.14). The claim follows immediately from Proposition 3.16, but would fail if Y were not truncated: in this case Y is not compact in $1\text{-}\widehat{\mathrm{Stk}}_k$, since e.g. id_Y does not factor through any truncation of Y .

Proposition 4.3. *Let $X \cong \mathrm{colim} X_\alpha$ be an ind-geometric presentation. Then for any truncated geometric stack Y , the natural map*

$$\mathrm{colim} \mathrm{Map}_{\mathrm{Stk}_k}(Y, X_\alpha) \rightarrow \mathrm{Map}_{\mathrm{Stk}_k}(Y, X)$$

is an isomorphism.

To discuss ind-geometric stacks more intrinsically, without referring to particular ind-geometric presentations, the following notion is useful.

Definition 4.4. Let X be an ind-geometric stack. A *truncated* (resp. *reasonable*) *geometric substack* of X is a truncated geometric stack X' equipped with a closed immersion $X' \rightarrow X$ (resp. an almost finitely presented closed immersion $X' \rightarrow X$).

Proposition 4.5. *Let $X \cong \mathrm{colim} X_\alpha$ be an ind-geometric (resp. reasonable) presentation. Then for all α , the structure morphism $i_\alpha : X_\alpha \rightarrow X$ realizes X_α as a truncated (resp. reasonable) geometric substack of X . Any other truncated (resp. reasonable) geometric substack $X' \rightarrow X$ can be factored as $X' \xrightarrow{j_\alpha} X_\alpha \xrightarrow{i_\alpha} X$ for some α , and in any such factorization j_α is a closed immersion (resp. almost finitely presented closed immersion), hence affine.*

Proof. To show i_α is a closed immersion, fix $\mathrm{Spec} A \rightarrow X$ and let $Z := X_\alpha \times_X \mathrm{Spec} A$. Since $\tau_{\leq 0} Z \cong \tau_{\leq 0}(X_\alpha \times_X \tau_{\leq 0} \mathrm{Spec} A)$ we may assume A is classical. By Proposition 4.3 we can then factor $\mathrm{Spec} A \rightarrow X$ through some $X_{\alpha'}$, which we may assume satisfies $\alpha' \geq \alpha$. For $\alpha'' \geq \alpha'$ let $Z_{\alpha''} := X_{\alpha'} \times_{X_{\alpha'}} \mathrm{Spec} A$. We have $Z \cong \mathrm{colim}_{\alpha'' \geq \alpha'} Z_{\alpha''}$ since filtered colimits in $\widehat{\mathrm{Stk}}_k$ are left exact [Lur09, Ex. 7.3.4.7]. Moreover, $\tau_{\leq 0} Z \cong \mathrm{colim}_{\alpha'' \geq \alpha'} \tau_{\leq 0} Z_{\alpha''}$ since by Proposition 3.16 and its proof all terms are in $1\text{-}\widehat{\mathrm{Stk}}_k$ and $Y \mapsto \tau_{\leq 0} Y$ preserves filtered colimits in $1\text{-}\widehat{\mathrm{Stk}}_k$. To show $\tau_{\leq 0} Z \rightarrow \tau_{\leq 0} \mathrm{Spec} A$ is a closed immersion it then suffices to show $\tau_{\leq 0} Z_{\alpha'} \rightarrow \tau_{\leq 0} Z_{\alpha''}$ is an isomorphism for any $\alpha'' \geq \alpha'$, since $\tau_{\leq 0} Z_{\alpha'} \rightarrow \tau_{\leq 0} \mathrm{Spec} A$ is a closed immersion by hypothesis. But this follows from Proposition 3.19.

Now suppose the given presentation is reasonable, and let $A \cong \mathrm{colim} A_\beta$ be a filtered colimit in $\tau_{\leq n} \mathrm{CAlg}_k$ for some n . By Proposition 4.3 we have $X(A_\beta) \cong \mathrm{colim}_{\gamma \geq \alpha} X_\gamma(A_\beta)$ for

all β , and likewise $X(A) \cong \operatorname{colim}_{\gamma \geq \alpha} X_\gamma(A)$. We then have

$$X_\alpha(A) \times_{X(A)} \operatorname{colim}_{\beta} X(A_\beta) \cong \operatorname{colim}_{\gamma \geq \alpha} \left(X_\alpha(A) \times_{X_\gamma(A)} \operatorname{colim}_{\beta} X_\gamma(A_\beta) \right)$$

since filtered colimits of spaces are left exact [Lur09, Prop. 5.3.3.3]. But the right hand colimit is isomorphic to $\operatorname{colim}_{\beta} X_\alpha(A_\beta)$ since each individual term is by hypothesis.

Finally, by Proposition 4.3 we can factor $X' \rightarrow X$ through a morphism $j_\alpha : X' \rightarrow X_\alpha$ for some α . Then j is a closed immersion by Proposition 3.17, hence is affine by Proposition 3.18, and is almost of finite presentation in the reasonable case by Proposition 3.9. \square

4.2. Properties of morphisms. Notions such as ind-properness extend from ind-schemes to ind-geometric stacks in the obvious way.

Definition 4.6. A morphism $f : X \rightarrow Y$ of ind-geometric stacks is ind-proper (resp. an ind-closed immersion, of ind-finite cohomological dimension) if for every diagram

$$(4.7) \quad \begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

in which $X' \rightarrow X$ and $Y' \rightarrow Y$ are truncated geometric substacks, the map f' is proper (resp. a closed immersion, of finite cohomological dimension).

If $f : X \rightarrow Y$ is of ind-finite cohomological dimension and there exists an n such that any morphism f' as in Definition 4.6 is of cohomological dimension $\leq n$, then we say f is of finite cohomological dimension. For example, an ind-closed immersion is of finite cohomological dimension (with $n = 0$), while the projection $\mathbb{P}^\infty := \bigcup \mathbb{P}^n \rightarrow \operatorname{Spec} k$ is of ind-finite, but not finite, cohomological dimension.

Proposition 4.8. *Let $f : X \rightarrow Y$ be a morphism of ind-geometric stacks and $X \cong \operatorname{colim}_{\alpha} X_\alpha$ an ind-geometric presentation. Then f is ind-proper (resp. an ind-closed immersion, of ind-finite cohomological dimension) if and only if for every X_α there exists a diagram*

$$(4.9) \quad \begin{array}{ccc} X_\alpha & \longrightarrow & X \\ f_\alpha \downarrow & & \downarrow f \\ Y_\alpha & \longrightarrow & Y \end{array}$$

in which Y_α is a truncated geometric substack of Y and f_α is proper (resp. a closed immersion, of finite cohomological dimension).

Proof. Fix an ind-geometric presentation $Y \cong \operatorname{colim} Y_\beta$. The only if direction follows since $f \circ i_\alpha$ factors through some Y_β by Proposition 4.3. Now consider the if direction, and fix a

diagram as in Definition 4.6. By hypothesis and Proposition 4.3 there exists a diagram of the left-hand form for some α ,

$$\begin{array}{ccc}
 X' & \xrightarrow{f'} & Y' \\
 \downarrow & \searrow & \searrow \\
 & X & \xrightarrow{f} Y \\
 \uparrow & \nearrow & \nearrow \\
 X_\alpha & \xrightarrow{f_\alpha} & Y_\alpha
 \end{array}
 \qquad
 \begin{array}{ccc}
 X' & \xrightarrow{f'} & Y' \\
 \downarrow & & \downarrow \\
 & & Y_\beta \xrightarrow{\quad} Y, \\
 \uparrow & & \uparrow \\
 X_\alpha & \xrightarrow{f_\alpha} & Y_\alpha
 \end{array}$$

where f_α is proper and $Y_\alpha \rightarrow Y$ is a truncated geometric substack. We claim this extends to a diagram of the right-hand form for some Y_β .

To see this, note that for any finite diagram $p : K \rightarrow \mathbf{GStk}^+$, the natural map

$$\mathrm{colim} \mathrm{Map}_{\mathrm{Fun}(K, \mathbf{Stk}_k)}(p, Y_\beta) \rightarrow \mathrm{Maps}_{\mathrm{Fun}(K, \mathbf{Stk}_k)}(p, Y)$$

is an isomorphism, where we let Y and Y_β denote the associated constant diagrams. This follows since p is compact in $\mathrm{Fun}(K, 1\text{-}\widehat{\mathbf{Stk}}_k)$ by [Lur09, Prop. 5.3.4.13] and Proposition 3.16. The claim at hand follows by taking p to be the subdiagram on the left spanned by X' , Y' , X_α , and Y_α .

In the right-hand diagram, the vertical maps are closed immersions by Proposition 4.5, hence f' is proper by Proposition 3.7. The other classes of morphisms are treated the same way, using Propositions 3.5 and 3.17, and the following observation: if f and g are composable morphisms in \mathbf{GStk}_k such that $g \circ f$ is of finite cohomological dimension and g is affine (hence g_* conservative and t-exact), then f is of finite cohomological dimension. \square

Proposition 4.10. *Ind-proper morphisms, ind-closed immersions, and morphisms of ind-finite cohomological dimension are stable under composition in $\mathrm{indGStk}_k$.*

Proof. Let X , Y , and Z be ind-geometric stacks, $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ ind-proper morphisms, and $X \cong \mathrm{colim} X_\alpha$ an ind-geometric presentation. For any α , Proposition 4.3 implies there exist truncated geometric substacks $Y_\alpha \rightarrow Y$ and $Z_\alpha \rightarrow Z$ such that the restrictions of f and g factor through proper morphisms $f_\alpha : X_\alpha \rightarrow Y_\alpha$ and $g_\alpha : Y_\alpha \rightarrow Z_\alpha$. But then $g_\alpha \circ f_\alpha$ is proper, hence $g \circ f$ is ind-proper by the other direction of Proposition 4.8. The other classes of morphisms are treated the same way. \square

Proposition 4.11. *Let $f : X \rightarrow Y$ be a morphism of geometric stacks. Then f is ind-proper (resp. an ind-closed immersion, of ind-finite cohomological dimension) if and only if it is proper (resp. a closed immersion, of finite cohomological dimension).*

Proof. Recall that $X \cong \mathrm{colim} \tau_{\leq n} X$ and $Y \cong \mathrm{colim} \tau_{\leq n} Y$ are ind-geometric presentations. Properness of f is equivalent to properness of $\tau_{\leq 0} f$ [Lur18, Rem. 5.1.2.2], hence to properness of each $\tau_{\leq n} f$, hence to ind-properness (Proposition 4.8). The corresponding claim for

closedness is immediate, while for finiteness of cohomological dimension it follows from [HLP14, Lem. A.1.6] and the fact that $\mathrm{QCoh}(X)^\heartsuit \cong \mathrm{QCoh}(\tau_{\leq 0}X)^\heartsuit$. \square

Let $f : X \rightarrow Y$ be a morphism of reasonable ind-geometric stacks. We will often say f is almost ind-finitely presented if it is almost finitely presented (i.e. in the sense of (3.8)). This is justified by the following result.

Proposition 4.12. *Let f be a morphism of reasonable ind-geometric stacks, and $X \cong \mathrm{colim}_\alpha X_\alpha$ a reasonable presentation. The following conditions are equivalent.*

- (1) *The morphism f is almost finitely presented.*
- (2) *For every diagram (4.7) in which $X' \rightarrow X$ and $Y' \rightarrow Y$ are reasonable geometric substacks, the map f' is almost finitely presented.*
- (3) *For every X_α there exists a diagram (4.9) in which Y_α is a reasonable geometric substack of Y and f_α is almost finitely presented.*

Proof. That (1) implies (2) follows from Propositions 3.9 and 4.5, and that (2) implies (3) is immediate. To show (3) implies (1) let $A \cong \mathrm{colim}_\beta A_\beta$ be a filtered colimit in $\tau_{\leq n}\mathrm{CAlg}_k$ for some n . Then we have

$$\mathrm{colim}_\beta X(A_\beta) \cong \mathrm{colim}_{\alpha, \beta} X_\alpha(A_\beta) \cong \mathrm{colim}_\alpha \left(X_\alpha(A) \times_{Y(A)} \mathrm{colim}_\beta Y(A_\beta) \right),$$

the first isomorphism using Proposition 4.3 and the second Proposition 4.5. But the last expression is then isomorphic to $X(A) \times_{Y(A)} \mathrm{colim}_\beta Y(A_\beta)$ by the left exactness of filtered colimits of spaces. \square

Ind-closed immersions have the following closure property. Here $\mathrm{indGStk}_{k,cl} \subset \mathrm{indGStk}_k$ (resp. $\mathrm{indGStk}_{k,cl,afp}^{reas} \subset \mathrm{indGStk}_k^{reas}$) denotes the 1-full subcategory which only includes ind-closed immersions (resp. almost finitely presented ind-closed immersions), similarly for $\mathrm{GStk}_{k,cl}^+ \subset \mathrm{GStk}_k^+$ (resp. $\mathrm{GStk}_{k,cl,afp}^+ \subset \mathrm{GStk}_k^+$). Recall that a subcategory is 1-full if for $n > 1$ it includes all n -simplices whose edges belong to the indicated class of morphisms.

Proposition 4.13. *The canonical continuous functor $\mathrm{Ind}(\mathrm{GStk}_{k,cl}^+) \rightarrow \widehat{\mathrm{Stk}}_k$ factors through an equivalence $\mathrm{Ind}(\mathrm{GStk}_{k,cl}^+) \cong \mathrm{indGStk}_{k,cl}$, and likewise $\mathrm{Ind}(\mathrm{GStk}_{k,cl,afp}^+) \rightarrow \widehat{\mathrm{Stk}}_k$ factors through an equivalence $\mathrm{Ind}(\mathrm{GStk}_{k,cl,afp}^+) \cong \mathrm{indGStk}_{k,cl,afp}^{reas}$. In particular, $\mathrm{indGStk}_k$ (resp. $\mathrm{indGStk}_k^{reas}$) is closed in $\widehat{\mathrm{Stk}}_k$ under filtered colimits along ind-closed immersions (resp. almost ind-finitely presented ind-closed immersions).*

Proof. By definition $\mathrm{indGStk}_k$ is the essential image of $\mathrm{Ind}(\mathrm{GStk}_{k,cl}^+)$. Let $X \cong \mathrm{colim}_\alpha X_\alpha$, $Y \cong \mathrm{colim}_\beta Y_\beta$ be ind-geometric presentations. By abuse we denote the corresponding objects of $\mathrm{Ind}(\mathrm{GStk}_{k,cl}^+)$ by X and Y as well, so that

$$\mathrm{Map}_{\mathrm{Ind}(\mathrm{GStk}_{k,cl}^+)}(X, Y) \cong \lim_{\alpha} \mathrm{colim}_{\beta} \mathrm{Map}_{\mathrm{GStk}_{k,cl}^+}(X_\alpha, Y_\beta).$$

Now the natural map

$$\lim_{\alpha} \operatorname{colim}_{\beta} \operatorname{Map}_{\operatorname{GStk}_{k,cl}^+}(X_{\alpha}, Y_{\beta}) \rightarrow \lim_{\alpha} \operatorname{colim}_{\beta} \operatorname{Map}_{\operatorname{GStk}_k^+}(X_{\alpha}, Y_{\beta}) \cong \operatorname{Map}_{\operatorname{indGStk}_k}(X, Y)$$

is a monomorphism since monomorphisms are stable under limits and filtered colimits (note that the isomorphism on the right follows from Proposition 3.16). It thus suffices to show its image is exactly the subspace of ind-closed immersions, but this follows from the definitions and Proposition 4.8. The other case is proved the same way. \square

Remark 4.14. Note that a closed immersion of non-truncated geometric stacks is also an ind-closed morphism of ind-geometric stacks. It follows from Proposition 4.13 that $\operatorname{indGStk}_k$ is the essential image of the (not fully faithful) functor $\operatorname{Ind}(\operatorname{GStk}_{k,cl}) \rightarrow \widehat{\operatorname{Stk}}_k$, where $\operatorname{GStk}_{k,cl} \subset \operatorname{GStk}_k$ is the 1-full subcategory which only includes closed immersions. In other words, we obtain the same class of objects if in Definition 4.1 we do not require the X_{α} to be truncated.

Recall that a morphism $f : X \rightarrow Y$ of stacks is geometric if for any morphism $Y' \rightarrow Y$ with Y' an affine scheme, $Y' \times_Y X$ is a geometric stack. By Proposition 3.3 this implies more generally that $Y' \times_Y X$ is a geometric stack whenever Y' is.

Definition 4.15. Let $f : X \rightarrow Y$ be a morphism of reasonable ind-geometric stacks. We say f has semi-universal coherent pullback if it is geometric and for every truncated geometric stack Y' and every ind-tamely presented morphism $Y' \rightarrow Y$, the base change $f' : Y' \times_Y X \rightarrow Y'$ has coherent pullback.

Likewise we will say a morphism of arbitrary ind-geometric stacks is of Tor-dimension $\leq n$ (resp. of finite Tor-dimension) if it is geometric and its base change to any truncated geometric stack is of Tor-dimension $\leq n$ (resp. of finite Tor-dimension) in the sense of Section ???. The following criterion for semi-universal coherent pullback entails in particular that the above definition is consistent with Definition ??? when Y is a truncated geometric stack.

Proposition 4.16. *Let $f : X \rightarrow Y$ be a geometric morphism of reasonable (resp. arbitrary) ind-geometric stacks and $Y \cong \operatorname{colim}_{\alpha} Y_{\alpha}$ a reasonable (resp. ind-geometric) presentation of Y . Then f has semi-universal coherent pullback (resp. is of finite Tor-dimension) if and only if its base change to every Y_{α} has semi-universal coherent pullback (resp. is of finite Tor-dimension).*

Proof. Propositions 4.5 and 4.12 imply the only if direction. Let Y' be a truncated geometric stack and $j : Y' \rightarrow Y$ a tamely presented morphism. By Proposition 4.5 and the definitions we can factor j through some $i_{\alpha} : Y_{\alpha} \rightarrow Y$ via a tamely presented morphism $j_{\alpha} : Y' \rightarrow Y_{\alpha}$, hence the base change of f to Y' has coherent pullback since its base change to Y_{α} has semi-universal coherent pullback. The finite Tor-dimension case is proved the same way. \square

Proposition 4.17. *Morphisms with semi-universal coherent pullback (resp. of finite Tor-dimension) are stable under composition in $\text{indGStk}_k^{\text{reas}}$ (resp. indGStk_k).*

Proof. Let X , Y , and Z be reasonable ind-geometric stacks, $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ morphisms with semi-universal coherent pullback, and $Z \cong \text{colim } Z_\alpha$ a reasonable presentation. Define $g_\alpha : Y_\alpha \rightarrow Z_\alpha$ and $f_\alpha : X_\alpha \rightarrow Y_\alpha$ by base change. Then each Y_α is a truncated geometric stack since g_α has coherent pullback, the maps $Y_\alpha \rightarrow Y_\beta$ are almost finitely presented closed immersions by base change, and $Y \cong \text{colim } Y_\alpha$ by left exactness of filtered colimits in $\widehat{\text{Stk}}_k$. In particular each f_α has semi-universal coherent pullback, hence each $g_\alpha \circ f_\alpha$ does by Proposition ??, hence $g \circ f$ does by Proposition 4.16. The finite Tor-dimension case is proved the same way. \square

4.3. Fiber Products. Now we consider fiber products of ind-geometric stacks, and the base change properties of the classes of morphisms considered above.

Proposition 4.18. *Ind-geometric stacks are closed under finite limits in $\widehat{\text{Stk}}_k$ (and Stk_k).*

Proof. Note that $\widehat{\text{Stk}}_k$ is closed under limits in Stk_k , so the two claims are equivalent. Since indGStk_k contains the terminal object $\text{Spec } k$, it suffices to show closure under fiber products [Lur09, Prop. 4.4.2.6].

Let $f : X \rightarrow Y$ and $h : Y' \rightarrow Y$ be morphisms of ind-geometric stacks, and let $X' := X \times_Y Y'$. Suppose first that X and Y' are truncated geometric stacks, and let $Y \cong \text{colim } Y_\alpha$ be an ind-geometric presentation. By Proposition 4.3 we can factor f and h through Y_α for some α . We have $X' \cong \text{colim}_{\beta \geq \alpha} X'_\beta$, where $X'_\beta := X \times_{Y_\beta} Y'$, by left exactness of filtered colimits in $\widehat{\text{Stk}}_k$ [Lur09, Ex. 7.3.4.7]. The transition maps are closed immersions of not necessarily truncated geometric stacks by Proposition 3.19. It follows they are ind-closed as morphisms of ind-geometric stacks, hence X' is ind-geometric by Proposition 4.13. Now suppose $X \cong \text{colim } X_\alpha$ and $Y' \cong \text{colim } Y'_\beta$ are ind-geometric presentations. Then as above $X' \cong \text{colim } X_\alpha \times_Y Y'_\beta$ expresses X' as a filtered colimit in $\widehat{\text{Stk}}_k$ of ind-geometric stacks along ind-closed immersions, so again X' is ind-geometric by Proposition 4.13. \square

Already the self-intersection of the origin in \mathbb{A}^∞ illustrates that reasonable ind-geometric stacks are not closed under arbitrary fiber products. Instead we have the following more limited result.

Proposition 4.19. *Let the following be a Cartesian diagram of ind-geometric stacks.*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ h' \downarrow & & \downarrow h \\ X & \xrightarrow{f} & Y \end{array}$$

Suppose that X , Y , and Y' are reasonable. Suppose also that h has semi-universal coherent pullback and f is ind-tamely presented (resp. that h is of finite Tor-dimension and f is

arbitrary). Then X' is reasonable. Moreover, h' has semi-universal coherent pullback and f' is ind-tamely presented (resp. h' is of finite Tor-dimension), and f' is almost ind-finitely presented if f is.

Proof. Let $X \cong \operatorname{colim} X_\alpha$ be a reasonable presentation. For any α , there is by hypothesis a reasonable geometric substack $Y_\alpha \rightarrow Y$ and a commutative cube

$$\begin{array}{ccccc}
 & & X'_\alpha & \xrightarrow{\phi'} & Y'_\alpha \\
 & \swarrow & \downarrow f' & \swarrow & \downarrow \psi \\
 X' & \xrightarrow{\quad} & Y' & & \\
 \downarrow & \swarrow \psi' & \downarrow \phi & \downarrow h & \\
 & X_\alpha & \xrightarrow{\quad} & Y_\alpha & \\
 \downarrow & \swarrow f & \downarrow & \swarrow & \\
 X & \xrightarrow{\quad} & Y & &
 \end{array}$$

in $\operatorname{indGStk}_k$ such that all but the top and bottom faces are Cartesian, and such that ϕ and ϕ' are tamely presented.

By hypothesis ψ has semi-universal coherent pullback, and by Proposition ?? so does ψ' . In particular X'_α is a truncated geometric stack. We have $X' \cong \operatorname{colim} X'_\alpha$ in $\widehat{\operatorname{Stk}}_k$ since filtered colimits are left exact [Lur09, Ex. 7.3.4.7], and this is a reasonable presentation since almost finitely presented closed immersions are stable under base change. The rest of the claim now follows from Propositions 4.16 and 4.8. The other claims follow the same way. \square

Proposition 4.20. *Ind-proper morphisms (resp. ind-closed immersions, morphisms of ind-finite cohomological dimension) are stable under base change in $\operatorname{indGStk}_k$.*

Proof. Let $f : X \rightarrow Y$ and $h : Y' \rightarrow Y$ be morphisms in $\operatorname{indGStk}_k$ such that f is ind-proper. If $X \cong \operatorname{colim} X_\alpha$ is an ind-geometric presentation, we have for all α a commutative cube

$$\begin{array}{ccccc}
 & & X'_\alpha & \xrightarrow{\phi'} & Y'_\alpha \\
 & \swarrow & \downarrow f' & \swarrow & \downarrow \psi \\
 X' & \xrightarrow{\quad} & Y' & & \\
 \downarrow & \swarrow \psi' & \downarrow \phi & \downarrow h & \\
 & X_\alpha & \xrightarrow{\quad} & Y_\alpha & \\
 \downarrow & \swarrow f & \downarrow & \swarrow & \\
 X & \xrightarrow{\quad} & Y & &
 \end{array}$$

in $\operatorname{indGStk}_k$ such that all but the top and bottom faces are Cartesian, $Y_\alpha \rightarrow Y$ is a truncated geometric substack, and ϕ is proper. Let $Y'_\alpha \cong \operatorname{colim}_\beta Y'_{\alpha\beta}$ be an ind-geometric presentation. Then, letting $X'_{\alpha\beta} := X'_\alpha \times_{Y'_\alpha} Y'_{\alpha\beta}$, we have $X'_\alpha \cong \operatorname{colim}_\beta X'_{\alpha\beta}$ by left exactness of filtered colimits in $\widehat{\operatorname{Stk}}_k$. Note that for all β the morphisms $X'_{\alpha\beta} \rightarrow X'$ and $Y'_{\alpha\beta} \rightarrow Y'$ are closed immersions since $Y'_{\alpha\beta} \rightarrow Y'_\alpha$ and $Y_\alpha \rightarrow Y$ are, and in particular $Y'_{\alpha\beta}$ is a truncated geometric substack of Y' .

