ON THE COMBINATORICS OF EXACT LAGRANGIAN SURFACES

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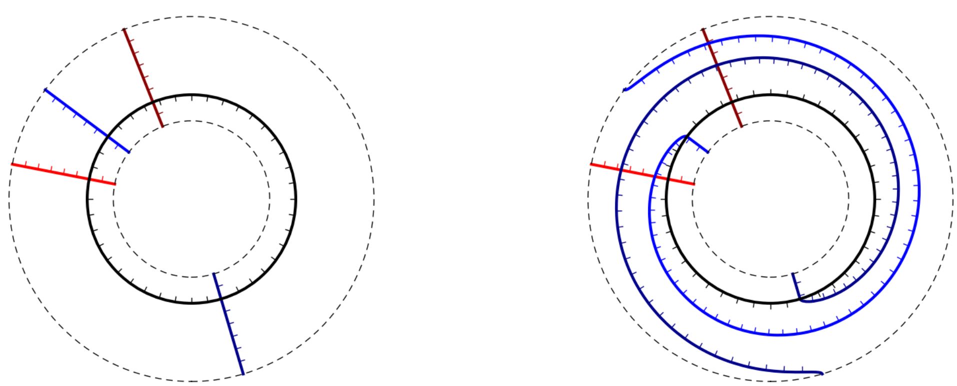
ABSTRACT. We study Lagrangian skeleta given by attaching disks to a surface along a collection of simple closed curves. In terms of these curves, we describe surgeries that change the skeleton but preserve the corresponding Weinstein 4-manifold. The surgeries can be iterated to produce more such skeleta — in many cases, infinitely many more.

The space of rank one local systems on the surface determines an algebraic torus in the moduli of A-branes in the Weinstein manifold. We show that the skeletal surgeries induce cluster transformations on these tori — and noncommutative cluster transformations on the corresponding moduli arising from higher rank local systems.

In particular, the problem of producing and distinguishing such Lagrangians maps to a combination of combinatorial-geometric questions about curve configurations on surfaces and algebraic questions about exchange graphs of cluster algebras.



1. Introduction

The zero section of a cotangent bundle is the prototypical example of an exact Lagrangian embedded in an exact symplectic manifold. Producing and distinguishing such Lagrangians is a basic problem in symplectic geometry. One benchmark is the standing conjecture of Arnol'd that the zero section gives the only compact exact Lagrangian in a cotangent bundle, up to Hamiltonian isotopy. Together with the Weinstein neighborhood theorem, this Arnol'd conjecture suggests that exact Lagrangians have a discrete nature: up to Hamiltonian isotopy, they should have no moduli.

Our purpose here is to introduce a mechanism for producing and distinguishing large collections of exact Lagrangian surfaces. We assume we are given one exact Lagrangian surface, $\mathcal{L}$, to which a collection of Lagrangian disks are attached along smooth circles, forming a singular Lagrangian $\mathbb{L}$. We work in a Weinstein manifold W retracting to this skeleton $\mathbb{L}$. After collapsing one of the attached disks so that $\mathcal{L}$ acquires a singularity, there are two choices of Lagrangian surgery [LS, Pol] — one returning the original surface $\mathcal{L}$, and one yielding a new exact Lagrangian surface $\mathcal{L}'$ which is smoothly, but not Hamiltonianly, isotopic to the original surface. The transition $\mathcal{L} \rightsquigarrow \mathcal{L}'$ is the Lagrangian disk surgery of M.-L. Yau [Yau].

The basic geometric contribution of this paper is to explain how the entire skeleton may be carried through this transition, giving a new skeleton $\mathbb{L}'$ extending the new Lagrangian surface. This skeletal surgery $\mathbb{L} \rightsquigarrow \mathbb{L}'$ has a combinatorial description in terms of operations on configurations of curves in $\mathcal{L}$ — the projections of the attaching Legendrians used to build the skeleta. For constructing exact Lagrangians, the point is that this procedure can now be iterated: we can use the disks in $\mathbb{L}'$ to perform surgeries on $\mathcal{L}'$, and get more exact Lagrangians.

This branching production of Lagrangians by a local surgery procedure — potential sequences of surgeries are indexed by an n-ary tree, if n disks were attached — maps to the combinatorial notion of quiver mutation. Indeed, $\mathbb{L}$ defines a quiver: its vertices index the curves along which disks are attached, and its arrows record the intersection numbers of their projections to $\mathcal{L}$. This quiver undergoes a mutation when we perform a skeletal surgery.

Associated to a quiver and its mutations is a cluster algebra [FZ] — the coordinate ring of a space built from open algebraic tori (cluster charts) labeling the vertices of the n-ary tree, glued along certain birational maps (cluster transformations). These arise in various contexts such as canonical bases and total positivity in Lie theory [Fom, GLS] and character varieties of punctured surfaces [FG, GSV2].

In our setting this structure appears as follows: $\mathbb{L}$ carries a sheaf of categories μloc; we will be interested in the global sections $\mu loc(\mathbb{L})$, which we term microlocal sheaves on $\mathbb{L}$. We will show that skeletal surgery $\mathbb{L} \rightsquigarrow \mathbb{L}'$ induces an equivalence $\mu loc(\mathbb{L}) \cong \mu loc(\mathbb{L}')$. On the other hand, there is a natural inclusion of the category of local systems on the original surface into the category of microlocal sheaves on the skeleton. Of particular interest are the rank one local systems, the category of which we write as $Loc_1(\mathcal{L})$; note these are parameterized by an algebraic torus. Thus the skeletal surgery induces a comparison of algebraic tori:

$$Loc_1(\mathcal{L}) \subset \mu loc(\mathbb{L}) \cong \mu loc(\mathbb{L}') \supset Loc_1(\mathcal{L}'). \tag{1}$$

We show this to be the cluster $\mathcal{X}$-transformation associated to the quiver mutation described above. More generally, the corresponding comparison on higher rank local systems is given by a non-abelian version of a cluster transformation.

In particular, it follows that the images of $Loc_1(\mathcal{L})$ and $Loc_1(\mathcal{L}')$ in $\mu loc(\mathbb{L})$ are different.

This fact holds geometric significance. Indeed, according to a conjecture of Kontsevich [Kon2], perfect modules over the wrapped Fukaya category of W are the global sections of a certain sheaf of categories over $\mathbb{L}$. This is known in the case of cotangent bundles [NZ, N1, FSS]. More generally, the expected sheaf can be described explicitly in terms of the microlocalization theory of Kashiwara and Schapira [KS]; it is our sheaf μloc. The inclusion $loc(\mathcal{L}) \to \mu loc(\mathbb{L})$ corresponds to the pullback of perfect module categories along the Viterbo restriction functor [AS] for the inclusion $T^*\mathcal{L} \subset W$. For disk surgery on a torus, Equation 1 corresponds to the wall-crossing transformation computed in [Aur], and expressed explicitly as a cluster transformation in [Sei4, Prop. 11.8].

Accepting this conjectural package, it follows that $\mathcal{L}$ and $\mathcal{L}'$ cannot be Hamiltonian isotopic: otherwise the images of $Loc_1(\mathcal{L})$ and $Loc_1(\mathcal{L}')$ in the Fukaya category would necessarily coincide. Moreover, since we have established that skeletal surgeries induce cluster transformations, we can employ cluster algebra to compute — and, in particular, distinguish — the algebraic tori $Loc_1(\mathcal{L})$ and $Loc_1(\mathcal{L}'')$, even when $\mathcal{L}$ and $\mathcal{L}''$ are related by a longer sequence of surgeries. The (conjectural) Hamiltonian isotopy invariance of the cluster chart associated to a Lagrangian implies that solving the algebraic/combinatorial problem of distinguishing cluster charts in fact solves the symplecto-geometric problem of distinguishing the Lagrangians. A general cluster variety has infinitely many distinct cluster charts [FZ2], so the cluster chart associated to a Lagrangian is a strong enough invariant to distinguish infinitely many Lagrangians.

However, for a given $\mathbb{L}$ it may not be possible to lift an arbitrary sequence of quiver mutations to a corresponding sequence of skeletal surgeries. This is because of the following subtlety: while the surgery $\mathbb{L} \rightsquigarrow \mathbb{L}'$ always results in a skeleton which can be built from a surface by attaching handles along Legendrian lifts of curves, the surgery can create self-intersections in these curves. Our surgery does not apply to disks attached along curves with self-intersections, so this results in an obstruction to subsequent mutations. We will show that when L is a torus and the curve collection is geodesic, this issue never arises and arbitrary mutations can be performed; this is related to the constructions of [Sym, Via2].

In another direction, the present construction extends the reach of the dictionary established in [STWZ]. There, we gave a symplectic interpretation of the relation between cluster algebras and bicolored graphs on surfaces [Pos, FG, GK]. This went as follows: a surface Σ and a Legendrian knot Λ in the contact boundary of $T^*\Sigma$ together determine a certain moduli space; an exact

Lagrangian filling of Λ determines a toric chart; and a bicolored graph Γ on Σ determines a Legendrian knot Λ together with a canonical filling $\mathcal{L}$. It is well known that, for the resulting cluster structures, not all cluster charts can be realized by bicolored graphs. In particular, the vertices of the quiver associated to a bicolored graph are named by the faces of the graph; one can perform an abstract quiver mutation at any of these, but the new cluster comes from another bicolored graph only when the face is a square.

In [STWZ], we raised the possibility that the remaining charts come from Lagrangian fillings which do not arise from bicolored graphs. The present technology allows us to construct such fillings. The first step is to change perspective from that of the surface Σ containing the bicolored graph, to that of the associated Lagrangian filling $\mathcal{L}$. We recover $T^*\Sigma$ as the result of attaching handles to $T^*\mathcal{L}$; the associated Lagrangian disks are exactly the faces of the bicolored graph. The filling $\mathcal{L}$ has boundary Λ, but our constructions still make sense in this context. When $\mathcal{L}$ has boundary, $\mu loc(\mathbb{L})$ should correspond to the "partially wrapped" Fukaya category where $\partial\mathcal{L} \subset \partial W$ serves as a Legendrian stopper.

In particular, in the framework of the present paper we can perform a skeletal surgery on any face, square or not, giving a symplecto-geometric description of the chart resulting from the corresponding mutation. Though beyond the scope of bicolored graphs, the resulting theory is still combinatorially explicit in the sense of being completely encoded in configurations of curves on $\mathcal{L}$. Moreover, by allowing Weinstein manifolds more general than cotangent bundles, the present framework captures cluster structures (i.e., equivalence classes of quivers) more general than those realized by bicolored graphs.

1.1. **Main results.** We outline more formally the main definitions and results of the article. Our basic data-set is a configuration of curves on a surface:

Definition 1.1. Let $\mathcal{L}$ be a topological surface. A *curve configuration* on $\mathcal{L}$ will mean a set of properly immersed, co-oriented, pairwise transverse curves on $\mathcal{L}$. Here a co-orientation of a curve is a choice of one of the two orientations of its conormal bundle. If $\mathcal{L}$ has boundary, then we allow the curves to end on the boundary of $\mathcal{L}$.

This data encodes a Lagrangian skeleton of a Weinstein 4-manifold. The underlying topological space of the skeleton is the following:

Definition 1.2. Let $\mathcal{C}$ be a curve configuration on $\mathcal{L}$. We write $\mathbb{L}$ for the topological space formed by gluing one disk to $\mathcal{L}$ along each curve in $\mathcal{C}$. In the case that $\mathcal{L}$ has boundary along which a curve C ends, we glue a half-disk to that C.

The 4-manifold is formed by attaching Weinstein handles [Wei] – note that a co-orientation of an immersed curve is the same data as a lift to a Legendrian in the contact boundary $T^\infty\mathcal{L}$ of $T^*\mathcal{L}$.

Definition 1.3. For a curve configuration $\mathcal{C}$ on a surface $\mathcal{L}$, we write $W := W_{\mathcal{C}}$ for the Weinstein 4-manifold formed by attaching Weinstein handles to $T^*\mathcal{L}$ along the Legendrian lifts of closed curves in $\mathcal{C}$ to their co-orientations.

We write $T^+_{\mathcal{C}}\mathcal{L} \subset T^*\mathcal{L}$ for the zero section together with the cones over the Legendrian lifts of the C_i, and realize $\mathbb{L}$ inside W as the union of $T^+_{\mathcal{C}}\mathcal{L}$ with the cores of the handles. It is a Lagrangian skeleton of W in the sense that it is the complement in W of the locus escaping to infinity under a natural Liouville flow; in particular, it is a retract of W. In case $\mathcal{L}$ has boundary, then $\partial\mathbb{L}$ has boundary coming from the union of the boundary of $\mathcal{L}$ with the boundaries of the half-disks attached along any curves which end along $\partial\mathcal{L}$. This is naturally viewed as a singular Legendrian in ∂W. See [N2] for more discussion of skeleta in this context.

We refer to $\mathbb{L}$ as a seed skeleton, as it is a singular Lagrangian incarnation of a seed of a cluster algebra. We follow Fock and Goncharov, for whom this denotes a collection of elements in a lattice equipped with a skew-symmetric form [FG]. Often one encodes this data as a quiver without oriented 2-cycles, whose vertices are the given lattice elements and whose arrows record the pairings between them.

The configuration $\mathcal{C}$ naturally gives rise to a seed: the lattice is $H_1(\mathcal{L};\mathbb{Z})$ with its intersection pairing, and the elements are the classes of the C_i. Note that this seed and the Weinstein manifold W depend only on the Legendrian isotopy classes of the lifts of the C_i. With this in mind by an isotopy of a curve configuration we mean an isotopy of its constituent curves that arises from a Legendrian isotopy of their lifts. Concretely this means whenever two curves become tangent during the isotopy, their co-orientations at any point of tangency are opposite.

In cluster algebra, there is a fundamental operation on seeds called *mutation*, determined by the choice of one of the lattice elements determining the seed, or equivalently, a vertex of the quiver. We lift this to an operation of mutation at any simple closed curve C_k in $\mathcal{C}$ (Theorem 2.12). The result is a new curve configuration $\mu_k(\mathcal{C}) = \{C'_i\}$ obtained from $\mathcal{C}$ by twisting the other curves around C_k according to the orientations of their intersections with C_k; see Definition 2.3 and Figure 1. We write $\mathcal{C}'$ for $\mu_k(\mathcal{C})$ when k is understood, and for clarity denote the surface on which the new configuration sits as $\mathcal{L}'$. Writing $\mathbb{L}' := \mu_k(\mathbb{L})$, and $W' := \mu_k(W)$ for the seed skeleton and Weinstein manifold associated to $\mathcal{C}'$ as above, we show:

Theorem 1.4. *(cf. Theorem 2.22) There is a symplectomorphism $W \cong W'$ such that the preimage of $\mathcal{L}' \subset W'$ is related to $\mathcal{L}$ by Lagrangian disk surgery along the disk D_k attached to C_k.*

In Section 3.2 we define a sheaf μloc of dg categories on $\mathbb{L}$. It is glued together from sheaves of categories on conical models of local pieces of $\mathbb{L}$; these local sheaves of categories are themselves microlocalizations of constructible sheaf categories as in [KS, Chap. 6], [N2, N3, N4], [Gui]. Following the discussion above we expect that the global section category $\mu loc(\mathbb{L})$ captures some appropriate version of the Fukaya category of W. Given such a result, we could deduce from Theorem 2.22 that $\mu loc(\mathbb{L}) \cong Fuk(W) \cong Fuk(W') \cong \mu loc(\mathbb{L}')$.

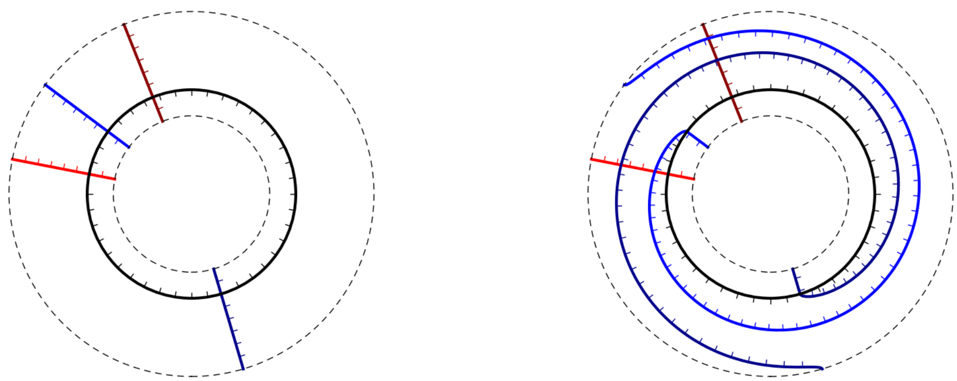


FIGURE 1. Local pictures before (left) and after (right) mutation at an embedded curve (here, the black circular curve). We represent the co-orientations of the curves as hairs pointing to one side. By convention we say, for example, that on the left the blue curve intersects the black circle positively. Mutation twists the curves around the black circle wherever they intersect it positively, and leaves them alone wherever they intersect it negatively.

Instead, we construct such a composition directly using sheaf-theoretic results of [GKS]. We refer to the resulting equivalence as a mutation functor.

Theorem 1.5. *(cf. Theorem 4.6) If C_k is a simple closed curve in the curve collection $\mathcal{C}$, there is an equivalence $Mut_k : \mu loc(\mathbb{L}) \cong \mu loc(\mathbb{L}')$.*

The equivalence is induced by a local construction in a neighborhood of the surgery, which does not depend on the remaining geometry of the skeleton. Implicit in the above statement, and explicit in the proof of the theorem, is the freedom to pass between different conical models used to locally describe μloc.

Let $loc(\mathcal{L})$ be the category of local systems on $\mathcal{L}$.

Lemma 1.6. *(cf. Proposition 3.15) There is a fully faithful inclusion $loc(\mathcal{L}) \hookrightarrow \mu loc(\mathbb{L})$ whose essential image is the full subcategory of microlocal sheaves supported on $\mathcal{L} \subset \mathbb{L}$..*

The relation between $\mu loc(\mathbb{L})$ and cluster algebra comes from the comparison

$$loc(\mathcal{L}) \subset \mu loc(\mathbb{L}) \cong \mu loc(\mathbb{L}') \supset loc(\mathcal{L}')$$

Let $Loc_1(\mathcal{L}) \subset loc(\mathcal{L})$ denote the full subcategory of local systems whose stalks are free of rank one and concentrated in cohomological degree zero, similarly for $Loc_1(\mathcal{L}')$. The objects of $Loc_1(\mathcal{L})$ are determined by their holonomies, hence are parametrized by an algebraic torus.

Since $\mathcal{L}'$ comes with a homeomorphism to $\mathcal{L}$, it makes sense to ask how the holonomies of a rank one local system transform under surgery.

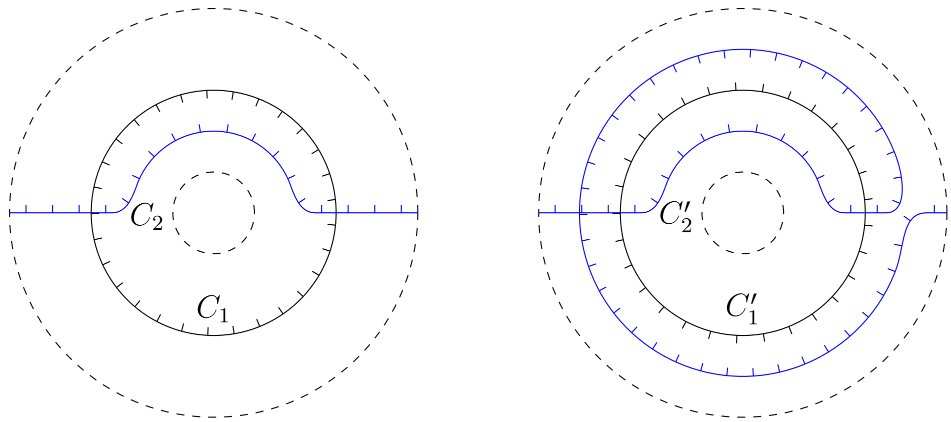


FIGURE 2. On the left are two curves with intersections of opposite signs. When we mutate at C_2, the curve C_1 is twisted into the self-intersecting curve C_1'.

Theorem 1.7. *(cf. Theorem 4.15) The image of an object of $Loc_1(\mathcal{L})$ under Mut_k is an object of $Loc_1(\mathcal{L}')$ if and only if their holonomies differ by the (signed) cluster $\mathcal{X}$-transformation at $[C_k]$.*

We reduce the general proof of Theorem 1.7 to a special case proved in [BK] (see the discussion below). The signs indicated depend on a choice made in defining μloc, which we have omitted from the notation. The choice is classified by an element of $H^2(\mathbb{L}, \mathbb{Z}/2\mathbb{Z})$ which vanishes upon restriction to $\mathcal{L}$. Depending on the choice made one obtains a mix of the usual positive formulae or similar ones with minus signs. Typically in the cluster literature one assumes the lattice elements in a seed are linearly independent; in this case the sign choice is cosmetic. We expect that this choice can be identified with the corresponding one made in defining $Fuk(W)$ (for which see e.g. [Sei5, Sec. 12] or [FOOO]).

We let $\mathcal{M}_n(\mathbb{L})$ denote the closure in the moduli space of $\mu loc(\mathbb{L})$ of the locus of objects whose stalks on $\mathcal{L}$ have cohomology of rank n concentrated in degree zero. The precise meaning of moduli space here is clarified in Section 3.3. Translated to a statement about spaces, Theorem 1.7 becomes:

Theorem 1.8. *(cf. Theorem 5.7) The rank one moduli space $\mathcal{M}_1(\mathbb{L})$ has a partial cluster $\mathcal{X}$-structure with initial seed $(H_1(\mathcal{L};\mathbb{Z}), \{[C_i]\})$.*

For $n > 1$, the moduli space $\mathcal{M}_n(\mathbb{L})$ carries likewise a "nonabelian cluster structure" whose charts are spaces of rank n local systems on $\mathcal{L}$. Their transition functions are determined by the same computation as in Theorem 1.7, see Section 5 for discussion.

When $\mathcal{C}$ consists of pairwise nonintersecting simple closed curves, the seed skeleton $\mathbb{L}$ is the arborealization of a nodal surface [N3, N4]. The homotopy category of $\mu loc(\mathbb{L})$ contains the category of microlocal perverse sheaves associated to the nodal surface by Bezrukavnikov and Kapranov [BK]. They used the conical model of the nodal singularity given by the union of $\mathbb{R}^2$ and the cotangent fiber over the origin. Their moduli coincide with our $\mathcal{M}_1(\mathbb{L})$ and its higher-rank, framed versions, recovering the multiplicative quiver varieties of [CBS, Yam]. The avatar of disk surgery from this perspective is the action of the Fourier-Sato transform on perverse sheaves. That its action on local systems is a cluster transformation is computed in [BK].

The notion of cluster structure appearing in Theorem 1.8 is more general than that usually encountered in the literature. We have seen that choices made in defining μloc can lead to the appearance of certain signs. But even making choices to avoid such signs, the precise notion of cluster $\mathcal{X}$-structure introduced in [FG2] applies only when the classes of the C_i are a basis for $H_1(\mathcal{L}, \mathbb{Z})$. In general, each toric chart on $\mathcal{M}_1(\mathbb{L})$ has a map to a corresponding chart on the usual $\mathcal{X}$-variety, dual to the natural homomorphism of character lattices (see Section 5 for discussion).

