TAMELY PRESENTED MORPHISMS AND COHERENT PULLBACK

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Abstract. We study two classes of morphisms in infinite type: tamely presented morphisms and morphisms with coherent pullback. These are generalizations of finitely presented morphisms and morphisms of finite Tor-dimension, respectively. The class of tamely presented stacks is restricted enough to retain the key features of finite-type schemes from the point of view of coherent sheaf theory, but wide enough to encompass many infinite-type examples of interest in geometric representation theory. The condition that a diagonal morphism has coherent pullback is a natural generalization of smoothness to the tamely presented setting, and we show such objects retain many of the good cohomological properties of smooth varieties. Our results are motivated by applications to the double affine Hecke category and its relatives from the theory of Coulomb branches.

Contents

1. Introduction

Integral (or Fourier-Mukai) transforms play a well-established role in algebraic geometry. Recall that if $p : X \to Z$, $q : Y \to Z$ are proper maps of smooth varieties, then the integral transform associated to $\mathcal{K} \in \mathrm{Coh}(X \times_Z Y)$ is the functor $\mathrm{Coh}(X) \to \mathrm{Coh}(Y)$ given by $\mathcal{F} \mapsto q_*(p^*(\mathcal{F}) \otimes \mathcal{K})$. Here $\mathrm{Coh}(-)$ denotes the bounded derived category of coherent sheaves.

In geometric representation theory integral transforms often appear in the guise of convolution products. For example, let G be a reductive group, $X = T^*\mathcal{B} \to \mathcal{N} \subset \mathfrak{g} = Z$ the Springer resolution, and $St_G = X \times_Z X$ the (derived) Steinberg variety. Then the affine Hecke algebra of $G^\vee$ is categorified by $\mathrm{Coh}^{G \times \mathbb{G}_m}(St_G)$. Its multiplication corresponds to convolution on St_G, which in turn corresponds to the composition of integral transforms.

This example has an important class of relatives introduced by Braverman-Finkelberg-Nakajima [BFN18]. Here the scheme St_G is replaced by an ind-scheme $\mathcal{R}_{G,N}$ which also depends on a representation N, and which has an action of the group $G_{\mathcal{O}}$ of maps $\operatorname{Spec}\mathcal{O} \to G$ (where $\mathcal{O} = \mathbb{C}[[t]]$). The affine Hecke algebra is replaced by the quantized Coulomb branch of the 4d $\mathcal{N} = 2$ gauge theory associated to G and N. In [CW23a] we construct a canonical basis in this Coulomb branch by constructing a nonstandard t-structure on the category $\operatorname{Coh}^{G_{\mathcal{O}}}(\mathcal{R}_{G,N})$.

A central difficulty in studying $\operatorname{Coh}^{G_{\mathcal{O}}}(\mathcal{R}_{G,N})$ as opposed to $\operatorname{Coh}^{G\times\mathbb{G}_m}(St_G)$ is that $\mathcal{R}_{G,N}$ and the objects appearing in its definition are no longer of finite type. In particular, the analogue of $T^*\mathcal{B}$ is by definition not smooth in the strict sense. The main goal of this paper is to develop a framework for infinite-type algebraic geometry which is limited enough to allow a robust extension of the theory of integral transforms of coherent sheaves, but wide enough to encompass these examples.

A basic requirement for such a framework is that at the affine level it should involve only coherent rings (i.e. rings whose category of finitely presented modules is abelian). However, general coherent rings are poorly behaved with respect to base change, hence are not limited enough for our purposes. Instead, let us say an algebra A is *tamely presented* if it is the union of the finitely presented subalgebras over which it is flat. It is a classical result that over a Noetherian base such an algebra is coherent [Bou72, Sec. I.2 Ex. 12]. Moreover, tamely presented algebras are stable under composition and base change. Geometrically, this means that the resulting class of tamely presented schemes and stacks is well-behaved with respect to both coherent sheaf theory and basic constructions such as fiber products.

Smoothness is important in the theory of integral transforms because of its interpretation as a cohomological finiteness condition. One formulation of this is that a variety X is smooth if and only if its diagonal $\Delta_X : X \to X \times X$ has *coherent pullback*. That is, letting $\operatorname{QCoh}(-)$ denote the undbounded derived category of quasicoherent sheaves, the functor $\Delta_X^* : \operatorname{QCoh}(X \times X) \to \operatorname{QCoh}(X)$ takes $\operatorname{Coh}(X \times X)$ to $\operatorname{Coh}(X)$. Since X is of finite type, this is equivalent to Δ_X being of finite Tor-dimension, i.e. to Δ_X^* taking $\operatorname{QCoh}(X \times X)^{\geq 0}$ to $\operatorname{QCoh}(X)^{\geq n}$ for some n.

However, when X is tamely presented this is no longer the case in general. A simple example is $\mathbb{A}^\infty = \operatorname{Spec}\mathbb{C}[x_1, x_2, \dots]$. The image of $\Delta_{\mathbb{A}^\infty}$ is of infinite codimension, and using this one can show that $\Delta_{\mathbb{A}^\infty}^*\Delta_{\mathbb{A}^\infty *}(\mathcal{O}_{\mathbb{A}^\infty})$ is not bounded below, hence that $\Delta_{\mathbb{A}^\infty}$ is of infinite Tor-dimension. On the other hand, we can present $\mathbb{A}^\infty$ as an inverse limit of smooth varieties along flat maps (i.e. the projections $\mathbb{A}^{n+1} \to \mathbb{A}^n$), and using this one can show that $\Delta_{\mathbb{A}^\infty}$ has coherent pullback. Thus coherent pullback of the diagonal, unlike finite Tor-dimension of the diagonal, is a flexible enough cohomological notion of smoothness to remain relevant in infinite type. Moreover, such examples are directly relevant to the case of $\mathcal{R}_{G,N}$, whose construction involves $N_{\mathcal{O}} \cong \mathbb{A}^\infty$.

To have a good theory of integral transforms in the tamely presented setting, we need to know that morphisms with coherent pullback retain enough of the good features they possess in the finite type setting, where they can be controlled by their Tor-dimension. These features include stability under base change and compatibility with !-pullback and sheaf Hom. This turns out to be subtle: already the unboundedness of $\Delta^*_{\mathbb{A}^\infty}\Delta_{\mathbb{A}^\infty *}(\mathcal{O}_{\mathbb{A}^\infty})$ shows that the derived fiber product of $\mathbb{A}^\infty$ with itself over $\mathbb{A}^\infty \times \mathbb{A}^\infty$ has a structure sheaf which is unbounded, hence not coherent. In particular, while morphisms of finite Tor-dimension are stable under arbitrary (derived) base change, morphisms with coherent pullback are not. Nonetheless, we will show that the above properties do extend under suitable hypotheses. Such results are essential to show that adjoints of integral transforms are well-behaved, and in particular to show $\mathrm{Coh}^{G_\mathcal{O}}(\mathcal{R}_{G,N})$ is rigid as a monoidal category.

1.1. **Summary of main definitions and results.** We refer to Section 2 for detailed conventions. For now the reader may take k to be a field of characteristic zero and CAlg_k the (enhanced homotopy) category of nonpositively graded commutative dg k-algebras.

In Section 3.1 we define the notion of a *strictly tamely presented* algebra. If $A \in \mathrm{CAlg}_k$ is an ordinary Noetherian ring, an ordinary A-algebra B is strictly tamely presented if it can be written as a filtered colimit $B \cong \operatorname{colim} B_\alpha$ of finitely presented A-algebras with B flat over each B_α. If A and B are more general, we ask that each truncation of B satisfy a similar condition. Beyond basic stability properties, the key results of this section are that tamely presented k-algebras are coherent (Proposition 3.8), that morphisms with coherent pullback are stable under strictly tamely presented base change (Proposition 3.11), and that a strictly tamely presented algebra over a Noetherian base is of finite Tor-dimension if and only if it has coherent pullback (Proposition 3.17).

As noted before, the notion of strictly tamely presented algebra extends a classical construction of coherent rings [Bou72, Sec. I.2 Ex. 12]. It is closely related to the notion of almost smoothness [KV04, Def. 3.2.4] or placidity [Ras14, Def. 16.29.1], and a similar definition is considered in [Ras19, Sec. 6.36]. We note that B is flat over each B_α if the structure morphisms $B_\alpha \to B_\beta$ are themselves flat. But it is only the relationship between B and the terms in the expression $\operatorname{colim} B_\alpha$ which is invoked in proofs, and emphasizing this leads to a more intrinsic notion.

In Section 4 we define the notion of a tamely presented geometric stack. Following [Lur18, Sec. 9] a geometric stack will mean a functor $X : \mathrm{CAlg}_k \to \mathcal{S}$ which satisfies flat descent, has affine diagonal, and admits a flat cover $\operatorname{Spec} A \to X$ (here $\mathcal{S}$ is the category of spaces). We say X is tamely presented (over k) if this flat cover can be chosen so that it is strictly tamely presented and so that A is strictly tamely presented as a k-algebra. A key example relevant to the space $\mathcal{R}_{G,N}$ is the quotient $N_\mathcal{O}/G_\mathcal{O}$. Per the previous paragraph, tamely presented geometric stacks are a variant of placid stacks considered in [BKV22].

We say a morphism has *semi-universal coherent pullback* if it has coherent pullback after base change along any tamely presented morphism. For a morphism $f : X \to Y$ of tamely presented geometric stacks, this is equivalent (via Propositions 3.11 and 4.12) to asking that the base change f' along any fixed tamely presented flat cover $\operatorname{Spec} A \to Y$ has coherent pullback. A further key result is that, under these hypotheses, f' can be approximated by morphisms of finite Tor-dimension in a suitable sense (Proposition 4.14).

In Section 5 we extend our main notions to the setting of ind-geometric stacks. Ind-geometric stacks generalize ind-schemes, in particular the derived notion of ind-scheme introduced in [GR14], and their basic theory is developed in [CW23b]. As a key motivating example, the quotient $\mathcal{R}_{G,N}/G_{\mathcal{O}}$ is an ind-tamely presented ind-geometric stack.

In Section 6 we study the interaction between coherent pullback and !-pullback in the tamely presented setting. The main result, Proposition 6.6, is the following. Suppose we have a Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

of coherent, ind-tamely presented ind-geometric stacks in which h has semi-universal coherent pullback and f is ind-proper and almost ind-finitely presented. Then the Beck-Chevalley transformation $h'^* f^! \to f'^! h^*$ of functors $\operatorname{IndCoh}(Y) \to \operatorname{IndCoh}(X')$ is an isomorphism.

In Section 7 we study the interaction between coherent pullback and sheaf Hom in the tamely presented setting. The main result, Proposition 7.23, is illustrated by the following corollary. Suppose $h : X \to Y$ is a morphism with semi-universal coherent pullback between coherent, ind-tamely presented ind-geometric stacks such that $X \times Y$ and $Y \times Y$ are coherent. Then for any $\mathcal{F} \in \operatorname{Coh}(Y)$ the natural transformation

$$h^* \mathcal{H}om(\mathcal{F}, -) \to \mathcal{H}om(h^*(\mathcal{F}), h^*(-))$$

of functors $\operatorname{IndCoh}(Y) \to \operatorname{IndCoh}(X)$ is an isomorphism.

Acknowledgements.

2. Conventions and notation

Our notation generally follows [Lur18] unless otherwise specified. We write Cat_∞ (resp. $\widehat{\mathrm{Cat}}_\infty$) for the ∞-category of small (resp. not necessarily small) ∞-categories, $\mathcal{S} \subset \mathrm{Cat}_\infty$ for the ∞-category of spaces, Sp for the ∞-category of spectra, and $\mathcal{P}\mathrm{r}^{\mathrm{L}} \subset \widehat{\mathrm{Cat}}_\infty$ for the category of presentable ∞-categories and functors which preserve small colimits. We use the terms category and ∞-category interchangeably, and say ordinary category when we specifically mean a category in the traditional sense. Given morphisms $f : X \to Y$, $g : Y \to Z$ in an

∞-category $\mathcal{C}$, we often refer by abuse to the composition $g \circ f$, with the understanding that this is only well-defined up to homotopy.

Our most significant departure from [Lur18] is that we use cohomological indexing and notation for t-structures. Thus if $\mathcal{C}$ has a t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ with heart $\mathcal{C}^{\heartsuit}$ we write $\tau^{\leq n} : \mathcal{C} \to \mathcal{C}^{\leq n}$, $H^n : \mathcal{C} \to \mathcal{C}^{\heartsuit}$, etc., for the associated functors. This is of course more consistent with the general conventions in algebraic geometry, but it does create some awkwardness in that it remains most convenient to write $\tau_{\leq n}$ for the subcategory of n-truncated objects in an ∞-category $\mathcal{D}$. Thus, for example, $\tau_{\leq n}(\mathcal{C}^{\leq 0})$ and $\mathcal{C}^{[-n,0]}$ refer to the same subcategory of $\mathcal{C}$. The reader can remain oriented by distinguishing subscripts from superscripts, which should respectively be read homologically and cohomologically.

If $\mathcal{C} \in \widehat{\mathrm{Cat}}_\infty$ is monoidal we write $\mathrm{CAlg}(\mathcal{C})$ for the category of commutative algebra objects of $\mathcal{C}$. When $\mathcal{C} \cong \mathrm{Sp}^{\leq 0}$ is the category of connective spectra we omit it from the notation, so that CAlg denotes the category of connective $\mathbb{E}_\infty$-rings (we omit the superscript used in [Lur18] since we never consider nonconnective $\mathbb{E}_\infty$-rings). Its full subcategory $\tau_{\leq 0}\mathrm{CAlg}$ is equivalently the ordinary category of ordinary commutative rings.

Given $A \in \mathrm{CAlg}$, we write $\mathrm{CAlg}_A := \mathrm{CAlg}_{A/}$ for the category of commutative A-algebras, $\tau_{\leq n}\mathrm{CAlg}_A$ for the subcategory of n-truncated algebras, and $\tau_{<\infty}\mathrm{CAlg}_A := \cup_n \tau_{\leq n}\mathrm{CAlg}_A$ for its subcategory of truncated algebras (i.e. n-truncated for some n). If A is an ordinary ring containing $\mathbb{Q}$ then CAlg_A is equivalently the (enhanced homotopy) category of nonpositively graded commutative dg algebras. We write Mod_A for the category of A-modules (i.e. A-module objects in the category of spectra). If A is an ordinary ring this is the (enhanced) unbounded derived category of ordinary A-modules (i.e. of $\mathrm{Mod}_A^{\heartsuit}$).

Throughout the text we fix a Noetherian base $k \in \mathrm{CAlg}$. That is, $H^0(k)$ is an ordinary Noetherian ring and $H^n(k)$ is finitely generated over $H^0(k)$ for all $n < 0$.

We write PStk_k for the category of prestacks over $\mathrm{Spec}\, k$, i.e. functors $\mathrm{CAlg}_k \to \mathcal{S}$. Similarly we write $\widehat{\mathrm{PStk}}_k$, $\mathrm{PStk}_{k,\leq n}$ for the categories of functors from $\tau_{<\infty}\mathrm{CAlg}_k$, $\tau_{\leq n}\mathrm{CAlg}_k$ to $\mathcal{S}$. A stack will mean a prestack which is a sheaf for the fpqc topology on CAlg_k [Lur18, Prop. B.6.1.3], and we denote the category of stacks by $\mathrm{Stk}_k \subset \mathrm{PStk}_k$. Similarly we write $\widehat{\mathrm{Stk}}_k \subset \widehat{\mathrm{PStk}}_k$, $\mathrm{Stk}_{k,\leq n} \subset \mathrm{PStk}_{k,\leq n}$ for the subcategories of fpqc sheaves on $\tau_{<\infty}\mathrm{CAlg}_k$, $\tau_{\leq n}\mathrm{CAlg}_k$ (note that $\tau_{<\infty}\mathrm{CAlg}_k$ does not admit arbitrary pushouts, but for the application of [Lur18, Prop. A.3.2.1] in defining the fpqc topology closure under flat pushouts is sufficient). We write Spec for the Yoneda embedding $\mathrm{CAlg}_k \to \mathrm{Stk}_k$.

Given $k' \in \mathrm{CAlg}_k$, the natural functor $\mathrm{PStk}_{k'} \to (\mathrm{PStk}_k)_{/\mathrm{Spec}\, k'}$ is an equivalence by [Lur09, Cor. 5.1.6.12], and we record the following analogue for stacks.

Lemma 2.1. *The equivalence* $\mathrm{PStk}_{k'} \to (\mathrm{PStk}_k)_{/\mathrm{Spec}\, k'}$ *restricts to an equivalence* $\mathrm{Stk}_{k'} \to (\mathrm{Stk}_k)_{/\mathrm{Spec}\, k'}$.

We will make use several times of the following standard fact (c.f. [Gai, Sec. 1.3.4]).

Lemma 2.2. *Let $\mathcal{C} \cong \operatorname{colim} \mathcal{C}_\alpha$ be a small colimit in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, with $F_\alpha : \mathcal{C}_\alpha \to \mathcal{C}$ the canonical functors and $G_\alpha : \mathcal{C} \to \mathcal{C}_\alpha$ their right adjoints. Then any $X \in \mathcal{C}$ can be written as $X \cong \operatorname{colim} F_\alpha G_\alpha(X)$.*

3. Affine schemes

We begin by studying strictly tamely presented morphisms and coherent pullback in the setting of affine schemes. We first establish some basic stability and coherence properties of the latter (Propositions 3.4 and 3.8). We then show that morphisms with coherent pullback and strictly tamely presented source are stable under strictly tamely presented base change (Theorem 3.11). We also show that a strictly tamely presented algebra over a Noetherian base has coherent pullback if and only if it is of finite Tor-dimension (Proposition 3.17).

3.1. **Strictly tame presentations.** Given $A \in \mathrm{CAlg}_k$, an A-algebra B is finitely n-presented if it is n-truncated and compact in $\tau_{\leq n}\mathrm{CAlg}_A$, and is almost of finite presentation if $\tau_{\leq n} B$ is finitely n-presented for all n. In particular, an A-algebra B is finitely zero-presented if and only if it is an ordinary commutative ring and is finitely presented over $\tau_{\leq 0} A = H^0(A)$. We say that B is flat over A if $H^0(B)$ is flat over $H^0(A)$ in the ordinary sense and the natural map $H^0(B) \otimes_{H^0(A)} H^n(A) \to H^n(B)$ is an isomorphism for all n.

Definition 3.1. Given $A \in \mathrm{CAlg}_k$, we say $B \in \mathrm{CAlg}_A$ is *strictly tamely n-presented* if it can be written as a filtered colimit $B \cong \operatorname{colim} B_\alpha$ of finitely n-presented A-algebras such that B is flat over each B_α. We call such an expression a *strictly tame presentation of order n*. We say B is *strictly tamely presented* if $\tau_{\leq n} B$ is strictly tamely n-presented for all n.

A strictly tamely n-presented algebra is n-truncated, since $\tau_{\leq n}\mathrm{CAlg}_A$ is stable under filtered colimits by [Lur17, Prop. 7.2.4.27], [Lur09, Cor. 5.5.7.4]. If A is Noetherian, [Lur17, Prop. 7.2.4.31] implies that an ordinary A-algebra B is strictly tamely presented if and only if it is strictly tamely zero-presented. Definition 3.1 extends from algebras to morphisms in CAlg_k in the obvious way. It would be more precise to say "almost strictly tamely presented" instead of "strictly tamely presented", but for simplicity we use the shorter terminology.

Example 3.2. If A and B are ordinary rings and B is placid over A in the sense of [Ras14, Def. 16.29.1], or equivalently almost smooth over A in the sense of [KV04, Def. 3.2.4], then B is strictly tamely zero-presented.

Strictly tamely n-presented algebras have the following more intrinsic characterization. In the setting of ordinary rings, a (more elementary) variant of the proof characterizes strictly tamely zero-presented algebras as those which are the union of the finitely zero-presented subalgebras over which they are flat.

Proposition 3.3. *Given $A \in \mathrm{CAlg}_k$, an n-truncated A-algebra B is strictly tamely n-presented if and only if it satisfies the following condition. For every finitely n-presented*

A-algebra C, every morphism $C \to B$ in CAlg_A admits a factorization $C \to C' \to B$ such that C' is finitely n-presented over A and B is flat over C'.

Proof. The only if direction follows from compactness of C in $\tau_{\leq n}\mathrm{CAlg}_A$. For the if direction, let $\mathrm{CAlg}_A^{n\text{-}fp}$ denote the category of finitely n-presented A-algebras, and $(\mathrm{CAlg}_A^{n\text{-}fp})_{/f\text{-}B} \subset (\mathrm{CAlg}_A^{n\text{-}fp})_{/B} = \mathrm{CAlg}_A^{n\text{-}fp} \times_{\mathrm{CAlg}_A} (\mathrm{CAlg}_A)_{/B}$ the full subcategory of algebras over which B is flat. It suffices to show $(\mathrm{CAlg}_A^{n\text{-}fp})_{/f\text{-}B}$ is filtered and that B is the colimit over its forgetful functor to CAlg_A.

Since CAlg_A is compactly generated so is $\tau_{\leq n}\mathrm{CAlg}_A$ [Lur09, Cor. 5.5.7.4], hence $(\mathrm{CAlg}_A^{n\text{-}fp})_{/B}$ is filtered and B is the colimit over its forgetful functor to $\tau_{\leq n}\mathrm{CAlg}_A$ [Lur09, Cor. 5.3.5.4, Cor. 5.5.7.4]. For any $B' \in \mathrm{CAlg}_A^{n\text{-}fp}$, we have $(\mathrm{CAlg}_A^{n\text{-}fp})_{B'/} \cong \mathrm{CAlg}_{B'}^{n\text{-}fp}$ by [Lur18, Prop. 4.1.3.1] and the fact that being finitely n-presented is equivalent to being n-truncated and of finite generation to order $n+1$ [Lur18, Rem. 4.1.1.9]. It thus suffices to show $(\mathrm{CAlg}_{B'}^{n\text{-}fp})_{/f\text{-}B}$ is filtered for all $B' \in (\mathrm{CAlg}_A^{n\text{-}fp})_{/B}$, since then $(\mathrm{CAlg}_A^{n\text{-}fp})_{/f\text{-}B} \to (\mathrm{CAlg}_A^{n\text{-}fp})_{/B}$ is left cofinal by [Lur09, Thm. 4.1.3.1, Lem. 5.3.1.18], and $(\mathrm{CAlg}_A^{n\text{-}fp})_{/f\text{-}B}$ is filtered as a special case.

We must show any finite diagram $K \to (\mathrm{CAlg}_{B'}^{n\text{-}fp})_{/f\text{-}B}$ extends to a diagram $K^{\triangleright} \to (\mathrm{CAlg}_{B'}^{n\text{-}fp})_{f\text{-}B}$. Since $(\mathrm{CAlg}_{B'}^{n\text{-}fp})_{/B}$ is filtered, we have an extension $K^{\triangleright} \to (\mathrm{CAlg}_{B'}^{n\text{-}fp})_{/B}$. The image of the cone point is finitely n-presented over A, hence by hypothesis its morphism to B factors through some $C \in (\mathrm{CAlg}_A^{n\text{-}fp})_{/f\text{-}B}$. But C is also finitely n-presented over B' (again by [Lur18, Prop. 4.1.3.1]), hence composing with this we obtain an extension $K^{\triangleright} \to (\mathrm{CAlg}_{B'}^{n\text{-}fp})_{/f\text{-}B}$. $\square$

Morphisms of strictly tame presentation have the following stability properties.