Now let $X' \cong \operatorname{colim} X'_\gamma$ be an ind-geometric presentation and fix some γ . By Proposition 4.3 we can choose α so that $X'_\gamma \rightarrow X$ factors through X_α , hence so that $X'_\gamma \rightarrow X'$ factors through X'_α . Proposition 3.16 then implies that $X'_\gamma \rightarrow X'$ factors through $X'_{\alpha\beta}$ for some β . This map $X'_\gamma \rightarrow X'_{\alpha\beta}$ is a closed immersion since $X'_\gamma \rightarrow X'$ and $X'_{\alpha\beta} \rightarrow X'$ are, while $X'_{\alpha\beta} \rightarrow Y'_{\alpha\beta}$ is proper since it is a base change of ϕ . Thus the composition $X'_\gamma \rightarrow X'_{\alpha\beta} \rightarrow Y'_{\alpha\beta}$ is proper, hence f' is ind-proper by Proposition 4.8. The other classes of morphisms are treated the same way. \square

4.4. Truncated ind-geometric stacks. We say an ind-geometric stack is n -truncated if it admits an ind-geometric presentation $X \cong \operatorname{colim} X_\alpha$ in which each X_α is an n -truncated geometric stack. Recall that $(-)_{\leq n} : \widehat{\operatorname{Stk}}_k \rightarrow \operatorname{Stk}_{k, \leq n}$ identifies the category of n -truncated geometric stacks with a full subcategory of $\operatorname{Stk}_{k, \leq n}$, the inverse equivalence being given by the left adjoint $\widehat{i}_{\leq n} : \operatorname{Stk}_{k, \leq n} \rightarrow \widehat{\operatorname{Stk}}_k$. Note that $(-)_{\leq n}$ takes filtered colimits in $1\text{-}\widehat{\operatorname{Stk}}_k$ to filtered colimits in $\tau_{\leq n+1}\operatorname{Stk}_{k, \leq n}$, since it restricts from a continuous functor $\widehat{\operatorname{PStk}}_k \rightarrow \operatorname{PStk}_{k, \leq n}$, and since as in Proposition 3.16 and its proof $1\text{-}\widehat{\operatorname{Stk}}_k \subset \widehat{\operatorname{PStk}}_k$ and $\tau_{\leq n+1}\operatorname{Stk}_{k, \leq n} \subset \operatorname{PStk}_{k, \leq n}$ are closed under filtered colimits. Since $\widehat{i}_{\leq n}$ is continuous it follows that $(-)_{\leq n}$ identifies the category of n -truncated ind-geometric stacks with the obvious subcategory of $\operatorname{Stk}_{k, \leq n}$. Again letting $\widehat{\tau}_{\leq n} : \widehat{\operatorname{Stk}}_k \rightarrow \widehat{\operatorname{Stk}}_k$ denote the composition of $(-)_{\leq n}$ and $\widehat{i}_{\leq n}$, the following result states in particular that Definition 4.1 is indeed the obvious extension of [GR14, Def. 1.4.2] from schemes to geometric stacks (it is stated differently so that reasonableness may be introduced more easily).

Proposition 4.21. *A convergent stack X is ind-geometric if and only if $\widehat{\tau}_{\leq n}X$ is an n -truncated ind-geometric stack for all n .*

Proof. The only if direction follows since $\widehat{\tau}_{\leq n}$ preserves closed immersions of geometric stacks, and since by the above discussion its restriction to $1\text{-}\widehat{\operatorname{Stk}}_k$ is continuous. The if direction follows from Proposition 4.13, since each $\widehat{\tau}_{\leq n}X \rightarrow \widehat{\tau}_{\leq n+1}X$ is classically an isomorphism, hence is an ind-closed immersion. \square

Example 4.22. A typical example of a classical (i.e. zero-truncated) ind-geometric stack is the following. Suppose $X \cong \operatorname{colim} X_\alpha$ is a presentation of a classical ind-scheme as a filtered colimit of quasi-compact, semi-separated ordinary schemes (regarded as objects of $\widehat{\operatorname{Stk}}_k$) along closed immersions, and that G is a classical affine group scheme acting on X . For each α the induced map $X_\alpha \times G \rightarrow X$ factors through some X_β . The closure $X'_\alpha \subset X_\beta$ of its image is a G -invariant closed subscheme of X . We obtain a presentation $X \cong \operatorname{colim} X'_\alpha$ by closed G -invariant subschemes, and it follows that the quotient X/G (in $\widehat{\operatorname{Stk}}_k$) is a classical ind-geometric stack with ind-geometric presentation $X/G \cong \operatorname{colim} X'_\alpha/G$. Note that the quotients X'_α/G taken in Stk_k are again geometric [Lur18, Prop. 9.3.1.3], hence convergent (Proposition 3.13), hence they coincide with the quotients X'_α/G taken in $\widehat{\operatorname{Stk}}_k$.

5. COHERENT SHEAVES

In this section we consider coherent sheaves on reasonable ind-geometric stacks. We construct the functors of almost ind-finitely presented, ind-proper pushforward and finite Tor-dimension pullback, and show that these are compatible with base change.

5.1. Coherent sheaves and pushforward. We now define the category $\mathrm{Coh}(X)$ of coherent sheaves on a reasonable ind-geometric stack X . Explicitly, if $X \cong \mathrm{colim} X_\alpha$ is a reasonable presentation we will have $\mathrm{Coh}(X) \cong \mathrm{colim} \mathrm{Coh}(X_\alpha)$, where the colimit is taken in Cat_∞ along the pushforward functors. Such expressions will follow from a more canonical definition of $\mathrm{Coh}(X)$, which will also make manifest its functoriality under proper pushforward.

Let $\mathrm{GStk}_{k,prop}^+ \subset \mathrm{GStk}_k^+$ denote the 1-full subcategory which only includes proper, almost finitely presented morphisms, similarly for $\mathrm{indGStk}_{k,prop}^{reas} \subset \mathrm{indGStk}_k^{reas}$. Note that $\mathrm{GStk}_{k,prop}^+$ is a full subcategory of $\mathrm{indGStk}_{k,prop}^{reas}$ by Proposition 4.11. We have a canonical functor

$$(5.1) \quad \mathrm{Coh} : \mathrm{GStk}_{k,prop}^+ \rightarrow \mathrm{Cat}_\infty$$

which takes X to $\mathrm{Coh}(X)$ and $f : X \rightarrow Y$ to $f_* : \mathrm{Coh}(X) \rightarrow \mathrm{Coh}(Y)$. This is obtained by restriction from the corresponding functor $\mathrm{QCoh} : \mathrm{GStk}_k \rightarrow \widehat{\mathrm{Cat}}_\infty$, given that proper, almost finitely presented pushforward preserves coherence (Proposition 3.7).

Definition 5.2. We write

$$(5.3) \quad \mathrm{Coh} : \mathrm{indGStk}_{k,prop}^{reas} \rightarrow \mathrm{Cat}_\infty$$

for the left Kan extension of (5.1) along the inclusion $\mathrm{GStk}_{k,prop}^+ \subset \mathrm{indGStk}_{k,prop}^{reas}$. We write $f_* : \mathrm{Coh}(X) \rightarrow \mathrm{Coh}(Y)$ for the functor assigned to an ind-proper, almost ind-finitely presented morphism $f : X \rightarrow Y$.

Existence of the indicated left Kan extension follows from [Lur09, Lem. 4.3.2.13]. The following result implies that $\mathrm{Coh}(X)$ can be expressed in terms of reasonable presentations as stated earlier.

Proposition 5.4. *The functor $\mathrm{Coh} : \mathrm{indGStk}_{k,prop}^{reas} \rightarrow \mathrm{Cat}_\infty$ preserves filtered colimits along almost ind-finitely presented ind-closed immersions.*

Proof. For the proof we distinguish (5.1) and (5.3) by writing them as Coh_{geom} and Coh_{ind} , respectively. The proof of Proposition 4.13 adapts to show that the canonical continuous functor $\mathrm{Ind}(\mathrm{GStk}_{k,prop}^+) \rightarrow \widehat{\mathrm{Stk}}_k$ is faithful and that $\mathrm{indGStk}_{k,prop}^{reas}$ is the intersection of its image with $\mathrm{indGStk}_k^{reas}$. In particular, the inclusion $\mathrm{indGStk}_{k,prop}^{reas} \subset \widehat{\mathrm{Stk}}_k$ factors through a fully faithful functor $\mathrm{indGStk}_{k,prop}^{reas} \rightarrow \mathrm{Ind}(\mathrm{GStk}_{k,prop}^+)$. By the transitivity of left Kan extensions [Lur09, Prop. 4.3.2.8], Coh_{ind} is the restriction to $\mathrm{indGStk}_{k,prop}^{reas}$ of Coh_{Ind} , the left Kan extension of Coh_{geom} to $\mathrm{Ind}(\mathrm{GStk}_{k,prop}^+)$.

Recall from Proposition 4.13 that $\mathrm{indGStk}_{k,cl,afp}^{reas}$ admits filtered colimits and its inclusion into $\widehat{\mathrm{Stk}}_k$ is continuous. By the previous paragraph this inclusion factors through a faithful

functor $\mathrm{indGStk}_{k,cl,afp}^{reas} \rightarrow \mathrm{Ind}(\mathrm{GStk}_{k,prop}^+)$, which is then also continuous. But Coh_{Ind} is continuous [Lur09, Lem. 5.3.5.8], hence so is its restriction to $\mathrm{indGStk}_{k,cl,afp}^{reas}$. \square

Suppose $f : X \rightarrow Y$ is an ind-proper, almost ind-finitely presented morphism and $X \cong \mathrm{colim} X_\alpha$ a reasonable presentation. Proposition 5.4 implies in particular that any $\mathcal{F} \in \mathrm{Coh}(X)$ can be written as $\mathcal{F} \cong i_{\alpha*}(\mathcal{F}_\alpha)$ for some α and some $\mathcal{F}_\alpha \in \mathrm{Coh}(X_\alpha)$. The behavior of ind-proper pushforward on objects is thus determined from the geometric setting by the following compatibility: we can factor $f \circ i_\alpha$ through a proper, almost finitely presented morphism $f_\alpha : X_\alpha \rightarrow Y_\alpha$ to a reasonable geometric substack $j_\alpha : Y_\alpha \rightarrow Y$, and we then have $f_*(\mathcal{F}) \cong j_{\alpha*}f_{\alpha*}(\mathcal{F}_\alpha)$.

5.2. Pullback and base change. Next we consider the pullback of coherent sheaves along suitable morphisms of reasonable ind-geometric stacks. For explicitness we first define this directly in terms of presentations, and then give a more global description that will make its compatibilities more manifest.

Let $f : X \rightarrow Y$ be a morphism with semi-universal coherent pullback, which we emphasize includes any morphism of finite Tor-dimension. Let $Y \cong \mathrm{colim} Y_\alpha$ a reasonable presentation, $f_\alpha : X_\alpha \rightarrow Y_\alpha$ the base change of f , and $i_{\alpha\beta} : X_\alpha \rightarrow X_\beta$, $j_{\alpha\beta} : Y_\alpha \rightarrow Y_\beta$ the induced maps. By proper base change we have isomorphisms $f_\beta^* j_{\alpha\beta*} \cong i_{\alpha\beta*} f_\alpha^*$ of functors $\mathrm{Coh}(Y_\alpha) \rightarrow \mathrm{Coh}(X_\beta)$. The identity $\mathrm{Coh}(Y) \cong \mathrm{colim} \mathrm{Coh}(Y_\alpha)$ implies there is an essentially unique functor $f^* : \mathrm{Coh}(Y) \rightarrow \mathrm{Coh}(X)$ equipped with coherent isomorphisms $f^* j_{\alpha*} \cong i_{\alpha*} f_\alpha^*$ of functors $\mathrm{Coh}(Y_\alpha) \rightarrow \mathrm{Coh}(X)$ for all α . More formally, we can construct this functor as follows.

Write A for the index category of the presentation $Y \cong \mathrm{colim} Y_\alpha$, so that Y and the Y_α together define a diagram $A^\triangleright \rightarrow \mathrm{indGStk}_k^{reas}$. Taking Cartesian products with X we obtain a diagram $A^\triangleright \times \Delta^1 \rightarrow \mathrm{indGStk}_k^{reas}$ and by restriction a diagram $A \times \Delta^1 \rightarrow \mathrm{GStk}_k^+$. We then obtain a diagram $A \times \Delta^1 \rightarrow \mathrm{Cat}_\infty$ taking f_α to $f_\alpha^* : \mathrm{Coh}(Y_\alpha) \rightarrow \mathrm{Coh}(X_\alpha)$ and $i_{\alpha\beta}$ to $i_{\alpha\beta*} : \mathrm{Coh}(X_\alpha) \rightarrow \mathrm{Coh}(X_\beta)$. More precisely, this is constructed from the more primitive functor $A \times \Delta^1 \rightarrow \widehat{\mathrm{Cat}}_\infty$ that takes each map to the associated pullback of quasi-coherent sheaves, passing to adjoints along morphisms in A via [Lur17, Cor. 4.7.5.18] and then passing to coherent subcategories. This is equivalent to the data of the associated functor $A \rightarrow \mathrm{Cat}_\infty^{\Delta^1} := \mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$ taking α to $f_\alpha^* : \mathrm{Coh}(Y_\alpha) \rightarrow \mathrm{Coh}(X_\alpha)$. By [Lur09, Cor. 5.1.2.3] and Proposition 5.4 the colimit of this diagram is a functor $f^* : \mathrm{Coh}(Y) \rightarrow \mathrm{Coh}(X)$ with the desired compatibilities.

To make this manifestly canonical, observe that $(\mathrm{GStk}_{k,cl,afp}^+)_Y$, the category of reasonable geometric substacks of Y , provides a canonical reasonable presentation $Y \cong \mathrm{colim}(\mathrm{GStk}_{k,cl,afp}^+)_Y$.

Definition 5.5. Let $f : X \rightarrow Y$ be a morphism with semi-universal pullback between reasonable ind-geometric stacks. Then we define $f^* : \mathrm{Coh}(Y) \rightarrow \mathrm{Coh}(X)$ by applying the above construction to the canonical reasonable presentation $Y \cong \mathrm{colim}(\mathrm{GStk}_{k,cl,afp}^+)_Y$. That is, f^* is the unique functor which admits a coherent family of isomorphisms $f^* i_* \cong i'_* f'^*$ for any reasonable geometric substack $i : Y' \rightarrow Y$.

That this agrees with the above construction applied to an arbitrary reasonable presentation follows since the diagram $A \rightarrow \mathrm{GStk}_k^+$ associated to any other presentation factors through $(\mathrm{GStk}_{k,cl,afp}^+)/Y$.

Proposition 5.6. *Let the following be a Cartesian diagram of ind-geometric stacks.*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ h' \downarrow & & \downarrow h \\ X & \xrightarrow{f} & Y \end{array}$$

*Suppose that X , Y , and Y' are reasonable, that h has semi-universal coherent pullback, and that f is ind-proper and almost ind-finitely presented. Then X' is reasonable, h' and f' have the same properties as h and f , and there is a canonical isomorphism $h^*f_* \cong f'_*h'^*$ of functors $\mathrm{Coh}(X) \rightarrow \mathrm{Coh}(Y')$.*

Proof. The conclusions about X' , h' , and f' are Propositions 4.19 and 4.20. It follows from the proof of Proposition 5.4 and a straightforward variant of [Lur09, 5.3.5.15] that we can write $f \in (\mathrm{indGStk}_{k,prop}^{reas})^{\Delta^1}$ as the colimit of a filtered diagram $\{f_\alpha : X_\alpha \rightarrow Y_\alpha\}$ whose restrictions to the vertices of Δ^1 define reasonable presentations $X \cong \mathrm{colim} X_\alpha$, $Y \cong \mathrm{colim} Y_\alpha$. Writing A for the index category of this diagram, we repeat the above construction to obtain a functor $A \times \Delta^1 \times \Delta^1 \dots$ \square

Note that if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are morphisms with semi-universal coherent pullback, we obtain a canonical isomorphism $(g \circ f)^* \cong f^*g^*$ by repeating the construction of each individual functor with Δ^2 in place of Δ^1 . Similarly, one shows in this way that the base change isomorphisms of Proposition 5.6 are compatible with composition. We can encode such compatibilities more systematically using the formalism of correspondences.

Recall that given a category \mathcal{C} with Cartesian products, we have a category $\mathrm{Corr}(\mathcal{C})$ with the same objects as \mathcal{C} , and in which a morphism from X to Z is a diagram $X \xleftarrow{f} Y \xrightarrow{h} Z$ [GR17a, Sec. 7.1.2.5]. Composition of correspondences is given by taking Cartesian products in the standard way. We note that we will never refer to $(\infty, 2)$ -categories of correspondences in this text.

Let $\mathrm{Corr}(\mathrm{indGStk}_k^{reas})_{prop;coh}$ denote the 1-full subcategory of $\mathrm{Corr}(\mathrm{indGStk}_k)$ which only includes correspondences $X \xleftarrow{f} Y \xrightarrow{h} Z$ such that X , Y , and Z are reasonable, h has semi-universal coherent pullback, and f is ind-proper and almost of ind-finite presentation (these are stable under composition of correspondences by Propositions 4.10, 4.17, 4.20, and 4.19). We define $\mathrm{Corr}(\mathrm{GStk}_k^+)_{prop;coh}$ similarly, noting that it is a full subcategory of $\mathrm{Corr}(\mathrm{indGStk}_k)_{prop;coh}$ by Proposition 4.11.

Note now that the functor $\mathrm{Coh} : \mathrm{GStk}_{k,prop}^+ \rightarrow \mathrm{Cat}_\infty$ of (5.1) extends to a functor

$$(5.7) \quad \mathrm{Coh} : \mathrm{Corr}(\mathrm{GStk}_k^+)_{prop;coh} \rightarrow \mathrm{Cat}_\infty.$$

To construct it, consider the 1-full subcategory $\text{Corr}(\text{GStk}_k)_{fcd;all} \subset \text{Corr}(\text{GStk}_k)$ in which we only include correspondences $X \xleftarrow{f} Y \xrightarrow{h} Z$ such that f is of finite cohomological dimension. By Proposition 3.5 and [GR17a, Thm. 3.2.2] there exists a canonical functor $\text{QCoh} : \text{Corr}(\text{GStk}_k)_{fcd;all} \rightarrow \widehat{\text{Cat}}_\infty$ which takes a correspondence $X \xleftarrow{f} Y \xrightarrow{h} Z$ to the functor $f_* h^* : \text{QCoh}(Z) \rightarrow \text{QCoh}(X)$. We obtain the desired functor (5.7) by restricting along the inclusion $\text{Corr}(\text{GStk}_k^+)_{prop;coh} \subset \text{Corr}(\text{GStk}_k)_{fcd;all}$, and observing that the associated functors $f_* h^*$ preserve coherence.

Proposition 5.8. *Consider the left Kan extension*

$$(5.9) \quad \text{Coh} : \text{Corr}(\text{indGStk}_k^{reas})_{prop;coh} \rightarrow \text{Cat}_\infty$$

of (5.7) along the inclusion $\text{Corr}(\text{GStk}_k^+)_{prop;coh} \subset \text{Corr}(\text{indGStk}_k^{reas})_{prop;coh}$. The restriction of this extension to $\text{indGStk}_{k,prop}^{reas}$ is the functor of Definition 5.2, and its value on a morphism $h : X \rightarrow Y$ with semi-universal coherent pullback is the functor $h^* : \text{Coh}(Y) \rightarrow \text{Coh}(X)$ of Definition 5.5.

Proof. The first claim follows from [GR17a, Thm. 6.1.5] (in its opposite form for left Kan extensions). Now let $Y \cong \text{colim } Y_\alpha$ be a reasonable presentation with index set A , so that as before we have a diagram $A \times \Delta^1 \rightarrow \text{Cat}_\infty$ taking $\{\alpha\} \times \Delta^1$ to h_α^* , where $h_\alpha : X_\alpha \rightarrow Y_\alpha$ is the base change of h . It follows from the first claim, from Proposition ??, and from the fact that $X \cong \text{colim } X_\alpha$ is also a reasonable presentation that the restriction of (5.9) along the induced functor $A^\triangleright \times \Delta^1 \rightarrow \text{Corr}(\text{indGStk}_k^{reas})_{prop;coh}$ is a left Kan extension of its restriction to $A \times \Delta^1$. The second claim follows since this is the definition of h^* . \square

5.3. External products. When k is an ordinary ring of finite global dimension, the closure of GStk_k^+ under products implies the same for indGStk_k^{reas} (again using left exactness of filtered colimits in $\widehat{\text{Stk}}_k$). Proposition ?? then extends as follows.

Proposition 5.10. *If k is an ordinary ring of finite global dimension, then the functor*

$$\text{Coh} : \text{Corr}(\text{indGStk}_k^{reas})_{prop;coh} \rightarrow \text{Cat}_\infty$$

of Definition 5.2 is canonically lax symmetric monoidal.

Proof. Applying [Lur17, Prop. 3.1.1.20, Thm. 3.1.2.3] we may consider the lax symmetric monoidal functor $\text{Corr}(\text{indGStk})_{prop;coh} \rightarrow \widehat{\text{Cat}}_\infty$ defined by operadic left Kan extension (more precisely, in the notation of [Lur17, Rem. 3.1.3.15] we take $\mathcal{A} = \text{Corr}(\text{GStk}^+)_{prop;coh}$, $\mathcal{B} = \text{Corr}(\text{indGStk})_{prop;coh}$, $\mathcal{C} = \widehat{\text{Cat}}_\infty$, and $\mathcal{O}^\otimes = \text{Fin}_*$). That its underlying functor is obtained by (non-operadic) left Kan extension follows from comparing the diagrams in [Lur17, Def. 3.1.1.2] and [Lur09, Def. 4.3.2.2] after pulling back along $\mathcal{D}' = \{\langle 1 \rangle\} \hookrightarrow \mathcal{O}^\otimes = \text{Fin}_*$ (trivial Kan fibrations being stable under pullback). In particular, this extension a posteriori takes values in Cat_∞ . \square

6. IND-COHERENT SHEAVES

We now expand our attention from coherent to ind-coherent sheaves. Even if one is interested primarily in coherent sheaves (as in our motivating application), ind-coherent sheaves are necessary in order to discuss adjoint functorialities such as sheaf Hom and ind-proper $!$ -pullback.

If X is a geometric stack we define $\mathrm{IndCoh}(X)$ as the left anticompletion of $\mathrm{QCoh}(X)$, following a construction of [Lur18, App. C]. This characterizes $\mathrm{IndCoh}(X)$ in terms of a universal property satisfied by bounded colimit-preserving functors out of it (see (6.4)). The definition is then extended to the ind-geometric setting so that $\mathrm{IndCoh}(X) \cong \mathrm{colim} \mathrm{IndCoh}(X_\alpha)$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ for any ind-geometric presentation $X \cong \mathrm{colim} X_\alpha$ (Proposition 6.11), following the case of ind-schemes [GR14, Sec. 2.4], [Ras19, Sec. 6.10]. In particular $\mathrm{Coh}(X)$ is naturally a subcategory of $\mathrm{IndCoh}(X)$ (Proposition 6.26), though unless X is geometric $\mathrm{Coh}(X)$ is not generally a subcategory of $\mathrm{QCoh}(X)$ in any natural way.

