TAMELY PRESENTED MORPHISMS AND COHERENT PULLBACK

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Abstract. We study two classes of morphisms in infinite type: tamely presented morphisms and morphisms with coherent pullback. These are generalizations of finitely presented morphisms and morphisms of finite Tor-dimension, respectively. The class of tamely presented stacks is restricted enough to retain the key features of finite-type schemes from the point of view of coherent sheaf theory, but wide enough to encompass many infinite-type examples of interest in geometric representation theory. The condition that a diagonal morphism has coherent pullback is a natural generalization of smoothness to the tamely presented setting, and we show such objects retain many of the good cohomological properties of smooth varieties. Our results are motivated by applications to Coulomb branches and their categorification. To this end, we extend our framework to the setting of ind-geometric stacks, the foundations of which we develop along the way.

Contents

1. Introduction

Given a complex algebraic variety X, the condition that X is smooth can be interpreted as a cohomological condition on its diagonal $\Delta_X : X \to X \times X$. One formulation of this is that X is smooth if and only if Δ_X is of finite Tor-dimension. That is, if $\mathrm{QCoh}(X)$ is the unbounded derived category of quasi-coherent sheaves on X and $\mathrm{QCoh}(X)^+$ its bounded below subcategory, the derived pullback $\Delta_X^* : \mathrm{QCoh}(X \times X) \to \mathrm{QCoh}(X)$ takes $\mathrm{QCoh}(X \times X)^+$ to $\mathrm{QCoh}(X)^+$. Being of finite Tor-dimension implies a number of other useful properties, and is inherited from Δ_X by any regular map $f : Y \to X$. Thus the smoothness of a variety guarantees good cohomological control over maps into it.

For example, if $f : Y \to X$ is of finite Tor-dimension, it follows immediately that f^* takes $\mathrm{Coh}(X) \subset \mathrm{QCoh}(X)$, the subcategory of coherent sheaves (i.e. bounded complexes with coherent cohomology in the abelian sense), to $\mathrm{Coh}(Y)$. We say that a morphism with this property has *coherent pullback*. Less obviously, Δ_X (or any regular map of varieties) is of finite Tor-dimension if it has coherent pullback, hence X is smooth if and only if Δ_X has coherent pullback.

In representation theory and mathematical physics, one naturally encounters complex schemes which are not of finite type, hence by definition cannot be smooth in the strict sense, but are smooth in an approximate sense. For example, associated to the variety X is the jet scheme $X_{\mathcal{O}}$ parametrizing maps $\operatorname{Spec} \mathcal{O} \to X$, where $\mathcal{O} := \mathbb{C}[[t]]$. Except in trivial cases $X_{\mathcal{O}}$ is not of finite type. But if X is affine and étale over $\mathbb{A}^n$, $X_{\mathcal{O}}$ can be written as an inverse limit of smooth schemes along smooth affine morphisms [KV04, Prop. 1.7.1]. How is this reflected in the cohomological properties of $\Delta_{X_{\mathcal{O}}}$?

As an example, if $X \cong \mathbb{A}^n$ then $X_{\mathcal{O}} \cong \mathbb{A}^\infty := \operatorname{Spec} \mathbb{C}[x_1, x_2, \dots]$. The image of $\Delta_{\mathbb{A}^\infty}$ is of infinite codimension, and it follows that $\Delta^*_{\mathbb{A}^\infty}\Delta_{\mathbb{A}^\infty *}(\mathcal{O}_{\mathbb{A}^\infty})$ is not bounded below, hence that $\Delta_{\mathbb{A}^\infty}$ is not of finite Tor-dimension. On the other hand, suppose $\mathcal{F} \in \mathrm{Coh}(\mathbb{A}^\infty \times \mathbb{A}^\infty)$. This category may be defined the same way as in the Noetherian case since $\mathbb{C}[x_1, x_2, \dots]$ is coherent (i.e. its finitely presented modules form an abelian category). We have $\mathbb{A}^\infty \times \mathbb{A}^\infty \cong \lim \mathbb{A}^k \times \mathbb{A}^k$, and flatness of the projections implies that $\mathcal{F}$ is the pullback of a coherent sheaf on $\mathbb{A}^k \times \mathbb{A}^k$ for some k. But each $\mathbb{A}^k \times \mathbb{A}^k$ is smooth, hence $\Delta_{\mathbb{A}^\infty}$ has finite Tor-dimension after composing with each of the projections, hence $\Delta^*_{\mathbb{A}^\infty}$ still has coherent pullback. In particular, coherent pullback of the diagonal, unlike finite Tor-dimension of the diagonal, is a flexible enough cohomological notion of smoothness to remain relevant in infinite type.

On the other hand, having coherent pullback is a more delicate property than being of finite Tor-dimension. For example, the unboundedness of $\Delta^*_{\mathbb{A}^\infty}\Delta_{\mathbb{A}^\infty *}(\mathcal{O}_{\mathbb{A}^\infty})$ implies that the derived fiber product of $\mathbb{A}^\infty$ with itself over $\mathbb{A}^\infty \times \mathbb{A}^\infty$ has a structure sheaf which is unbounded, hence not coherent. In particular, while being of finite Tor-dimension is stable under arbitrary (derived) base change, having coherent pullback is not. Nonetheless, in cases such as $\Delta_{\mathbb{A}^\infty}$ we show it is stable under base change along the class of morphisms we call *tamely presented*.

A simple example of such a morphism is the projection $\mathbb{A}^\infty \to \operatorname{Spec} \mathbb{C}$. While any $\mathbb{C}$-algebra is the union of its finitely generated subalgebras, a special feature of $\mathbb{C}[x_1, x_2, \dots]$ is that it is the union of the finitely generated subalgebras over which it is flat (such as $\mathbb{C}[x_1, \dots, x_n]$ for any n). It is a classical fact that over a Noetherian base any such algebra is coherent. Moreover, such algebras enjoy better formal properties than arbitrary coherent rings, in particular satisfying a generalized Hilbert basis theorem.

Geometrically, this means that coherent sheaf theory on the resulting class of tamely presented schemes is mostly as well behaved as in the Noetherian case. Indeed, the fact that a morphism with coherent pullback can have infinite Tor-dimension is one of the few genuinely new complications that arise in tamely presented coherent sheaf theory. The basic purpose of this paper is to quantify the extent to which such morphisms are as well behaved in the tamely presented world as they are in the finitely presented world, where they can be controlled by their Tor-dimension. Implicitly, this quantifies the extent to which morphisms into pro-smooth schemes retain the good cohomological properties of morphisms into smooth varieties. To explain what properties we wish to consider, and to motivate the wider geometric context in which we will actually work, let us briefly describe our main application.

1.1. **Convolution on ind-geometric stacks.** Associated to a complex reductive group G and a finite-dimensional representation N is an infinite-dimensional space $\mathcal{R}_{G,N}$, introduced by Braverman-Finkelberg-Nakajima in [Nak16, BFN18]. Its $G_{\mathcal{O}}$-equivariant Borel-Moore and K-homology, endowed with their natural convolution products, model regular functions on the Coulomb branches of certain gauge theories associated to G and N. In [CW23] we study the corresponding convolution product on the category $\mathrm{Coh}^{G_{\mathcal{O}}}(\mathcal{R}_{G,N})$ of $G_{\mathcal{O}}$-equivariant coherent sheaves on $\mathcal{R}_{G,N}$.

Convolution in $\mathrm{Coh}^{G_{\mathcal{O}}}(\mathcal{R}_{G,N})$ involves pullback along a morphism $\delta_{G,N}$ obtained by base change from the diagonal of the quotient $N_{\mathcal{O}}/G_{\mathcal{O}}$. Various key properties of convolution ultimately hinge on $\delta_{G,N}$ behaving sufficiently like a morphism of finite Tor-dimension, despite merely having coherent pullback. In particular, to show $\mathrm{Coh}^{G_{\mathcal{O}}}(\mathcal{R}_{G,N})$ is rigid as a monoidal category we need to know that pullback along $\delta_{G,N}$ commutes with sheaf Hom and proper !-pullback, just as it would if $\delta_{G,N}$ were of finite Tor-dimension.

To this end we must consider such properties in a more general context than that of ordinary schemes. First, the space $\mathcal{R}_{G,N}$ is really an object of derived algebraic geometry, and we will always implicitly mean e.g. derived scheme when we say scheme. Second, $\mathcal{R}_{G,N}$ is not a scheme, but rather an ind-scheme. And third, in working equivariantly with respect to the infinite-type group scheme $G_{\mathcal{O}}$, we are implicitly discussing coherent sheaves on the quotient stack $\mathcal{R}_{G,N}/G_{\mathcal{O}}$.

We will formalize the nature of this quotient by viewing it as an *ind-geometric stack*. Here we use the term geometric stack in the sense of [Lur18, Ch. 9]. This generalizes the notion of a quasi-compact Artin stack with affine diagonal, allowing for derived structures, and for quotients by groups which are not of finite type. An ind-geometric stack is then an inductive limit of geometric stacks along closed immersions. A secondary purpose of this paper is to document the basic theory of ind-geometric stacks, extending various constructions and results from the more familiar theory ind-schemes. This formalism will justify itself by letting us navigate various technical issues that arise in infinite-dimensional equivariant sheaf theory as efficiently as possible.

1.2. **Definitions and results.** We now summarize our main definitions and results more precisely. We fix for the introduction a base field k of characteristic zero, though in the main text we allow more general bases (see our detailed conventions in Appendix A). We write CAlg_k for the ∞-category of nonpositively graded commutative dg k-algebras, or equivalently of connective $\mathbb{E}_\infty$-algebras over k.

Given $A \in \mathrm{CAlg}_k$ we write CAlg_A for the ∞-category of A-algebras in CAlg_k. We write $\tau_{\leq n}\mathrm{CAlg}_A \subset \mathrm{CAlg}_A$ for the subcategory of n-truncated algebras (i.e. dg algebras with vanishing cohomology below degree $-n$) and $\tau_{\leq n}B \in \tau_{\leq n}\mathrm{CAlg}_A$ for the n-truncation of $B \in \mathrm{CAlg}_A$. In particular, $\tau_{\leq 0}\mathrm{CAlg}_k$ is the category of ordinary commutative k-algebras. Recall that an A-algebra B is finitely n-presented if it is n-truncated and compact in $\tau_{\leq n}\mathrm{CAlg}_A$, and is almost finitely presented if $\tau_{\leq n}B$ is finitely n-presented for all n. If A is an ordinary commutative k-algebra, then $B \in \tau_{\leq 0}\mathrm{CAlg}_A$ is finitely zero-presented if and only if it is a finitely presented A-algebra in the ordinary sense.

Definition 1.1. Given $A \in \mathrm{CAlg}_k$, we say $B \in \mathrm{CAlg}_A$ is *strictly tamely n-presented* if it can be written as a filtered colimit $B \cong \operatorname{colim} B_\alpha$ of finitely n-presented A-algebras such that B is flat over B_α for all α. We say B is *strictly tamely presented* if $\tau_{\leq n}B$ is strictly tamely n-presented for all n.

This definition extends a classical construction of coherent rings [Bou72, Sec. I.2 Ex. 12]. It is closely related to the notion of almost smoothness [KV04, Def. 3.2.4] or placidity [Ras14, Def. 16.29.1], and a similar definition is considered in [Ras19, Sec. 6.36]. In particular, B is flat over each B_α if the structure morphisms $B_\alpha \to B_\beta$ are themselves flat (though the reverse implication does not hold, see Example 2.5). But it is only the relationship between B and the terms in the expression $\operatorname{colim} B_\alpha$, rather than among these terms, which will be directly invoked in proofs, and emphasizing this leads to a more intrinsic notion (Proposition 2.3). As discussed before, strictly tamely presented k-algebras are better behaved than arbitrary coherent k-algebras, for example being closed under tensor products (Proposition 2.4).

We write Stk_k for the ∞-category of stacks, i.e. functors $X : \mathrm{CAlg}_k \to \mathcal{S}$ satisfying faithfully flat descent (where $\mathcal{S}$ is the ∞-category of spaces). A stack is geometric if it admits a faithfully flat affine morphism $\operatorname{Spec} A \to X$ for some $A \in \mathrm{CAlg}_k$, and we write $\mathrm{GStk}_k \subset \mathrm{Stk}_k$ for the category of geometric stacks. A morphism of stacks is geometric if its base change to any affine scheme has geometric source. Definition 1.1 extends to affine morphisms in the obvious way.

Definition 1.2. A geometric morphism $X \to Y$ in Stk_k is *tamely presented* if for any $\operatorname{Spec} A \to Y$, there exists a strictly tamely presented flat cover $\operatorname{Spec} B \to \operatorname{Spec} A \times_Y X$ such that B is strictly tamely presented over A. A geometric stack is tamely presented if it is so over $\operatorname{Spec} k$.

Per the discussion above, the notion of tamely presented geometric stack is a variant of the notion of placid stack considered in [BKV22]. Tamely presented morphisms are stable under composition and base change, and are local on the target with respect to tamely presented flat covers (Proposition 3.6). Any representable, almost finitely presented morphism of geometric stacks is tamely presented (Proposition 3.4), and the quotient $N_{\mathcal{O}}/G_{\mathcal{O}}$ discussed above is tamely presented over $\operatorname{Spec}\mathbb{C}$.

A geometric stack is truncated if it admits a flat cover $\operatorname{Spec} A \to X$ such that A is n-truncated for some n, and we write $\mathrm{GStk}_k^+ \subset \mathrm{GStk}_k$ for the subcategory of such. Given $X \in \mathrm{GStk}_k^+$, we write $\mathrm{Coh}(X) \subset \mathrm{QCoh}(X)$ for the subcategory of bounded almost perfect sheaves. When X is an ordinary Noetherian scheme this is the usual (enhanced) bounded derived category of coherent sheaves, and when X is an ordinary quasi-compact scheme it is the (enhanced) homotopy category of bounded pseudo-coherent complexes.

Definition 1.3. A morphism $f : X \to Y$ in GStk_k^+ has *coherent pullback* if $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ takes $\mathrm{Coh}(Y)$ to $\mathrm{Coh}(X)$. It has *semi-universal coherent pullback* if whenever $h : Y' \to Y$ is a tamely presented morphism in GStk_k^+, the base change $f' : X' \to Y'$ has coherent pullback (and its source X' is truncated).

Morphisms with semi-universal coherent pullback are stable under composition, tamely presented base change, and are flat-local on the target (Proposition 3.11). We have the following key result in the affine case, which implies as a special case that $\Delta_{X_{\mathcal{O}}}$ has semi-universal coherent pullback when X is an affine variety étale over $\mathbb{A}^n$ (Proposition 3.14).

Theorem 1.4. *(Theorem 2.11) Let $U = \operatorname{Spec} A$, $V = \operatorname{Spec} B$ be affine schemes such that A is strictly tamely presented over k. Then a morphism $f : V \to U$ has semi-universal coherent pullback if and only if it has coherent pullback.*

In particular, we emphasize that semi-universal coherent pullback has the following simpler local characterization in the setting of tamely presented geometric stacks: if Y is tamely presented and $\operatorname{Spec} A \to Y$ is a flat cover demonstrating this, then $f : X \to Y$ has semi-universal coherent pullback if and only if its base change to $\operatorname{Spec} A$ has coherent pullback.

The above notions extend to ind-geometric stacks, defined as follows. Here $\widehat{\mathrm{Stk}}_k \subset \mathrm{Stk}_k$ denotes the subcategory of convergent (or nilcomplete) stacks, which in particular contains GStk_k (Proposition 3.24).