Proposition 3.4. *Let $\phi : A \to B, \psi : B \to C$, and $\eta : A \to A'$ be morphisms in CAlg_k.*

(1) If ϕ is of strictly tame presentation (resp. is strictly tamely n-presented) then so is $\phi' : A' \to B \otimes_A A'$ (resp. $\tau_{\leq n}\phi' : \tau_{\leq n}A' \to \tau_{\leq n}(B \otimes_A A')$).

(2) If ϕ and ψ are of strictly tame presentation (resp. are strictly tamely n-presented) then so is $\psi \circ \phi$.

(3) If ϕ is almost of finite presentation (resp. is finitely n-presented) then $\psi \circ \phi$ is of strictly tame presentation (resp. is strictly tamely n-presented) if and only if ψ is.

Proof. Note that in each case it suffices to prove the claim about strictly tamely n-presented morphisms, as it implies the claim about strictly tamely presented morphisms. For (1), let $B \cong \operatorname{colim} B_\alpha$ be a strictly tame presentation of order n. Then $\tau_{\leq n}(B_\alpha \otimes_A A')$ is finitely n-presented over A' for all α [Lur18, Prop. 4.1.3.2]. Since $\tau_{\leq n}$ is continuous and compatible with the symmetric monoidal structure on $\mathrm{Mod}_A^{\leq 0}$ [Lur17, Prop. 7.1.3.15], hence on CAlg_A, we have

$$\tau_{\leq n}(B \otimes_A A') \cong \tau_{\leq n}((\tau_{\leq n}B) \otimes_{\tau_{\leq n}A} \tau_{\leq n}A') \cong \operatorname{colim} \tau_{\leq n}(B_\alpha \otimes_{\tau_{\leq n}A} \tau_{\leq n}A') \cong \operatorname{colim} \tau_{\leq n}(B_\alpha \otimes_A A').$$

Since flatness is preserved by base change and $\tau_{\leq n}$, it follows that $\tau_{\leq n}(B \otimes_A A')$ is strictly tamely n-presented over A'.

For (2), let $B \cong \operatorname{colim} B_\alpha$ and $C \cong \operatorname{colim} C_\beta$ be strictly tame presentations of order n over A and B, respectively. Given a finitely n-presented A-algebra A and a morphism $D \to C$ in CAlg_A, we claim the criterion of Proposition 3.3 is satisfied. Note first that $D \to C$ factors through some C_β since D is compact in $\tau_{\leq n}\mathrm{CAlg}_A$. By Noetherian approximation [Lur18, Cor. 4.4.1.4] we have $C_\beta \cong \tau_{\leq n}(B \otimes_{B_\alpha} C_{\alpha\beta})$ for some α and some finitely n-presented B_α-algebra $C_{\alpha\beta}$. Since B is flat over B_α we in fact have $C_\beta \cong B \otimes_{B_\alpha} C_{\alpha\beta}$. Letting $C_{\gamma\beta} := B_\gamma \otimes_{B_\alpha} C_{\alpha\beta}$ for $\gamma \geq \alpha$, we moreover have $C_\beta \cong \tau_{\leq n} C_\beta \cong \operatorname{colim}_{\gamma \geq \alpha} \tau_{\leq n} C_{\gamma\beta}$.

Again by compactness $D \to C_\beta$ factors through some $\tau_{\leq n} C_{\gamma\beta}$. Now $B_\gamma \to \tau_{\leq n} C_{\gamma\beta}$ is finitely n-presented since $B_\alpha \to C_{\alpha\beta}$ is [Lur18, Prop. 4.1.3.2], hence the composition $A \to B_\gamma \to \tau_{\leq n} C_{\gamma\beta}$ is finitely n-presented [Lur18, Prop. 4.1.3.1]. But $\tau_{\leq n} C_{\gamma\beta} \to C_\beta$ is flat since $B_\gamma \to B$ is and since flatness is preserved by $\tau_{\leq n}$, hence the composition $\tau_{\leq n} C_{\gamma\beta} \to C_\beta \to C$ is flat and we are done.

For (3), suppose $C \cong \operatorname{colim} C_\alpha$ is a strictly tame presentation of order n over A. By compactness of B in $\tau_{\leq n}\mathrm{CAlg}_A$ we have that $B \to C$ factors through some C_α. By [Lur18, Prop. 4.1.3.1] the morphism $B \to C_\alpha$ is finitely n-presented, as is the composition $B \to C_\alpha \to C_\beta$ for all $\beta \geq \alpha$. But then $C \cong \operatorname{colim}_{\beta \geq \alpha} C_\beta$ is a strictly tame presentation of order n over B. This proves the only if direction, and the if direction follows from (2). $\square$

Note that if $B \cong \operatorname{colim} B_\alpha$ is a filtered colimit of finitely n-presented algebras such that the structure maps $B_\alpha \to B_\beta$ are flat, it follows that B is flat over each B_α. We do not know if every strictly tamely n-presented algebra admits a presentation with this stronger property, but it is easy to construct presentations which do not have it.

Example 3.5. Given $\{a_n\}_{n \in \mathbb{N}} \subset \mathbb{C}$, let B_n be the localization of $\mathbb{C}[x,y]/((x-a_n)y)$ by the elements $\{x - a_m\}_{m<n}$. In other words, $\operatorname{Spec} B_n$ is $\mathbb{A}^1$ with the points $a_1, \ldots, a_{n-1}$ removed and with another $\mathbb{A}^1$ intersecting at the point a_n. If $B_n \to B_{n+1}$ takes x to x and y to 0, then $B := \operatorname{colim} B_n$ is the localization of $\mathbb{C}[x]$ by the elements $\{x - a_n\}_{n \in \mathbb{N}}$. In particular it is a localization of each B_n, hence each $B_n \to B$ is flat even though no $B_m \to B_n$ is.

On the other hand, if $B \cong \operatorname{colim} B_\alpha$ is a strictly tame presentation of order n and B is *faithfully* flat over each B_α, it follows that the structure morphisms $B_\alpha \to B_\beta$ are also faithfully flat [Lur18, Lem. B.1.4.2]. But many natural examples, in particular those of the following class, do not admit such presentations.

Example 3.6. If A is an ordinary ring, recall that an ordinary A-algebra B is essentially finitely presented if it is a localization $B \cong S^{-1}C$ of a finitely zero-presented A-algebra C. In this case $B \cong \operatorname{colim} S_{fin}^{-1} C$, where the colimit is over all finite subsets $S_{fin} \subset S$. This is a strictly tame presentation of order zero, hence any essentially finitely presented algebra is strictly tamely zero-presented.

Often we can write an A-algebra of strictly tame presentation as a filtered colimit $B \cong \operatorname{colim} B_\alpha$ such that each B_α is almost finitely presented over A and B is flat over each B_α.

However, not every example is of this form. Moreover, the associated class of morphisms in CAlg_k is not obviously stable under composition, since almost finitely presented algebras cannot be directly controlled by Noetherian approximation the way finitely n-presented algebras can be. The situation is improved by suitable truncatedness hypotheses, but these are in turn not stable under base change. Thus Proposition 3.4 does not extend robustly to such morphisms.

Example 3.7. Let $A_0 = \mathbb{C}[x_1, x_2, \dots]$, $A_n = A_0/(x_1, \dots, x_n)$, and A the trivial square-zero extension of A_0 by $\oplus_{n>0} A_n[n]$. Then A is a strictly tamely presented $\mathbb{C}$-algebra, but cannot be written as a filtered colimit of almost finitely presented $\mathbb{C}$-algebras along flat morphisms: A_0 has no finitely presented subalgebra from which each A_n is obtained by base change, which flatness would require.

Recall that an ordinary commutative ring A is coherent if every finitely generated ideal is finitely presented. More generally, $A \in \mathrm{CAlg}_k$ is coherent if $H^0(A)$ is coherent in the above sense and $H^n(A)$ is a finitely presented $H^0(A)$-module for all n. In general this is a brittle property, and coherence of an ordinary ring A does not even imply coherence of $A[x]$. But tamely presentedness over a Noetherian base implies a more robust form of coherence.

Proposition 3.8. *Let A be a strictly tamely presented k-algebra. Then A is coherent, as is any strictly tamely presented A-algebra.*

Proof. The second claim follows from the first by the stability of strictly tamely presented morphisms under composition (Proposition 3.4). That $H^0(A)$ is coherent is essentially [Bou72, Sec. I.2 Ex. 12], but we repeat the argument. Let $H^0(A) \cong \operatorname{colim} A_\alpha$ be a strictly tame presentation of order zero over k. If $I \subset H^0(A)$ is a finitely generated ideal, we can write it as the image of a morphism $\phi : H^0(A)^n \to H^0(A)$ for some n. This is obtained by extension of scalars from some $\phi_\alpha : A_\alpha^n \to A_\alpha$ for some α. The kernel of ϕ_α is finitely generated since k and hence A_α are Noetherian, but $\ker \phi \cong (\ker \phi_\alpha) \otimes_{A_\alpha} H^0(A)$ since $H^0(A)$ is flat over A_α.

Now fix n, let $\tau_{\leq n} A \cong \operatorname{colim} A_\alpha$ be a strictly tame presentation of order n, and choose some α. Each A_α is Noetherian [Lur18, Prop. 4.2.4.1], hence $H^n(A_\alpha)$ is a finitely presented $H^0(A_\alpha)$-module. Since $A_\alpha \to \tau_{\leq n} A$ is flat $H^n(A) \cong H^n(A_\alpha) \otimes_{H^0(A_\alpha)} H^0(A)$, hence $H^n(A)$ is a finitely presented $H^0(A)$-module. The claim follows since n was arbitrary. □

Remark 3.9. Recall that an ordinary commutative ring is stably coherent if any finitely generated algebra over it is coherent. The above proof almost adapts to show that B is coherent if it is strictly tamely presented over an $A \in \mathrm{CAlg}_k$ such that A is coherent and $H^0(A)$ is stably coherent. No changes are needed if B is an ordinary ring, while [Lur18, Prop. 5.2.2.1] can be leveraged if $H^0(A)$ is of characteristic zero, or more generally if $A \to B$ arises from a morphism of animated/simplicial commutative rings (as do the terms in the needed strictly tame presentations). Plausibly these restrictions are unnecessary, the only question being whether the hypotheses on A imply the free algebra $A_m := A\{x_1, \dots, x_m\}$ is coherent (i.e. if each $H^n(A_m)$ is finitely presented over $H^0(A_m)$). On the other hand, we do not know an A satisfying these hypotheses which is not strictly tamely presented over a Noetherian ring (possibly after passing to a flat cover).

3.2. **Coherent pullback.** Recall that an A-module M is almost perfect if $\tau^{\geq n}M$ is compact in $\mathrm{Mod}_A^{\geq n}$ for all n [Lur18, Rem. 2.7.0.5]. If M is almost perfect it is right bounded, and if A is coherent M is almost perfect if and only if it is right bounded and $H^n(M)$ is a finitely presented $H^0(A)$-module for all n. If A is an ordinary ring, M is almost perfect if and only if it is pseudo-coherent in the sense of [Ill71], see [Lur18, Rem. 2.8.4.6].

We say $M \in \mathrm{Mod}_A$ is coherent if it is almost perfect and (left) bounded, and denote the full subcategory of coherent modules by $\mathrm{Coh}_A \subset \mathrm{Mod}_A$. We recall the following definition from the introduction (we will use the same terminology for algebra morphisms as for the associated morphisms of affine schemes).

Definition 3.10. A morphism $A \to B$ in CAlg_k has coherent pullback if $M \otimes_A B$ is a coherent B-module for every coherent A-module M.

Equivalently, $A \to B$ has coherent pullback if and only if $M \otimes_A B$ is (left) bounded for every coherent A-module M, since $M \otimes_A B$ is almost perfect over B if M is almost perfect over A [Lur18, Prop. 2.7.3.1]. We note that morphisms with coherent pullback are called eventually coconnective morphisms in [Gai13, Def. 3.5.2].

We have seen in the introduction that morphisms with coherent pullback are not stable under arbitrary base change: the diagonal of $\mathbb{A}^\infty$ has coherent pullback but its base change along itself does not. More basically, this is true of the inclusion $\{0\} \hookrightarrow \mathbb{A}^\infty$. The following result says that this only happens because these maps are "too far" from being finitely presented, and that such pathologies do not appear if we only consider strictly tamely presented base change.

Theorem 3.11. *Let the following be a coCartesian diagram in* CAlg_k*.*

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\phi"'] & A' \arrow[d, "\phi'"] \\ B \arrow[r, "\psi'"] & B \otimes_A A' \end{tikzcd} \tag{3.12}$$

Suppose that ϕ has coherent pullback, ψ is strictly tamely presented, and A is strictly tamely presented over k. Then ϕ' has coherent pullback.

The proof will use the following reformulation of [Swa19, Thm. 7.1].

Lemma 3.13. *Let A be an ordinary commutative ring such that A and $A[x]$ are coherent. Then $\mathrm{Coh}_{A[x]}$ is the smallest full stable subcategory of $\mathrm{Mod}_{A[x]}$ which contains the essential image of Coh_A under $- \otimes_A A[x]$.*

Proof. Let $\mathcal{C} \subset \mathrm{Mod}_{A[x]}$ be the smallest full stable subcategory containing the essential image of Coh_A, or equivalently of $\mathrm{Coh}_A^{\heartsuit}$. Since $A[x]$ is coherent it suffices to show $\mathrm{Coh}_{A[x]}^{\heartsuit} \subset \mathcal{C}$.

Given $M \in \mathrm{Coh}_{A[x]}^{\heartsuit}$, choose an exact sequence $0 \to N \to F \to F' \to M \to 0$ with F and F' free of finite rank. Certainly $F, F' \in \mathcal{C}$, so it suffices to show $N \in \mathcal{C}$. But this follows from the proof of [Swa19, Thm. 7.1], which shows that N fits into an exact sequence $0 \to N' \otimes_A A[x] \to N'' \otimes_A A[x] \to N \to 0$ with $N', N'' \in \mathrm{Coh}_A^{\heartsuit}$. □

Proof of Theorem 3.11. Set $B' := B \otimes_A A'$. Consider the following diagram in CAlg_k, where all but the top and bottom faces are coCartesian.

$$\begin{tikzcd}
& A \arrow[rr] \arrow[dl] \arrow[dd, "\phi"] & & A' \arrow[dl] \arrow[dd, "\phi'"] \\
H^0(A) \arrow[rr, crossing over] \arrow[dd, "\xi"'] & & H^0(A') & \\
& B \arrow[rr] \arrow[dl] & & B' \arrow[dl] \\
H^0(A) \otimes_A B \arrow[rr] & & H^0(A') \otimes_{A'} B' \arrow[from=uu, crossing over, "\xi'"] &
\end{tikzcd} \tag{3.14}$$

Proposition 3.8 implies A and A' are coherent. In particular, restriction of scalars along $A \to H^0(A)$ preserves coherence, hence ξ has coherent pullback since ϕ does. Moreover, it suffices to show $M \otimes_{A'} B'$ is bounded for $M \in \mathrm{Coh}_{A'}^{\heartsuit}$. But any such M is obtained by restriction of scalars along $A' \to H^0(A')$, hence it suffices to show ξ' has coherent pullback.

Since k is Noetherian, $k \to H^0(A)$ is strictly tamely presented since $k \to A$ is [Lur17, Prop. 7.2.4.31]. Similarly $H^0(\psi)$ is strictly tamely presented: if $H^0(A') \cong \mathrm{colim}\, A'_\alpha$ is a strictly tame presentation of order zero, each A'_α is almost finitely presented over $H^0(A)$ by [Lur18, Cor. 5.2.2.3] (note that any polynomial ring over $H^0(A)$ is coherent by Proposition 3.8). Replacing (3.12) with the front face of (3.14), we may thus assume A and A' are ordinary commutative rings.

Next suppose that ψ is finitely zero-presented. For some n we can factor (3.12) as

$$\begin{tikzcd}
A \arrow[r, "\theta"] \arrow[d, "\phi"'] & A[x_1, \ldots, x_n] \arrow[r, "\xi"] \arrow[d, "\phi''"] & A' \arrow[d, "\phi'"] \\
B \arrow[r, "\theta'"] & B \otimes_A A[x_1, \ldots, x_n] \arrow[r, "\xi'"] & B',
\end{tikzcd} \tag{3.15}$$

where ξ is surjective and finitely presented. Since $A[x_1, \ldots, x_n]$ and A' are coherent, restriction of scalars along ξ preserves coherence. But restriction of scalars along ξ' is conservative and t-exact, hence ϕ' has coherent pullback if ϕ'' does. Thus we may replace (3.12) with the left square of (3.15) and assume $A' \cong A[x_1, \ldots, x_n]$, and by induction we may then assume $n = 1$.

Now let $\mathcal{C} \subset \mathrm{Mod}_{A'}$ be the full subcategory of M such that $M \otimes_{A'} B'$ is bounded. Since $- \otimes_{A'} B'$ is exact, $\mathcal{C}$ is stable. Since ϕ has coherent pullback and ψ' is flat, $\mathcal{C}$ contains the essential image of Coh_A under $- \otimes_A A'$. Thus $\mathrm{Coh}_{A'} \subset \mathcal{C}$ by Lemma 3.13, hence ϕ' has coherent pullback.

Now suppose $A' \cong \operatorname{colim} A'_\alpha$ is a strictly tame presentation of order zero. Since A' is coherent, it suffices to show that $M \otimes_{A'} B'$ is bounded for any $M \in \mathrm{Coh}^\heartsuit_{A'}$. For each α we write the induced factorization of (3.12) as

$$\begin{tikzcd} A \arrow[r, "\theta_\alpha"] \arrow[d, "\phi"'] & A'_\alpha \arrow[r, "\xi_\alpha"] \arrow[d, "\phi_\alpha"] & A' \arrow[d, "\phi'"] \\ B \arrow[r, "\theta'_\alpha"] & B'_\alpha \arrow[r, "\xi'_\alpha"] & B'. \end{tikzcd}$$

By flatness of the ξ_α and by e.g. [Lur18, Cor. 4.5.1.10] or [TT90, Sec. C.4], there exists an α and $M_\alpha \in \mathrm{Coh}^\heartsuit_{A'_\alpha}$ such that $M \cong M_\alpha \otimes_{A'_\alpha} A'$. Since θ_α is finitely zero-presented, we have already shown that $M_\alpha \otimes_{A'_\alpha} B'_\alpha$ is bounded. But ξ'_α is flat since ξ_α is, hence $M \otimes_{A'} B' \cong M_\alpha \otimes_{A'_\alpha} A' \otimes_{A'} B' \cong M_\alpha \otimes_{A'_\alpha} B'_\alpha \otimes_{B'_\alpha} B'$ is also bounded. $\square$

Remark 3.16. The conclusion of Theorem 3.11 holds if instead of A being strictly tamely presented over k we assume that A and A' are coherent and that $H^0(A)$ is stably coherent. This is because the proof only uses this hypothesis on A in order to apply Proposition 3.8.

Recall that a morphism $\phi : A \to B$ in CAlg_k is of Tor-dimension $\leq n$ if $B \otimes_A M \in \mathrm{Mod}^{\geq -n}_B$ for all $M \in \mathrm{Mod}^\heartsuit_A$, and is of finite Tor-dimension if it is of Tor-dimension $\leq n$ for some n. Clearly ϕ has coherent pullback if it is of finite Tor-dimension. We have the following partial converse, which generalizes [Gai13, Lem. 3.6.3].

Proposition 3.17. *Suppose $\phi : A \to B$ is a morphism in CAlg_k with coherent pullback. If A is Noetherian and ϕ of strictly tame presentation, then ϕ is of finite Tor-dimension.*

Proof. Note that ϕ is of finite Tor-dimension if and only if its base change $\phi' : H^0(A) \to H^0(A) \otimes_A B$ is, since every discrete A-module is obtained by restriction of scalars from $H^0(A)$. Similarly ϕ has coherent pullback if and only if ϕ' does, since A is Noetherian (and in particular coherent). Since ϕ is of strictly tame presentation so is ϕ' (Proposition 3.4), hence we may replace ϕ with ϕ' and assume A is classical.

In particular, A is now coherent over itself, hence B is truncated since ϕ has coherent pullback. Since ϕ is of strictly tame presentation it admits a factorization

$$A \xrightarrow{\phi_1} A' \xrightarrow{\phi_2} B$$

such that ϕ_2 is flat and ϕ_1 is finitely n-presented for some n. Since A is Noetherian, ϕ_1 is then almost finitely presented and A' is also Noetherian [Lur18, Prop. 4.2.4.1].

Recall that ϕ_1 is of Tor-dimension $\leq n$ at a prime ideal $P' \subset H^0(A')$ if the localization $A'_{P'}$ is of Tor-dimension $\leq n$ over A. For any n the set of such prime ideals forms a Zariski open subset U_n of $|\mathrm{Spec}\, A'|$, the underlying topological space of $\mathrm{Spec}\, A'$ [Lur18, Lem. 6.1.5.5].

Since A' is Noetherian we can increase n as needed so that U_n is equal to the union of the U_m for all $m \in \mathbb{N}$. We claim that ϕ is of Tor-dimension $\leq n$.

It suffices to show that for any prime ideal $P \subset H^0(B)$, B_P is of Tor-dimension $\leq n$ over A [Lur18, Prop. 6.1.4.4]. Write $P' \subset H^0(A')$ and $Q \subset H^0(A)$ for the preimages of P under $H^0(\phi_2)$ and $H^0(\phi)$, and write κ for the residue field of A at Q. The morphism $\psi : A \to B_P$ can be factored as the middle row of the following diagram.

$$\begin{tikzcd}
 & A' \arrow[r, "\phi_2"] \arrow[d] & B \arrow[d] \\
A \arrow[ur, "\phi_1"] \arrow[r, "\psi_1"] \arrow[d] & A'_{P'} \arrow[r, "\psi_2"] \arrow[d] & B_P \arrow[d] \\
\kappa \arrow[r, "\psi_1'"] & A'_{P'} \otimes_A \kappa \arrow[r, "\psi_2'"] & B_P \otimes_A \kappa
\end{tikzcd}$$

Since ϕ_2 is flat so is ψ_2 [Lur18, Rem. 6.1.4.3], hence ψ is of Tor-dimension $\leq n$ if ψ_1 is, or equivalently if $P' \in U_n$.