In most cases of interest one can be more explicit. If X is a classical geometric stack then $\mathrm{IndCoh}(X)$ is (the dg nerve of) the category of injective complexes in $\mathrm{QCoh}(X)^\heartsuit$, introduced in [Kra05]. If X is coherent (for example, if it is a Noetherian scheme as in [Gai13]) then $\mathrm{IndCoh}(X)$ is the ind-completion of $\mathrm{Coh}(X)$. This is the primary case of interest (as our terminology abusively reflects) and is considered in detail in Section 7. As the focal results of Sections ?? and 8 are about this case, the reader concerned only with these may safely define $\mathrm{IndCoh}(X)$ via ind-completion and bypass the current section. But it is often convenient to have $\mathrm{IndCoh}(X)$ defined in greater generality, since for example the class of coherent ind-geometric stacks (or even of coherent affine schemes) is not closed under fiber products.

6.1. Anticompletion. We begin by recalling the notion of anticompleteness from [Lur18, App. C] in slightly adapted form. A t-structure on a stable ∞ -category \mathcal{C} is left complete if the natural functor $\mathcal{C} \rightarrow \lim_n \mathcal{C}^{\geq n}$ is an equivalence, and right complete if $\mathcal{C} \rightarrow \lim_n \mathcal{C}^{\leq n}$ is an equivalence. The category $\widehat{\mathcal{C}} := \lim_n \mathcal{C}^{\geq n}$ is called the left completion of \mathcal{C} . It has a canonical t-structure such that $\mathcal{C} \rightarrow \widehat{\mathcal{C}}$ is t-exact and restricts to an equivalence $\mathcal{C}^{\geq 0} \xrightarrow{\sim} \widehat{\mathcal{C}}^{\geq 0}$ [Lur17, Prop. 1.2.1.17]. If \mathcal{D} is another stable ∞ -category with a t-structure, an exact functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is bounded if there exist m, n such that $F(\mathcal{C}^{\geq 0}) \subset \mathcal{D}^{\geq m}$ and $F(\mathcal{C}^{\leq 0}) \subset \mathcal{D}^{\leq n}$. If \mathcal{C} and \mathcal{D} are presentable, we write $\mathrm{LFun}^b(\mathcal{C}, \mathcal{D}) \subset \mathrm{LFun}(\mathcal{C}, \mathcal{D})$ for the full subcategory of bounded colimit-preserving functors. In the presentable case, a t-structure on \mathcal{C} is accessible if $\mathcal{C}^{\geq 0}$ is also presentable, and is compatible with filtered colimits if $\mathcal{C}^{\geq 0}$ is closed under filtered colimits in \mathcal{C} .

Definition 6.1. Let \mathcal{C} be a presentable stable ∞ -category equipped with an accessible t-structure which is right complete and compatible with filtered colimits. We say \mathcal{C} is left anticomplete if composition with $\mathcal{D} \rightarrow \widehat{\mathcal{D}}$ induces an equivalence

$$\mathrm{LFun}^b(\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \mathrm{LFun}^b(\mathcal{C}, \widehat{\mathcal{D}})$$

for any other presentable stable ∞ -category \mathcal{D} equipped with an accessible t-structure which is right complete and compatible with filtered colimits.

Now let $\mathcal{P}_{\text{rcpl}}^{\text{St},b}$ denote the ∞ -category whose objects are presentable stable ∞ -categories equipped with accessible t-structures which are right complete and compatible with filtered colimits, and whose morphisms are bounded colimit-preserving functors. Explicitly, given $\mathcal{C} \in \mathcal{P}^{\text{St}}$ we consider the set of cores $\mathcal{C}^{\leq 0}$ (subcategories closed under small colimits and extensions), partially ordered by $\mathcal{C}_1^{\leq 0} < \mathcal{C}_2^{\leq 0}$ if $\mathcal{C}_1^{\leq 0} \subset \mathcal{C}_2^{\leq 0}[n]$ for some n . These posets are contravariantly functorial under taking preimages along exact functors, and $\mathcal{P}_{\text{rcpl}}^{\text{St},b}$ is a full subcategory of the associated Cartesian fibration over \mathcal{P}^{St} .

We further let $\mathcal{P}_{\text{acpl}}^{\text{St},b}$ and $\mathcal{P}_{\text{cpl}}^{\text{St},b}$ denote the full subcategories of $\mathcal{P}_{\text{rcpl}}^{\text{St},b}$ defined by only including t-structures which are respectively left anticomplete and left complete. We then have the following variant of [Lur18, Cor. C.3.6.4, Cor. C.5.5.11, Prop. C.5.9.2].

Proposition 6.2. *The inclusion $\mathcal{P}_{\text{cpl}}^{\text{St},b} \hookrightarrow \mathcal{P}_{\text{rcpl}}^{\text{St},b}$ admits a left adjoint, which acts on objects by $\mathcal{C} \mapsto \widehat{\mathcal{C}}$. The inclusion $\mathcal{P}_{\text{acpl}}^{\text{St},b} \hookrightarrow \mathcal{P}_{\text{rcpl}}^{\text{St},b}$ admits a right adjoint, which we denote by $\mathcal{C} \mapsto \check{\mathcal{C}}$. These restrictions of these adjoints define inverse equivalences between $\mathcal{P}_{\text{cpl}}^{\text{St},b}$ and $\mathcal{P}_{\text{acpl}}^{\text{St},b}$.*

Proof. Given $\mathcal{C}, \mathcal{D} \in \mathcal{P}_{\text{rcpl}}^{\text{St},b}$, write $\text{LFun}^{b,\leq n}(\mathcal{C}, \mathcal{D}) \subset \text{LFun}^b(\mathcal{C}, \mathcal{D})$ for the full subcategory of functors which take $\mathcal{C}^{\leq 0}$ to $\mathcal{D}^{\leq n}$. Since $\mathcal{C} \rightarrow \widehat{\mathcal{C}}$ is t-exact, composition with it induces a functor $\text{LFun}^{b,\leq n}(\widehat{\mathcal{C}}, \mathcal{D}) \rightarrow \text{LFun}^{b,\leq n}(\mathcal{C}, \mathcal{D})$. The existence of the desired adjoint follows if this is an equivalence for all n and for all $\mathcal{D} \in \mathcal{P}_{\text{rcpl}}^{\text{St},b}$ [Lur09, Prop. 5.2.4.2]. By shifting we can reduce to $n = 0$. That composition with $\mathcal{C} \rightarrow \widehat{\mathcal{C}}$ induces an equivalence between right t-exact functors follows from [Lur18, Prop. C.3.1.1., Prop. C.3.6.3] (noting that $\mathcal{D} \cong \text{Sp}(\mathcal{D}^{\leq 0})$ [Lur18, Rem. C.3.1.5]). But this further identifies right t-exact functors which are bounded since $\mathcal{C}^{\geq 0} \xrightarrow{\sim} \widehat{\mathcal{C}}^{\geq 0}$.

Now let $\check{\mathcal{C}} := \text{Sp}(\check{\mathcal{C}}^{\leq 0}) \in \mathcal{P}_{\text{rcpl}}^{\text{St},b}$, where $\check{\mathcal{C}}^{\leq 0}$ is the anticompletion of $\mathcal{C}^{\leq 0}$ in the sense of [Lur18, Prop. C.5.5.9]. We claim $\check{\mathcal{C}}$ is left anticomplete in the sense of Definition 6.1. It suffices to show that for any $\mathcal{D} \in \mathcal{P}_{\text{rcpl}}^{\text{St},b}$, the functor $\text{LFun}^{b,\leq n}(\check{\mathcal{C}}, \mathcal{D}) \rightarrow \text{LFun}^{b,\leq n}(\check{\mathcal{C}}, \widehat{\mathcal{D}})$ given by composition with $\mathcal{D} \rightarrow \widehat{\mathcal{D}}$ is an equivalence for all n . By shifting we can reduce to the case $n = 0$, which follows from [Lur18, Prop. C.3.1.1, Def. C.5.5.4].

The left-exact functor $\check{\mathcal{C}}^{\leq 0} \rightarrow \mathcal{C}^{\leq 0}$ of [Lur18, Prop. C.5.5.9] induces a t-exact functor $\check{\mathcal{C}} \rightarrow \mathcal{C}$ [Lur18, Prop. C.3.1.1, Prop. C.3.2.1]. The existence of the desired adjoint follows if the induced functor $\text{LFun}^b(\mathcal{D}, \check{\mathcal{C}}) \rightarrow \text{LFun}^b(\mathcal{D}, \mathcal{C})$ is an equivalence for all $\mathcal{D} \in \mathcal{P}_{\text{rcpl}}^{\text{St},b}$ [Lur09, Prop. 5.2.4.2]. But this follows from Definition 6.1, given that the left completions of $\check{\mathcal{C}}$ and \mathcal{C} are equivalent [Lur18, Prop. C.5.5.9].

Finally, it follows by adjunction and Definition 6.1 that $\mathcal{C} \mapsto \widehat{\mathcal{C}}$ has fully faithful restriction to $\mathcal{P}_{\text{acpl}}^{\text{St},b}$, likewise for $\mathcal{C} \mapsto \check{\mathcal{C}}$ and $\mathcal{P}_{\text{cpl}}^{\text{St},b}$. That these restrictions are inverse equivalences now follows since $\mathcal{C} \in \mathcal{P}_{\text{cpl}}^{\text{St},b}$ implies \mathcal{C} is the left completion of $\check{\mathcal{C}}$ [Lur18, Prop. C.5.5.9]. \square

6.2. Ind-coherent sheaves. Recall that if X is a geometric stack, the standard t-structure on $\mathrm{QCoh}(X)$ is accessible, left and right complete, and compatible with filtered colimits [Lur18, Cor. 9.1.3.2].

Definition 6.3. The category of ind-coherent sheaves on a geometric stack X is

$$\mathrm{IndCoh}(X) := \widetilde{\mathrm{QCoh}(X)},$$

the left anticompletion of its category of quasicoherent sheaves.

Conversely, $\mathrm{QCoh}(X)$ is the left completion of $\mathrm{IndCoh}(X)$, and in particular the natural functor $\mathrm{IndCoh}(X) \rightarrow \mathrm{QCoh}(X)$ induces an equivalence

$$\mathrm{IndCoh}(X)^+ \xrightarrow{\sim} \mathrm{QCoh}(X)^+.$$

Unwinding the definitions, we see that $\mathrm{IndCoh}(X)$ is uniquely characterized by the following universal property: for all $\mathcal{C} \in \mathcal{P}\mathrm{r}_{\mathrm{repl}}^{\mathrm{St}, \mathrm{b}}$ we have

$$(6.4) \quad \mathrm{LFun}^b(\mathrm{IndCoh}(X), \mathcal{C}) \cong \mathrm{LFun}^b(\mathrm{QCoh}(X), \widehat{\mathcal{C}}).$$

By Proposition 6.2 ind-coherent sheaves inherit all bounded, colimit-preserving functorialities of quasicoherent sheaves. Explicitly, recall from the proof of Proposition ?? the functor $\mathrm{QCoh} : \mathrm{Corr}(\mathrm{GStk})_{fcd;all} \rightarrow \mathcal{P}\mathrm{r}^{\mathrm{St}}$, which takes a correspondence $X \xleftarrow{f} Y \xrightarrow{h} Z$ such that f is of finite cohomological dimension to the functor $f_*h^* : \mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(Z)$. By construction its restriction to $\mathrm{Corr}(\mathrm{GStk}_k)_{fcd;ftd}$, the subcategory which only includes correspondences in which h is of finite Tor-dimension, lifts to a functor

$$(6.5) \quad \mathrm{QCoh} : \mathrm{Corr}(\mathrm{GStk}_k)_{fcd;ftd} \rightarrow \mathcal{P}\mathrm{r}_{\mathrm{cpl}}^{\mathrm{St}, \mathrm{b}}.$$

Definition 6.6. We define a functor

$$(6.7) \quad \mathrm{IndCoh} : \mathrm{Corr}(\mathrm{GStk}_k)_{fcd;ftd} \rightarrow \mathcal{P}\mathrm{r}_{\mathrm{acpl}}^{\mathrm{St}, \mathrm{b}}.$$

by composing (6.5) with the equivalence $\mathcal{P}\mathrm{r}_{\mathrm{cpl}}^{\mathrm{St}, \mathrm{b}} \xrightarrow{\sim} \mathcal{P}\mathrm{r}_{\mathrm{acpl}}^{\mathrm{St}, \mathrm{b}}$ of Proposition 6.2.

If $f : X \rightarrow Y$ is a morphism of finite cohomological dimension in GStk_k , we denote the associated functor $\mathrm{IndCoh}(X) \rightarrow \mathrm{IndCoh}(Y)$ simply by f_* when the context is clear. If we need to distinguish it from the associated functor $\mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(Y)$ we denote them respectively by f_{IC*} and f_{QC*} . The two are related by a commutative diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(X) & \longrightarrow & \mathrm{QCoh}(X) \\ f_{IC*} \downarrow & & \downarrow f_{QC*} \\ \mathrm{IndCoh}(Y) & \longrightarrow & \mathrm{QCoh}(Y), \end{array}$$

where the horizontal functors are the canonical ones. A similar discussion applies to the $*$ -pullback functor associated to a morphism of finite Tor-dimension.

To extend the definitions to ind-geometric stacks, first consider the functor

$$(6.8) \quad \text{IndCoh} : \text{Corr}(\text{GStk}_k^+)_{fcd;ftd} \rightarrow \mathcal{P}\mathbf{r}^{\mathbf{L}}$$

obtained by restricting (6.7) to correspondences of truncated geometric stacks and composing with the forgetful functor $\mathcal{P}\mathbf{r}_{\text{acpl}}^{\text{St},b} \rightarrow \mathcal{P}\mathbf{r}^{\mathbf{L}}$. Now let $\text{Corr}(\text{indGStk}_k)_{fcd;ftd}$ denote the subcategory of $\text{Corr}(\text{indGStk}_k)$ which only includes correspondences $X \xleftarrow{f} Y \xrightarrow{h} Z$ such that f is of ind-finite cohomological dimension and h is geometric and of finite Tor-dimension (these are stable under composition of correspondences by Propositions 4.10 and 4.20). By Proposition 4.11 $\text{Corr}(\text{GStk}_k^+)_{fcd;ftd}$ is a full subcategory of $\text{Corr}(\text{indGStk}_k)_{fcd;ftd}$.

Definition 6.9. We define a functor

$$(6.10) \quad \text{IndCoh} : \text{Corr}(\text{indGStk}_k)_{fcd;ftd} \rightarrow \mathcal{P}\mathbf{r}^{\mathbf{L}}$$

by left Kan extending (6.8) along $\text{Corr}(\text{GStk}_k^+)_{fcd;ftd} \subset \text{Corr}(\text{indGStk}_k)_{fcd;ftd}$.

If $f : X \rightarrow Y$ is a morphism of ind-geometric stacks which is of ind-finite cohomological dimension (resp. geometric and of finite Tor dimension), we write $f_* : \text{IndCoh}(X) \rightarrow \text{IndCoh}(Y)$ (resp. $f^* : \text{IndCoh}(Y) \rightarrow \text{IndCoh}(X)$) for the functor defined by Definition 6.9. We have the following variant of Proposition 5.4, which is proved the same way. Here we write $\text{indGStk}_{k,fcd} \subset \text{indGStk}_k$ for the subcategory which only includes morphisms of ind-finite cohomological dimension, indentifying it with a subcategory $\text{Corr}(\text{indGStk}_k)_{fcd;ftd}$ as before.

Proposition 6.11. *The restriction of IndCoh to $\text{indGStk}_{k,fcd}$ preserves filtered colimits along ind-closed immersions.*

In particular, if $X \cong \text{colim } X_\alpha$ is an ind-geometric presentation, then the functors $i_{\alpha*} : \text{IndCoh}(X_\alpha) \rightarrow \text{IndCoh}(X)$ induce an isomorphism

$$(6.12) \quad \text{IndCoh}(X) \cong \text{colim } \text{IndCoh}(X_\alpha)$$

in $\mathcal{P}\mathbf{r}^{\mathbf{L}}$.

6.3. !-pullback and t-structures. If a morphism $f : X \rightarrow Y$ ind-geometric stacks is ind-proper (hence of ind-finite cohomological dimension by Proposition 3.5), we write $f^! : \text{IndCoh}(Y) \rightarrow \text{IndCoh}(X)$ for the right adjoint of f_* . For ind-geometric X we define a standard t-structure on $\text{IndCoh}(X)$ in terms of these adjoints, following [GR17b, Sec. I.3.1.2].

Proposition 6.13. *If X is an ind-geometric stack, $\text{IndCoh}(X)$ has a t-structure defined by*

$$\text{IndCoh}(X)^{\geq 0} := \left\{ \mathcal{F} \in \text{IndCoh}(X) \text{ such that } i^!(\mathcal{F}) \in \text{IndCoh}(X')^{\geq 0} \text{ for any } \right. \\ \left. \text{truncated geometric substack } i : X' \rightarrow X \right\}.$$

This t-structure is accessible, compatible with filtered colimits, right complete, and left anticomplete. If $X \cong \text{colim}_\alpha X_\alpha$ is an ind-geometric presentation, the functors i_{α} are t-exact*

and induce equivalences

$$(6.14) \quad \mathrm{IndCoh}(X)^{\leq 0} \cong \operatorname{colim}_{\alpha} \mathrm{IndCoh}(X_{\alpha})^{\leq 0}, \quad \mathrm{IndCoh}(X)^{\geq 0} \cong \operatorname{colim}_{\alpha} \mathrm{IndCoh}(X_{\alpha})^{\geq 0}$$

in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.

Lemma 6.15. *Let $\mathcal{P}\mathrm{r}_{\mathrm{rcpl}}^{\mathrm{St}, \mathrm{t-ex}} \subset \mathcal{P}\mathrm{r}_{\mathrm{rcpl}}^{\mathrm{St}, \mathrm{b}}$ denote the subcategory which only includes t-exact functors. Then $\mathcal{P}\mathrm{r}_{\mathrm{rcpl}}^{\mathrm{St}, \mathrm{t-ex}}$ admits filtered colimits, and these are preserved by the functors to $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ given by $\mathcal{C} \mapsto \mathcal{C}$, $\mathcal{C} \mapsto \mathcal{C}^{\leq 0}$, and $\mathcal{C} \mapsto \mathcal{C}^{\geq 0}$. Moreover, the subcategory $\mathcal{P}\mathrm{r}_{\mathrm{acpl}}^{\mathrm{St}, \mathrm{t-ex}} \subset \mathcal{P}\mathrm{r}_{\mathrm{rcpl}}^{\mathrm{St}, \mathrm{t-ex}}$ which only includes left anticomplete t-structures is closed under filtered colimits.*

Proof. Let $\mathrm{Groth}_{\infty}^{\mathrm{lex}}$ denote the category of Grothendieck prestable ∞ -categories and left-exact functors. By [Lur18, Rem. C.3.1.5, Prop. C.3.2.1] $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$ and $\mathcal{C} \mapsto \mathcal{C}^{\leq 0}$ induce inverse equivalences of $\mathrm{Groth}_{\infty}^{\mathrm{lex}}$ and $\mathcal{P}\mathrm{r}_{\mathrm{rcpl}}^{\mathrm{St}, \mathrm{t-ex}}$. The claims about $\mathcal{C} \mapsto \mathcal{C}$ and $\mathcal{C} \mapsto \mathcal{C}^{\leq 0}$ now follow since $\mathrm{Groth}_{\infty}^{\mathrm{lex}}$ is closed under filtered colimits in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ [Lur18, Prop. C.3.3.5], and since $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$ preserves small colimits in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ [Lur17, Ex. 4.8.1.22].

Let $\mathcal{C} \cong \operatorname{colim} \mathcal{C}_{\alpha}$ be a filtered colimit in $\mathcal{P}\mathrm{r}_{\mathrm{rcpl}}^{\mathrm{St}, \mathrm{t-ex}}$. Since the structure functors $F_{\alpha\beta} : \mathcal{C}_{\alpha} \rightarrow \mathcal{C}_{\beta}$, $F_{\alpha} : \mathcal{C}_{\alpha} \rightarrow \mathcal{C}$ are t-exact their right adjoints are left t-exact. Then since $\mathcal{C} \cong \lim \mathcal{C}_{\alpha}$ in $\mathcal{P}\mathrm{r}^{\mathrm{R}}$ and since the inclusions $\mathcal{C}_{\alpha}^{\geq 0} \subset \mathcal{C}_{\alpha}$ are morphisms in $\mathcal{P}\mathrm{r}^{\mathrm{R}}$, they identify the limit of the $\mathcal{C}_{\alpha}^{\geq 0}$ in $\mathcal{P}\mathrm{r}^{\mathrm{R}}$ as the full subcategory of $X \in \mathcal{C}$ such that $F_{\alpha}^{\mathrm{R}}(X) \in \mathcal{C}_{\alpha}^{\geq 0}$ for all α . This is equivalent to $\mathrm{Map}_{\mathcal{C}}(F_{\alpha}(Y_{\alpha}), X) \cong 0$ for all α and all $Y_{\alpha} \in \mathcal{C}_{\alpha}^{\leq 0}$. But this is equivalent to $\mathrm{Map}_{\mathcal{C}}(Y, X) \cong 0$ for all $Y \in \mathcal{C}^{\leq 0}$, since by Lemma 2.2 and the previous paragraph $\mathcal{C}^{\leq 0}$ is generated under small colimits by objects of the form $F_{\alpha}(Y_{\alpha})$ with $Y_{\alpha} \in \mathcal{C}_{\alpha}^{\leq 0}$. It follows that $\mathcal{C}^{\geq 0} \cong \operatorname{colim} \mathcal{C}_{\alpha}^{\geq 0}$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Finally, it follows from [Lur18, Cor. C.5.5.10] and the discussion above that $\mathcal{P}\mathrm{r}_{\mathrm{acpl}}^{\mathrm{St}, \mathrm{t-ex}}$ is closed under all colimits that exist in $\mathcal{P}\mathrm{r}_{\mathrm{rcpl}}^{\mathrm{St}, \mathrm{t-ex}}$. \square

Proof of Proposition 6.13. Let $\mathrm{indGStk}_{k, cl} \subset \mathrm{indGStk}_k$ denote the subcategory which only includes ind-closed immersions, similarly for $\mathrm{GStk}_{k, cl}^+ \subset \mathrm{GStk}_k^+$. By construction the restriction of (6.7) to $\mathrm{GStk}_{k, cl}^+$ factors through $\mathcal{P}\mathrm{r}_{\mathrm{acpl}}^{\mathrm{St}, \mathrm{t-ex}}$, and we write $\mathrm{IndCoh}^t : \mathrm{indGStk}_{k, cl} \rightarrow \mathcal{P}\mathrm{r}_{\mathrm{acpl}}^{\mathrm{St}, \mathrm{t-ex}}$ for its left Kan extension. Adapting again the proof of Proposition 5.4, we find that $\mathrm{IndCoh}^t(X) \cong \operatorname{colim} \mathrm{IndCoh}^t(X_{\alpha})$ in $\mathcal{P}\mathrm{r}_{\mathrm{acpl}}^{\mathrm{St}, \mathrm{t-ex}}$. But by Proposition 6.11 $\mathrm{IndCoh}(X)$ is the underlying category of $\mathrm{IndCoh}^t(X)$. The claims now follow from Lemma 6.15 and Proposition 4.5 (the Lemma ensures each $i_{\alpha}^!$ is left t-exact, the Proposition ensures $i^!$ is left t-exact for all $i : X' \rightarrow X$). \square

Remark 6.16. Since $\mathrm{IndCoh}(X)$ is left anticomplete, it can be recovered functorially from $\mathrm{IndCoh}(X)^+$. By contrast, let $\mathrm{QCoh}'(X)$ denote the colimit of the categories $\mathrm{QCoh}(X_{\alpha})$ in $\mathcal{P}\mathrm{r}_{\mathrm{rcpl}}^{\mathrm{St}, \mathrm{t-ex}}$. Left completeness is stable under limits rather than colimits in $\mathcal{P}\mathrm{r}_{\mathrm{rcpl}}^{\mathrm{St}, \mathrm{t-ex}}$, so $\mathrm{QCoh}'(X)$ will in general be neither left complete nor left anticomplete. In particular, $\mathrm{QCoh}'(X)$ is wild in so far as it cannot be recovered functorially from its bounded below objects.