Definition 1.5. An *ind-geometric stack* (resp. *reasonable ind-geometric stack*) is a convergent stack which can be written as a filtered colimit in $\widehat{\mathrm{Stk}}_k$ of truncated geometric stacks along closed immersions (resp. almost finitely presented closed immersions).

This is the obvious extension of the derived notion of ind-scheme introduced in [GR14] (Proposition 4.30) and the derived notion of reasonableness introduced in [Ras19]. For example, the quotient $\mathrm{Gr}_G/G_{\mathcal{O}}$ of the affine Grassmannian is a reasonable ind-geometric stack (Example 4.31), and $\mathrm{Gr}_G/G_{\mathcal{O}} \cong \operatorname{colim} \overline{\mathrm{Gr}}^{\lambda}/G_{\mathcal{O}}$ presents it as a colimit of truncated geometric stacks if G is semisimple (where $\overline{\mathrm{Gr}}^{\lambda} \subset \mathrm{Gr}_G$ denotes a $G_{\mathcal{O}}$-orbit closure). Definition 1.2 extends in a standard way to give the notion of an ind-tamely presented morphism of reasonable ind-geometric stacks. The notion of a morphism with semi-universal coherent pullback likewise extends to the reasonable ind-geometric setting.

If X is a reasonable ind-geometric stack, the category of coherent sheaves on X is $\mathrm{Coh}(X) \cong \operatorname{colim} \mathrm{Coh}(X')$, where the colimit is over all reasonable geometric substacks (Proposition 4.23). By definition, any $\mathcal{F} \in \mathrm{Coh}(X)$ is thus a pushforward from some reasonable geometric substack of X. In addition to pullback along morphisms with semi-universal coherent pullback, coherent sheaves admit pushforward along ind-proper, almost ind-finitely presented morphisms, and these functors satisfy base change with respect to each other.

To discuss !-pullback and sheaf Hom we must also consider ind-coherent sheaves. When X is an ordinary Noetherian scheme $\mathrm{IndCoh}(X)$ is defined either as the ind-completion of $\mathrm{Coh}(X)$ or as the (enhanced) homotopy category of injective complexes on X [Kra05, Gai13]. A more general reformulation of the latter is that $\mathrm{IndCoh}(X)$ is the left anticompletion of $\mathrm{QCoh}(X)$ [Lur18, Thm. C.5.8.8]. This characterizes $\mathrm{IndCoh}(X)$ by a universal property with respect to bounded, colimit-preserving functors. We take this as the definition of $\mathrm{IndCoh}(X)$ when X is a geometric stack, and extend the definition to ind-geometric X by taking colimits over truncated geometric substacks. We caution that despite the notation $\mathrm{IndCoh}(X)$ is not obviously compactly generated in general.

Definition 1.6. A geometric stack X is coherent if it has a flat cover $\operatorname{Spec} A \to X$ such that A is coherent, and if $\mathrm{QCoh}(X)^{\heartsuit}$ (the heart of the standard t-structure) is compactly generated. An ind-geometric stack is coherent if it is reasonable and every reasonable geometric substack is coherent.

The assumption that X is coherent is sufficient to guarantee that $\mathrm{IndCoh}(X)$ is indeed compactly generated by the image of the natural embedding $\mathrm{Coh}(X) \subset \mathrm{IndCoh}(X)$ (Proposition 6.5). In particular, the above definitions are consistent with those of [GR14, Ras19] when X is a reasonable ind-scheme whose reasonable subschemes are locally coherent.

Given an ind-proper morphism $f : X \to Y$, we let $f^! : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$ denote the right adjoint of f_*. In a diagram such as (1.8) below, we then obtain from the isomorphism $h^* f_* \cong f'_* h'^*$ a Beck-Chevalley transformation $h'^* f^! \to f'^! h^*$ of functors $\mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X')$.

Theorem 1.7. *(Proposition 7.5) Let the following be a Cartesian diagram of coherent ind-geometric stacks.*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{1.8}$$

Suppose that Y and Y' are ind-tamely presented, that h has semi-universal coherent pullback, and that f is ind-proper and almost of ind-finite presentation. Then the Beck-Chevalley transformation $h'^* f^! \to f'^! h^*$ is an isomorphism.

We caution that coherence of X' does not follow from the other hypotheses, but does if we further assume h is affine (Proposition 6.8). More standard compatibilities of !-pullback also generalize to the ind-geometric setting (under suitable hypotheses), such as its compatibility with pushforward (Propositions 7.11 and 7.13) and finite Tor-dimension pullback (under wider conditions than above, Proposition 7.8).

While $\mathrm{IndCoh}(X)$ does not generally have a well-behaved tensor product, it does admit external products, and this is sufficient for a good notion of ind-coherent sheaf Hom. If X is reasonable and $\mathcal{F} \in \mathrm{Coh}(X)$, we define $\mathcal{H}om(\mathcal{F}, -) := (\mathcal{F} \boxtimes -)^R \Delta_{X*}$, where $(\mathcal{F} \boxtimes -)^R$ is the right adjoint of $\mathcal{F} \boxtimes - : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(X \times X)$. When X is geometric, this is compatible with the usual quasi-coherent sheaf Hom (Proposition 8.13), the latter being the right adjoint of $\mathcal{F} \otimes - \cong \Delta_X^*(\mathcal{F} \boxtimes -)$. In the ind-geometric case, $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is determined by this compatibility together with the fact that if $\mathcal{F} \cong i_*(\mathcal{F}')$ for some reasonable geometric substack $i : X' \to X$ and $\mathcal{F}' \in \mathrm{Coh}(X')$, then $\mathcal{H}om(\mathcal{F}, \mathcal{G}) \cong i_* \mathcal{H}om(\mathcal{F}', i^!(\mathcal{G}))$ (Corollary 8.43).

External products are compatible with pushforward and pullback along morphisms of finite Tor-dimension. This extends to morphisms with coherent pullback under suitable hypotheses. Under such hypotheses, a morphism $f : X \to Y$ with coherent pullback canonically induces a natural transformation

$$f^* \mathcal{H}om(\mathcal{F}, -) \to \mathcal{H}om(f^*(\mathcal{F}), f^*(-)) \tag{1.9}$$

of functors $\mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$ for any $\mathcal{F} \in \mathrm{Coh}(Y)$, as in the quasi-coherent setting.

Theorem 1.10. *(Corollary 8.34) Let $f : X \to Y$ be a morphism with semi-universal coherent pullback between coherent, ind-tamely presented ind-geometric stacks, and assume $X \times X$, $X \times Y$, and $Y \times Y$ are also coherent. Then the natural transformation (1.9) is an isomorphism.*

We note that coherence of the indicated products is automatic if X and Y are each tamely presented and affine over an ind-geometric stack which is ind-locally Noetherian (Proposition 6.9). Other more standard properties of sheaf Hom (and more fundamentally, of $(\mathcal{F} \boxtimes -)^R$) also extend to the ind-geometric setting, such as compatiblity with pushforward (Proposition 8.40), compatibility with finite Tor-dimension pullback (under wider hypotheses than above, Proposition 8.25), and almost continuity (Proposition 8.16).

1.3. **Further comments.** A precise definition of the category $\mathrm{Coh}^{G_\mathcal{O}}(\mathcal{R}_{G,N})$ first appears in [Ras19]. In the terminology of [Ras19], the quotient $\mathcal{R}_{G,N}/G_\mathcal{O}$ is a flat, affine inverse limit of renormalizable prestacks. The main difference between this formalism and the one taken here is whether $\mathcal{R}_{G,N}/G_\mathcal{O}$ is viewed primarily as a quotient of an inductive limit of schemes or as an inductive limit of quotients of schemes. The latter view turns out to be more efficient for our purposes.

On one hand, this lets results about coherent pullback be reduced to the affine case in fewer steps, non-affine schemes being bypassed entirely. On the other, compared to quotients, inductive limits have a more dramatic effect on coherent sheaf theory, insofar as they break the close relationship between quasi-coherent and ind-coherent sheaves. The result is that certain technical complications (e.g. the appearance of multiple anticompletion/renormalization steps) are avoided if inductive limits are delayed until after all quotients have been taken.

A related point is that the theory of ind-geometric stacks is logically parallel to, rather than logically dependent on, the theory of ind-schemes. A significant portion of the present paper is devoted to setting up this theory, which we expect to be a convenient language for other applications. But the conceptual framework and technical tools needed to carry this out are mostly the same as in the case of ind-schemes, and thus are due in the derived setting to Gaitsgory, Rozenblyum, and Raskin [GR14, Ras19]. Except for issues specifically related to coherent pullback, the arguments and overall approach of Sections 4 and 7, in particular, largely follow their counterparts for ind-schemes in [GR14, Ras19].

In light of the previous paragraph, let us highlight a few technical complications presented by ind-geometric stacks that are not presented by ind-schemes. First, if X is a geometric stack $\mathrm{QCoh}(X)$ and (as far as we know) $\mathrm{QCoh}(X)^\heartsuit$ need not be compactly generated. The former pathology is in fact typical when working equivariantly with respect to an infinite-type group such as $G_\mathcal{O}$. The latter implies that the ind-completion of $\mathrm{Coh}(X)$ need not be well-behaved in general, hence the importance of defining $\mathrm{IndCoh}(X)$ via anticompletion. A related pathology is that when X and Y are schemes (and k is a perfect field) the external product functor $\mathrm{IndCoh}(X) \otimes_{\mathrm{Mod}_k} \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X \times Y)$ is an equivalence [Gai13, Prop. 4.6.2], but this is not obviously the case when X and Y are merely geometric.

Additionally, while a quasi-compact, quasi-separated scheme is compact as an object of Stk_k (as follows from the existence of scallop decompositions [Lur18, Thm. 3.4.2.1]), this is not obviously the case for a general geometric stack. Compactness in Stk_k is used for certain arguments in [GR14], but here we get by with the fact that a truncated geometric stack is compact in $1\text{-}\widehat{\mathrm{Stk}}_k \subset \widehat{\mathrm{Stk}}_k$, the subcategory of convergent stacks such that $X(A)$ is $(n+1)$-truncated when A is n-truncated (Proposition 3.27). On the other hand, the result of [GR14, Prop. 1.4.4] that an ind-scheme can be written as a filtered colimit of schemes in Stk_k (as opposed to $\widehat{\mathrm{Stk}}_k$) does not obviously generalize to the ind-geometric case.

As alluded to earlier, in the main text we allow k to be any Noetherian $\mathbb{E}_\infty$-ring, except when discussing external products — these we only consider when k is an ordinary Noetherian ring of finite global dimension. In particular, we formally work with spectral rather than derived stacks when the two notions are distinct (they agree when k is an ordinary ring of characteristic zero). This is a natural choice given our focus on quasi-coherent sheaf theory: any derived stack has an underlying spectral stack, and the latter determines the category of quasi-coherent sheaves on the former. It also allows us to make use of spectral Tannaka duality in addressing certain issues about convergence and closed immersions (i.e. in Propositions 3.24 and 3.29), whereas in the derived setting we would need other tools.

It is clear that most, and perhaps all, notions and results in this paper extend from geometric stacks to quasi-geometric stacks (as in [Lur18, Ch. 9]) with only minor modifications of the proofs. We have not checked this systematically, though we note that considering ind-quasi-geometric stacks sidesteps the minor nuisance that an ind-scheme is ind-geometric if and only if it can be written as a filtered colimit of semi-separated, rather than merely quasi-separated, schemes. With more work much of our discussion also presumably extends to higher geometric stacks (as in [Sim96, TV08]). We do caution that what we call a geometric stack following [Lur18, Ch. 9] corresponds to a zero-geometric stack in the terminology of [TV08, Sec. 1.3.3], where a stack is called geometric if it is n-geometric for some n (and where the homotopical context is left as a variable, while we have fixed one). Similarly, what we call a strictly tamely presented morphism would be called a (-1)-tamely presented morphism following [TV08, Def. 1.3.3.1].

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2. Affine schemes

We begin by studying (strictly) tamely presented morphisms and coherent pullback in the setting of affine schemes. We first establish some basic stability and coherence properties of the latter (Propositions 2.4 and 2.8). We then show that morphisms with coherent pullback and strictly tamely presented source are stable under strictly tamely presented base change (Theorem 2.11). We also show that a strictly tamely presented algebra over a Noetherian base has coherent pullback if and only if it is of finite Tor-dimension (Proposition 2.17).

2.1. **Strictly tame presentations.** Given $A \in \mathrm{CAlg}_k$, an A-algebra B is finitely n-presented if it is n-truncated and compact in $\tau_{\leq n}\mathrm{CAlg}_A$, and is almost of finite presentation if $\tau_{\leq n}B$ is finitely n-presented for all n. An A-algebra B is finitely zero-presented if and only if it is an ordinary commutative ring and is finitely presented over $\tau_{\leq 0}A = H^0(A)$ in the sense of ordinary commutative algebra. We recall that B is flat over A if $H^0(B)$ is flat over $H^0(A)$ in the ordinary sense and the natural map $H^0(B) \otimes_{H^0(A)} H^n(A) \to H^n(B)$ is an isomorphism for all n.

Definition 2.1. Given $A \in \mathrm{CAlg}_k$, we say $B \in \mathrm{CAlg}_A$ is *strictly tamely n-presented* if it can be written as a filtered colimit $B \cong \operatorname{colim} B_\alpha$ of finitely n-presented A-algebras such that B is flat over each B_α. We call such an expression a *strictly tame presentation of order* n. We say B is *strictly tamely presented* if $\tau_{\leq n}B$ is strictly tamely n-presented for all n.

A strictly tamely n-presented algebra is n-truncated, since $\tau_{\leq n}\mathrm{CAlg}_A$ is stable under filtered colimits by [Lur17, Prop. 7.2.4.27], [Lur09, Cor. 5.5.7.4]. If A is Noetherian, [Lur17, Prop.

7.2.4.31] implies that an ordinary A-algebra B is strictly tamely presented if and only if it is strictly tamely zero-presented. Definition 2.1 extends from algebras to morphisms in CAlg_k in the obvious way. It would be more precise to say “almost strictly tamely presented” instead of “strictly tamely presented”, but for simplicity we use the shorter terminology.

Example 2.2. If A and B are ordinary rings and B is placid over A in the sense of [Ras14, Def. 16.29.1], or equivalently almost smooth over A in the sense of [KV04, Def. 3.2.4], then B is strictly tamely zero-presented.

Strictly tamely n-presented algebras have the following more intrinsic characterization. In the setting of ordinary rings, a (more elementary) variant of the proof characterizes strictly tamely zero-presented algebras as those which are the union of the finitely zero-presented subalgebras over which they are flat.

Proposition 2.3. *Given $A \in \mathrm{CAlg}_k$, an n-truncated A-algebra B is strictly tamely n-presented if and only if it satisfies the following condition. For every finitely n-presented A-algebra C, every morphism $C \to B$ in CAlg_A admits a factorization $C \to C' \to B$ such that C' is finitely n-presented over A and B is flat over C'.*

Proof. The only if direction follows from compactness of C in $\tau_{\leq n}\mathrm{CAlg}_A$. For the if direction, let $\mathrm{CAlg}_A^{n\text{-}fp}$ denote the category of finitely n-presented A-algebras, and $(\mathrm{CAlg}_A^{n\text{-}fp})_{/f\text{-}B} \subset (\mathrm{CAlg}_A^{n\text{-}fp})_{/B} = \mathrm{CAlg}_A^{n\text{-}fp} \times_{\mathrm{CAlg}_A} (\mathrm{CAlg}_A)_{/B}$ the full subcategory of algebras over which B is flat. It suffices to show $(\mathrm{CAlg}_A^{n\text{-}fp})_{/f\text{-}B}$ is filtered and that B is the colimit over its forgetful functor to CAlg_A.