Suppose $P' \notin U_n$. Then $P' \notin U_m$ for any m, hence $A'_{P'} \otimes_A \kappa$ is not truncated [Lur18, Lem. 6.1.5.2]. But ψ_2 is in fact faithfully flat since $H^0(\psi_2) : H^0(A'_{P'}) \to H^0(B_P)$ is a local ring homomorphism, so $B_P \otimes_A \kappa$ is also not truncated. Now note that ψ has coherent pullback since ϕ does, given that it is the composition of ϕ with the flat morphism $B \to B_P$. Since A is Noetherian, $A \to H^0(A)/Q$ is almost finitely presented, hence $A \to \kappa$ is strictly tamely presented (as in Example 3.6). Thus $\psi' := \psi_2' \circ \psi_1'$ also has coherent pullback (Theorem 3.11), and we have a contradiction since then $B_P \otimes_A \kappa$ must be truncated. □

4. Geometric stacks

We now consider tamely presented morphisms and coherent pullback in the setting of geometric stacks. Again we begin with basic stability properties (Propositions 4.7, 4.8, 4.12). Among tamely presented geometric stacks, we show morphisms with semi-universal coherent pullback can be approximated by morphisms of finite Tor-dimension (Proposition 4.14). Conversely, certain pro-smoothness conditions guarantee that the diagonal of a geometric stack has semi-universal coherent pullback (Proposition 4.15).

4.1. **Definitions.** Recall our convention that a stack means a functor $\mathrm{CAlg}_k \to \mathcal{S}$ satisfying fpqc descent (here k is our fixed Noetherian base), and that the category of stacks is denoted by Stk_k. Our usage of the term geometric stack follows [Lur18, Ch. 9] (up to the presence of the base k), though we caution again that the terminology varies in the literature.

Definition 4.1. A stack X is geometric if its diagonal $X \to X \times X$ is affine and there exists faithfully flat morphism $\operatorname{Spec} B \to X$ in Stk_k. A morphism $X \to Y$ in Stk_k is geometric if for any morphism $\operatorname{Spec} A \to Y$, the fiber product $X \times_Y \operatorname{Spec} A$ is geometric. We write $\mathrm{GStk}_k \subset \mathrm{Stk}_k$ for the full subcategory of geometric stacks.

Note here that products are taken in Stk_k, hence are implicitly over $\operatorname{Spec} k$. Also note that affineness of $X \to X \times X$ implies that any morphism $\operatorname{Spec} B \to X$ is affine. In particular, (faithful) flatness of such a morphism is defined by asking that its base change to any affine scheme is such. More generally, a morphism $X \to Y$ in GStk_k is (faithfully) flat if its composition with any faithfully flat $\operatorname{Spec} A \to X$ is (faithfully) flat. A faithfully flat morphism of geometric stacks will also be called a flat cover.

Definition 4.2. A geometric stack X is n-truncated if it admits a flat cover $\operatorname{Spec} A \to X$ such that A is n-truncated. We say $X \in \mathrm{GStk}_k$ is truncated if it is n-truncated for some n, and denote by $\mathrm{GStk}_k^+ \subset \mathrm{GStk}_k$ the full subcategory of truncated geometric stacks.

Alternatively, note that the restriction functor $(-)_{\leq n} : \mathrm{PStk}_k \to \mathrm{PStk}_{k,\leq n}$ takes Stk_k to $\mathrm{Stk}_{k,\leq n}$ [Lur18, Prop. A.3.3.1], and write $i_{\leq n} : \mathrm{Stk}_{k,\leq n} \to \mathrm{Stk}_k$ for the left adjoint of this restriction and $\tau_{\leq n} : \mathrm{Stk}_k \to \mathrm{Stk}_k$ for their composition. Then if X is geometric $\tau_{\leq n} X$ is an n-truncated geometric stack called the n-truncation of X, and X is n-truncated if and only if the natural map $\tau_{\leq n} X \to X$ is an isomorphism [Lur18, Cor. 9.1.6.8, Prop. 9.1.6.9].

Our terminology follows [Lur18, Def. 9.1.6.2], but we caution that what we call n-truncatedness is called n-coconnectedness in [GR17]. We also note that this use of the symbol $\tau_{\leq n}$ and of the term truncation are different from their usual meaning in terms of truncatedness of mapping spaces, but in practice no ambiguity will arise (and this abuse has the pleasant feature that $\tau_{\leq n} \operatorname{Spec} A \cong \operatorname{Spec} \tau_{\leq n} A$).

Proposition 4.3. *Geometric morphisms are stable under composition and base change in* Stk_k. *If $f : X \to Y$ is a morphism in* Stk_k, *then f is geometric if X and Y are, and X is geometric if f and Y are. In particular,* GStk_k *is closed under fiber products in* Stk_k.

Proof. When $k = S$ is the sphere spectrum this is [Lur18, Prop. 9.3.1.2, Ex. 9.3.1.10]. In general it suffices to show GStk_k is the preimage of $\mathrm{GStk} := \mathrm{GStk}_S$ under $\mathrm{Stk}_k \cong \mathrm{Stk}_{/\operatorname{Spec} k} \to \mathrm{Stk}$ (see Lemma 2.1). Clearly $X \in \mathrm{Stk}_k$ has a flat cover $\operatorname{Spec} A \to X$ over $\operatorname{Spec} k$ if and only if it does over $\operatorname{Spec} S$. Now note that if $f : Y \to Z$, $g : Z \to W$ are morphisms in Stk and g is affine, f is affine if and only if $g \circ f$ is (since any $\operatorname{Spec} B \to Y$ factors through $Y \times_Z \operatorname{Spec} B \to Y$). The morphism $X \times_{\operatorname{Spec} k} X \to X \times_{\operatorname{Spec} S} X$ is affine since it is a base change of $\operatorname{Spec} k \to \operatorname{Spec} (k \otimes_S k)$, hence $X \to X \times_{\operatorname{Spec} k} X$ is affine if and only if $X \to X \times_{\operatorname{Spec} S} X$ is. $\square$

4.2. **Tamely presented morphisms.** We now consider the geometric counterparts to the notion of strictly tamely presented algebra.

Definition 4.4. An affine morphism $X \to Y$ in Stk_k is strictly tamely presented if for any $\operatorname{Spec} A \to Y$, the coordinate ring of $X \times_Y \operatorname{Spec} A$ is strictly tamely presented as an A-algebra. A geometric morphism $X \to Y$ in Stk_k is tamely presented if for any $\operatorname{Spec} A \to Y$, there exists a strictly tamely presented flat cover $\operatorname{Spec} B \to \operatorname{Spec} A \times_Y X$ such that B is strictly tamely presented over A. A geometric stack X is tamely presented if it is so over $\operatorname{Spec} k$.

Recall following [Lur18, Def. 17.4.1.1] that $f : X \to Y$ is (locally) almost of finite presentation if, for any $n \in \mathbb{N}$ and any filtered colimit $A \cong \operatorname{colim} A_\alpha$ in $\tau_{\leq n}\mathrm{CAlg}_k$, the canonical map

$$\operatorname{colim} X(A_\alpha) \to X(A) \times_{Y(A)} \operatorname{colim} Y(A_\alpha)$$

is an isomorphism (we omit the word locally by default, as we mostly consider quasi-compact morphisms). We follow [Lur18, Def. 6.3.2.1] and say a morphism $f : X \to Y$ is representable if for any $\operatorname{Spec} A \to Y$, the fiber product $X \times_Y \operatorname{Spec} A$ is a (spectral) Deligne-Mumford stack.

Proposition 4.5. *If a geometric morphism $f : X \to Y$ in Stk_k is representable and almost of finite presentation, then it is tamely presented.*

Proof. Follows from [Lur18, Prop. 17.4.3.1], by which for any $\operatorname{Spec} A \to Y$ there is an étale cover $\operatorname{Spec} B \to X \times_Y \operatorname{Spec} A$ such that B is almost of finite presentation over A. □

We recall the following stability properties of almost finitely presented morphisms, then consider their generalizations to the tamely presented setting.

Proposition 4.6 ([Lur18, Rem. 17.4.1.3, Rem. 17.4.1.5])**.** *Almost finitely presented morphisms are stable under composition and base change in Stk_k. If f and g are composable morphisms in Stk_k such that $g \circ f$ and g are almost of finite presentation, then so is f.*

Proposition 4.7. *Tamely presented geometric morphisms are stable under composition and base change in Stk_k. Suppose we have a Cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

of geometric morphisms in Stk_k. If f' and h are tamely presented and h is faithfully flat, then f is tamely presented.

Proof. Stability under base change follows by construction. To see stability under composition let $f : X \to Y$ and $g : Y \to Z$ be tamely presented. Given $\operatorname{Spec} C \to Z$, there exists by hypothesis a diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "h''"] & X'' \arrow[r, "h'"] \arrow[d] & X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & \operatorname{Spec} B \arrow[r, "h"] & Y' \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ & & \operatorname{Spec} C \arrow[r] & Z \end{tikzcd}$$

in which all squares are Cartesian, h and h'' are strictly tamely presented flat covers, and A, B are strictly tamely presented over B, C. Proposition 3.4 then implies $h' \circ h''$ is a strictly tamely presented flat cover and A is strictly tamely presented over C.

To prove the last claim let $\operatorname{Spec} B \to Y$ be any morphism. By our hypotheses on h there exists a strictly tamely presented flat cover $\operatorname{Spec} B' \to Y' \times_Y \operatorname{Spec} B$ such that B' is strictly tamely presented over B. We then have a commutative cube

$$\begin{tikzcd}
& Z' \arrow[rr] \arrow[dl] \arrow[dd] & & \operatorname{Spec} B' \arrow[dl] \arrow[dd] \\
X' \arrow[rr, crossing over, "f'" near end] \arrow[dd] & & Y' & \\
& Z \arrow[rr] \arrow[dl] & & \operatorname{Spec} B \arrow[dl] \\
X \arrow[rr, "f"] & & Y \arrow[from=uu, crossing over, "h" near start] &
\end{tikzcd}$$

in which all but the left and right faces are Cartesian. Since f' is tamely presented there exists a strictly tamely presented flat cover $\operatorname{Spec} A \to Z'$ such that A is strictly tamely presented over B'. By Proposition 3.4 (and the stability of faithful flatness under composition and base change) it now follows that $\operatorname{Spec} A \to Z$ is a strictly tamely presented flat cover and that A is strictly tamely presented over B. $\square$

Proposition 4.8. *If f and g are composable geometric morphisms in Stk_k such that $g \circ f$ is tamely presented and g is almost of finite presentation, then f is tamely presented.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be the given morphisms. For any $\phi : \operatorname{Spec} A \to Y$, there exists by hypothesis a strictly tamely presented flat cover $h : \operatorname{Spec} B \to X' := X \times_Z \operatorname{Spec} A$ such that B is strictly tamely presented over A. Consider then the following diagram of Cartesian squares,

$$\begin{tikzcd}
\operatorname{Spec} C \arrow[r, "h'"] \arrow[d, "\psi'"] & X'' \arrow[r, "f''"] \arrow[d] & \operatorname{Spec} A \arrow[d, "\psi"] & \\
\operatorname{Spec} B \arrow[r, "h"] & X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[r, "g'"] \arrow[d] & \operatorname{Spec} A \arrow[d, "g \circ \phi"] \\
& X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

where ψ is the canonical section of g'. That ψ and hence ψ' are affine follows from g being geometric. That h' is a strictly tamely presented flat cover follows from h being so. Since g and hence g' are almost of finite presentation and $g' \circ \psi$ is the identity, it follows that ψ and hence ψ' are almost of finite presentation (Proposition 4.6). That C is strictly tamely presented over A now follows from Proposition 3.4. $\square$

Remark 4.9. The definition of tamely presented morphism has two obvious variants, where respectively the condition that $\operatorname{Spec} B \to \operatorname{Spec} A \times_Y X$ or $\operatorname{Spec} B \to \operatorname{Spec} A$ is strictly tamely presented is dropped. Some results we state extend to one or the other of these variants, but our most central results do not. Thus for the sake of uniformity we formulate all statements in terms of tamely presented morphisms, even when this obscures their generality somewhat.

4.3. **Coherent pullback.** Recall that any stack X has an associated category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves, defined as the limit of the categories Mod_A over all maps $\operatorname{Spec} A \to X$. If X is geometric this is equivalent to the corresponding limit over the Cech nerve of any flat cover [Lur18, Prop. 9.1.3.1]. In the truncated geometric case we define coherent sheaves as follows.

Definition 4.10. If $X \in \mathrm{GStk}_k^+$, then $\mathcal{F} \in \mathrm{QCoh}(X)$ is coherent if $f^*(\mathcal{F})$ is a coherent A-module for some (equivalently, any) flat cover $\operatorname{Spec} A \to X$. We write $\mathrm{Coh}(X) \subset \mathrm{QCoh}(X)$ for the full subcategory of coherent sheaves.

While the above definition makes sense when X is not truncated, without additional hypotheses the resulting category $\mathrm{Coh}(X)$ may be degenerate (for example, it may contain no nonzero objects). It will be convenient to exclude such degenerate cases from our discussion, though our treatment of coherent sheaves on ind-geometric stacks will include well-behaved non-truncated geometric stacks in its scope. We also caution that the above notion of coherence differs from that of [Lur18, Def. 6.4.3.1].

Definition 4.11. Let $f : X \to Y$ be a morphism in GStk_k such that Y is truncated. We say f has coherent pullback if X is truncated and $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ takes $\mathrm{Coh}(Y)$ to $\mathrm{Coh}(X)$. We say f has semi-universal coherent pullback if for any truncated Y' and any tamely presented morphism $Y' \to Y$, the base change $f' : X \times_Y Y' \to Y'$ has coherent pullback.

In practice, to show a morphism of geometric stacks has semi-universal coherent pullback we reduce to the affine case (where Theorem 3.11 can be applied) using the locality statement in the following result.

Proposition 4.12. *Morphisms with semi-universal coherent pullback are stable under composition and base change along tamely presented morphisms in* GStk^+. *Let* Y *and* Y' *be truncated geometric stacks and*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

a Cartesian diagram in GStk_k. *If* f' *has semi-universal coherent pullback and* h *is faithfully flat, then* f *has semi-universal coherent pullback.*

Proof. Stability under composition and tamely presented base change follow from Proposition 4.7. Suppose that $Z \to Y$ is tamely presented and Z is truncated. Consider the commutative cube

$$\begin{tikzcd}
& W' \arrow[rr, "\phi'"] \arrow[dl] \arrow[dd] & & Z' \arrow[dl] \arrow[dd] \\
X' \arrow[rr, "f'" near start] \arrow[dd] & & Y' \arrow[dd, "h" near start] & \\
& W \arrow[rr, "\phi" near start] \arrow[dl] & & Z \arrow[dl] \\
X \arrow[rr, "f"] & & Y &
\end{tikzcd}$$

with all faces Cartesian. By Proposition 4.7 the morphism $Z' \to Y'$ is tamely presented, hence ϕ' has coherent pullback. But since h is faithfully flat so are the other vertical morphisms, and it follows that ϕ has coherent pullback. □

Next we show that if a morphism with semi-universal coherent pullback has tamely presented source and target, it can be locally approximated by morphisms of finite Tor-dimension. A morphism $f : X \to Y$ in GStk_k is of Tor-dimension $\leq n$ if $f^*(\mathrm{QCoh}(Y)^{\geq 0}) \subset \mathrm{QCoh}(Y)^{\geq n}$, and is of finite Tor-dimension if it is of Tor-dimension $\leq n$ for some n. We have the following variant of standard results.

Proposition 4.13. *Morphisms of Tor-dimension $\leq n$ are stable under composition and base change in GStk_k. A morphism $f : X \to Y$ of geometric stacks is of Tor-dimension $\leq n$ if and only if its base change along any given flat cover $h : \operatorname{Spec} A \to Y$ is.*

Proof. Stability under composition is immediate. Flat locality on the target follows since if h' and f' are defined by base change, then h'^* is conservative, t-exact, and satisfies $h'^* f^* \cong f'^* h^*$. If $h : \operatorname{Spec} A \to Y$ is arbitrary, then h' is affine, hence h'_* is conservative, t-exact, and satisfies $f^* h_* \cong h'_* f'^*$ [Lur18, Prop. 9.1.5.7]. Stability under base change along affine morphisms follows, and arbitrary base change now follows by composing an arbitrary $Y' \to Y$ with a flat cover $\operatorname{Spec} B \to Y'$. □

By extension, we then say a geometric morphism in Stk_k is of Tor-dimension $\leq n$ if its base change to any geometric stack is so, and is of finite Tor-dimension if it is of Tor-dimension $\leq n$ for some n. We now have the following approximation result, where we note that an expression $A \cong \operatorname{colim} A_\alpha$ of the indicated kind always exists for some n.

Proposition 4.14. *Let $f : X \to Y$ be a morphism in GStk^+ with semi-universal coherent pullback, and suppose that X and Y are tamely presented. Choose a strictly tamely presented flat cover $\operatorname{Spec} A \to Y$ such that A is strictly tamely presented over k, and let $f' : X' \to \operatorname{Spec} A$ denote the base change of f. If $A \cong \operatorname{colim} A_\alpha$ exhibits A as a strictly tamely n-presented*

k-algebra for some n, then the composition

$$X' \xrightarrow{f'} \operatorname{Spec} A \xrightarrow{u_\alpha} \operatorname{Spec} A_\alpha$$

is of finite Tor-dimension for all α.

Proof. Fix a strictly tamely presented flat cover $g : \operatorname{Spec} B \to X$ such that B is strictly tamely presented over k. If $g' : \operatorname{Spec} B' \to X'$ is its base change (whose source is affine since Y is geometric), then B' is strictly tamely presented over B and thus k by Proposition 3.4. Since k is Noetherian, A_α is almost finitely presented over k for any α [Lur18, Prop. 4.1.2.1]. Proposition 3.4 then further implies that B' is strictly tamely presented over A_α.

It follows from the hypotheses that f' has coherent pullback, hence by flatness of g' and u_α the composition $u_\alpha \circ f' \circ g'$ also has coherent pullback. This composition then has finite Tor-dimension by Proposition 3.17, hence so does $u_\alpha \circ f'$ by faithful flatness of g'. □

Now suppose k is an ordinary Noetherian ring, and consider a strictly tame presentation $A \cong \operatorname{colim} A_\alpha$ of order zero. If each A_α is smooth over k we call this a weakly pro-smooth presentation. We say $A \in \tau_{\leq 0}\mathrm{CAlg}_k$ is weakly pro-smooth if it admits a weakly pro-smooth presentation, and we say $X \in \mathrm{GStk}_k$ is locally weakly pro-smooth if it admits a flat cover $\operatorname{Spec} A \to X$ such that A is weakly pro-smooth. We then have the following fundamental source of morphisms with semi-universal coherent pullback. It implies in particular that the jet scheme of an affine variety étale over $\mathbb{A}^n$ has a diagonal with semi-universal coherent pullback [KV04, Prop. 1.2.1, Prop. 1.7.1].

Proposition 4.15. *Suppose k is an ordinary Noetherian ring of finite global dimension, and suppose $X \in \mathrm{GStk}_k$ is locally weakly pro-smooth. Then the diagonal $\Delta_X : X \to X \times X$ has semi-universal coherent pullback, as does any tamely presented geometric morphism $f : Y \to X$.*

Proof. Note that the second claim follows from the first. Any such f factors as $Y \to X \times Y \to X$. The second factor is a base change of $X \to \operatorname{Spec} k$, hence is of finite Tor-dimension by our hypothesis on k. The first is the base change of Δ_X along the tamely presented morphism f, hence it and f have semi-universal coherent pullback if Δ_X does (Proposition 4.12).

Let $A \cong \operatorname{colim} A_\alpha$ be a weakly pro-smooth presentation and set $U = \operatorname{Spec} A$, $U_\alpha = \operatorname{Spec} A_\alpha$. Since $U \times U \to X \times X$ is a flat cover, it suffices to show the base change $\Delta'_X : U \times_X U \to U \times U$ of Δ_X has semi-universal coherent pullback (Proposition 4.12). Proposition 3.4 implies $U \times U$ is strictly tamely presented since U is, hence it suffices to show Δ'_X has coherent pullback (Theorem 3.11). Since $U \times U$ is coherent (Proposition 3.8) it suffices to show $\Delta'^*_X(\mathcal{F})$ is coherent for any $\mathcal{F} \in \mathrm{Coh}(U \times U)^\heartsuit$.

By flatness of the $p_\alpha \times p_\alpha : U \times U \to U_\alpha \times U_\alpha$ and by e.g. [Lur18, Cor. 4.5.1.10] or [TT90, Sec. C.4], $\mathcal{F} \cong (p_\alpha \times p_\alpha)^*(\mathcal{F}_\alpha)$ for some α and some $\mathcal{F}_\alpha \in \mathrm{Coh}(U_\alpha \times U_\alpha)^\heartsuit$. But since $U_\alpha \times U_\alpha$

is smooth its diagonal is of finite Tor-dimension [Sta, Lem. 0FDP], hence $(p_\alpha \times p_\alpha) \circ \Delta'_X$ is of finite Tor-dimension by the reasoning of the first paragraph, hence $\Delta'^*_X(\mathcal{F})$ is coherent. □

4.4. **Pushforward and base change.** Given a morphism $f : X \to Y$ in GStk_k, the pushforward $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is defined as the right adjoint of f^*. For f_* to be well-behaved one needs additional hypotheses on f. Recall that a morphism $f : X \to Y$ in GStk_k is of cohomological dimension $\leq n$ if $f_*(\mathrm{QCoh}(X)^{\leq 0}) \subset \mathrm{QCoh}(Y)^{\leq n}$, and is of finite cohomological dimension if it is of cohomological dimension $\leq n$ for some n.

Proposition 4.16. *Morphisms of finite cohomological dimension are stable under composition and base change in* GStk_k. *A morphism* $f : X \to Y$ *of geometric stacks is of finite cohomological dimension if and only if its base change along any given flat cover* $\mathrm{Spec}\, A \to Y$ *is. In this case* $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ *is continuous, and for any Cartesian square*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{4.17}$$

in GStk_k *the Beck-Chevalley transformation* $h^* f_*(\mathcal{F}) \to f'_* h'^*(\mathcal{F})$ *is an isomorphism for all* $\mathcal{F} \in \mathrm{QCoh}(X)$.

Proof. Stability under composition is immediate, while stability under base change and flat locality on the target follow from [HLP14, Prop. A.1.9] (whose proof applies to geometric stacks, not just algebraic stacks with affine diagonal). The remaining properties then follow from [Lur18, Prop. 9.1.5.3, Prop. 9.1.5.7] or [HLP14, Prop. A.1.5]. □

A key case is that of proper morphisms. We say $f : X \to Y$ is proper if it is representable and for any $\mathrm{Spec}\, A \to Y$, the fiber product $X \times_Y \mathrm{Spec}\, A$ is proper over $\mathrm{Spec}\, A$ in the sense of [Lur18, Def. 5.1.2.1]. In particular, this requires that $X \times_Y \mathrm{Spec}\, A$ be a quasi-compact separated algebraic space.