Remark 6.17. Let X be a classical ind-geometric stack and $X \cong \operatorname{colim}_{\alpha} X_{\alpha}$ an ind-geometric presentation by classical geometric stacks. By construction we have $\mathrm{IndCoh}(X_{\alpha})^{\heartsuit} \cong$

$\mathrm{QCoh}(X_\alpha)^\heartsuit$ for all α , and by Proposition 6.13 we have $\mathrm{IndCoh}(X)^\heartsuit \cong \mathrm{colim}_\alpha \mathrm{IndCoh}(X_\alpha)^\heartsuit$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. It follows from [Lur18, Cor. 10.4.6.8, Prop. C.5.5.20, Thm. C.5.8.8] that $\mathrm{IndCoh}(X)$ is the dg nerve of the category of injective complexes in $\mathrm{IndCoh}(X)^\heartsuit$, a construction first considered in [Kra05].

Using t-structures we can address the potential ambiguity in the definition of $\mathrm{IndCoh}(X)$ when X is a non-truncated geometric stack.

Proposition 6.18. *Given a non-truncated geometric stack X , the categories $\mathrm{IndCoh}(X)$ defined by Definitions 6.3 and 6.9 are canonically equivalent, and this equivalence identifies the t-structure of the former with that of Proposition 6.13.*

Proof. Temporarily denote the two categories by $\mathrm{IndCoh}_{\mathrm{geom}}(X)$ and $\mathrm{IndCoh}_{\mathrm{ind}}(X)$. For each n the morphism $i_n : \tau_{\leq n}X \rightarrow X$ is affine, hence yields a t-exact functor $i_{n*} : \mathrm{IndCoh}(\tau_{\leq n}X) \rightarrow \mathrm{IndCoh}_{\mathrm{geom}}(X)$. By Propositions 6.11 and 6.13 it suffices to show the induced t-exact functor

$$(6.19) \quad \mathrm{IndCoh}_{\mathrm{ind}}(X) \cong \mathrm{colim} \mathrm{IndCoh}(\tau_{\leq n}X) \rightarrow \mathrm{IndCoh}_{\mathrm{geom}}(X)$$

is an equivalence. Now for all $a \leq b$ (6.19) restricts to an equivalence $\mathrm{IndCoh}_{\mathrm{ind}}(X)^{[a,b]} \cong \mathrm{IndCoh}_{\mathrm{geom}}(X)^{[a,b]}$, since for all $n \geq b - a$ we have

$$\mathrm{IndCoh}(\tau_{\leq n}X)^{[a,b]} \cong \mathrm{QCoh}(\tau_{\leq n}X)^{[a,b]} \cong \mathrm{QCoh}(X)^{[a,b]} \cong \mathrm{IndCoh}(X)_{\mathrm{geom}}^{[a,b]}.$$

By right completeness of the two t-structures it follows that (6.19) restricts to an equivalence $\mathrm{IndCoh}_{\mathrm{ind}}(X)^{\geq a} \cong \mathrm{IndCoh}_{\mathrm{geom}}(X)^{\geq a}$ for any a . But then (6.19) induces an equivalence of left completions, hence by Proposition 6.2 is itself an equivalence since its source and target are left anticomplete. \square

If $f : X \rightarrow Y$ is ind-proper, the functor $f^!$ is not obviously continuous in general, but its failure to be so can be controlled by t-structures.

Definition 6.20. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between presentable stable ∞ -categories equipped with t-structures that are compatible with filtered colimits. Then F is almost continuous if its restriction to $\mathcal{C}^{\geq n}$ is continuous for all n .

Proposition 6.21. *Let $f : X \rightarrow Y$ be an ind-proper morphism of semi-reasonable ind-geometric stacks. Suppose either that f is geometric and almost of finite presentation, or that X is reasonable and f is of finite cohomological dimension and almost of ind-finite presentation. Then $f^!$ is almost continuous.*

The claim results from the following variant of [Lur18, Prop. 6.4.1.4], which we generalize further in Section ???. Here the Beck-Chevalley transformation $h'^* f^! \rightarrow f'^! h^*$ is the result of passing to adjoints from the natural isomorphism $f'_* h'^* \cong h^* f_*$.

Lemma 6.22. *Let the following be a Cartesian diagram of geometric stacks.*

$$(6.23) \quad \begin{array}{ccc} X' & \xrightarrow{f'} & \mathrm{Spec} A \\ h' \downarrow & & \downarrow h \\ X & \xrightarrow{f} & Y \end{array}$$

Suppose h is a flat cover and that f is proper and almost of finite presentation. Then for any $\mathcal{F} \in \mathrm{IndCoh}(Y)^+$ the Beck-Chevalley map $h'^ f^!(\mathcal{F}) \rightarrow f'^! h^*(\mathcal{F})$ is an isomorphism.*

Proof. Write $\mathrm{Spec} A^\bullet$ for the Čech nerve of h . Let $h_k : \mathrm{Spec} A^k \rightarrow Y$ denote the natural map, $h'_k : X_k \rightarrow X$ its base change along f , and $f_k : X_k \rightarrow \mathrm{Spec} A^k$ the base change of f along h_k (so $h_0 = h$, $h'_0 = h'$, and $f_0 = f'$). Finally, let $h_p : \mathrm{Spec} A^j \rightarrow \mathrm{Spec} A^i$ denote the map associated to a morphism $p : i \rightarrow j$ in Δ_s , and $h'_p : X_j \rightarrow X_i$ its base change along f_i .

Choose m so that f_* takes $\mathrm{IndCoh}(X)^{\leq 0}$ to $\mathrm{IndCoh}(Y)^{\leq m}$. The same holds for each f_{j*} since h_{j*} is t-exact and its restriction to $\mathrm{IndCoh}(X)^+$ is conservative, hence $f_j^!$ takes $\mathrm{IndCoh}(\mathrm{Spec} A^j)^{\geq 0}$ to $\mathrm{IndCoh}(X_j)^{\geq -m}$. Choosing n so that $\mathcal{F} \in \mathrm{IndCoh}(Y)^{\geq n}$, the Beck-Chevalley map $h_p^* f_i^! \rightarrow f_j^! h_p^*$ in $\mathrm{Fun}(\mathrm{IndCoh}(\mathrm{Spec} A^i)^{\geq n}, \mathrm{IndCoh}(X_j)^{\geq n-m})$ is an isomorphism for any p , since h_p is a map of affine schemes [Lur18, Prop. 6.4.1.4]. Since h is faithfully flat, we have $\mathrm{IndCoh}(Y)^{\geq n} \cong \lim_{\Delta_s} \mathrm{IndCoh}(\mathrm{Spec} A^i)^{\geq n}$ and $\mathrm{IndCoh}(X)^{\geq n-m} \cong \lim_{\Delta_s} \mathrm{IndCoh}(X_i)^{\geq n-m}$. The claim now follows from [Lur17, Cor. 4.7.5.18]. \square

Proof of Proposition 6.21. First suppose X and Y are geometric and f is almost of finite presentation, and fix a Cartesian square of the form (6.23). Since $f^!$ is left bounded, and since h'^* is conservative on $\mathrm{IndCoh}(X)^+$ in addition to being continuous, it suffices to show $h'^* f^!$ is almost continuous. It then suffices to show $f'^! h^*$ is almost continuous, since by Lemma 6.22 their restrictions to $\mathrm{IndCoh}(Y)^+$ are the same. But this follows since h^* is continuous and left t-exact, and since $f'^!$ is almost continuous by [Lur18, Lem. 6.4.1.5].

Now let $Y \cong \mathrm{colim} Y_\alpha$ be a semi-reasonable presentation. By the proof of [Lur09, Prop. 5.5.3.13] categories admitting filtered colimits and continuous functors among them form a subcategory of $\widehat{\mathrm{Cat}}_\infty$ which is closed under limits. By the previous paragraph $i_{\alpha\beta}^!$ restricts to a continuous functor $\mathrm{IndCoh}(Y_\beta)^{\geq n} \rightarrow \mathrm{IndCoh}(Y_\alpha)^{\geq n}$ for all $\beta \geq \alpha$ and any n . But the proof of Proposition 6.11 extends to show $\mathrm{IndCoh}(Y)^{\geq n} \cong \lim \mathrm{IndCoh}(Y_\alpha)^{\geq n}$, hence $i_\alpha^!$ is almost continuous for any α .

If f is geometric and almost of finite presentation, then letting $X_\alpha := X \times_Y Y_\alpha$ we have a semi-reasonable presentation $X \cong \mathrm{colim} X_\alpha$ by left exactness of filtered colimits in $\widehat{\mathrm{Stk}}_k$. Let $\mathcal{F} \cong \mathrm{colim} \mathcal{F}_\beta$ be a filtered colimit in $\mathrm{IndCoh}(Y)^{\geq n}$ for some n . Since $\mathrm{IndCoh}(X) \cong \lim \mathrm{IndCoh}(X_\alpha)$ it suffices to show the second factor of

$$\mathrm{colim}_\beta i_\alpha^! f^!(\mathcal{F}_\beta) \rightarrow i_\alpha^! \mathrm{colim}_\beta f^!(\mathcal{F}_\beta) \rightarrow i_\alpha^! f^!(\mathrm{colim}_\beta \mathcal{F}_\beta)$$

is an isomorphism for all α , where $i'_\alpha : X_\alpha \rightarrow X$ is the natural map. Since i'_α is almost continuous by the previous paragraph, and since $f^!$ is left bounded, the first factor is an isomorphism. It now suffices to show $i'_\alpha f^!$ is almost continuous, since then the composition is an isomorphism. But $i'_\alpha f^! \cong f'_\alpha i'_\alpha$, where $f'_\alpha : X_\alpha \rightarrow Y_\alpha$ is the base change of f , and $f'_\alpha i'_\alpha$ is almost continuous by the previous paragraph and left t-exactness of i'_α .

Finally, suppose $X \cong \text{colim } X_\gamma$ is a reasonable presentation and f is of finite cohomological dimension and almost of ind-finite presentation. Then for any γ we can factor $f \circ i_\gamma$ through a proper, almost finitely presented map $f_{\gamma\alpha} : X_\gamma \rightarrow Y_\alpha$ for some α , and the claim follows by the same argument as above. \square

6.4. Pushforward and !-pullback. We turn next to the commutation of pushforward and !-pullback. The proofs below largely follow those of corresponding results for ind-schemes [Gai13, Prop. 3.4.2], [GR14, Prop. 2.9.2], [Ras19, Lem. 6.17.2].

Proposition 6.24. *Let the following be a Cartesian diagram of ind-geometric stacks.*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ h' \downarrow & & \downarrow h \\ X & \xrightarrow{f} & Y \end{array}$$

Suppose that X and Y are each geometric or reasonable, that h is of ind-finite cohomological dimension, and that f is ind-proper, almost finitely presented, and of finite cohomological dimension. Then for any $\mathcal{F} \in \text{IndCoh}(Y')^+$ the Beck-Chevalley map $h'_ f^!(\mathcal{F}) \rightarrow f^! h_*(\mathcal{F})$ is an isomorphism.*

Proof. Suppose first that X' and Y' are geometric. Passing through the equivalences $\text{IndCoh}(-)^+ \cong \text{QCoh}(-)^+$ it suffices to show the natural transformation $h'_{QC*} f'_{QC}{}^! \rightarrow f_{QC}^! h_{QC*}$ in $\text{Fun}(\text{QCoh}(Y'), \text{QCoh}(X))$ is an isomorphism, $f'_{QC}{}^!$ and $f_{QC}^!$ denoting the right adjoints of f_{QC*} and f_{QC*} . But this follows since it is obtained from the isomorphism $f'_{QC*} h_{QC}^* \xrightarrow{\sim} h_{QC}^* f_{QC*}$ in $\text{Fun}(\text{QCoh}(X), \text{QCoh}(Y'))$ by passing to right adjoints.

Now let $Y' \cong \text{colim } Y'_\alpha$ be an ind-geometric presentation. For every α we have a diagram

$$\begin{array}{ccccc} Y'_\alpha & \xrightarrow{i_\alpha} & Y' & \xrightarrow{h} & Y \\ f_\alpha \uparrow & & \uparrow f' & & \uparrow f \\ X'_\alpha & \xrightarrow{i'_\alpha} & X' & \xrightarrow{h'} & X \end{array}$$

of Cartesian squares. Expanding the Beck-Chevalley maps in terms of units and counits, it follows from the basic properties of these that the counits $i_{\alpha*} i'_\alpha{}^! \rightarrow \text{id}_{IC(Y')}$ and $i'_{\alpha*} i'_\alpha{}^! \rightarrow \text{id}_{IC(X')}$

fit into diagrams

$$(6.25) \quad \begin{array}{ccccc} h'_* i'_{\alpha*} i'^!_{\alpha} f^! (\mathcal{F}) & \xrightarrow{\sim} & h'_* i'_{\alpha*} f^! i'^!_{\alpha} (\mathcal{F}) & \longrightarrow & f^! h_* i_{\alpha*} i^!_{\alpha} (\mathcal{F}) \\ \downarrow & & & & \downarrow \\ h'_* f^! (\mathcal{F}) & \longrightarrow & & & f^! h_* (\mathcal{F}), \end{array}$$

compatibly for $\beta \geq \alpha$. That is, the counits $i_{\alpha*} i^!_{\alpha} \cong i_{\beta*} i_{\alpha\beta*} i^!_{\alpha\beta} i^!_{\beta} \rightarrow i_{\beta*} i^!_{\beta}$ and $i'_{\alpha*} i'^!_{\alpha} \cong i'_{\beta*} i'_{\alpha\beta*} i'^!_{\alpha\beta} i'^!_{\beta} \rightarrow i'_{\beta*} i'^!_{\beta}$ intertwine the top compositions in (6.25) for $\beta \geq \alpha$. Passing to colimits we obtain a diagram

$$\begin{array}{ccc} \operatorname{colim} h'_* i'_{\alpha*} i'^!_{\alpha} f^! (\mathcal{F}) & \longrightarrow & \operatorname{colim} f^! h_* i_{\alpha*} i^!_{\alpha} (\mathcal{F}) \\ \downarrow & & \downarrow \\ h'_* f^! (\mathcal{F}) & \longrightarrow & f^! h_* (\mathcal{F}). \end{array}$$

The vertical arrows are isomorphisms by Lemma 2.2, the continuity of h_* and h'_* , and the left t-exactness of $h_* i_{\alpha*} i^!_{\alpha}$ together with the almost continuity of $f^!$ (Proposition 6.21). But the top arrow is an isomorphism since the top right arrow in (6.25) is by the first paragraph, hence so is the bottom arrow.

First note that the coherence hypotheses imply that $f^!$ and $f^!$ are continuous. If X and Y are geometric, the claim now follows immediately from Proposition 6.24 since $\operatorname{IndCoh}(Y')$ is compactly generated by $\operatorname{Coh}(Y')$.

Now suppose that X is truncated and geometric, and let $Y \cong \operatorname{colim} Y_{\alpha}$ be a reasonable presentation. Define $h_{\alpha} : Y'_{\alpha} \rightarrow Y_{\alpha}$, $i'_{\alpha} : Y'_{\alpha} \rightarrow Y'$ by base change, similarly $i'_{\alpha\beta} : Y'_{\alpha} \rightarrow Y'_{\beta}$ for $\beta \geq \alpha$. Each Y'_{α} is coherent by Proposition 7.7, so by the previous paragraph we have $h_{\alpha*} i'^!_{\alpha\beta} \cong i'^!_{\alpha\beta} h_{\beta*}$ in $\operatorname{Fun}(\operatorname{IndCoh}(Y'_{\beta}), \operatorname{IndCoh}(Y_{\alpha}))$. For any α we then have $h_{\alpha*} i'^!_{\alpha} \cong i'^!_{\alpha} h_*$ in $\operatorname{Fun}(\operatorname{IndCoh}(Y'), \operatorname{IndCoh}(Y_{\alpha}))$ by [Lur17, Prop. 4.7.5.19]. By Proposition 4.3 we can factor f as $X \xrightarrow{f_{\alpha}} Y_{\alpha} \xrightarrow{i_{\alpha}} Y$ for some α . Consider the diagram

$$\begin{array}{ccccc} X' & \xrightarrow{f'_{\alpha}} & Y'_{\alpha} & \xrightarrow{i'_{\alpha}} & Y' \\ h' \downarrow & & \downarrow h_{\alpha} & & \downarrow h \\ X & \xrightarrow{f_{\alpha}} & Y_{\alpha} & \xrightarrow{i_{\alpha}} & Y \end{array}$$

of Cartesian squares. The map in the statement now factors as

$$h'_* f^! (\mathcal{F}) \cong h'_* f'_{\alpha} i'^!_{\alpha} (\mathcal{F}) \rightarrow f'_{\alpha} h_{\alpha*} i'^!_{\alpha} (\mathcal{F}) \rightarrow f'_{\alpha} i'^!_{\alpha} h_* (\mathcal{F}) \cong f^! h_* (\mathcal{F}),$$

and we have already shown both factors are isomorphisms.

Now suppose $X \cong \operatorname{colim} X_\alpha$ is a reasonable presentation. For any α we have a diagram

$$\begin{array}{ccccc} X'_\alpha & \xrightarrow{i'_\alpha} & X' & \xrightarrow{f'} & Y' \\ h'_\alpha \downarrow & & \downarrow h' & & \downarrow h \\ X_\alpha & \xrightarrow{i_\alpha} & X & \xrightarrow{f} & Y \end{array}$$

of Cartesian squares. Each X'_α is coherent by Proposition 7.7, and $X' \cong \operatorname{colim} X'_\alpha$ since filtered colimits are left exact in $\widehat{\operatorname{Stk}}_k$. Since the functors i'_α determine an isomorphism $\operatorname{IndCoh}(X') \cong \lim_\alpha \operatorname{IndCoh}(X'_\alpha)$ in $\widehat{\operatorname{Cat}}_\infty$, it suffices to show that $i'_\alpha h'_* f'^!(\mathcal{F}) \rightarrow i_\alpha f^! h_*(\mathcal{F})$ is an isomorphism for all α . But by the previous paragraph both the composition

$$h'_{\alpha*} i'_\alpha f'^!(\mathcal{F}) \rightarrow i'_\alpha h'_* f'^!(\mathcal{F}) \rightarrow i'_\alpha f^! h_*(\mathcal{F})$$

and its first factor are isomorphisms, hence the second is as well. \square

6.5. Relation to coherent sheaves. We now turn to the relationship between $\operatorname{Coh}(X)$ and $\operatorname{IndCoh}(X)$ when X is reasonable. We recall from [Lur18, Prop. 9.1.5.1] that when X is geometric, $\operatorname{Coh}(X)$ can be characterized as the full subcategory of bounded, almost compact objects in $\operatorname{QCoh}(X)$ (i.e. of $\mathcal{F} \in \operatorname{QCoh}(X)^+$ such that $\tau^{\geq n} \mathcal{F}$ is compact in $\operatorname{QCoh}(X)^{\geq n}$ for all n). The equivalence $\operatorname{IndCoh}(X)^+ \cong \operatorname{QCoh}(X)^+$ thus also identifies $\operatorname{Coh}(X)$ with the full subcategory of bounded, almost compact objects in $\operatorname{IndCoh}(X)$.

Proposition 6.26. *For any reasonable ind-geometric stack X there is a canonical fully faithful functor $\operatorname{Coh}(X) \hookrightarrow \operatorname{IndCoh}(X)$. Its essential image consists of bounded, almost compact objects. It is induced from a canonical natural transformation between the functors $\operatorname{Corr}(\operatorname{indGStk}_k^{\operatorname{reas}})_{\operatorname{prop}; \operatorname{ftd}} \rightarrow \widehat{\operatorname{Cat}}_\infty$ given by restricting the domains of (5.9) and (6.10) and composing them with the natural functors to $\widehat{\operatorname{Cat}}_\infty$. In particular, we have a diagram*

$$\begin{array}{ccc} \operatorname{Coh}(X') & \hookrightarrow & \operatorname{IndCoh}(X') \\ i_* \downarrow & & \downarrow i_* \\ \operatorname{Coh}(X) & \hookrightarrow & \operatorname{IndCoh}(X), \end{array}$$

for any reasonable geometric substack $i : X' \rightarrow X$.

Proof. By construction the inclusion $\operatorname{Coh}(X) \subset \operatorname{IndCoh}(X)$ for geometric X enhances to a natural transformation of functors $\operatorname{Corr}(\operatorname{GStk}_k^+)_{\operatorname{prop}; \operatorname{ftd}} \rightarrow \widehat{\operatorname{Cat}}_\infty$. The variant of (5.9) appearing in the statement is the left Kan extension of its restriction to $\operatorname{Corr}(\operatorname{GStk}_k^+)_{\operatorname{prop}; \operatorname{ftd}}$, since following the proof of Proposition 5.4 the restrictions of both to $\operatorname{indGStk}_{k, \operatorname{prop}}$ are left Kan extended from $\operatorname{GStk}_{k, \operatorname{prop}}^+$. The desired natural transformation and the pictured diagram then follow from the characteristic adjunction of left Kan extensions [Lur09, Prop. 4.3.2.17].

Fix a reasonable presentation $X \cong \operatorname{colim}_\alpha X_\alpha$. Given $\mathcal{F} \in \operatorname{Coh}(X)$ we can write $\mathcal{F} \cong i_{\alpha*}(\mathcal{F}_\alpha)$ for some α and some $\mathcal{F}_\alpha \in \operatorname{Coh}(X_\alpha)$. The t-exactness of $i_{\alpha*}$ implies \mathcal{F} is bounded in $\operatorname{IndCoh}(X)$, while the almost continuity of $i_\alpha^!$ (Proposition 6.21) implies \mathcal{F} is almost compact.

Given $\mathcal{G} \in \text{Coh}(X)$ we can write $\mathcal{G} \cong i_{\alpha*}(\mathcal{G}_{\alpha})$ for some $\mathcal{G}_{\alpha} \in \text{Coh}(X_{\alpha})$, increasing α if needed. Identifying \mathcal{F}, \mathcal{G} with their images in $\text{IndCoh}(X)$ we then have

$$\begin{aligned} \text{Map}_{IC(X)}(\mathcal{F}, \mathcal{G}) &\cong \text{Map}_{IC(X_{\alpha})}(\mathcal{F}_{\alpha}, i_{\alpha}^! i_{\alpha*}(\mathcal{G}_{\alpha})) \\ &\cong \text{Map}_{IC(X_{\alpha})}(\mathcal{F}_{\alpha}, \text{colim}_{\beta \geq \alpha} i_{\alpha\beta}^! i_{\alpha\beta*}(\mathcal{G}_{\alpha})) \\ &\cong \text{colim}_{\beta \geq \alpha} \text{Map}_{IC(X_{\alpha})}(\mathcal{F}_{\alpha}, i_{\alpha\beta}^! i_{\alpha\beta*}(\mathcal{G}_{\alpha})) \\ &\cong \text{colim}_{\beta \geq \alpha} \text{Map}_{IC(X_{\beta})}(i_{\alpha\beta*}(\mathcal{F}_{\alpha}), i_{\alpha\beta*}(\mathcal{G}_{\alpha})). \end{aligned}$$

Here the second isomorphism follows from Proposition 6.11 and [GR17b, Lemma 1.1.10], and the third follows since \mathcal{F}_{α} and \mathcal{G}_{α} are coherent and each $i_{\alpha\beta}^! i_{\alpha\beta*}$ is left t-exact. But since $\text{Coh}(X_{\beta})$ is a full subcategory of $\text{IndCoh}(X_{\beta})$ for all $\beta \geq \alpha$, the last expression is equivalent to $\text{Map}_{\text{Coh}(X)}(\mathcal{F}, \mathcal{G})$ [Roz, Lem. 0.2.1]. \square

6.6. The pushforward/ $*$ -pullback adjunction. Suppose $f : X \rightarrow Y$ is both of finite Tor-dimension and ind-finite cohomological dimension. Then we have separately defined functors f_{IC*} and f_{IC}^* , but the definition does not explicitly entail any direct relationship between them. Nonetheless, the two functors are adjoint in the expect way.