A more fundamental issue is that while a cluster structure on a space consists of a system of toric charts related by all possible sequences of mutations, in the present context we are forced to also consider partial cluster structures; that is, structures involving only the tori obtained from a subset of possible mutation sequences. This is because we only have a sensible notion of mutation at a simple closed curve, while a mutation at one curve may create a self-intersection in another. This occurs exactly when the two curves have intersections of opposite signs as in Figure 2. Since this condition is *not* invariant under (Legendrian) isotopies of the curve configuration, it is possible that by choosing suitable isotopies between mutations one can realize arbitrary sequences of surgeries:

Definition 1.9. A configuration $\mathcal{C}$ of embedded co-oriented curves is nondegenerate if it admits arbitrary sequences of mutations. That is, for every list $(i_1, \ldots, i_n)$ of curve indices, there is a sequence $(\iota_1, \ldots, \iota_{n-1})$, where ι_k is an isotopy from $\mu_{i_k} \circ \iota_{k-1} \circ \mu_{i_{k-1}} \cdots \iota_1 \circ \mu_{i_1}(\mathcal{C})$ to a configuration such that $\mu_{i_{k+1}} \circ \iota_k \circ \mu_{i_k} \cdots \mu_{i_1}$ consists entirely of simple closed curves.

It is easy to find curve configurations which are clearly degenerate, and in general we do not know how to determine whether a given configuration is nondegenerate. However, we have one important case when the existence of a complete cluster structure is guaranteed:

Theorem 1.10. *(cf. Theorem 2.15) If the surface $\mathcal{L}$ is a torus and all curves in $\mathcal{C}$ are geodesics in a flat metric, then $\mathcal{C}$ is nondegenerate. In particular, $\mathcal{M}_1(\mathbb{L})$ carries a complete cluster $\mathcal{X}$-structure.*

Such examples arise from almost toric fibrations, and include the complements of anti-canonical divisors in complex toric surfaces; we review their theory in Section 6.

1.2. **Degeneracy and potentials.** The theory of curve configurations described here appears to be related to the theory of quivers with potential [DWZ]. A potential on a quiver is the extra data of a formal sum of oriented cycles. It encodes an algebra, the Jacobian algebra, which is the quotient of the path algebra by the cyclic derivates of the potential. One is generally only interested in potentials up to automorphisms of the path algebra, as the isomorphism class of the Jacobian algebra is invariant under such transformations.

One can mutate a quiver with potential at a vertex not meeting oriented 2-cycles, but this may create 2-cycles elsewhere in the quiver. The potential dictates when to erase these new 2-cycles and mutate further: exactly when it can be done without changing the Jacobian algebra.

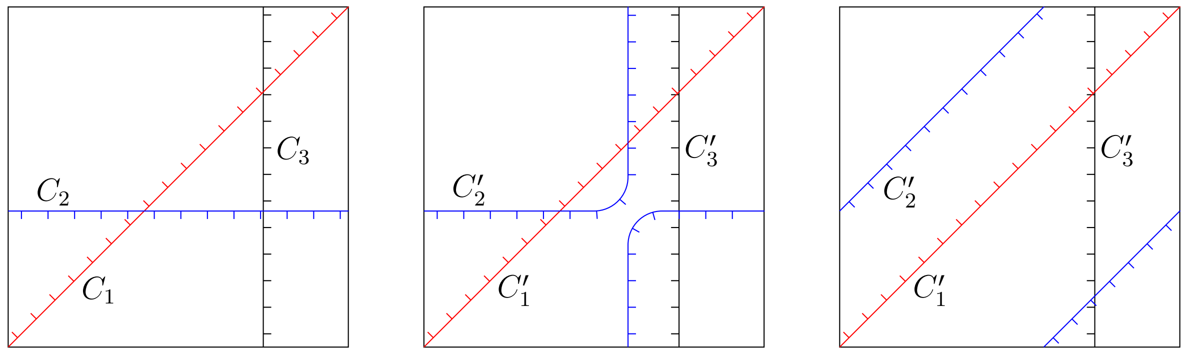


FIGURE 3. The left frame shows a configuration of three geodesics on T^2. The middle frame shows the result of mutating at C_3. This twists C_2 into a new curve C_2', which has two intersections with $C_1' = C_1$ of opposite orientations; mutating at C_2' would thus create a self-intersection in C_1'. But there is an isotopy of the Legendrian lifts which pulls C_2' straight, allowing mutation at C_2' without obstructing further mutation.

In our case, to a configuration $\mathcal{C}$ of simple closed curves we associate a quiver $Q_{\mathcal{C}}$. Its vertices are the curves in $\mathcal{C}$ and its arrows are their intersections. The orientations of the arrows correspond to the orientations of the intersections; if we erase all 2-cycles we get the 2-acyclic quiver obtained from the the seed associated to $\mathcal{C}$.

The condition that mutation at C_k does not produce a self-intersection in C_j is precisely the condition that C_k and C_j are not the vertices of an oriented 2-cycle in $Q_{\mathcal{C}}$. The notion of equivalence we have defined allows us to erase all such 2-cycles exactly when there is an isotopy of $\mathcal{C}$ that cancels out intersections of C_k and C_j until those remaining are of the same orientation. Thus the data of the curve configuration is formally analogous to that of a potential on $Q_{\mathcal{C}}$, insofar as it controls the erasing of 2-cycles needed to iterate the mutation process.

With this in mind, we have borrowed the terminology of nondegenerate configurations from the analogous notion for quivers with potential. We expect this relation can be made precise:

Problem 1.11. For any configuration $\mathcal{C}$ construct a potential on $Q_{\mathcal{C}}$ combinatorially — e.g. by counting some polygons in $\mathcal{L}$ with edges on the curves and corners at their intersections. Show that Legendrian isotopies of curve configurations induce equivalences of potentials, and that mutations of curve configurations induce mutations of potentials (up to equivalence).

Following [Smi], such a potential should have a geometric origin. To the 4-manifold W one should associate a Calabi-Yau 6-manifold fibered over W, with the disks in $\mathbb{L}$ being the images of an associated a collection of Lagrangian 3-spheres. The quiver $Q_{\mathcal{C}}$ then records the intersections of these 3-spheres, and the potential should record Floer-theoretic relations among them:

Problem 1.12. For any configuration $\mathcal{C}$ construct a potential on $Q_{\mathcal{C}}$ geometrically, using the Fukaya category of a Calabi-Yau 6-manifold which fibers over W.

A solution to either of the above problems should help address the more general question of how curve configurations can fail to be nondegenerate:

Problem 1.13. Given a curve configuration $\mathcal{C}$, compute which sequences of mutations can be performed on it without creating self-intersections. Generalize Theorem 1.10 by describing other explicit classes on nondegenerate configurations. For example, show that $\mathcal{C}$ is nondegenerate when $Q_{\mathcal{C}}$ has no directed cycles.

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Conventions and notation. Throughout $\Bbbk$ will denote a field of characteristic zero. Most categorical constructions we perform will take place in the ∞-category of cocomplete $\Bbbk$-linear dg categories and colimit-preserving functors, which we denote by $\mathrm{DGCat}_{\mathrm{cont}}$ as in [GR, Sec. I.1.10]. In particular our main object of study in Sections 3 and 4 — the sheaf μloc of microlocal sheaf categories on the skeleton $\mathbb{L}$ — will take values in $\mathrm{DGCat}_{\mathrm{cont}}$ and satisfy a sheaf property with respect to limits taken therein. To this end we recall that the natural functor from $\mathrm{DGCat}_{\mathrm{cont}}$ to $\mathrm{DGCat}^{\mathrm{non-cocompl}}$, the ∞-category of not necessarily cocomplete dg categories and not necessarily colimit-preserving functors, preserves limits (cf. [GR, Sec. I.1.5.3.3, I.1.8.1.3]). Moreover, the latter is the ∞-category obtained by inverting the weak equivalences in the (Dwyer-Kan) model category of strict dg categories [Hau, Ex. 5.11].

Given a quiver Q, we write $\Bbbk Q$-mod for the unbounded dg category of Q-representations. Given a smooth manifold M, we let $sh(M)$ denote the unbounded dg category of sheaves of $\Bbbk_M$-modules on M which are weakly constructible with respect to a Whitney stratification (i.e. sheaves which satisfy the requisite singular support condition but have stalks of arbitrary size); we simply refer to an object of $sh(M)$ as a sheaf. We write $loc(M) \subset sh(M)$ for the full subcategory of dg local systems (i.e. objects whose cohomology sheaves are local systems in the classical sense). We write $Loc_n(M) \subset loc(M)$ for the full subcategory of objects whose cohomology is concentrated in degree zero and whose stalks are of rank n.

Given a conical Lagrangian $L \subset T^*M$ we denote by $sh_L(M) \subset sh(M)$ the full subcategory of sheaves whose singular support is contained in L, see [KS]. This category is denoted by $sh_L^{\diamond}(M)$ in [N5], where its compact objects were termed wrapped constructible sheaves; the full subcategory of such we denote by $sh_L^w(M)$. There is a full (and generally proper) subcategory $sh_L^{sm}(M) \subset sh_L^w(M)$ of (ordinary, or small) constructible sheaves, i.e. objects whose stalks are perfect $\Bbbk$-modules.

2. Geometry of Mutation

This section contains the detailed description of our skeletal surgery. We proceed on two levels: first in terms of a combinatorial operation on curve configurations, then in terms of a geometric operation on Lagrangian skeleta. We then show the ambient symplectic manifold W is preserved by skeletal surgery.

2.1. Mutation of curve configurations.

Fix a surface $\mathcal{L}$ and a collection $\mathcal{C}$ of co-oriented, immersed curves, as in Definition 1.3. Choose a simple closed curve C_k among them. We give here a combinatorial description of a new collection $\mathcal{C}'$ of curves on $\mathcal{L}$.

Convention 2.1. If α and β are co-oriented curves on a surface Σ that meet transversely at $x \in \Sigma$, the ordered pair $(T^+_\alpha\Sigma, T^+_\beta\Sigma)$ of rays in $T^*_x\Sigma$ determines an orientation of $T^*_x\Sigma$. If Σ is oriented, we write $\langle\alpha,\beta\rangle_+$ and $\langle\alpha,\beta\rangle_-$ for the number of intersections that agree, resp. disagree with the orientation.

Remark 2.2. Note that $\langle\beta,\alpha\rangle_+ = \langle\alpha,\beta\rangle_-$, and that the algebraic intersection number $\langle\alpha,\beta\rangle$ is equal to $\langle\alpha,\beta\rangle_+ - \langle\alpha,\beta\rangle_-$.

We write $\mathcal{L}_{(k)}$ for a neighborhood of C_k in $\mathcal{L}$, and $\mathcal{C}_{(k)}$ for the intersection of the curve configuration with this cylinder. More precisely, $\mathcal{C}_{(k)}$ consists of one closed curve C_k in the center of an annulus, and a number of pairwise noncrossing curves B_i running from one end of the annulus to the other.

We may assume each B_i intersects C_k only once.

Warning. Multiple B_i can come from the same curve C_j in $\mathcal{C}$: if C_j intersects C_k n times, then $C_j \cap \mathcal{L}_{(k)}$ will have n components.

Definition 2.3. Let $tw : \mathcal{L}_{(k)} \to \mathcal{L}_{(k)}$ be a positive Dehn twist in a very small collar neighborhood of a translation of C_k a short distance in the direction opposite its co-orientation. Let $\mathcal{C}_+ := \{B_i \,|\, \langle B_i, C_k\rangle = 1\}$ be the subcollection of curves which intersect C_k positively, and let $\mathcal{C}_- := \{B_i \,|\, \langle B_i, C_k\rangle = -1\}$ be the subcollection which intersect it negatively. Let C'_k be obtained from C_k by reversing the co-orientation.

We define the mutated curve collection $\mathcal{C}'_{(k)}$ on $\mathcal{L}_{(k)}$ by:

$$\mathcal{C}'_{(k)} := tw(\mathcal{C}_+) \cup C'_k \cup \mathcal{C}_-$$

The definition is best understood by staring at Figure 4, which should be interpreted according to the following convention.

Convention 2.4. (Drawing hairs.) Let X be a manifold. Fix a submanifold $V \subset X$. Choosing a metric on X, we can identify the conormal bundle, normal bundle, and a tubular neighborhood of V. Thus we can describe subvarieties of the conormal bundle of V in terms of subvarieties of the tubular neighborhood. In the present case, X is always two dimensional. Local pictures of X are

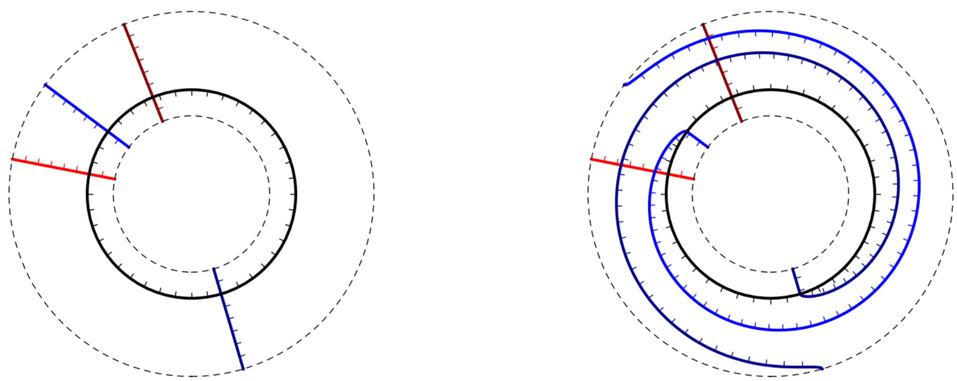

FIGURE 4. Before (left) and after (right) pictures of a mutation at a curve met positively by two lines and negatively by two lines. Here $\Sigma_{(k)}$, $\Sigma'_{(k)}$ are the left and right annuli whose boundaries are the dashed circles, and the curves C_k, C'_k are the black circles. Convention 2.1 says that in the left picture $\langle \text{black}, \text{red} \rangle_+ = 1$.

drawn on the piece of paper, which gives a choice of local metric. We indicate a conical subvariety of the conormal bundle to a manifold by drawing "hairs" in the tubular neighborhood. Generally we will draw the hairs in the same color as the submanifold. Note that when V is codimension one, drawing the hairs on one side or the other is the same as choosing a co-orientation of V.

Remark 2.5. Mutating at C_k, and then mutating at C'_k results in a curve configuration whose Legendrian lifts are Legendrian isotopic (relative the boundary) to the configuration which would result from applying a single Dehn twist.

Proposition 2.6. *Let B_1, B_2 be segments, and C_k the central curve in the collection $\mathcal{C}_{(k)}$; let B'_1, B'_2 and C'_k be their counterparts in $\mathcal{C}'_{(k)}$. Then*

$$\begin{aligned} \langle B'_i, C'_k \rangle_\pm &= \langle B_i, C_k \rangle_\mp \\ \langle B'_1, B'_2 \rangle_+ &= \langle B_1, C_k \rangle_+ \langle C_k, B_2 \rangle_+ \end{aligned}$$

Proof. By inspection of Figure 4.

Or, in symbols: the statement regarding intersections with C_k holds because we are reversing C_k to get C'_k. For the second statement, there are different cases according as the values of $\langle B_1, C_k \rangle_+$ and $\langle B_2, C_k \rangle_+$.

When B_1, B_2 have the same orientation, i.e. $\langle B_1, C_k \rangle_+ = \langle B_2, C_k \rangle_+$, we have $\langle B'_1, B'_2 \rangle_+ = 0$ and also one of $\langle B_1, C_k \rangle_+$, $\langle C_k, B_2 \rangle_+$ must be zero, giving the desired equality.

In case $\langle B_1, C_k \rangle_+ = 1 = \langle C_k, B_2 \rangle_+$, we have to show that $\langle B'_1, B'_2 \rangle_+ = 1$. In homology $B'_1 = B + C_k$ and $B'_2 = B_2$, so $\langle B'_1, B'_2 \rangle_+ = \langle B_1 + C_k, B_2 \rangle_+ = \langle B_1, B_2 \rangle_+ + \langle C_k, B_2 \rangle_+ = 1$.

Finally, in case $\langle B_1, C_k \rangle_+ = 0 = \langle C_k, B_2 \rangle_+$ we have to show that $\langle B'_1, B'_2 \rangle_+ = 0$. Here $B'_1 = B_1$ and $B'_2 = B_2 + C_k$, and so $\langle B'_1, B'_2 \rangle_+ = \langle B_1, B_2 + C_k \rangle_+ = \langle B_1, B_2 \rangle_+ + \langle B_1, C_k \rangle_+ = 0$. □

We define the mutation globally by gluing the mutation on $\mathcal{L}_{(k)}$ to the identity elsewhere. That is, writing $\mathcal{L}^{(k)}$ for an ϵ-neighborhood of the closure of the complement of $\mathcal{L}_{(k)}$.

Definition 2.7. Let $\mathcal{C}$ be a curve collection on a surface $\mathcal{L}$, and $C_k \in \mathcal{C}$ a simple closed curve. We define the mutation of $\mathcal{C}$ at C_k to be the curve configuration which coincides with $\mathcal{C}$ inside of $\mathcal{L}^{(k)}$ and coincides with $\mathcal{C}'_{(k)}$ inside $\mathcal{L}_{(k)}$. We denote this mutation by $\mathcal{C}'$.

Remark 2.8. To ensure that $\mathcal{C}'_{(k)}$ and $\mathcal{C}_{(k)}$ agree in a collar sufficiently close to the boundary of $\mathcal{L}_{(k)}$, we choose ϵ small enough that $\mathcal{L}_{(k)} \cap \mathcal{L}^{(k)}$ lies in this collar.

The following globalizes Proposition 2.6:

Proposition 2.9. *For any curve $C \neq C_k$, we have $\langle C', C'_k \rangle_+ = \langle C_k, C \rangle_+$ and $\langle C'_k, C' \rangle_+ = \langle C, C_k \rangle_+$. For $C_i, C_j \neq C_k$,*

$$\langle C'_i, C'_j \rangle_+ = \langle C_i, C_j \rangle_+ + (\langle C_i, C_k \rangle_+ \cdot \langle C_k, C_j \rangle_+)$$

Proof. The change of intersections happens in the collar $\mathcal{L}_{(k)}$, so we restrict attention here; the result now follows from Proposition 2.6. □

2.2. **Iteration of curve mutation.** Mutating a curve configuration as in Definition 2.3 returns another curve configuration. However, even when the original curve configuration contain only embedded curves, this may no longer be the case in the mutated configuration; see Figure 2. Since we only mutate at embedded curves, this constrains the possibility of iterating the mutation procedure. We introduce notation to name the difficulty:

Definition 2.10. Let $\mathcal{C}$ be a curve configuration. Its *intersection quiver* $Q_{\mathcal{C}}$ has for vertices the curves the curves of $\mathcal{C}$. Arrows from C_i to C_j are geometric intersections contributing to $\langle C_i, C_j \rangle_+$. The *algebraic intersection quiver* $Q_{[\mathcal{C}]}$ is the quiver whose vertices are the curves, and which has $\max(\langle C_i, C_j \rangle, 0)$ arrows from C_i to C_j.

The quiver $Q_{\mathcal{C}}$ can have loops, corresponding to self-intersections of the curves. It can also have oriented two cycles, when the absolute value of the algebraic intersection number of the curves is smaller than the number of geometric intersections. If we take $Q_{\mathcal{C}}$ and erase all self-loops and cancel out 2-cycles until none remain, we obtain $Q_{[\mathcal{C}]}$. While the notion of quiver mutation is usually formulated for quivers without self-loops or oriented 2-cycles [FZ], consideration of quivers such as $Q_{\mathcal{C}}$ leads to the following generalization:

Definition 2.11. Let Q be any quiver and v_k a vertex with no self-loops. The mutation $Mut_k(Q)$ of Q at v_k has the same vertices as Q and

- an arrow $a : v_i \to v_j$ for each such arrow of Q with $v_i, v_j \neq v_k$,
- an arrow $a^{op} : v_j \to v_i$ for each arrow $a : v_i \to v_j$ of Q with either $v_i = v_k$ or $v_j = v_k$,
- an arrow $[ab] : v_i \to v_j$ for each pair of arrows $a : v_i \to v_k$, $b : v_k \to v_j$ in Q.

Warning. Given a quiver without self-loops or oriented 2-cycles, the above notion of mutation does *not* agree with the usual notion of quiver mutation, which is defined only for quivers of this kind. Rather, if we take such a quiver, perform the above mutation operation, then erase all 2-cycles created, we obtain the result of the standard notion of mutation of a 2-acyclic quiver. Note the general mutation is exactly what mutation of a quiver *with potential* does to the underlying quiver [DWZ]. In practice no ambiguity will result: we mean the above notion when we refer to mutation of a quiver such as $Q_{\mathcal{C}}$ that may in principle have 2-cycles, we mean the standard notion when we refer to mutation of a quiver such as $Q_{[\mathcal{C}]}$ that by definition cannot.

Proposition 2.12. *If C_k is a simple closed curve in $\mathcal{C}$, $Q_{Mut_k(\mathcal{C})}$ (resp. $Q_{[Mut_k(\mathcal{C})]}$) is the mutation of $Q_{\mathcal{C}}$ (resp. $Q_{[\mathcal{C}]}$), at C_k.*

Proof. This is a restatement of Proposition 2.9. □

Note in particular that vertex v_i which participates in a 2-cycle with v_j creates a self-loop at v_j. Mutating at the corresponding curve C_i creates a self-intersection in the resulting C_j'. While this is well defined, it is not desirable, since we do not then know how to mutate at C_j'.

Definition 2.13. A curve configuration $\mathcal{C}$ is *simple* if $Q_{\mathcal{C}}$ has no loops or oriented two-cycles. In other words, if all curves are embedded, and the algebraic and geometric intersection numbers agree up to sign.

However, in some cases we can make use of the freedom that, while for definiteness we have defined the curve configuration as co-oriented immersed curves in a surface, in fact we only care about the Legendrian lifts of these curves, up to Legendrian isotopy. In terms of the curves in the surface, we may isotope them past each other, as long as we do not in the process pass through a tangency of curves *with the same co-orientation*.

In some cases, it is possible to Legendrian isotope the curve configuration to cancel a pair of oppositely oriented intersections between curves. For example, a mutation at C_k followed by a mutation C_k' generally creates pairs of intersections which can be cancelled by Legendrian isotopy.

Definition 2.14. A simple configuration $\mathcal{C}$ of embedded co-oriented curves is *nondegenerate* if it admits arbitrary sequences of mutations. That is, for every list $(i_1, \dots, i_n)$ of indices of circles, there is a sequence of Legendrian isotopies $(\iota_1, \dots, \iota_{n-1})$ such that $\iota_k \circ \mu_{i_k} \circ \iota_{k-1} \circ \mu_{i_{k-1}} \cdots \iota_1 \circ \mu_{i_1}(\mathcal{C})$ is a simple curve configuration, for all k.

We know one family of such examples.

Theorem 2.15. *A configuration $\mathcal{C} = \{C_i\}$ of co-oriented geodesics on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, equipped with its standard Euclidean metric, is nondegenerate.*

Proof. First note that for any i, j all intersections between C_i and C_j are of the same sign; that is, the number of intersections between C_i and C_j is exactly the absolute value of $\langle C_i, C_j \rangle$. Indeed, the sign of all such intersections is determined by the slopes of any lifts of C_i and C_j to $\mathbb{R}^2$. Thus the quiver $Q_{\mathcal{C}}$ does not itself contain any 2-cycles.

We claim that the configuration obtained by mutating at some C_k is again equivalent to one consist entirely of geodesics. Inductively applying the observation of the previous paragraph, it will then follow that $\mathcal{C}$ is nondegenerate.

Choosing coordinates appropriately on the universal cover $\mathbb{R}^2$, we may assume C_k lifts to a rightwardly co-oriented vertical line. The Dehn twist around C_k lifts to a homeomorphism from $\mathbb{R}^2$ to itself which is isotopic to the linear homeomorphism given by the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

On T^2 this isotopy descends to one which simultaneously straightens the curves C_j' which were twisted by mutation at C_k (that is, the C_j' for which $\langle C_j, C_k\rangle > 0$).