Since CAlg_A is compactly generated so is $\tau_{\leq n}\mathrm{CAlg}_A$ [Lur09, Cor. 5.5.7.4], hence $(\mathrm{CAlg}_A^{n\text{-}fp})_{/B}$ is filtered and B is the colimit over its forgetful functor to $\tau_{\leq n}\mathrm{CAlg}_A$ [Lur09, Cor. 5.3.5.4, Cor. 5.5.7.4]. For any $B' \in \mathrm{CAlg}_A^{n\text{-}fp}$, we have $(\mathrm{CAlg}_A^{n\text{-}fp})_{B'/} \cong \mathrm{CAlg}_{B'}^{n\text{-}fp}$ by [Lur18, Prop. 4.1.3.1] and the fact that being finitely n-presented is equivalent to being n-truncated and of finite generation to order $n+1$ [Lur18, Rem. 4.1.1.9]. It thus suffices to show $(\mathrm{CAlg}_{B'}^{n\text{-}fp})_{/f\text{-}B}$ is filtered for all $B' \in (\mathrm{CAlg}_A^{n\text{-}fp})_{/B}$, since then $(\mathrm{CAlg}_A^{n\text{-}fp})_{/f\text{-}B} \to (\mathrm{CAlg}_A^{n\text{-}fp})_{/B}$ is left cofinal by [Lur09, Thm. 4.1.3.1, Lem. 5.3.1.18], and $(\mathrm{CAlg}_A^{n\text{-}fp})_{/f\text{-}B}$ is filtered as a special case.

We must show any finite diagram $K \to (\mathrm{CAlg}_{B'}^{n\text{-}fp})_{/f\text{-}B}$ extends to a diagram $K^{\triangleright} \to (\mathrm{CAlg}_{B'}^{n\text{-}fp})_{f\text{-}B}$. Since $(\mathrm{CAlg}_{B'}^{n\text{-}fp})_{/B}$ is filtered, we have an extension $K^{\triangleright} \to (\mathrm{CAlg}_{B'}^{n\text{-}fp})_{/B}$. The image of the cone point is finitely n-presented over A, hence by hypothesis its morphism to B factors through some $C \in (\mathrm{CAlg}_A^{n\text{-}fp})_{/f\text{-}B}$. But C is also finitely n-presented over B' (again by [Lur18, Prop. 4.1.3.1]), hence composing with this we obtain an extension $K^{\triangleright} \to (\mathrm{CAlg}_{B'}^{n\text{-}fp})_{/f\text{-}B}$. □

Morphisms of strictly tame presentation have the following stability properties.

Proposition 2.4. *Let $\phi : A \to B, \psi : B \to C$, and $\eta : A \to A'$ be morphisms in CAlg_k.*

(1) If ϕ is of strictly tame presentation (resp. is strictly tamely n-presented) then so is $\phi' : A' \to B \otimes_A A'$ (resp. $\tau_{\leq n}\phi' : \tau_{\leq n}A' \to \tau_{\leq n}(B \otimes_A A')$).

(2) *If ϕ and ψ are of strictly tame presentation (resp. are strictly tamely n-presented) then so is $\psi \circ \phi$.*

(3) *If ϕ is almost of finite presentation (resp. is finitely n-presented) then $\psi \circ \phi$ is of strictly tame presentation (resp. is strictly tamely n-presented) if and only if ψ is.*

Proof. Note that in each case it suffices to prove the claim about strictly tamely n-presented morphisms, as it implies the claim about strictly tamely presented morphisms. For (1), let $B \cong \operatorname{colim} B_\alpha$ be a strictly tame presentation of order n. Then $\tau_{\leq n}(B_\alpha \otimes_A A')$ is finitely n-presented over A' for all α [Lur18, Prop. 4.1.3.2]. Since $\tau_{\leq n}$ is continuous and compatible with the symmetric monoidal structure on $\mathrm{Mod}_A^{\leq 0}$ [Lur17, Prop. 7.1.3.15], hence on CAlg_A, we have

$$\tau_{\leq n}(B \otimes_A A') \cong \tau_{\leq n}((\tau_{\leq n} B) \otimes_{\tau_{\leq n} A} \tau_{\leq n} A') \cong \operatorname{colim} \tau_{\leq n}(B_\alpha \otimes_{\tau_{\leq n} A} \tau_{\leq n} A') \cong \operatorname{colim} \tau_{\leq n}(B_\alpha \otimes_A A').$$

Since flatness is preserved by base change and $\tau_{\leq n}$, it follows that $\tau_{\leq n}(B \otimes_A A')$ is strictly tamely n-presented over A'.

For (2), let $B \cong \operatorname{colim} B_\alpha$ and $C \cong \operatorname{colim} C_\beta$ be strictly tame presentations of order n over A and B, respectively. Given a finitely n-presented A-algebra A and a morphism $D \to C$ in CAlg_A, we claim the criterion of Proposition 2.3 is satisfied. Note first that $D \to C$ factors through some C_β since D is compact in $\tau_{\leq n}\mathrm{CAlg}_A$. By Noetherian approximation [Lur18, Cor. 4.4.1.4] we have $C_\beta \cong \tau_{\leq n}(B \otimes_{B_\alpha} C_{\alpha\beta})$ for some α and some finitely n-presented B_α-algebra $C_{\alpha\beta}$. Since B is flat over B_α we in fact have $C_\beta \cong B \otimes_{B_\alpha} C_{\alpha\beta}$. Letting $C_{\gamma\beta} := B_\gamma \otimes_{B_\alpha} C_{\alpha\beta}$ for $\gamma \geq \alpha$, we moreover have $C_\beta \cong \tau_{\leq n} C_\beta \cong \operatorname{colim}_{\gamma \geq \alpha} \tau_{\leq n} C_{\gamma\beta}$.

Again by compactness $D \to C_\beta$ factors through some $\tau_{\leq n} C_{\gamma\beta}$. Now $B_\gamma \to \tau_{\leq n} C_{\gamma\beta}$ is finitely n-presented since $B_\alpha \to C_{\alpha\beta}$ is [Lur18, Prop. 4.1.3.2], hence the composition $A \to B_\gamma \to \tau_{\leq n} C_{\gamma\beta}$ is finitely n-presented [Lur18, Prop. 4.1.3.1]. But $\tau_{\leq n} C_{\gamma\beta} \to C_\beta$ is flat since $B_\gamma \to B$ is and since flatness is preserved by $\tau_{\leq n}$, hence the composition $\tau_{\leq n} C_{\gamma\beta} \to C_\beta \to C$ is flat and we are done.

For (3), suppose $C \cong \operatorname{colim} C_\alpha$ is a strictly tame presentation of order n over A. By compactness of B in $\tau_{\leq n}\mathrm{CAlg}_A$ we have that $B \to C$ factors through some C_α. By [Lur18, Prop. 4.1.3.1] the morphism $B \to C_\alpha$ is finitely n-presented, as is the composition $B \to C_\alpha \to C_\beta$ for all $\beta \geq \alpha$. But then $C \cong \operatorname{colim}_{\beta \geq \alpha} C_\beta$ is a strictly tame presentation of order n over B. This proves the only if direction, and the if direction follows from (2). ∎

Note that if $B \cong \operatorname{colim} B_\alpha$ is a filtered colimit of finitely n-presented algebras such that the structure maps $B_\alpha \to B_\beta$ are flat, it follows that B is flat over each B_α. We do not know if every strictly tamely n-presented algebra admits a presentation with this stronger property, but it is easy to construct presentations which do not have it.

Example 2.5. Given $\{a_n\}_{n \in \mathbb{N}} \subset \mathbb{C}$, let B_n be the localization of $\mathbb{C}[x, y]/((x - a_n)y)$ by the elements $\{x - a_m\}_{m<n}$. In other words, $\operatorname{Spec} B_n$ is $\mathbb{A}^1$ with the points $a_1, \ldots, a_{n-1}$ removed and with another $\mathbb{A}^1$ intersecting at the point a_n. If $B_n \to B_{n+1}$ takes x to x and y to 0,

then $B := \operatorname{colim} B_n$ is the localization of $\mathbb{C}[x]$ by the elements $\{x - a_n\}_{n\in\mathbb{N}}$. In particular it is a localization of each B_n, hence each $B_n \to B$ is flat even though no $B_m \to B_n$ is.

On the other hand, if $B \cong \operatorname{colim} B_\alpha$ is a strictly tame presentation of order n and B is *faithfully* flat over each B_α, it follows that the structure morphisms $B_\alpha \to B_\beta$ are also faithfully flat [Lur18, Lem. B.1.4.2]. But many natural examples, in particular those of the following class, do not admit such presentations.

Example 2.6. If A is an ordinary ring, recall that an ordinary A-algebra B is essentially finitely presented if it is a localization $B \cong S^{-1}C$ of a finitely zero-presented A-algebra C. In this case $B \cong \operatorname{colim} S_{fin}^{-1}C$, where the colimit is over all finite subsets $S_{fin} \subset S$. This is a strictly tame presentation of order zero, hence any essentially finitely presented algebra is strictly tamely zero-presented.

Often we can write an A-algebra of strictly tame presentation as a filtered colimit $B \cong \operatorname{colim} B_\alpha$ such that each B_α is almost finitely presented over A and B is flat over each B_α. However, not every example is of this form. Moreover, the associated class of morphisms in CAlg_k is not obviously stable under composition, since almost finitely presented algebras cannot be directly controlled by Noetherian approximation the way finitely n-presented algebras can be. The situation is improved by suitable truncatedness hypotheses, but these are in turn not stable under base change. Thus Proposition 2.4 does not extend robustly to such morphisms.

Example 2.7. Let $A_0 = \mathbb{C}[x_1, x_2, \dots]$, $A_n = A_0/(x_1, \dots, x_n)$, and A the trivial square-zero extension of A_0 by $\oplus_{n>0} A_n[n]$. Then A is a strictly tamely presented $\mathbb{C}$-algebra, but cannot be written as a filtered colimit of almost finitely presented $\mathbb{C}$-algebras along flat morphisms: A_0 has no finitely presented subalgebra from which each A_n is obtained by base change, which flatness would require.

Recall that an ordinary commutative ring A is coherent if every finitely generated ideal is finitely presented. More generally, $A \in \mathrm{CAlg}_k$ is coherent if $H^0(A)$ is coherent in the above sense and $H^n(A)$ is a finitely presented $H^0(A)$-module for all n. In general this is a brittle property, and coherence of an ordinary ring A does not even imply coherence of $A[x]$. But tamely presentedness over a Noetherian base implies a more robust form of coherence.

Proposition 2.8. *Let A be a strictly tamely presented k-algebra. Then A is coherent, as is any strictly tamely presented A-algebra.*

Proof. The second claim follows from the first by the stability of strictly tamely presented morphisms under composition (Proposition 2.4). That $H^0(A)$ is coherent is essentially [Bou72, Sec. I.2 Ex. 12], but we repeat the argument. Let $H^0(A) \cong \operatorname{colim} A_\alpha$ be a strictly tame presentation of order zero over k. If $I \subset H^0(A)$ is a finitely generated ideal, we can write it as the image of a morphism $\phi : H^0(A)^n \to H^0(A)$ for some n. This is obtained by extension of scalars from some $\phi_\alpha : A_\alpha^n \to A_\alpha$ for some α. The kernel of ϕ_α is finitely generated since k and hence A_α are Noetherian, but $\ker \phi \cong (\ker \phi_\alpha) \otimes_{A_\alpha} H^0(A)$ since $H^0(A)$ is flat over A_α.

Now fix n, let $\tau_{\leq n} A \cong \operatorname{colim} A_\alpha$ be a strictly tame presentation of order n, and choose some α. Each A_α is Noetherian [Lur18, Prop. 4.2.4.1], hence $H^n(A_\alpha)$ is a finitely presented $H^0(A_\alpha)$-module. Since $A_\alpha \to \tau_{\leq n} A$ is flat $H^n(A) \cong H^n(A_\alpha) \otimes_{H^0(A_\alpha)} H^0(A)$, hence $H^n(A)$ is a finitely presented $H^0(A)$-module. The claim follows since n was arbitrary. □

Remark 2.9. Recall that an ordinary commutative ring is stably coherent if any finitely generated algebra over it is coherent. The above proof almost adapts to show that B is coherent if it is strictly tamely presented over an $A \in \mathrm{CAlg}_k$ such that A is coherent and $H^0(A)$ is stably coherent. No changes are needed if B is an ordinary ring, while [Lur18, Prop. 5.2.2.1] can be leveraged if $H^0(A)$ is of characteristic zero, or more generally if $A \to B$ arises from a morphism of animated/simplicial commutative rings (as do the terms in the needed strictly tame presentations). Plausibly these restrictions are unnecessary, the only question being whether the hypotheses on A imply the free algebra $A_m := A\{x_1, \ldots, x_m\}$ is coherent (i.e. if each $H^n(A_m)$ is finitely presented over $H^0(A_m)$). On the other hand, we do not know an A satisfying these hypotheses which is not strictly tamely presented over a Noetherian ring (possibly after passing to a flat cover).

2.2. **Coherent pullback.** Recall that an A-module M is almost perfect if $\tau^{\geq n} M$ is compact in $\mathrm{Mod}_A^{\geq n}$ for all n [Lur18, Rem. 2.7.0.5]. If M is almost perfect it is right bounded, and if A is coherent M is almost perfect if and only if it is right bounded and $H^n(M)$ is a finitely presented $H^0(A)$-module for all n. If A is an ordinary ring, M is almost perfect if and only if it is pseudo-coherent in the sense of [Ill71], see [Lur18, Rem. 2.8.4.6].

We say $M \in \mathrm{Mod}_A$ is coherent if it is almost perfect and (left) bounded, and denote the full subcategory of coherent modules by $\mathrm{Coh}_A \subset \mathrm{Mod}_A$. We recall the following definition from the introduction (we will use the same terminology for algebra morphisms as for the associated morphisms of affine schemes).

Definition 2.10. A morphism $A \to B$ in CAlg_k has coherent pullback if $M \otimes_A B$ is a coherent B-module for every coherent A-module M.

Equivalently, $A \to B$ has coherent pullback if and only if $M \otimes_A B$ is (left) bounded for every coherent A-module M, since $M \otimes_A B$ is almost perfect over B if M is almost perfect over A [Lur18, Prop. 2.7.3.1]. We note that morphisms with coherent pullback are called eventually coconnective morphisms in [Gai13, Def. 3.5.2].

We have seen in the introduction that morphisms with coherent pullback are not stable under arbitrary base change: the diagonal of $\mathbb{A}^\infty$ has coherent pullback but its base change along itself does not. More basically, this is true of the inclusion $\{0\} \hookrightarrow \mathbb{A}^\infty$. The following result says that this only happens because these maps are "too far" from being finitely presented, and that such pathologies do not appear if we only consider strictly tamely presented base change.

Theorem 2.11. *Let the following be a coCartesian diagram in* CAlg_k*.*

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\phi"'] & A' \arrow[d, "\phi'"] \\ B \arrow[r, "\psi'"] & B \otimes_A A' \end{tikzcd} \tag{2.12}$$

Suppose that ϕ has coherent pullback, ψ is strictly tamely presented, and A is strictly tamely presented over k. Then ϕ' has coherent pullback.

The proof will use the following reformulation of [Swa19, Thm. 7.1].

Lemma 2.13. *Let A be an ordinary commutative ring such that A and $A[x]$ are coherent. Then $\mathrm{Coh}_{A[x]}$ is the smallest full stable subcategory of $\mathrm{Mod}_{A[x]}$ which contains the essential image of Coh_A under $- \otimes_A A[x]$.*

Proof. Let $\mathcal{C} \subset \mathrm{Mod}_{A[x]}$ be the smallest full stable subcategory containing the essential image of Coh_A, or equivalently of $\mathrm{Coh}_A^{\heartsuit}$. Since $A[x]$ is coherent it suffices to show $\mathrm{Coh}_{A[x]}^{\heartsuit} \subset \mathcal{C}$.