Proposition 6.27. *Let X and Y be ind-geometric stacks and $f : X \rightarrow Y$ a morphism which is both of finite Tor-dimension and of ind-finite cohomological dimension. Then f_{IC*} is right adjoint to f_{IC}^* .*

Lemma 6.28. *Let $\widehat{\mathcal{C}}, \widehat{\mathcal{D}}$ be the left completions of $\check{\mathcal{C}}, \check{\mathcal{D}} \in \mathcal{P}\text{r}_{\text{acpl}}^{\text{St}, b}$. Let $\check{F} : \check{\mathcal{C}} \rightarrow \check{\mathcal{D}}$ and $\check{G} : \check{\mathcal{D}} \rightarrow \check{\mathcal{C}}$ be bounded colimit-preserving functors, and let $\widehat{F} : \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}$ and $\widehat{G} : \widehat{\mathcal{D}} \rightarrow \widehat{\mathcal{C}}$ be their images under the equivalences $\text{LFun}^b(\check{\mathcal{C}}, \check{\mathcal{D}}) \cong \text{LFun}^b(\widehat{\mathcal{C}}, \widehat{\mathcal{D}})$ and $\text{LFun}^b(\check{\mathcal{D}}, \check{\mathcal{C}}) \cong \text{LFun}^b(\widehat{\mathcal{D}}, \widehat{\mathcal{C}})$. Then \check{G} is right adjoint to \check{F} if and only if \widehat{G} is right adjoint to \widehat{F} . If this is the case, the equivalences $\text{LFun}^b(\check{\mathcal{C}}, \check{\mathcal{C}}) \cong \text{LFun}^b(\widehat{\mathcal{C}}, \widehat{\mathcal{C}})$ and $\text{LFun}^b(\check{\mathcal{D}}, \check{\mathcal{D}}) \cong \text{LFun}^b(\widehat{\mathcal{D}}, \widehat{\mathcal{D}})$ together identify pairs of a compatible unit and counit for the two adjunctions.*

Proof. We consider the if direction, the other being symmetric. Recall that \widehat{G} being right adjoint to \widehat{F} is equivalent to the existence of unit and counit transformations $\widehat{u} : id_{\widehat{\mathcal{D}}} \rightarrow \widehat{G}\widehat{F}$, $\widehat{\epsilon} : \widehat{F}\widehat{G} \rightarrow id_{\widehat{\mathcal{C}}}$ and diagrams

$$\begin{array}{ccc} & \widehat{F}\widehat{G}\widehat{F} & \\ id_{\widehat{F}} \cdot \widehat{u} \nearrow & & \searrow \widehat{\epsilon} \cdot id_{\widehat{F}} \\ \widehat{F} & \xrightarrow{id_{\widehat{F}}} & \widehat{F} \end{array} \qquad \begin{array}{ccc} & \widehat{G}\widehat{F}\widehat{G} & \\ \widehat{u} \cdot id_{\widehat{G}} \nearrow & & \searrow id_{\widehat{G}} \cdot \widehat{\epsilon} \\ \widehat{G} & \xrightarrow{id_{\widehat{G}}} & \widehat{G} \end{array}$$

This is a reformulation of [RV22, Def. 2.1.1], which is equivalent to [Lur09, Def. 5.2.2.1] by [RV22, Prop. F.5.6].

By hypotheses \widehat{u} and \widehat{e} are morphisms in $\mathrm{LFun}^b(\widehat{\mathcal{D}}, \widehat{\mathcal{D}})$ and $\mathrm{LFun}^b(\widehat{\mathcal{C}}, \widehat{\mathcal{C}})$, and we write \check{u} and \check{e} for the corresponding morphisms in $\mathrm{LFun}^b(\check{\mathcal{D}}, \check{\mathcal{D}})$ and $\mathrm{LFun}^b(\check{\mathcal{C}}, \check{\mathcal{C}})$. The above diagrams are respectively in $\mathrm{LFun}^b(\widehat{\mathcal{C}}, \widehat{\mathcal{D}})$ and $\mathrm{LFun}^b(\widehat{\mathcal{D}}, \widehat{\mathcal{C}})$, and we claim the corresponding diagrams in $\mathrm{LFun}^b(\check{\mathcal{C}}, \check{\mathcal{D}})$ and $\mathrm{LFun}^b(\check{\mathcal{D}}, \check{\mathcal{C}})$ witness \check{u} and \check{e} as the unit and counit of an adjunction between \check{F} and \check{G} . In other words, we claim the equivalence $\mathrm{LFun}^b(\widehat{\mathcal{C}}, \widehat{\mathcal{D}}) \cong \mathrm{LFun}^b(\check{\mathcal{C}}, \check{\mathcal{D}})$ takes $\widehat{F}\widehat{G}\widehat{F}$, $id_{\widehat{F}} \cdot \widehat{u}$, and $\widehat{e} \cdot id_{\widehat{F}}$ respectively to $\check{F}\check{G}\check{F}$, $id_{\check{F}} \cdot \check{u}$, and $\check{e} \cdot id_{\check{F}}$, similarly for the right diagram.

Write $\Psi_{\mathcal{C}} : \check{\mathcal{C}} \rightarrow \widehat{\mathcal{C}}$ and $\Psi_{\mathcal{D}} : \check{\mathcal{D}} \rightarrow \widehat{\mathcal{D}}$ for the canonical functors. Then \check{F} is characterized in $\mathrm{LFun}^b(\check{\mathcal{C}}, \check{\mathcal{D}})$ by the condition $\Psi_{\mathcal{D}}\check{F} \cong \widehat{F}\Psi_{\mathcal{C}}$, similarly for \check{G} . It follows that $\Psi_{\mathcal{D}}\check{F}\check{G}\check{F} \cong \widehat{F}\widehat{G}\widehat{F}\Psi_{\mathcal{C}}$, which likewise characterizes $\check{F}\check{G}\check{F}$ as the functor corresponding to $\widehat{F}\widehat{G}\widehat{F}$. Now consider the following diagram, in which all horizontal arrows are equivalences.

$$\begin{array}{ccccc} \mathrm{LFun}^b(\widehat{\mathcal{C}}, \widehat{\mathcal{C}}) & \xrightarrow{-\Psi_{\mathcal{C}}} & \mathrm{LFun}^b(\check{\mathcal{C}}, \widehat{\mathcal{C}}) & \xleftarrow{\Psi_{\mathcal{C}}-} & \mathrm{LFun}^b(\check{\mathcal{C}}, \check{\mathcal{C}}) \\ \widehat{F}-\downarrow & & \downarrow \widehat{F}- & & \downarrow \check{F}- \\ \mathrm{LFun}^b(\widehat{\mathcal{C}}, \widehat{\mathcal{D}}) & \xrightarrow{-\Psi_{\mathcal{C}}} & \mathrm{LFun}^b(\check{\mathcal{C}}, \widehat{\mathcal{D}}) & \xleftarrow{\Psi_{\mathcal{D}}-} & \mathrm{LFun}^b(\check{\mathcal{C}}, \check{\mathcal{D}}) \end{array}$$

Here the compositions around the left square are evidently isomorphic, while those around the right square are because of the isomorphism $\Psi_{\mathcal{D}}\check{F} \cong \widehat{F}\Psi_{\mathcal{C}}$. The morphism $id_{\widehat{F}} \cdot \widehat{u}$ is the image of \widehat{u} under the left vertical map, while $id_{\check{F}} \cdot \check{u}$ is the image of \check{u} under the right. But by definition \check{u} is the image of \widehat{u} under the overall top equivalence, hence $id_{\check{F}} \cdot \check{u}$ is the image of $id_{\widehat{F}} \cdot \widehat{u}$ under the overall bottom equivalence. The remaining conditions are checked the same way. \square

Proof of Proposition 6.27. We let IC subscripts be implicit in the proof, f_* always meaning f_{IC*} , etc. If X and Y are truncated and geometric, the claim follows immediately from Lemma 6.28. In general, let $Y \cong \mathrm{colim} Y_{\alpha}$ be an ind-geometric presentation and $f_{\alpha} : X_{\alpha} \rightarrow Y_{\alpha}$ the base change of f . We then have an ind-geometric presentation $X \cong \mathrm{colim} X_{\alpha}$ since f is of finite Tor-dimension and filtered colimits in $\widehat{\mathrm{Stk}}_k$ are left exact. Let A denote the index category and $\Delta_{\alpha\beta}^1 \subset A$ the morphism associated to $\alpha \leq \beta$.

By construction we have a functor $\Delta^1 \times A \rightarrow \mathcal{P}r^L$ taking $\Delta^1 \times \Delta_{\alpha\beta}^1$ to the diagram witnessing the isomorphism $i_{\alpha\beta*}f_{\alpha*} \cong f_{\beta*}i'_{\alpha\beta*}$, and a similar functor $(\Delta^1)^{\mathrm{op}} \times A \rightarrow \mathcal{P}r^L$ packaging the isomorphisms $f_{\beta}^*i_{\alpha\beta*} \cong i'_{\alpha\beta*}f_{\alpha}^*$. Passing to right adjoints, these are equivalent to the data of a functor $A^{\mathrm{op}} \rightarrow \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \widehat{\mathrm{Cat}}_{\infty})$ taking α to $f_{\alpha*}^R$ and a functor $A^{\mathrm{op}} \rightarrow \mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_{\infty})$ taking α to f_{α}^{*R} .

Note that the unit/counit compatibility of Lemma 6.28 implies more precisely that the isomorphism $f_{\beta}^*i_{\alpha\beta*} \cong i'_{\alpha\beta*}f_{\alpha}^*$ is the Beck-Chevalley transformation associated to the isomorphism $i_{\alpha\beta*}f_{\alpha*} \cong f_{\beta*}i'_{\alpha\beta*}$. In the notation of [Lur17, Def. 4.7.5.16], the above functors thus take values in $\mathrm{Fun}^{\mathrm{RAd}}((\Delta^1)^{\mathrm{op}}, \widehat{\mathrm{Cat}}_{\infty})$ and $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \widehat{\mathrm{Cat}}_{\infty})$, respectively, and correspond under the equivalence $\mathrm{Fun}^{\mathrm{RAd}}((\Delta^1)^{\mathrm{op}}, \widehat{\mathrm{Cat}}_{\infty}) \cong \mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \widehat{\mathrm{Cat}}_{\infty})$ of [Lur17, Cor. 4.7.5.18].

By the same result these subcategories are closed under limits in $\text{Fun}((\Delta^1)^{\text{op}}, \widehat{\text{Cat}}_\infty)$ and $\text{Fun}(\Delta^1, \widehat{\text{Cat}}_\infty)$. But by Proposition 6.11 and [Lur09, Cor. 5.1.2.3] we have $f_*^R \cong \lim f_{\alpha*}^R$ and $f^{*R} \cong \lim f_{\alpha}^{*R}$, hence f_*^R is right adjoint to f^{*R} , hence f_* is right adjoint to f^* . \square

The definition of ind-coherent pushforward and $*$ -pullback also defines base change isomorphisms for suitable Cartesian squares. These isomorphisms are compatible with the adjunction of Proposition 6.27 in the following sense.

Proposition 6.29. *Let the following be a Cartesian diagram of ind-geometric stacks in which f is both of finite Tor-dimension and of ind-finite cohomological dimension.*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ h' \downarrow & & \downarrow h \\ X & \xrightarrow{f} & Y \end{array}$$

*If h is of ind-finite cohomological dimension (resp. of finite Tor-dimension), the isomorphism $f^*h_* \cong h'_*f'^*$ of functors $\text{IndCoh}(Y') \rightarrow \text{IndCoh}(X)$ is the Beck-Chevalley transformation associated to the isomorphism $f_*h'_* \cong h_*f'_*$ of functors $\text{IndCoh}(X') \rightarrow \text{IndCoh}(Y)$ (resp. the isomorphism $h'^*f^* \cong f'^*h^*$ of functors $\text{IndCoh}(Y) \rightarrow \text{IndCoh}(X')$).*

Proof. Consider the case with h of ind-finite cohomological dimension, the finite Tor-dimension case following by a variation of the same argument. If X , Y , and Y' are truncated and geometric the claim follows immediately from Lemma 6.28. If h is the immersion of an ind-geometric substack the claim was established during the proof of Proposition 6.27.

We pass to right adjoints and identify the isomorphisms $f^{*R}h_*^R \cong h'^R_*f'^{*R}$ and $f_*^R h'^R_* \cong h_*^R f'^R_*$ respectively with a morphism $f^{*R} \rightarrow f'^{*R}$ in $\text{Fun}(\Delta^1, \widehat{\text{Cat}}_\infty)$ and a morphism $f_*^R \rightarrow f'^R_*$ in $\text{Fun}((\Delta^1)^{\text{op}}, \widehat{\text{Cat}}_\infty)$, performing such identifications without comment in the rest of the proof. In the notation of [Lur17, Def. 4.7.5.16], we want to show these belong to $\text{Fun}^{\text{LAd}}(\Delta^1, \widehat{\text{Cat}}_\infty)$ and $\text{Fun}^{\text{RAAd}}((\Delta^1)^{\text{op}}, \widehat{\text{Cat}}_\infty)$, respectively, and correspond to each other under the equivalence $\text{Fun}^{\text{LAd}}(\Delta^1, \widehat{\text{Cat}}_\infty) \cong \text{Fun}^{\text{RAAd}}((\Delta^1)^{\text{op}}, \widehat{\text{Cat}}_\infty)$ of [Lur17, Cor. 4.7.5.18].

First suppose X and Y are truncated and geometric, let $Y' \cong \text{colim } Y'_\alpha$ be an ind-geometric presentation, and write $f'_\alpha : X'_\alpha \rightarrow Y'_\alpha$ for the base change of f' . As in the proof of Proposition 6.27, we have $f'^{*R} \cong \lim f'^{*R}_\alpha$ in both $\text{Fun}^{\text{LAd}}(\Delta^1, \widehat{\text{Cat}}_\infty)$ and $\text{Fun}(\Delta^1, \widehat{\text{Cat}}_\infty)$, hence $f^{*R} \rightarrow f'^{*R}$ is in $\text{Fun}^{\text{LAd}}(\Delta^1, \widehat{\text{Cat}}_\infty)$ since each $f^{*R} \rightarrow f'^{*R}_\alpha$ is. Similarly, $f_*^R \rightarrow f'^R_* \cong \lim f'^R_{\alpha*}$ is in $\text{Fun}^{\text{RAAd}}((\Delta^1)^{\text{op}}, \widehat{\text{Cat}}_\infty)$ since each $f_*^R \rightarrow f'^R_{\alpha*}$ is, and it corresponds to $f^{*R} \rightarrow f'^{*R}$ under [Lur17, Cor. 4.7.5.18] since each $f_*^R \rightarrow f'^R_{\alpha*}$ corresponds to $f^{*R} \rightarrow f'^{*R}$.

Now let $Y \cong \operatorname{colim} Y_\alpha$ be an ind-geometric presentation. For any α we have a diagram

$$\begin{array}{ccccc}
 & & X'_\alpha & \xrightarrow{f'_\alpha} & Y'_\alpha \\
 & \swarrow j'_\alpha & \downarrow h'_\alpha & & \swarrow j_\alpha \\
 X' & \xrightarrow{\quad} & Y' & & \\
 \downarrow i'_\alpha & & \downarrow f_\alpha & & \downarrow h_\alpha \\
 & \swarrow i'_\alpha & X_\alpha & \xrightarrow{\quad} & Y_\alpha \\
 & & \downarrow & & \swarrow i_\alpha \\
 X & \xrightarrow{\quad} & Y & &
 \end{array}$$

with Cartesian faces. We have already shown that the compositions $f_*^{*R} \rightarrow f_\alpha^{*R} \rightarrow f_\alpha'^{*R}$ and $f_*^R \rightarrow f_{\alpha*}^R \rightarrow f_{\alpha*}'^R$ belong to $\operatorname{Fun}^{\operatorname{LAd}}(\Delta^1, \widehat{\operatorname{Cat}}_\infty)$ and $\operatorname{Fun}^{\operatorname{RAAd}}((\Delta^1)^{\operatorname{op}}, \widehat{\operatorname{Cat}}_\infty)$, and correspond under [Lur17, Cor. 4.7.5.18]. By closure of these subcategories under limits, and by Proposition 6.11, it suffices to show $f_\beta'^{*R} \rightarrow f_\alpha'^{*R}$ and $f_{\beta*}'^R \rightarrow f_{\alpha*}'^R$ belong to $\operatorname{Fun}^{\operatorname{LAd}}(\Delta^1, \widehat{\operatorname{Cat}}_\infty)$ and $\operatorname{Fun}^{\operatorname{RAAd}}((\Delta^1)^{\operatorname{op}}, \widehat{\operatorname{Cat}}_\infty)$ and correspond under [Lur17, Cor. 4.7.5.18].

Consider the following diagram in $\operatorname{Fun}(\operatorname{IndCoh}(X_\beta), \operatorname{IndCoh}(Y'_\alpha))$.

$$\begin{array}{ccccc}
 j_{\alpha\beta*}^R h_{\beta*}^R f_\beta^{*R} & \xrightarrow{\sim} & j_{\alpha\beta*}^R f_\beta'^{*R} h_{\beta*}'^R & \xrightarrow{\sim} & f_\alpha'^{*R} j_{\alpha\beta*}'^R h_{\beta*}'^R \\
 \downarrow \wr & & & & \downarrow \wr \\
 h_{\alpha*}^R j_{\alpha\beta*}^R f_\beta^{*R} & \xrightarrow{\sim} & h_{\alpha*}^R f_\alpha'^{*R} j_{\alpha\beta*}'^R & \xrightarrow{\sim} & f_\alpha'^{*R} h_{\alpha*}'^R j_{\alpha\beta*}'^R
 \end{array}$$

We have already shown $f_\alpha'^{*R}$ and $f_{\alpha*}'^R$, etc., are adjoint, and the claim at hand is equivalent to the top right arrow being the Beck-Chevalley transformation associated to the isomorphism $j_{\alpha\beta*}'^R f_\beta'^{*R} \cong f_{\alpha*}'^R j_{\alpha\beta*}'^R$. But this follows since we have shown the corresponding claim for the top left and bottom arrows, and since all arrows in the diagram are isomorphisms. \square

7. COHERENT IND-GEOMETRIC STACKS

Given an arbitrary ind-geometric stack X , we have noted that in general the category $\operatorname{IndCoh}(X)$ defined in the previous section is not necessarily compactly generated by $\operatorname{Coh}(X)$ (despite the notation). This is true, however, in the main cases of interest. This includes, of course, Noetherian schemes as considered in [Gai13], as well as locally Noetherian geometric stacks (in particular geometric stacks which are almost of finite type and QCA as in [DG13, Thm. 3.3.5]) and reasonable inductive limits of these. More generally it includes the class of coherent ind-geometric stacks considered in this section (Proposition 7.5), which also includes examples such as the quotient $\mathcal{R}_{G,N}/G_{\mathcal{O}}$ studied in [BFN18, CW23a].

7.1. Definitions and Key Properties. Recall that a geometric stack X is locally coherent if there exists a flat cover $\operatorname{Spec} A \rightarrow X$ such that A is coherent.

Definition 7.1. A geometric stack X is coherent if it is locally coherent and $\mathrm{QCoh}(X)^\heartsuit$ is compactly generated. An ind-geometric stack X is coherent if it is reasonable and every reasonable geometric substack is coherent.

If X is a coherent geometric stack, it follows from [Lur18, Prop. 9.1.5.1] that more specifically $\mathrm{QCoh}(X)^\heartsuit$ is compactly generated by $\mathrm{Coh}(X)^\heartsuit$. As a basic example, recall that a geometric stack X is locally Noetherian if it admits a flat cover $\mathrm{Spec} A \rightarrow X$ such that A is Noetherian. In this case $\mathrm{QCoh}(X)^\heartsuit$ is compactly generated, hence X is coherent, by [Lur18, Prop. 9.5.2.3]. More generally, the following result implies $\mathrm{QCoh}(X)^\heartsuit$ is compactly generated if X is affine over a locally Noetherian geometric stack.

Proposition 7.2. *Let $f : X \rightarrow Y$ be an affine morphism of geometric stacks. If $\mathrm{QCoh}(Y)^\heartsuit$ is compactly generated, then so is $\mathrm{QCoh}(X)^\heartsuit$.*

Proof. Since f is affine $f_* : \mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(Y)$ is t-exact and conservative, hence restricts to a conservative functor $\mathrm{QCoh}(X)^\heartsuit \rightarrow \mathrm{QCoh}(Y)^\heartsuit$. This restriction is continuous and has a left adjoint, the restriction of $\tau^{\geq 0} \circ f^*$. Thus compact generation of $\mathrm{QCoh}(Y)^\heartsuit$ implies that of $\mathrm{QCoh}(X)^\heartsuit$ by [Lur17, Prop. 7.1.4.12] (whose proof applies to compact generation, not just compact projective generation). \square

Together with Proposition 4.2, the following result ensures the definition of coherence is consistent when we regard a geometric stack as an ind-geometric stack.

Proposition 7.3. *Let $f : X \rightarrow Y$ be an almost finitely presented closed immersion of geometric stacks. If Y is locally coherent (resp. coherent), then so is X .*

Proof. Let $\mathrm{Spec} A \rightarrow Y$ be a flat cover with A coherent, and $f' : \mathrm{Spec} B \rightarrow \mathrm{Spec} A$ the base change of f (recall that f is affine by Proposition 3.18). By [Lur18, Cor. 5.2.2.2] B is almost perfect as an A -module, hence $H^n(B)$ is finitely presented over $H^0(A)$ for all $n \leq 0$. Moreover $H^0(B)$ is a quotient of $H^0(A)$ by a finitely generated ideal, so $H^0(B)$ is coherent and the $H^n(B)$ are finitely presented over $H^0(B)$ [Gla89, Thm. 2.4.1]. Moreover, if $\mathrm{QCoh}(Y)^\heartsuit$ is compactly generated then so is $\mathrm{QCoh}(X)^\heartsuit$ by Proposition 7.2. \square

Proposition 7.4. *Coherent ind-geometric stacks are closed under filtered colimits along almost ind-finitely presented closed immersions in $\widehat{\mathrm{Stk}}_k$. In particular, an ind-geometric stack is coherent if and only if it has a reasonable presentation whose terms are coherent.*

Proof. If $X \cong \mathrm{colim} X_\alpha$ is a filtered colimit as in the statement, then X is a reasonable ind-geometric stack by Proposition 4.13. If $i : X' \rightarrow X$ is a reasonable geometric substack, then i factors through some X_α by Proposition 3.16. Moreover, if $X_\alpha \cong \mathrm{colim}_\beta X_{\alpha\beta}$ is a reasonable presentation i then factors through an almost finitely presented closed immersion to some $X_{\alpha\beta}$ by Proposition 4.5. But $X_{\alpha\beta}$ is coherent by hypothesis, hence X' is coherent by Proposition 7.3. \square

We have the following key property of coherent ind-geometric stacks.

Proposition 7.5. *If X is a coherent ind-geometric stack then the canonical functor $\mathrm{Ind}(\mathrm{Coh}(X)) \rightarrow \mathrm{IndCoh}(X)$ is an equivalence, and induces equivalences $\mathrm{Ind}(\mathrm{Coh}(X)^{\leq 0}) \cong \mathrm{IndCoh}(X)^{\leq 0}$, $\mathrm{Ind}(\mathrm{Coh}(X)^{\geq 0}) \cong \mathrm{IndCoh}(X)^{\geq 0}$.*

Proof. For a coherent geometric stack this follows from [Lur18, Thm. C.6.7.1]. The ind-geometric case then follows from Propositions 5.4, 6.11, and 6.26, since ind-completion of idempotent-complete categories admitting finite colimits commutes with filtered colimits [Lur17, Lem. 7.3.5.11]. \square

Remark 7.6. In light of Proposition 7.5, the notion of coherent ind-geometric stack is in a sense formally dual to the notion of perfect stack considered in [BZFN10].

7.2. Fiber Products. Coherent ind-geometric stacks are not closed under general fiber products (e.g. [Gla89, Sec. 7.3.13]), but we record a few cases of interest where they are (beyond more elementary ones with Noetherian hypotheses).

Proposition 7.7. *Let the following be a Cartesian diagram of ind-geometric stacks.*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ h' \downarrow & & \downarrow h \\ X & \xrightarrow{f} & Y \end{array}$$

Suppose that X and Y are reasonable, that Y' is coherent, and that f is an almost ind-finitely presented ind-closed immersion. Then X' is coherent and f' is an almost ind-finitely presented ind-closed immersion.