It remains to argue that this isotopy of the twisted curves lifts to an isotopy of the Legendrian lift of the entire configuration. This happens provided that whenever a curve being isotoped becomes tangent to a curve which is not moved, their co-orientations are opposite. The stationary C_j' are those C_j' for which $\langle C_j, C_k\rangle \leq 0$, in which case $C_j' = C_j$. These have the property that any lift to $\mathbb{R}^2$ is a straight line which is either vertical or whose co-orientation points downward. On the other hand, for the moving C_j', the geodesic at the end of the isotopy lifts to a straight line which is co-oriented upward. The twisted curve C_j' does not lift to a straight line, but nonetheless can be chosen so that its lift is upwardly co-oriented away from its vertical tangents. The straightening isotopy can be chosen to preserve this property, thus only creating tangencies between a downward co-oriented curve and an upward co-oriented curve (see Figure 3). □

2.3. **Mutation as seen by the disk.** We have drawn the mutation from the point of view of curves on the surface $\mathcal{L}$. While our discussion made it seem as if it was a discrete process, in fact there is a natural interpolation between the before and after configurations. However, it cannot be seen from the point of view of the surface. Instead, we describe it from the point of view of the disk being attached to the curve C_k at which we are mutating.

First we describe the neighborhood of the disk D_k inside the skeleton $\mathbb{L}$, supposing no other curves met C_k. In this case, the skeleton looks locally like the union of a cylinder — $\mathcal{L}_{(k)}$ from the previous discussion — with a disk D_k, glued in along C_k. Observe that while this cannot be drawn conically inside the $T^*\mathcal{L}_{(k)}$, it can be drawn conically inside the cotangent bundle of an $\mathbb{R}^2$ which contains D_k as the unit disk — we take $\mathcal{L}_{(k)}$ to be the conormal bundle of the boundary of the disk.

Another picture we shall use, and denote by $\perp\!\!\!\perp$, is the union of the zero section of $T^*\mathbb{R}^2$ with the "inward" conormals to the disk. Topologically, this is again a disk glued to the cylinder, although it is not diffeomorphic to the previous one. (There is no reason it should be: a skeleton is the union of downward flows of a Morse function, and the natural relation between them as the Morse function varies is not diffeomorphism.)

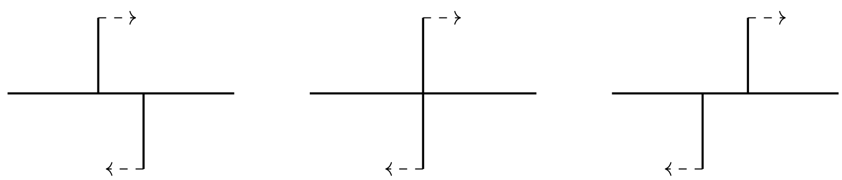

FIGURE 5. The disk surgery as the cone over a Legendrian isotopy

Under the disk surgery [Yau], we should see the disk shrink and then regrow in another way. The key insight in this section is that the movie of disk surgery can be seen as the cone over a certain Legendrian isotopy, at least in the conical model $\perp\!\!\!\perp$. This fact will ultimately allow us to define a mutation functor on sheaf categories, using [GKS].

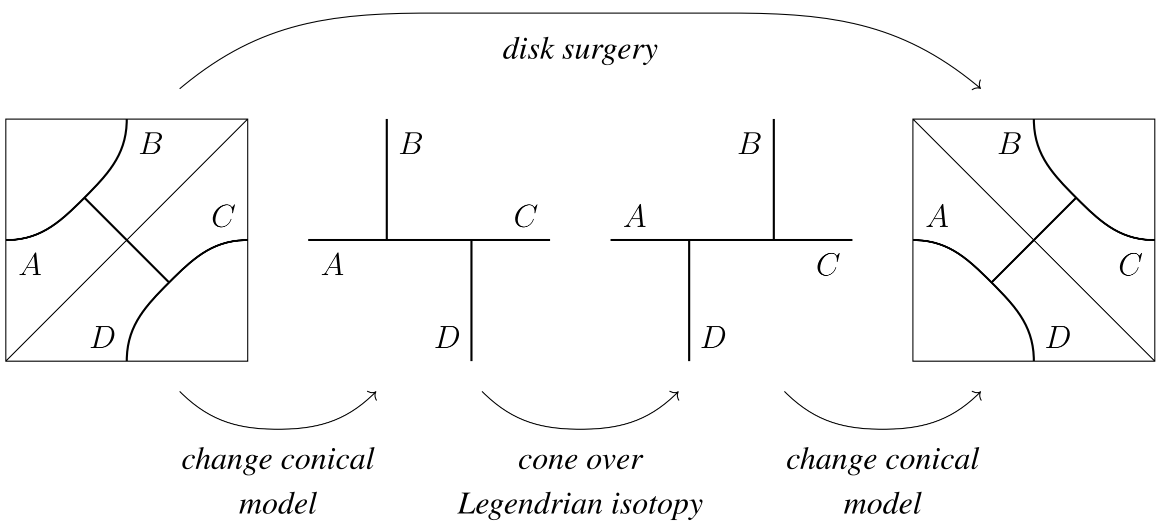

FIGURE 6. Disk surgery in Weinstein 2-manifolds as a change of triangulation. In 4-dimensions we describe a skeletal surgery by passing through certain local conical models, whose analogues are pictured here.

When no other curves meet C_k, we can take the isotopy to be the geodesic flow (i.e. negative Reeb flow) — i.e., flowing each point of the cosphere bundle in the dual tangent direction along the base, while preserving the cotangent vector. The curve C_k lies at infinity along the inward conormal to the disk; after time 1 geodesic flow, it becomes the cocircle over the origin, and after time 2, it becomes the outward conormal to the disk. Meanwhile, the disk bounded by the image of C_k contracts and then expands. This makes sense in any dimension; we give the one-dimensional picture in 5. We denote the union of the outward conormal to the disk and the zero section as $\top\!\!\!\top$.

Remark 2.16. For 1-dimensional skeleta of 2-dimensional Weinstein manifolds, the disk surgery is dual to flipping the triangles in a triangulation. See Figure 6, and see [DyK] for a corresponding account of the Fukaya category.

We will now give a similar, albeit more complicated, account of the general situation — when the curve C_k may meet other curves.

Definition 2.17. We define $\perp\!\!\!\perp \subset T^*\mathbb{R}^2$ as the union of the zero section and the cone over a singular Lagrangian $\perp\!\!\!\perp^\infty$. We specify $\perp\!\!\!\perp^\infty$ by giving its front projection with co-orientation. This is the disjoint union of an inwardly co-oriented unit circle, γ_k, with pairwise disjoint open intervals $\{\gamma_p\}_{p\in I_k}$, where I_k is set of intersections of C_k with other curves in $\mathcal{C}$. We fix a homeomorphism of γ_k and C_k and use this to freely identify $p \in I_k$ with points of γ_k. The position and co-orientation of each γ_p are chosen so that $p \in \gamma_k$ is contained in the closure of γ_p and so that the Legendrian lift $\perp\!\!\!\perp^\infty$ is connected.

The result of mutation also has a conical model. We again describe it in terms of geodesic flow near the unit disk, but far away from the disk we should not disturb the other arcs $\{\gamma_p\}_{p\in I_k}$. More precisely, we consider a compactly supported contact isotopy $\{\varphi_t\}_{t\in[0,1]}$ of $T^\infty\mathbb{R}^2$ (i.e. a family of contactomorphisms of $T^\infty\mathbb{R}^2$ with φ_0 the identity) which acts on some open disk of radius more than 1 by a scaling of geodesic flow such that the inward conormal lift of the unit circle at $t = 0$ is carried to its outward conormal lift at $t = 1$.

For example, let (x, y) be the usual coordinates on $\mathbb{R}^2$ and θ an additional angular coordinate for the cocircle bundle. Let $f : \mathbb{R}^2 \to [0, 1]$ be a smooth function that vanishes outside a disk of large radius and is identically 1 on a slightly smaller disk. Then we may take $\{\varphi_t\}_{t\in[0,1]}$ to be the flow along the contact vector field

$$-2(f(x,y)\cos(\theta)\partial_x + f(x,y)\sin(\theta)\partial_y + (f_x(x,y)\sin(\theta) + f_y(x,y)\cos(\theta))\partial_\theta). \tag{2}$$

The family $\varphi_t(\perp\!\!\!\perp^\infty)$ is illustrated in Figure 7.

Definition 2.18. We define $\top\!\!\!\top^\infty := \varphi_1(\perp\!\!\!\perp^\infty)$, and $\top\!\!\!\top$ as the union of the cone over $\top\!\!\!\top^\infty$ with the zero section.

Recall that we write $\mathcal{L}_{(k)}{}'$ and $\mathcal{L}^{(k)}{}'$ *not* for a neighborhood of C_k' and its complement, but instead for the parts of $\mathcal{L}'$ which are the images of $\mathcal{L}_{(k)}$ and $\mathcal{L}^{(k)}$ under the fixed identification $\mathcal{L} \cong \mathcal{L}'$ used in defining the mutation of curve configurations. In particular, recall that the restricted curve configuration $\mathcal{C}_{(k)}{}'$ will generally contain intersections amongst the curves ending on the boundary of $\mathcal{L}_{(k)}{}'$, whereas a neighborhood of C_k' would not.

Remark 2.19. The "neighborhood of C_k'" would be naturally denoted by $\mathcal{L}'_{(k)}$, similarly for $\mathcal{L}'^{(k)}, \mathcal{C}'_{(k)}$. Compare the ordering of the prime and the k with the above. However, we never use these subsets in this paper: any occurences below of the these symbols are misprints for the other ordering.

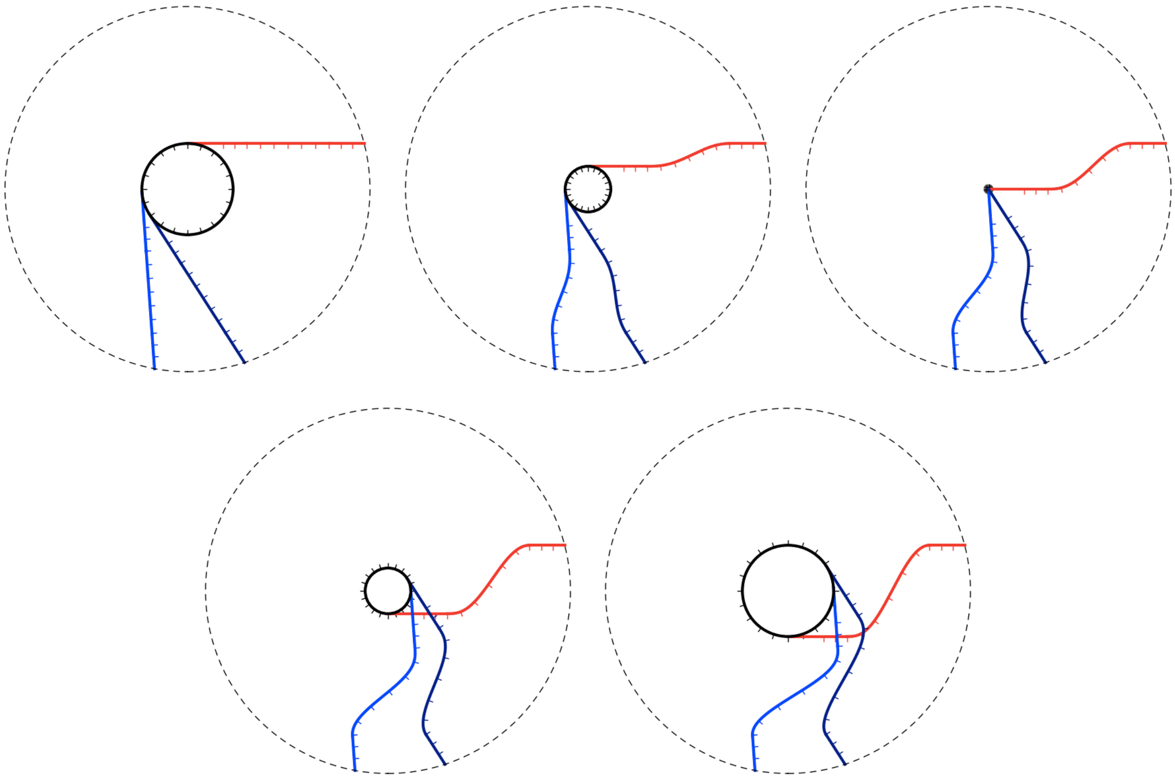

FIGURE 7. The contact isotopy φ_t carrying $⫫^\infty$ (first frame) to $⫪^\infty$ (last frame). Note the distinction from Figure 4: in the first frame $\mathcal{L}_{(k)}$ does not lie entirely in the page, but rather is the union of the annulus outside C_k (the black circle) and the conormal to C_k. The disk inside C_k as drawn on the page is the disk D_k. The colored strands are other C_i which intersect C_k.

Definition 2.20. Let $\mathbb{L}_{(k)}$, resp. $\mathbb{L}_{(k)}{}'$, be the skeleton resulting from applying Definition 1.2 to $(\mathcal{L}_{(k)}, \mathcal{C}_{(k)})$, resp. $(\mathcal{L}_{(k)}{}', \mathcal{C}_{(k)}{}')$.

Proposition 2.21. *There are homeomorphisms, respecting the obvious identifications at the boundary,* $\mathbb{L}_{(k)} \cong ⫫$ *and* $\mathbb{L}_{(k)} \cong ⫪$.

Proof. The real content here is the assertion that the flow F_t creates the correct intersections in the projections of $⫪^\infty$. This can be seen by inspection of Figure 7. $\square$

2.4. **Mutation inside the 4-manifold.** A curve configuration $\mathcal{C}$ and its mutation $\mathcal{C}'$ give rise to two symplectic manifolds W and W' via the construction of Definition 1.3. In this section we consider the relation between W and W'.

As we have constructed W by Weinstein handle attachment, it is an exact symplectic manifold — i.e. the symplectic form has a primitive, $\omega = d\theta$ — with convex contact boundary — i.e., along the boundary, the vector X with $\omega(X, \cdot) = \theta$ points outward. Actually, we shall prefer to attach an infinite conical end along this boundary, which we do without changing the notation for W.

More generally, our conventions regarding exact symplectic manifolds follow [Sei3]. Our notion of isomorphism of such manifolds is Liouville isomorphism: a symplectomorphism $f : W \to W'$ such that moreover $\theta - f^*\theta'$ is exact and compactly supported.

Previously, we have viewed both $\mathcal{C}$ and $\mathcal{C}'$ as living on the same surface. However for our present purposes, we take the view that $\mathcal{C}'$ sits on a surface $\mathcal{L}'$ which is different from but diffeomorphic to the original surface $\mathcal{L}$, by a diffeomorphism which we have chosen in the process of defining it. Similarly for the local curve configuration $\mathcal{C}_{(k)}, \mathcal{C}'_{(k)}$. This choice could have been made in other ways; but a best choice will be ultimately dictated by Theorem 4.15.

Theorem 2.22. *There is a Liouville isomorphism of W and W' which identifies $\mathbb{L}$ and $\mathbb{L}'$ outside a neighborhood of the disk D_k. The preimage of $\mathcal{L}'$ under this symplectomorphism is obtained from $\mathcal{L}$ by Lagrangian disk surgery on D_k.*

The proof of this theorem occupies the remainder of this section. As noted in [Sei3, Lem. 2.2], Moser's lemma implies that in any isotopy of Liouville structures preserves the Liouville isomorphism type. We construct such an isotopy by describing an isotopy of skeleta; which implicitly names an isotopy of the corresponding flows. We will be somewhat informal, omitting in particular all rounding-of-corners arguments. We refer to the treatise [CE] for many methods of manipulating such manifolds.

Remark 2.23. Strictly speaking, the theorems asserted in the remainder of the paper concern microlocal sheaf categories on skeleta, and do not logically depend on Theorem 2.22.

Recall that we write $\mathcal{L}_{(k)}$ for a small neighborhood in $\mathcal{L}$ of C_k, and $\mathcal{L}^{(k)}$ for a complementary chart; we abusively write $\partial\mathcal{L}_{(k)} = \partial\mathcal{L}^{(k)}$ for the overlap of these charts; equivalently a collar neighborhood of either one of their boundaries. We likewise denote by $\mathcal{C}_{(k)}, \mathcal{C}^{(k)}, \partial\mathcal{C}_{(k)} = \partial\mathcal{C}^{(k)}$ the restrictions of the curve configuration to these spaces.

Evidently we can apply Definition 1.3 to any of these; we denote the resulting manifolds by $W_{(k)}, W^{(k)}, \partial W_{(k)} = \partial W^{(k)}$. Evidently

$$W = W_{(k)} \bigcup_{\partial W_{(k)}} W^{(k)}$$

Let us describe $W_{(k)}$ more explicitly. First we make the co-disk bundle $D^*\mathcal{L}_{(k)} = D^2 \times \mathcal{L}_{(k)}$. This is a manifold with corners: it has boundary components given by the cocircle bundle $S^*\mathcal{L}_{(k)} = S^1 \times \mathcal{L}_{(k)}$, and $D^*\mathcal{L}|_{\partial\mathcal{L}_{(k)}} = \partial\mathcal{L}_{(k)} \times D^2 = S^1 \times S^0 \times D^2$. These intersect along $S^*\mathcal{L}|_{\partial\mathcal{L}_{(k)}} = S^1 \times S^0 \times S^1$. We now attach a Weinstein handle along the lift Λ_k of C_k.

We write the resulting manifold-with-corners as $W_{(k)}$. The intersection of the corresponding Lagrangian skeleton $\mathbb{L}_{(k)}$ with $\partial W_{(k)}$ is as follows. In each component of $D^*\Sigma|_{\partial\Sigma_{(k)}}$, the intersection is a circle of $\partial\Sigma_{(k)}$, emanating radial spokes for corresponding to the disk fragments being attached along the $C_i \cap \Sigma_{(k)}$ for $i \neq k$. This disk fragments also intersect the remaining boundary component, so in all $\mathbb{L}_{(k)} \cap \partial W_{(k)}$ is two circles, joined by several lines.

We smooth the corners of $W_{(k)}$ to get a space $\widetilde{W}_{(k)}$; alternatively it might be taken as an $\epsilon-$neighborhood of the disk D_k. The space $\widetilde{W}_{(k)}$ is symplectically a ball.

We now observe that our conical models can be glued in place of $\widetilde{W}_{(k)}$ and $\widetilde{W}'_{(k)}$.

Proposition 2.24. *There is a neighborhood U of $\perp\!\!\!\perp$ and a symplectomorphism respecting the boundary*

$$(U, \partial U, \perp\!\!\!\perp \cap \partial U) \cong (\widetilde{W}_{(k)}, \partial\widetilde{W}_{(k)}, \mathbb{L} \cap \partial\widetilde{W}_{(k)})$$

There is a neighborhood U' of $\top\!\!\!\top$ and a symplectomorphism respecting the boundary

$$(U', \partial U', \top\!\!\!\top \cap \partial U') \cong (\widetilde{W}'_{(k)}, \partial\widetilde{W}'_{(k)}, \mathbb{L}' \cap \partial\widetilde{W}'_{(k)})$$

Remark 2.25. The above symplectomorphism certainly does not identify $\perp\!\!\!\perp$ and $\mathbb{L}_{(k)}$: no diffeomorphism can, since these spaces have different singularities. However, one should not expect them to be identified: they implicitly name different Morse functions, and the appropriate relation between such functions is one of isotopy. Correspondingly, it can be shown that there are deformations $\perp\!\!\!\perp \sim \mathbb{L}_{(k)}$ and $\top\!\!\!\top \sim {\mathbb{L}_{(k)}}'$ which are noncharacteristic in the sense of [N4].

Finally, we can describe the desired symplectomorphism:

Definition 2.26. There is a symplectomorphism $\mu_k : W \to W'$ restricting to the evident identification on $W \smallsetminus \widetilde{W}_{(k)} = W' \smallsetminus \widetilde{W}'_{(k)}$. Identifying $\widetilde{W}_{(k)}$ and $\widetilde{W}'_{(k)}$ to standard balls via Propositions 2.24, we define the rest of the map on $\widetilde{W}_{(k)} \to \widetilde{W}'_{(k)}$ to be the identity sufficiently far from the boundary, and a movie of contact isotopy of Definition 2.18 near the boundary.

For a picture of what is meant, one dimension down, and beginning and ending with the conormal-to-disk conical model rather than $\perp\!\!\!\perp$ and $\top\!\!\!\top$, see Figures 8, 9, and 10.

Finally, we have a Lagrangian surface named $\mathcal{L}$ inside W and a Lagrangian surface named $\mathcal{L}'$ inside W'. To see that they are related by the Lagrangian attaching disk surgery of [Yau], recall that the local model of the disk surgery is the passage between the hyperbolas $xy = -\epsilon$ and $xy = \epsilon$ for $\epsilon \in \mathbb{R}$. A conical model of this transition is given by the collapsing and re-expanding of the disk in the base in the transition of Definition 2.18.

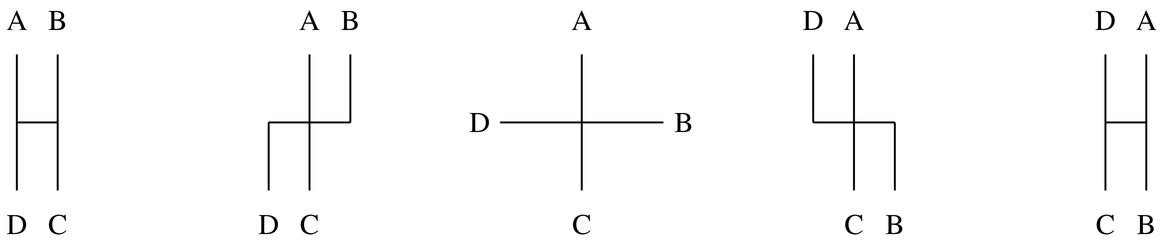

FIGURE 8. The disk surgery, realized as a cone over a Legendrian isotopy. In this 1-dimensional picture the Legendrian is 4 points at the boundary of $T^*\mathbb{R}$.

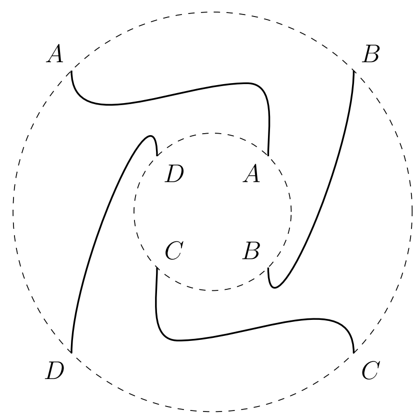

FIGURE 9. The movie of the Legendrian isotopy from Figure 8.

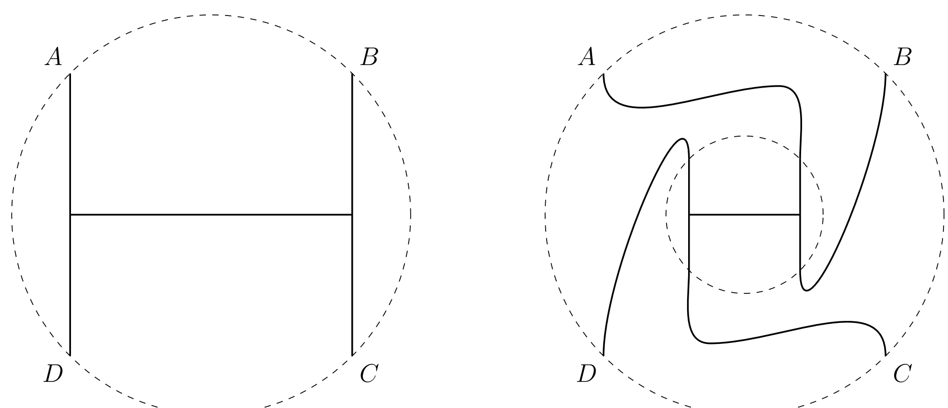

FIGURE 10. The disk surgery, realized by gluing a transformed conical model into the movie of the contact isotopy realizing the transformation, as in Definition 2.26.