Given $M \in \mathrm{Coh}_{A[x]}^{\heartsuit}$, choose an exact sequence $0 \to N \to F \to F' \to M \to 0$ with F and F' free of finite rank. Certainly $F, F' \in \mathcal{C}$, so it suffices to show $N \in \mathcal{C}$. But this follows from the proof of [Swa19, Thm. 7.1], which shows that N fits into an exact sequence $0 \to N' \otimes_A A[x] \to N'' \otimes_A A[x] \to N \to 0$ with $N', N'' \in \mathrm{Coh}_A^{\heartsuit}$. □

Proof of Theorem 2.11. Set $B' := B \otimes_A A'$. Consider the following diagram in CAlg_k, where all but the top and bottom faces are coCartesian.

$$\begin{tikzcd} & A \arrow[rr] \arrow[dl] \arrow[dd, "\phi"] & & A' \arrow[dl] \arrow[dd, "\phi'"] \\ H^0(A) \arrow[rr] \arrow[dd, "\xi"'] & & H^0(A') \arrow[dd, "\xi'"] & \\ & B \arrow[rr] \arrow[dl] & & B' \arrow[dl] \\ H^0(A) \otimes_A B \arrow[rr] & & H^0(A') \otimes_{A'} B' & \end{tikzcd} \tag{2.14}$$

Proposition 2.8 implies A and A' are coherent. In particular, restriction of scalars along $A \to H^0(A)$ preserves coherence, hence ξ has coherent pullback since ϕ does. Moreover, it suffices to show $M \otimes_{A'} B'$ is bounded for $M \in \mathrm{Coh}_{A'}^{\heartsuit}$. But any such M is obtained by restriction of scalars along $A' \to H^0(A')$, hence it suffices to show ξ' has coherent pullback.

Since k is Noetherian, $k \to H^0(A)$ is strictly tamely presented since $k \to A$ is [Lur17, Prop. 7.2.4.31]. Similarly $H^0(\psi)$ is strictly tamely presented: if $H^0(A') \cong \operatorname{colim} A'_\alpha$ is a strictly tame presentation of order zero, each A'_α is almost finitely presented over $H^0(A)$ by [Lur18, Cor. 5.2.2.3] (note that any polynomial ring over $H^0(A)$ is coherent by Proposition 2.8). Replacing (2.12) with the front face of (2.14), we may thus assume A and A' are ordinary commutative rings.

Next suppose that ψ is finitely zero-presented. For some n we can factor (2.12) as

$$\begin{tikzcd} A \arrow[r, "\theta"] \arrow[d, "\phi"'] & A[x_1,\dots,x_n] \arrow[r, "\xi"] \arrow[d, "\phi''"] & A' \arrow[d, "\phi'"] \\ B \arrow[r, "\theta'"] & B \otimes_A A[x_1,\dots,x_n] \arrow[r, "\xi'"] & B', \end{tikzcd} \tag{2.15}$$

where ξ is surjective and finitely presented. Since $A[x_1,\dots,x_n]$ and A' are coherent, restriction of scalars along ξ preserves coherence. But restriction of scalars along ξ' is conservative and t-exact, hence ϕ' has coherent pullback if ϕ'' does. Thus we may replace (2.12) with the left square of (2.15) and assume $A' \cong A[x_1,\dots,x_n]$, and by induction we may then assume $n = 1$.

Now let $\mathcal{C} \subset \mathrm{Mod}_{A'}$ be the full subcategory of M such that $M \otimes_{A'} B'$ is bounded. Since $- \otimes_{A'} B'$ is exact, $\mathcal{C}$ is stable. Since ϕ has coherent pullback and ψ' is flat, $\mathcal{C}$ contains the essential image of Coh_A under $- \otimes_A A'$. Thus $\mathrm{Coh}_{A'} \subset \mathcal{C}$ by Lemma 2.13, hence ϕ' has coherent pullback.

Now suppose $A' \cong \operatorname{colim} A'_\alpha$ is a strictly tame presentation of order zero. Since A' is coherent, it suffices to show that $M \otimes_{A'} B'$ is bounded for any $M \in \mathrm{Coh}^{\heartsuit}_{A'}$. For each α we write the induced factorization of (2.12) as

$$\begin{tikzcd} A \arrow[r, "\theta_\alpha"] \arrow[d, "\phi"'] & A'_\alpha \arrow[r, "\xi_\alpha"] \arrow[d, "\phi_\alpha"] & A' \arrow[d, "\phi'"] \\ B \arrow[r, "\theta'_\alpha"] & B'_\alpha \arrow[r, "\xi'_\alpha"] & B'. \end{tikzcd}$$

By flatness of the ξ_α and by e.g. [Lur18, Cor. 4.5.1.10] or [TT90, Sec. C.4], there exists an α and $M_\alpha \in \mathrm{Coh}^{\heartsuit}_{A'_\alpha}$ such that $M \cong M_\alpha \otimes_{A'_\alpha} A'$. Since θ_α is finitely zero-presented, we have already shown that $M_\alpha \otimes_{A'_\alpha} B'_\alpha$ is bounded. But ξ'_α is flat since ξ_α is, hence $M \otimes_{A'} B' \cong M_\alpha \otimes_{A'_\alpha} A' \otimes_{A'} B' \cong M_\alpha \otimes_{A'_\alpha} B'_\alpha \otimes_{B'_\alpha} B'$ is also bounded. □

Remark 2.16. The conclusion of Theorem 2.11 holds if instead of A being strictly tamely presented over k we assume that A and A' are coherent and that $H^0(A)$ is stably coherent. This is because the proof only uses this hypothesis on A in order to apply Proposition 2.8.

Recall that a morphism $\phi : A \to B$ in CAlg_k is of Tor-dimension $\leq n$ if $B \otimes_A M \in \mathrm{Mod}_B^{\geq -n}$ for all $M \in \mathrm{Mod}_A^{\heartsuit}$, and is of finite Tor-dimension if it is of Tor-dimension $\leq n$ for some n. Clearly ϕ has coherent pullback if it is of finite Tor-dimension. We have the following partial converse, which generalizes [Gai13, Lem. 3.6.3].

Proposition 2.17. *Suppose $\phi : A \to B$ is a morphism in CAlg_k with coherent pullback. If A is Noetherian and ϕ of strictly tame presentation, then ϕ is of finite Tor-dimension.*

Proof. Note that ϕ is of finite Tor-dimension if and only if its base change $\phi' : H^0(A) \to H^0(A) \otimes_A B$ is, since every discrete A-module is obtained by restriction of scalars from $H^0(A)$. Similarly ϕ has coherent pullback if and only if ϕ' does, since A is Noetherian (and in particular coherent). Since ϕ is of strictly tame presentation so is ϕ' (Proposition 2.4), hence we may replace ϕ with ϕ' and assume A is classical.

In particular, A is now coherent over itself, hence B is truncated since ϕ has coherent pullback. Since ϕ is of strictly tame presentation it admits a factorization

$$A \xrightarrow{\phi_1} A' \xrightarrow{\phi_2} B$$

such that ϕ_2 is flat and ϕ_1 is finitely n-presented for some n. Since A is Noetherian, ϕ_1 is then almost finitely presented and A' is also Noetherian [Lur18, Prop. 4.2.4.1].

Recall that ϕ_1 is of Tor-dimension $\leq n$ at a prime ideal $P' \subset H^0(A')$ if the localization $A'_{P'}$ is of Tor-dimension $\leq n$ over A. For any n the set of such prime ideals forms a Zariski open subset U_n of $|\mathrm{Spec}\, A'|$, the underlying topological space of $\mathrm{Spec}\, A'$ [Lur18, Lem. 6.1.5.5]. Since A' is Noetherian we can increase n as needed so that U_n is equal to the union of the U_m for all $m \in \mathbb{N}$. We claim that ϕ is of Tor-dimension $\leq n$.

It suffices to show that for any prime ideal $P \subset H^0(B)$, B_P is of Tor-dimension $\leq n$ over A [Lur18, Prop. 6.1.4.4]. Write $P' \subset H^0(A')$ and $Q \subset H^0(A)$ for the preimages of P under $H^0(\phi_2)$ and $H^0(\phi)$, and write κ for the residue field of A at Q. The morphism $\psi : A \to B_P$ can be factored as the middle row of the following diagram.

$$\begin{tikzcd}
 & A' \arrow[r, "\phi_2"] \arrow[d] & B \arrow[d] \\
A \arrow[ur, "\phi_1"] \arrow[r, "\psi_1"] \arrow[d] & A'_{P'} \arrow[r, "\psi_2"] \arrow[d] & B_P \arrow[d] \\
\kappa \arrow[r, "\psi'_1"] & A'_{P'} \otimes_A \kappa \arrow[r, "\psi'_2"] & B_P \otimes_A \kappa
\end{tikzcd}$$

Since ϕ_2 is flat so is ψ_2 [Lur18, Rem. 6.1.4.3], hence ψ is of Tor-dimension $\leq n$ if ψ_1 is, or equivalently if $P' \in U_n$.

Suppose $P' \notin U_n$. Then $P' \notin U_m$ for any m, hence $A'_{P'} \otimes_A \kappa$ is not truncated [Lur18, Lem. 6.1.5.2]. But ψ_2 is in fact faithfully flat since $H^0(\psi_2) : H^0(A'_{P'}) \to H^0(B_P)$ is a local ring homomorphism, so $B_P \otimes_A \kappa$ is also not truncated. Now note that ψ has coherent pullback since ϕ does, given that it is the composition of ϕ with the flat morphism $B \to B_P$. Since A is Noetherian, $A \to H^0(A)/Q$ is almost finitely presented, hence $A \to \kappa$ is strictly tamely presented (as in Example 2.6). Thus $\psi' := \psi'_2 \circ \psi'_1$ also has coherent pullback (Theorem 2.11), and we have a contradiction since then $B_P \otimes_A \kappa$ must be truncated. $\square$

3. Geometric stacks

We now consider tamely presented morphisms and coherent pullback in the setting of geometric stacks. Again we begin with basic stability properties (Propositions 3.6, 3.7, 3.11). Among tamely presented geometric stacks, we show morphisms with semi-universal coherent pullback can be approximated by morphisms of finite Tor-dimension (Proposition 3.13). Conversely, certain pro-smoothness conditions guarantee that the diagonal of a geometric stack has semi-universal coherent pullback (Proposition 3.14). We also consider the interaction of coherent pullback with proper base change and external tensor products (Propositions 3.18, 3.22), and show that geometric stacks are convergent (Proposition 3.24) and are compact in a suitable subcategory of convergent stacks (Proposition 3.27).

3.1. Definitions.

Recall our convention that a stack means a functor $\mathrm{CAlg}_k \to \mathcal{S}$ satisfying fpqc descent (here k is our fixed Noetherian base), and that the category of stacks is denoted by Stk_k. Our usage of the term geometric stack follows [Lur18, Ch. 9] (up to the presence of the base k), though we caution again that the terminology varies in the literature.

Definition 3.1. A stack X is geometric if its diagonal $X \to X \times X$ is affine and there exists faithfully flat morphism $\operatorname{Spec} B \to X$ in Stk_k. A morphism $X \to Y$ in Stk_k is geometric if for any morphism $\operatorname{Spec} A \to Y$, the fiber product $X \times_Y \operatorname{Spec} A$ is geometric. We write $\mathrm{GStk}_k \subset \mathrm{Stk}_k$ for the full subcategory of geometric stacks.

Note here that products are taken in Stk_k, hence are implicitly over $\operatorname{Spec} k$. Also note that affineness of $X \to X \times X$ implies that any morphism $\operatorname{Spec} B \to X$ is affine. In particular, (faithful) flatness of such a morphism is defined by asking that its base change to any affine scheme is such. More generally, a morphism $X \to Y$ in GStk_k is (faithfully) flat if its composition with any faithfully flat $\operatorname{Spec} A \to X$ is (faithfully) flat. A faithfully flat morphism of geometric stacks will also be called a flat cover.

A geometric stack X is n-truncated if it admits a flat cover $\operatorname{Spec} A \to X$ such that A is n-truncated [Lur18, Def. 9.1.6.2]. Alternatively, note that the restriction functor $(-)_{\leq n} : \mathrm{PStk}_k \to \mathrm{PStk}_{k,\leq n}$ takes Stk_k to $\mathrm{Stk}_{k,\leq n}$ [Lur18, Prop. A.3.3.1], and write $i_{\leq n} : \mathrm{Stk}_{k,\leq n} \to \mathrm{Stk}_k$ for the left adjoint of this restriction and $\tau_{\leq n} : \mathrm{Stk}_k \to \mathrm{Stk}_k$ for their composition. Then if X is geometric $\tau_{\leq n}X$ is an n-truncated geometric stack called the n-truncation of X, and X is n-truncated if and only if the natural map $\tau_{\leq n}X \to X$ is an isomorphism [Lur18, Cor. 9.1.6.8, Prop. 9.1.6.9]. We say $X \in \mathrm{GStk}_k$ is truncated if it is n-truncated for some n, and denote by $\mathrm{GStk}_k^+ \subset \mathrm{GStk}_k$ the full subcategory of truncated geometric stacks.

We do caution that this use of the symbol $\tau_{\leq n}$ and of the term truncation are different from their usual meaning in terms of truncatedness of mapping spaces, but in practice no ambiguity will arise (and this use results in the pleasant feature that $\tau_{\leq n}\operatorname{Spec} A \cong \operatorname{Spec} \tau_{\leq n} A$). We also note that what we call n-truncatedness is called n-coconnectedness in [GR17a].

Proposition 3.2. *Geometric morphisms are stable under composition and base change in* Stk_k*. If* $f : X \to Y$ *is a morphism in* Stk_k*, then* f *is geometric if* X *and* Y *are, and* X *is geometric if* f *and* Y *are. In particular,* GStk_k *is closed under fiber products in* Stk_k*.*

Proof. When $k = S$ is the sphere spectrum this is [Lur18, Prop. 9.3.1.2, Ex. 9.3.1.10]. In general it suffices to show GStk_k is the preimage of $\mathrm{GStk} := \mathrm{GStk}_S$ under $\mathrm{Stk}_k \cong \mathrm{Stk}_{/\operatorname{Spec} k} \to \mathrm{Stk}$ (see Lemma A.1). Clearly $X \in \mathrm{Stk}_k$ has a flat cover $\operatorname{Spec} A \to X$ over $\operatorname{Spec} k$ if and only if it does over $\operatorname{Spec} S$. Now note that if $f : Y \to Z$, $g : Z \to W$ are morphisms in Stk and g is affine, f is affine if and only if $g \circ f$ is (since any $\operatorname{Spec} B \to Y$ factors through $Y \times_Z \operatorname{Spec} B \to Y$). The morphism $X \times_{\operatorname{Spec} k} X \to X \times_{\operatorname{Spec} S} X$ is affine since it is a base change of $\operatorname{Spec} k \to \operatorname{Spec}(k \otimes_S k)$, hence $X \to X \times_{\operatorname{Spec} k} X$ is affine if and only if $X \to X \times_{\operatorname{Spec} S} X$ is. $\square$

3.2. **Tamely presented morphisms.** We now consider the geometric counterparts to the notion of strictly tamely presented algebra.