Proof. Suppose first that X and Y' are truncated geometric stacks, and let $Y \cong \mathrm{colim} Y_\alpha$ be a reasonable presentation. We may assume f and h factor through maps $f_\alpha : X \rightarrow Y_\alpha$, $h_\alpha : Y' \rightarrow Y_\alpha$ for all α . Letting $X'_\alpha := X \times_{Y_\alpha} Y'$, each $f'_\alpha : X'_\alpha \rightarrow Y'$ is an almost finitely presented closed immersion by base change and Proposition 4.12, hence X'_α is coherent by Proposition 7.3. For any $\beta \geq \alpha$ the induced map $i'_{\alpha\beta} : X'_\alpha \rightarrow X'_\beta$ is an almost finitely presented closed immersion since $f'_\beta \circ i'_{\alpha\beta} \cong f'_\alpha$ (Proposition 3.9). Since $X' \cong \mathrm{colim} X \times_{Y_\alpha} Y'$ in $\widehat{\mathrm{Stk}}_k$ by left exactness of filtered colimits [Lur09, Ex. 7.3.4.7], it follows that X' is coherent by Proposition 7.4.

In general, fix reasonable presentations $X \cong \mathrm{colim} X_\alpha$ and $Y' \cong \mathrm{colim} Y'_\beta$. Then as above $X' \cong \mathrm{colim} X_\alpha \times_Y Y'_\beta$ presents X' as a filtered colimit of coherent ind-geometric stacks along almost ind-finitely presented ind-closed immersions, hence X' is coherent by Proposition 7.4. That f' is an almost ind-finitely presented ind-closed immersion follows from Propositions 4.12 and 4.20. \square

Proposition 7.8. *Let the following be a Cartesian diagram of ind-geometric stacks.*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ h' \downarrow & & \downarrow h \\ X & \xrightarrow{f} & Y \end{array}$$

Suppose that X is coherent, Y is reasonable, and Y' is ind-tamely presented. Suppose also that f is ind-tamely presented and that h is affine and has coherent pullback. Then X' is coherent.

Proof. If $X \cong \operatorname{colim} X_\alpha$ is a reasonable presentation, then as in the proof of Proposition 4.19 $X' \cong \operatorname{colim} X'_\alpha$ is a reasonable presentation, where $X'_\alpha := X_\alpha \times_Y Y'$. By Propositions 4.10 and 4.11 each X'_α is tamely presented, hence locally coherent by Proposition ???. But X_α is coherent, hence by Proposition 7.2 so is X'_α since h is affine. \square

We say an ind-geometric stack is ind-locally Noetherian if it is reasonable and every reasonable geometric substack is locally Noetherian.

Proposition 7.9. *Suppose $X \rightarrow X'$ and $Y \rightarrow Y'$ are tamely presented affine morphisms of ind-geometric stacks such that X' and Y' are ind-locally Noetherian. Then X , Y , and $X \times Y$ are coherent.*

Proof. Let $X' \cong \operatorname{colim} X'_\alpha$ be a reasonable presentation. Then $X \cong \operatorname{colim} X_\alpha$, where $X_\alpha := X \times_{X'} X'_\alpha$, by left exactness of filtered colimits in $\widehat{\operatorname{Stk}}_k$. Each X'_α is locally Noetherian, hence each X_α is coherent since $X_\alpha \rightarrow X'_\alpha$ is tamely presented and affine (Propositions ??? and 7.2). Coherence of X follows from Proposition 7.4. The other claims are the same, noting that $X' \times Y'$ is ind-locally Noetherian and that $X \times Y$ is tamely presented and affine over it. \square

7.3. Pushforward and !-pullback. When the stacks appearing in Proposition 6.24 are coherent, it immediately follows that its conclusion holds for all $\mathcal{F} \in \operatorname{IndCoh}(Y')$, as $f^!$ and $f'^!$ are continuous and $\operatorname{IndCoh}(Y')$ is compactly generated by $\operatorname{Coh}(Y')$. When X and Y are ind-geometric, the coherent case differs from the general case more substantially in terms of what other hypotheses are needed, so we isolate these into separate results below.

Proposition 7.10. *Let the following be a Cartesian diagram of ind-geometric stacks.*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ h' \downarrow & & \downarrow h \\ X & \xrightarrow{f} & Y \end{array}$$

Suppose that all stacks in the diagram are coherent, that f is ind-proper and almost of ind-finite presentation, and that h is of ind-finite cohomological dimension. Then for any $\mathcal{F} \in \operatorname{IndCoh}(Y')$ the Beck-Chevalley map $h'_ f'^!(\mathcal{F}) \rightarrow f^! h_*(\mathcal{F})$ is an isomorphism.*

Proof. First note that the coherence hypotheses imply that $f^!$ and $f^!$ are continuous. If X and Y are geometric, the claim now follows immediately from Proposition 6.24 since $\text{IndCoh}(Y')$ is compactly generated by $\text{Coh}(Y')$.

Now suppose that X is truncated and geometric, and let $Y \cong \text{colim } Y_\alpha$ be a reasonable presentation. Define $h_\alpha : Y'_\alpha \rightarrow Y_\alpha$, $i'_\alpha : Y'_\alpha \rightarrow Y'$ by base change, similarly $i'_{\alpha\beta} : Y'_\alpha \rightarrow Y'_\beta$ for $\beta \geq \alpha$. Each Y'_α is coherent by Proposition 7.7, so by the previous paragraph we have $h_{\alpha*}i'^!_{\alpha\beta} \cong i'^!_{\alpha\beta}h_{\beta*}$ in $\text{Fun}(\text{IndCoh}(Y'_\beta), \text{IndCoh}(Y_\alpha))$. For any α we then have $h_{\alpha*}i'^!_\alpha \cong i'^!_\alpha h_*$ in $\text{Fun}(\text{IndCoh}(Y'), \text{IndCoh}(Y_\alpha))$ by [Lur17, Prop. 4.7.5.19]. By Proposition 4.3 we can factor f as $X \xrightarrow{f_\alpha} Y_\alpha \xrightarrow{i_\alpha} Y$ for some α . Consider the diagram

$$\begin{array}{ccccc} X' & \xrightarrow{f'_\alpha} & Y'_\alpha & \xrightarrow{i'_\alpha} & Y' \\ h' \downarrow & & \downarrow h_\alpha & & \downarrow h \\ X & \xrightarrow{f_\alpha} & Y_\alpha & \xrightarrow{i_\alpha} & Y \end{array}$$

of Cartesian squares. The map in the statement now factors as

$$h'_*f^!(\mathcal{F}) \cong h'_*f'_\alpha i'^!_\alpha(\mathcal{F}) \rightarrow f'_\alpha h_{\alpha*}i'^!_\alpha(\mathcal{F}) \rightarrow f'_\alpha i'^!_\alpha h_*(\mathcal{F}) \cong f^!h_*(\mathcal{F}),$$

and we have already shown both factors are isomorphisms.

Now suppose $X \cong \text{colim } X_\alpha$ is a reasonable presentation. For any α we have a diagram

$$\begin{array}{ccccc} X'_\alpha & \xrightarrow{i'_\alpha} & X' & \xrightarrow{f'} & Y' \\ h'_\alpha \downarrow & & \downarrow h' & & \downarrow h \\ X_\alpha & \xrightarrow{i_\alpha} & X & \xrightarrow{f} & Y \end{array}$$

of Cartesian squares. Each X'_α is coherent by Proposition 7.7, and $X' \cong \text{colim } X'_\alpha$ since filtered colimits are left exact in $\widehat{\text{Stk}}_k$. Since the functors $i'^!_\alpha$ determine an isomorphism $\text{IndCoh}(X') \cong \lim_\alpha \text{IndCoh}(X'_\alpha)$ in $\widehat{\text{Cat}}_\infty$, it suffices to show that $i'^!_\alpha h'_*f^!(\mathcal{F}) \rightarrow i'^!_\alpha f^!h_*(\mathcal{F})$ is an isomorphism for all α . But by the previous paragraph both the composition

$$h'_{\alpha*}i'^!_\alpha f^!(\mathcal{F}) \rightarrow i'^!_\alpha h'_*f^!(\mathcal{F}) \rightarrow i'^!_\alpha f^!h_*(\mathcal{F})$$

and its first factor are isomorphisms, hence the second is as well. \square

7.4. Pushforward and *-pullback. In the context of coherent ind-geometric stacks, Proposition 7.5 lets us extend both of the basic functorialities of IndCoh .

Definition 7.11. Let X and Y be reasonable ind-geometric stacks such that Y is coherent, and let $f : X \rightarrow Y$ be a morphism with coherent pullback. We write $f^* : \text{IndCoh}(Y) \rightarrow \text{IndCoh}(X)$ for the unique continuous functor whose restriction to $\text{Coh}(Y)$ factors through the functor $f^* : \text{Coh}(Y) \rightarrow \text{Coh}(X)$ of Definition 5.2.

When f is of finite Tor-dimension this is indeed consistent with Definition 6.9, since the previously defined f^* is continuous and preserves coherence. To describe the pushforward counterpart of Definition 7.11 first note that if $f : X \rightarrow Y$ is any morphism of ind-geometric stacks such that X is reasonable, there is a canonical functor $f_* : \text{Coh}(X) \rightarrow \text{IndCoh}(Y)$ defined as follows. Write $\text{IndCoh}_{naive}^+ : \text{indGStk}_k \rightarrow \widehat{\text{Cat}}_\infty$ for the left Kan extension of the evident functor $\text{IndCoh}^+ : \text{GStk}_k^+ \rightarrow \widehat{\text{Cat}}_\infty$. Explicitly, $\text{IndCoh}_{naive}^+(X)$ is the full subcategory of $\mathcal{F} \in \text{IndCoh}^+(X)$ which are pushed forward from some ind-geometric substack of X . By construction we have a functor $f_* : \text{IndCoh}(X)_{naive}^+ \rightarrow \text{IndCoh}(Y)_{naive}^+$, while by the universal property of left Kan extensions we have canonical functors $\text{Coh}(X) \rightarrow \text{IndCoh}(X)_{naive}^+$ and $\text{IndCoh}(Y)_{naive}^+ \rightarrow \text{IndCoh}(Y)^+$, and we let f_* be the composition of these.

Definition 7.12. Let $f : X \rightarrow Y$ be a morphism of ind-geometric stacks, and suppose that X is coherent. Then we write $f_* : \text{IndCoh}(X) \rightarrow \text{IndCoh}(Y)$ for the unique continuous functor whose restriction to $\text{Coh}(X)$ is the functor above.

Suppose $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are morphisms of ind-geometric stacks such that X is coherent, and such that either Y is coherent or g is of ind-finite cohomological dimension. Then have an isomorphism $g_* f_* \cong (g \circ f)_*$ of functors $\text{IndCoh}(X) \rightarrow \text{IndCoh}(Z)$, since both are continuous and have their restrictions to $\text{Coh}(X)$ are isomorphic by construction.

If the source and target of $f : X \rightarrow Y$ are coherent, we have the following extension of Proposition 6.27.

Proposition 7.13. Let X and Y be coherent ind-geometric stacks and $f : X \rightarrow Y$ a morphism with coherent pullback. Then $f_* : \text{IndCoh}(X) \rightarrow \text{IndCoh}(Y)$ is right adjoint to $f^* : \text{IndCoh}(Y) \rightarrow \text{IndCoh}(X)$. In particular, suppose f sits in a Cartesian diagram of the following form, where h and h' have coherent pullback and X' and Y' are coherent.

$$(7.14) \quad \begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ h' \downarrow & & \downarrow h \\ X & \xrightarrow{f} & Y \end{array}$$

Then we obtain a Beck-Chevalley transformation $h^* f_* \rightarrow f'_* h'^*$ from the the isomorphism $h_* f'_* \cong f_* h'_*$ of functors $\text{IndCoh}(X') \rightarrow \text{IndCoh}(Y)$, and the former is itself an isomorphism if f is proper and almost of finite presentation.

Lemma 7.15. Let $\widehat{\mathcal{C}}, \widehat{\mathcal{D}}$ be the left completions of $\check{\mathcal{C}}, \check{\mathcal{D}} \in \mathcal{P}\mathbf{r}_{\text{acpl}}^{\text{St}, \text{b}}$, and let $\Psi_{\mathcal{C}} : \check{\mathcal{C}} \rightarrow \widehat{\mathcal{C}}$ and $\Psi_{\mathcal{D}} : \check{\mathcal{D}} \rightarrow \widehat{\mathcal{D}}$ be the canonical functors. Let $\check{F} : \check{\mathcal{C}} \rightarrow \check{\mathcal{D}}, \widehat{F} : \widehat{\mathcal{C}} \rightarrow \widehat{\mathcal{D}}$ be colimit-preserving functors such that \widehat{F} is right bounded and $\Psi_{\mathcal{D}} \check{F} \cong \widehat{F} \Psi_{\mathcal{C}}$, and let $\check{G} : \check{\mathcal{D}} \rightarrow \check{\mathcal{C}}, \widehat{G} : \widehat{\mathcal{D}} \rightarrow \widehat{\mathcal{C}}$ be their right adjoints. Then the Beck-Chevalley map $\Psi_{\mathcal{C}} \check{G}(X) \rightarrow \widehat{G} \Psi_{\mathcal{D}}(X)$ is an isomorphism for all $X \in \check{\mathcal{D}}^+$.

Proof. By definition the Beck-Chevalley map is the composition

$$(7.16) \quad \Psi_{\mathcal{C}} \check{G}(X) \rightarrow \Psi_{\mathcal{C}} \check{G} \Psi_{\mathcal{D}}^R \Psi_{\mathcal{D}}(X) \cong \Psi_{\mathcal{C}} \Psi_{\mathcal{C}}^R \widehat{G} \Psi_{\mathcal{D}}(X) \rightarrow \widehat{G} \Psi_{\mathcal{D}}(X)$$

of unit and counit maps. Since $\Psi_{\mathcal{C}}$ is t-exact and restricts to an equivalence $\check{\mathcal{C}}^+ \xrightarrow{\sim} \widehat{\mathcal{C}}^+$, its right adjoint $\Psi_{\mathcal{C}}^R$ is left t-exact and restricts to the inverse equivalence $\widehat{\mathcal{C}}^+ \xrightarrow{\sim} \check{\mathcal{C}}^+$, likewise for $\Psi_{\mathcal{D}}^R$. In particular, the first map in (7.16) is an isomorphism since $X \in \check{\mathcal{D}}^+$. But \widehat{G} is left bounded since \widehat{F} is right bounded, hence $\widehat{G} \Psi_{\mathcal{D}}(X) \in \widehat{\mathcal{C}}^+$, hence the last map in (7.16) is an isomorphism. \square

Proof. First suppose X and Y are truncated and geometric. Since $\text{IndCoh}(X)$ is compactly generated and f^* preserves compactness, the right adjoint f^{*R} is continuous. But f_* and f^{*R} have isomorphic restrictions to $\text{Coh}(X)$ by Lemma 7.15 and the definitions, hence by continuity they are themselves isomorphic. When X' and Y' are truncated and geometric, the final claim follows immediately since $h^* f_*$ and $f'_* h'^*$ are continuous and the transformation restricts to an isomorphism of functors $\text{Coh}(X) \rightarrow \text{Coh}(Y')$.

Now let $Y \cong \text{colim } Y_{\alpha}$ be a reasonable presentation, let $f_{\alpha} : X_{\alpha} \rightarrow Y_{\alpha}$ be the base change of f , and let $i'_{\alpha\beta} : X_{\alpha} \rightarrow X_{\beta}$ the base change of $i_{\alpha\beta} : Y_{\alpha} \rightarrow Y_{\beta}$. Unwinding the definition of f_* , it follows from Proposition 6.11 and [Lur09, Cor. 5.1.2.3] that $f_* \cong \text{colim } f_{\alpha*}$ in $\text{Fun}(\Delta^1, \mathcal{P}\text{r}^{\text{L}})$, the structure maps being given by the isomorphisms $i'_{\alpha\beta*} f_{\beta*} \cong f_{\alpha*} i_{\alpha\beta*}$. Likewise, we have $f_* \cong \text{colim } f_{\alpha*}$ in $\text{Fun}((\Delta^1)^{\text{op}}, \mathcal{P}\text{r}^{\text{L}})$, the structure maps being given by the Beck-Chevalley isomorphisms $f_{\beta}^* i_{\alpha\beta*} \cong i'_{\alpha\beta*} f_{\alpha}^*$. Passing to right adjoints we have $f_*^R \cong \lim f_{\alpha*}^R$ in $\text{Fun}((\Delta^1)^{\text{op}}, \widehat{\text{Cat}}_{\infty})$ and $f^{*R} \cong \lim f_{\alpha}^{*R}$ in $\text{Fun}(\Delta^1, \widehat{\text{Cat}}_{\infty})$. In the notation of [Lur17, Def. 4.7.5.16], it follows from [Lur17, Cor. 4.7.5.18] and the previous paragraph that $f^{*R} \in \text{Fun}^{\text{LAd}}(\Delta^1, \widehat{\text{Cat}}_{\infty})$, $f_*^R \in \text{Fun}^{\text{RAAd}}(\Delta^1, \widehat{\text{Cat}}_{\infty})$, and that f^{*R} and f_*^R correspond under the equivalence $\text{Fun}^{\text{LAd}}(\Delta^1, \widehat{\text{Cat}}_{\infty}) \cong \text{Fun}^{\text{RAAd}}(\Delta^1, \widehat{\text{Cat}}_{\infty})$ of [Lur17, Cor. 4.7.5.18]. But then f_* is right adjoint to f^* since f_*^R is to f^{*R} . The general case of the final claim now follows as in the geometric case. \square

8. EXTERNAL PRODUCTS AND SHEAF HOM

Given a geometric stack Y , the sheaf Hom out of $\mathcal{F} \in \text{QCoh}(Y)$ is defined by the adjunction

$$- \otimes \mathcal{F} : \text{QCoh}(Y) \rightleftarrows \text{QCoh}(Y) : \mathcal{H}om(\mathcal{F}, -).$$

For any $f : X \rightarrow Y$, the isomorphism $f^*(- \otimes \mathcal{F}) \cong - \otimes f^*(\mathcal{F})$ then gives rise to a map

$$(8.1) \quad f^* \mathcal{H}om(\mathcal{F}, \mathcal{G}) \rightarrow \mathcal{H}om(f^*(\mathcal{F}), f^*(\mathcal{G}))$$

which is natural in $\mathcal{G} \in \text{QCoh}(Y)$. This is not an isomorphism in general, but is when f is of finite Tor-dimension under certain hypotheses on \mathcal{F} and \mathcal{G} . The basic goal of this section is to generalize this and related results, in particular allowing f to have coherent pullback, X and Y to be ind-geometric, and \mathcal{F} and \mathcal{G} to be ind-coherent.

Since $\mathrm{IndCoh}(X)$ does not generally have a tensor product, external products instead play the primary role. That is, for suitable $\mathcal{F} \in \mathrm{IndCoh}(Y)$ we have an adjunction

$$-\boxtimes \mathcal{F} : \mathrm{IndCoh}(X) \rightleftarrows \mathrm{IndCoh}(X \times Y) : (-\boxtimes \mathcal{F})^R.$$

To make explicit their dependence on X we will denote these functors by $e_{\mathcal{F},X}$ and $e_{\mathcal{F},X}^R$. When X and Y are geometric, we have an isomorphism $\mathcal{H}om(\mathcal{F}, -) \cong e_{\mathcal{F},Y}^R \Delta_{Y*}$, letting us subsume results about $\mathcal{H}om(\mathcal{F}, -)$ in corresponding results about $e_{\mathcal{F},Y}^R$. This formula moreover provides a useful definition of sheaf Hom in the ind-geometric setting.

On a technical level, there is a close analogy between $e_{\mathcal{F},X}^R$ for coherent \mathcal{F} and the functor $g^!$ associated to a morphism $g : X' \rightarrow Y'$ which is ind-proper and almost of ind-finite presentation — the two functors have similar formal properties for similar reasons. In particular, many proofs about $e_{\mathcal{F},X}^R$ in this section closely follow corresponding proofs in Section ???. The main difference is that the role of the map g is now played by the projection $X \times Y \rightarrow X$, so a contravariant functoriality has been replaced with covariant one.

8.1. Ind-coherent external products. We begin by defining the the external product of a pair of ind-coherent sheaves, at least provided one of them is bounded. *We assume that k is an ordinary ring of finite global dimension for the rest of the paper.* Suppose first that X and Y are geometric stacks and $\mathcal{F} \in \mathrm{IndCoh}(Y)^b$. By Lemma ??? the assignment $\mathcal{G} \mapsto \mathcal{G} \boxtimes \Psi_Y(\mathcal{F})$ defines a bounded colimit-preserving functor $e_{\Psi_Y(\mathcal{F}),X} : \mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(X \times Y)$. The universal property of $\mathrm{IndCoh}(-)$ guarantees that this functor has a unique ind-coherent lift in the following sense.

Definition 8.2. If X and Y are geometric stacks and $\mathcal{F} \in \mathrm{IndCoh}(Y)^b$, we let $e_{\mathcal{F},X}$ denote the unique bounded colimit-preserving functor fitting into a diagram of the following form.

$$(8.3) \quad \begin{array}{ccc} \mathrm{IndCoh}(X) & \xrightarrow{e_{\mathcal{F},X}} & \mathrm{IndCoh}(X \times Y) \\ \Psi_X \downarrow & & \downarrow \Psi_{X \times Y} \\ \mathrm{QCoh}(X) & \xrightarrow{e_{\Psi_Y(\mathcal{F}),X}} & \mathrm{QCoh}(X \times Y) \end{array}$$

When X and Y are truncated this is compatible with our earlier definition of external products of coherent sheaves (Proposition ???) in the obvious way, given the identification of $\mathrm{Coh}(X)$ and $\mathrm{Coh}(X \times Y)$ with full subcategories of $\mathrm{IndCoh}(X)$ and $\mathrm{IndCoh}(X \times Y)$. When X is coherent Definition 8.2 is determined by this compatibility, as $e_{\mathcal{F},X}$ is then the left Kan extension of its restriction to $\mathrm{Coh}(X)$ (Proposition 7.5). The functoriality of Definition 8.2 in X , Y , and \mathcal{F} is described by the following result.

Proposition 8.4. *There exists a diagram*

$$(8.5) \quad \begin{array}{ccc} \mathrm{IndCoh}(-) \times \mathrm{IndCoh}(-)^b & \longrightarrow & \mathrm{IndCoh}(- \times -) \\ \Psi_{(-)} \times \Psi_{(-)} \downarrow & & \downarrow \Psi_{(- \times -)} \\ \mathrm{QCoh}(-) \times \mathrm{QCoh}(-)^b & \longrightarrow & \mathrm{QCoh}(- \times -) \end{array}$$

of functors $\mathrm{Corr}(\mathrm{GStk}_k)^{\times 2}_{fcd;ftd} \rightarrow \widehat{\mathrm{Cat}}_\infty$ which specializes to the diagram (8.3) when evaluated on any $X, Y \in \mathrm{GStk}_k$ and any $\mathcal{F} \in \mathrm{IndCoh}(Y)^b$.

We postpone the proof of Proposition 8.4 while we extend Definition 8.2 to ind-geometric stacks. To simplify the needed constructions we restrict our attention to the case where Y is reasonable and \mathcal{F} is coherent. This is not strictly essential, but most good properties of $e_{\mathcal{F},X}^R$ (e.g. almost continuity) will require \mathcal{F} to be coherent anyway.

Note first that the top arrow of (8.5) can be encoded as a functor $\mathrm{Corr}(\mathrm{GStk}_k)^{\times 2}_{fcd;ftd} \rightarrow \widehat{\mathrm{Cat}}_\infty^{\Delta^1}$. Restricting its domain and values we then obtain a functor $\mathrm{Corr}(\mathrm{GStk}_k^+)_{fcd;ftd} \times \mathrm{Corr}(\mathrm{GStk}_k^+)_{prop;ftd} \rightarrow \widehat{\mathrm{Cat}}_\infty^{\Delta^1}$ of the form

$$\mathrm{IndCoh}(-) \times \mathrm{Coh}(-) \rightarrow \mathrm{IndCoh}(- \times -).$$

For any $X, Y \in \mathrm{GStk}_k$ the specialization of this expression preserves small colimits in $\mathrm{IndCoh}(X)$, hence there exists a unique extension to a functor $\mathrm{Corr}(\mathrm{GStk}_k^+)_{fcd;ftd} \times \mathrm{Corr}(\mathrm{GStk}_k^+)_{prop;ftd} \rightarrow (\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\Delta^1}$ of the form

$$(8.6) \quad \mathrm{IndCoh}(-) \otimes \mathrm{Ind}(\mathrm{Coh}(-)) \rightarrow \mathrm{IndCoh}(- \times -).$$

We now define a functor $\mathrm{Corr}(\mathrm{indGStk}_k)_{fcd;ftd} \times \mathrm{Corr}(\mathrm{indGStk}_k^{reas})_{prop;ftd} \rightarrow (\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\Delta^1}$ via left Kan extension. This extension exists, and moreover is of the same form (8.6), since $(\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\Delta^1}$ admits small colimits and $(\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\Delta^1} \rightarrow (\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\times 2}$ preserves them [Lur09, Cor. 5.1.2.3], since the tensor product in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ preserves small colimits in each variable [Lur17, Rem. 4.8.1.23], and since ind-completion of idempotent-complete categories admitting finite colimits commutes with filtered colimits [Lur17, Lem. 7.3.5.11].