Proposition 4.18. *Proper morphisms are stable under composition and base change in* Stk_k. *If* f *and* g *are morphisms in* Stk_k *such that* $g \circ f$ *and* g *are proper, then so is* f. *Proper morphisms of geometric stacks are of finite cohomological dimension. If* f *is a proper, almost finitely presented morphism of truncated geometric stacks, then* f_* *takes* $\mathrm{Coh}(X)$ *to* $\mathrm{Coh}(Y)$.

Proof. Stablity under base change is immediate, and the claims about composition follow from [Lur18, Prop. 5.1.4.1, Prop. 6.3.2.2]. By Proposition 4.16 finiteness of cohomological dimension can be checked after base change along a flat cover $h : \mathrm{Spec}\, A \to Y$, where it follows from [Lur18, Prop. 2.5.4.4, Prop. 3.2.3.1]. If f is almost of finite presentation and f' is its base change along h, then f'_* preserves coherence by [Lur18, Thm. 5.6.0.2]. Then so does $h^* f_*$ by Proposition 4.16, hence so does f_* by definition. □

5. Ind-geometric stacks

We now consider tamely presented morphisms and coherent pullback in the setting of ind-geometric stacks. We begin by reviewing the basic theory of such objects, developed in [CW23b]. We then consider the stability properties of our main classes of morphisms, as well as their interaction with coherent sheaves.

5.1. **Definitions.** The definition of ind-geometric stack is built on the notion of a convergent (or nilcomplete) stack. A stack X is convergent if for all $A \in \mathrm{CAlg}_k$ the natural morphism $X(A) \to \lim X(\tau_{\leq n} A)$ is an isomorphism, and we write $\widehat{\mathrm{Stk}}_k \subset \mathrm{Stk}_k$ for the subcategory of convergent stacks. Geometric stacks are examples of convergent stacks [CW23b].

Convergent stacks play a central role in our discussion because colimits in $\widehat{\mathrm{Stk}}_k$ are typically more natural than colimits in Stk_k. For example, a geometric stack X is the colimit of its truncations $\tau_{\leq n} X$ in $\widehat{\mathrm{Stk}}_k$ [CW23b], but not in Stk_k unless X is itself truncated. In particular, the inclusion $\widehat{\mathrm{Stk}}_k \subset \mathrm{Stk}_k$ does not preserve colimits in general.

Definition 5.1. An *ind-geometric stack* is a convergent stack X which admits an expression $X \cong \operatorname{colim}_\alpha X_\alpha$ as a filtered colimit in $\widehat{\mathrm{Stk}}_k$ of truncated geometric stacks along closed immersions. We call such an expression an *ind-geometric presentation* of X. We call it a *reasonable presentation* if the structure maps are almost finitely presented, and say X is *reasonable* if it admits a reasonable presentation.

We write $\mathrm{indGStk}_k \subset \widehat{\mathrm{Stk}}_k$ (resp. $\mathrm{indGStk}_k^{reas} \subset \widehat{\mathrm{Stk}}_k$) for the full subcategory of ind-geometric (resp. reasonable ind-geometric) stacks. Any geometric stack X is ind-geometric with $X \cong \operatorname{colim} \tau_{\leq n} X$ being an ind-geometric presentation. On the other hand, not every geometric stack is reasonable, a basic example being the self-intersection of the origin in $\mathbb{A}^\infty$.

Truncatedness plays an essential role in our discussion due to the following variant of [GR14, Lem. 1.3.6]. Note that it would fail if Y were not truncated: in this case Y is not compact in 1-$\widehat{\mathrm{Stk}}_k$, since e.g. id_Y does not factor through any truncation of Y.

Proposition 5.2. *Let $X \cong \operatorname{colim} X_\alpha$ be an ind-geometric presentation. Then for any truncated geometric stack Y, the natural map*

$$\operatorname{colim} \mathrm{Map}_{\mathrm{Stk}_k}(Y, X_\alpha) \to \mathrm{Map}_{\mathrm{Stk}_k}(Y, X)$$

is an isomorphism.

To discuss ind-geometric stacks more intrinsically, without referring to particular ind-geometric presentations, the following notion is useful.

Definition 5.3. Let X be an ind-geometric stack. A *truncated* (resp. *reasonable*) *geometric substack* of X is a truncated geometric stack X' equipped with a closed immersion $X' \to X$ (resp. an almost finitely presented closed immersion $X' \to X$).

Proposition 5.4. *([CW23b]) Let $X \cong \operatorname{colim} X_\alpha$ be an ind-geometric (resp. reasonable) presentation. Then for all α, the structure morphism $i_\alpha : X_\alpha \to X$ realizes X_α as a truncated (resp. reasonable) geometric substack of X. Any other truncated (resp. reasonable) geometric substack $X' \to X$ can be factored as $X' \xrightarrow{j_\alpha} X_\alpha \xrightarrow{i_\alpha} X$ for some α, and in any such factorization j_α is a closed immersion (resp. almost finitely presented closed immersion), hence affine.*

Our main results in later sections will focus on the following class of stacks. Here we say a geometric stack X is locally coherent (resp. locally Noetherian) if there exists a flat cover $\operatorname{Spec} A \to X$ such that A is coherent (resp. Noetherian).

Definition 5.5. A geometric stack X is coherent if it is locally coherent and $\operatorname{QCoh}(X)^\heartsuit$ is compactly generated. An ind-geometric stack X is coherent if it is reasonable and every reasonable geometric substack is coherent.

An ind-geometric stack is coherent if and only if it has a reasonable presentation whose terms are coherent [CW23b]. Any locally Noetherian geometric stack is coherent by [Lur18, Prop. 9.5.2.3]. Tamely presented affine morphisms give rise to the following wider class of examples. Here we say an ind-geometric stack is locally Noetherian if every reasonable geometric substack is so (equivalently, every term in some reasonable presentation is so).

Proposition 5.6. *Let X' be a locally Noetherian ind-geometric stack and $X \to X'$ a tamely presented affine morphism. Then X is coherent. If $Y \to Y'$ is another tamely presented affine morphism with Y' locally Noetherian, then $X \times Y$ is also coherent.*

Proof. Let $X' \cong \operatorname{colim} X'_\alpha$ be a reasonable presentation. Then $X \cong \operatorname{colim} X_\alpha$, where $X_\alpha := X \times_{X'} X'_\alpha$, by left exactness of filtered colimits in $\widehat{\operatorname{Stk}}_k$. Each X'_α is locally Noetherian, hence each X_α is coherent since $X_\alpha \to X'_\alpha$ is tamely presented and affine (Propositions 3.8 and **??**). Coherence of X follows from Proposition **??**. The other claims are the same, noting that $X' \times Y'$ is locally Noetherian and that $X \times Y$ is tamely presented and affine over it. $\square$

5.2. **Ind-tamely presented morphisms.** The notion of tamely presented morphism may be extended to the ind-geometric setting following the usual pattern for notions such as ind-properness. The basic properties of the latter, together with ind-closedness and ind-finite cohomological dimension, are established in the ind-geometric setting in [CW23b].

Definition 5.7. A morphism $f : X \to Y$ of reasonable ind-geometric stacks is ind-tamely presented if for every commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

such that $X' \to X$ and $Y' \to Y$ are reasonable geometric substacks, the map f' is tamely presented. An ind-geometric stack is ind-tamely presented if it is reasonable and is ind-tamely presented over $\operatorname{Spec} k$.

Proposition 5.8. *Let $f : X \to Y$ be a morphism of ind-geometric stacks and $X \cong \operatorname{colim}_\alpha X_\alpha$ a reasonable presentation. Then f is ind-tamely presented if and only if for every X_α there exists a diagram*

$$\begin{tikzcd} X_\alpha \arrow[r] \arrow[d, "f_\alpha"'] & X \arrow[d, "f"] \\ Y_\alpha \arrow[r] & Y \end{tikzcd}$$

in which Y_α is a truncated geometric substack of Y and f_α is tamely presented.

Proof. Same as [CW23b]. ☐

Proposition 5.9. *Ind-tamely presented morphisms are stable under composition in* $\mathrm{indGStk}_k$.

Proof. Same as [CW23b]. ☐

Proposition 5.10. *A $f : X \to Y$ morphism of truncated geometric stacks is tamely presented if and only if it is ind-tamely presented.*

. Then f is ind-proper (resp. an ind-closed immersion, of ind-finite cohomological dimension) if and only if it is proper (resp. a closed immersion, of finite cohomological dimension). If X and Y are truncated, then f is ind-tamely presented (resp. almost ind-finitely presented) if and only if it is tamely presented (resp. almost finitely presented).

Proof. Same as [CW23b]. ☐

We recall the following closure property of ind-closed immersions. Here $\mathrm{indGStk}_{k,cl} \subset \mathrm{indGStk}_k$ (resp. $\mathrm{indGStk}^{reas}_{k,cl,afp} \subset \mathrm{indGStk}^{reas}_k$) denotes the 1-full subcategory which only includes ind-closed immersions (resp. almost finitely presented ind-closed immersions), similarly for $\mathrm{GStk}^+_{k,cl} \subset \mathrm{GStk}^+_k$ (resp. $\mathrm{GStk}^+_{k,cl,afp} \subset \mathrm{GStk}^+_k$). Recall that a subcategory is 1-full if for $n > 1$ it includes all n-simplices whose edges belong to the indicated class of morphisms.

Proposition 5.11. *([CW23b]) The canonical continuous functor $\mathrm{Ind}(\mathrm{GStk}^+_{k,cl}) \to \widehat{\mathrm{Stk}}_k$ factors through an equivalence $\mathrm{Ind}(\mathrm{GStk}^+_{k,cl}) \cong \mathrm{indGStk}_{k,cl}$, and likewise $\mathrm{Ind}(\mathrm{GStk}^+_{k,cl,afp}) \to \widehat{\mathrm{Stk}}_k$ factors through an equivalence $\mathrm{Ind}(\mathrm{GStk}^+_{k,cl,afp}) \cong \mathrm{indGStk}^{reas}_{k,cl,afp}$. In particular, $\mathrm{indGStk}_k$ (resp. $\mathrm{indGStk}^{reas}_k$) is closed in $\widehat{\mathrm{Stk}}_k$ under filtered colimits along ind-closed immersions (resp. almost ind-finitely presented ind-closed immersions).*

5.3. **Coherent pullback.** Recall that a morphism $f : X \to Y$ of stacks is geometric if for any morphism $Y' \to Y$ with Y' an affine scheme, $Y' \times_Y X$ is a geometric stack. By Proposition 4.3 this implies more generally that $Y' \times_Y X$ is a geometric stack whenever Y' is.

Definition 5.12. Let $f : X \to Y$ be a morphism of reasonable ind-geometric stacks. We say f has semi-universal coherent pullback if it is geometric and for every truncated geometric stack Y' and every ind-tamely presented morphism $Y' \to Y$, the base change $f' : Y' \times_Y X \to Y'$ has coherent pullback.

Likewise we will say a morphism of arbitrary ind-geometric stacks is of Tor-dimension $\leq n$ (resp. of finite Tor-dimension) if it is geometric and its base change to any truncated geometric stack is of Tor-dimension $\leq n$ (resp. of finite Tor-dimension) in the sense of Section 4.3. The following criterion for semi-universal coherent pullback entails in particular that the above definition is consistent with Definition 4.11 when Y is a truncated geometric stack.

Proposition 5.13. *Let $f : X \to Y$ be a geometric morphism of reasonable (resp. arbitrary) ind-geometric stacks and $Y \cong \operatorname{colim}_\alpha Y_\alpha$ a reasonable (resp. ind-geometric) presentation of Y. Then f has semi-universal coherent pullback (resp. is of finite Tor-dimension) if and only if its base change to every Y_α has semi-universal coherent pullback (resp. is of finite Tor-dimension).*

Proof. Propositions 5.4 and **??** imply the only if direction. Let Y' be a truncated geometric stack and $j : Y' \to Y$ a tamely presented morphism. By Proposition 5.4 and the definitions we can factor j through some $i_\alpha : Y_\alpha \to Y$ via a tamely presented morphism $j_\alpha : Y' \to Y_\alpha$, hence the base change of f to Y' has coherent pullback since its base change to Y_α has semi-universal coherent pullback. The finite Tor-dimension case is proved the same way. □

Proposition 5.14. *Morphisms with semi-universal coherent pullback (resp. of finite Tor-dimension) are stable under composition in $\mathrm{indGStk}_k^{reas}$ (resp. $\mathrm{indGStk}_k$).*

Proof. Let X, Y, and Z be reasonable ind-geometric stacks, $f : X \to Y$ and $g : Y \to Z$ morphisms with semi-universal coherent pullback, and $Z \cong \operatorname{colim} Z_\alpha$ a reasonable presentation. Define $g_\alpha : Y_\alpha \to Z_\alpha$ and $f_\alpha : X_\alpha \to Y_\alpha$ by base change. Then each Y_α is a truncated geometric stack since g_α has coherent pullback, the maps $Y_\alpha \to Y_\beta$ are almost finitely presented closed immersions by base change, and $Y \cong \operatorname{colim} Y_\alpha$ by left exactness of filtered colimits in $\widehat{\mathrm{Stk}}_k$. In particular each f_α has semi-universal coherent pullback, hence each $g_\alpha \circ f_\alpha$ does by Proposition 4.12, hence $g \circ f$ does by Proposition 5.13. The finite Tor-dimension case is proved the same way. □

5.4. **Fiber Products.** Now we consider fiber products of ind-geometric stacks, and the base change properties of the classes of morphisms considered above.

Proposition 5.15. *Ind-geometric stacks are closed under finite limits in $\widehat{\mathrm{Stk}}_k$ (and Stk_k).*

Already the self-intersection of the origin in $\mathbb{A}^\infty$ illustrates that reasonable ind-geometric stacks are not closed under arbitrary fiber products. Instead we have the following more limited result.

Proposition 5.16. *Let the following be a Cartesian diagram of ind-geometric stacks.*

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Suppose that X, Y, and Y' are reasonable. Suppose also that h has semi-universal coherent pullback and f is ind-tamely presented (resp. that h is of finite Tor-dimension and f is arbitrary). Then X' is reasonable. Moreover, h' has semi-universal coherent pullback and f' is ind-tamely presented (resp. h' is of finite Tor-dimension), and f' is almost ind-finitely presented if f is.

Proof. Let $X \cong \operatorname{colim} X_\alpha$ be a reasonable presentation. For any α, there is by hypothesis a reasonable geometric substack $Y_\alpha \to Y$ and a commutative cube

$$\begin{tikzcd}
& X'_\alpha \arrow[rr, "\phi'"] \arrow[dl] \arrow[dd, "\psi'"' near start] & & Y'_\alpha \arrow[dl] \arrow[dd, "\psi"] \\
X' \arrow[rr, "f'" near end, crossing over] \arrow[dd] & & Y' & \\
& X_\alpha \arrow[rr, "\phi" near start] \arrow[dl] & & Y_\alpha \arrow[dl] \\
X \arrow[rr, "f"] & & Y \arrow[from=uu, "h" near start, crossing over] &
\end{tikzcd}$$

in $\mathrm{indGStk}_k$ such that all but the top and bottom faces are Cartesian, and such that ϕ and ϕ' are tamely presented.

By hypothesis ψ has semi-universal coherent pullback, and by Proposition 4.12 so does ψ'. In particular X'_α is a truncated geometric stack. We have $X' \cong \operatorname{colim} X'_\alpha$ in $\widehat{\mathrm{Stk}}_k$ since filtered colimits are left exact [Lur09, Ex. 7.3.4.7], and this is a reasonable presentation since almost finitely presented closed immersions are stable under base change. The rest of the claim now follows from Propositions 5.13 and 5.8. The other claims follow the same way. $\square$

Proposition 5.17. *Ind-proper morphisms (resp. ind-closed immersions, morphisms of ind-finite cohomological dimension) are stable under base change in* $\mathrm{indGStk}_k$.

Coherent ind-geometric stacks are not closed under general fiber products (e.g. [Gla89, Sec. 7.3.13]), but we record a few cases of interest where they are (beyond more elementary ones with Noetherian hypotheses).

Proposition 5.18. *Let the following be a Cartesian diagram of ind-geometric stacks.*

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Suppose that X is coherent, Y is reasonable, and Y' is ind-tamely presented. Suppose also that f is ind-tamely presented and that h is affine and has semi-universal coherent pullback. Then X' is coherent.

Proof. If $X \cong \operatorname{colim} X_\alpha$ is a reasonable presentation, then as in the proof of Proposition 5.16 $X' \cong \operatorname{colim} X'_\alpha$ is a reasonable presentation, where $X'_\alpha := X_\alpha \times_Y Y'$. By Propositions 5.9 and 5.10 each X'_α is tamely presented, hence locally coherent by Proposition 3.8. But X_α is coherent, hence by Proposition **??** so is X'_α since h is affine. □

5.5. **Coherent sheaves and pushforward.** We now define the category $\mathrm{Coh}(X)$ of coherent sheaves on a reasonable ind-geometric stack X. Explicitly, if $X \cong \operatorname{colim} X_\alpha$ is a reasonable presentation we will have $\mathrm{Coh}(X) \cong \operatorname{colim} \mathrm{Coh}(X_\alpha)$, where the colimit is taken in Cat_∞ along the pushforward functors. Such expressions will follow from a more canonical definition of $\mathrm{Coh}(X)$, which will also make manifest its functoriality under proper pushforward.

Let $\mathrm{GStk}^+_{k,prop} \subset \mathrm{GStk}^+_k$ denote the 1-full subcategory which only includes proper, almost finitely presented morphisms, similarly for $\mathrm{indGStk}^{reas}_{k,prop} \subset \mathrm{indGStk}^{reas}_k$. Note that $\mathrm{GStk}^+_{k,prop}$ is a full subcategory of $\mathrm{indGStk}^{reas}_{k,prop}$ by Proposition 5.10. We have a canonical functor

$$\mathrm{Coh} : \mathrm{GStk}^+_{k,prop} \to \mathrm{Cat}_\infty \tag{5.19}$$

which takes X to $\mathrm{Coh}(X)$ and $f : X \to Y$ to $f_* : \mathrm{Coh}(X) \to \mathrm{Coh}(Y)$. This is obtained by restriction from the corresponding functor $\mathrm{QCoh} : \mathrm{GStk}_k \to \widehat{\mathrm{Cat}}_\infty$, given that proper, almost finitely presented pushforward preserves coherence (Proposition 4.18).

Definition 5.20. We write

$$\mathrm{Coh} : \mathrm{indGStk}^{reas}_{k,prop} \to \mathrm{Cat}_\infty \tag{5.21}$$

for the left Kan extension of (5.19) along the inclusion $\mathrm{GStk}^+_{k,prop} \subset \mathrm{indGStk}^{reas}_{k,prop}$. We write $f_* : \mathrm{Coh}(X) \to \mathrm{Coh}(Y)$ for the functor assigned to an ind-proper, almost ind-finitely presented morphism $f : X \to Y$.

Existence of the indicated left Kan extension follows from [Lur09, Lem. 4.3.2.13]. The following result implies that $\mathrm{Coh}(X)$ can be expressed in terms of reasonable presentations as stated earlier.

Proposition 5.22. *The functor* $\mathrm{Coh} : \mathrm{indGStk}^{reas}_{k,prop} \to \mathrm{Cat}_\infty$ *preserves filtered colimits along almost ind-finitely presented ind-closed immersions.*

Proof. For the proof we distinguish (5.19) and (5.21) by writing them as Coh_{geom} and Coh_{ind}, respectively. The proof of Proposition 5.11 adapts to show that the canonical continuous functor $\mathrm{Ind}(\mathrm{GStk}^+_{k,prop}) \to \widehat{\mathrm{Stk}}_k$ is faithful and that $\mathrm{indGStk}^{reas}_{k,prop}$ is the intersection of its image with $\mathrm{indGStk}^{reas}_k$. In particular, the inclusion $\mathrm{indGStk}^{reas}_{k,prop} \subset \widehat{\mathrm{Stk}}_k$ factors through a fully faithful functor $\mathrm{indGStk}^{reas}_{k,prop} \to \mathrm{Ind}(\mathrm{GStk}^+_{k,prop})$. By the transitivity of left Kan extensions [Lur09, Prop. 4.3.2.8], Coh_{ind} is the restriction to $\mathrm{indGStk}^{reas}_{k,\,prop}$ of Coh_{Ind}, the left Kan extension of Coh_{geom} to $\mathrm{Ind}(\mathrm{GStk}^{+}_{k,\,prop})$.

Recall from Proposition 5.11 that $\mathrm{indGStk}^{reas}_{k,\,cl,\,afp}$ admits filtered colimits and its inclusion into $\widehat{\mathrm{Stk}}_k$ continuous. By the previous paragraph this inclusion factors through a faithful functor $\mathrm{indGStk}^{reas}_{k,\,cl,\,afp} \to \mathrm{Ind}(\mathrm{GStk}^{+}_{k,\,prop})$, which is then also continuous. But Coh_{Ind} is continuous [Lur09, Lem. 5.3.5.8], hence so is its restriction to $\mathrm{indGStk}^{reas}_{k,\,cl,\,afp}$. $\square$

Suppose $f : X \to Y$ is an ind-proper, almost ind-finitely presented morphism and $X \cong \operatorname{colim} X_\alpha$ a reasonable presentation. Proposition 5.22 implies in particular that any $\mathcal{F} \in \mathrm{Coh}(X)$ can be written as $\mathcal{F} \cong i_{\alpha *}(\mathcal{F}_\alpha)$ for some α and some $\mathcal{F}_\alpha \in \mathrm{Coh}(X_\alpha)$. The behavior of ind-proper pushforward on objects is thus determined from the geometric setting by the following compatibility: we can factor $f \circ i_\alpha$ through a proper, almost finitely presented morphism $f_\alpha : X_\alpha \to Y_\alpha$ to a reasonable geometric substack $j_\alpha : Y_\alpha \to Y$, and we then have $f_*(\mathcal{F}) \cong j_{\alpha *} f_{\alpha *}(\mathcal{F}_\alpha)$.

5.6. **Pullback and base change.** Next we consider the pullback of coherent sheaves along suitable morphisms of reasonable ind-geometric stacks. For explicitness we first define this directly in terms of presentations, and then give a more global description that will make its compatibilities more manifest.

Let $f : X \to Y$ be a morphism with semi-universal coherent pullback, which we emphasize includes any morphism of finite Tor-dimension. Let $Y \cong \operatorname{colim} Y_\alpha$ a reasonable presentation, $f_\alpha : X_\alpha \to Y_\alpha$ the base change of f, and $i_{\alpha\beta} : X_\alpha \to X_\beta$, $j_{\alpha\beta} : Y_\alpha \to Y_\beta$ the induced maps. By proper base change we have isomorphisms $f^*_\beta j_{\alpha\beta *} \cong i_{\alpha\beta *} f^*_\alpha$ of functors $\mathrm{Coh}(Y_\alpha) \to \mathrm{Coh}(X_\beta)$. The identity $\mathrm{Coh}(Y) \cong \operatorname{colim} \mathrm{Coh}(Y_\alpha)$ implies there is an essentially unique functor $f^* : \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$ equipped with coherent isomorphisms $f^* j_{\alpha *} \cong i_{\alpha *} f^*_\alpha$ of functors $\mathrm{Coh}(Y_\alpha) \to \mathrm{Coh}(X)$ for all α. More formally, we can construct this functor as follows.