Definition 3.3. An affine morphism $X \to Y$ in Stk_k is strictly tamely presented if for any $\operatorname{Spec} A \to Y$, the coordinate ring of $X \times_Y \operatorname{Spec} A$ is strictly tamely presented as an A-algebra. A geometric morphism $X \to Y$ in Stk_k is tamely presented if for any $\operatorname{Spec} A \to Y$, there exists a strictly tamely presented flat cover $\operatorname{Spec} B \to \operatorname{Spec} A \times_Y X$ such that B is strictly tamely presented over A. A geometric stack X is tamely presented if it is so over $\operatorname{Spec} k$.

Recall following [Lur18, Def. 17.4.1.1] that $f : X \to Y$ is (locally) almost of finite presentation if, for any $n \in \mathbb{N}$ and any filtered colimit $A \cong \operatorname{colim} A_\alpha$ in $\tau_{\leq n}\mathrm{CAlg}_k$, the canonical map

$$\operatorname{colim} X(A_\alpha) \to X(A) \times_{Y(A)} \operatorname{colim} Y(A_\alpha)$$

is an isomorphism (we omit the word locally by default, as we mostly consider quasi-compact morphisms). We follow [Lur18, Def. 6.3.2.1] and say a morphism $f : X \to Y$ is representable if for any $\operatorname{Spec} A \to Y$, the fiber product $X \times_Y \operatorname{Spec} A$ is a (spectral) Deligne-Mumford stack.

Proposition 3.4. *If a geometric morphism* $f : X \to Y$ *in* Stk_k *is representable and almost of finite presentation, then it is tamely presented.*

Proof. Follows from [Lur18, Prop. 17.4.3.1], by which for any $\operatorname{Spec} A \to Y$ there is an étale cover $\operatorname{Spec} B \to X \times_Y \operatorname{Spec} A$ such that B is almost of finite presentation over A. $\square$

We recall the following stability properties of almost finitely presented morphisms, then consider their generalizations to the tamely presented setting.

Proposition 3.5 ([Lur18, Rem. 17.4.1.3, Rem. 17.4.1.5])**.** *Almost finitely presented morphisms are stable under composition and base change in* Stk_k*. If* f *and* g *are composable morphisms in* Stk_k *such that* $g \circ f$ *and* g *are almost of finite presentation, then so is* f*.*

Proposition 3.6. *Tamely presented geometric morphisms are stable under composition and base change in* Stk_k. *Suppose we have a Cartesian diagram*

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

of geometric morphisms in Stk_k. *If* f' *and* h *are tamely presented and* h *is faithfully flat, then* f *is tamely presented.*

Proof. Stability under base change follows by construction. To see stability under composition let $f : X \to Y$ and $g : Y \to Z$ be tamely presented. Given $\operatorname{Spec} C \to Z$, there exists by hypothesis a diagram

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r, "h''"] & X'' \arrow[r, "h'"] \arrow[d] & X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& \operatorname{Spec} B \arrow[r, "h"] & Y' \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\
& & \operatorname{Spec} C \arrow[r] & Z
\end{tikzcd}$$

in which all squares are Cartesian, h and h'' are strictly tamely presented flat covers, and A, B are strictly tamely presented over B, C. Proposition 2.4 then implies $h' \circ h''$ is a strictly tamely presented flat cover and A is strictly tamely presented over C.

To prove the last claim let $\operatorname{Spec} B \to Y$ be any morphism. By our hypotheses on h there exists a strictly tamely presented flat cover $\operatorname{Spec} B' \to Y' \times_Y \operatorname{Spec} B$ such that B' is strictly tamely presented over B. We then have a commutative cube

$$\begin{tikzcd}
& Z' \arrow[rr] \arrow[dl] \arrow[dd] & & \operatorname{Spec} B' \arrow[dl] \arrow[dd] \\
X' \arrow[rr, "f'"] \arrow[dd] & & Y' \arrow[dd, "h"] & \\
& Z \arrow[rr] \arrow[dl] & & \operatorname{Spec} B \arrow[dl] \\
X \arrow[rr, "f"] & & Y &
\end{tikzcd}$$

in which all but the left and right faces are Cartesian. Since f' is tamely presented there exists a strictly tamely presented flat cover $\operatorname{Spec} A \to Z'$ such that A is strictly tamely presented over B'. By Proposition 2.4 (and the stability of faithful flatness under composition and base change) it now follows that $\operatorname{Spec} A \to Z$ is a strictly tamely presented flat cover and that A is strictly tamely presented over B. $\square$

Proposition 3.7. *If f and g are composable geometric morphisms in Stk_k such that $g \circ f$ is tamely presented and g is almost of finite presentation, then f is tamely presented.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be the given morphisms. For any $\phi : \operatorname{Spec} A \to Y$, there exists by hypothesis a strictly tamely presented flat cover $h : \operatorname{Spec} B \to X' := X \times_Z \operatorname{Spec} A$ such that B is strictly tamely presented over A. Consider then the following diagram of Cartesian squares,

$$\begin{tikzcd}
\operatorname{Spec} C \arrow[r, "h'"] \arrow[d, "\psi'"] & X'' \arrow[r, "f''"] \arrow[d] & \operatorname{Spec} A \arrow[d, "\psi"] & \\
\operatorname{Spec} B \arrow[r, "h"] & X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[r, "g'"] \arrow[d] & \operatorname{Spec} A \arrow[d, "g \circ \phi"] \\
& X \arrow[r, "f"] & Y \arrow[r, "g"] & Z
\end{tikzcd}$$

where ψ is the canonical section of g'. That ψ and hence ψ' are affine follows from g being geometric. That h' is a strictly tamely presented flat cover follows from h being so. Since g and hence g' are almost of finite presentation and $g' \circ \psi$ is the identity, it follows that ψ and hence ψ' are almost of finite presentation (Proposition 3.5). That C is strictly tamely presented over A now follows from Proposition 2.4. $\square$

Remark 3.8. The definition of tamely presented morphism has two weaker variants, where respectively the condition that $\operatorname{Spec} B \to \operatorname{Spec} A \times_Y X$ or $\operatorname{Spec} B \to \operatorname{Spec} A$ is strictly tamely presented is dropped. Some results we state extend to one or the other of these variants, but our most central results do not. Thus for the sake of uniformity we formulate all statements in terms of tamely presented morphisms, even when this obscures their generality somewhat.

3.3. **Coherent pullback.** Recall that any stack X has an associated category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves, defined as the limit of the categories Mod_A over all maps $\operatorname{Spec} A \to X$. If X is geometric this is equivalent to the corresponding limit over the Cech nerve of any flat cover [Lur18, Prop. 9.1.3.1]. In the truncated geometric case we define coherent sheaves as follows.

Definition 3.9. If $X \in \mathrm{GStk}_k^+$, then $\mathcal{F} \in \mathrm{QCoh}(X)$ is coherent if $f^*(\mathcal{F})$ is a coherent A-module for some (equivalently, any) flat cover $\operatorname{Spec} A \to X$. We write $\mathrm{Coh}(X) \subset \mathrm{QCoh}(X)$ for the full subcategory of coherent sheaves.

While the above definition makes sense when X is not truncated, without additional hypotheses the resulting category $\mathrm{Coh}(X)$ may be degenerate (for example, it may contain no nonzero objects). It will be convenient to exclude such degenerate cases from our discussion, though our treatment of coherent sheaves on ind-geometric stacks will include well-behaved non-truncated geometric stacks in its scope. We also caution that the above notion of coherence differs from that of [Lur18, Def. 6.4.3.1].

Definition 3.10. Let $f : X \to Y$ be a morphism in GStk_k such that Y is truncated. We say f has coherent pullback if X is truncated and $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ takes $\mathrm{Coh}(Y)$ to $\mathrm{Coh}(X)$. We say f has semi-universal coherent pullback if for any truncated Y' and any tamely presented morphism $Y' \to Y$, the base change $f' : X \times_Y Y' \to Y'$ has coherent pullback.

In practice, to show a morphism of geometric stacks has semi-universal coherent pullback we reduce to the affine case (where Theorem 2.11 can be applied) using the locality statement in the following result.

Proposition 3.11. *Morphisms with semi-universal coherent pullback are stable under composition and base change along tamely presented morphisms in* GStk^+*. Let* Y *and* Y' *be truncated geometric stacks and*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

a Cartesian diagram in GStk_k*. If* f' *has semi-universal coherent pullback and* h *is faithfully flat, then* f *has semi-universal coherent pullback.*

Proof. Stability under composition and tamely presented base change follow from Proposition 3.6. Suppose that $Z \to Y$ is tamely presented and Z is truncated. Consider the commutative cube

$$\begin{tikzcd}[row sep=small, column sep=small] & W' \arrow[rr, "\phi'"] \arrow[dl] \arrow[dd] & & Z' \arrow[dl] \arrow[dd] \\ X' \arrow[rr, "f'", near start] \arrow[dd] & & Y' \arrow[dd, "h", near start] & \\ & W \arrow[rr, "\phi", near start] \arrow[dl] & & Z \arrow[dl] \\ X \arrow[rr, "f"] & & Y & \end{tikzcd}$$

with all faces Cartesian. By Proposition 3.6 the morphism $Z' \to Y'$ is tamely presented, hence ϕ' has coherent pullback. But since h is faithfully flat so are the other vertical morphisms, and it follows that ϕ has coherent pullback. □

Next we show that if a morphism with semi-universal coherent pullback has tamely presented source and target, it can be locally approximated by morphisms of finite Tor-dimension. A morphism $f : X \to Y$ in GStk_k is of Tor-dimension $\leq n$ if $f^*(\mathrm{QCoh}(Y)^{\geq 0}) \subset \mathrm{QCoh}(Y)^{\geq n}$, and is of finite Tor-dimension if it is of Tor-dimension $\leq n$ for some n. We have the following variant of standard results.

Proposition 3.12. *Morphisms of Tor-dimension $\leq n$ are stable under composition and base change in* GStk_k*. A morphism $f : X \to Y$ of geometric stacks is of Tor-dimension $\leq n$ if and only if its base change along any given flat cover $h : \operatorname{Spec} A \to Y$ is.*

Proof. Stability under composition is immediate. Flat locality on the target follows since if h' and f' are defined by base change, then h'^* is conservative, t-exact, and satisfies $h'^* f^* \cong f'^* h^*$. If $h : \operatorname{Spec} A \to Y$ is arbitrary, then h' is affine, hence h'_* is conservative, t-exact, and satisfies $f^* h_* \cong h'_* f'^*$ [Lur18, Prop. 9.1.5.7]. Stability under base change along affine morphisms follows, and arbitrary base change now follows by composing an arbitrary $Y' \to Y$ with a flat cover $\operatorname{Spec} B \to Y'$. □

By extension, we then say a geometric morphism in Stk_k is of Tor-dimension $\leq n$ if its base change to any geometric stack is so, and is of finite Tor-dimension if it is of Tor-dimension $\leq n$ for some n. We now have the following approximation result, where we note that an expression $A \cong \operatorname{colim} A_\alpha$ of the indicated kind always exists for some n.

Proposition 3.13. *Let $f : X \to Y$ be a morphism in* GStk^+ *with semi-universal coherent pullback, and suppose that X and Y are tamely presented. Choose a strictly tamely presented flat cover* $\operatorname{Spec} A \to Y$ *such that A is strictly tamely presented over k, and let $f' : X' \to \operatorname{Spec} A$ denote the base change of f. If $A \cong \operatorname{colim} A_\alpha$ exhibits A as a strictly tamely n-presented k-algebra for some n, then the composition*

$$X' \xrightarrow{f'} \operatorname{Spec} A \xrightarrow{u_\alpha} \operatorname{Spec} A_\alpha$$

is of finite Tor-dimension for all α.

Proof. Fix a strictly tamely presented flat cover $g : \operatorname{Spec} B \to X$ such that B is strictly tamely presented over k. If $g' : \operatorname{Spec} B' \to X'$ is its base change (whose source is affine since Y is geometric), then B' is strictly tamely presented over B and thus k by Proposition 2.4. Since k is Noetherian, A_α is almost finitely presented over k for any α [Lur18, Prop. 4.1.2.1]. Proposition 2.4 then further implies that B' is strictly tamely presented over A_α.

It follows from the hypotheses that f' has coherent pullback, hence by flatness of g' and u_α the composition $u_\alpha \circ f' \circ g'$ also has coherent pullback. This composition then has finite Tor-dimension by Proposition 2.17, hence so does $u_\alpha \circ f'$ by faithful flatness of g'. □

Now suppose k is an ordinary Noetherian ring, and consider a strictly tame presentation $A \cong \operatorname{colim} A_\alpha$ of order zero. If each A_α is smooth over k we call this a weakly pro-smooth presentation. We say $A \in \tau_{\leq 0}\mathrm{CAlg}_k$ is weakly pro-smooth if it admits a weakly pro-smooth presentation, and we say $X \in \mathrm{GStk}_k$ is locally weakly pro-smooth if it admits a flat cover $\operatorname{Spec} A \to X$ such that A is weakly pro-smooth. We then have the following fundamental source of morphisms with semi-universal coherent pullback. It implies in particular that the jet scheme of an affine variety étale over $\mathbb{A}^n$ has a diagonal with semi-universal coherent pullback [KV04, Prop. 1.2.1, Prop. 1.7.1].

Proposition 3.14. *Suppose k is an ordinary Noetherian ring of finite global dimension, and suppose $X \in \mathrm{GStk}_k$ is locally weakly pro-smooth. Then the diagonal $\Delta_X : X \to X \times X$ has semi-universal coherent pullback, as does any tamely presented geometric morphism $f : Y \to X$.*

Proof. Note that the second claim follows from the first. Any such f factors as $Y \to X \times Y \to X$. The second factor is a base change of $X \to \operatorname{Spec} k$, hence is of finite Tor-dimension by our hypothesis on k. The first is the base change of Δ_X along the tamely presented morphism f, hence it and f have semi-universal coherent pullback if Δ_X does (Proposition 3.11).

Let $A \cong \operatorname{colim} A_\alpha$ be a weakly pro-smooth presentation and set $U = \operatorname{Spec} A$, $U_\alpha = \operatorname{Spec} A_\alpha$. Since $U \times U \to X \times X$ is a flat cover, it suffices to show the base change $\Delta'_X : U \times_X U \to U \times U$ of Δ_X has semi-universal coherent pullback (Proposition 3.11). Proposition 2.4 implies $U \times U$ is strictly tamely presented since U is, hence it suffices to show Δ'_X has coherent pullback (Theorem 2.11). Since $U \times U$ is coherent (Proposition 2.8) it suffices to show $\Delta'^*_X(\mathcal{F})$ is coherent for any $\mathcal{F} \in \mathrm{Coh}(U \times U)^\heartsuit$.

By flatness of the $p_\alpha \times p_\alpha : U \times U \to U_\alpha \times U_\alpha$ and by e.g. [Lur18, Cor. 4.5.1.10] or [TT90, Sec. C.4], $\mathcal{F} \cong (p_\alpha \times p_\alpha)^*(\mathcal{F}_\alpha)$ for some α and some $\mathcal{F}_\alpha \in \mathrm{Coh}(U_\alpha \times U_\alpha)^\heartsuit$. But since $U_\alpha \times U_\alpha$ is smooth its diagonal is of finite Tor-dimension [Sta, Lem. 0FDP], hence $(p_\alpha \times p_\alpha) \circ \Delta'_X$ is of finite Tor-dimension by the reasoning of the first paragraph, hence $\Delta'^*_X(\mathcal{F})$ is coherent. □

3.4. **Pushforward and base change.** Given a morphism $f : X \to Y$ in GStk_k, the pushforward $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is defined as the right adjoint of f^*. For f_* to be well-behaved one needs additional hypotheses on f. Recall that a morphism $f : X \to Y$ in GStk_k is of cohomological dimension $\leq n$ if $f_*(\mathrm{QCoh}(X)^{\leq 0}) \subset \mathrm{QCoh}(Y)^{\leq n}$, and is of finite cohomological dimension if it is of cohomological dimension $\leq n$ for some n.