Definition 8.7. We define a functor $\mathrm{Corr}(\mathrm{indGStk}_k)_{fcd;ftd} \times \mathrm{Corr}(\mathrm{indGStk}_k^{reas})_{prop;ftd} \rightarrow \widehat{\mathrm{Cat}}_\infty^{\Delta^1}$ of the form

$$\mathrm{IndCoh}(-) \times \mathrm{Coh}(-) \rightarrow \mathrm{IndCoh}(- \times -)$$

by taking the functor $\mathrm{Corr}(\mathrm{indGStk}_k)_{fcd;ftd} \times \mathrm{Corr}(\mathrm{indGStk}_k^{reas})_{prop;ftd} \rightarrow (\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\Delta^1}$ defined above, passing to its underlying $\widehat{\mathrm{Cat}}_\infty^{\Delta^1}$ -valued functor, and then composing with the canonical natural transformation $\mathrm{IndCoh}(-) \times \mathrm{Coh}(-) \rightarrow \mathrm{IndCoh}(-) \otimes \mathrm{Ind}(\mathrm{Coh}(-))$. Given ind-geometric stacks X and Z such that Z is reasonable, and given $\mathcal{F} \in \mathrm{Coh}(Z)$, we write

$$e_{\mathcal{F},X} : \mathrm{IndCoh}(X) \rightarrow \mathrm{IndCoh}(X \times Y)$$

for the associated functor.

Explicitly, Definition 8.7 says that if $\mathcal{F} \cong i_*(\mathcal{F}')$ for some reasonable geometric substack $i : Z' \rightarrow Z$, and if $X \cong \operatorname{colim} X_\alpha$ is an ind-geometric presentation, then we have a diagram

$$\begin{array}{ccc} \operatorname{IndCoh}(X_\alpha) & \xrightarrow{e_{\mathcal{F}', X_\alpha}} & \operatorname{IndCoh}(X_\alpha \times Z') \\ i_{\alpha*} \downarrow & & \downarrow (i_\alpha \times i)_* \\ \operatorname{IndCoh}(X) & \xrightarrow{e_{\mathcal{F}, X}} & \operatorname{IndCoh}(X \times Z) \end{array}$$

for all α . The functor $e_{\mathcal{F}, X}$ is determined by these diagrams together with the fact that by construction it preserves small colimits.

If X and $X \times Z$ are reasonable and $\mathcal{F} \in \operatorname{Coh}(Z)$, it follows from the definitions that $e_{\mathcal{F}, X}$ takes $\operatorname{Coh}(X)$ to $\operatorname{Coh}(X \times Z)$. In particular, suppose X and $X \times Z$ are coherent, X' and Z' are reasonable, and $h : X' \rightarrow X$, $\phi : Z' \rightarrow Z$ are morphisms with coherent pullback. Then there is a canonical isomorphism

$$(8.8) \quad (h \times \phi)^* e_{\mathcal{F}, X} \cong e_{\phi^*(\mathcal{F}), X'} h^*$$

since both sides are continuous and have canonically isomorphic restrictions to $\operatorname{Coh}(X)$.

Proposition 8.9. *Let X and Z be ind-geometric stacks such that Z is reasonable, and let $\mathcal{F} \in \operatorname{Coh}(Z)$. Then $e_{\mathcal{F}, X}$ is bounded.*

Proof. Fix an ind-geometric presentation $X \cong \operatorname{colim} X_\alpha$ and write $\mathcal{F} \cong i_*(\mathcal{F}')$, where $i : Z' \rightarrow Z$ is a reasonable geometric substack and $\mathcal{F}' \in \operatorname{Coh}(Z')^{[m, n]}$. If $\mathcal{G}_\alpha \in \operatorname{IndCoh}(X_\alpha)^{\geq 0}$ for some α , then by t-exactness of $(i_\alpha \times i)_*$ and the proof of Lemma ?? we have $e_{\mathcal{F}, X} i_{\alpha*}(\mathcal{G}_\alpha) \cong (i_\alpha \times i)_* e_{\mathcal{F}', X_\alpha}(\mathcal{G}_\alpha) \in \operatorname{IndCoh}(X)^{\geq m'}$, where m' is m minus the global dimension of k . Similarly, if $\mathcal{G}_\alpha \in \operatorname{IndCoh}(X_\alpha)^{\leq 0}$ then $e_{\mathcal{F}, X} i_{\alpha*}(\mathcal{G}_\alpha) \in \operatorname{IndCoh}(X)^{\leq n}$. Given that $\mathcal{G} \cong \operatorname{colim} i_{\alpha*} i_\alpha^!(\mathcal{G})$ for any $\mathcal{G} \in \operatorname{IndCoh}(X)$ (by Proposition 6.11 and Lemma 2.2), it follows that $e_{\mathcal{F}, X}$ takes $\operatorname{IndCoh}(X)^{\geq 0}$ to $\operatorname{IndCoh}(X \times Z)^{\geq m'}$ since $i_{\alpha*} i_\alpha^!$ is left t-exact and $\operatorname{IndCoh}(X \times Z)^{\geq m'}$ is closed under filtered colimits. If $\mathcal{G} \in \operatorname{IndCoh}(X)^{\leq 0}$, then we additionally have $\mathcal{G} \cong \operatorname{colim} i_{\alpha*} \tau^{\leq 0} i_\alpha^!(\mathcal{G})$ (by Proposition 6.13 and Lemma 2.2, given that $\operatorname{IndCoh}(X)^{\leq 0}$ is closed under colimits and $\tau^{\leq 0} i_\alpha^!$ is right adjoint to the restriction $i_{\alpha*} : \operatorname{IndCoh}(X_\alpha)^{\leq 0} \rightarrow \operatorname{IndCoh}(X)^{\leq 0}$). It now follows that $e_{\mathcal{F}, X}$ takes $\operatorname{IndCoh}(X)^{\leq 0}$ to $\operatorname{IndCoh}(X \times Z)^{\leq n}$. \square

We note the following consistency between Definitions 8.2 and 8.7, which is true by construction when X and Z are truncated, and whose statement implicitly uses Proposition 6.18.

Proposition 8.10. *Let X and Z be geometric stacks such that Z is reasonable, and let $\mathcal{F} \in \operatorname{Coh}(Z)$. Then the functors $e_{\mathcal{F}, X}$ of Definitions 8.2 and 8.7 are canonically isomorphic.*

Proof. Since the functor $e_{\mathcal{F}, X}$ of Definition 8.7 is bounded (Proposition 8.9), it suffices to show it fits into a diagram of the form (8.3). Let $X \cong \operatorname{colim} X_\alpha$, $Z \cong \operatorname{colim} Z_\beta$ be respectively an ind-geometric and a reasonable presentation, and write $\mathcal{F} \cong i_{\beta*}(\mathcal{F}_\beta)$ for some β and $\mathcal{F}_\beta \in \operatorname{Coh}(Z_\beta)$. By Proposition 8.4 the functors $e_{i_{\beta*}(\mathcal{F}_\beta), X_\alpha}$ form a filtered system in $(\mathcal{P}_r^L)^{\Delta^1}$

which lifts to a filtered system in $(\mathcal{P}r^L)_{/e_{\Psi_Z(\mathcal{F}),X}}^{\Delta^1}$ (i.e. given termwise by taking $e_{i_{\beta\gamma*}(\mathcal{F}_\beta),X_\alpha}$ to the diagram realizing the isomorphism $(i_\alpha \times i_\gamma)_{*,QC} \Psi_{X_\alpha \times Z_\gamma} e_{i_{\beta\gamma*}(\mathcal{F}_\beta),X_\alpha} \cong e_{\Psi_Z(\mathcal{F}),X} i_{\alpha*,QC} \Psi_{X_\alpha}$). By Proposition 6.11 and [Lur09, Prop. 1.2.13.8] its colimit in $(\mathcal{P}r^L)_{/e_{\Psi_Z(\mathcal{F}),X}}^{\Delta^1}$ is a diagram whose top and bottom arrows are $e_{\mathcal{F},X}$ and $e_{\Psi_Z(\mathcal{F}),X}$. But the vertical arrows in this diagram are t-exact and induce equivalences of left completions by Proposition 6.13 and t-exactness of the $\Psi(-)$ functors and the pushforward functors in the filtered system, hence they are isomorphic to Ψ_X and $\Psi_{X \times Z}$. \square

We now return to the (tedious but ultimately straightforward) proof of Proposition 8.4.

Proof of Proposition 8.4. We can regard the natural transformation on the bottom of (8.5) as a functor $\text{Corr}(\text{GStk}_k)^{\times 2}_{fcd;ftd} \rightarrow \widehat{\text{Cat}}_\infty^{\Delta^1}$ taking (X, Y) to $\text{QCoh}(X) \times \text{QCoh}(Y)^b \rightarrow \text{QCoh}(X \times Y)$. Here we write $\widehat{\text{Cat}}_\infty^{\Delta^1} := \text{Fun}(\Delta^1, \widehat{\text{Cat}}_\infty)$, similarly in other cases below. By construction this functor factors through the category $\widehat{\text{Cat}}_{cc}^{b,\Delta^1}$ defined as follows. We set $\mathcal{P}r_{cc} := \mathcal{P}r_{\text{cpl}}^{\text{St},b} \times \widehat{\text{Cat}}_\infty \times \mathcal{P}r_{\text{acpl}}^{\text{St},b}$, regarding it as a category over $\widehat{\text{Cat}}_\infty^{\times 2}$ via $(\widehat{\mathcal{C}}, \mathcal{D}, \widehat{\mathcal{E}}) \mapsto (\widehat{\mathcal{C}} \times \mathcal{D}, \widehat{\mathcal{E}})$. We then write $\widehat{\text{Cat}}_{cc}^{\Delta^1} := \widehat{\text{Cat}}_\infty^{\Delta^1} \times_{\widehat{\text{Cat}}_\infty^{\times 2}} \mathcal{P}r_{cc}$ for the category of tuples $(\widehat{\mathcal{C}}, \mathcal{D}, \widehat{\mathcal{E}}, F)$, where $\widehat{\mathcal{C}}, \widehat{\mathcal{E}} \in \mathcal{P}r_{\text{cpl}}^{\text{St},b}$, $\mathcal{D} \in \widehat{\text{Cat}}_\infty$, and $F : \widehat{\mathcal{C}} \times \mathcal{D} \rightarrow \widehat{\mathcal{E}}$. Finally, we write $\widehat{\text{Cat}}_{cc}^{b,\Delta^1}$ for the full subcategory of such tuples whose associated functor $\mathcal{D} \rightarrow \text{Fun}(\widehat{\mathcal{C}}, \widehat{\mathcal{E}})$ takes values in $\text{LFun}^b(\widehat{\mathcal{C}}, \widehat{\mathcal{E}})$.

We will prove the claim by constructing a functor $\widehat{\text{Cat}}_{cc}^{b,\Delta^1} \rightarrow \widehat{\text{Cat}}_\infty^{\Delta^1 \times \Delta^1}$ which takes the bottom arrow in (8.5), evaluated on any (X, Y) , to the entire diagram. Let us set $\mathcal{P}r_{aa} := \mathcal{P}r_{\text{acpl}}^{\text{St},b} \times \widehat{\text{Cat}}_\infty \times \mathcal{P}r_{\text{acpl}}^{\text{St},b}$ and $\mathcal{P}r_{ac} := \mathcal{P}r_{\text{acpl}}^{\text{St},b} \times \widehat{\text{Cat}}_\infty \times \mathcal{P}r_{\text{cpl}}^{\text{St},b}$, defining $\widehat{\text{Cat}}_{aa}^{\Delta^1}$, etc., as above. The main step will be to first construct a diagram

$$(8.11) \quad \begin{array}{ccccc} \widehat{\text{Cat}}_{aa}^{\Delta^1} & \longrightarrow & \widehat{\text{Cat}}_{ac}^{\Delta^1} & \longleftarrow & \widehat{\text{Cat}}_{cc}^{\Delta^1} \\ \uparrow & & \uparrow & & \uparrow \\ \widehat{\text{Cat}}_{aa}^{b,\Delta^1} & \xrightarrow{\sim} & \widehat{\text{Cat}}_{ac}^{b,\Delta^1} & \xleftarrow{\sim} & \widehat{\text{Cat}}_{cc}^{b,\Delta^1} \end{array}$$

in which the bottom functors are equivalences, and such that under these equivalences the bottom arrow in (8.5) (as a $\widehat{\text{Cat}}_{cc}^{b,\Delta^1}$ -valued functor) corresponds to the top arrow and overall composition of (8.5) (respectively as a $\widehat{\text{Cat}}_{aa}^{b,\Delta^1}$ -valued and a $\widehat{\text{Cat}}_{ac}^{b,\Delta^1}$ -valued functor).

Let us explicitly construct the top left functor in (8.11) and show that it restricts to the equivalence on the bottom left; the construction of the right square is parallel. To do this we introduce the following pair of diagrams.

$$(8.12) \quad \begin{array}{ccccc} \widehat{\text{Cat}}_\infty^{\Delta^2} & \longrightarrow & \widehat{\text{Cat}}_\infty^{\Delta_2^2} & \longrightarrow & \widehat{\text{Cat}}_\infty^{\Delta_{02}^1} \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{\text{Cat}}_\infty^{\Delta_1^2} & \longrightarrow & \widehat{\text{Cat}}_\infty^{\Delta_0^0 \cup \Delta_{12}^1} & \longrightarrow & \widehat{\text{Cat}}_\infty^{\Delta_0^0 \cup \Delta_2^0} \\ \downarrow & & \downarrow & \nearrow & \downarrow \\ \widehat{\text{Cat}}_\infty^{\Delta_{01}^1} & \longrightarrow & \widehat{\text{Cat}}_\infty^{\Delta_0^0 \cup \Delta_1^0} & \longleftarrow & \mathcal{P}r_{aa} \end{array} \quad \begin{array}{ccccc} \widehat{\text{Cat}}_{aac}^{\Delta^2} & \longrightarrow & \widehat{\text{Cat}}_{aac}^{\Delta_2^2} & \longrightarrow & \widehat{\text{Cat}}_{ac}^{\Delta_{02}^1} \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{\text{Cat}}_{aac}^{\Delta_1^2} & \longrightarrow & & & \\ \downarrow & & & & \\ \widehat{\text{Cat}}_{aa}^{\Delta_{01}^1} & \longrightarrow & & & \mathcal{P}r_{aa} \end{array}$$

Here the subscripts in e.g. Δ_{02}^1 indicate a particular 1-simplex of Δ^2 , and the arrows in the left diagram not involving $\mathcal{P}r_{aa}$ are induced by restriction. The unit of the localization $\mathcal{C} \mapsto \widehat{\mathcal{C}}$ on $\mathcal{P}r_{\text{repl}}^{\text{St},b}$ induces a functor $\mathcal{P}r_{\text{acpl}}^{\text{St},b} \rightarrow \widehat{\text{Cat}}_{\infty}^{\Delta^1}$ taking $\check{\mathcal{C}}$ to $\check{\mathcal{C}} \rightarrow \widehat{\mathcal{C}}$, and the diagonal arrow out of $\mathcal{P}r_{aa}$ is the induced functor $(\check{\mathcal{C}}, \mathcal{D}, \check{\mathcal{E}}) \mapsto (\check{\mathcal{C}} \times \mathcal{D}, \check{\mathcal{E}} \rightarrow \widehat{\mathcal{E}})$. The horizontal and vertical arrows out of $\mathcal{P}r_{aa}$ thus take $(\check{\mathcal{C}}, \mathcal{D}, \check{\mathcal{E}})$ to $(\check{\mathcal{C}} \times \mathcal{D}, \check{\mathcal{E}})$ and $(\check{\mathcal{C}} \times \mathcal{D}, \widehat{\mathcal{E}})$, respectively.

In the right diagram, $\widehat{\text{Cat}}_{aa}^{\Delta_{01}^1}$ and $\widehat{\text{Cat}}_{ac}^{\Delta_{02}^1}$ are respectively the fiber products of the bottom row and right column of the left diagram (which is consistent with our existing notation after forgetting subscripts). The remaining three categories are the fiber products of their counterparts on the left with $\mathcal{P}r_{aa}$ over $\widehat{\text{Cat}}_{\infty}^{\Delta_{00}^1 \cup \Delta_{12}^1}$. Note that their natural maps to $\mathcal{P}r_{aa}$ indeed factor through those of $\widehat{\text{Cat}}_{aa}^{\Delta_{01}^1}$ and $\widehat{\text{Cat}}_{ac}^{\Delta_{02}^1}$ as indicated.

We claim the leftmost vertical functors in the right diagram are equivalences. For the top, this follows since it is base changed from its counterpart on the left, which is an equivalence by [Lur09, Cor. 2.3.2.2]. For the bottom, this follows from the bottom left square of the left diagram being Cartesian. Composing the inverse equivalences with the top arrows we obtain a functor $\widehat{\text{Cat}}_{aa}^{\Delta_{01}^1} \rightarrow \widehat{\text{Cat}}_{ac}^{\Delta_{01}^1}$ as desired.

The fiber of this functor over a particular $(\check{\mathcal{C}}, \mathcal{D}, \check{\mathcal{E}}) \in \mathcal{P}r_{aa}$ is the map $\text{Fun}(\check{\mathcal{C}} \times \mathcal{D}, \check{\mathcal{E}})^{\cong} \rightarrow \text{Fun}(\check{\mathcal{C}} \times \mathcal{D}, \widehat{\mathcal{E}})^{\cong}$ given by composition with $\check{\mathcal{E}} \rightarrow \widehat{\mathcal{E}}$ (the superscripts indicate that non-invertible natural transformations are excluded). Since the corresponding map $\text{LFun}^b(\check{\mathcal{C}}, \check{\mathcal{E}}) \rightarrow \text{LFun}^b(\check{\mathcal{C}}, \widehat{\mathcal{E}})$ is an equivalence by definition, it follows that $\widehat{\text{Cat}}_{aa}^{\Delta_{01}^1} \rightarrow \widehat{\text{Cat}}_{ac}^{\Delta_{01}^1}$ restricts to a functor $\widehat{\text{Cat}}_{aa}^{b, \Delta_{01}^1} \rightarrow \widehat{\text{Cat}}_{ac}^{b, \Delta_{01}^1}$ which in turns restricts to an isomorphism of fibers over $\mathcal{P}r_{aa}$.

We recall that $\widehat{\text{Cat}}_{\infty}^{\Delta^1}$ is a bifibration over $\widehat{\text{Cat}}_{\infty}^2$ [Lur09, Cor. 2.4.7.11]. It follows from the definitions that bifibrations are stable under pullback along products of maps and under restriction to full subcategories. In particular, $\widehat{\text{Cat}}_{aa}^{b, \Delta_{01}^1}$ and $\widehat{\text{Cat}}_{ac}^{b, \Delta_{01}^1}$ are bifibrations over $\mathcal{P}r_{aa}$, factored as the product of $\mathcal{P}r_{\text{acpl}}^{\text{St},b} \times \widehat{\text{Cat}}_{\infty}$ and $\mathcal{P}r_{\text{acpl}}^{\text{St},b}$. It now follows from [Lur09, Prop. 2.4.7.6] and the previous paragraph that $\widehat{\text{Cat}}_{aa}^{b, \Delta_{01}^1} \rightarrow \widehat{\text{Cat}}_{ac}^{b, \Delta_{01}^1}$ is an equivalence.

To complete the proof, note that by construction the bottom row of (8.11) factors as

$$\widehat{\text{Cat}}_{aa}^{b, \Delta_{01}^1} \xleftarrow{\sim} \widehat{\text{Cat}}_{aac}^{b, \Delta^2} \xrightarrow{\sim} \widehat{\text{Cat}}_{ac}^{b, \Delta_{02}^1} \xleftarrow{\sim} \widehat{\text{Cat}}_{acc}^{b, \Delta^2} \xrightarrow{\sim} \widehat{\text{Cat}}_{cc}^{b, \Delta_{12}^1}.$$

Here we again use subscripts to indicate edges in Δ^2 , $\widehat{\text{Cat}}_{acc}^{\Delta^2}$ is the evident counterpart of $\widehat{\text{Cat}}_{aac}^{\Delta^2}$, and $\widehat{\text{Cat}}_{aac}^{\Delta^2}$, $\widehat{\text{Cat}}_{acc}^{\Delta^2}$ are the full subcategories corresponding to $\widehat{\text{Cat}}_{aa}^{b, \Delta^1}$. The middle terms in this factorization map to $\widehat{\text{Cat}}_{\infty}^{\Delta^2}$, $\widehat{\text{Cat}}_{\infty}^{\Delta^1}$, and $\widehat{\text{Cat}}_{\infty}^{\Delta^2}$ compatibly with the relevant maps, hence we obtain a functor

$$\widehat{\text{Cat}}_{cc}^{b, \Delta^1} \rightarrow \widehat{\text{Cat}}_{\infty}^{\Delta^2} \times_{\widehat{\text{Cat}}_{\infty}^{\Delta_{02}^1}} \widehat{\text{Cat}}_{\infty}^{\Delta^2} \cong \widehat{\text{Cat}}_{\infty}^{\Delta^1 \times \Delta^1}. \quad \square$$

8.2. External and internal adjoints. Let X and Z be ind-geometric stacks such that Z is reasonable, and let $\mathcal{F} \in \text{Coh}(Z)$. We denote the right adjoint of $e_{\mathcal{F}, X}$ by

$$e_{\mathcal{F}, X}^R : \text{IndCoh}(X \times Z) \rightarrow \text{IndCoh}(X).$$

When X and Z are geometric and $\mathcal{F} \in \mathrm{QCoh}(X)$ we define $e_{\mathcal{F},X}^R : \mathrm{QCoh}(X \times Z) \rightarrow \mathrm{QCoh}(X)$ similarly. These are external counterparts of the internal sheaf Hom , which we define in the setting of ind-coherent sheaves on a reasonable ind-geometric stack X as

$$(8.13) \quad \mathcal{H}em(\mathcal{F}, -) := e_{\mathcal{F},X}^R \Delta_{X*} : \mathrm{IndCoh}(X) \rightarrow \mathrm{IndCoh}(X).$$

Here we again assume $\mathcal{F} \in \mathrm{Coh}(X)$. This definition is justified in part by the following result, and will be more fully justified by Corollary 8.30.