Write A for the index category of the presentation $Y \cong \operatorname{colim} Y_\alpha$, so that Y and the Y_α together define a diagram $A^{\triangleright} \to \mathrm{indGStk}^{reas}_k$. Taking Cartesian products with X we obtain a diagram $A^{\triangleright} \times \Delta^1 \to \mathrm{indGStk}^{reas}_k$ and by restriction a diagram $A \times \Delta^1 \to \mathrm{GStk}^{+}_k$. We then obtain a diagram $A \times \Delta^1 \to \mathrm{Cat}_\infty$ taking f_α to $f^*_\alpha : \mathrm{Coh}(Y_\alpha) \to \mathrm{Coh}(X_\alpha)$ and $i_{\alpha\beta}$ to $i_{\alpha\beta *} : \mathrm{Coh}(X_\alpha) \to \mathrm{Coh}(X_\beta)$. More precisely, this is constructed from the more primitive functor $A \times \Delta^1 \to \widehat{\mathrm{Cat}}_\infty$ that takes each map to the associated pullback of quasi-coherent sheaves, passing to adjoints along morphisms in A via [Lur17, Cor. 4.7.5.18] and then passing to coherent subcategories. This is equivalent to the data of the associated functor $A \to \mathrm{Cat}^{\Delta^1}_\infty := \mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$ taking α to $f^*_\alpha : \mathrm{Coh}(Y_\alpha) \to \mathrm{Coh}(X_\alpha)$. By [Lur09, Cor. 5.1.2.3] and Proposition 5.22 the colimit of this diagram is a functor $f^* : \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$ with the desired compatibilities.

To make this manifestly canonical, observe that $(\mathrm{GStk}^{+}_{k,\,cl,\,afp})_{/Y}$, the category of reasonable geometric substacks of Y, provides a canonical reasonable presentation $Y \cong \operatorname{colim}(\mathrm{GStk}^{+}_{k,\,cl,\,afp})_{/Y}$.

Definition 5.23. Let $f : X \to Y$ be a morphism with semi-universal pullback between reasonable ind-geometric stacks. Then we define $f^* : \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$ by applying the above construction to the canonical reasonable presentation $Y \cong \mathrm{colim}(\mathrm{GStk}^+_{k,\,cl,\,afp})_{/Y}$. That is, f^* is the unique functor which admits a coherent family of isomorphisms $f^* i_* \cong i'_* f'^*$ for any reasonable geometric substack $i : Y' \to Y$.

That this agrees with the above construction applied to an arbitrary reasonable presentation follows since the diagram $A \to \mathrm{GStk}^+_k$ associated to any other presentation factors through $(\mathrm{GStk}^+_{k,\,cl,\,afp})_{/Y}$.

Proposition 5.24. *Let the following be a Cartesian diagram of ind-geometric stacks.*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Suppose that X, Y, and Y' are reasonable, that h has semi-universal coherent pullback, and that f is ind-proper and almost ind-finitely presented. Then X' is reasonable, h' and f' have the same properties as h and f, and there is a canonical isomorphism $h^* f_* \cong f'_* h'^*$ of functors $\mathrm{Coh}(X) \to \mathrm{Coh}(Y')$.

Proof. The conclusions about X', h', and f' are Propositions 5.16 and 5.17. It follows from the proof of Proposition 5.22 and a straightforward variant of [Lur09, 5.3.5.15] that we can write $f \in (\mathrm{indGStk}^{reas}_{k,\,prop})^{\Delta^1}$ as the colimit of a filtered diagram $\{f_\alpha : X_\alpha \to Y_\alpha\}$ whose restrictions to the vertices of Δ^1 define reasonable presentations $X \cong \mathrm{colim}\, X_\alpha$, $Y \cong \mathrm{colim}\, Y_\alpha$. Writing A for the index category of this diagram, we repeat the above construction to obtain a functor $A \times \Delta^1 \times \Delta^1$... $\square$

Note that if $f : X \to Y$ and $g : Y \to Z$ are morphisms with semi-universal coherent pullback, we obtain a canonical isomorphism $(g \circ f)^* \cong f^* g^*$ by repeating the construction of each individual functor with Δ^2 in place of Δ^1. Similarly, one shows in this way that the base change isomorphisms of Proposition 5.24 are compatible with composition. We can encode such compatibilities more systematically using the formalism of correspondences.

Recall that given a category $\mathcal{C}$ with Cartesian products, we have a category $\mathrm{Corr}(\mathcal{C})$ with the same objects as $\mathcal{C}$, and in which a morphism from X to Z is a diagram $X \xleftarrow{f} Y \xrightarrow{h} Z$ [GR17, Sec. 7.1.2.5]. Composition of correspondences is given by taking Cartesian products in the standard way. We note that we will never refer to $(\infty, 2)$-categories of correspondences in this text.

Let $\mathrm{Corr}(\mathrm{indGStk}^{reas}_k)_{prop;coh}$ denote the 1-full subcategory of $\mathrm{Corr}(\mathrm{indGStk}_k)$ which only includes correspondences $X \xleftarrow{f} Y \xrightarrow{h} Z$ such that X, Y, and Z are reasonable, h has semi-universal coherent pullback, and f is ind-proper and almost of ind-finite presentation (these are stable under composition of correspondences by Propositions 5.9, 5.14, 5.17, and 5.16). We define $\mathrm{Corr}(\mathrm{GStk}_k^+)_{prop;coh}$ similarly, noting that it is a full subcategory of $\mathrm{Corr}(\mathrm{indGStk}_k)_{prop;coh}$ by Proposition 5.10.

Note now that the functor $\mathrm{Coh} : \mathrm{GStk}^+_{k,\, prop} \to \mathrm{Cat}_\infty$ of (5.19) extends to a functor

$$\mathrm{Coh} : \mathrm{Corr}(\mathrm{GStk}_k^+)_{prop;coh} \to \mathrm{Cat}_\infty. \tag{5.25}$$

To construct it, consider the 1-full subcategory $\mathrm{Corr}(\mathrm{GStk}_k)_{fcd;all} \subset \mathrm{Corr}(\mathrm{GStk}_k)$ in which we only include correspondences $X \xleftarrow{f} Y \xrightarrow{h} Z$ such that f is of finite cohomological dimension. By Proposition 4.16 and [GR17, Thm. 3.2.2] there exists a canonical functor $\mathrm{QCoh} : \mathrm{Corr}(\mathrm{GStk}_k)_{fcd;all} \to \widehat{\mathrm{Cat}}_\infty$ which takes a correspondence $X \xleftarrow{f} Y \xrightarrow{h} Z$ to the functor $f_*h^* : \mathrm{QCoh}(Z) \to \mathrm{QCoh}(X)$. We obtain the desired functor (5.25) by restricting along the inclusion $\mathrm{Corr}(\mathrm{GStk}_k^+)_{prop;coh} \subset \mathrm{Corr}(\mathrm{GStk}_k)_{fcd;all}$, and observing that the associated functors f_*h^* preserve coherence.

Proposition 5.26. *Consider the left Kan extension*

$$\mathrm{Coh} : \mathrm{Corr}(\mathrm{indGStk}_k^{reas})_{prop;coh} \to \mathrm{Cat}_\infty \tag{5.27}$$

of (5.25) along the inclusion $\mathrm{Corr}(\mathrm{GStk}^+)_{prop;coh} \subset \mathrm{Corr}(\mathrm{indGStk}_k^{reas})_{prop;coh}$. *The restriction of this extension to* $\mathrm{indGStk}^{reas}_{k,\, prop}$ *is the functor of Definition 5.20, and its value on a morphism* $h : X \to Y$ *with semi-universal coherent pullback is the functor* $h^* : \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$ *of Definition 5.23.*

Proof. The first claim follows from [GR17, Thm. 6.1.5] (in its opposite form for left Kan extensions). Now let $Y \cong \operatorname{colim} Y_\alpha$ be a reasonable presentation with index set A, so that as before we have a diagram $A \times \Delta^1 \to \mathrm{Cat}_\infty$ taking $\{\alpha\} \times \Delta^1$ to h_α^*, where $h_\alpha : X_\alpha \to Y_\alpha$ is the base change of h. It follows from the first claim, from Proposition **??**, and from the fact that $X \cong \operatorname{colim} X_\alpha$ is also a reasonable presentation that the restriction of (5.27) along the induced functor $A^\triangleright \times \Delta^1 \to \mathrm{Corr}(\mathrm{indGStk}_k^{reas})_{prop;coh}$ is a left Kan extension of its restriction to $A \times \Delta^1$. The second claim follows since this is the definition of h^*. ☐

5.7. **Pushforward and $*$-pullback.** In the context of coherent ind-geometric stacks, Proposition 6.1 lets us extend both of the basic functorialities of IndCoh.

Definition 5.28. Let X and Y be reasonable ind-geometric stacks such that Y is coherent, and let $f : X \to Y$ be a morphism with coherent pullback. We write $f^* : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$ for the unique continuous functor whose restriction to $\mathrm{Coh}(Y)$ factors through the functor $f^* : \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$ of Definition 5.20.

When f is of finite Tor-dimension this is indeed consistent with Definition **??**, since the previously defined f^* is continuous and preserves coherence. To describe the pushforward counterpart of Definition 5.28 first note that if $f : X \to Y$ is any morphism of ind-geometric stacks such that X is reasonable, there is a canonical functor $f_* : \mathrm{Coh}(X) \to \mathrm{IndCoh}(Y)$

defined as follows. Write $\mathrm{IndCoh}^+_{naive} : \mathrm{indGStk}_k \to \widehat{\mathrm{Cat}}_\infty$ for the left Kan extension of the evident functor $\mathrm{IndCoh}^+ : \mathrm{GStk}_k^+ \to \widehat{\mathrm{Cat}}_\infty$. Explicitly, $\mathrm{IndCoh}^+_{naive}(X)$ is the full subcategory of $\mathcal{F} \in \mathrm{IndCoh}^+(X)$ which are pushed forward from some ind-geometric substack of X. By construction we have a functor $f_* : \mathrm{IndCoh}(X)^+_{naive} \to \mathrm{IndCoh}(Y)^+_{naive}$, while by the universal property of left Kan extensions we have canonical functors $\mathrm{Coh}(X) \to \mathrm{IndCoh}(X)^+_{naive}$ and $\mathrm{IndCoh}(Y)^+_{naive} \to \mathrm{IndCoh}(Y)^+$, and we let f_* be the composition of these.

Definition 5.29. Let $f : X \to Y$ be a morphism of ind-geometric stacks, and suppose that X is coherent. Then we write $f_* : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$ for the unique continuous functor whose restriction to $\mathrm{Coh}(X)$ is the functor above.

Suppose $f : X \to Y$ and $g : Y \to Z$ are morphisms of ind-geometric stacks such that X is coherent, and such that either Y is coherent or g is of ind-finite cohomological dimension. Then have an isomorphism $g_* f_* \cong (g \circ f)_*$ of functors $\mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Z)$, since both are continuous and have their restrictions to $\mathrm{Coh}(X)$ are isomorphic by construction.

If the source and target of $f : X \to Y$ are coherent, we have the following extension of Proposition **??**.

Proposition 5.30. *Let X and Y be coherent ind-geometric stacks and $f : X \to Y$ a morphism with coherent pullback. Then $f_* : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$ is right adjoint to $f^* : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$. In particular, suppose f sits in a Cartesian diagram of the following form, where h and h' have coherent pullback and X' and Y' are coherent.*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{5.31}$$

Then we obtain a Beck-Chevalley transformation $h^* f_* \to f'_* h'^*$ from the the isomorphism $h_* f'_* \cong f_* h'_*$ of functors $\mathrm{IndCoh}(X') \to \mathrm{IndCoh}(Y)$, and the former is itself an isomorphism if f is proper and almost of finite presentation.

Lemma 5.32. *Let $\widehat{\mathcal{C}}, \widehat{\mathcal{D}}$ be the left completions of $\check{\mathcal{C}}, \check{\mathcal{D}} \in \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{acpl}}$, and let $\Psi_{\mathcal{C}} : \check{\mathcal{C}} \to \widehat{\mathcal{C}}$ and $\Psi_{\mathcal{D}} : \check{\mathcal{D}} \to \widehat{\mathcal{D}}$ be the canonical functors. Let $\check{F} : \check{\mathcal{C}} \to \check{\mathcal{D}}$, $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ be colimit-preserving functors such that $\widehat{F}$ is right bounded and $\Psi_{\mathcal{D}} \check{F} \cong \widehat{F} \Psi_{\mathcal{C}}$, and let $\check{G} : \check{\mathcal{D}} \to \check{\mathcal{C}}$, $\widehat{G} : \widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$ be their right adjoints. Then the Beck-Chevalley map $\Psi_{\mathcal{C}} \check{G}(X) \to \widehat{G} \Psi_{\mathcal{D}}(X)$ is an isomorphism for all $X \in \check{\mathcal{D}}^+$.*

Proof. By definition the Beck-Chevalley map is the composition

$$\Psi_{\mathcal{C}} \check{G}(X) \to \Psi_{\mathcal{C}} \check{G} \Psi_{\mathcal{D}}^R \Psi_{\mathcal{D}}(X) \cong \Psi_{\mathcal{C}} \Psi_{\mathcal{C}}^R \widehat{G} \Psi_{\mathcal{D}}(X) \to \widehat{G} \Psi_{\mathcal{D}}(X) \tag{5.33}$$

of unit and counit maps. Since $\Psi_{\mathcal{C}}$ is t-exact and restricts to an equivalence $\check{\mathcal{C}}^+ \xrightarrow{\sim} \widehat{\mathcal{C}}^+$, its right adjoint $\Psi_{\mathcal{C}}^R$ is left t-exact and restricts to the inverse equivalence $\widehat{\mathcal{C}}^+ \xrightarrow{\sim} \check{\mathcal{C}}^+$, likewise for

$\Psi_{\mathcal{D}}^R$. In particular, the first map in (5.33) is an isomorphism since $X \in \check{\mathcal{D}}^+$. But $\widehat{G}$ is left bounded since $\widehat{F}$ is right bounded, hence $\widehat{G}\Psi_{\mathcal{D}}(X) \in \widehat{\mathcal{C}}^+$, hence the last map in (5.33) is an isomorphism. $\square$

Proof. First suppose X and Y are truncated and geometric. Since $\mathrm{IndCoh}(X)$ is compactly generated and f^* preserves compactness, the right adjoint f^{*R} is continuous. But f_* and f^{*R} have isomorphic restrictions to $\mathrm{Coh}(X)$ by Lemma 5.32 and the definitions, hence by continuity they are themselves isomorphic. When X' and Y' are truncated and geometric, the final claim follows immediately since $h^* f_*$ and $f'_* h'^*$ are continuous and the transformation restricts to an isomorphism of functors $\mathrm{Coh}(X) \to \mathrm{Coh}(Y')$.

Now let $Y \cong \operatorname{colim} Y_\alpha$ be a reasonable presentation, let $f_\alpha : X_\alpha \to Y_\alpha$ be the base change of f, and let $i'_{\alpha\beta} : X_\alpha \to X_\beta$ the base change of $i_{\alpha\beta} : Y_\alpha \to Y_\beta$. Unwinding the definition of f_*, it follows from Proposition **??** and [Lur09, Cor. 5.1.2.3] that $f_* \cong \operatorname{colim} f_{\alpha*}$ in $\mathrm{Fun}(\Delta^1, \mathcal{P}\mathrm{r}^{\mathrm{L}})$, the structure maps being given by the isomorphisms $i'_{\alpha\beta*} f_{\beta*} \cong f_{\alpha*} i_{\alpha\beta*}$. Likewise, we have $f_* \cong \operatorname{colim} f_{\alpha*}$ in $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{P}\mathrm{r}^{\mathrm{L}})$, the structure maps being given by the Beck-Chevalley isomorphisms $f_\beta^* i_{\alpha\beta*} \cong i'_{\alpha\beta*} f_\alpha^*$. Passing to right adjoints we have $f_*^R \cong \lim f_{\alpha*}^R$ in $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \widehat{\mathrm{Cat}}_\infty)$ and $f^{*R} \cong \lim f_\alpha^{*R}$ in $\mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$. In the notation of [Lur17, Def. 4.7.5.16], it follows from [Lur17, Cor. 4.7.5.18] and the previous paragraph that $f^{*R} \in \mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$, $f_*^R \in \mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$, and that f^{*R} and f_*^R correspond under the equivalence $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty) \cong \mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$ of [Lur17, Cor. 4.7.5.18]. But then f_* is right adjoint to f^* since f_*^R is to f^{*R}. The general case of the final claim now follows as in the geometric case. $\square$

6. Coherent pullback and !-pullback

Let $f : X \to Y$ be a morphism of ind-geometric stacks. In this section we study the right adjoint $f^! : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$ of the functor f_* when X and Y are reasonable and f is almost of ind-finite presentation. The main result (Proposition 6.6) studies conditions under which !-pullback commutes with $*$-pullback inside Cartesian diagrams.

Proposition 6.1. *If X is a coherent ind-geometric stack then the canonical functor* $\mathrm{Ind}(\mathrm{Coh}(X)) \to \mathrm{IndCoh}(X)$ *is an equivalence, and induces equivalences* $\mathrm{Ind}(\mathrm{Coh}(X)^{\leq 0}) \cong \mathrm{IndCoh}(X)^{\leq 0}$, $\mathrm{Ind}(\mathrm{Coh}(X)^{\geq 0}) \cong \mathrm{IndCoh}(X)^{\geq 0}$.

6.1. !-pullback and $*$-pullback.

We first consider morphisms of finite Tor-dimension between geometric stacks, the proofs closely following similar results with different finiteness conditions [Lur18, Prop. 6.4.1.4], [Gai13, Prop. 7.1.6], [Ras19, Lem. 6.14.2]. Having the ind-geometric case in mind we formulate the statement in terms of ind-coherent sheaves, but this is essentially a notational choice since $\mathrm{IndCoh}(X)^+ \cong \mathrm{QCoh}(X)^+$.

Proposition 6.2. *Let the following be a Cartesian diagram of geometric stacks.*

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Suppose h is of finite Tor dimension and f is proper and almost of finite presentation. Then for any $\mathcal{F} \in \mathrm{IndCoh}(Y)^+$ the Beck-Chevalley map $h'^*f^!(\mathcal{F}) \to f'^!h^*(\mathcal{F})$ is an isomorphism.

Proof. First note that we may assume that Y' is affine. Otherwise choose a flat cover $u : \operatorname{Spec} B \to Y'$ and consider the following diagram of Cartesian squares.

$$\begin{tikzcd}
X'' \arrow[r, "u'"] \arrow[d, "f''"'] & X' \arrow[r, "h'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
\operatorname{Spec} B \arrow[r, "u"] & Y' \arrow[r, "h"] & Y
\end{tikzcd}$$

We have $h'^*f^!(\mathcal{F}), f'^!h^*(\mathcal{F}) \in \mathrm{IndCoh}(X')^+$ since h and h' are of finite Tor dimension, and since f is proper and almost of finite presentation. By faithful flatness the restriction of u'^* to $\mathrm{IndCoh}(X')^+$ is conservative, so it suffices to show the first factor of

$$u'^*h'^*f^!(\mathcal{F}) \to u'^*f'^!h^*(\mathcal{F}) \to f''^!u^*h^*(\mathcal{F}),$$

is an isomorphism. But the second factor is an isomorphism by Lemma **??**, hence it suffices to show the composition is.

Now fix a flat cover $g : \operatorname{Spec} A \to Y$. We have a commutative cube

$$\begin{tikzcd}
& Z' \arrow[rr, "\phi'"] \arrow[dl, "\theta'"'] \arrow[dd, "\psi'"] & & \operatorname{Spec} A' \arrow[dl, "\theta"] \arrow[dd, "\psi"] \\
X' \arrow[rr] \arrow[dd] & & Y' \arrow[dd] & \\
& Z \arrow[rr, "\phi"] \arrow[dl, "g'"'] & & \operatorname{Spec} A \arrow[dl, "g"] \\
X \arrow[rr] & & Y &
\end{tikzcd} \tag{6.3}$$

with Cartesian faces (here ψ is affine since Y' is). Again by faithful flatness the restriction of θ'^* to $\mathrm{IndCoh}(X')^+$ is conservative, hence it suffices to show $\theta'^*h'^*f^!(\mathcal{F}) \to \theta'^*f'^!h^*(\mathcal{F})$ is an isomorphism. Since $\theta'^*f'^!h^*(\mathcal{F}) \to \phi'^!\theta^*h^*(\mathcal{F})$ is an isomorphism by Lemma **??**, it suffices to show their composition is an isomorphism. But this can be refactored as

$$\theta'^*h'^*f^!(\mathcal{F}) \cong \psi'^*g'^*f^!(\mathcal{F}) \to \psi'^*\phi^!g^*(\mathcal{F}) \to \phi'^!\psi^*g^*(\mathcal{F}) \cong \phi'^!\theta^*h^*(\mathcal{F}), \tag{6.4}$$

whose first factor is an isomorphism by Lemma **??**, and whose second factor is an isomorphism since ψ is a map of affine schemes [Lur18, Prop. 6.4.1.4]. $\square$

We now turn to morphisms with coherent pullback, first in the setting of geometric stacks. Note that Proposition 6.6 will show the truncatedness hypothesis in the following statement is ultimately unnecessary.

Proposition 6.5. *Let the following be a Cartesian diagram of geometric stacks.*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Suppose that all stacks in the diagram are coherent and that Y and Y' are truncated and tamely presented. Suppose also that h has semi-universal coherent pullback and that f is proper and almost of finite presentation. Then for any $\mathcal{F} \in \mathrm{IndCoh}(Y)$ the Beck-Chevalley map $h'^* f^!(\mathcal{F}) \to f'^! h^*(\mathcal{F})$ is an isomorphism.

Proof. Note that f_* and f'_* preserve compact objects and have compactly generated source by Proposition 6.1, hence $f^!$ and $f'^!$ are continuous [Lur09, Prop. 5.5.7.2]. By compact generation of $\mathrm{IndCoh}(Y)$ it thus suffices to assume $\mathcal{F} \in \mathrm{Coh}(Y)$. The proof of Proposition 6.2 then carries over except we must argue differently at two points. First, it is no longer immediate that $h'^* f^!(\mathcal{F})$ is bounded below, since $f^!(\mathcal{F})$ need not be coherent and h'^* need not be left bounded. Second, to show (6.4) is an isomorphism we can no longer directly apply [Lur18, Prop. 6.4.1.4], since ψ need not be of finite Tor dimension.