Proposition 3.15. *Morphisms of finite cohomological dimension are stable under composition and base change in GStk_k. A morphism $f : X \to Y$ of geometric stacks is of finite cohomological dimension if and only if its base change along any given flat cover $\operatorname{Spec} A \to Y$ is. In this case $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ is continuous, and for any Cartesian square*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{3.16}$$

in GStk_k the Beck-Chevalley transformation $h^* f_*(\mathcal{F}) \to f'_* h'^*(\mathcal{F})$ is an isomorphism for all $\mathcal{F} \in \mathrm{QCoh}(X)$.

Proof. Stability under composition is immediate, while stability under base change and flat locality on the target follow from [HLP14, Prop. A.1.9] (whose proof applies to geometric stacks, not just algebraic stacks with affine diagonal). The remaining properties then follow from [Lur18, Prop. 9.1.5.3, Prop. 9.1.5.7] or [HLP14, Prop. A.1.5]. □

A key case is that of proper morphisms. We say $f : X \to Y$ is proper if it is representable and for any $\operatorname{Spec} A \to Y$, the fiber product $X \times_Y \operatorname{Spec} A$ is proper over $\operatorname{Spec} A$ in the sense of [Lur18, Def. 5.1.2.1]. In particular, this requires that $X \times_Y \operatorname{Spec} A$ be a quasi-compact separated algebraic space.

Proposition 3.17. *Proper morphisms are stable under composition and base change in* Stk_k*. If f and g are morphisms in* Stk_k *such that $g \circ f$ and g are proper, then so is f. Proper morphisms of geometric stacks are of finite cohomological dimension. If f is a proper, almost finitely presented morphism of truncated geometric stacks, then f_* takes* $\mathrm{Coh}(X)$ *to* $\mathrm{Coh}(Y)$*.*

Proof. Stablity under base change is immediate, and the claims about composition follow from [Lur18, Prop. 5.1.4.1, Prop. 6.3.2.2]. By Proposition 3.15 finiteness of cohomological dimension can be checked after base change along a flat cover $h : \operatorname{Spec} A \to Y$, where it follows from [Lur18, Prop. 2.5.4.4, Prop. 3.2.3.1]. If f is almost of finite presentation and f' is its base change along h, then f'_* preserves coherence by [Lur18, Thm. 5.6.0.2]. Then so does $h^* f_*$ by Proposition 3.15, hence so does f_* by definition. □

The functorialities of coherent sheaves considered so far, and moreover the homotopy coherence of these functorialities, can be packaged using correspondence categories as follows. Let $\mathrm{Corr}(\mathrm{GStk}_k^+)_{prop;coh}$ denote the subcategory of $\mathrm{Corr}(\mathrm{GStk}_k)$ which only includes correspondences $X \xleftarrow{f} Y \xrightarrow{h} Z$ such that X, Y, and Z are truncated, h has semi-universal coherent pullback, and f is proper and almost of finite presentation. These are stable under composition of correspondences by Propositions 3.4, 3.11, and 3.17.

Proposition 3.18. *There exists a canonical functor*

$$\mathrm{Coh} : \mathrm{Corr}(\mathrm{GStk}_k^+)_{prop;coh} \to \mathrm{Cat}_\infty \tag{3.19}$$

which takes a correspondence $X \xleftarrow{f} Y \xrightarrow{h} Z$ to the functor $f_* h^* : \mathrm{Coh}(Z) \to \mathrm{Coh}(X)$.

Proof. We begin with the functor $\mathrm{QCoh} : \mathrm{GStk}_k^{\mathrm{op}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ which takes a morphism $h : Y \to Z$ to the functor $h^* : \mathrm{QCoh}(Z) \to \mathrm{QCoh}(Y)$. Let $\mathrm{Corr}(\mathrm{GStk}_k)_{fcd;all}$ denote the subcategory of $\mathrm{Corr}(\mathrm{GStk}_k)$ where we only allow correspondences $X \xleftarrow{f} Y \xrightarrow{h} Z$ such that f is of finite cohomological dimension. By Proposition 3.15 and [GR17a, Thm. 3.2.2] the isomorphisms $h^* f_* \xrightarrow{\sim} f'_* h'^*$ associated to the squares (3.16) extend canonically to the data of a functor

$$\mathrm{QCoh} : \mathrm{Corr}(\mathrm{GStk}_k)_{fcd;all} \to \mathcal{P}\mathrm{r}^{\mathrm{L}} \tag{3.20}$$

which takes a correspondence $X \xleftarrow{f} Y \xrightarrow{h} Z$ to the functor $f_* h^* : \mathrm{QCoh}(Z) \to \mathrm{QCoh}(X)$.

Now restrict QCoh along the inclusion $\mathrm{Corr}(\mathrm{GStk}_k^+)_{prop;coh} \to \mathrm{Corr}(\mathrm{GStk}_k)_{fcd;all}$. For any morphism $X \xleftarrow{f} Y \xrightarrow{h} Z$ in $\mathrm{Corr}(\mathrm{GStk}_k^+)_{prop;coh}$, the associated functor $f_* h^* : \mathrm{QCoh}(Z) \to \mathrm{QCoh}(X)$ takes $\mathrm{Coh}(Z)$ to $\mathrm{Coh}(X)$ by Proposition 3.17 and the definitions. Thus we obtain (3.19) by restriction. □

Next we discuss external products. Given $X, Y \in \mathrm{GStk}_k$, the external product of $\mathcal{F} \in \mathrm{QCoh}(X)$ and $\mathcal{G} \in \mathrm{QCoh}(Y)$ is defined by

$$\mathcal{F} \boxtimes \mathcal{G} := \pi_X^*(\mathcal{F}) \otimes \pi_Y^*(\mathcal{G}) \in \mathrm{QCoh}(X \times Y).$$

In this paper we only consider external products under the hypothesis that k is an ordinary (Noetherian) ring of finite global dimension, in which case the following result holds.

Lemma 3.21. *Suppose k is an ordinary ring of finite global dimension. If $X, Y \in \mathrm{GStk}_k$, $\mathcal{F} \in \mathrm{QCoh}(X)^+$, and $\mathcal{G} \in \mathrm{QCoh}(Y)^+$, then $\mathcal{F} \boxtimes \mathcal{G} \in \mathrm{QCoh}(X \times Y)^+$. In particular, $X \times Y$ is truncated if X and Y are, and $\mathcal{F} \boxtimes \mathcal{G}$ is coherent if $\mathcal{F}$ and $\mathcal{G}$ are.*

Proof. It suffices to check the claim for the pullbacks of $\mathcal{F}$, $\mathcal{G}$ to flat covers of X and Y, hence we may assume X and Y are affine. If $\mathcal{F} \in \mathrm{QCoh}(X)^{\geq m}$ and $\mathcal{G} \in \mathrm{QCoh}(Y)^{\geq n}$, then since the underlying k-module of $\mathcal{F} \boxtimes \mathcal{G}$ is $\mathcal{F} \otimes_k \mathcal{G}$ we have $\mathcal{F} \boxtimes \mathcal{G} \in \mathrm{QCoh}(X \times Y)^{\geq \ell}$, where ℓ is $m+n$ minus the global dimension of k. The last claims follow by taking $\mathcal{F} \cong \mathcal{O}_X$ and $\mathcal{G} \cong \mathcal{O}_Y$, and by recalling that almost perfect sheaves are closed under pullbacks and tensor products. $\square$

Under the above hypotheses on k, the external product thus restricts to a functor

$$- \boxtimes - : \mathrm{Coh}(X) \times \mathrm{Coh}(Y) \to \mathrm{Coh}(X \times Y).$$

Its functoriality in X and Y is captured by the following statement. Here we regard GStk_k^+ and Cat_∞ as symmetric monoidal categories under Cartesian products.

Proposition 3.22. *Suppose k is classical and of finite global dimension. Then the functor*

$$\mathrm{Coh} : \mathrm{Corr}(\mathrm{GStk}_k^+)_{prop;coh} \to \mathrm{Cat}_\infty$$

of Proposition 3.18 is canonically lax symmetric monoidal.

Proof. The functor QCoh of (3.20) can be factored through a symmetric monoidal functor $\mathrm{Corr}(\mathrm{GStk}_k)_{fcd;all} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ by [GR17a, Sec. 5.5.3] (noting that it is harmless to replace schematic maps with representable maps in its statement). The inclusion $\mathcal{P}\mathrm{r}^{\mathrm{L}} \hookrightarrow \widehat{\mathrm{Cat}}_\infty$ is lax symmetric monoidal [Lur17, Cor. 4.8.1.4, Prop. 4.8.1.14], hence QCoh obtains a lax symmetric monoidal structure. This structure then restricts to Coh by Lemma 3.21. $\square$

3.5. **Convergence.** Recall that $\widehat{\mathrm{PStk}}_k$ denotes the category of functors $\tau_{<\infty}\mathrm{CAlg}_k \to \mathcal{S}$, and $\widehat{\mathrm{Stk}}_k \subset \widehat{\mathrm{PStk}}_k$ the subcategory of functors satisfying fpqc descent. The restriction functor $(-)_{<\infty} : \mathrm{PStk}_k \to \widehat{\mathrm{PStk}}_k$ has a fully faithful right adjoint, by which we generally regard $\widehat{\mathrm{PStk}}_k$ as a subcategory of PStk_k. One says a prestack is convergent (or nilcomplete) if it is contained in this subcategory. Explicitly, $X \in \mathrm{PStk}_k$ is convergent if for all $A \in \mathrm{CAlg}_k$ the natural morphism $X(A) \to \lim X(\tau_{\leq n} A)$ is an isomorphism. We have an induced notion of convergent stack, which is unambiguous in the following sense.

Lemma 3.23. *The inclusion $\widehat{\mathrm{PStk}}_k \subset \mathrm{PStk}_k$ identifies $\widehat{\mathrm{Stk}}_k$ with $\mathrm{Stk}_k \cap \widehat{\mathrm{PStk}}_k$.*

Proof. That $\mathrm{Stk}_k \cap \widehat{\mathrm{PStk}}_k \subset \widehat{\mathrm{Stk}}_k$ follows from the definition of the fpqc topology and from [Lur18, Prop. A.3.3.1] (closure of $\tau_{<\infty}\mathrm{CAlg}_k$ under pushouts along flat morphisms is sufficient to apply this to $\widehat{\mathrm{PStk}}_k$). Now suppose $X \in \widehat{\mathrm{Stk}}_k \subset \widehat{\mathrm{PStk}}_k \subset \mathrm{PStk}_k$. Note that $\tau_{\leq n} : \mathrm{Sp}^{\leq 0} \to \tau_{\leq n}\mathrm{Sp}^{\leq 0}$ preserves finite products since it preserves colimits and $\mathrm{Sp}^{\leq 0}$ is additive, hence $\tau_{\leq n} : \mathrm{CAlg}_k \to \tau_{\leq n}\mathrm{CAlg}_k$ preserves finite products by [Lur17, Cor. 3.2.2.5]. Then if $A \cong \prod_{i=1}^n A_i$ is a finite product in CAlg_k, we have $X(A) \cong \lim_n X(\tau_{\leq n}A) \cong \lim_{n,i} X(\tau_{\leq n}A_i) \cong \lim_i X(A_i)$. Similarly, if $A \to A^0$ in CAlg_k is faithfully flat and $A^\bullet$ its Cech nerve, then $X(A) \cong \lim_n X(\tau_{\leq n}A) \cong \lim_{n,i} X(\tau_{\leq n}A^i) \cong \lim_i X(A^i)$. Thus $X \in \mathrm{Stk}_k$ by [Lur18, Prop. A.3.3.1]. $\square$

We now have the following result in the geometric case.

Proposition 3.24. *Geometric stacks are convergent.*

Proof. Suppose $X \in \mathrm{GStk}_k$ and $A \in \mathrm{CAlg}_k$. It suffices to treat the case where $k = S$ is the sphere spectrum. In general, $\mathrm{Map}_{\mathrm{Stk}_k}(\mathrm{Spec}\,A, X)$ is the fiber of the map $\mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\,A, X) \to \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\,A, \mathrm{Spec}\,k)$ induced by composition with $X \to \mathrm{Spec}\,k$ over the point corresponding to $\mathrm{Spec}\,A \to \mathrm{Spec}\,k$ [Lur09, Lem. 5.5.5.12] (recall $\mathrm{Stk}_k \cong \mathrm{Stk}_{/\mathrm{Spec}\,k}$ per Lemma A.1). Since the same holds for each $\tau_{\leq n}A$, it follows that X is convergent over $\mathrm{Spec}\,k$ if it is convergent over $\mathrm{Spec}\,S$ (since $\mathrm{Spec}\,k$ is).

Consider the natural diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\,A, X) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)}(\mathrm{QCoh}(X)^{\leq 0}, \mathrm{Mod}_A^{\leq 0}) \arrow[d] \\ \lim \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\,\tau_{\leq n}A, X) \arrow[r] & \lim \mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)}(\mathrm{QCoh}(X)^{\leq 0}, \mathrm{Mod}_{\tau_{\leq n}A}^{\leq 0}). \end{tikzcd} \tag{3.25}$$

By [Lur, Lem. 5.4.6] the top map is a monomorphism if the corresponding map from $\mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\,A, X)$ to $\mathrm{Map}_{\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)}(\mathrm{QCoh}(X), \mathrm{Mod}_A)$ is a monomorphism, which in turn follows from Tannaka duality [Lur18, Prop. 9.3.0.3]. Since this also holds replacing A with $\tau_{\leq n}A$, and since monomorphisms are stable under limits, it follows similarly that the bottom map is a monomorphism. The right map is an isomorphism, since we have

$$\mathrm{Mod}_A^{\leq 0} \cong \lim_m \mathrm{Mod}_A^{[m,0]} \cong \lim_{m,n} \mathrm{Mod}_{\tau_{\leq n}A}^{[m,0]} \cong \lim_n \mathrm{Mod}_{\tau_{\leq n}A}^{\leq 0},$$

in $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty)$. Here the first and third equivalences follow from the relevant t-structures being left complete and $\mathrm{CAlg}(\widehat{\mathrm{Cat}}_\infty) \to \widehat{\mathrm{Cat}}_\infty$ preserving limits [Lur17, Cor. 3.2.2.5], the second from $\mathrm{Mod}_A^{[m,0]} \to \mathrm{Mod}_{\tau_{\leq n}A}^{[m,0]}$ being an equivalence for $m \leq -n$. It follows that the left map in (3.25) is a monomorphism, so we must show it is essentially surjective.