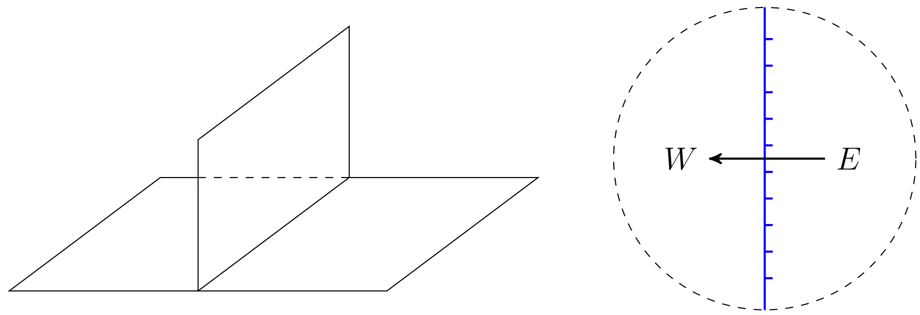

FIGURE 11. A conic Lagrangian $L \subset T^*\mathbb{R}^2$ such that $sh_L(\mathbb{R}^2) \cong \Bbbk A_2$-mod. On the right we indicate L as the cone over the Legendrian lift of the y-axis, co-oriented rightward. Note the non-isomorphic restriction maps of a sheaf with stalks W and E go "against the grain" of the co-oriented curve. The topology of L is pictured on the left: a plane with a single fin attached to it.

3. MICROLOCAL SHEAVES

In this section we define the category $\mu loc(\mathbb{L})$ of microlocal sheaves on the Lagrangian seed skeleton $\mathbb{L}$ of Definition 1.3. We expect this to be equivalent to a Fukaya category of W, following a proposal of [Kon2] and the results of [NZ, N1] in the case of cotangent bundles. We glue this category together from Kashiwara-Schapira sheaves of conical local models.

3.1. **Microlocal sheaves on locally conical Lagrangians.** A Lagrangian in a cotangent bundle is conical if it is invariant under the fiberwise rescaling action. Recall that such a Lagrangian $\perp \subset T^*M$ defines a category $sh_\perp(M)$ of sheaves on M with microsupport contained in $\perp$ [KS]. We very briefly review the notion of microsupport and definition of $sh_\perp(M)$ in Appendix A (see also the account of our conventions at the end of Section 1). In fact, the only geometries required for our purposes are the following three local examples, which admit simple algebraic descriptions in terms of quiver representations.

Example 3.1. Locally constant sheaves are characterized by having singular support equal to the zero section. Thus, regarding M as a conical Lagrangian in its cotangent bundle, $sh_M(M)$ is just $loc(M)$, the category of locally constant sheaves.

Example 3.2. Let $\perp \subset T^*\mathbb{R}^2$ be the union of the zero section with the cone over $\Lambda = dx|_{\{x=0\}}$, the Legendrian whose front projection is the y-axis, cooriented to the right (see Figure 11). Then $sh_\perp(\mathbb{R}^2)$ is equivalent to $\Bbbk A_2$-mod, the unbounded dg derived category of representations of the A_2-quiver, as follows. We write W and E for the stalks of a sheaf in the open left half-plane $\{x < 0\}$ and closed right half-disk $\{x \geq 0\}$, respectively (all stalks in either region are canonically isomorphic up to homotopy). There is a map $E \to W$, referred to as a generization map, given by restricting from a neighborhood of a point on the y-axis to a smaller open set lying entirely to the

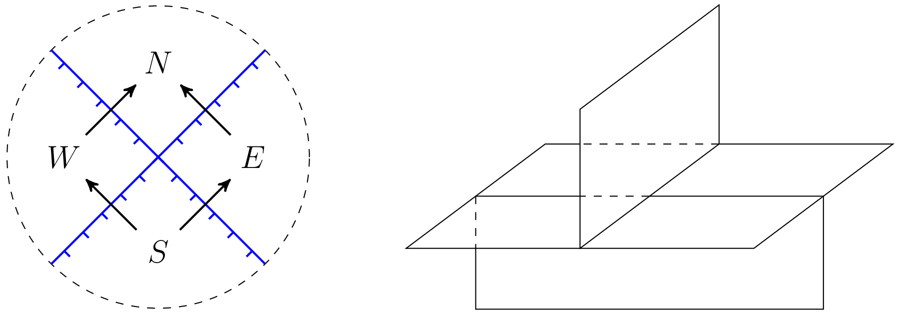

FIGURE 12. A conic Lagrangian $L \subset T^*\mathbb{R}^2$ such that $sh_L(\mathbb{R}^2) \cong \Bbbk A_3$-mod. On the left we indicate L as the cone over the Legendrian lift of a pair of co-oriented curves. The topology of L is pictured on the right: a plane with a pair of fins attached to it.

left of the y-axis. We also have the microlocal stalk at a point of L lying off the zero section, hence above the y-axis: it is the cone over this map.

Example 3.3. Let $\perp \subset T^*\mathbb{R}^2$ be the union of the zero section with the cone over $\Lambda = (dx - dy)|_{\{x=y\}} \cup (-dx - dy)|_{\{x=-y\}}$, the Legendrian whose front projection is the union of the lines $x = y$ and $x = -y$, co-oriented downwards (see Figure 12). Then $sh_\perp(\mathbb{R}^2)$ can be described in terms of the dg category of quadruples N, W, E, S of (complexes of) $\Bbbk$-modules, with a commuting square of maps as pictured. Such data gives rise to an object of $sh_\perp(\mathbb{R}^2)$ under the following crossing condition: the total complex $S \to W \oplus E \to N$ must be acyclic [STZ, Theorem 3.12].

The restrictions $sh_\perp(\mathbb{R}^2) \to sh_\perp(\mathbb{R}^2 \cap \{y > \epsilon\})$, $sh_\perp(\mathbb{R}^2) \to sh_\perp(\mathbb{R}^2 \cap \{y < -\epsilon\})$ to the regions above and below the x-axis are equivalences. These categories are equivalent to the representation categories of two different orientations of the A_3 quiver, which we obtain by forgetting S, N, respectively. The induced equivalence $sh_\perp(\mathbb{R}^2 \cap \{y > \epsilon\}) \cong sh_\perp(\mathbb{R}^2 \cap \{y < -\epsilon\})$ is a reflection functor.

To glue together sheaf categories associated to conical Lagrangians, it is crucial that these categories localize over the entire Lagrangian — not merely the base manifold. That is, we want to realize these sheaf categories as the global sections of a (homotopy) sheaf of dg categories over $\perp$. While working with sheaves of dg categories requires the homotopical foundations of, for example, [Lur1, Toe, Tab], the treatise [KS] predates these texts by a number of years. We briefly sketch how the relevant geometric results of [KS] may be adapted to the differential graded setting at hand (see also [JT, Sec. 2], [SiTZ, Sec. 2.1], [N2, Sec. 5.2]).

Kashiwara and Schapira study T^*M in the conic topology, i.e., the open sets are $\mathbb{R}_+$-invariant. For each conic open subset $U \subset T^*M$, they define a category — we denote it by $\mathrm{KS}^{pre}(U)$ — as the quotient of the category of sheaves on M by the category of sheaves with microsupport not meeting U (though we emphasize again that here we will impose no size restriction on the stalks of sheaves on M). This is a presheaf of dg categories on T^*M; that is, the restriction and quotient functors make the assignment $U \mapsto \mathrm{KS}^{pre}(U)$ into a contravariant functor from the category of open subsets of T^*M to $\mathrm{DGCat}_{\mathrm{cont}}$.

Each object of $\mathrm{KS}^{pre}(U)$ has a well-defined (conic, co-isotropic) support in U. Thus for a conic Lagrangian subset $\perp \subset T^*M$, there is a presheaf of full subcategories $\mathrm{KS}^{pre}_{\perp}(U) \subset \mathrm{KS}^{pre}(U)$ of objects supported on $\perp$, which by definition vanishes when $U \cap \perp$ is empty. The sheafification $\mathrm{KS}_{\perp}$ of $\mathrm{KS}^{pre}_{\perp}$ is therefore supported on $\perp$.

Definition 3.4. For a conic Lagrangian $\perp \subset T^*M$, we write μloc for the $\mathrm{DGCat}_{\mathrm{cont}}$-valued sheaf of dg categories on $\perp$ given by the restriction of $\mathrm{KS}_{\perp}$ to $\perp$.

The stalks of $KS_{\perp}$ and μloc are determined by [KS, Thm. 6.1.2]. In particular,

- If $U \subset T^*M$ is a conic open set of the form $T^*\pi(U)$, then $\mu loc(\perp \cap U) \cong sh_{\perp \cap U}(\pi(U))$.
- If $U \subset T^*M$ is a conic open set that does not meet the zero section and is sufficiently small, $\mu loc(\perp \cap U)$ is the quotient of $sh_{\pi(U) \cup (\perp \cap U)}(\pi(U))$ by $loc(\pi(U))$.

These results allow us to conclude that μloc is indeed the desired localization:

Proposition 3.5. *For a conical Lagrangian $\perp \subset T^*M$, the global section category $\mu loc(\perp)$ is equivalent to $sh_{\perp}(M)$.*

Proof. The global sections of μloc can be calculated by first pushing forward along $\perp \hookrightarrow T^*M \to M$ and then pushing forward to a point. But the pushforward to M admits a map from the "sheaf of sheaves" on M — i.e. the sheaf whose value over $U \subset M$ is $sh_{\perp \cap T^*U}(U)$. It follows from the above results that this map is an isomorphism at stalks, hence an isomorphism of sheaves, hence induces an isomorphism of global sections. $\square$

We are interested in gluing conical Lagrangians along open neighborhoods of their boundary components — both components in the base manifold M, and components at contact infinity $T^\infty M$. Here we write $T^\infty M$ for the cosphere bundle of M, embedded as a large contact hypersurface in T^*M. If $\perp$ is a conical Lagrangian subset of T^*M, we have an associated Legendrian subset $\Lambda = \perp \cap T^\infty M$ of $T^\infty M$. As μloc is $\mathbb{R}_+$-invariant in the fiber direction, its restriction to Λ contains the same data as its restriction to the open subset $\perp \setminus (\perp \cap M)$ or to any collar neighborhood of Λ in $\perp$. With this in mind we sometimes conflate Λ with such a collar neighborhood in our notation, when no confusion should arise.

It will be useful to have the following "Morse trivialization" of the restrction of μloc to Λ.

Lemma 3.6. *Let $\perp \subset T^*M$ be a conical Lagrangian such that $\Lambda = \perp \cap T^\infty M$ is smooth and $\pi : \Lambda \to M$ is an immersion. Let loc_Λ denote the sheaf of categories on Λ whose sections give categories of local systems. On Λ, there is a canonical equivalence $\mu loc|_\Lambda \cong loc_\Lambda$.*

Proof. Since π restricts to an immersion on Λ, it restricts to an embedding on any sufficiently small open ball $B \subset \Lambda$. Let f be a real-valued smooth function on a neighborhood of $\pi(B)$ such that $f^{-1}(0) = \pi(B)$ and $df|_{\pi(B)}$ is nonzero and contained in the cone over Λ. To an object F

of $\mu loc(B)$ we assign the local system on B whose global sections are $Cone(F(f^{-1}(-\infty,\epsilon)) \to F(f^{-1}(-\infty,-\epsilon)))$. This defines an equivalence $\mu loc(B) \xrightarrow{\sim} loc(B)$ [KS, Chap. 3], compatible with restrictions hence inducing an isomorphism of sheaves. □

Note that to a point $p \in \perp$ away from the zero section the KS sheaf assigns a stalk category $\mu loc(\perp)_p$. Given a sheaf $\mathcal{F} \in sh_\perp(M)$ we refer to its image $\mathcal{F}_p \in \mu loc(\perp)_p$ as the microlocal stalk of $\mathcal{F}$ at p. Lemma 3.6 says in particular that when $\perp$ satisfies the given hypotheses we have a canonical identification $\mu loc(\perp)_p \cong \Bbbk\text{-mod}$, hence can view the microlocal stalk $\mathcal{F}_p$ as a $\Bbbk$-module (just like an ordinary stalk at a point in M).

Remark 3.7. When Σ is a surface and Λ a Legendrian knot such that $\Lambda \to \Sigma$ is not necessarily an immersion, the rotation number of Λ measures the failure of $\mu loc|_\Lambda$ to be trivializable [STZ]. See [Gui] for some considerations in the general case.

We are now ready to consider microlocal sheaves on spaces which are only locally conical.

Definition 3.8. A Kashiwara-Schapira (KS) sheaf on a topological space $\mathbb{T}$ is a $\mathrm{DGCat}_{\mathrm{cont}}$-valued sheaf μloc of dg categories such that there exists

- an open cover $\mathbb{T} = \bigcup \mathbb{T}_i$
- embeddings $\iota_i : \mathbb{T}_i \hookrightarrow T^*M_i$ whose images are conical Lagrangians
- equivalences $\mu loc|_{\mathbb{T}_i} \cong \mu loc|_{\iota_i(\mathbb{T}_i)}$, where the left-hand side refers to the given sheaf of categories on $\mathbb{T}$ and the right-hand side to the KS sheaf of the conical Lagrangian $\iota_i(\mathbb{T}_i) \subset T^*M_i$ given by Definition 3.4.

We call a category of the form $\mu loc(\mathbb{T})$ for some KS sheaf μloc a "category of microlocal sheaves on $\mathbb{T}$".

A space $\mathbb{T}$ generally does not have a unique KS sheaf — but when $\mathbb{T}$ is presented as the skeleton of a Weinstein manifold W, it should be possible to specify a choice of μloc through some further trivializations of topological structures on W, or equivalently $\mathbb{T}$. We will instead content ourselves with constructing an explicit choice in the case at hand: given a cover $\mathbb{T} = \bigcup \mathbb{T}_i$ and conic Lagrangian embeddings $\iota_i : \mathbb{T}_i \hookrightarrow T^*M_i$, one can glue together a KS sheaf from the categories $sh_{\iota_i(\mathbb{T}_i)}(M_i)$. This requires specifying descent data, in particular equivalences $\mu loc|_{\iota_i(\mathbb{T}_i\cap\mathbb{T}_j)} \cong \mu loc|_{\iota_j(\mathbb{T}_i\cap\mathbb{T}_j)}$.

For a fixed μloc, there will also generally be many other choices of open cover and conical embeddings witnessing the fact that μloc is a KS sheaf. We use the freedom to pass between such models in an essential way.

3.2. **Microlocal sheaves on seed skeleta.** We return to the setting of Section 2, letting $\mathcal{C}$ be a curve configuration on a surface $\mathcal{L}$, and $\mathbb{L} \subset W$ the Lagrangian skeleton of the associated 4-manifold. By construction $\mathbb{L}$ is equipped with an open cover by conic Lagrangians, namely the positive conormal bundle $T^+_{\mathcal{C}}\mathcal{L} \subset T^*\mathcal{L}$ of the curve configuration and, for the closed curves C_i, cotangent bundles of open disks $D_i \subset T^*D_i$.

Convention 3.9. We write $\mathcal{C}^\circ \subset \mathcal{C}$ to denote the subset of closed curves.

Definition 3.10. We define a KS sheaf on $\mathbb{L}$, by gluing the KS sheaves of the conical Lagrangians $T^+_\mathcal{C}\mathcal{L} \subset T^*\mathcal{L}$ and $D_i \subset T^*D_i$ along the annuli $T^+_\mathcal{C}\mathcal{L} \cap D_i$. This requires gluing data on the annuli (and no more, since there are no multiple overlaps). We identify each restriction with $loc|_{T^+_\mathcal{C}\mathcal{L}\cap D_i}$. For $\mu loc|_{D_i} = loc|_{D_i}$ this is immediate. For $\mu loc|_{T^+_\mathcal{C}\mathcal{L}}$ we use the Morse trivialization of Lemma 3.6, composed with the autoequivalence of $loc|_{T^+_\mathcal{C}\mathcal{L}\cap D_i}$ given multiplying the monodromy by $\sigma(C_i)$, for some fixed choice $\sigma : \mathcal{C}^\circ \to \mathbb{Z}/2\mathbb{Z}$. We denote the resulting sheaf as μloc_σ, and generally abbreviate this just to μloc since the dependence on σ will play only a minor role.

The abstract construction of Definition 3.10 can be made concrete in practice since the local categories involved are sufficiently simple. For example, we will work out a crucial example in detail in Section 4.3.

There is a natural identification between functions from $\mathcal{C}^\circ$ to $\mathbb{Z}/2\mathbb{Z}$ and $H^2(\mathbb{L},\mathcal{L};\mathbb{Z}/2\mathbb{Z})$. Note the long exact sequence of cohomology

$$H^1(\mathcal{L};\mathbb{Z}/2\mathbb{Z}) \to H^2(\mathbb{L},\mathcal{L};\mathbb{Z}/2\mathbb{Z}) \to H^2(\mathbb{L};\mathbb{Z}/2\mathbb{Z}) \to H^2(\mathcal{L};\mathbb{Z}/2\mathbb{Z})$$

Proposition 3.11. *The category $\mu loc_\sigma(\mathbb{L})$ only depends on the image of σ in $H^2(\mathbb{L};\mathbb{Z}/2\mathbb{Z})$.*

Proof. If σ_1 and σ_2 differ by an element of $H^1(\mathcal{L},\mathbb{Z}/2\mathbb{Z})$, tensoring with the associated local system is an autoequivalence of $\mu loc(T^+_\mathcal{C}\mathcal{L})$ inducing an equivalence $\mu loc_{\sigma_1}(\mathbb{L}) \cong \mu loc_{\sigma_2}(\mathbb{L})$. □

Remark 3.12. The Fukaya category depends on a class in $H^2(W;\mathbb{Z}/2\mathbb{Z}) = H^2(\mathbb{L};\mathbb{Z}/2\mathbb{Z})$; for $\mathcal{L}$ to support any branes in this category, the restriction of the class to $\mathcal{L}$ must vanish [Sei3, Sec. 12].

Definition 3.13. For $p \in \mathbb{L}$, we write $\mathcal{F}_p$ for the image of $\mathcal{F}$ under the functor to the stalk category $\mu loc(\mathbb{L}) \to \mu loc(\mathbb{L})_p$. The *support* of $\mathcal{F}$ is the set of points $p \in \mathbb{L}$ at which $\mathcal{F}_p$ is nonzero.

Remark 3.14. Note that to discuss the degrees in which $\mathcal{F}_p$ has cohomology we would need to specify a trivialization of $\mu loc(\mathbb{L})_p$, which in general cannot be done canonically. For example if $p \in T^+_\mathcal{C}\mathcal{L}\cap D_i$ the natural trivializations $\mu loc(\mathbb{L})_p \cong \Bbbk\text{-mod}$ inherited from the inclusions of $T^+_\mathcal{C}\mathcal{L}\cap D_i$ into D_i and $T^+_\mathcal{C}\mathcal{L}$ (the latter via Lemma 3.6) differ by a cohomological shift. However discussing when $\mathcal{F}_p$ is zero does not depend on such a choice.

Up to equivalence $\mu loc(\mathbb{L})_p$ is classified by the local models discussed in Section 3.1: when p is a smooth point, $\mu loc(\mathbb{L})_p$ is (non-canonically) equivalent to the dg derived category of $\Bbbk$-modules [KS, Chap. 6]; when p lies on one of the C_i, but not at a crossing, $\mu loc(\mathbb{L})_p$ is equivalent to $\Bbbk A_2$-mod (cf. Example 3.2); when p lies at a crossing of the C_i, $\mu loc(\mathbb{L})_p$ is equivalent to $\Bbbk A_3$-mod (cf. Example 3.3).

Proposition 3.15. *There is a fully faithful functor $loc(\mathcal{L}) \hookrightarrow \mu loc(\mathbb{L})$ whose essential image is the full subcategory of microlocal sheaves supported on $\mathcal{L} \subset \mathbb{L}$.*

Proof. We have $loc(\mathcal{L}) \subset \mu loc(\mathcal{L} \cup \bigcup_i T^+_{C_i}\mathcal{L})$. Local systems have vanishing microlocal stalks away from the zero section, so the restriction morphism $loc(\mathcal{L}) \to loc(\coprod \Lambda_i)$ is the zero morphism. The only element of $\coprod D_i$ which restricts to 0 is the zero sheaf. There is a well-defined functor $loc(\mathcal{L}) \to \mu loc(\mathbb{L})$ given by

$$loc(\mathcal{L}) = loc(\mathcal{L}) \times_0 0 \subset \mu loc(\mathcal{L} \cup N^+C_i) \times_{loc(\coprod \Lambda_i)} loc(D_i).$$

It is clear this functor is fully faithful. □

With this in mind we regard $loc(\mathcal{L})$ as a full subcategory of $\mu loc(\mathbb{L})$ from now on without comment. More generally, by the same sort of argument we have the following.

Proposition 3.16. *If $\mathcal{D}$ is any subset of the curve configuration $\mathcal{C}$, and $\iota : \mathbb{L}_{\mathcal{D}} \subset \mathbb{L}_{\mathcal{C}}$ is the associated inclusion of skeleta, then there is a canonical, locally fully faithful functor $\iota_* \mu loc \to \mu loc$ whose global sections are a fully faithful inclusion $\mu loc(\mathbb{L}_{\mathcal{D}}) \to \mu loc(\mathbb{L}_{\mathcal{C}})$ with image exactly equal the microlocal sheaves supported on $\mathbb{L}_{\mathcal{D}}$.*

There are various size restrictions on objects of $\mu loc(\mathbb{L})$ that we will sometimes want to consider. First note that the restriction functors in μloc preserve limits in addition to colimits [N5, Sec. 3.6]. Thus on general categorical grounds we have

(1) for each inclusion $U \hookrightarrow U'$ of open subsets of $\mathbb{L}$ there is a corestriction functor $\rho^L : \mu loc(U) \to \mu loc(U')$ left adjoint to the restriction functor $\rho : \mu loc(U') \to \mu loc(U)$ [GR, Lem. 5.3.2],
(2) μloc has the structure of a $\mathrm{DGCat}_{\mathrm{cont}}$-valued cosheaf with respect to the corestriction functors [GR, Cor. 5.3.4], and
(3) the corestriction functors preserve compact objects and $\mu loc(U)$ is compactly generated for any open subset $U \subset \mathbb{L}$ [GR, Lem. 7.1.5, Cor. 7.2.7].

Given an open subset $U \subset \mathbb{L}$ we write $\mu loc^w(U) \subset \mu loc(U)$ for the full subcategory of compact objects and refer to these as wrapped microlocal sheaves on U. The subcategory $\mu loc^{sm}(U) \subset \mu loc^w(U)$ of objects with perfect stalks can be identified with the full subcategory of exact functors $\mu loc^w(U) \to \Bbbk\text{-mod}$ which take values in perfect complexes [N5, Thm. 3.21]; we refer to objects of $\mu loc^{sm}(U)$ as ordinary or small microlocal sheaves. Thus $\mu loc^{sm}(U)$ is the category of pseudo-perfect $\mu loc^w(U)$-modules in the terminology of [TVa].