By hypothesis we may assume the flat cover $g : \operatorname{Spec} A \to Y$ is strictly tamely presented and that A is strictly tamely presented over k. Since Y is truncated, there exists for some n a strictly tame presentation $A \cong \operatorname{colim} A_\alpha$ of order n over k. Using Noetherian approximation [Lur18, Prop. 4.2.1.5, Thm. 4.4.2.2, Prop. 5.5.4.1] and faithful flatness of the $u_\alpha : \operatorname{Spec} A \to \operatorname{Spec} A_\alpha$ we can, for some α, extend (7.16) to a diagram

$$\begin{tikzcd} & Z' \arrow[rr, "\phi'"] \arrow[dl, "\theta'"'] \arrow[dd, "\psi'"] & & W \arrow[dl, "\theta"] \arrow[dd, "\psi"] \\ X' \arrow[rr] \arrow[dd] & & Y' \arrow[dd] & \\ & Z \arrow[rr, "\phi"] \arrow[dl, "g'"'] \arrow[dd, "u'_\alpha"] & & \operatorname{Spec} A \arrow[dl, "g"] \arrow[dd, "u_\alpha"] \\ X \arrow[rr] & & Y & \\ & Z_\alpha \arrow[rr, "\phi_\alpha"] & & \operatorname{Spec} A_\alpha \end{tikzcd}$$

in which all squares are Cartesian and $\phi_\alpha : Z_\alpha \to \operatorname{Spec} A_\alpha$ is proper and almost of finite presentation. Moreover, increasing α if needed, we may assume by [Lur18, Cor. 4.5.1.10], flatness of the u_α, and coherence of the A_α that there exists $\mathcal{F}_\alpha \in \mathrm{Coh}(\operatorname{Spec} A_\alpha)$ such that $g^*(\mathcal{F}) \cong u_\alpha^*(\mathcal{F}_\alpha)$.

We now claim that $\tau^{\leq n} f^!(\mathcal{F})$ is coherent for all n. It follows from Lemma **??** and [Lur18, Prop. 6.4.1.4] that $g'^* f^!(\mathcal{F}) \cong u'^*_\alpha \phi^!_\alpha(\mathcal{F}_\alpha)$. By flatness of g' and u'_α we then have

$$g'^* \tau^{\leq n} f^!(\mathcal{F}) \cong \tau^{\leq n} g'^* f^!(\mathcal{F}) \cong \tau^{\leq n} u'^*_\alpha \phi^!_\alpha(\mathcal{F}_\alpha) \cong u'^*_\alpha \tau^{\leq n} \phi^!_\alpha(\mathcal{F}_\alpha).$$

Thus $\tau^{\leq n} f^!(\mathcal{F})$ is coherent if $\tau^{\leq n} \phi^!_\alpha(\mathcal{F}_\alpha)$ is, since g' is faithfully flat and $\tau^{\leq n} f^!(\mathcal{F})$ is bounded below. But $\tau^{\leq n} \phi^!_\alpha(\mathcal{F}_\alpha)$ is coherent for all n by [Lur18, Prop. 6.4.3.4].

Since the standard t-structure is right complete we have $f^!(\mathcal{F}) \cong \operatorname{colim} \tau^{\leq n} f^!(\mathcal{F})$ by Lemma 2.2. Since it is compatible with filtered colimits, and since h'^* is continuous, $h'^* f^!(\mathcal{F})$ is then bounded below if the sheaves $h'^* \tau^{\leq n} f^!(\mathcal{F})$ are uniformly bounded below. Since θ' is faithfully flat, hence θ'^* conservative on $\operatorname{IndCoh}(X')^+$, it suffices to show this for the sheaves $\theta'^* h'^* \tau^{\leq n} f^!(\mathcal{F})$. But since Y and Y' are truncated and tamely presented, $u_\alpha \circ \psi$ is of finite Tor-dimension by Proposition 4.14, hence so is $u'_\alpha \circ \psi'$. The claim then follows since

$$\theta'^* h'^* \tau^{\leq n} f^!(\mathcal{F}) \cong \psi'^* g'^* \tau^{\leq n} f^!(\mathcal{F}) \cong \psi'^* u'^*_\alpha \tau^{\leq n} \phi^!_\alpha(\mathcal{F}_\alpha).$$

Now we show (6.4) is an isomorphism, which as before reduces to showing $\psi'^* \phi^! g^*(\mathcal{F}) \to \phi'^! \psi^* g^*(\mathcal{F})$ is. The composition

$$\psi'^* u'^*_\alpha \phi^!_\alpha(\mathcal{F}_\alpha) \to \psi'^* \phi^! u^*_\alpha(\mathcal{F}_\alpha) \to \phi'^! \psi^* u^*_\alpha(\mathcal{F}_\alpha),$$

and its first factor are now isomorphisms by Proposition 6.2, since again $u_\alpha \circ \psi$ is of finite Tor-dimension by Proposition 4.14 (note that Z_α is geometric since Z is and since u'_α is faithfully flat [Lur18, Prop. 9.3.1.3]). But then the second factor, which is a rewriting of $\psi'^* \phi^! g^*(\mathcal{F}) \to \phi'^! \psi^* g^*(\mathcal{F})$, is also an isomorphism. ☐

The extension to ind-geometric stacks now follows the corresponding result for flat morphisms among ind-schemes [Ras19, Lem. 6.17.2].

Proposition 6.6. *Let the following be a Cartesian diagram of ind-geometric stacks.*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Suppose that all stacks in the diagram are coherent and that Y and Y' are ind-tamely presented. Suppose also that h has semi-universal coherent pullback and that f is ind-proper and almost of ind-finite presentation. Then for any $\mathcal{F} \in \operatorname{IndCoh}(Y)$ the Beck-Chevalley map $h'^* f^!(\mathcal{F}) \to f'^! h^*(\mathcal{F})$ is an isomorphism.

Proof. Suppose first that X is truncated and geometric. Let $Y \cong \operatorname{colim} Y_\alpha$ be a reasonable presentation, and define $h_\alpha : Y'_\alpha \to Y_\alpha$, $i'_\alpha : Y'_\alpha \to Y'$ by base change, similarly $i'_{\alpha\beta} : Y'_\alpha \to Y'_\beta$ for $\beta \geq \alpha$. Note that since h has semi-universal coherent pullback each Y'_α is a reasonable geometric substack of Y', and in particular is tamely presented. By Proposition 6.5 we have $h'^*_\alpha i^!_{\alpha\beta} \cong i'^!_{\alpha\beta} h^*_\beta$ in $\operatorname{Fun}(\operatorname{IndCoh}(Y_\beta), \operatorname{IndCoh}(Y'_\alpha))$. For any α we then have $h'^*_\alpha i^!_\alpha \cong i'^!_\alpha h^*$ in

$\mathrm{Fun}(\mathrm{IndCoh}(Y), \mathrm{IndCoh}(Y'_\alpha))$ by (**??**) and [Lur17, Prop. 4.7.5.19]. By Proposition 5.2 we can factor f as $X \xrightarrow{f_\alpha} Y_\alpha \xrightarrow{i_\alpha} Y$ for some α. Consider the diagram

$$\begin{tikzcd} X' \arrow[r, "f'_\alpha"] \arrow[d, "h'"'] & Y'_\alpha \arrow[r, "i'_\alpha"] \arrow[d, "h_\alpha"] & Y' \arrow[d, "h"] \\ X \arrow[r, "f_\alpha"] & Y_\alpha \arrow[r, "i_\alpha"] & Y \end{tikzcd}$$

of Cartesian squares. The map in the statement now factors as

$$h'^* f^!(\mathcal{F}) \cong h^* f^!_\alpha i^!_\alpha(\mathcal{F}) \to f'^!_\alpha h^*_\alpha i^!_\alpha(\mathcal{F}) \to f'^!_\alpha i'^!_\alpha h^*(\mathcal{F}) \cong f'^! h^*(\mathcal{F}),$$

and we have shown both factors are isomorphisms (again using Proposition 6.5).

Now suppose $X \cong \operatorname{colim} X_\alpha$ is a reasonable presentation. For any α we have a diagram

$$\begin{tikzcd} X'_\alpha \arrow[r, "i'_\alpha"] \arrow[d, "h'_\alpha"'] & X' \arrow[r, "f'"] \arrow[d, "h'"] & Y' \arrow[d, "h"] \\ X_\alpha \arrow[r, "i_\alpha"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

of Cartesian squares. By Proposition 5.16 each h'_α has semi-universal coherent pullback, hence we have a reasonable presentation $X' \cong \operatorname{colim} X'_\alpha$ since filtered colimits are left exact in $\widehat{\mathrm{Stk}}_k$. The functors $i'^!_\alpha$ thus determine an isomorphism $\mathrm{IndCoh}(X') \cong \lim_\alpha \mathrm{IndCoh}(X'_\alpha)$ in $\widehat{\mathrm{Cat}}_\infty$, hence it suffices to show that $i'^!_\alpha h'^* f^!(\mathcal{F}) \to i'^!_\alpha f'^! h^*(\mathcal{F})$ is an isomorphism for all α. But by the previous paragraph both the composition

$$h'^*_\alpha i^!_\alpha f^!(\mathcal{F}) \to i'^!_\alpha h'^* f^!(\mathcal{F}) \to i'^!_\alpha f'^! h^*(\mathcal{F}) \tag{6.7}$$

and its first factor are isomorphisms, hence the second is as well. ☐

We can relax the coherence hypotheses in Proposition 6.6 if we assume h is of finite Tor-dimension and impose suitable boundedness conditions on $\mathcal{F}$. However, the proof does not quite work with the hypothesis $\mathcal{F} \in \mathrm{IndCoh}(Y)^+$, for the following reason: ind-properness of f does not imply that $f^!$ takes $\mathrm{IndCoh}(Y)^+$ to $\mathrm{IndCoh}(X)^+$.

Instead, consider the full subcategory $\mathrm{IndCoh}(Y)^+_{lim} \subset \mathrm{IndCoh}(Y)$ of $\mathcal{F}$ such that $i^!(\mathcal{F}) \in \mathrm{IndCoh}(Y')^+$ for every truncated geometric substack $i : Y' \to Y$. By Proposition 5.4 this is equivalent to $i^!_\alpha(\mathcal{F}) \in \mathrm{IndCoh}(Y_\alpha)^+$ for all α, where $Y \cong \operatorname{colim} Y_\alpha$ is any ind-geometric presentation. In particular, the equivalance $\mathrm{IndCoh}(Y) \cong \lim \mathrm{IndCoh}(Y_\alpha)$ in $\widehat{\mathrm{Cat}}_\infty$ restricts to an equivalence

$$\mathrm{IndCoh}(Y)^+_{lim} \cong \lim \mathrm{IndCoh}(Y_\alpha)^+. \tag{6.8}$$

We have $\mathrm{IndCoh}(Y)^+ \subset \mathrm{IndCoh}(Y)^+_{lim}$ since each $i^!$ is left t-exact, and it follows from the definitions that $f^!$ takes $\mathrm{IndCoh}(Y)^+_{lim}$ to $\mathrm{IndCoh}(X)^+_{lim}$. We now have the following variant of Proposition 6.6, proved the same way with Proposition 6.2 in place of Proposition 6.5 and

(6.8) in place of (**??**) (and [Lur17, Cor. 4.7.5.18] in place of [Lur17, Prop. 4.7.5.19], since bounded below categories are not presentable).

Proposition 6.9. *Let the following be a Cartesian diagram of ind-geometric stacks.*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Suppose that X and Y are reasonable. Suppose also that h is geometric and of finite Tor-dimension, and that f is ind-proper and almost of ind-finite presentation. Then for any $\mathcal{F} \in \mathrm{IndCoh}(Y)^+_{lim}$ the Beck-Chevalley map $h'^* f^!(\mathcal{F}) \to f'^! h^*(\mathcal{F})$ is an isomorphism.

7. Coherent pullback and sheaf Hom

Given a geometric stack Y, the sheaf Hom out of $\mathcal{F} \in \mathrm{QCoh}(Y)$ is defined by the adjunction

$$- \otimes \mathcal{F} : \mathrm{QCoh}(Y) \leftrightarrows \mathrm{QCoh}(Y) : \mathcal{H}om(\mathcal{F}, -).$$

For any $f : X \to Y$, the isomorphism $f^*(- \otimes \mathcal{F}) \cong - \otimes f^*(\mathcal{F})$ then gives rise to a map

$$f^* \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(f^*(\mathcal{F}), f^*(\mathcal{G})) \tag{7.1}$$

which is natural in $\mathcal{G} \in \mathrm{QCoh}(Y)$. This is not an isomorphism in general, but is when f is of finite Tor-dimension under certain hypotheses on $\mathcal{F}$ and $\mathcal{G}$. The basic goal of this section is to generalize this and related results, in particular allowing f to have coherent pullback, X and Y to be ind-geometric, and $\mathcal{F}$ and $\mathcal{G}$ to be ind-coherent.

Since $\mathrm{IndCoh}(X)$ does not generally have a tensor product, external products instead play the primary role. That is, for suitable $\mathcal{F} \in \mathrm{IndCoh}(Y)$ we have an adjunction

$$- \boxtimes \mathcal{F} : \mathrm{IndCoh}(X) \leftrightarrows \mathrm{IndCoh}(X \times Y) : (- \boxtimes \mathcal{F})^R.$$

To make explicit their dependence on X we will denote these functors by $e_{\mathcal{F},X}$ and $e^R_{\mathcal{F},X}$. When X and Y are geometric, we have an isomorphism $\mathcal{H}om(\mathcal{F}, -) \cong e^R_{\mathcal{F},Y} \Delta_{Y*}$, letting us subsume results about $\mathcal{H}om(\mathcal{F}, -)$ in corresponding results about $e^R_{\mathcal{F},Y}$. This formula moreover provides a useful definition of sheaf Hom in the ind-geometric setting.

On a technical level, there is a close analogy between $e^R_{\mathcal{F},X}$ for coherent $\mathcal{F}$ and the functor $g^!$ associated to a morphism $g : X' \to Y'$ which is ind-proper and almost of ind-finite presentation — the two functors have similar formal properties for similar reasons. In particular, many proofs about $e^R_{\mathcal{F},X}$ in this section closely follow corresponding proofs in Section 6. The main difference is that the role of the map g is now played by the projection $X \times Y \to X$, so a contravariant functoriality has been replaced with covariant one.

7.1. **External products.** We begin by recalling some basic results about external products. *We assume that k is an ordinary ring of finite global dimension for the rest of the paper.* Suppose first that X and Y are geometric stacks and $\mathcal{F} \in \mathrm{IndCoh}(Y)^b$. By Lemma **??** the assignment $\mathcal{G} \mapsto \mathcal{G} \boxtimes \Psi_Y(\mathcal{F})$ defines a bounded colimit-preserving functor $e_{\Psi_Y(\mathcal{F}),X}$: $\mathrm{QCoh}(X) \to \mathrm{QCoh}(X \times Y)$. The universal property of $\mathrm{IndCoh}(-)$ guarantees that this functor has a unique ind-coherent lift in the following sense.

Definition 7.2. If X and Y are geometric stacks and $\mathcal{F} \in \mathrm{IndCoh}(Y)^b$, we let $e_{\mathcal{F},X}$ denote the unique bounded colimit-preserving functor fitting into a diagram of the following form.

$$\begin{tikzcd} \mathrm{IndCoh}(X) \arrow[r, "e_{\mathcal{F},X}"] \arrow[d, "\Psi_X"'] & \mathrm{IndCoh}(X \times Y) \arrow[d, "\Psi_{X \times Y}"] \\ \mathrm{QCoh}(X) \arrow[r, "e_{\Psi_Y(\mathcal{F}),X}"] & \mathrm{QCoh}(X \times Y) \end{tikzcd} \tag{7.3}$$

In [CW23b] we show this is functorial in that there exists a diagram

$$\begin{tikzcd} \mathrm{IndCoh}(-) \times \mathrm{IndCoh}(-)^b \arrow[r] \arrow[d, "\Psi_{(-)} \times \Psi_{(-)}"'] & \mathrm{IndCoh}(- \times -) \arrow[d, "\Psi_{(- \times -)}"] \\ \mathrm{QCoh}(-) \times \mathrm{QCoh}(-)^b \arrow[r] & \mathrm{QCoh}(- \times -) \end{tikzcd} \tag{7.4}$$

of functors $\mathrm{Corr}(\mathrm{GStk}_k)^{\times 2}_{fcd;ftd} \to \widehat{\mathrm{Cat}}_\infty$ which specializes to the diagram (7.3) when evaluated on any $X, Y \in \mathrm{GStk}_k$ and any $\mathcal{F} \in \mathrm{IndCoh}(Y)^b$.

We postpone the proof of Proposition **??** while we extend Definition 7.2 to ind-geometric stacks. To simplify the needed constructions we restrict our attention to the case where Y is reasonable and $\mathcal{F}$ is coherent. This is not strictly essential, but most good properties of $e^R_{\mathcal{F},X}$ (e.g. almost continuity) will require $\mathcal{F}$ to be coherent anyway.

The top arrow can be extended to a functor $\mathrm{Corr}(\mathrm{indGStk}_k)_{fcd;ftd} \times \mathrm{Corr}(\mathrm{indGStk}_k^{reas})_{prop;ftd} \to \widehat{\mathrm{Cat}}_\infty^{\Delta^1}$ of the form

$$\mathrm{IndCoh}(-) \times \mathrm{Coh}(-) \to \mathrm{IndCoh}(- \times -).$$

We refer to [CW23b] for details, but its behavior on objects is determined as follows. If $\mathcal{F} \cong i_*(\mathcal{F}')$ for some reasonable geometric substack $i : Z' \to Z$, and if $X \cong \mathrm{colim}\, X_\alpha$ is an ind-geometric presentation, then we have a diagram

$$\begin{tikzcd} \mathrm{IndCoh}(X_\alpha) \arrow[r, "e_{\mathcal{F}',X_\alpha}"] \arrow[d, "i_{\alpha *}"'] & \mathrm{IndCoh}(X_\alpha \times Z') \arrow[d, "(i_\alpha \times i)_*"] \\ \mathrm{IndCoh}(X) \arrow[r, "e_{\mathcal{F},X}"] & \mathrm{IndCoh}(X \times Z) \end{tikzcd}$$

for all α. The functor $e_{\mathcal{F},X}$ is determined by these diagrams together with the fact that by construction it preserves small colimits. Moreover, $e_{\mathcal{F},X}$ is bounded [CW23b].

If X and $X \times Z$ are reasonable and $\mathcal{F} \in \mathrm{Coh}(Z)$, it follows from the definitions that $e_{\mathcal{F},X}$ takes $\mathrm{Coh}(X)$ to $\mathrm{Coh}(X \times Z)$. In particular, suppose X and $X \times Z$ are coherent, X' and Z' are reasonable, and $h : X' \to X$, $\phi : Z' \to Z$ are morphisms with coherent pullback. Then there is a canonical isomorphism

$$(h \times \phi)^* e_{\mathcal{F},X} \cong e_{\phi^*(\mathcal{F}),X'} h^* \tag{7.5}$$

since both sides are continuous and have canonically isomorphic restrictions to $\mathrm{Coh}(X)$.

7.2. **Adjoints and sheaf Hom.** Let X and Z be ind-geometric stacks such that Z is reasonable, and let $\mathcal{F} \in \mathrm{Coh}(Z)$. We denote the right adjoint of $e_{\mathcal{F},X}$ by

$$e^R_{\mathcal{F},X} : \mathrm{IndCoh}(X \times Z) \to \mathrm{IndCoh}(X).$$

When X and Z are geometric and $\mathcal{F} \in \mathrm{QCoh}(X)$ we define $e^R_{\mathcal{F},X} : \mathrm{QCoh}(X \times Z) \to \mathrm{QCoh}(X)$ similarly. These are external counterparts of the internal sheaf Hom, which we define in the setting of ind-coherent sheaves on a reasonable ind-geometric stack X as

$$\mathcal{H}om(\mathcal{F}, -) := e^R_{\mathcal{F},X} \Delta_{X*} : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(X). \tag{7.6}$$

Here we again assume $\mathcal{F} \in \mathrm{Coh}(X)$. This definition is justified in part by the following result, and will be more fully justified by Corollary 7.32.

Proposition 7.7. *Let X and Z be ind-geometric stacks such that Z is reasonable, and let $\mathcal{F} \in \mathrm{Coh}(Z)$. Then $e^R_{\mathcal{F},X}$ is left bounded. If X and Z are geometric, the Beck-Chevalley map $\Psi_X e^R_{\mathcal{F},X}(\mathcal{G}) \to e^R_{\Psi_Z(\mathcal{F}),X} \Psi_{X \times Z}(\mathcal{G})$ is an isomorphism for all $\mathcal{G} \in \mathrm{IndCoh}(X \times Z)^+$, and the induced map $\Psi_X \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(\Psi_X(\mathcal{F}), \Psi_X(\mathcal{G}))$ is an isomorphism for all $\mathcal{G} \in \mathrm{IndCoh}(X)^+$.*

Proof. Since $e_{\mathcal{F},X}$ is bounded (Proposition **??**), $e^R_{\mathcal{F},X}$ is left bounded and the two functors restrict to an adjunction between $\mathrm{IndCoh}(X)^+$ and $\mathrm{IndCoh}(X \times Z)^+$. The analogous statement holds for $e_{\Psi_Z(\mathcal{F}),X}$ and $e^R_{\Psi_Z(\mathcal{F}),X}$, and the second claim follows and since $\Psi_{(-)}$ restricts to an equivalence $\mathrm{IndCoh}(-)^+ \xrightarrow{\sim} \mathrm{QCoh}(-)^+$ and since $\Psi_{X \times Z} e_{\mathcal{F},X} \cong e_{\Psi_Z(\mathcal{F}),X} \Psi_X$ (Proposition **??**). The third follows since Δ_{X*} is also compatible with the $\Psi_{(-)}$ functors, and since we have an isomorphism $- \otimes \Psi_Z(\mathcal{F}) \cong \Delta^*_X e_{\Psi_Z(\mathcal{F}),X}$ of functors $\mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$. □

Suppose that $f : X' \to X$, $g : Z' \to Z$ are morphisms of ind-finite cohomological dimension between ind-geometric stacks, and that Z' and Z are reasonable. Suppose also that either f and g are of finite Tor-dimension, or that they have coherent pullback and X, X', $X \times Z$, and $X' \times Z'$ are coherent. Then for $\mathcal{F} \in \mathrm{Coh}(Z)$ the isomorphism $(f \times g)^* e_{\mathcal{F},X} \cong e_{g^*(\mathcal{F}),X'} f^*$ of functors $\mathrm{IndCoh}(X) \to \mathrm{IndCoh}(X' \times Z')$ yields an isomorphism

$$e^R_{\mathcal{F},X}(f \times g)_* \cong f_* e^R_{g^*(\mathcal{F}),X'} \tag{7.8}$$

of right adjoints.

Similarly, suppose instead that f and g are ind-proper and that g is almost of ind-finite presentation. Then for $\mathcal{F} \in \mathrm{Coh}(Z')$ the isomorphism $(f \times g)_* e_{\mathcal{F},X'} \cong e_{g_*(\mathcal{F}),X} f_*$ of functors $\mathrm{IndCoh}(X') \to \mathrm{IndCoh}(X \times Z)$ yields an isomorphism

$$e^R_{\mathcal{F},X'}(f \times g)^! \cong f^! e^R_{g_*(\mathcal{F}),X} \tag{7.9}$$

of right adjoints.