Since the horizontal maps are monomorphisms, it suffices to show the right isomorphism restricts to an essential surjection between their essential images. By Tannaka duality [Lur18, Prop. 9.3.0.3] the essential image of the bottom map consists of systems $\{G_n : \mathrm{QCoh}(X)^{\leq 0} \to \mathrm{Mod}_{\tau_{\leq n}A}^{\leq 0}\}$ of symmetric monoidal functors which preserve small colimits and flat objects, similarly for the top map. Let $G : \mathrm{QCoh}(X)^{\leq 0} \to \mathrm{Mod}_A^{\leq 0}$ be the functor associated to some such system $\{G_n\}$ under the right isomorphism. Then G preserves small colimits since $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ is closed under limits in $\widehat{\mathrm{Cat}}_\infty$ [Lur09, Prop. 5.5.3.13]. Moreover, if $G_n(\mathcal{F}) \cong \tau_{\leq n}A \otimes_A G(\mathcal{F})$ is flat over $\tau_{\leq n}A$ for all n, then since $\tau_{\leq n}G(\mathcal{F}) \cong \tau_{\leq n}(\tau_{\leq n}A \otimes_A G(\mathcal{F}))$ by [Lur17, Prop. 7.1.3.15] it follows from the definition of flatness that $G(\mathcal{F})$ is flat over A. Thus G is in the image of the top map in (3.25), establishing the claim. □

Since $\tau_{\leq n}\mathrm{CAlg}_k$ is closed under products and targets of flat morphisms in $\tau_{<\infty}\mathrm{CAlg}_k$, the restriction functor $(-)_{\leq n} : \widehat{\mathrm{PStk}}_k \to \mathrm{PStk}_{k,\leq n}$ takes $\widehat{\mathrm{Stk}}_k$ to $\mathrm{Stk}_{k,\leq n}$ [Lur18, Prop. A.3.3.1]. We write $\widehat{i}_{\leq n} : \mathrm{Stk}_{k,\leq n} \to \widehat{\mathrm{Stk}}_k$ for the resulting left adjoint and $\widehat{\tau}_{\leq n} : \widehat{\mathrm{Stk}}_k \to \widehat{\mathrm{Stk}}_k$ for their composition.

The functors $(-)_{\leq n}$ induce an equivalence $\widehat{\mathrm{Stk}}_k \cong \lim \mathrm{Stk}_{k,\leq n}$ in $\widehat{\mathrm{Cat}}_\infty$ [Lur18, Ex. A.7.1.6], hence for any $X \in \widehat{\mathrm{Stk}}_k$ the natural map $\operatorname{colim} \widehat{\tau}_{\leq n}X \to X$ is an isomorphism by Lemma A.2. More explicitly, if $\widehat{\tau}_{\leq n}^{pre} : \widehat{\mathrm{PStk}}_k \to \widehat{\mathrm{PStk}}_k$ denotes the composition of $(-)_{\leq n} : \widehat{\mathrm{PStk}}_k \to \mathrm{PStk}_{k,\leq n}$ and its left adjoint, then $\operatorname{colim} \widehat{\tau}_{\leq n}^{pre}X \to X$ is an isomorphism since $\tau_{<\infty}\mathrm{CAlg}_k = \cup_n \tau_{\leq n}\mathrm{CAlg}_k$ and since colimits in $\widehat{\mathrm{PStk}}_k$ are computed objectwise. But $\widehat{\tau}_{\leq n}$ is the sheafification of $\widehat{\tau}_{\leq n}^{pre}$, so $\operatorname{colim} \widehat{\tau}_{\leq n}X \to X$ is an isomorphism since sheafification is continuous. In particular, if X is a geometric stack Proposition 3.24 implies $\tau_{\leq n}X \cong \widehat{\tau}_{\leq n}X$, hence we obtain the following corollary.

Proposition 3.26. *For any geometric stack X we have $X \cong \operatorname{colim} \tau_{\leq n}X$ in $\widehat{\mathrm{Stk}}_k$.*

Now let $1\text{-}\widehat{\mathrm{Stk}}_k \subset \widehat{\mathrm{Stk}}_k$ denote the full subcategory consisting of X such that $X(A)$ is an $(n+1)$-truncated space for all $A \in \tau_{\leq n}\mathrm{CAlg}_k$. Then Proposition 3.24 is refined by the following result, the first half of which is a variant of [GR17a, Cor. I.2.4.3.4], the second of [GR14, Lem. 1.3.6].

Proposition 3.27. *Geometric stacks are objects of $1\text{-}\widehat{\mathrm{Stk}}_k$. Moreover, $1\text{-}\widehat{\mathrm{Stk}}_k$ is closed under filtered colimits in $\widehat{\mathrm{PStk}}_k$, and truncated geometric stacks are compact as objects of $1\text{-}\widehat{\mathrm{Stk}}_k$.*

Proof. Let X be a geometric stack. To show $X \in 1\text{-}\widehat{\mathrm{Stk}}_k$, it suffices to show that $X_{\leq n}$ belongs to $\tau_{\leq n+1}\mathrm{Stk}_{k,\leq n}$, the category of $(n+1)$-truncated objects of $\mathrm{Stk}_{k,\leq n}$, since for $A \in \tau_{\leq n}\mathrm{CAlg}_k$ we have $X(A) \cong \mathrm{Map}_{\mathrm{Stk}_{k,\leq n}}((\operatorname{Spec} A)_{\leq n}, X_{\leq n})$.

Let $\operatorname{Spec} B \to X$ be a flat cover, so that $(\operatorname{Spec} B)_{\leq n} \to X_{\leq n}$ is a flat cover in $\mathrm{Stk}_{k,\leq n}$. If $(\operatorname{Spec} B)^\bullet_{\leq n}$ is its Cech nerve, we have $X_{\leq n} \cong \operatorname{colim}(\operatorname{Spec} B)^m_{\leq n}$ in $\mathrm{Stk}_{k,\leq n}$. Equivalently, $X_{\leq n}$ is the sheafification of the same colimit taken in $\mathrm{PStk}_{k,\leq n}$. Since sheafification is left exact it preserves $(n+1)$-truncated objects [Lur09, Prop. 5.5.6.16], hence it suffices to show the colimit taken in $\mathrm{PStk}_{k,\leq n}$ is $(n+1)$-truncated. For any m an object of $\mathrm{PStk}_{k,\leq n}$ is m-truncated if its values on $\tau_{\leq n}\mathrm{CAlg}_k$ are m-truncated spaces [Lur09, Rem. 5.5.8.26]. Each $(\operatorname{Spec} B)^m_{\leq n}$ is affine since X is geometric, hence is then n-truncated in $\mathrm{PStk}_{k,\leq n}$ since $\tau_{\leq n}\mathrm{CAlg}_k$ is an $(n+1)$-category. The claim then follows since values of colimits in $\mathrm{PStk}_{k,\leq n}$

are computed objectwise, and since the geometric realization of a groupoid of n-truncated spaces is $(n+1)$-truncated.

We now claim $X_{\leq n}$ is compact in $\tau_{\leq n+1}\mathrm{Stk}_{k,\leq n}$. By the argument of [GR14, Lem. 1.3.3] we have that $\tau_{\leq n+1}\mathrm{Stk}_{k,\leq n}$ is closed under filtered colimits in $\mathrm{PStk}_{k,\leq n}$ (noting that by [Lur09, Prop. 5.5.6.16] we have $\tau_{\leq n+1}\mathrm{Stk}_{k,\leq n} = \mathrm{Stk}_{k,\leq n} \cap \tau_{\leq n+1}\mathrm{PStk}_{k,\leq n}$ since $\mathrm{Stk}_{k,\leq n}$ is closed under limits in $\mathrm{PStk}_{k,\leq n}$). It follows that each $(\mathrm{Spec}\, B)^m_{\leq n}$ is compact in $\tau_{\leq n+1}\mathrm{Stk}_{k,\leq n}$ since it is so in $\mathrm{PStk}_{k,\leq n}$. Moreover, $X_{\leq n}$ is their colimit in $\tau_{\leq n+1}\mathrm{Stk}_{k,\leq n}$ since it is their colimit in $\mathrm{Stk}_{k,\leq n}$. But $\tau_{\leq n+1}\mathrm{Stk}_{k,\leq n}$ is an $(n+2)$-category, so by the proof of [Lur17, Lem. 1.3.3.10] $X_{\leq n}$ is also the colimit in $\tau_{\leq n+1}\mathrm{Stk}_{k,\leq n}$ of the $(\mathrm{Spec}\, B)^m_{\leq n}$ over the finite subdiagam $\Delta^{op}_{s,\leq n+2} \subset \Delta^{op}_s$. It follows that $X_{\leq n}$ is compact in $\tau_{\leq n+1}\mathrm{Stk}_{k,\leq n}$ [Lur09, Cor. 5.3.4.15].

Next note that [Lur18, Prop. A.3.3.1] also implies $\widehat{\mathrm{Stk}}_k$ is the full subcategory of $X \in \widehat{\mathrm{PStk}}_k$ such that $X_{\leq n} \in \mathrm{Stk}_{k,\leq n}$ for all n. But then $\tau_{\leq n+1}\mathrm{Stk}_{k,\leq n} = \mathrm{Stk}_{k,\leq n} \cap \tau_{\leq n+1}\mathrm{PStk}_{k,\leq n}$ implies 1-$\widehat{\mathrm{Stk}}_k$ is the full subcategory of $X \in \widehat{\mathrm{PStk}}_k$ such that $X_{\leq n} \in \tau_{\leq n+1}\mathrm{Stk}_{k,\leq n}$ for all n. Closure of 1-$\widehat{\mathrm{Stk}}_k$ under filtered colimits then follows since $(-)_{\leq n} : \widehat{\mathrm{PStk}}_k \to \mathrm{PStk}_{k,\leq n}$ is continuous and since as recalled above $\tau_{\leq n+1}\mathrm{Stk}_{k,\leq n}$ is closed under filtered colimits in $\mathrm{PStk}_{k,\leq n}$.

Now suppose X is an n-truncated geometric stack, let $Y \cong \operatorname{colim} Y_\alpha$ be a filtered colimit in 1-$\widehat{\mathrm{Stk}}_k$, and consider the following diagram.

$$\begin{tikzcd}
\operatorname{colim} \mathrm{Map}_{1\text{-}\widehat{\mathrm{Stk}}_k}(X, Y_\alpha) \arrow[r] \arrow[dd] & \operatorname{colim} \mathrm{Map}_{\tau_{\leq n+1}\mathrm{Stk}_{k,\leq n}}(X_{\leq n}, (Y_\alpha)_{\leq n}) \arrow[d] \\
 & \mathrm{Map}_{\tau_{\leq n+1}\mathrm{Stk}_{k,\leq n}}(X_{\leq n}, \operatorname{colim}(Y_\alpha)_{\leq n}) \arrow[d] \\
\mathrm{Map}_{1\text{-}\widehat{\mathrm{Stk}}_k}(X, Y) \arrow[r] & \mathrm{Map}_{\tau_{\leq n+1}\mathrm{Stk}_{k,\leq n}}(X_{\leq n}, Y_{\leq n})
\end{tikzcd}$$

The bottom right map is an isomorphism since by the preceding paragraph $(-)_{\leq n}$ restricts to a continuous functor 1-$\widehat{\mathrm{Stk}}_k \to \tau_{\leq n+1}\mathrm{Stk}_{k,\leq n}$. Since X is n-truncated it is the image of $X_{\leq n}$ under the left adjoint $\widehat{i}_{\leq n} : \mathrm{Stk}_{k,\leq n} \to \widehat{\mathrm{Stk}}_k$ of $(-)_{\leq n}$, hence the horizontal maps are isomorphisms. The top right map is an isomorphism since $X_{\leq n} \in \tau_{\leq n+1}\mathrm{Stk}_{k,\leq n}$ is compact, hence the left map is an isomorphism and X is compact in 1-$\widehat{\mathrm{Stk}}_k$. □

3.6. **Closed immersions.** We conclude this section with some minor results on closed immersions we will use in treating ind-geometric stacks. Following [GR14, Sec. 1.6] we say a map $f : X \to Y$ in Stk_k is a closed immersion if for any $\mathrm{Spec}\, A \to Y$, the map $\tau_{\leq 0}(X \times_Y \mathrm{Spec}\, A) \to \tau_{\leq 0}\mathrm{Spec}\, A$ is a closed immersion of ordinary affine schemes. The following statement follows immediately from its classical counterpart.

Proposition 3.28. *Closed immersions are stable under composition and base change in* Stk_k*. If* f *and* g *are composable morphisms in* Stk_k *such that* $g \circ f$ *and* g *are closed immersions, then so is* f*.*

A closed immersion need not be affine (for example, the inclusion of a closed subscheme of a classical ind-scheme is typically not affine in the derived sense), but in GStk_k we have the following.

Proposition 3.29. *Closed immersions between geometric stacks are affine.*

Proof. Let $f : X \to Y$ be a closed immersion between geometric stacks, and let $\operatorname{Spec} A \to Y$ be arbitrary. The base change $f' : X' \to \operatorname{Spec} A$ of f factors through a map $f'' : X' \to \operatorname{Spec} B$, where $B := f'_*(\mathcal{O}_{X'}) \in \mathrm{CAlg}_A$. We claim f'' is an isomorphism. By Tannaka duality [Lur18, Thm. 9.2.0.2] it suffices to show $f''_* : \mathrm{QCoh}(X') \to \mathrm{Mod}_B$ is an equivalence. By Barr-Beck-Lurie [Lur17, Thm. 4.7.4.5] it further suffices to show $f'_* : \mathrm{QCoh}(X') \to \mathrm{Mod}_A$ is conservative and preserves small colimits.

Since f is a closed immersion $\tau_{\leq 0} f'' : \tau_{\leq 0} X' \to \tau_{\leq 0}\operatorname{Spec} B$ is an isomorphism, hence f''_* restricts to an equivalence $\mathrm{QCoh}(X')^\heartsuit \cong \mathrm{Mod}_B^\heartsuit$. It follows that the restriction of f'_* to $\mathrm{QCoh}(X')^\heartsuit$ is conservative, continuous, and factors through $\mathrm{Mod}_A^\heartsuit$. It follows as in [HLP14, Lem. A.1.6] that f'_* is t-exact.

Suppose now that $f'_*(\mathcal{F}) \cong 0$ for some $\mathcal{F} \in \mathrm{QCoh}(X')$. It follows from the preceding paragraph that $\mathcal{H}^n(\mathcal{F}) \cong 0$ for all $n \in \mathbb{Z}$. Since $\mathrm{QCoh}(X')$ is t-complete we then have $\mathcal{F} \cong 0$, hence f'_* is conservative. Similarly, suppose $\mathcal{F} \cong \operatorname{colim} \mathcal{F}_\alpha$ is a filtered colimit in $\mathrm{QCoh}(X')$, and let $\phi : \operatorname{colim} f'_*(\mathcal{F}_\alpha) \to f'_*(\mathcal{F})$ denote the natural map. Since the t-structures on $\mathrm{QCoh}(X')$ and Mod_A are compatible with filtered colimits, it follows from the preceding paragraph that $\mathcal{H}^n(\phi)$ is an isomorphism for all $n \in \mathbb{Z}$. That ϕ is an isomorphism, hence that f'_* is continuous, now follows from the t-completeness of Mod_A. But then f'_* preserves all small colimits since it is exact [Lur09, Prop. 5.5.1.9]. $\square$

Proposition 3.30. *Let $f : X \to Y_\alpha$, $h : Y' \to Y_\alpha$, and $i : Y_\alpha \to Y_\beta$ be morphisms of geometric stacks. If i is a closed immersion, the map $\tau_{\leq 0}(X \times_{Y_\alpha} Y') \to \tau_{\leq 0}(X \times_{Y_\beta} Y')$ is an isomorphism.*

Proof. Let $X'_\alpha := X \times_{Y_\alpha} Y'$, $X'_\beta := X \times_{Y_\beta} Y'$, and suppose first that X and Y' are affine. Fixing a flat cover $\operatorname{Spec} B_\beta \to Y_\beta$ we obtain flat covers $\operatorname{Spec} A'_\alpha \to X'_\alpha$, $\operatorname{Spec} A'_\beta \to X'_\beta$ by base change. It suffices to show the induced morphism $\tau_{\leq 0}\operatorname{Spec} A'_\alpha \to \tau_{\leq 0}\operatorname{Spec} A'_\beta$ is an isomorphism [Lur09, Lem. 6.2.3.16]. If $\operatorname{Spec} A \to X$, $\operatorname{Spec} B' \to Y'$, and $\operatorname{Spec} B_\alpha \to Y_\alpha$ are also obtained by base change from $\operatorname{Spec} B_\beta \to Y_\beta$, then $\tau_{\leq 0}\operatorname{Spec} B_\alpha \to \tau_{\leq 0}\operatorname{Spec} B_\beta$ is a closed immersion since i is. But then $\tau_{\leq 0}\operatorname{Spec} A'_\alpha \to \tau_{\leq 0}\operatorname{Spec} A'_\beta$ is an isomorphism since $A'_\alpha \cong A \otimes_{B_\alpha} B'$ and $A'_\beta \cong A \otimes_{B_\beta} B'$.