Remark 3.17. In general the categories $\mu loc(\mathbb{L})$ and $\mu loc^w(\mathbb{L})$ cannot be recovered from $\mu loc^{sm}(\mathbb{L})$. However, they can be recovered from the entire sheaf μloc^{sm} given by $U \mapsto \mu loc^{sm}(U)$. This follows from the fact that $\mathbb{L}$ is locally arboreal [N5], hence it admits an open cover by subsets U_i such that $\mu loc^{sm}(U_i) \cong \Bbbk A_n\text{-mod} \cong \mu loc^w(U_i)$. In particular, the sheaf μloc is the sheafification of the presheaf $U_i \mapsto \mathrm{Ind}\, \mu loc^{sm}(U_i)$.

We will also wish to discuss objects in $\mu loc(\mathbb{L})$ coming from sheaves in the underived sense, i.e., with cohomology concentrated in degree zero. More precisely, consider the restriction functor

$\mu loc(\mathbb{L}) \to \mu loc(T^+_{\mathcal{C}}\mathcal{L}) = sh_{T^+_{\mathcal{C}}\mathcal{L}}(\mathcal{L})$. We define $\mu Loc(\mathbb{L})$ to be the full subcategory whose objects are the preimage of objects in $sh_{T^+_{\mathcal{C}}\mathcal{L}}(\mathcal{L})$ whose cohomology is concentrated in degree zero.

3.3. **Moduli Spaces.** The moduli theory of objects in dg categories was developed in [TVa]. The setting is that of derived algebraic geometry; for general background we refer to [Lur3, Toe2, TV2, GR]. Our main interest here is in moduli spaces of Lagrangian branes (in the guise of microlocal sheaves), especially in the subspace parametrizing branes supported on a fixed exact Lagrangian: we can conclude two Lagrangians are not Hamiltonian isotopic if these subspaces do not coincide. However, as this question can be decided by considering only the truncations of the moduli spaces involved, it is essentially one of ordinary algebraic geometry. With this in mind the reader will lose little in bypassing the discussion of derived moduli spaces and proceding with Definition 3.21 in mind.

By construction $\mathbb{L}$ admits a finite cover by open subsets U such that $\mu loc(U) \cong \Bbbk A_n$-mod for $n \in \{1,2,3\}$. It follows that the category $\mu loc^w(\mathbb{L})$ of wrapped microlocal sheaves is a finite colimit of finite type dg categories in the sense of [TVa], hence is itself finite type. Recall that $\mu loc^w(\mathbb{L})$ is by definition the subcategory of compact objects in $\mu loc(\mathbb{L})$, and that the subcategory $\mu loc^c(U) \subset \mu loc^w(U)$ of small microlocal sheaves can be identified with the full subcategory of exact functors $\mu loc^w(U) \to \Bbbk$-mod which take values in perfect complexes. Thus as a special case of [TVa, Thm. 0.2] we have the following.

Proposition 3.18. ([TVa, Thm. 0.2]) *Let $\mathbb{L}$ be the Lagrangian skeleton attached to a curve configuration $\mathcal{C}$. There exists a derived moduli stack $\mathbb{R}\mathcal{M}(\mathbb{L})$ whose $\Bbbk$-points parametrize objects of $\mu loc^c(\mathbb{L})$, and which is locally geometric and locally of finite presentation.*

The infinitesimal study of derived moduli spaces is generally more accessible than that of ordinary moduli spaces. For example, writing $\mathbb{R}loc(\mathcal{L})$ for the derived moduli stack of local systems (of perfect $\Bbbk$-modules) on $\mathcal{L}$, we have the following consequence of Proposition 3.15.

Proposition 3.19. *The inclusion $loc(\mathcal{L}) \hookrightarrow \mu loc(\mathbb{L})$ associated to the embedding $\mathcal{L} \subset \mathbb{L}$ induces an open inclusion $\mathbb{R}loc(\mathcal{L}) \hookrightarrow \mathbb{R}\mathcal{M}(\mathbb{L})$.*

Proof. This follows formally from the fact that $loc(\mathcal{L}) \hookrightarrow \mu loc(\mathbb{L})$ is a fully faithful inclusion of dg categories. Indeed, it follows from this that the morphism is injective on points, and since the tangent complexes to the moduli spaces are given by self-ext algebras [TVa, Thm 0.2], it follows that the map is étale. ☐

With this in mind we will primarily restrict our attention to the following subspace of $\mathbb{R}\mathcal{M}(\mathbb{L})$, where $\mathbb{R}Loc_n(\mathcal{L}) \subset \mathbb{R}loc(\mathcal{L})$ denotes the substack of local systems whose stalks have cohomology concentrated in degree zero and of rank n.

Definition 3.20. The moduli space $\mathbb{R}\mathcal{M}_n(\mathbb{L})$ of rank n microlocal sheaves on $\mathbb{L}$ is the component of $\mathbb{R}\mathcal{M}(\mathbb{L})$ containing the image of $\mathbb{R}Loc_n(\mathcal{L})$.

As noted before, our present interest in the moduli space $\mathbb{R}\mathcal{M}(\mathbb{L})$ is largely in that organizes subspaces of the form $\mathbb{R}Loc_n(\mathcal{L}')$ for Lagrangians $\mathcal{L}'$ obtained by iterated surgery on $\mathcal{L}$. While the higher and derived structures on $\mathbb{R}\mathcal{M}(\mathbb{L})$ are important for many purposes, the question of distinguishing these subspaces can studied at the level of truncations without losing any information.

Proposition 3.21. *We let $\mathcal{M}_n(\mathbb{L})$ denote the substack of the truncation $t_0\mathbb{R}\mathcal{M}_n(\mathbb{L})$ parametrizing objects without negative self-extensions. It is an Artin stack in the classical sense, and the functor $loc(\mathcal{L}) \hookrightarrow \mu loc(\mathbb{L})$ induces an open map $Loc_n(\mathcal{L}) \hookrightarrow \mathcal{M}_n(\mathbb{L})$ from the classical moduli stack of rank n local systems on $\mathcal{L}$.*

Proof. That the locus in $t_0\mathbb{R}\mathcal{M}_n(\mathbb{L})$ without negative self-extensions is an Artin 1-stack follows from [TVa, Sec. 3.4]. That we have an open map of ordinary stacks follows from the fact that local systems on $\mathcal{L}$ do not have negative self-extenstions, and that the truncation of an étale map is étale [TV2, Sec. 2.2.4]. □

4. MUTATION FUNCTORS

Given a skeletal surgery $\mathbb{L} \rightsquigarrow \mathbb{L}'$ at a disk D_k, we now construct an equivalence of categories $Mut_k : \mu loc(\mathbb{L}) \xrightarrow{\sim} \mu loc(\mathbb{L}')$. Just as the surgery $\mathbb{L} \rightsquigarrow \mathbb{L}'$ is local to the disk D_k, so too is the mutation functor Mut_k.

The mutation functor should be the microlocal counterpart of the equivalence $Fuk(W) \cong Fuk(W')$ associated to the symplectomorphism $W \cong W'$ of Theorem 2.22, under the expected equivalence $\mu loc(\mathbb{L}) \cong Fuk(W)$. We compute how $loc(\mathcal{L}) \subset \mu loc(\mathbb{L})$ transforms under surgery; our calculation is modeled on a computation in [BK] of the effect of the Fourier transform on perverse sheaves. The result will be interpreted in Section 5 as showing this transformation to be a cluster $\mathcal{X}$-transformation or a nonabelian version thereof.

4.1. Sheaf category equivalences from contact isotopy.

The basic tool in our construction of equivalences between categories of microlocal sheaves is the quantization theorem of Guillermou, Kashiwara, and Schapira [GKS]. Informally, this asserts that a contact isotopy of $T^\infty M$ induces an autoequivalence of the constructible sheaf category which respects the action of the contact isotopy on microsupport.

Recall that a contact isotopy is a smooth family $\{\varphi_t\}_{t\in I}$ of contactomorphisms of $T^\infty M$ such that φ_0 is the identity. Here we have written I for the unit interval, and below we write M_t for $M \times \{t\}$ and π_I for the projection $T^*(M \times I) \twoheadrightarrow (T^*M) \times I$.

From a Legendrian subset $\Lambda_0 \subset T^\infty M$ the contact isotopy $\{\varphi_t\}_{t\in I}$ produces a family of Legendrian subsets $\{\Lambda_t := \varphi_t(\Lambda)\}_{t\in I}$. This family is equivalent to the data of an induced Legendrian subset $\Lambda_I \subset T^\infty(M \times I)$. Namely, the cone $\mathbb{R}_+\Lambda_I$ over Λ_I is the unique conical Lagrangian subset such that the intersection $T^*M_t \cap \pi_I(\mathbb{R}_+\Lambda_I)$ is equal to the cone over Λ_t (see [GKS, Sec. A.2] — though we allow Λ to be singular the discussion applies equally to its flow under a contact isotopy).

We then have the following result of [GKS], as reformulated in [Zho, Thm. 3.1].

Theorem 4.1. ([GKS]) *Let $\Lambda_I \subset T^\infty(M \times I)$ be the Legendrian associated to the flow of a Legendrian subset $\Lambda_0 \subset T^\infty M$ under a contact isotopy $\{\varphi_t\}_{t\in I}$. Then for any $t \in I$ the restriction functor $i^*_{M_t} : sh_{\Lambda_I}(M \times I) \to sh_{\Lambda_t}(M)$ is an equivalence. In particular, there is a canonical equivalence $i^*_{M_1} \circ (i^*_{M_0})^{-1} : sh_{\Lambda_0}(M) \xrightarrow{\sim} sh_{\Lambda_1}(M)$.*

Remark 4.2. This is stated in [GKS, Zho] for the ordinary constructible sheaf category $sh^{sm}_{\Lambda_t}(M) \subset sh_{\Lambda_t}(M)$, but is known to hold without size restrictions on stalks [JT, Sec. 2.11].

Recall the singular Legendrians $\perp\!\!\!\perp^\infty, \top\!\!\!\top^\infty \subset T^\infty\mathbb{R}^2$ of Definitions 2.17, 2.18; the unions of the cones over each together with the zero section were the conical Lagrangians $\perp\!\!\!\perp, \top\!\!\!\top \subset T^*\mathbb{R}^2$. By definition $\top\!\!\!\top^\infty$ was obtained from $\perp\!\!\!\perp^\infty$ by applying a contact isotopy $\{\varphi_t\}_{t\in I}$. We thus have the following special case of Theorem 4.1.

Proposition 4.3. *The isotopy $\{\varphi_t\}_{t\in I}$ of Definition 2.18 induces an equivalence $sh_{\perp\!\!\!\perp}(\mathbb{R}^2) \xrightarrow{\sim} sh_{\top\!\!\!\top}(\mathbb{R}^2)$.*

We will also need the following locality property of the preceding equivalence. Recall that $\partial\perp\!\!\!\perp \subset \perp\!\!\!\perp$ denotes a neighborhood of the asymptotic boundary of $\perp\!\!\!\perp$, for example the intersection of $\perp\!\!\!\perp$ with the complement of any sufficiently large ball in $T^*\mathbb{R}^2$. The subset $\partial\top\!\!\!\top \subset \top\!\!\!\top$ is defined similarly, and without loss of generality we may assume that away from the zero section $\partial\top\!\!\!\top = \varphi_1(\partial\perp\!\!\!\perp)$ (where φ_1 is extended to a homogeneous symplectomorphism of $(T^*\mathbb{R}^2) \smallsetminus \mathbb{R}^2$ in the canonical way). Also recall that microlocalization provides canonical functors $sh_{\perp\!\!\!\perp}(\mathbb{R}^2) \cong \mu loc(\perp\!\!\!\perp) \to \mu loc(\partial\perp\!\!\!\perp)$, $sh_{\top\!\!\!\top}(\mathbb{R}^2) \cong \mu loc(\top\!\!\!\top) \to \mu loc(\partial\top\!\!\!\top)$.

Proposition 4.4. *The equivalence of Proposition 4.3 extends to a diagram of functors*

$$\begin{tikzcd} \mu loc(\perp\!\!\!\perp) \arrow[r, "\cong"] \arrow[d] & \mu loc(\top\!\!\!\top) \arrow[d] \\ \mu loc(\partial\perp\!\!\!\perp) \arrow[r, "\cong"] & \mu loc(\partial\top\!\!\!\top) \end{tikzcd}$$

which commutes via a canonical isomorphism.

Proof. Consider the conical Lagrangian subset $\perp\!\!\!\perp_I \subset T^*(\mathbb{R}^2 \times I)$ associated as above to the flow of $\perp\!\!\!\perp^\infty$ under $\{\varphi_t\}_{t\in I}$. In particular, $T^*\mathbb{R}^2_0 \cap \pi_I(\perp\!\!\!\perp_I)$ coincides with $\perp\!\!\!\perp$ and $T^*\mathbb{R}^2_1 \cap \pi_I(\perp\!\!\!\perp_I)$ coincides with $\top\!\!\!\top$. We let $\partial\perp\!\!\!\perp_I$ denote a neighborhood of the boundary of $\perp\!\!\!\perp_I$ in the same sense as $\partial\perp\!\!\!\perp$, $\partial\top\!\!\!\top$. In particular, we can assume that away from the zero section $T^*\mathbb{R}^2_t \cap \pi_I(\partial\perp\!\!\!\perp_I)$ coincides with $\varphi_t(\partial\perp\!\!\!\perp)$.

We have the following diagram of restriction functors between microlocal sheaves on various subsets of $\partial\perp\!\!\!\perp_I$.

$$\begin{tikzcd} \mu loc(\perp\!\!\!\perp) \arrow[d] & \mu loc(\perp\!\!\!\perp_I) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d] & \mu loc(\top\!\!\!\top) \arrow[d] \\ \mu loc(\partial\perp\!\!\!\perp) & \mu loc(\partial\perp\!\!\!\perp_I) \arrow[l, "\cong"'] \arrow[r, "\cong"] & \mu loc(\partial\top\!\!\!\top) \end{tikzcd} \tag{3}$$

The top functors are equivalences by Theorem 4.1. The bottom functors are equivalences since by assumption $\partial\amalg$ is far enough from the unit disk that its topology and local symplectic geometry are unchanged during the isotopy $\{\varphi_t\}_{t\in I}$. Note that each square commutes via a canonical isomorphism; this is part of the statement that the assignment $U \mapsto \mu loc(U)$ is a sheaf of dg categories on $\amalg_I$. In particular, we can replace the horizontal functors in the left-hand square with their inverses to obtain another canonically commuting square whose horizontal functors are directed to the right. Combining this square with the right-hand one now yields a canonically commuting square as in the statement of the Proposition. □

Remark 4.5. Note however that the above isomorphism *does not* respect the Morse trivialization procedure of Lemma 3.6, even when it is defined (i.e., when C_k is the only curve). This is because at one point during the isotopy — when C_k is collapsed to a point — the front projection of the Legendrian will fail to be an immersion. In fact the result is that there is a cohomological shift in degree with respect to this trivialization.

4.2. **Construction of the mutation functor.** We now use the local isomorphism $\mu loc(\amalg) \cong \mu loc(\Pi)$ to build an equivalence $Mut_k : \mu loc(\mathbb{L}) \xrightarrow{\sim} \mu loc(\mathbb{L}')$. We fix once more the data of a curve configuration $\mathcal{C}$ on a surface $\mathcal{L}$, with $\mathbb{L} \subset W$ the associated Lagrangian skeleton and 4-manifold, let $\mathcal{C}'$, $\mathcal{L}'$, and $\mathbb{L}'$ denote their counterparts under mutation at a fixed embedded curve $C_k \in \mathcal{C}$.

Keeping the notation of Section 2, we write $\mathcal{L} = \mathcal{L}_{(k)} \cup \mathcal{L}^{(k)}$ for an open cover consisting of a neighborhood $\mathcal{L}_{(k)}$ of C_k and a complementary open set. Similarly, $\mathbb{L} = \mathbb{L}_{(k)} \cup \mathbb{L}^{(k)}$. We also somewhat abusively write $\partial\mathbb{L}_{(k)}$ for the intersection $\mathbb{L}_{(k)} \cap \mathbb{L}^{(k)} \subset \mathbb{L}$, which is an open subset of both $\mathbb{L}_{(k)}$ and $\mathbb{L}^{(k)}$. This retracts onto the standard 1-dimensional interpretation of $\partial\mathbb{L}_{(k)}$, which would be two circles (the inside and outside translates of C_k on $\mathcal{L}$) connected by several segments (from other disks and half-disks whose boundaries meet C_k).

Recall however $\mathcal{L}_{(k)}{}'$ and $\mathcal{L}^{(k)}{}'$ *do not* mean a neighborhood of the curve C_k' and its complement, but instead for the parts of $\mathcal{L}'$ which are the images of $\mathcal{L}_{(k)}$ and $\mathcal{L}^{(k)}$ under the fixed identification $\mathcal{L} \cong \mathcal{L}'$ used in defining the mutation of curve configurations. In particular, recall that the restricted curve configuration $\mathcal{C}_{(k)}{}'$ will generally contain intersections amongst the curves ending on the boundary of $\mathcal{L}_{(k)}{}'$, whereas a neighborhood of C_k' would not.

The canonical homeomorphism $\mathbb{L}^{(k)} \cong \mathbb{L}^{(k)}{}'$ is a priori compatible with the conical local models used to define the KS sheaves of each side. It follows that we have a canonically commutative diagram of functors

$$\begin{tikzcd}
\mu loc(\mathbb{L}^{(k)}) \arrow[r, "\cong"] \arrow[d] & \mu loc(\mathbb{L}^{(k)}{}') \arrow[d] \\
\mu loc(\partial\mathbb{L}_{(k)}) \arrow[r, "\cong"] & \mu loc(\partial\mathbb{L}_{(k)}{}').
\end{tikzcd}$$

To construct Mut_k it suffices to construct an equivalence $\mu loc(\mathbb{L}_{(k)}) \cong \mu loc(\mathbb{L}_{(k)}{}')$ and specify an isomorphism making the corresponding square of boundary restriction functors commute.

In Section 4.1 we constructed a corresponding equivalence for the conical models $\perp\!\!\!\perp$ and $\top\!\!\!\top$. It remains to construct equivalences $\mu loc(\mathbb{L}_{(k)}) \xrightarrow{\sim} \mu loc(\perp\!\!\!\perp)$. One should expect such an equivalence, since one expects both these categories to model the Fukaya categories of B^4 whose objects have the same prescribed asymptotics in S^3. On the sheaf side, we construct the equivalence just by computing directly.

Theorem 4.6. *There is an equivalence $\mu loc(\mathbb{L}_{(k)}) \xrightarrow{\sim} \mu loc(\mathbb{L}_{(k)}{}')$ together with a canonical isomorphism making the following diagram commute.*

$$\begin{tikzcd} \mu loc(\mathbb{L}_{(k)}) \arrow[r, "\cong"] \arrow[d] & \mu loc(\mathbb{L}_{(k)}{}') \arrow[d] \\ \mu loc(\partial\mathbb{L}_{(k)}) \arrow[r, "\cong"] & \mu loc(\partial\mathbb{L}_{(k)}{}') \end{tikzcd} \tag{4}$$

This induces a global equivalence $Mut_k : \mu loc(\mathbb{L}) \xrightarrow{\sim} \mu loc(\mathbb{L}')$ that is intertwined with this local equivalence by the restriction functors $\mu loc(\mathbb{L}) \to \mu loc(\mathbb{L}_{(k)})$, $\mu loc(\mathbb{L}') \to \mu loc(\mathbb{L}_{(k)}{}')$.

Proof. We build the desired local mutation functor together with the commutative square (4) by building the following larger diagram which passes through the conical models $\perp\!\!\!\perp$, $\top\!\!\!\top$.

$$\begin{tikzcd} \mu loc(\mathbb{L}_{(k)}) \arrow[r, "\cong"] \arrow[d] & \mu loc(\perp\!\!\!\perp) \arrow[r, "\cong"] \arrow[d] & \mu loc(\top\!\!\!\top) \arrow[r, "\cong"] \arrow[d] & \mu loc(\mathbb{L}_{(k)}{}') \arrow[d] \\ \mu loc(\partial\mathbb{L}_{(k)}) \arrow[r, "\cong"] & \mu loc(\partial\perp\!\!\!\perp) \arrow[r, "\cong"] & \mu loc(\partial\top\!\!\!\top) \arrow[r, "\cong"] & \mu loc(\partial\mathbb{L}_{(k)}{}') \end{tikzcd} \tag{5}$$

While all the sheaves of categories have been denoted by the same symbol μloc, the above identifications are by no means tautological: the categories in each column are defined independently of the others using a priori different local conical Lagrangian models. The functors and commuting isomorphism appearing in the central square are those treated in Proposition 4.4, so it remains to discuss the outside squares.

To construct the leftmost commutative square, let $D \subset \mathbb{R}^2$ be the open unit disk and D° an open disk whose closure is contained in D. Recall that the front projection of $\perp\!\!\!\perp^\infty$ contains the boundary of D but does not meet D itself. By the sheaf axiom we have

$$\mu loc(\perp\!\!\!\perp) = \mu loc(\perp\!\!\!\perp \smallsetminus D^\circ) \times_{loc(D\smallsetminus D^\circ)} loc(D).$$

On the other hand, by construction we have

$$\mu loc(\mathbb{L}_{(k)}) = \mu loc(T^+_{\mathcal{C}_{(k)}}\mathcal{L}_{(k)}) \times_{loc(D_k \smallsetminus D^\circ_k)} loc(D_k),$$

where D°_k is the core of the handle attachment above C_k.

We can thus specify an equivalence $\mu loc(\perp\!\!\!\perp) \cong \mu loc(\mathbb{L}_{(k)})$ by providing vertical equivalences in the following diagram, together with natural isomorphisms of functors making the left and right

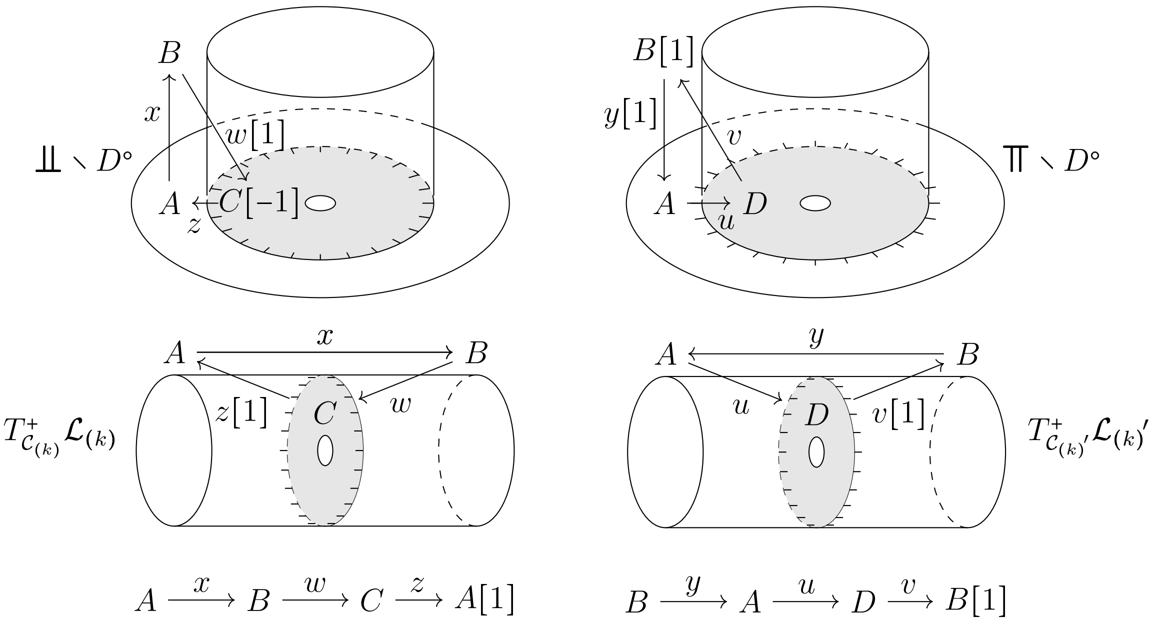

FIGURE 13. On the left, the equivalence $\mu loc(T^+_{\mathcal{C}_{(k)}}\mathcal{L}_{(k)}) \xrightarrow{\sim} \mu loc(\amalg \smallsetminus D^\circ)$, on the right, its counterpart for $\top\!\!\top$. Each category is canonically equivalent to $\Bbbk A_2$-mod $\times$ $loc(S^1)$: the $\Bbbk$-modules at either A_2 vertex are stalks of a sheaf on the zero section, the cone over the A_2 arrow is a microlocal stalk away from the zero section (we omit the S^1-monodromy from the picture). The autoequivalence of $\Bbbk A_2$-mod $\times$ $loc(S^1)$ induced from $\mu loc(T^+_{\mathcal{C}_{(k)}}\mathcal{L}_{(k)}) \xrightarrow{\sim} \mu loc(\amalg \smallsetminus D^\circ)$ is a reflection functor in the $\Bbbk A_2$-mod factor, i.e. it rotates the given exact triangle.

squares commute.

$$\begin{tikzcd}
\mu loc(\amalg \smallsetminus D^\circ) \arrow[r] \arrow[d, "\cong"'] & loc(D \smallsetminus D^\circ) \arrow[d, "\cong"'] & loc(D) \arrow[l] \arrow[d, "\cong"'] \\
\mu loc(T^+_{\mathcal{C}_{(k)}}\mathcal{L}_{(k)}) \arrow[r] & loc(D_k \smallsetminus D_k^\circ) & loc(D_k) \arrow[l]
\end{tikzcd} \tag{6}$$

Recall that we fixed before a homeomorphism $\amalg \cong \mathbb{L}_{(k)}$. The equivalences on the middle and right, and the corresponding commuting isomorphism, are the canonical ones from the induced homeomorphism of D_k with the unit disk.