Proposition 7.10. *Let X and Z be ind-geometric stacks such that Z is reasonable and X is semi-reasonable, and let $\mathcal{F} \in \mathrm{Coh}(Z)$. Then $e^R_{\mathcal{F},X}$ is almost continuous.*

7.3. **External products and $*$-pullback.** Suppose that X, Y, and Z are ind-geometric stacks, that Z is reasonable, that $h : X \to Y$ is a morphism of finite Tor-dimension, and that $\mathcal{F} \in \mathrm{Coh}(Z)$. We have an isomorphism $e_{\mathcal{F},X} h^* \cong (h \times id_Z)^* e_{\mathcal{F},Y}$, hence an associated Beck-Chevalley map

$$h^* e^R_{\mathcal{F},Y}(\mathcal{G}) \to e^R_{\mathcal{F},X}(h \times id_Z)^*(\mathcal{G}) \tag{7.11}$$

for $\mathcal{G} \in \mathrm{IndCoh}(Y \times Z)$. We also have a corresponding map when h has coherent pullback and suitable coherence hypotheses are satisfied, and this section studies when (7.11) and related maps are isomorphisms.

In particular, results about these maps immediately imply more familiar statements about sheaf Hom. Suppose X and Y are reasonable ind-geometric stacks and $h : X \to Y$ is a morphism of finite Tor-dimension and ind-finite cohomological dimension. Then for any $\mathcal{F} \in \mathrm{Coh}(Y)$ and $\mathcal{G} \in \mathrm{IndCoh}(Y)$ there is a natural map

$$h^* \mathscr{H}\!om(\mathcal{F}, \mathcal{G}) \to \mathscr{H}\!om(h^*(\mathcal{F}), h^*(\mathcal{G})) \tag{7.12}$$

given by the composition

$$h^* e^R_{\mathcal{F},Y} \Delta_{Y*}(\mathcal{G}) \to e^R_{\mathcal{F},X}(h \times id_Y)^* \Delta_{Y*}(\mathcal{G}) \to e^R_{\mathcal{F},X} \Delta'_{Y*} h^*(\mathcal{G}) \cong e^R_{h^*(\mathcal{F}),X} \Delta_{X*} h^*(\mathcal{G}). \tag{7.13}$$

Here $\Delta'_Y : Y \to X \times Y$ is the base change of Δ_Y, and the last isomorphism follows from (7.8) and $\Delta'_{Y*} \cong (id_X \times h)_* \Delta_{X*}$. When X and Y are geometric and $\mathcal{G} \in \mathrm{IndCoh}(Y)^+$, one can check using Proposition 7.7 that the equivalences $\Psi^+_{(-)} : \mathrm{IndCoh}(-)^+ \cong \mathrm{QCoh}(-)^+$ identify (7.12) with the Beck-Chevalley map (7.1) associated to the isomorphism $h^*(- \otimes \Psi_Y(\mathcal{F})) \cong - \otimes h^* \Psi_Y(\mathcal{F})$.

If h has coherent pullback, we have a map (7.12) given by the same formula provided X, Y, $X \times X$, $X \times Y$, and $Y \times Y$ are coherent. In our cases of interest X and Y are coherent because they are tamely presented and affine over an ind-locally Noetherian geometric stack, and in this case $X \times X$, $X \times Y$, and $Y \times Y$ are automatically coherent (Proposition 5.6).

Remark 7.14. The finite cohomological dimension hypotheses above and throughout this section are satisfied in our applications, but can mostly be relaxed. Note that (7.8) remains true with such hypothesis, replacing $(f \times g)_*$ and f_* with the (not necessarily continuous) right adjoints $(f \times g)^{*R}$ and f^{*R}. However, even when these are continuous because of coherence hypotheses, this relaxation raises subtleties it is convenient to avoid.

Proposition 7.15. *Let X, Y, and Z be geometric stacks such that Z is reasonable, and let $\mathcal{F} \in \mathrm{Coh}(Z)$. If $h : X \to Y$ is a morphism of finite Tor-dimension, then for any $\mathcal{G} \in \mathrm{IndCoh}(Y \times Z)^+$ the Beck-Chevalley map $h^* e^R_{\mathcal{F},Y}(\mathcal{G}) \to e^R_{\mathcal{F},X}(h \times id_Z)^*(\mathcal{G})$ is an isomorphism.*

Proof. Set $h' = h \times id_Z$, and note first that we may assume X is affine. Otherwise choose a flat cover $g : U = \operatorname{Spec} A \to X$ and consider the following diagram of Cartesian squares.

$$\begin{tikzcd}
U \times Z \arrow[r, "g'"] \arrow[d] & X \times Z \arrow[r, "h'"] \arrow[d] & Y \times Z \arrow[d] \\
U \arrow[r, "g"] & X \arrow[r, "h"] & Y
\end{tikzcd}$$

We have $h^* e^R_{\mathcal{F},Y}(\mathcal{G}), e^R_{\mathcal{F},X} h'^*(\mathcal{G}) \in \mathrm{IndCoh}(X)^+$ since $h^* e^R_{\mathcal{F},Y}$ and $e^R_{\mathcal{F},X} h'^*$ are left bounded (Proposition 7.7). By faithful flatness the restriction of g^* to $\mathrm{IndCoh}(X)^+$ is conservative, so it suffices to show the first factor of

$$g^* h^* e^R_{\mathcal{F},Y}(\mathcal{G}) \to g^* e^R_{\mathcal{F},X} h'^*(\mathcal{G}) \to e^R_{\mathcal{F},U} g'^* h'^*(\mathcal{G}),$$

is an isomorphism. But the second factor is an isomorphism by Lemma **??**, hence it suffices to show the composition is.

Now fix a flat cover $\phi : V = \operatorname{Spec} B \to Y$. We have a diagram

$$\begin{tikzcd}
& & U \arrow[dll, "\theta"'] \arrow[dd, "\psi"] & & U \times Z \arrow[ll] \arrow[dl, "\theta'"] \arrow[dd, "\psi'"] \\
X \arrow[dd, "h"'] & & & X \times Z \arrow[lll, crossing over] \arrow[dd, "h'"] & \\
& & V \arrow[dll, "\phi"'] & & V \times Z \arrow[ll, crossing over] \arrow[dl, "\phi'"] \\
Y & & & Y \times Z \arrow[lll] &
\end{tikzcd} \tag{7.16}$$

with Cartesian faces, and an associated diagram

$$\begin{tikzcd}
\theta^* h^* e^R_{\mathcal{F},Y}(\mathcal{G}) \arrow[r] \arrow[d, "\wr"] & \theta^* e^R_{\mathcal{F},X} h'^*(\mathcal{G}) \arrow[r] & e^R_{\mathcal{F},U} \theta'^* h'^*(\mathcal{G}) \arrow[d, "\wr"] \\
\psi^* \phi^* e^R_{\mathcal{F},Y}(\mathcal{G}) \arrow[r] & \psi^* e^R_{\mathcal{F},V} \phi'^*(\mathcal{G}) \arrow[r] & e^R_{\mathcal{F},U} \psi'^* \phi'^*(\mathcal{G})
\end{tikzcd}$$

in $\mathrm{IndCoh}(U)$. Again by faithful flatness the restriction of θ^* to $\mathrm{IndCoh}(X)^+$ is conservative, hence it suffices to show the top left map is an isomorphism. Noting that U is affine since X and V are, the bottom right map is an isomorphism by Lemma **??** and flatness of ϕ'. But the top right and bottom left maps are isomorphisms by Lemma **??** and flatness of h', so the top left map must be as well. $\square$

Corollary 7.17. *Let X and Y be reasonable geometric stacks and $h : X \to Y$ a morphism of finite Tor-dimension and ind-finite cohomological dimension. Then for any $\mathcal{F} \in \mathrm{Coh}(Y)$, $\mathcal{G} \in \mathrm{IndCoh}(Y)^+$ the natural map $h^* \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(h^*(\mathcal{F}), h^*(\mathcal{G}))$ is an isomorphism.*

Proof. Follows from Proposition 7.15 and the definition (7.13), as Δ_{Y*} is left t-exact. □

In relaxing the finite Tor-dimension hypothesis of Proposition 7.15 we will need to replace $e^R_{\mathcal{F},X}$ by its composition with a suitable generalized diagonal map. The example to keep in mind in the following statement is $Z = W = Y$, in which case Y' is Y and δ_Y is Δ_Y.

Proposition 7.18. *Let the following be a diagram of geometric stacks in which both squares are Cartesian, X and Y are truncated and coherent, h and g are of finite cohomological dimension and have semi-universal coherent pullback, and f is proper and almost of finite presentation.*

$$\begin{tikzcd} X' \arrow[r, "h'"] \arrow[d, "f''"'] & Y' \arrow[r, "g'"] \arrow[d, "f'"] & Z \arrow[d, "f"] \\ X \arrow[r, "h"] & Y \arrow[r, "g"] & W \end{tikzcd}$$

Suppose that $X \times Z$ and $Y \times Z$ are coherent and that X, Y, and Z are tamely presented. Then X' and Y' are coherent, and for any $\mathcal{F} \in \mathrm{Coh}(Z)$ and $\mathcal{G} \in \mathrm{IndCoh}(Y')$ the composition

$$h^* e^R_{\mathcal{F},Y} \delta_{Y*}(\mathcal{G}) \to e^R_{\mathcal{F},X} (h \times id_Z)^* \delta_{Y*}(\mathcal{G}) \to e^R_{\mathcal{F},X} \delta_{X*} h'^*(\mathcal{G}) \tag{7.19}$$

is an isomorphism, where $\delta_X : X' \to X \times Z$ and $\delta_Y : Y' \to Y \times Z$ are the natural maps.

Lemma 7.20. *Let X be a locally coherent geometric stack and $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)$. Then $\tau^{\leq n} \mathcal{H}om(\mathcal{F}, \mathcal{G})$ is coherent for all n.*

Proof. First suppose $A \in \mathrm{CAlg}_k$ is coherent and $M, N \in \mathrm{Coh}_A$. For any m there exists an exact triangle $P \to M \to Q$ such that P is perfect and $Q \in \mathrm{Mod}_A^{\leq m}$ [Lur18, Cor. 2.7.2.2], yielding an exact triangle

$$\mathcal{H}om(Q, N) \to \mathcal{H}om(M, N) \to \mathcal{H}om(P, N).$$

If $N \in \mathrm{Mod}^{\geq i}$ then $\mathcal{H}om(Q, N) \in \mathrm{Mod}^{\geq i-m}$, hence it follows from the associated long exact sequence that $H^j \mathcal{H}om(M, N) \cong H^j \mathcal{H}om(P, N)$ for $j < i - m - 1$. Since P is perfect $\mathcal{H}om(P, N) \cong P^\vee \otimes N$, which is coherent since N is and since $P^\vee$ is perfect ([Lur17, Prop. 7.2.4.11, Prop. 7.2.4.23]). Since A is coherent $H^j \mathcal{H}om(M, N)$ is then finitely presented over $H^0(A)$ for $j < i - m - 1$. But since m was arbitrary and $\mathcal{H}om(M, N)$ is bounded below, coherence of A then also implies $\tau^{\leq n} \mathcal{H}om(M, N)$ is coherent for all n.

Now let $f : \mathrm{Spec}\, A \to X$ be a flat cover such that A is coherent. By Proposition 7.7 we can conflate $\mathcal{F}$ and $\mathcal{G}$ with their images in $\mathrm{QCoh}(X)$. It suffices to show $f^* \tau^{\leq n} \mathcal{H}om(\mathcal{F}, \mathcal{G})$ is coherent. We have $f^* \tau^{\leq n} \mathcal{H}om(\mathcal{F}, \mathcal{G}) \cong \tau^{\leq n} \mathcal{H}om(f^*(\mathcal{F}), f^*(\mathcal{G}))$ by flatness and Corollary 7.17, hence the claim follows from the previous paragraph. □

Proof of Proposition 7.18. First note that X' and Y' are tamely presented by our hypotheses on X, Y, and f. The maps δ_X and δ_Y are affine since they are base changes of Δ_W (along $gh \times f$ and $g \times f$), hence X' and Y' are coherent by Proposition **??**. In particular, the second factor of (7.19) is well-defined by Proposition 5.30. Moreover, $\operatorname{IndCoh}(Y')$ is compactly generated and all functors in the statement are continuous, so it suffices to assume $\mathcal{G} \in \operatorname{Coh}(Y')$.

We first claim that $h^* e^R_{\mathcal{F},Y} \delta_{Y*}(\mathcal{G})$ and $e^R_{\mathcal{F},X} \delta_{X*} h'^*(\mathcal{G})$ are bounded below. This is immediate for the latter but not the former, since $e^R_{\mathcal{F},Y} \delta_{Y*}(\mathcal{G})$ need not be coherent and h^* need not be left bounded. Fix a strictly tamely presented flat cover $\phi : V = \operatorname{Spec} A \to Y$ such that A is strictly tamely presented over k. Since Y is truncated there exists for some n a strictly tame presentation $A \cong \operatorname{colim} A_\alpha$ of order n over k. Set $V_\alpha = \operatorname{Spec} A_\alpha$. Using Noetherian approximation [Lur18, Prop. 4.2.1.5, Thm. 4.4.2.2, Prop. 5.5.4.1] and faithful flatness of the $u_\alpha : V \to V_\alpha$ we have, for some α, a diagram

$$
\begin{tikzcd}
 & W \arrow[dl, "\theta"'] \arrow[dd, "\psi"] & & W' \arrow[ll, "\xi'"'] \arrow[dl, "\theta'"] \arrow[dd, "\psi'"] \\
X \arrow[dd] & & X' \arrow[ll] \arrow[dd] & \\
 & V \arrow[dl, "\phi"'] \arrow[dd, "u_\alpha"] & & V' \arrow[ll, "\xi"'] \arrow[dl, "\phi'"] \arrow[dd, "u'_\alpha"] \\
Y & & Y' \arrow[ll] & \\
 & V_\alpha & & V'_\alpha \arrow[ll, "\xi_\alpha"']
\end{tikzcd}
\tag{7.21}
$$

in which all faces are Cartesian and ξ_α is proper and almost of finite presentation. Since f is representable and almost of finite presentation g' has semi-universal coherent pullback and $g'^*(\mathcal{F}) \in \operatorname{Coh}(Y')$. Increasing α, we may then assume by [Lur18, Thm. 4.5.12.3], flatness of the u_α, and coherence of the A_α that there exist $\mathcal{F}_\alpha, \mathcal{G}_\alpha \in \operatorname{Coh}(Y'_\alpha)$ such that $\phi'^* g'^*(\mathcal{F}) \cong u'^*_\alpha(\mathcal{F}_\alpha)$ and $\phi'^*(\mathcal{G}) \cong u'^*_\alpha(\mathcal{G}_\alpha)$.

Letting $\delta'_Y : Y' \to Y \times Y'$ denote the natural map, we have isomorphisms

$$e^R_{\mathcal{F},Y} \delta_{Y*}(\mathcal{G}) \cong e^R_{g'^*(\mathcal{F}),Y} \delta'_{Y*}(\mathcal{G}) \cong f'_* e^R_{g'^*(\mathcal{F}),Y'} \Delta_{Y'*}(\mathcal{G}) \cong f'_* \mathcal{H}om(g'^*(\mathcal{F}), \mathcal{G}).$$

Here the first uses (7.8) and the fact that $\delta_Y \cong (id_Y \times g') \circ \delta'_Y$, the second uses Proposition 7.30 (which does not depend on the current Proposition) and the fact that $\delta'_Y \cong (f' \times id_{Y'}) \circ \Delta_{Y'}$, and the last is by definition. Since $\operatorname{IndCoh}(Y')$ is right complete and $h^* f'_*$ is continuous we then have $h^* e^R_{\mathcal{F},Y} \delta_{Y*}(\mathcal{G}) \cong \operatorname{colim} h^* f'_* \tau^{\leq n} \mathcal{H}om(g'^*(\mathcal{F}), \mathcal{G})$.

Now $h^* f'_*$ preserves coherence, hence $h^* f'_* \tau^{\leq n} \mathcal{H}om(g'^*(\mathcal{F}), \mathcal{G})$ is coherent for all n by Lemma 7.20. Since the t-structure on $\operatorname{IndCoh}(X)$ is compatible with filtered colimits, $h^* e^R_{\mathcal{F},Y} \delta_{Y*}(\mathcal{G})$ is then bounded below if $h^* f'_* \tau^{\leq n} \mathcal{H}om(g'^*(\mathcal{F}), \mathcal{G})$ is uniformly bounded below in n. Since θ is faithfully flat, hence θ^* t-exact and conservative on $\operatorname{IndCoh}(X)^+$, it suffices to show $\theta^* h^* f'_* \tau^{\leq n} \mathcal{H}om(g'^*(\mathcal{F}), \mathcal{G})$ is uniformly bounded below in n. But we have

$$\begin{aligned} \theta^* h^* f'_* \tau^{\leq n} \mathcal{H}om(g'^*(\mathcal{F}), \mathcal{G}) &\cong \psi^* \xi_* \phi'^* \tau^{\leq n} \mathcal{H}om(g'^*(\mathcal{F}), \mathcal{G}) \\ &\cong \psi^* \xi_* \tau^{\leq n} \mathcal{H}om(\phi'^*(g'^*(\mathcal{F})), \phi'^*(\mathcal{G})) \\ &\cong \psi^* \xi_* \tau^{\leq n} \mathcal{H}om(u'^*_\alpha(\mathcal{F}_\alpha), u'^*_\alpha(\mathcal{G}_\alpha)) \\ &\cong \psi^* \xi_* u'^*_\alpha \tau^{\leq n} \mathcal{H}om(\mathcal{F}_\alpha, \mathcal{G}_\alpha) \\ &\cong \psi^* u^*_\alpha \xi_{\alpha *} \tau^{\leq n} \mathcal{H}om(\mathcal{F}_\alpha, \mathcal{G}_\alpha), \end{aligned}$$

where the second and fourth isomorphisms use Corollary 7.17 and flatness of ϕ' and u'_α. The claim now follows since $u_\alpha \circ \psi$ is of finite Tor-dimension by Proposition 4.14.

Note that we may now assume X is affine and strictly tamely presented over $\operatorname{Spec} k$. Otherwise choose a flat cover $u : U = \operatorname{Spec} B \to X$ by such an affine, letting $u' : U' \to X'$ denote its base change and $\delta_U : U' \to U \times Z$ the base change of δ_Y. By faithful flatness the restriction of u^* to $\operatorname{IndCoh}(X)^+$ is conservative, so it suffices to show the first factor of

$$u^* h^* e^R_{\mathcal{F},Y} \delta_{Y*}(\mathcal{G}) \to u^* e^R_{\mathcal{F},X} \delta_{X*} h'^*(\mathcal{G}) \to e^R_{\mathcal{F},U} \delta_{U*} u'^* h'^*(\mathcal{G}),$$

is an isomorphism. But the second factor is an isomorphism by Lemma **??**, hence it suffices to show the composition is.

Returing to (7.21), it now follows that W is affine and strictly tamely presented over $\operatorname{Spec} k$, and that $W \times Z$, $V \times Z$, W', and V' are tamely presented. Proposition **??** now implies $W \times Z$, $V \times Z$, W', and V' are coherent since $X \times Z$, $Y \times Z$, X', and Y' are, and since ϕ and its base changes are affine. We then have a diagram

$$\begin{tikzcd} \theta^* h^* e^R_{\mathcal{F},Y} \delta_{Y*}(\mathcal{G}) \arrow[r] \arrow[d, "\wr"] & \theta^* e^R_{\mathcal{F},X} \delta_{X*} h'^*(\mathcal{G}) \arrow[r] & e^R_{\mathcal{F},W} \delta_{W*} \theta'^* h'^*(\mathcal{G}) \arrow[d, "\wr"] \\ \psi^* \phi^* e^R_{\mathcal{F},Y} \delta_{Y*}(\mathcal{G}) \arrow[r] & \psi^* e^R_{\mathcal{F},V} \delta_{V*} \phi'^*(\mathcal{G}) \arrow[r] & e^R_{\mathcal{F},W} \delta_{W*} \psi'^* \phi'^*(\mathcal{G}), \end{tikzcd} \tag{7.22}$$

where $\delta_W : W' \to W \times Z$ and $\delta_V : V' \to V \times Z$ are the base changes of δ_Y, and where the bottom right map is well-defined by Proposition 5.30. As in the previous paragraph, it suffices by faithful flatness of θ to show the top left map is an isomorphism. The top right and bottom left maps are by Lemma **??**, so it further suffices to show the bottom right is.

Let $\delta'_V : V' \to V \times V'$ and $\delta'_W : W' \to W \times V'$ denote the natural maps, noting that $\delta_V \cong (id_V \times g'\phi') \circ \delta'_V$ and $\delta_W \cong (id_W \times g'\phi') \circ \delta'_W$. Since $V \times V'$ is an algebraic space, $\operatorname{QCoh}(V \times V')$ is compactly generated by perfect sheaves [Lur18, Prop. 9.6.1.1]. But $V \times V'$ is also tamely presented, hence locally coherent. Thus $\mathcal{H}^0 : \operatorname{QCoh}(V \times V') \to \operatorname{QCoh}(V \times V')^\heartsuit$ preserves compactness, hence $\operatorname{QCoh}(V \times V')^\heartsuit$ is compactly generated, hence $V \times V'$ is coherent. Similarly $W \times V'$ is coherent. We then have a diagram

$$\begin{tikzcd} \psi^* e^R_{\mathcal{F},V}\delta_{V*} \arrow[r, "\sim"] \arrow[d] & \psi^* e^R_{\mathcal{F},V}(id_V \times g'\phi')_*\delta'_{V*} \arrow[r, "\sim"] \arrow[d] & \psi^* e^R_{\phi'^* g'^*(\mathcal{F}),V}\delta'_{V*} \arrow[d] \\ e^R_{\mathcal{F},W}\delta_{W*}\psi'^* \arrow[r, "\sim"] & e^R_{\mathcal{F},W}(id_W \times g'\phi')_*\delta'_{W*}\psi'^* \arrow[r, "\sim"] & e^R_{\phi'^* g'^*(\mathcal{F}),W}\delta'_{W*}\psi'^*, \end{tikzcd}$$

where the right square is well-defined by (7.8) and the left by Proposition 5.30.