Proposition 8.14. *Let X and Z be ind-geometric stacks such that Z is reasonable, and let $\mathcal{F} \in \mathrm{Coh}(Z)$. Then $e_{\mathcal{F},X}^R$ is left bounded. If X and Z are geometric, the Beck-Chevalley map $\Psi_X e_{\mathcal{F},X}^R(\mathcal{G}) \rightarrow e_{\Psi_Z(\mathcal{F}),X}^R \Psi_{X \times Z}(\mathcal{G})$ is an isomorphism for all $\mathcal{G} \in \mathrm{IndCoh}(X \times Z)^+$, and the induced map $\Psi_X \mathcal{H}em(\mathcal{F}, \mathcal{G}) \rightarrow \mathcal{H}em(\Psi_X(\mathcal{F}), \Psi_X(\mathcal{G}))$ is an isomorphism for all $\mathcal{G} \in \mathrm{IndCoh}(X)^+$.*

Proof. Since $e_{\mathcal{F},X}$ is bounded (Proposition 8.9), $e_{\mathcal{F},X}^R$ is left bounded and the two functors restrict to an adjunction between $\mathrm{IndCoh}(X)^+$ and $\mathrm{IndCoh}(X \times Z)^+$. The analogous statement holds for $e_{\Psi_Z(\mathcal{F}),X}$ and $e_{\Psi_Z(\mathcal{F}),X}^R$, and the second claim follows and since $\Psi_{(-)}$ restricts to an equivalence $\mathrm{IndCoh}(-)^+ \xrightarrow{\sim} \mathrm{QCoh}(-)^+$ and since $\Psi_{X \times Z} e_{\mathcal{F},X} \cong e_{\Psi_Z(\mathcal{F}),X} \Psi_X$ (Proposition 8.10). The third follows since Δ_{X*} is also compatible with the $\Psi_{(-)}$ functors, and since we have an isomorphism $- \otimes \Psi_Z(\mathcal{F}) \cong \Delta_{X*} e_{\Psi_Z(\mathcal{F}),X}$ of functors $\mathrm{QCoh}(X) \rightarrow \mathrm{QCoh}(X)$. \square

Suppose that $f : X' \rightarrow X$, $g : Z' \rightarrow Z$ are morphisms of ind-finite cohomological dimension between ind-geometric stacks, and that Z' and Z are reasonable. Suppose also that either f and g are of finite Tor-dimension, or that they have coherent pullback and X , X' , $X \times Z$, and $X' \times Z'$ are coherent. Then for $\mathcal{F} \in \mathrm{Coh}(Z)$ the isomorphism $(f \times g)^* e_{\mathcal{F},X} \cong e_{g^*(\mathcal{F}),X'} f^*$ of functors $\mathrm{IndCoh}(X) \rightarrow \mathrm{IndCoh}(X' \times Z')$ yields an isomorphism

$$(8.15) \quad e_{\mathcal{F},X}^R(f \times g)_* \cong f_* e_{g^*(\mathcal{F}),X'}^R$$

of right adjoints.

Similarly, suppose instead that f and g are ind-proper and that g is almost of ind-finite presentation. Then for $\mathcal{F} \in \mathrm{Coh}(Z')$ the isomorphism $(f \times g)_* e_{\mathcal{F},X'} \cong e_{g_*(\mathcal{F}),X} f_*$ of functors $\mathrm{IndCoh}(X') \rightarrow \mathrm{IndCoh}(X \times Z)$ yields an isomorphism

$$(8.16) \quad e_{\mathcal{F},X'}^R(f \times g)^! \cong f^! e_{g_*(\mathcal{F}),X}^R$$

of right adjoints.

Proposition 8.17. *Let X and Z be ind-geometric stacks such that Z is reasonable and X is semi-reasonable, and let $\mathcal{F} \in \mathrm{Coh}(Z)$. Then $e_{\mathcal{F},X}^R$ is almost continuous.*

Lemma 8.18. *Proposition 8.17 is true when X is affine and Z is geometric.*

Proof. In this case $\mathrm{QCoh}(X)$ is compactly generated by perfect sheaves. Thus $e_{\Psi_Z(\mathcal{F}),X}^R$ is almost continuous since $e_{\Psi_Z(\mathcal{F}),X}$ takes compact objects to almost compact objects by the

proof of Lemma ?? (if X is not truncated and $\mathcal{G} \in \mathrm{QCoh}(X)$ is perfect, $e_{\Psi_Z(\mathcal{F}),X}(\mathcal{G})$ may be unbounded but is still almost perfect, hence almost compact [Lur18, Prop. 9.1.5.1]). It follows that $e_{\mathcal{F},X}^R$ is almost continuous since it and $e_{\Psi_Z(\mathcal{F}),X}^R$ are left bounded and intertwined by the equivalences $\Psi_{(-)}^+$ (Proposition 8.14). \square

Lemma 8.19. *Let X and Y be affine schemes, Z a reasonable geometric stack, and $\mathcal{F} \in \mathrm{Coh}(Z)$. If $h : X \rightarrow Y$ is a morphism of finite Tor-dimension and $h' = h \times id_Z$, then for any $\mathcal{G} \in \mathrm{IndCoh}(Y \times Z)^+$ the Beck-Chevalley map $h^*e_{\mathcal{F},Y}^R(\mathcal{G}) \rightarrow e_{\mathcal{F},X}^Rh'^*(\mathcal{G})$ is an isomorphism.*

Proof. Write $X \cong \mathrm{Spec} A$ and $Y \cong \mathrm{Spec} B$, and assume first that Z is truncated and geometric. Since h is affine and $h^*e_{\mathcal{F},Y}^R(\mathcal{G}), e_{\mathcal{F},X}^Rh'^*(\mathcal{G}) \in \mathrm{IndCoh}(X)^+$, it suffices to show the given map is an isomorphism after composing with $\Psi_Y h_*$. Since the second factor of

$$\Psi_Y h_* h^* e_{\mathcal{F},Y}^R(\mathcal{G}) \rightarrow \Psi_Y h_* e_{\mathcal{F},X}^R h'^*(\mathcal{G}) \rightarrow \Psi_Y e_{\mathcal{F},Y}^R h'_* h'^*$$

is an isomorphism by (8.15), it suffices to show the composition is. After commuting the given functors with Ψ_Y (hence using Proposition 8.14) and applying the projection formula, the composition becomes identified with the Beck-Chevalley map

$$(8.20) \quad \theta_M : e_{\Psi_Z(\mathcal{F}),Y}^R \Psi_{Y \times Z}(\mathcal{G}) \otimes M \rightarrow e_{\Psi_Z(\mathcal{F}),Y}^R(\Psi_{Y \times Z}(\mathcal{G}) \otimes p_Y^*(M)),$$

where $p_Y : Y \times Z \rightarrow Y$ is the projection and we substitute A for $M \in \mathrm{Mod}_B \cong \mathrm{QCoh}(Y)$.

Write \mathcal{C} for the full subcategory of $M \in \mathrm{Mod}_B$ such that θ_M is an isomorphism. The assignment $M \mapsto \theta_M$ extends to a functor $\mathrm{Mod}_B \rightarrow \mathrm{Mod}_B^{\Delta^1}$, which is exact since both terms in (8.20) are exact in M . It follows that \mathcal{C} is stable and closed under retracts, as isomorphisms form a stable subcategory closed under retracts in $\mathrm{Mod}_B^{\Delta^1}$. Clearly $B \in \mathcal{C}$, hence \mathcal{C} contains all perfect B -modules. If A is of Tor-dimension $\leq n$ over B , then we can write it as a filtered colimit $A \cong \mathrm{colim}_\alpha M_\alpha$ of perfect B -modules of Tor-dimension $\leq n$ [Lur18, Prop. 9.6.7.1]. The claim now follows since tensoring is continuous, since the $\mathcal{G} \otimes p_Y^*(M_\alpha)$ are uniformly bounded below, and since $e_{\mathcal{F},Y}^R$ is almost continuous by Lemma 8.18. \square

Lemma 8.21. *Let X and Z be geometric stacks such that Z is reasonable, and let $\mathcal{F} \in \mathrm{Coh}(Z)$. If U is an affine scheme, $h : U \rightarrow X$ a flat cover, and $h' = h \times id_Z$, then for any $\mathcal{G} \in \mathrm{IndCoh}(X \times Z)^+$ the Beck-Chevalley map $h^*e_{\mathcal{F},X}^R(\mathcal{G}) \rightarrow e_{\mathcal{F},U}^Rh'^*(\mathcal{G})$ is an isomorphism.*

Proof. Write U_\bullet for the Čech nerve of h . Let $h_k : U_k \rightarrow X$ denote the natural map and let $h'_k := h_k \times id_Z$. Given a morphism $p : i \rightarrow j$ in Δ_s , let $h_p : U_j \rightarrow U_i$ denote the associated map and let $h'_p := h_p \times id_Z$. By construction we have compatible isomorphisms $e_{\mathcal{F},U_j} h_p^*(\mathcal{G}') \cong h_p'^* e_{\mathcal{F},U_i}(\mathcal{G}')$ for any p and any $\mathcal{G}' \in \mathrm{IndCoh}(U_i)$. By Lemma 8.19 the Beck-Chevalley map $h_p^* e_{\mathcal{F},U_i}^R(\mathcal{G}') \rightarrow e_{\mathcal{F},U_j}^R h_p'^*(\mathcal{G}')$ is an isomorphism for any p and any $\mathcal{G}' \in \mathrm{IndCoh}(U_i \times Z)^+$. Since h is faithfully flat, we have $\mathrm{IndCoh}(X)^+ \cong \lim_{\Delta_s} \mathrm{IndCoh}(U_i)^+$ and $\mathrm{IndCoh}(X \times Z)^+ \cong \lim_{\Delta_s} \mathrm{IndCoh}(U_i \times Z)^+$. The claim now follows from [Lur17, Cor. 4.7.5.18]. \square

Proof of Proposition 8.17. Suppose first that X and Z are geometric, and let $h : U \cong \operatorname{Spec} A \rightarrow X$ be a flat cover. Since $e_{\mathcal{F},X}^R$ is left bounded, and since h^* is conservative on $\operatorname{IndCoh}(X)^+$ in addition to being continuous, it suffices to show $h^*e_{\mathcal{F},X}^R$ is almost continuous. It then suffices to show $e_{\mathcal{F},U}^R h'^*$ is almost continuous, since by Lemma 8.21 it has the same restriction to $\operatorname{IndCoh}(X \times Z)^+$. But this follows since h'^* is continuous and left bounded, and since $e_{\mathcal{F},U}^R$ is almost continuous by 8.18.

Still assuming Z is geometric, let $X \cong \operatorname{colim} X_\alpha$ be a semi-reasonable presentation and $\mathcal{G} \cong \operatorname{colim} \mathcal{G}_\beta$ a filtered colimit in $\operatorname{IndCoh}(X \times Z)^{\geq n}$ for some n . Since $\operatorname{IndCoh}(X) \cong \lim \operatorname{IndCoh}(X_\alpha)$ in $\widehat{\operatorname{Cat}}_\infty$, it suffices to show the second factor in

$$\operatorname{colim}_\beta i_\alpha^! e_{\mathcal{F},X}^R(\mathcal{G}_\beta) \rightarrow i_\alpha^! \operatorname{colim}_\beta e_{\mathcal{F},X}^R(\mathcal{G}_\beta) \rightarrow i_\alpha^! e_{\mathcal{F},X}^R(\operatorname{colim}_\beta \mathcal{G}_\beta)$$

is an isomorphism for all α . The first factor is an isomorphism since $e_{\mathcal{F},X}^R$ is left bounded and since $i_\alpha^!$ is almost continuous by the proof of Proposition 6.21. But $i_\alpha^! e_{\mathcal{F},X}^R \cong e_{\mathcal{F},X_\alpha}^R (i_\alpha \times id_Z)^!$ by (8.16), so the composition is an isomorphism by the first paragraph and the almost continuity and left t-exactness of $(i_\alpha \times id_Z)^!$.

Finally, suppose $Z \cong \operatorname{colim} Z_\alpha$ is a reasonable presentation, and write $\mathcal{F} \cong i_{\alpha*}(\mathcal{F}_\alpha)$ for some α and $\mathcal{F}_\alpha \in \operatorname{Coh}(Z_\alpha)$. By (8.16) we have $e_{\mathcal{F},X}^R \cong e_{\mathcal{F}_\alpha,X}^R i_\alpha^!$, and the claim follows since $i_\alpha^!$ is left t-exact and since $e_{\mathcal{F}_\alpha,X}^R$ and $i_\alpha^!$ are almost continuous. \square

8.3. External products and pushforward. If X , Y , and Z are geometric stacks and $\mathcal{F} \in \operatorname{QCoh}(Z)$, then for any $f : X \rightarrow Y$ the isomorphism $(f \times id_Z)^* e_{\mathcal{F},Y} \cong e_{\mathcal{F},X} f^*$ of functors $\operatorname{QCoh}(Y) \rightarrow \operatorname{QCoh}(X \times Z)$ yields an isomorphism

$$(8.22) \quad f_* e_{X,\mathcal{F}}^R \cong e_{Y,\mathcal{F}}^R (f \times id_Z)_*$$

of right adjoints $\operatorname{QCoh}(X \times Z) \rightarrow \operatorname{QCoh}(Y)$. This is an external counterpart of the isomorphism

$$\mathcal{H}em(\mathcal{F}, f_*(-)) \cong f_* \mathcal{H}em(f^*(\mathcal{F}), -)$$

obtained for $\mathcal{F} \in \operatorname{QCoh}(X)$ by taking right adjoints of the isomorphism $f^*(- \otimes \mathcal{F}) \cong f^*(-) \otimes f^*(\mathcal{F})$. Similarly, if f is proper the projection isomorphism $f_*(\mathcal{F} \otimes f^*(-)) \cong f_*(\mathcal{F}) \otimes -$ yields an isomorphism

$$(8.23) \quad f_* \mathcal{H}em(\mathcal{F}, f^!(-)) \cong \mathcal{H}em(f_*(\mathcal{F}), -)$$

of right adjoints.

This section generalizes these isomorphisms to ind-coherent sheaves under suitable hypotheses, letting X , Y , and Z be ind-geometric and $f : X \rightarrow Y$ a morphism of ind-finite cohomological dimension. In this setting $f_* : \operatorname{IndCoh}(X) \rightarrow \operatorname{IndCoh}(Y)$ typically does not have a left adjoint. Instead, if Z is reasonable and $\mathcal{F} \in \operatorname{Coh}(Z)$, we may take the isomorphism $(f \times id_Z)^* e_{\mathcal{F},X} \cong e_{\mathcal{F},Y} f^*$ of functors $\operatorname{IndCoh}(X) \rightarrow \operatorname{IndCoh}(Y \times Z)$ and consider the associated

Beck-Chevalley transformation

$$(8.24) \quad f_* e_{\mathcal{F}, X}^R \rightarrow e_{\mathcal{F}, Y}^R (f \times id_Z)_*$$

of functors $\text{IndCoh}(X \times Z) \rightarrow \text{IndCoh}(Y)$.

Suppose in addition that X and Y are reasonable and that f is ind-proper and almost of ind-finite presentation. Then for $\mathcal{F} \in \text{Coh}(Y)$ and $\mathcal{G} \in \text{IndCoh}(Y)$ we have a transformation

$$(8.25) \quad f_* \mathcal{H}em(\mathcal{F}, f^!(-)) \rightarrow \mathcal{H}em(f_*(\mathcal{F}), -)$$

of functors $\text{IndCoh}(Y) \rightarrow \text{IndCoh}(Y)$ given by the composition

$$(8.26) \quad f_* e_{\mathcal{F}, X}^R \Delta_X f^! \rightarrow e_{\mathcal{F}, Y}^R (f \times id_X)_* \Delta_X f^! \rightarrow e_{\mathcal{F}, Y}^R (id_Y \times f)^! \Delta_Y \cong e_{f_*(\mathcal{F}), Y}^R \Delta_Y.$$

Here the last isomorphism is given by (8.16), and we have used the fact that $\Delta_X \circ (f \times id_X)$ is the base change of Δ_Y along $id_Y \times f$. In the geometric case one can check that if we restrict to bounded below subcategories, (8.25) and (8.24) are indeed identified with the isomorphisms (8.22) and (8.23) under the equivalences $\text{IndCoh}(-)^+ \cong \text{QCoh}(-)^+$.

Proposition 8.27. *Let X , Y , and Z be ind-geometric stacks such that Z is reasonable and Y is semi-reasonable. Let $f : X \rightarrow Y$ be a morphism of ind-finite cohomological dimension and $f' = f \times id_Z$. Then for any $\mathcal{F} \in \text{Coh}(Z)$ and $\mathcal{G} \in \text{IndCoh}(X \times Z)^+$ the Beck-Chevalley map $f_* e_{\mathcal{F}, X}^R(\mathcal{G}) \rightarrow e_{\mathcal{F}, Y}^R f'_*(\mathcal{G})$ is an isomorphism.*

Proof. First assume X , Y , and Z are geometric and Z is truncated. By Proposition 8.14 all functors involved are left bounded and compatible with the equivalences $\text{IndCoh}(-)^+ \cong \text{QCoh}(-)^+$, hence the claim follows from (8.22).

Now let $X \cong \text{colim } X_\alpha$ be an ind-geometric presentation, still supposing Y and Z are geometric and Z is truncated. For every α we have a diagram

$$\begin{array}{ccccc} X_\alpha \times Z & \xrightarrow{i'_\alpha} & X \times Z & \xrightarrow{f'} & Y \times Z \\ \downarrow & & \downarrow & & \downarrow \\ X_\alpha & \xrightarrow{i_\alpha} & X & \xrightarrow{f} & Y \end{array}$$

of Cartesian squares. Expanding the Beck-Chevalley maps in terms of units and counits, it follows from the basic properties of these that the counits $i_{\alpha*} i_\alpha^! \rightarrow id_{IC(X)}$ and $i'_{\alpha*} i_\alpha^! \rightarrow id_{IC(X \times Z)}$ fit into diagrams

$$(8.28) \quad \begin{array}{ccccc} f_* i_{\alpha*} i_\alpha^! e_{\mathcal{F}, X}^R(\mathcal{G}) & \rightarrow & f_* i_{\alpha*} e_{\mathcal{F}, X_\alpha}^R i_\alpha^! (\mathcal{G}) & \rightarrow & e_{\mathcal{F}, Y}^R f'_* i'_{\alpha*} i_\alpha^! (\mathcal{G}) \\ \downarrow & & & & \downarrow \\ f_* e_{\mathcal{F}, X}^R(\mathcal{G}) & \xrightarrow{\quad\quad\quad} & & & e_{\mathcal{F}, Y}^R f'_*(\mathcal{G}) \end{array}$$

compatibly for $\beta \geq \alpha$. That is, the counits $i_{\alpha*} i_\alpha^! \cong i_{\beta*} i_{\alpha\beta*} i_{\alpha\beta}^! i_\beta^! \rightarrow i_{\beta*} i_\beta^!$ and $i'_{\alpha*} i_\alpha^! \cong i'_{\beta*} i_{\alpha\beta*} i_{\alpha\beta}^! i_\beta^! \rightarrow i'_{\beta*} i_\beta^!$ intertwine the top compositions in (8.28) for $\beta \geq \alpha$. Passing to colimits

we obtain a diagram

$$\begin{array}{ccc} \operatorname{colim} f_* i_{\alpha*} i_{\alpha}^! e_{\mathcal{F},X}^R(\mathcal{G}) & \longrightarrow & \operatorname{colim} e_{\mathcal{F},Y}^R f'_* i_{\alpha*} i_{\alpha}^! (\mathcal{G}) \\ \downarrow & & \downarrow \\ f_* e_{\mathcal{F},X}^R(\mathcal{G}) & \longrightarrow & e_{\mathcal{F},Y}^R f'_*(\mathcal{G}). \end{array}$$

The vertical arrows are isomorphisms by Lemma 2.2, the continuity of f_* and f'_* , and the left t-exactness of $f'_* i_{\alpha*} i_{\alpha}^!$ together with the almost continuity of $e_{\mathcal{F},Y}^R$ (Proposition 8.17). But the top arrow is an isomorphism since the top arrows in (8.28) are by (8.16) and the previous paragraph, hence so is the bottom arrow.

Now let $Y \cong \operatorname{colim} Y_{\alpha}$ be a semi-reasonable presentation and write $\mathcal{F} \cong i_*(\mathcal{F}')$ for some truncated geometric substack $i : Z' \rightarrow Z$ and $\mathcal{F}' \in \operatorname{Coh}(Z')$. For any α we have a diagram

$$\begin{array}{ccccc} & & X_{\alpha} & \longleftarrow & X_{\alpha} \times Z \\ & \swarrow j_{\alpha} & \downarrow f_{\alpha} & \swarrow j'_{\alpha} & \downarrow f'_{\alpha} \\ X & \longleftarrow & X \times Z & & \\ \downarrow f & & \downarrow f' & & \\ & \swarrow i_{\alpha} & Y_{\alpha} & \longleftarrow & Y_{\alpha} \times Z \\ & & \downarrow i'_{\alpha} & & \\ Y & \longleftarrow & Y \times Z & & \end{array}$$

with all faces but the top and bottom Cartesian, and with $i'_{\alpha} = i_{\alpha} \times i$. We have a diagram

$$\begin{array}{ccccc} f_{\alpha*} j_{\alpha}^! e_{\mathcal{F},X}^R(\mathcal{G}) & \longrightarrow & i_{\alpha}^! f_* e_{\mathcal{F},X}^R(\mathcal{G}) & \longrightarrow & i_{\alpha}^! e_{\mathcal{F},Y}^R f'_*(\mathcal{G}) \\ \downarrow \wr & & & & \downarrow \wr \\ f_{\alpha*} e_{\mathcal{F},X_{\alpha}}^R j_{\alpha}^! (\mathcal{G}) & \longrightarrow & e_{\mathcal{F},Y_{\alpha}}^R f'_{\alpha*} j_{\alpha}^! (\mathcal{G}) & \longrightarrow & e_{\mathcal{F},Y_{\alpha}}^R i_{\alpha}^! f'_*(\mathcal{G}) \end{array}$$

in $\operatorname{IndCoh}(Y_{\alpha})$, where the vertical isomorphisms are given by (8.16). Since the functors $i_{\alpha}^!$ determine an isomorphism $\operatorname{IndCoh}(Y) \cong \lim \operatorname{IndCoh}(Y_{\alpha})$ in $\widehat{\operatorname{Cat}}_{\infty}$, it suffices to show the top right map is an isomorphism for all α . The top left and bottom right maps are isomorphisms by Proposition ??, right t-exactness of $i_{\alpha*}$ and $i'_{\alpha*}$, and left boundedness of $e_{\mathcal{F},X}^R$. But the bottom left is an isomorphism by the previous paragraph and left t-exactness of $j_{\alpha}^!$, hence the top right is as well. \square

Corollary 8.29. *Let X , Y , and Z be ind-geometric stacks such that Z is reasonable and X , Y , $X \times Z$, and $Y \times Z$ are coherent. Let $f : X \rightarrow Y$ be a morphism of ind-finite cohomological dimension and $f' = f \times \operatorname{id}_Z$. Then for any $\mathcal{F} \in \operatorname{Coh}(Z)$ and $\mathcal{G} \in \operatorname{IndCoh}(X \times Z)$ the Beck-Chevalley map $f_* e_{\mathcal{F},X}^R(\mathcal{G}) \rightarrow e_{\mathcal{F},Y}^R f'_*(\mathcal{G})$ is an isomorphism.*

Proof. Under these hypotheses $f_* e_{\mathcal{F},X}^R(\mathcal{G})$ and $e_{\mathcal{F},Y}^R f'_*$ are continuous and $\operatorname{IndCoh}(X \times Z)$ is compactly generated by $\operatorname{Coh}(X \times Z) \subset \operatorname{IndCoh}(X \times Z)^+$, hence the claim follows from Proposition 8.27. \square

Note that the extension of sheaf $\mathcal{H}om$ from the geometric to the ind-geometric setting is uniquely determined by the following corollary, since we can always write $\mathcal{F} \in \mathrm{Coh}(X)$ as $i_*(\mathcal{F}')$ for some reasonable geometric substack $i : X' \rightarrow X$ and $\mathcal{F}' \in \mathrm{Coh}(X')$.

Corollary 8.30. *Let X and Y be reasonable ind-geometric stacks, $f : X \rightarrow Y$ an ind-proper, almost ind-finitely presented morphism of finite cohomological dimension, and $\mathcal{F} \in \mathrm{Coh}(X)$. Then for any $\mathcal{G} \in \mathrm{IndCoh}(Y)^+$ the natural map $f_* \mathcal{H}om(\mathcal{F}, f^!(\mathcal{G})) \rightarrow \mathcal{H}om(f_*(\mathcal{F}), \mathcal{G})$ is an isomorphism.*

Proof. Under these hypotheses $\Delta_{X*} f^!$ is left bounded, hence the claim follows from Propositions ?? and 8.27. \square

Corollary 8.31. *Let X and Y be coherent ind-geometric stacks such that $X \times X$, $X \times Y$, and $Y \times Y$ are coherent, $f : X \rightarrow Y$ an ind-proper, almost of ind-finitely presented morphism, and $\mathcal{F} \in \mathrm{Coh}(X)$. Then for any $\mathcal{G} \in \mathrm{IndCoh}(Y)$ the natural map $f_* \mathcal{H}om(\mathcal{F}, f^!(\mathcal{G})) \rightarrow \mathcal{H}om(f_*(\mathcal{F}), \mathcal{G})$ is an isomorphism.*

Proof. Follows from Proposition 7.10 and Corollary 8.29. \square

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