Having reduced to the setting of cotangent bundles, we can now apply Lemma 3.6 to $T^+_{\mathcal{C}_{(k)}}\mathcal{L}_{(k)}$ and $\amalg \smallsetminus D^\circ$ to obtain natural trivializations of the stalks of their KS sheaves at any smooth point (i.e. these stalk categories are canonically equivalent to $\Bbbk$-mod, even for smooth points away from the zero section). Moreover, the local geometry of $T^+_{\mathcal{C}_{(k)}}\mathcal{L}_{(k)}$ and $\amalg \smallsetminus D^\circ$ is described by the A_2 and A_3 arboreal singularities discussed in Examples 3.2 and 3.3 — in particular, an object of $\mu loc(T^+_{\mathcal{C}_{(k)}}\mathcal{L}_{(k)})$ or $\mu loc(\amalg \smallsetminus D^\circ)$ is determined by its stalks at smooth points together with natural maps among them as in these local models.

By inspection one sees there is a canonical equivalence $\mu loc(T^+_{\mathcal{C}_{(k)}}\mathcal{L}_{(k)}) \xrightarrow{\sim} \mu loc(\perp\!\!\!\perp \smallsetminus D^\circ)$ which

(1) identifies up to a degree shift the natural trivializations of stalks at every smooth point,
(2) intertwines up to a degree shift the natural maps between these, and
(3) which is compatible with the cohomological degrees of the natural trivializations at smooth points of $\perp\!\!\!\perp$ far out in $\mathbb{R}^2$ and smooth points of $\mathbb{L}_{(k)}$ at the corresponding end of $\mathcal{L}_{(k)}$.

This is illustrated explicitly in Figure 13 for the case where C_k meets no other curves in $\mathcal{C}$, and the general case can be described equally explicitly. Locally we can cover $T^+_{\mathcal{C}_{(k)}}\mathcal{L}_{(k)} \cong \perp\!\!\!\perp \smallsetminus D^\circ$ with open subsets U such that we have a natural trivialization $\mu loc(U) \cong \Bbbk A_n$-mod for some n, and in these trivializations the equivalence acts by a reflection functor if U meets $C_k \cong \partial D$ and by some cohomological shift otherwise (the overall normalization of these shifts being determined by condition (3)).

Note that, as a subset of $\mathbb{L}_{(k)}$, $\partial\mathbb{L}_{(k)}$ is contained in $T^+_{\mathcal{C}_{(k)}}\mathcal{L}_{(k)}$, similarly for $\partial\perp\!\!\!\perp$ and $\perp\!\!\!\perp \smallsetminus D^\circ$. Moreover, constructing the equivalence $\mu loc(T^+_{\mathcal{C}_{(k)}}\mathcal{L}_{(k)}) \to \mu loc(\perp\!\!\!\perp \smallsetminus D^\circ)$ as we have in terms of local trivializations and local maps among them implicitly specifies a commuting isomorphism for the square

$$\begin{tikzcd} \mu loc(\perp\!\!\!\perp \smallsetminus D^\circ) \arrow[r] \arrow[d, "\cong"'] & \mu loc(\partial\perp\!\!\!\perp) \times loc(D \smallsetminus D^\circ) \arrow[d, "\cong"'] \\ \mu loc(T^+_{\mathcal{C}_{(k)}}\mathcal{L}_{(k)}) \arrow[r] & \mu loc(\partial\mathbb{L}_{(k)}) \times loc(D_k \smallsetminus D_k^\circ) \end{tikzcd} \tag{7}$$

since the boundary sheaf categories can be locally trivialized in the same fashion. Thus we obtain on one hand the isomorphism needed to make the left-hand square in (6) commute, and on the other the isomorphism needed to make the leftmost square in (5) commute when its top functor $\mu loc(\perp\!\!\!\perp) \cong \mu loc(\mathbb{L}_{(k)})$ is the one we have just constructed.

The functors and commuting isomorphism in the rightmost square of (5) are now obtained similarly. Note however that the Morse trivialization of $\mu loc(\top\!\!\!\top^\infty)$ differs from the corresponding trivializations of $\mu loc(\perp\!\!\!\perp^\infty)$ and $\partial\mathbb{L}_{(k)}'$ by a cohomological degree shift (as is visible on the right side of Figure 13).

To construct a global equivalence $\mu loc(\mathbb{L}) \xrightarrow{\sim} \mu loc(\mathbb{L}')$ it suffices to construct vertical equivalences in the diagram below, together with isomorphisms making the left and right squares commute.

$$\begin{tikzcd} \mu loc(\mathbb{L}^{(k)}) \arrow[r] \arrow[d] & \mu loc(\partial\mathbb{L}_{(k)}) \arrow[d] & \mu loc(\mathbb{L}_{(k)}) \arrow[l] \arrow[d] \\ \mu loc(\mathbb{L}^{(k)\prime}) \arrow[r] & \mu loc(\partial\mathbb{L}_{(k)}') & \mu loc(\mathbb{L}_{(k)}') \arrow[l] \end{tikzcd} \tag{8}$$

We have just finished constructing the commutative square on the left. But, as discussed before the proof, the right-hand square canonically commutes by construction. $\square$

Remark 4.7. The choice of $\sigma : \mathcal{C} \to \mathbb{Z}/2\mathbb{Z}$ made in defining μloc is immaterial in the preceding theorem. While different choices may change the global category $\mu loc(\mathbb{L})$, the restriction of μloc to $\mathbb{L}_{(k)}$ is always the same by Proposition 3.11, since $\mathbb{L}_{(k)}$ is contractible.

4.3. **A disk glued to a cylinder.** In this section, we study in detail the category of microlocal sheaves on the skeleton $\mathbb{L}_0$ obtained from Definition 1.3 by taking $\mathcal{L}$ to be an annulus and $\mathcal{C}$ to consist of a single noncontractible curve C.

Let A_2 be the quiver $\bullet \to \bullet$. We write

$$\begin{aligned} \mathfrak{c} : \mathbb{k}A_2\text{-mod} &\to \mathbb{k}\text{-mod} \\ A \to B &\mapsto \mathrm{Cone}(A \to B) \end{aligned}$$

for the functor that maps a representation to the cone over its defining map.

Recall that an isomorphism in a (strict) dg category is, by definition, a closed degree zero map, invertible in the homotopy category. Note we can describe $loc(S^1)$ as pairs $X \in \mathbb{k}$-mod and an isomorphism $m : X \to X$. In general, we denote a pair consisting of an object X and an isomorphism $m : X \to X$ as $X \circlearrowleft m$.

Proposition 4.8. *Up to equivalence, the objects of* $\mu loc(\mathbb{L}_0)$ *are tuples* $\{(X, m, y)\}$*, where*

(1) $X \in A_2$*-mod,*
(2) $m : X \to X$ *is an automorphism, and*
(3) $y : \mathfrak{c}(X) \to \mathfrak{c}(X)$ *is a degree -1 map;* $dy = 1 - (-1)^k \mathfrak{c}(m)$ *on degree* k *elements of* $\mathfrak{c}(X)$*.*

Proof. The starting point for the calculation is the defining decomposition $\mathbb{L}_0 = T^+_C\mathcal{L} \cup D$ into local conical models. By definition $\mu loc(\mathbb{L}_0)$ is the (homotopy) fiber product

$$\begin{tikzcd} \mu loc(\mathbb{L}_0) \arrow[r] \arrow[d] & loc(D) \arrow[d] \\ \mu loc(T^+_C\mathcal{L}) \arrow[r] & \mu loc(T^+_C\mathcal{L} \cap D) \end{tikzcd} \tag{9}$$

in $\mathrm{DGCat}_{\mathrm{cont}}$.

Let us first describe explicitly the terms in this fiber product. The Morse trivialization of Lemma 3.6 identifies $\mu loc(T^+_C\mathcal{L} \cap D)$ with $loc(T^+_C\mathcal{L} \cap D) \cong loc(S^1)$. As $T^+_C\mathcal{L} \cap D$ is a collar neighborhood of the boundary of D, the right-hand functor is the restriction of a local system on the disk to its boundary, or equivalently the inclusion of $\mathbb{k}$-mod into $loc(S^1)$ as trivial local systems. The space $T^+_C\mathcal{L}$ is the product of a circle and an A_2 arboreal singularity [N3], and one easily sees that an object of $\mu loc(T^+_C\mathcal{L})$ is an object $X \in A_2$-mod together with an automorphism m (cf. Example 3.2). Note that $\mathfrak{c} : A_2\text{-mod} \to \mathbb{k}\text{-mod}$ extends to a functor $\mu loc(T^+_C\mathcal{L}) \to loc(S^1)$ in a straightforward way.

Recall that $\mathrm{DGCat}_{\mathrm{cont}} \hookrightarrow \mathrm{DGCat}^{\mathrm{non-cocompl}}$ preserves limits (e.g. [GR, Sec. I.1.5.3.3, I.1.8.1.3]), and that $\mathrm{DGCat}^{\mathrm{non-cocompl}}$ can be realized by inverting weak equivalences in the (Dwyer-Kan) model category of strict dg categories [Hau, Ex. 5.11]. Thus we can compute the homotopy fiber product (9) by replacing the bottom right corner with a path space and taking the corresponding strict limit. With respect to the Dwyer-Kan model structure, the path space $\mathcal{P}(T)$ of a dg category T is the dg category whose objects are isomorphisms in T, and whose morphisms are pairs of maps intertwining the source and target isomorphisms up to a chosen homotopy [Tab, Lem. 4.1]. More precisely, on underlying vector spaces we have

$$\mathrm{Hom}_{\mathcal{P}(T)}(X \xrightarrow{f} Y, W \xrightarrow{g} Z) = \mathrm{Hom}_T(X,W) \oplus \mathrm{Hom}_T(Y,Z) \oplus \mathrm{Hom}_T(X,Z)[-1],$$

so a degree k map from f to g is given by a triple $m_1 \in \mathrm{Hom}^k_T(X,W)$, $m_2 \in \mathrm{Hom}^k_T(Y,Z)$, $h \in \mathrm{Hom}^{k-1}_T(X,Z)$, and the differential $d_{\mathcal{P}(T)}$ is defined by

$$d_{\mathcal{P}(T)} \begin{pmatrix} m_1 & 0 \\ h & m_2 \end{pmatrix} = \begin{pmatrix} d_T m_1 & 0 \\ d_T h + g \circ m_1 - (-1)^k m_2 \circ f & d_T m_2 \end{pmatrix}.$$

Thus we replace $loc(S^1)$ with its path space, and compute $\mu loc(\mathbb{L}_0)$ as the ordinary limit of the diagram

$$\begin{tikzcd} \mu loc(T^+_C\mathcal{L}) \arrow[dr] & & \mathcal{P}(loc(S^1)) \arrow[dl] \arrow[dr] & & loc(D) \arrow[dl] \\ & loc(S^1) & & loc(S^1) & \end{tikzcd} \tag{10}$$

Thus an object of $\mu loc(\mathbb{L}_0)$ is an object of $\mu loc(T^+_C\mathcal{L})$, which we represent as a pair $X \circlearrowleft m$ with $X \in A_2$-mod, an object Y of $loc(D) \cong \mathbb{k}$-mod, and an isomorphism

$$y : \mathfrak{c}(X \circlearrowleft m) \to Y \circlearrowleft 1 \in loc(S^1).$$

The morphisms in $\mu loc(\mathbb{L}_0)$ can similarly be computed from the above diagram and the description of morphisms in $\mathcal{P}(loc(S^1))$.

We can simplify this description of $\mu loc(\mathbb{L}_0)$ further. An isomorphism $f : (M \circlearrowleft m) \to (N \circlearrowleft n)$ in $loc(S^1)$ can be expressed in terms of $\mathbb{k}$-module maps: a degree zero morphism $f_0 : M \to N$ and a degree -1 morphism $f_{-1} : M \to N$ such that $df_{-1} = f_0 m \pm n f_0$ (the signs depending on degrees in the usual way).

In particular, the above map $y : \mathfrak{c}(X \circlearrowleft m) \to Y \circlearrowleft 1$ decomposes as a pair y_0, y_{-1} of degree zero and degree -1 maps from $\mathfrak{c}(X)$ to Y in $\mathbb{k}$-mod. The above stated condition that y is an isomorphism amounts to $dy_{-1} = y_0 \circ (1 - (-1)^k \mathfrak{c}(m))$ on the degree k part of $\mathfrak{c}(X)$. But we may consider a full subcategory where Y is $\mathfrak{c}(X)$ itself and y_0 is the identity. The inclusion of this subcategory is easily seen to be essentially surjective, leading to the description given in the statement of the Proposition. □

Recall that we write $\mu Loc(\mathbb{L})$ for the full subcategory of $\mu loc(\mathbb{L})$ whose objects, when restricted to $sh_{T^+_C\mathcal{L}}(\mathcal{L})$, have cohomology sheaves concentrated in degree zero.

Proposition 4.9. *Let Q_0 denote the quiver pictured in Figure 14. The category $\mu Loc(\mathbb{L}_0)$ is equivalent to the full subcategory of $\mathbb{k}Q_0$-mod such that 1) $m_B = 1 - xy$ and $m_A = 1 - yx$, 2) m_A and m_B are invertible, 3) x and y intertwine m_A and m_B, and 4) A and B have cohomology concentrated in degree zero.*

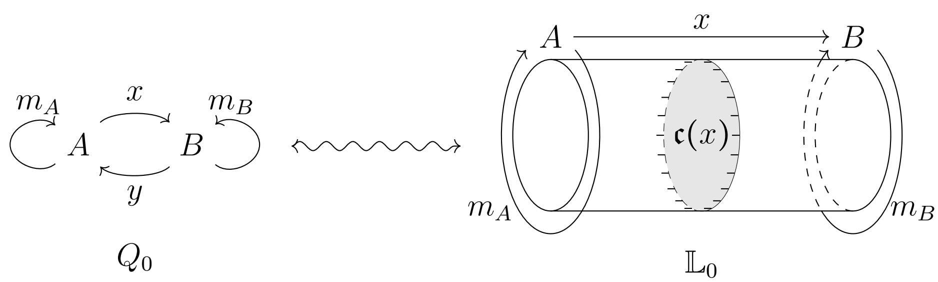

FIGURE 14. The correspondence between representations of Q_0 and objects of $\mu Loc(\mathbb{L}_0)$. A and B are identified with the stalks at either end of $\mathbb{L}_0$, and the cone $\mathfrak{c}(x)$ is the stalk on the disk. The map x is a generazation map in $sh_{T^+_{\mathcal{C}}\mathcal{L}}(\mathcal{L})$ while y trivializes the monodromy on $\mathfrak{c}(x)$.

Proof. The desired functor from this subcategory of $\Bbbk Q_0$-mod to $\mu loc(\mathbb{L}_0)$ takes the data pictured in Figure 14 to the $\Bbbk A_2$-module X given by $A \xrightarrow{x} B$ and the automorphism given by m_A and m_B. Given that A and B have cohomology concentrated in degree zero, a degree -1 map y as in Proposition 4.8 on the cone over x is the same data as the degree zero map from $B \to A$ pictured in Figure 14. Moreover, the identity required by Proposition 4.8 follows from the relations we have imposed on our $\Bbbk Q_0$-modules. □

Remark 4.10. This algebraic description has historically been associated to something slightly different, namely perverse sheaves on $\mathbb{R}^2$ constructible with respect to the stratification $0, \mathbb{R}^2 \smallsetminus 0$ [Bei, GGM]. The relation to the present context is that such constructible sheaves are the same as those microsupported in the union $+$ of the zero section and the cotangent fiber at zero; this conical Lagrangian appears at the midpoint of the oisotopy taking $\perp\!\!\!\perp$ to $\top\!\!\!\top$ (cf. Definition 2.18), hence by [GKS] determines the same category of sheaves.

Example 4.11. The trivial local system on $\mathbb{L}_0$ is given by the representation

$$\begin{tikzcd} \Bbbk \arrow[loop left, "1"] \arrow[r, bend left, "1"] & \Bbbk \arrow[l, bend left, "0"] \arrow[loop right, "1"] \end{tikzcd}$$

We now assume $\Bbbk$ is an algebraically closed field, and recall some classical facts about the representation theory of this quiver.

Lemma 4.12. *The full subcategory of $\Bbbk Q_0$-mod on objects such that $m_A = 1 - xy$ and $m_B = 1 - yx$ and both of these are invertible is generated by the objects S^1_A, S^1_B, and P^m_A of Figure 15. Up to a shift they are the only simple objects in the category.*

Proof. We reproduce the classical argument. It suffices to show that any representation has one of these as a subrepresentation. If $yx \neq 0$, then choose an eigenspace V; we can do this since $\Bbbk$ is an algebraically closed field. Evidently V is also an eigenspace of $m_A = 1 - yx$. If $xV = 0$, then $(V, 0)$ determines a subrepresentation on which m_A acts as the identity; it is thus a sum of S^1_A's.

Otherwise observe $(xy)xV = x(yx)V \subset xV$. Choose an eigenspace $W \subset xV$ for xy. If $yW = 0$, then $(0, W)$ determines a subrepresentation on which m_B acts as the identity; it is a sum of S^1_B's.

If not, then observe $xyW \subset W$, hence (yW, W) is a subrepresentation, on which yx and xy, hence m_A and m_B, act as scalars. We have $y(1 - m_B)W = yxyW = (1 - m_A)yW$, so $m_A = m_B := m$. By assumption moreover xy and yx were not zero, so $m \neq 1$ and thus xy and yx are nonzero scalars, hence x and y, are invertible. Finally, by an appropriate choice of basis, we can demand that x is represented by an identity matrix, and y by the scalar $1 - m$. □

4.4. **Mutation of local systems.** We want to understand the comparison

$$loc(\mathcal{L}) \subset \mu loc(\mathbb{L}) \cong \mu loc(\mathbb{L}') \supset loc(\mathcal{L}')$$

induced by Mut_k. As we have argued, because the inclusion $loc(\mathcal{L}) \subset \mu loc(\mathbb{L})$ is fully faithful, it induces an open inclusion of moduli spaces. To understand an open inclusion, it suffices to understand the geometric points. With this in mind, we assume in this section that $\Bbbk$ is an algebraically closed field.

It suffices to work with a subcategory containing both $loc(\mathcal{L})$ and its image under mutation at k. Writing temporarily $\widetilde{\mathbb{L}}$ for the skeleton built from the curve collection consisting only of C_k, i.e. as a topological space $\widetilde{\mathbb{L}} = \mathcal{L} \cup D_k$, observe that, by the local nature of the construction of the mutation functor, the inclusion $\mu loc(\widetilde{\mathbb{L}}) \subset \mu loc(\mathbb{L})$ of Proposition 3.16 intertwines mutation functors. In particular, the image of $loc(\mathcal{L}) \subset \mu loc(\widetilde{\mathbb{L}}) \subset \mu loc(\mathbb{L})$ remains, after mutation, inside $\mu loc(\widetilde{\mathbb{L}})$. It thus suffices to compute inside $\mu loc(\widetilde{\mathbb{L}})$. In short, we may assume without loss of generality that the curve collection only had a single curve to begin with. Henceforth we do this, and hence cease to distinguish between $\widetilde{\mathbb{L}}$ and $\mathbb{L}$.

Note that our assumption that there was only a single curve amounts to $\mathbb{L}^{(k)} = \mathcal{L}^{(k)}$. By the sheaf axiom $\mu loc(\mathbb{L}) = \mu loc(\mathbb{L}_{(k)}) \overset{h}{\underset{\mu loc(\partial \mathbb{L}_{(k)})}{\times}} loc(\mathcal{L}^{(k)})$. Thus $Mut_k : \mu loc(\mathbb{L}) \cong \mu loc(\mathbb{L}')$ is determined by the following commutative diagram, as in Equation 8:

$$\begin{tikzcd}
\mu loc(\mathcal{L}^{(k)}) \arrow[r] \arrow[d] & \mu loc(\partial \mathbb{L}_{(k)}) \arrow[d] & \mu loc(\mathbb{L}_{(k)}) \arrow[l] \arrow[d] \\
\mu loc(\mathcal{L}^{(k)\prime}) \arrow[r] & \mu loc(\partial \mathbb{L}_{(k)}{}') & \mu loc(\mathbb{L}_{(k)}{}') \arrow[l]
\end{tikzcd} \tag{11}$$

Let $\mathbb{L}_0$ be the cylinder-with-disk of Section 4.3. Fix a homeomorphism $\mathbb{L}_0 \cong \mathcal{L}_{(k)} \cup D_k \subset \mathbb{L}$, compatible with the co-orientation of C_k. By construction this induces $\mu loc(\mathbb{L}_{(k)}) \cong \mu loc(\mathbb{L}_0)$. We fix a similar homeomorphism $\mathbb{L}_0 \cong \mathcal{L}_{(k)}{}' \cup D'_k \subset \mathbb{L}'$ and equivalence $\mu loc(\mathbb{L}_{(k)}{}') \cong \mu loc(\mathbb{L}_0)$. Note, however, that the homeomorphism $\mathcal{L}_{(k)} \cong \mathcal{L}_{(k)}{}'$ arising from the comparison $\mathcal{L}_{(k)} \subset \mathbb{L}_0 \supset \mathcal{L}_{(k)}{}'$ is *not* the one corresponding to the homeomorphism $\mathcal{L} \cong \mathcal{L}'$ which we used to define the mutated

$$S^1_A = \begin{tikzcd} \Bbbk \arrow[loop left, "1"] \arrow[r, bend left] & 0 \arrow[l, bend left] \arrow[loop right] \end{tikzcd} \qquad P^m_A = \begin{tikzcd} \Bbbk \arrow[loop left, "m"] \arrow[r, bend left, "1"] & \Bbbk \arrow[l, bend left, "1-m"] \arrow[loop right, "m"] \end{tikzcd} \qquad \Pi_A = \begin{tikzcd} \Bbbk\mathbb{Z} \arrow[loop left, "t"] \arrow[r, bend left, "1"] & \Bbbk\mathbb{Z} \arrow[l, bend left, "1-t"] \arrow[loop right, "t"] \end{tikzcd}$$

$$S^1_B = \begin{tikzcd} 0 \arrow[loop left] \arrow[r, bend left] & \Bbbk \arrow[l, bend left] \arrow[loop right, "1"] \end{tikzcd} \qquad P^m_B = \begin{tikzcd} \Bbbk \arrow[loop left, "m"] \arrow[r, bend left, "1-m"] & \Bbbk \arrow[l, bend left, "1"] \arrow[loop right, "m"] \end{tikzcd} \qquad \Pi_B = \begin{tikzcd} \Bbbk\mathbb{Z} \arrow[loop left, "t"] \arrow[r, bend left, "1-t"] & \Bbbk\mathbb{Z} \arrow[l, bend left, "1"] \arrow[loop right, "t"] \end{tikzcd}$$

FIGURE 15. Notation for various objects of $\mu loc(\mathbb{L}_0) \subset \Bbbk Q_0$-mod. The objects P^m_A and P^m_B are isomorphic for $m \neq 1$.

curve configuration. Instead, since the co-orientations of C_k, C'_k are opposite, the composition of these is an automorphism of $\mathcal{L}_{(k)}$ that exchanges the two components of $\partial\mathcal{L}_{(k)}$.