In particular, the bottom right map in (7.22) is an isomorphism if and only if the map $\psi^* e^R_{\phi'^* g'^*(\mathcal{F}),V}\delta'_{V*}u'^*_\alpha(\mathcal{G}_\alpha) \to e^R_{\phi'^* g'^*(\mathcal{F}),W}\delta'_{W*}\psi'^* u'^*_\alpha(\mathcal{G}_\alpha)$ is an isomorphism (recall that $\phi'^*(\mathcal{G}) \cong u'^*_\alpha(\mathcal{G}_\alpha)$). Varying the above argument with $V \times V'_\alpha$ in place of $V \times Z$, we see that this is the case if and only if the second factor of

$$\psi^* u^*_\alpha e^R_{\mathcal{F}_\alpha,V_\alpha}\delta''_{V_\alpha *}(\mathcal{G}_\alpha) \to \psi^* e^R_{\mathcal{F}_\alpha,V}\delta''_{V*}u'^*_\alpha(\mathcal{G}_\alpha) \to e^R_{\mathcal{F}_\alpha,W}\delta''_{W*}\psi'^* u'^*_\alpha(\mathcal{G}_\alpha)$$

is an isomorphism, where $\delta''_W : W' \to W \times V'_\alpha$, $\delta''_V : V' \to V \times V'_\alpha$ are the natural maps. But the first factor and the composition are isomorphisms by Proposition 7.15, since u_α is flat and $u_\alpha \circ \psi$ is of finite Tor-dimension. □

In generalizing Proposition 7.18 to ind-geometric stacks we will use the following terminology. We say a (not necessarily geometric) morphism $f : X \to Y$ of ind-geometric stacks is ind-flat (resp. of ind-finite Tor-dimension) if X can be written as a filtered colimit $X \cong \operatorname{colim} X_\alpha$ of ind-geometric stacks along ind-closed immersions $i_{\alpha\beta} : X_\alpha \to X_\beta$ such that $f \circ i_\alpha : X_\alpha \to Y$ is flat (resp. of finite Tor-dimension) for all α. For example, the condition that k is an ordinary ring of finite global dimension implies that any ind-geometric stack X is of ind-finite Tor-dimension over $\operatorname{Spec} k$. Similarly, if X and Y are reasonable we say f has inductively semi-universal coherent pullback if X can be written as a filtered colimit $X \cong \operatorname{colim} X_\alpha$ of reasonable ind-geometric stacks along almost ind-finitely presented closed immersions such that $f \circ i_\alpha : X_\alpha \to Y$ has semi-universal coherent pullback for all α.

Proposition 7.23. *Let the following be a diagram of ind-geometric stacks in which both squares are Cartesian, X and Y are coherent, W and Z are reasonable, h and g are of ind-finite cohomological dimension, h has semi-universal coherent pullback, g has inductively semi-universal coherent pullback, and f is ind-proper and almost of ind-finite presentation.*

$$\begin{tikzcd} X' \arrow[r, "h'"] \arrow[d, "f''"'] & Y' \arrow[r, "g'"] \arrow[d, "f'"] & Z \arrow[d, "f"] \\ X \arrow[r, "h"] & Y \arrow[r, "g"] & W \end{tikzcd}$$

Suppose that $X \times Z$ and $Y \times Z$ are coherent and that X, Y, and Z are ind-tamely presented. Then X' and Y' are coherent, and for any $\mathcal{F} \in \operatorname{Coh}(Z)$ and $\mathcal{G} \in \operatorname{IndCoh}(Y')$ the composition

$$h^* e^R_{\mathcal{F},Y}\delta_{Y*}(\mathcal{G}) \to e^R_{\mathcal{F},X}(h \times id_Z)^*\delta_{Y*}(\mathcal{G}) \to e^R_{\mathcal{F},X}\delta_{X*}h'^*(\mathcal{G})$$

is an isomorphism, where $\delta_X : X' \to X \times Z$ *and* $\delta_Y : Y' \to Y \times Z$ *are the natural maps.*

Proof. Suppose first that Z is truncated and geometric, and that g has semi-universal coherent pullback (not just inductively so). If $W \cong \operatorname{colim} W_\alpha$ is a reasonable presentation, then for some α we can refine the given diagram to a diagram

$$\begin{tikzcd} X' \arrow[r, "h'"] \arrow[d, "f''_\alpha"'] & Y' \arrow[r, "g'"] \arrow[d, "f'_\alpha"] & Z \arrow[d, "f_\alpha"] \\ X_\alpha \arrow[r, "h_\alpha"] \arrow[d, "i''_\alpha"'] & Y_\alpha \arrow[r, "g_\alpha"] \arrow[d, "i'_\alpha"] & W_\alpha \arrow[d, "i_\alpha"] \\ X \arrow[r, "h"] & Y \arrow[r, "g"] & W \end{tikzcd}$$

of Cartesian squares in which f_α is proper and almost of finite presentation. Proposition 5.16, our hypotheses on k, and coherence of X, Y, $X \times Z$, and $Y \times Z$ imply that X_α, Y_α, $X_\alpha \times Z$, and $Y_\alpha \times Z$ are truncated and coherent, and that h_α and g_α have coherent pullback. Thus X' and Y' are coherent by Proposition 7.18. Letting $\delta_{X_\alpha} : X' \to X_\alpha \times Z$, $\delta_{Y_\alpha} : Y' \to Y_\alpha \times Z$ denote the natural maps, we have an associated diagram

$$\begin{tikzcd} h^* e^R_{\mathcal{F},Y} \delta_{Y*}(\mathcal{G}) \arrow[rr] \arrow[d, "\wr"] & & e^R_{\mathcal{F},X} \delta_{X*} h'^*(\mathcal{G}) \arrow[d, "\wr"] \\ h^* i'_{\alpha*} e^R_{\mathcal{F},Y_\alpha} \delta_{Y_\alpha *}(\mathcal{G}) \arrow[r] & i''_{\alpha*} h^*_\alpha e^R_{\mathcal{F},Y_\alpha} \delta_{Y_\alpha *}(\mathcal{G}) \arrow[r] & i''_{\alpha*} e^R_{\mathcal{F},X_\alpha} \delta_{X_\alpha *} h'^*(\mathcal{G}). \end{tikzcd}$$

Here the vertical isomorphisms are given by Proposition 7.30 (which does not depend on the current Proposition) and the identities $\delta_X \cong (i''_\alpha \times id_Z) \circ \delta_{X_\alpha}$, $\delta_Y \cong (i'_\alpha \times id_Z) \circ \delta_{Y_\alpha}$. But the bottom maps are isomorphisms by Propositions 5.30 and 7.18, hence so is the top map.

Now suppose only that g has semi-universal coherent pullback, and let $Z \cong \operatorname{colim} Z_\alpha$ be a reasonable presentation. For each α the given diagram extends to a diagram

$$\begin{tikzcd} X_\alpha \arrow[r, "h_\alpha"] \arrow[d, "i''_\alpha"'] & Y_\alpha \arrow[r, "g_\alpha"] \arrow[d, "i'_\alpha"] & Z_\alpha \arrow[d, "i_\alpha"] \\ X' \arrow[r, "h'"] \arrow[d, "f''"'] & Y' \arrow[r, "g'"] \arrow[d, "f'"] & Z \arrow[d, "f"] \\ X \arrow[r, "h"] & Y \arrow[r, "g"] & W \end{tikzcd}$$

of Cartesian squares. We have reasonable presentations $X' \cong \operatorname{colim} X_\alpha$ and $Y' \cong \operatorname{colim} Y_\alpha$ by left exactness of filtered colimits in $\widehat{\mathrm{Stk}}_k$ and since the X_α and Y_α are truncated (as g_α and $g_\alpha \circ h_\alpha$ have coherent pullback). Since $X \times Z$ and $Y \times Z$ are coherent so are $X \times Z_\alpha$ and $Y \times Z_\alpha$ by Proposition **??**. By the previous paragraph X_α and Y_α are then coherent for every α, hence X' and Y' are coherent by Proposition **??**.

Write $\mathcal{F} \cong i_{\alpha *}(\mathcal{F}_\alpha)$ for some α and some $\mathcal{F}_\alpha \in \mathrm{Coh}(Z_\alpha)$. Letting $\delta_{X_\alpha} : X_\alpha \to X \times Z_\alpha$, $\delta_{Y_\alpha} : Y_\alpha \to Y \times Z_\alpha$ denote the natural maps, we have a diagram

$$\begin{tikzcd}
h^* e^R_{\mathcal{F},Y} \delta_{Y*}(\mathcal{G}) \arrow[rr] \arrow[d, "\wr"] & & e^R_{\mathcal{F},X} \delta_{X*} h'^*(\mathcal{G}) \arrow[d, "\wr"] \\
h^* e^R_{\mathcal{F}_\alpha,Y} \delta_{Y_\alpha *} i'^!_\alpha(\mathcal{G}) \arrow[r] & e^R_{\mathcal{F}_\alpha,X} \delta_{X_\alpha *} h^*_\alpha i'^!_\alpha(\mathcal{G}) \arrow[r] & e^R_{\mathcal{F}_\alpha,X} \delta_{X_\alpha *} i''^!_\alpha h'^*(\mathcal{G}).
\end{tikzcd}$$

Here the vertical isomorphisms are given by (7.9) and Proposition **??**. But the bottom maps are isomorphisms by the first paragraph and Proposition 6.6, hence so is the top map.

Finally, suppose that $Y \cong \operatorname{colim} Y_\alpha$ expresses Y as a filtered colimit of reasonable ind-geometric stacks along almost ind-finitely presented ind-closed immerions such that $g \circ i_\alpha : Y_\alpha \to W$ has coherent pullback for all α. For any α we have a diagram

$$\begin{tikzcd}
& X_\alpha \arrow[dl, "j_\alpha"'] \arrow[dd, "h_\alpha" near start] & & X_\alpha \times Z \arrow[ll] \arrow[dl] \arrow[dd] & & X'_\alpha \arrow[ll, "\delta_{X_\alpha}"'] \arrow[dl, "j'_\alpha"] \arrow[dd, "h'_\alpha"] \\
X \arrow[dd, "h"'] & & X \times Z \arrow[ll] \arrow[dd] & & X' \arrow[ll, "\delta_X"' near start] \arrow[dd, "h'" near start] & \\
& Y_\alpha \arrow[dl, "i_\alpha"'] & & Y_\alpha \times Z \arrow[ll] \arrow[dl] & & Y'_\alpha \arrow[ll, "\delta_{Y_\alpha}"' near start] \arrow[dl, "i'_\alpha"] \\
Y & & Y \times Z \arrow[ll] & & Y' \arrow[ll, "\delta_Y"'] &
\end{tikzcd}$$

with Cartesian faces. Proposition 5.11 implies that i_α is an almost ind-finitely presented ind-closed immersion, and Proposition **??** implies that X_α, Y_α, $X_\alpha \times Z$, and $Y_\alpha \times Z$ are coherent and j_α, $(i_\alpha \times id_Z)$, and $(i_\alpha \times id_Z)$ are almost ind-finitely presented ind-closed immersions. It then follows from the previous paragraphs that X'_α and Y'_α are coherent for all α. We have $X' \cong \operatorname{colim} X'_\alpha$ and $Y' \cong \operatorname{colim} Y'_\alpha$ by left exactness of filtered colimits in $\widehat{\mathrm{Stk}}_k$, and since the maps $j'_{\alpha\beta} : X_\alpha \to X_\beta$, $i'_{\alpha\beta} : Y_\alpha \to Y_\beta$ are almost ind-finitely presented ind-closed immersions by Proposition **??**, X' and Y' are then coherent by Proposition **??**.

For every α we now have a diagram

$$\begin{tikzcd}
h^*_\alpha i^!_\alpha e^R_{\mathcal{F},Y} \delta_{Y*}(\mathcal{G}) \arrow[r] \arrow[d, "\wr"] & j^!_\alpha h^* e^R_{\mathcal{F},Y} \delta_{Y*}(\mathcal{G}) \arrow[r] & j^!_\alpha e^R_{\mathcal{F},X} \delta_{X*} h'^*(\mathcal{G}) \arrow[d, "\wr"] \\
h^*_\alpha e^R_{\mathcal{F},Y_\alpha} \delta_{Y_\alpha *} i'^!_\alpha(\mathcal{G}) \arrow[r] & e^R_{\mathcal{F},X_\alpha} \delta_{X_\alpha *} h'^*_\alpha i'^!_\alpha(\mathcal{G}) \arrow[r] & e^R_{\mathcal{F},X_\alpha} \delta_{X_\alpha *} j'^!_\alpha h'^*(\mathcal{G})
\end{tikzcd}$$

in $\mathrm{IndCoh}(X_\alpha)$. Here the vertical isomorphisms are given by (7.9) and Proposition **??**. Since the functors $j^!_\alpha$ determine an isomorphism $\mathrm{IndCoh}(X) \cong \lim \mathrm{IndCoh}(X_\alpha)$ in $\widehat{\mathrm{Cat}}_\infty$, it suffices to show that the top right map is an isomorphism for all α. But the top left and bottom right maps are isomorphisms by Proposition 6.6 and the bottom left is by the previous paragraphs, hence the top right is as well. $\square$

Corollary 7.24. *Let X and Y be coherent, ind-tamely presented ind-geometric stacks such that $X \times Y$ and $Y \times Y$ are coherent. Let $h : X \to Y$ be a morphism of ind-finite finite* *cohomological dimension with semi-universal coherent pullback. Then for any* $\mathcal{F} \in \mathrm{Coh}(Y)$*,* $\mathcal{G} \in \mathrm{IndCoh}(Y)$ *the natural map* $h^* \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(h^*(\mathcal{F}), h^*(\mathcal{G}))$ *is an isomorphism.*

Proof. Follows from Proposition 7.23, taking $Z = W = Y$. □

7.4. **External products and pushforward.** If X, Y, and Z are geometric stacks and $\mathcal{F} \in \mathrm{QCoh}(Z)$, then for any $f : X \to Y$ the isomorphism $(f \times id_Z)^* e_{\mathcal{F},Y} \cong e_{\mathcal{F},X} f^*$ of functors $\mathrm{QCoh}(Y) \to \mathrm{QCoh}(X \times Z)$ yields an isomorphism

$$f_* e^R_{X,\mathcal{F}} \cong e^R_{Y,\mathcal{F}} (f \times id_Z)_* \tag{7.25}$$

of right adjoints $\mathrm{QCoh}(X \times Z) \to \mathrm{QCoh}(Y)$. This is an external counterpart of the isomorphism

$$\mathcal{H}om(\mathcal{F}, f_*(-)) \cong f_* \mathcal{H}om(f^*(\mathcal{F}), -)$$

obtained for $\mathcal{F} \in \mathrm{QCoh}(X)$ by taking right adjoints of the isomorphism $f^*(- \otimes \mathcal{F}) \cong f^*(-) \otimes f^*(\mathcal{F})$. Similarly, if f is proper the projection isomorphism $f_*(\mathcal{F} \otimes f^*(-)) \cong f_*(\mathcal{F}) \otimes -$ yields an isomorphism

$$f_* \mathcal{H}om(\mathcal{F}, f^!(-)) \cong \mathcal{H}om(f_*(\mathcal{F}), -) \tag{7.26}$$

of right adjoints.

This section generalizes these isomorphisms to ind-coherent sheaves under suitable hypotheses, letting X, Y, and Z be ind-geometric and $f : X \to Y$ a morphism of ind-finite cohomological dimension. In this setting $f_* : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$ typically does not have a left adjoint. Instead, if Z is reasonable and $\mathcal{F} \in \mathrm{Coh}(Z)$, we may take the isomorphism $(f \times id_Z)_* e_{\mathcal{F},X} \cong e_{\mathcal{F},Y} f_*$ of functors $\mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y \times Z)$ and consider the associated Beck-Chevalley transformation

$$f_* e^R_{\mathcal{F},X} \to e^R_{\mathcal{F},Y} (f \times id_Z)_* \tag{7.27}$$

of functors $\mathrm{IndCoh}(X \times Z) \to \mathrm{IndCoh}(Y)$.

Suppose in addition that X and Y are reasonable and that f is ind-proper and almost of ind-finite presentation. Then for $\mathcal{F} \in \mathrm{Coh}(Y)$ and $\mathcal{G} \in \mathrm{IndCoh}(Y)$ we have a transformation

$$f_* \mathcal{H}om(\mathcal{F}, f^!(-)) \to \mathcal{H}om(f_*(\mathcal{F}), -) \tag{7.28}$$

of functors $\mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(Y)$ given by the composition

$$f_* e^R_{\mathcal{F},X} \Delta_{X*} f^! \to e^R_{\mathcal{F},Y} (f \times id_X)_* \Delta_{X*} f^! \to e^R_{\mathcal{F},Y} (id_Y \times f)^! \Delta_{Y*} \cong e^R_{f_*(\mathcal{F}),Y} \Delta_{Y*}. \tag{7.29}$$

Here the last isomorphism is given by (7.9), and we have used the fact that $\Delta_X \circ (f \times id_X)$ is the base change of Δ_Y along $id_Y \times f$. In the geometric case one can check that if we restrict to bounded below subcategories, (7.28) and (7.27) are indeed identified with the isomorphisms (7.25) and (7.26) under the equivalences $\mathrm{IndCoh}(-)^+ \cong \mathrm{QCoh}(-)^+$.

Proposition 7.30. *Let X, Y, and Z be ind-geometric stacks such that Z is reasonable and Y is semi-reasonable. Let $f : X \to Y$ be a morphism of ind-finite cohomological dimension and $f' = f \times id_Z$. Then for any $\mathcal{F} \in \mathrm{Coh}(Z)$ and $\mathcal{G} \in \mathrm{IndCoh}(X \times Z)^+$ the Beck-Chevalley map $f_* e^R_{\mathcal{F},X}(\mathcal{G}) \to e^R_{\mathcal{F},Y} f'_*(\mathcal{G})$ is an isomorphism.*

Corollary 7.31. *Let X, Y, and Z be ind-geometric stacks such that Z is reasonable and X, Y, $X \times Z$, and $Y \times Z$ are coherent. Let $f : X \to Y$ be a morphism of ind-finite cohomological dimension and $f' = f \times id_Z$. Then for any $\mathcal{F} \in \mathrm{Coh}(Z)$ and $\mathcal{G} \in \mathrm{IndCoh}(X \times Z)$ the Beck-Chevalley map $f_* e^R_{\mathcal{F},X}(\mathcal{G}) \to e^R_{\mathcal{F},Y} f'_*(\mathcal{G})$ is an isomorphism.*

Proof. Under these hypotheses $f_* e^R_{\mathcal{F},X}(\mathcal{G})$ and $e^R_{\mathcal{F},Y} f'_*$ are continuous and $\mathrm{IndCoh}(X \times Z)$ is compactly generated by $\mathrm{Coh}(X \times Z) \subset \mathrm{IndCoh}(X \times Z)^+$, hence the claim follows from Proposition 7.30. □

Note that the extension of sheaf Hom from the geometric to the ind-geometric setting is uniquely determined by the following corollary, since we can always write $\mathcal{F} \in \mathrm{Coh}(X)$ as $i_*(\mathcal{F}')$ for some reasonable geometric substack $i : X' \to X$ and $\mathcal{F}' \in \mathrm{Coh}(X')$.

Corollary 7.32. *Let X and Y be reasonable ind-geometric stacks, $f : X \to Y$ an ind-proper, almost ind-finitely presented morphism of finite cohomological dimension, and $\mathcal{F} \in \mathrm{Coh}(X)$. Then for any $\mathcal{G} \in \mathrm{IndCoh}(Y)^+$ the natural map $f_* \mathcal{H}om(\mathcal{F}, f^!(\mathcal{G})) \to \mathcal{H}om(f_*(\mathcal{F}), \mathcal{G})$ is an isomorphism.*

Proof. Under these hypotheses $\Delta_{X*} f^!$ is left bounded, hence the claim follows from Propositions **??** and 7.30. □

Corollary 7.33. *Let X and Y be coherent ind-geometric stacks such that $X \times X$, $X \times Y$, and $Y \times Y$ are coherent, $f : X \to Y$ an ind-proper, almost of ind-finitely presented morphism, and $\mathcal{F} \in \mathrm{Coh}(X)$. Then for any $\mathcal{G} \in \mathrm{IndCoh}(Y)$ the natural map $f_* \mathcal{H}om(\mathcal{F}, f^!(\mathcal{G})) \to \mathcal{H}om(f_*(\mathcal{F}), \mathcal{G})$ is an isomorphism.*

Proof. Follows from Proposition **??** and Corollary 7.31. □

References

[BFN18] A. Braverman, M. Finkelberg, and H. Nakajima, *Towards a mathematical definition of 3-dimensional $\mathcal{N} = 4$ gauge theories, II*, Adv. Theor. Math. Phys. **22** (2018), no. 5, 1071–1147. 2

[BKV22] A. Bouthier, D. Kazhdan, and Y. Varshavsky, *Perverse sheaves on infinite-dimensional stacks, and affine Springer theory*, Adv. Math. **408** (2022), Paper No. 108572, 132. 3

[Bou72] N. Bourbaki, *Commutative algebra. Chapters 1–7*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1972, Translated from the French. 2, 3, 9

[CW23a] S. Cautis and H. Williams, *Canonical bases for coulomb branches of 4d $\mathcal{N} = 2$ gauge theories.* 2

[CW23b] ———, *Ind-geometric stacks.* 4, 21, 22, 23, 37

[Gai] D. Gaitsgory, *Notes on geometric langlands: generalities on dg categories*, https://people.math.harvard.edu/ gaitsgde/GL/textDG.pdf. 5

[Gai13] Dennis Gaitsgory, *Ind-coherent sheaves*, Mosc. Math. J. **13** (2013), no. 3, 399–528, 553. 10, 12, 31

[Gla89] S. Glaz, *Commutative coherent rings*, Lecture Notes in Mathematics, vol. 1371, Springer-Verlag, Berlin, 1989. 25

[GR14] D. Gaitsgory and N. Rozenblyum, *DG indschemes*, Perspectives in representation theory, Contemp. Math., vol. 610, Amer. Math. Soc., Providence, RI, 2014, pp. 139–251. 4, 21

[GR17] ———, *A study in derived algebraic geometry. Vol. I. Correspondences and duality*, Mathematical Surveys and Monographs, vol. 221, American Mathematical Society, Providence, RI, 2017. 14, 28, 29

[HLP14] D. Halpern-Leistner and A. Preygel, *Mapping stacks and categorical notions of properness*, arXiv:1402.3204 (2014). 20

[Ill71] L. Illusie, *Généralités sur les conditions de finitude dans les catégories dérivées*, Théorie des intersections et théoréme de Riemann-Roch (SGA 6), Springer-Verlag, New York, 1971, pp. 78–273. 10

[KV04] M. Kapranov and E. Vasserot, *Vertex algebras and the formal loop space*, Publ. Math. Inst. Hautes Études Sci. (2004), no. 100, 209–269. 3, 6, 19

[Lur09] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. 5, 6, 7, 25, 26, 27, 28, 31, 33

[Lur17] ———, *Higher algebra*, 2017. 6, 7, 11, 27, 31, 35, 36, 41

[Lur18] ———, *Spectral algebraic geometry*, 2018. 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 22, 31, 32, 33, 34, 41, 42, 43

[Ras14] S. Raskin, *Chiral principal series categories*, ProQuest LLC, Ann Arbor, MI, 2014. 3, 6

[Ras19] ———, *Homological methods in semi-infinite contexts*. 3, 31, 34

[Sta] *The stacks project*, `http://stacks.math.columbia.edu`. 20

[Swa19] Richard G. Swan, *K-theory of coherent rings*, J. Algebra Appl. **18** (2019), no. 9, 1950161, 16. 10, 11

[TT90] R. Thomason and T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 247–435. 12, 19

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