We now leverage our algebraic understanding of $\mu loc(\mathbb{L}_0)$ from Section 4.3. The following proposition is a close analogue of [BK, Prop 4.5], which derives the same formula for the action of the Fourier transform on perverse sheaves.

Proposition 4.13. *The composition $\mu loc(\mathbb{L}_0) \cong \mu loc(\mathbb{L}_{(k)}) \cong \mu loc({\mathbb{L}_{(k)}}') \cong \mu loc(\mathbb{L}_0)$ interchanges S^1_A and S^1_B, and interchanges $P^m_A \to P^{1/m}_B$.*

Proof. This equivalence must permute simples (characterized in Lemma 4.12); these in turn are characterized by $A \circlearrowleft m_A$ and $B \circlearrowleft m_B$. But these monodromies at the boundary are determined by the above discussion of how $\mathcal{L}_{(k)}$ and ${\mathcal{L}_{(k)}}'$ include into $\mathbb{L}_0$. □

Proposition 4.14. *Assume $\mathcal{E} \in loc(\mathcal{L}) \subset \mu loc(\mathbb{L})$. Then the mutation $\mathcal{E}' := Mut_k(\mathcal{E}) \in \mu loc(\mathbb{L}')$ will in fact lie in $loc(\mathcal{L}')$ if and only if the monodromy around C_k does have 1 as an eigenvalue.*

Proof. It's enough to check the local case, for $\mathcal{L}_{(k)}$. An object lies in $loc(\mathcal{L}_{(k)})$ if and only if it's in the subcategory spanned by the P^m_A. Under mutation, these are sent to the $P^{1/m}_B$, which are isomorphic to $P^{1/m}_A$, except when $m = 1$. P^1_B is not isomorphic to any of the P^m_A. □

Recall that we write $Loc(\mathcal{L})$ for local systems in the ordinary sense, i.e., those which have cohomology sheaves only in degree zero. To describe the action of Mut_k on $Loc(\mathcal{L}) \subset \mu loc(\mathbb{L})$, we write an object of $Loc(\mathcal{L})$ as a representation of the fundamental groupoid of $\mathcal{L}$ on a fixed $\Bbbk$-module. That is, it is a $\Bbbk$-module $\mathcal{E}$ together with an endomorphism $\mathcal{E}_\gamma$ for every path $\gamma : I \to \mathcal{L}$, compatible with concatenation of paths. We write $\mathcal{E}_{C_k}$ for the holonomy around C_k, oriented so that its co-orientation points rightward.

Theorem 4.15. *There exists an identification of $\mathcal{L}$ with $\mathcal{L}'$ such that, whenever $\mathcal{E}$ and $\mathcal{E}' = Mut_k(\mathcal{E})$ are both local systems,*

- *If γ does not meet C'_k then $\mathcal{E}'_\gamma = \mathcal{E}_\gamma$.*
- *Suppose γ crosses C'_k exactly once, and that its tangent at the crossing pairs negatively with the co-orientation of C'_k. Let $\gamma_{<C'_k}$ be the subpath that starts at $\gamma(0)$ and ends at the*

crossing, $\gamma_{>C'_k}$ the subpath which starts at the crossing and ends at $\gamma(1)$. Then

$$\mathcal{E}'_\gamma = \mathcal{E}_{\gamma_{>C'_k}}(\mathrm{Id} - \mathcal{E}_{C_k})\mathcal{E}_{\gamma_{<C'_k}}.$$

The identification $\mathcal{L} \cong \mathcal{L}'$ here differs from the identification used to define $\mathcal{C}'$ by some fixed universal number of Dehn twists about C_k.

Proof. The first bullet point is obvious: $\mathcal{L}$ and $\mathcal{L}'$ are canonically identified away from C_k.

Let γ be a path as in the second bullet point – without loss of generality we take $\gamma(0)$ and $\gamma(1)$ to lie in opposite components of $\mathcal{L}_{(k)}{}'$. Using the canonical homeomorphism $\mathcal{L}^{(k)} \cong \mathcal{L}^{(k)\prime}$ (which gave the isomorphism making the left square in Equation 11 commute), we obtain an isomorphism between the stalks of $\mathcal{E}$ and $\mathcal{E}'$ at these points.

In Proposition 4.13 we determined $\mathcal{E}'|_{\mathbb{L}_{(k)}}$ up to isomorphism. If $\mathcal{E}|_{\partial\mathbb{L}_{(k)}} \cong P^m_A$, then $\mathcal{E}'_\gamma = a_\mathcal{E}\mathcal{E}_\gamma$ for some $a_\mathcal{E} \in \mathbb{k}^*$ since both are isomorphisms of 1-dimensional spaces. By functoriality, $a_\mathcal{E}$ only depends on $\mathcal{E}|_{\mathbb{L}_{(k)}}$ up to isomorphism, hence only on m; with this in mind we write a_m rather than $a_\mathcal{E}$.

On the other hand, suppose $\mathcal{E}|_{\partial\mathbb{L}_{(k)}} \cong \Pi_A$ – thus in passing we allow $\mathcal{E}$ to lie in the category $\mu loc^\infty(\mathbb{L}) \supset \mu loc(\mathbb{L})$ of Remark **??** (which embeds now into the category of not necessarily finite-dimensional Q_0-representations). Then $\mathcal{E}'|_{\partial\mathbb{L}_{(k)}} \cong \Pi_B$, since Π_A and Π_B are the projective covers of S^1_A and S^1_B, which are exchanged by Mut_k.

Fix a particular isomorphism $\mathcal{E}|_{\partial\mathbb{L}_{(k)}} \cong \Pi_A$ to identifys the stalks of $\mathcal{E}$ at $\gamma(0)$, $\gamma(1)$ with $\mathbb{k}\mathbb{Z}$, and $\mathcal{E}_\gamma$ with the identity. Independently, fix a particular isomorphism $\mathcal{E}'|_{\partial\mathbb{L}_{(k)}} \cong \Pi_B$, identifying the stalks of $\mathcal{E}'$ at $\gamma(0)$, $\gamma(1)$ with $\mathbb{k}\mathbb{Z}$, and $\mathcal{E}'_\gamma$ with multiplication by $1-t$.

As noted above, the homeomorphism $\mathcal{L}^{(k)} \cong \mathcal{L}^{(k)\prime}$ identifies the stalks of $\mathcal{E}$ and $\mathcal{E}'$ at $\gamma(0)$ with each other, and moreover does so in a way that intertwines the monodromy around $\partial\mathbb{L}_{(k)}$. With respect to the above trivializations of each by $\mathbb{k}\mathbb{Z}$, this identification is multiplication by a unit in $\mathbb{k}\mathbb{Z}$ (i.e. a monomial). The same holds for the trivializations at $\gamma(1)$.

As these units commute with multiplication by $1-t$, we combine them and conclude that with respect to the trivialization of the stalks of $\mathcal{E}$, we have $\mathcal{E}'_\gamma = at^n(1-t)\mathcal{E}_\gamma$ for some $a \in \mathbb{k}^*$ and some $n \in \mathbb{Z}$. Again, by functoriality these numbers depend only on $\mathcal{E}$ up to isomorphism. More importantly, we have a morphism $\Pi_A \to P^m_A$ for all m. Now functoriality tells us that $a_m = am^n(1-m)$, independently of m.

Note that for a given $\mathbb{L}$ it is possible there are no objects $\mathcal{E}$ whose restriction is isomorphic to Π_A. But the calculation is entirely about a gluing that takes place in a neighborhood of $\mathbb{L}_{(k)}$ hence we may take $\mathbb{L}$ to small enough to have such $\mathcal{E}$.

The ambiguity left in Mut_k comes from the choice of natural transformation making the right square in Equation 11 commute, and the fact that the homeomorphism $\mathcal{L} \cong \mathcal{L}'$ is only canonical up to Dehn twisting around C_k. There is an action of automorphisms of the identity functor of $\mu loc(\partial\mathbb{L}_{(k)})$ on the choice of natural transformation – with this we can fix a to be 1 (note that a priori $a = \pm 1$ since everything is defined over $\mathbb{Z}$, even if we have assumed $\mathbb{k}$ to be a field in our analysis). Changing the homeomorphism $\mathcal{L} \cong \mathcal{L}'$ by a Dehn twist changes n, so likewise we may take $n = 1$.

The reduction from the general case to the previous calculation follows by decomposing into eigenspaces of $\mathcal{E}_{C_k}$. If it is semisimple there is nothing to check, otherwise functoriality tells us the calculation is compatible with extensions, giving us the case when there are nontrivial Jordan blocks. □

Remark 4.16. It is possible to determine, rather than absorb, the ambiguities in the above proof, by a microlocal calculation involving the GKS kernel.

5. Relation to Cluster Theory

The language of cluster algebra [FZ] provides a natural setting in which to organize the results of the preceding sections. We begin by reviewing the general theory with certain extensions necessitated by the scope of our discussion. These are related to the implicit sign choices in the definition of the Kashiwara-Schapira sheaf, as well as the fact that a general curve configuration is not nondegenerate. We also discuss certain noncommutative analogues of cluster structures possessed by spaces of rank $n > 1$ Lagrangian branes.

We begin with the notion of a seed, the defining data of a cluster structure. Our notation follows that of [FG, GHK2].

Definition 5.1. A **seed** $s = (N, \{e_i\})$ is a lattice N with skew-symmetric integral form $\{,\}$ and a finite collection $\{e_i\}_{i \in I} \subset N$ of distinct primitive elements indexed by a set I.

We write $[a]_+$ for $\max(a, 0)$.

Definition 5.2. The mutation of s at $k \in I$ is the seed $\mu_k s = (N, \{\mu_k e_i\})$, where

$$\mu_k e_i = \begin{cases} e_i + [\{e_i, e_k\}]_+ e_k & i \neq k \\ -e_k & i = k. \end{cases} \tag{12}$$

To a seed we associate a quiver without oriented 2-cycles and with vertex set $\{v_i\}_{i \in I}$. The number of arrows from v_i to v_j is $[\{e_i, e_j\}]_+$, and if the e_i are a basis the seed is determined up to isomorphism by the quiver. Conversely, given such a quiver Q we have a seed given by taking the vector space whose standard basis vectors are enumerated by the vertices, and whose skew-symmetric form is given by the arrows. In the literature one often only considers seeds of this form. One can also consider seeds related to skew-symmetrizable matrices, but these do not arise in our setting. We also suppress any explicit discussion of "frozen indices"; this notion is already included by allowing the e_i to fail to generate N.

Given a seed $s = (N, \{e_i\})$, we write $M = \operatorname{Hom}(N, \mathbb{Z})$ and consider the dual algebraic tori

$$\mathcal{X}_s = \operatorname{Spec} \mathbb{Z}N, \quad \mathcal{A}_s = \operatorname{Spec} \mathbb{Z}M,$$

We let $z^n \in \mathbb{Z}N$ denote the monomial associated to $n \in N$, likewise $z^m \in \mathbb{Z}M$ for $m \in M$.

Definition 5.3. For $k \in I$, the **cluster $\mathcal{X}$- and $\mathcal{A}$-transformations** $\mu_k : \mathcal{X}_s \dashrightarrow \mathcal{X}_{\mu_k s}$, $\mu_k : \mathcal{A}_s \dashrightarrow \mathcal{A}_{\mu_k s}$ are the rational maps defined by

$$\mu_k^* z^n = z^n(1 + z^{e_k})^{\langle e_k, n\rangle}, \quad \mu_k^* z^m = z^m(1 + z^{\{e_k, -\}})^{-\langle e_k, m\rangle}, \tag{13}$$

where $\langle e_k, n\rangle$ denotes the skew-symmetric pairing on N and $\langle e_k, m\rangle$ the evaluation pairing.

Let T be an infinite $|I|$-ary tree with edges labeled by I so that the edges incident to a given vertex have distinct labels. Fix a root $t_0 \in T_0$ and label it by the seed s. Label the remaining $t \in T_0$ by seeds s_t such that if t and t' are connected by an edge labeled k, and t' is farther from t_0 than t, then $s_{t'} = \mu_k s_t$.

Definition 5.4. A **cluster $\mathcal{X}$-structure** on Y is a collection $\{\mathcal{X}_{s_t} \hookrightarrow Y\}_{t \in T_0}$ of open maps such that the images of $\mathcal{X}_{s_t}$ and $\mathcal{X}_{\mu_k s_t}$ are related by a cluster $\mathcal{X}$-transformation for all t, k. A partial cluster $\mathcal{X}$-structure is the same but with maps only for a subset of T_0, and a cluster $\mathcal{A}$-structure the same but with $\mathcal{A}$-tori and $\mathcal{A}$-transformations.

Remark 5.5. Though we allow the e_i to not be a basis of N, there is a map of seeds $\overline{s} := (\mathbb{Z}^n = \mathbb{Z}\{e_i\}, \{e_i\}) \to (N, \{e_i\})$, where $\mathbb{Z}^n$ carries the skew-symmetric form pulled back from N. The associated tori are $\mathcal{A}$- and $\mathcal{X}$-tori in the standard sense of [FG2], and are related to those of s by the commutative square

$$\begin{array}{ccc} \mathcal{A}_s & \longleftarrow & \mathcal{A}_{\overline{s}} \\ \downarrow & & \downarrow \\ \mathcal{X}_s & \longleftarrow & \mathcal{X}_{\overline{s}} \end{array}$$

We will require a slightly more general notion, in which signs may appear in the cluster transformations.

Definition 5.6. A signing on a seed is a function $\sigma : \{e_i\} \to \mathbb{Z}/2\mathbb{Z}$. Signings transform under mutation by $\sigma(e_i') = \sigma(e_i)$. A σ-signed cluster structure is like a cluster structure, but with the transformation law

$$\mu_k^* z^n = z^n(1 + (-1)^{\sigma(e_k)} z^{e_k})^{\{e_k, n\}}, \quad \mu_k^* z^m = z^m(1 + (-1)^{\sigma(e_k)} z^{\{e_k, -\}})^{-\langle e_k, m\rangle}$$

The set of signings on $(N, \{e_i\})$ can be identified with $\mathrm{Hom}(\mathbb{Z}\{e_i\}, \mathbb{Z}/2\mathbb{Z})$, hence it carries an action by $\mathrm{Hom}(N, \mathbb{Z}/2\mathbb{Z})$ via pullback along $\mathbb{Z}\{e_i\} \to N$. Acting by $\mathrm{Hom}(N, \mathbb{Z}/2\mathbb{Z})$ on the tori $\mathcal{X}_{s_t}$ identifies cluster structures with signings differing by this action. If the e_i are a basis of N all signings are equivalent in this sense, so a signed cluster $\mathcal{X}$-structure is only a nontrivial notion when this is not the case.

We now return to the setting of $\mathcal{C}$ a curve configuration on a surface $\mathcal{L}$, W the associated 4-manifold with seed skeleton $\mathbb{L}$. Our terminology for $\mathbb{L}$ is justified by having an obvious seed associated to it, namely $(H_1(\mathcal{L}, \mathbb{Z}), \{[C_i]\})$, where we take only the classes of the closed curves $C_i \in \mathcal{C}^\circ$ and $H_1(\mathcal{L}, \mathbb{Z})$ carries its intersection pairing. Recall that the definition of the Kashiwara-Schapira sheaf, hence the category $\mu loc(\mathbb{L})$, implicitly depends on a function $\sigma : \mathcal{C}^\circ \to \mathbb{Z}/2\mathbb{Z}$.

Theorem 5.7. *The moduli space $\mathcal{M}_1(\mathbb{L})$ carries a partial, σ-signed cluster $\mathcal{X}$-structure with initial seed $(H_1(\mathcal{L},\mathbb{Z}),\{[C_i]\})$.*

Proof. This is basically just a summary of the previous sections' results in different terminology. The association of seeds to curve configurations intertwines mutations of both objects by Proposition 2.9. Since $N = H_1(\mathcal{L},\mathbb{Z})$ we have $\mathcal{X}_s \cong Loc_1(\mathcal{L})$ (up to stabilizers), and there is an open map $Loc_1(\mathcal{L}) \to \mathcal{M}_1(\mathbb{L})$ by Definition 3.21. Mutation at a simple closed curve C_k induces another map $Loc_1(\mathcal{L}') \to \mathcal{M}_1(\mathbb{L})$, and the two are intertwined by a (possibly signed) cluster $\mathcal{X}$-transformation by Proposition 4.15. Iterated mutation of the curve configuration in general only induces a partial cluster structure since it may not be possible to perform arbitrary mutation sequences without creating self-intersections, following the discussion in Section 2.2. □

Remark 5.8. The choice of phrasing the previous theorem in terms of $\mathcal{M}_1(\mathbb{L})$ rather than $\mathbb{R}\mathcal{M}_1(\mathbb{L})$ is basically cosmetic; it would be perfectly natural to say the maps $\mathbb{R}Loc_1(\mathcal{L}) \to \mathbb{R}\mathcal{M}_1(\mathbb{L})$ assemble into a cluster structure on $\mathbb{R}\mathcal{M}_1(\mathbb{L})$.

While the classes of the closed curves in $\mathcal{C}$ are the only ones at which we can perform mutations, our definitions allow half-disks attached to open curves ending on $\partial\mathcal{L}$. This would be necessary to discuss gluing of skeleta, which would be the skeletal version of the amalgamation process of [FG2].

Just as the rank one moduli space $\mathcal{M}_1(\mathbb{L})$ is a recepticle for maps from the torus $Loc_1(\mathcal{L})$ of rank one local systems, the higher rank moduli spaces $\mathcal{M}_n(\mathbb{L})$ receive maps from spaces of higher rank local systems. The transition functions between these are determined by Proposition 4.15 as they are in the rank one case. Adopting the notation of Section 4.4, these are:

Definition 5.9. Let $C \subset \mathcal{L}$ be an oriented simple closed curve. The rank-n cluster $\mathcal{X}$-transformation $\mu_C : Loc_n(\mathcal{L}) \dashrightarrow Loc_n(\mathcal{L})$ is the following birational map. It is regular on the local systems $\mathcal{E}$ for which $\mathrm{Id} - \mathcal{E}_{C_k}$ is invertible. Given such an $\mathcal{E}$, its image $\mathcal{E}' := \mu_C(\mathcal{E})$ is determined by the following properties:

- If γ does not meet C then $\mathcal{E}'_\gamma = \mathcal{E}_\gamma$.
- Suppose γ crosses C exactly once, with C oriented to the right of γ. Let $\gamma_{<C}$ be the subpath that starts at $\gamma(0)$ and ends at the crossing, $\gamma_{>C}$ the subpath which starts at the crossing and ends at $\gamma(1)$. Then
$$\mathcal{E}'_\gamma = \mathcal{E}_{\gamma_{>C}}(\mathrm{Id} - \mathcal{E}_C)\mathcal{E}_{\gamma_{<C}}.$$

With this in mind, we say that $\mathcal{M}_n(\mathbb{L})$ has a rank-n (partial, signed) cluster $\mathcal{X}$-structure by analogy with Theorem 5.7.

6. Examples from Almost Toric Geometry

We recall the setting of almost toric geometry, a formalism for describing certain Liouville integrable systems. An example with no degenerate fibres is the restriction of the moment map of a toric variety to the interior of the moment polytope. More general almost-toric fibrations [LSy, Sym, KoS] describe the situation in which certain degenerate fibres are allowed. We restrict ourselves here to the case where the total space is 4 real dimensional.

One begins with the moment map of a toric algebraic surface, $\overline{W} \to \overline{\Delta}$, and possibly makes non-toric blowups along the boundary divisors. Thus the total space of the boundary divisor has many nodes. Degenerate fibres can be introduced into the interior by the so-called "nodal trade", in which these singularities are pushed into interior fibers, and the boundary divisor is correspondingly smoothed. We then restrict attention to the interior $W \to \Delta$.

Thus for us the data of an almost-toric fibration is: the interior of a polygon, a certain number of marked points $d_1, \ldots, d_n$, branch cuts from these to the boundary of the polytope, and specifications of how the integral affine structure of the polygon changes along the branch cuts. From this data, an almost toric fibration can be uniquely reconstructed: the d_i sit below the singular fibers, and under the identification of the connection on $H_1(\text{fibre})$ with the affine structure from the action coordinates, the changes across the branch cut specify the monodromy.

As we are working in the complement of the boundary of the polytope, the total space of the fibration is an exact symplectic manifold. There is a unique point $0 \in \Delta$ above which the fiber is a smooth exact Lagrangian Σ. The Lagrangian thimbles above the straight lines from d to the degenerate fibers give Lagrangian disks D_i which end on curves C_i on Σ. The total space of the almost-toric fibration is the same as the space built from the Σ and C_i according to Definition 1.3. The union of Σ and the D_i is our Lagrangian skeleton $\mathbb{L}$.

By appropriate change of presentation of the base as in [Sym, Sec. 5.3], it can be arranged that the points d_i and the branch cuts are in the complement of a neighborhood of 0. Beginning in this situation, we can consider the deformation of the integrable system which brings some given branch point d_i to 0 along the line connecting them, and then past. Watching from the point of view of the unique exact fibre, one sees the curve C_i collapse to a point and then re-expand – that is, one sees precisely the Lagrangian disk surgery of [Yau]. Afterward, the configuration no longer satisfies the constraint that all branch cuts stay away from 0; again a manipulation of the integral affine structure can restore this situation, at the cost of cutting and regluing the polytope Δ. For details, see [Via, Sec. 2.3].

Example 6.1 (Torus With One Disk)**.** Begin with the moment map $\mathbb{P}^2 \to \overline{\Delta}$, and make the "nodal trade" at one vertex of the closed triangle $\overline{\Delta}$. The resulting fibration, restricted to the interior of the triangle, has a single degenerate fibre. Taking the thimble over the line connecting the degenerate fibre to the exact fibre Σ, we find that the skeleton is a torus with a disk attached.

We are working in the complement of the boundary. Having made the nodal trade at one vertex has smoothed it out one of the boundary divisors, so the space W is the complement in $\mathbb{P}^2$ of the union of a quadric and a line. That is, it is the affine surface $W = \{(x,y) \,|\, xy - 1 \neq 0\} \subset \mathbb{C}^2$. We refer to [Sei4, Prop. 11.8] for an account of this space via a Lefschetz pencil presentation.

Example 6.2 (Vianna's Tori)**.** Begin again with the moment map $\mathbb{P}^2 \to \overline{\Delta}$, and make the "nodal trade" at all three vertices of the triangle. Having smoothed out all singularities of the boundary divisor, the fibration $W \to \Delta$ has total space equal to the complement of a smooth elliptic curve.

Taking thimbles from the three degenerate fibres to the exact fibre $\Sigma = T^2$ gives three curves. To determine which curves they are, consider the anticanonical moment map image triangle with vertices $(-1,-1),(2,-1),(-1,2)$. The thimble in the direction (a,b) of the affine structure on Δ is carried by the class $(-b,a)$ in $H_1(T^2,\mathbb{Z})$, up to some universal choice of sign conventions. Thus, our curves are in the classes $(1,-1),(1,2),(-2,-1)$.

Vianna constructs infinitely many monotone tori in $\mathbb{P}^2$ [Via2]. His construction can be identified with the iterated disk surgery we have given here, via the dictionary described above between the disk surgery prescription and degenerations of the almost toric picture.

Assuming $Fuk(W) \cong \mu loc(\mathbb{L})$, one can give a cluster-theoretic reason why there are infinitely many inequivalent tori here. Indeed, Hamiltonian isotopic tori necessarily give rise to the same objects in $Fuk(W)$, hence the same cluster charts on $\mathcal{M}_1(\mathbb{L})$. On the other hand, each cluster chart gives rise to a torus as we have described. The cluster structure on $\mathcal{M}_1(\mathbb{L})$ is determined by the quiver defined by the intersection pairings: an oriented 3-cycle with all of its edges tripled. As this is not a Dynkin quiver, there are infinitely many clusters hence infinitely many distinct tori [FZ2]. The above argument would show that these tori are non-isotopic in W. More precisely, one should also argue that distinct tori in the usual $\mathcal{X}$-variety have distinct intersections with the image of the $\mathcal{A}$-variety, as this image is what is directly related to $\mathcal{M}_1(\mathbb{L})$ by Theorem 5.7. We note that Vianna proves the stronger statement that the corresponding monotone tori are not Hamiltonian isotopic even in $\mathbb{P}^2$.

Example 6.3 (Keating's Tori)**.** Consider the algebraic surface singularity $x^p + y^q + z^r + axyz = 0$, where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$. We write $T_{p,q,r}$ for its Milnor fibre; this is a Stein space, hence a Weinstein 4-manifold. This space was studied in [Kea], where exact Lagrangian tori are constructed for the purpose of showing that vanishing cycles cannot split generate the Fukaya category of $T_{p,q,r}$.

These spaces can be constructed from the almost toric point of view as follows. Begin with $\mathbb{P}^2 \to \overline{\Delta}$. Blow up the three boundary divisors, respectively, p, q, and r times at distinct points. Make the nodal trades, so as to introduce $p+q+r$ degenerate fibres. On the corresponding exact Lagrangian fibre, the thimbles determine $p+q+r$ curves on Σ in the respective classes $(0,-1)$, $(1,0)$, $(-1,1)$. The corresponding quiver has $p+q+r$ vertices; one arrow from each of the p vertices to each of the q vertices, one arrow from each of the q vertices to each of the r vertices, and one arrow from each of the r vertices to each of the p vertices, for a total of $pq+qr+rp$ arrows, which participate in pqr 3-cycles. The tori resulting from the present construction are those of [Kea].

In these almost toric examples, we are working in the complement of a divisor linearly equivalent to the total transform under blowup of the toric boundary, hence the spaces W are (log) Calabi-Yau surfaces. By construction they come with fibrations by Lagrangian tori, i.e., are in the setting appropriate to [SYZ] mirror symmetry. This has been studied for these surfaces using both tropical [GHK] and to some extent symplectic [Pas] techniques.

From the above description, it can be seen that the moduli space $\mathcal{M}_1(\mathbb{L})$ is, in the almost toric case, meant to be the moduli space of objects in the Fukaya category of W in the class of a torus fibre. That is, it is the SYZ mirror, and our perspective matches that of [GHK2, GHKK].

Appendix A. Appendix: Background on constructible sheaves

A sheaf can be thought of as a family of (complexes of) $\Bbbk$-modules, parameterized by X; in particular, for a sheaf $\mathcal{F}$ and a point $x \in X$, there is a $\Bbbk$-module $\mathcal{F}_x$ called the stalk of $\mathcal{F}$ at x. Sheaves are the natural coefficients for Cech cohomology; i.e., it makes sense to write $H^*(X, \mathcal{F})$. Constructible sheaves are those for which there is some stratification of X such that the $\mathcal{F}_x$ remain locally constant along strata.

In this section, we very briefly recall from [KS] some formal manipulations which can be performed on constructible sheaves: functors between sheaf categories and their basic properties; integral kernels; microsupport; and from [GKS], the action of contact isotopy on the sheaf category.

Given a manifold or stratified space X and a commutative ring $\Bbbk$, we write $sh(X;\Bbbk)$ (or simply $sh(X)$) for the dg version of the derived category of constructible sheaves of $\Bbbk_X$-modules on X. That is, it is the quotient [Kel3, Dri] of the dg category of complexes of sheaves with constructible cohomology by the acyclic complexes. The formalism of [KS] makes good sense in this context; we refer to [N1] for some details on this point, which we ignore for the remainder of the appendix.

A.1. **Functors between sheaf categories.**

A.1.1. *Six functors.* Given a space X, and a sheaf $\mathcal{F}$, there are adjoint functors

$$\cdot \otimes \mathcal{F} : sh(X) \leftrightarrow sh(X) : \underline{\mathrm{Hom}}(\mathcal{F}, \cdot)$$

Given a map of spaces $f : Y \to X$, one obtains two pairs of adjoint functors

$$f^* : sh(X) \leftrightarrow sh(Y) : f_*$$

$$f_! : sh(Y) \leftrightarrow sh(X) : f^!$$

The left adjoints are easier to understand at the level of stalks: for a point p,

$$\begin{aligned} (\mathcal{F} \otimes \mathcal{F}')_p &= \mathcal{F}_p \otimes \mathcal{F}'_p \\ (f^*\mathcal{F})_p &= f^*(\mathcal{F}_p) \\ f_!(\mathcal{G})_p &= H^*_c(f^{-1}(p), \mathcal{G}) \end{aligned}$$

The right adjoints are easier to understand at the level of sections: for an open set U,

$$\begin{aligned} H^*(U, \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{F}')) &= \mathrm{Hom}(\mathcal{F}|_U, \mathcal{F}'|_U) \\ H^*(U, f_*\mathcal{G}) &= H^*(f^{-1}(U), \mathcal{G}) \\ H^*(U, f^!\mathcal{F}) &= \mathbb{D}H^*_c(U, f^*\mathbb{D}\mathcal{F}) \end{aligned}$$

The last line is written in terms of the Verdier duality operation — an anti-involution $\mathbb{D} : sh(M;\Bbbk) \to sh(M;\Bbbk)$. It interchanges shrieks and stars — $\mathbb{D}f_* = f_!\mathbb{D}$ and $\mathbb{D}f^* = f^!\mathbb{D}$ — so can be used to calculate the shriek pullback.

The shrieks and stars are directly related in two cases: when f is proper, we have $f_! = f_*$; when f is a smooth fibration, the sheafification of Poincaré duality asserts $f^! = f^*[\dim Z - \dim Y]$. It follows by considering the map to a point that $\mathbb{D}\Bbbk_M = \Bbbk_M[\dim M]$; one recovers the usual Poincaré duality from this as $H^*(M, \Bbbk_M[\dim M]) = \mathbb{D}H^*_c(U, \Bbbk_M)$, where now the operation $\mathbb{D}$ is just the linear duality of complexes of vector spaces.

If $\overline{M}$ is a manifold with boundary, and $j : M \to \overline{M}$ is the inclusion of its interior, then note that $j_*\Bbbk_M = \Bbbk_{\overline{M}}$. Taking Verdier duals, we see $j_!\Bbbk_M[\dim M] = \mathbb{D}\Bbbk_{\overline{M}}$. We use this in the following form: if $f : \overline{M} \to N$ is any map to a manifold (without boundary), then

$$f^!\Bbbk_N = f^!\mathbb{D}\Bbbk_N[\dim N] = \mathbb{D}f^*\Bbbk_N[\dim N] = \mathbb{D}\Bbbk_M[\dim N] = \Bbbk_M[\dim M - \dim N]$$

The contortions above to compute $f^!$ are inevitable – this operation has a certain irreducible complexity (which can be hidden inside the Poincaré-Verdier duality, but this in turn is nontrivial to compute). However, when f is the inclusion of a closed subset, $f^!$ extracts the sections supported on that subset.

A.1.2. *Base change.* Given another map $g : Z \to Y$, we write also $g : Z \times_Y X \to X$ and $f : Z \times_Y X \to Z$ for the maps induced on the fibre product. The base change theorems assert the following relations: $f_!g^* = f_!g^* : Sh(X) \to Sh(Z)$ and similarly the other three $f_*g^! = g^!f_*$, and $g_!f^* = f^*g_!$, and $g_*f^! = f^!g_*$.

E.g., if $g : U \to Y$ is an open inclusion, then

$$g^*f_*\mathcal{F} = g^!f_*\mathcal{F} = f_*g^!\mathcal{F} = f_*g^*\mathcal{F} = f_*(\mathcal{F}|_{f^{-1}(U)})$$

Taking global sections (i.e. cohomology, i.e. pushing forward to a point), one has $H^*(U, f_*\mathcal{F}) = H^*(f^{-1}(U), \mathcal{F})$. This is usually given as the definition of f_*. Taking $U = Y$ and $\mathcal{F} = \Bbbk$, we have $H^*(Y, f_*\Bbbk) = H^*(X, \Bbbk)$. Expanding out $f_*\Bbbk$ into its cohomology sheaves gives rise to the Leray spectral sequence; this ability to factor cohomological calculations is one of the main classical uses of sheaf theory.

A.1.3. *Recollement.* Consider the inclusion of an open subset U and its closed complement V into Y,

$$j : U \to Y \leftarrow V : i$$

Here, $j^* = j^!$ and $i_* = i_!$. Because $U \cap V = \varnothing$, all compositions involved in the base change formula vanish. Moreover (because U and V cover), we have exact triangles

$$i_! i^! \to \mathbf{1} \to j_* j^* \xrightarrow{[1]} \qquad\qquad j_! j^! \to \mathbf{1} \to i_* i^* \xrightarrow{[1]}$$

These sequences are the sheaf-theoretic incarnations of excision: applied to the constant sheaf on Y and pushed forward to a point, one recovers

$$H^*(V, i^! \Bbbk) \to H^*(Y, \Bbbk) \to H^*(U, \Bbbk) \xrightarrow{[1]} \qquad\qquad H^*_c(U, \Bbbk) \to H^*_c(Y, \Bbbk) \to H^*_c(V, \Bbbk) \xrightarrow{[1]}$$

These allow us to understand the category $sh(Y)$ in terms of the categories $sh(U)$, $sh(V)$, and the data prescribing the connecting morphism in one of the above exact triangles; see e.g. [BBD, 1.4.3]. Nadler has suggested of how this formalism may be applied to the Fukaya category in [N2].

Recollement has historically been used for gluing sheaf categories together from local pieces. Note however that our approach in the present article has been at least grammatically different: we view the global category as the global sections of a sheaf of categories, hence work with an open cover rather than with a decomposition as above.

A.1.4. *Functors from kernels.* Additional functors between sheaf categories can be constructed via the formalism of kernels [KS, Sec. 3.6]. This works as follows. Consider the product

$$Y \xleftarrow{\pi_Y} Y \times X \xrightarrow{\pi_X} X$$

Given any sheaf $\mathcal{K}$ on $Y \times X$, one gets two pairs of adjoint functors:

$$\mathcal{K}^* : sh(X) \leftrightarrow sh(Y) : \mathcal{K}_*$$
$$\mathcal{K}_! : sh(Y) \leftrightarrow sh(X) : \mathcal{K}^!$$

Their definitions are as follows. Let $\mathcal{G}$ be a sheaf on Y and $\mathcal{F}$ a sheaf on X.

$$\begin{aligned}
\mathcal{K}^* : \mathcal{F} &\mapsto \pi_{Y!}(\mathcal{K} \otimes \pi_X^* \mathcal{F}) \\
\mathcal{K}_! : \mathcal{G} &\mapsto \pi_{X!}(\mathcal{K} \otimes \pi_Y^* \mathcal{G}) \\
\mathcal{K}_* : \mathcal{G} &\mapsto \pi_{X*} \underline{Hom}(\mathcal{K}, \pi_Y^! \mathcal{G}) \\
\mathcal{K}^! : \mathcal{F} &\mapsto \pi_{Y*} \underline{Hom}(\mathcal{K}, \pi_X^! \mathcal{F})
\end{aligned}$$

Example A.1. Let $f : Y \to X$ be a map. Let $\Bbbk(f) \in sh(Y \times X)$ be the constant sheaf on the graph of f. Then $\Bbbk(f)^* = f^*$, $\Bbbk(f)_* = f_*$, $\Bbbk(f)^! = f^!$, $\Bbbk(f)_! = f_!$

Example A.2. (Fourier-Sato transform) Let Φ be the constant sheaf on the locus $\{\mathbf{x}\cdot\mathbf{y} \geq \mathbf{0} \,|\, (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^\mathbf{n} \times \mathbb{R}^\mathbf{n}\}$. This defines the so-called Fourier-Sato transforms

$$\Phi^* = \Phi_! : sh(\mathbb{R}^n) \to sh(\mathbb{R}^n)$$
$$\Phi_* = \Phi^! : sh(\mathbb{R}^n) \to sh(\mathbb{R}^n)$$

These transforms are generally considered restricted to the subcategory of conic sheaves, i.e., sheaves which are constant along any open ray emanating from the origin. Here, the Fourier transform squares to pull-back by the antipodal map. Its inverse is given by the kernel $\{\mathbf{x}\cdot\mathbf{y} \le \mathbf{0} \,|\, (\mathbf{x},\mathbf{y}) \in \mathbb{R}^{\mathbf{n}} \times \mathbb{R}^{\mathbf{n}}\}$ See [KS, Sec. 3.7] for more.

The functors induced by the kernel can be composed by convolving the kernels. That is, if one has $\mathcal{K}' \in sh(Z\times Y)$ and $\mathcal{K} \in sh(Y \times X)$, then with the evident notation for projection to the factors, one defines

$$\mathcal{K}' \circ \mathcal{K} := \pi_{ZX!}(\pi_{ZY}^* \mathcal{K}' \otimes \pi_{YX}^* \mathcal{K}) \in sh(Z \times X)$$

This has the properties

$$(\mathcal{K}' \circ \mathcal{K})_* = \mathcal{K}'_* \circ \mathcal{K}_*, \qquad (\mathcal{K}' \circ \mathcal{K})_! = \mathcal{K}'_! \circ \mathcal{K}_!, \qquad (\mathcal{K}' \circ \mathcal{K})^* = \mathcal{K}^* \circ \mathcal{K}'^*, \qquad (\mathcal{K}' \circ \mathcal{K})^! = \mathcal{K}^! \circ \mathcal{K}'^!.$$

A.1.5. *Cutoff functors.* Let $j : U \subset X$ be the inclusion of an open subset. Then $j_!$ and j_* are fully faithful.

Assume in addition that $X \smallsetminus \overline{U}$ is cylindrical, and equipped with the structure $X \smallsetminus \overline{U} = \mathbb{R}_+ \times \partial U$. We say a sheaf is *cylindrical past the boundary* if its restriction to any ray $\mathbb{R}_+ \times u \subset \mathbb{R}_+ \times \partial U = X \smallsetminus \overline{U}$ is constant.

Lemma A.3. *The restriction of $j^! = j^*$ to the subcategory of cylindrical sheaves is an equivalence of categories.*

Note however that $j_! j^!$ and $j_* j^*$ are not in this case the identity. We think of them as cutoff functors, and call the first the "soft cutoff" and the second the "hard cutoff". In case U has many boundary components, we can decide independently on each whether to use $j_!$ or j_*.

Informally: when we are describing microlocal sheaf categories on some manifold X with cylindrical end, it is equivalent to allow the sheaves to go to infinity, or to cut them off at some point in the cylinder, and moreover each cutoff can be co-oriented arbitrarily.

A.2. **Microsupport.** Let $\mathcal{F}$ be a sheaf on a manifold M. The microsupport $ss(\mathcal{F})$, introduced in [KS], is meant to capture the locus in T^*M of obstructions to the propagation of sections of $\mathcal{F}$. For instance, if $f : M \to \mathbb{R}$ is a function such that the graph of df avoids $ss(\mathcal{F})$ over the locus $f^{-1}((a,b])$, then the restriction of sections is an isomorphism [KS, Prop. 5.2.1]:

$$H^*(f^{-1}(-\infty,b],\mathcal{F}) \xrightarrow{\sim} H^*(f^{-1}(-\infty,a],\mathcal{F})$$

The formal definition is a local version of the above criterion:

Definition A.4. [KS, Chap. 5] A point $p = (x,\xi) \in T^*M$ is in the microsupport of a sheaf $\mathcal{F}$ if there are points (x',ξ') arbitrarily close to (x,ξ) and functions $f : M \to \mathbb{R}$ with $f(x') = 0, df(x') = \xi'$, such that: if $c_f : \{x \,|\, f(x) \ge 0\} \to M$ is the inclusion, then $(c_f^! \mathcal{F})_{x'} \ne 0$.

Shriek pullback to a closed subset gives the local sections supported on that subset. Thus the statement $(c_f^! \mathcal{F})_{x'} \ne 0$ is informally read as: "there is a section of $\mathcal{F}$ beginning at x' and propagating in the direction along which f increases." Note that, taking the zero function, the support of $\mathcal{F}$ is contained in its microsupport.

For us, microsupports are used as a way to specify certain categories of sheaves. For a subset (usually conical Lagrangian) L in T^*M, we write $sh_L(M;\Bbbk)$ for the category of sheaves on M with coefficients in $\Bbbk$ and microsupport in L. For instance, the category of local systems on M is $sh_{0_M}(M;\Bbbk)$.

A.2.1. *Properties of the microsupport.* For sheaves constructible with respect to a given stratification, it is straightforward to show that the microsupport is contained within the union of the conormals of the strata. Since the microsupport is co-isotropic, it is in this case Lagrangian, and necessarily a full dimensional subset of the union of conormals.

Finally, note that, per the definition, to show that (x,ξ) is not in the microsupport, one needs to check a property of every function vanishing near x with derivative near ξ, at every point near x. In fact, it is enough to check a function f which is stratified Morse at x. Such functions need not exist for all (x,ξ) with respect to a given stratification, but because they will exist for general points in each component of the microsupport. Since microsupports are closed, they can be computed with stratified Morse functions [GM].

Many properties of the microsupport are developed in [KS]. In particular: writing $\mathbb{D}\mathcal{F}$ for the Verdier dual of $\mathcal{F}$, the microsupports $ss(\mathcal{F})$ and $ss(\mathbb{D}\mathcal{F})$ are related by the antipodal map on cotangent fibres, and given an exact triangle, $A \to B \to C \xrightarrow{[1]}$, one has

$$(ss(A) \smallsetminus ss(B)) \cup (ss(B) \smallsetminus ss(A)) \subset ss(C) \subset ss(A) \cup ss(B)$$

Microsupport interacts well with integral kernels. Given a conical Lagrangian $M \subset T^*(Y \times X)$, we have the convolution

$$\begin{array}{rcl} M_\star : ConLag(T^*Y) & \to & ConLag(T^*X) \\ L & \mapsto & \pi_X(\pi_Y^{-1} L \cap M) \end{array}$$

and given a kernel $\mathcal{K} \in sh(Y \times X)$ satisfying certain properness and non-characteristic hypotheses (see [GKS, Eq. 1.1]) one has

$$ss(\mathcal{K}_! \mathcal{F}) \subset ss(\mathcal{K})_\star ss(\mathcal{F})$$

The effect on microsupport of the other functors can be determined from the above using Verdier duality and transposition. In particular, since the functors associated to a map $Y \to X$ are convolution with the conormal to the graph, the above formula determines their effect on microsupports.

We record the effect on microsupport of the hard and soft cutoff functors.

Lemma A.5. *Let $u : U \hookrightarrow M$ be the inclusion of an open set into a manifold extending to an inclusion of a manifold with boundary $\overline{u} : \overline{U} \hookrightarrow M$. Give ∂U the boundary coorientation – its positive conormal direction is out. Let $\mathcal{F} \in sh(M)$ be conical past the boundary. Then:*

$$ss(u_* u^* \mathcal{F}) \subset ss(\mathcal{F})|_{\overline{U}} \cup T^+_{\partial U} M$$

$$ss(u_! u^! \mathcal{F}) \subset ss(\mathcal{F})|_{\overline{U}} \cup T^-_{\partial U} M$$

A.3. **Contact isotopies.** Consistent with the expectation that constructible sheaves model the Fukaya category, contact isotopies of $T^\infty M$ act on $sh(M)$. We recall the result as formulated in [GKS].

Theorem A.6. [GKS] *Let M be a manifold, I an interval, $T^\circ M$ the cotangent bundle minus the zero section, and $\Phi : T^\circ M \times I \to T^\circ M$ a smooth map. Assume $\Phi(\cdot, 0)$ is the identity, and $\Phi(\cdot, t)$ is a homogenous (i.e. commutes with the scaling) symplectomorphism for each t.*

Then there exists a unique closed conic Lagrangian in $\Lambda \subset T^\circ(M \times M \times I)$ such that $\Lambda_\star T^*_t I \subset T^\circ(M \times M)$ is the graph of $\Phi(\cdot, t)$.

Moreover, there exists a unique locally bounded sheaf $K_\Phi \in sh(M \times M \times I)$ such that $ss(K_\Phi) = \Lambda$ and $K_\Phi|_{M \times M \times 0}$ is the constant sheaf on the diagonal.

Remark A.7. A homogenous symplectomorphism can be given (up to rescaling) by a contactomorphism at infinity.

Corollary A.8. [GKS] *Convolution with the kernel $K_\Phi|_t$ induces an equivalence of categories $sh(M) \to sh(M)$. Away from the zero section, the microsupport of the image of a sheaf under convolution is the image of its microsupport under $\Phi(\cdot, t)$.*

Proof. We indicate how to derive this from Theorem A.6. The inverse is given by the kernel coming from the inverse family of symplectomorphisms: convolving the two families of kernels gives a kernel whose microsupport must lie in the conormal to the diagonal and is the constant sheaf there at time zero. □

This is an extremely powerful tool, and was used in [GKS] to prove various non-displaceability theorems. In the present paper we use it to define mutation functors in Section 4: given a neighborhood of a Lagrangian skeleton small enough to be embeddable into a cotangent bundle, the result of [GKS] amounts to an assertion that the microlocal sheaf category depends only on the (singular) Legendrian skeleton at the boundary of this neighborhood. Thus we may isotope around this Legendrian at will, changing the topology of its Lagrangian cone in the process.

Example A.9. (Reeb flow) Let $\Phi(\cdot, t)$ be the flow of the Hamiltonian $H(\mathbf{q}, \mathbf{p}) = \mathbf{p}^2$ on $T^*\mathbb{R}^n$. Then for $t \geq 0$, the kernel $K_\Phi|_t$ is given by the constant sheaf on the locus $|\mathbf{x} - \mathbf{y}| \leq \mathbf{t}$ in $\mathbb{R}^n \times \mathbb{R}^n$. Convolution with this kernel acts as an averaging operator: the stalk after convolution at $\mathbf{x}$ is the global sections of the sheaf over the radius t ball around $\mathbf{x}$.

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