In the general case, fix flat covers $\operatorname{Spec} C \to X$ and $\operatorname{Spec} D' \to Y'$, and let $\operatorname{Spec} C'_\alpha := \operatorname{Spec} C \times_{Y_\alpha} \operatorname{Spec} D'$ and $\operatorname{Spec} C'_\beta := \operatorname{Spec} C \times_{Y_\beta} \operatorname{Spec} D'$. By the preceding paragraph $\tau_{\leq 0}\operatorname{Spec} C'_\alpha \to \tau_{\leq 0}\operatorname{Spec} C'_\beta$ is an isomorphism. But since it is the base change of $\tau_{\leq 0} X'_\alpha \to \tau_{\leq 0} X'_\beta$ along the flat cover $\tau_{\leq 0}\operatorname{Spec} C'_\beta \to \tau_{\leq 0} X'_\beta$, the claim follows. $\square$

4. Ind-geometric stacks

We now extend the discussion of the previous section from geometric stacks to ind-geometric stacks. Much of this section is devoted to setting up the basic theory of these, which largely follows the parallel theory of ind-schemes in [GR14, Ras19]. After showing some basic facts about reasonable geometric substacks (Proposition 4.6), we establish the basic stability properties of semi-universal coherent pullback and ind-tamely presented morphisms (Propositions 4.9, 4.16, 4.19), as well as the closure of ind-geometric stacks under fiber products (Proposition 4.17) and ind-closed filtered colimits (Proposition 4.11). We then consider coherent sheaves on reasonable ind-geometric stacks (Definition 4.21), the main functorialities of such, and consider a few special classes of ind-geometric stacks.

4.1. **Definitions.** A morphism $f : X \to Y$ in Stk_k is a closed immersion if for any $\operatorname{Spec} A \to Y$, the map $\tau_{\leq 0}(X \times_Y \operatorname{Spec} A) \to \tau_{\leq 0}\operatorname{Spec} A$ is a closed immersion of classical affine schemes. Recall that f is almost of finite presentation if, for any n and any filtered colimit $A \cong \operatorname{colim} A_\alpha$ in $\tau_{\leq n}\mathrm{CAlg}_k$, the canonical map

$$\operatorname{colim} X(A_\alpha) \to X(A) \times_{Y(A)} \operatorname{colim} Y(A_\alpha) \tag{4.1}$$

is an isomorphism. Finally, recall from Proposition 3.24 that GStk_k is contained in the full subcategory $\widehat{\mathrm{Stk}}_k \subset \mathrm{Stk}_k$ of convergent stacks. Colimits in this subcategory are in some ways more natural. For example, any $\operatorname{Spec} A$ is the colimit of its truncations $\operatorname{Spec} \tau_{\leq n} A$ in $\widehat{\mathrm{Stk}}_k$, but not in Stk_k unless A is itself truncated. In particular, the inclusion $\widehat{\mathrm{Stk}}_k \subset \mathrm{Stk}_k$ does not preserve colimits in general.

Definition 4.2. An *ind-geometric stack* is a convergent stack X which admits an expression $X \cong \operatorname{colim}_\alpha X_\alpha$ as a filtered colimit in $\widehat{\mathrm{Stk}}_k$ of truncated geometric stacks along closed immersions. We call such an expression an *ind-geometric presentation* of X. We call it a *reasonable presentation* if the structure maps are almost finitely presented, and say X is *reasonable* if it admits a reasonable presentation.

We write $\mathrm{indGStk}_k \subset \widehat{\mathrm{Stk}}_k$ (resp. $\mathrm{indGStk}_k^{reas} \subset \widehat{\mathrm{Stk}}_k$) for the full subcategory of ind-geometric (resp. reasonable ind-geometric) stacks. See Proposition 4.30 for a comparison with the definition of ind-scheme in [GR14, Def. 1.4.2]. The above notion of reasonableness was introduced by [Ras19, Def. 6.8.1] in the context of ind-schemes.

Any geometric stack X is indeed ind-geometric since $X \cong \operatorname{colim} \tau_{\leq n} X$ in $\widehat{\mathrm{Stk}}_k$ by Proposition 3.26. On the other hand, not every geometric stack is reasonable, a basic example being the self-intersection of the origin in $\mathbb{A}^\infty$. However, we have the following result, where we say a geometric stack is locally coherent if it admits a flat cover $\operatorname{Spec} A \to X$ such that A is coherent.

Proposition 4.3. *If a geometric stack X is locally coherent (in particular, if it is tamely presented), then it is reasonable.*

Proof. Let $\operatorname{Spec} A \to X$ be a flat cover such that A is coherent. Then $\operatorname{Spec} \tau_{\leq n} A \to \operatorname{Spec} \tau_{\leq n+1} A$ is almost of finite presentation for all n [Lur18, Cor. 5.2.2.2]. It follows that $\tau_{\leq n} X \to \tau_{n+1} X$ is as well [Lur18, Prop. 4.1.4.3], noting that $\operatorname{Spec} \tau_{\leq n} A$ is the base change of $\operatorname{Spec} A$ to $\tau_{\leq n} X$. Thus X is reasonable by Proposition 3.26, and the second claim follows from Proposition 2.8. □

Truncatedness plays an essential role in our discussion due to the following variant of [GR14, Lem. 1.3.6] (though see Remark 4.12). The claim follows immediately from Proposition 3.27, but would fail if Y were not truncated: in this case Y is not compact in 1-$\widehat{\mathrm{Stk}}_k$, since e.g. id_Y does not factor through any truncation of Y.

Proposition 4.4. *Let $X \cong \operatorname{colim} X_\alpha$ be an ind-geometric presentation. Then for any truncated geometric stack Y, the natural map*

$$\operatorname{colim} \operatorname{Map}_{\mathrm{Stk}_k}(Y, X_\alpha) \to \operatorname{Map}_{\mathrm{Stk}_k}(Y, X)$$

is an isomorphism.

To discuss ind-geometric stacks more intrinsically, without referring to particular ind-geometric presentations, the following notion is useful.

Definition 4.5. Let X be an ind-geometric stack. A *truncated* (resp. *reasonable*) *geometric substack* of X is a truncated geometric stack X' equipped with a closed immersion $X' \to X$ (resp. an almost finitely presented closed immersion $X' \to X$).

Proposition 4.6. *Let $X \cong \operatorname{colim} X_\alpha$ be an ind-geometric (resp. reasonable) presentation. Then for all α, the structure morphism $i_\alpha : X_\alpha \to X$ realizes X_α as a truncated (resp. reasonable) geometric substack of X. Any other truncated (resp. reasonable) geometric substack $X' \to X$ can be factored as $X' \xrightarrow{j_\alpha} X_\alpha \xrightarrow{i_\alpha} X$ for some α, and in any such factorization j_α is a closed immersion (resp. almost finitely presented closed immersion), hence affine.*

Proof. To show i_α is a closed immersion, fix $\operatorname{Spec} A \to X$ and let $Z := X_\alpha \times_X \operatorname{Spec} A$. Since $\tau_{\leq 0} Z \cong \tau_{\leq 0}(X_\alpha \times_X \tau_{\leq 0} \operatorname{Spec} A)$ we may assume A is classical. By Proposition 4.4 we can then factor $\operatorname{Spec} A \to X$ through some $X_{\alpha'}$, which we may assume satisfies $\alpha' \geq \alpha$. For $\alpha'' \geq \alpha'$ let $Z_{\alpha''} := X_\alpha \times_{X_{\alpha''}} \operatorname{Spec} A$. We have $Z \cong \operatorname{colim}_{\alpha'' \geq \alpha'} Z_{\alpha''}$ since filtered colimits in $\widehat{\mathrm{Stk}}_k$ are left exact [Lur09, Ex. 7.3.4.7]. Moreover, $\tau_{\leq 0} Z \cong \operatorname{colim}_{\alpha'' \geq \alpha'} \tau_{\leq 0} Z_{\alpha''}$ since by Proposition 3.27 and its the proof all terms are in 1-$\widehat{\mathrm{Stk}}_k$ and $Y \mapsto \tau_{\leq 0} Y$ preserves filtered colimits in 1-$\widehat{\mathrm{Stk}}_k$. To show $\tau_{\leq 0} Z \to \tau_{\leq 0} \operatorname{Spec} A$ is a closed immersion it then suffices to show $\tau_{\leq 0} Z_{\alpha'} \to \tau_{\leq 0} Z_{\alpha''}$ is an isomorphism for any $\alpha'' \geq \alpha'$, since $\tau_{\leq 0} Z_{\alpha'} \to \tau_{\leq 0} \operatorname{Spec} A$ is a closed immersion by hypothesis. But this follows from Proposition 3.30.

Now suppose the given presentation is reasonable, and let $A \cong \operatorname{colim} A_\beta$ be a filtered colimit in $\tau_{\leq n} \mathrm{CAlg}_k$ for some n. By Proposition 4.4 we have $X(A_\beta) \cong \operatorname{colim}_{\gamma \geq \alpha} X_\gamma(A_\beta)$ for all β, and likewise $X(A) \cong \mathrm{colim}_{\gamma \geq \alpha} X_\gamma(A)$. We then have

$$X_\alpha(A) \times_{X(A)} \operatorname*{colim}_\beta X(A_\beta) \cong \operatorname*{colim}_{\gamma \geq \alpha} \left(X_\alpha(A) \times_{X_\gamma(A)} \operatorname*{colim}_\beta X_\gamma(A_\beta) \right)$$

since filtered colimits of spaces are left exact [Lur09, Prop. 5.3.3.3]. But the right hand colimit is isomorphic to $\mathrm{colim}_\beta X_\alpha(A_\beta)$ since each individual term is by hypothesis.

Finally, by Proposition 4.4 we can factor $X' \to X$ through a morphism $j_\alpha : X' \to X_\alpha$ for some α. Then j is a closed immersion by Proposition 3.28, hence is affine by Proposition 3.29, and is almost of finite presentation in the reasonable case by Proposition 3.5. □

4.2. **Ind-tamely presented morphisms.** The notion of tamely presented morphism may be extended to the ind-geometric setting following the usual pattern for notions such as ind-properness.

Definition 4.7. A morphism $f : X \to Y$ of ind-geometric stacks is ind-proper (resp. an ind-closed immersion, of ind-finite cohomological dimension) if for every commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

such that $X' \to X$ and $Y' \to Y$ are truncated geometric substacks, the map f' is proper (resp. a closed immersion, of finite cohomological dimension). If X and Y are reasonable then f is ind-tamely presented (resp. almost ind-finitely presentation) if the corresponding condition holds when X' and Y' are reasonable geometric substacks. An ind-geometric stack is ind-tamely presented (resp. almost ind-finitely presented) if it is reasonable and is ind-tamely presented (resp. almost ind-finitely presented) over $\operatorname{Spec} k$.

If $f : X \to Y$ is of ind-finite cohomological dimension and there exists an n such that any morphism f' as in Definition 4.7 is of cohomological dimension $\leq n$, then we say f is of finite cohomological dimension. For example, an ind-closed immersion is of finite cohomological dimension (with $n = 0$), while the projection $\mathbb{P}^\infty := \bigcup \mathbb{P}^n \to \operatorname{Spec} k$ is of ind-finite, but not finite, cohomological dimension.

Proposition 4.8. *Let $f : X \to Y$ be a morphism of ind-geometric stacks and $X \cong \mathrm{colim}_\alpha X_\alpha$ an ind-geometric presentation. Then f is ind-proper (resp. an ind-closed immersion, of ind-finite cohomological dimension) if and only if for every X_α there exists a commutative diagram*

$$\begin{tikzcd} X_\alpha \arrow[r] \arrow[d, "f_\alpha"'] & X \arrow[d, "f"] \\ Y_\alpha \arrow[r] & Y \end{tikzcd}$$

such that Y_α is a truncated geometric substack of Y and f_α is proper (resp. a closed immersion, of finite cohomological dimension). If X and Y are reasonable and $X \cong \operatorname{colim}_\alpha X_\alpha$ is a reasonable presentation, then f is ind-tamely presented (resp. almost ind-finitely presented) if and only if the corresponding condition holds with Y_α a reasonable geometric substack.

Proof. Let us explicitly treat the case of ind-tamely presented morphisms. Fix a reasonable presentation $Y \cong \operatorname{colim} Y_\beta$. The only if direction follows since $f \circ i_\alpha$ factors through some Y_β by Proposition 4.4. Now consider the if direction, and fix a diagram as in Definition 4.7. By hypothesis and Proposition 4.4 there exists a diagram of the left-hand form for some α,

$$
\begin{tikzcd}
X' \arrow[rr, "f'"] \arrow[dr] \arrow[dd] & & Y' \arrow[dr] & \\
& X \arrow[rr, "f"] & & Y \\
X_\alpha \arrow[rr, "f_\alpha"] \arrow[ur] & & Y_\alpha \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[dd] & Y' \arrow[d] \arrow[dr] & \\
& Y_\beta \arrow[r] & Y, \\
X_\alpha \arrow[r, "f_\alpha"] & Y_\alpha \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

where f_α is tamely presented and $Y_\alpha \to Y$ is a reasonable geometric substack. We claim this extends to a diagram of the right-hand form for some Y_β.

To see this, note that for any finite diagram $p : K \to \mathrm{GStk}^+$, the natural map

$$\operatorname{colim} \mathrm{Map}_{\mathrm{Fun}(K,\mathrm{Stk}_k)}(p, Y_\beta) \to \mathrm{Maps}_{\mathrm{Fun}(K,\mathrm{Stk}_k)}(p, Y)$$

is an isomorphism, where we let Y and Y_β denote the associated constant diagrams. This follows since p is compact in $\mathrm{Fun}(K, 1\text{-}\widehat{\mathrm{Stk}}_k)$ by [Lur09, Prop. 5.3.4.13] and Proposition 3.27. The claim at hand follows by taking p to be the subdiagram on the left spanned by X', Y', X_α, and Y_α.

In the right-hand diagram, the vertical maps are almost finitely presented closed immersions (and affine) by Proposition 4.6. It follows that f' is tamely presented by Propositions 3.4, 3.6, and 3.7. The other classes of morphisms are treated the same way, respectively using Propositions 3.5, 3.15, 3.17, 3.28 and the following observation: if f and g are composable morphisms in GStk_k such that $g \circ f$ is of finite cohomological dimension and g is affine (hence g_* conservative and t-exact), then f is of finite cohomological dimension. □

Proposition 4.9. *Ind-proper morphisms, ind-closed immersions, and morphisms of ind-finite cohomological dimension (resp. ind-tamely presented morphisms and almost ind-finitely presented morphisms) are stable under composition in* $\mathrm{indGStk}_k$ *(resp.* $\mathrm{indGStk}_k^{reas}$*).*

Proof. Let X, Y, and Z be ind-geometric stacks, $f : X \to Y$ and $g : Y \to Z$ ind-proper morphisms, and $X \cong \operatorname{colim} X_\alpha$ an ind-geometric presentation. For any α, Proposition 4.4 implies there exist truncated geometric substacks $Y_\alpha \to Y$ and $Z_\alpha \to Z$ such that the restrictions of f and g factor through proper morphisms $f_\alpha : X_\alpha \to Y_\alpha$ and $g_\alpha : Y_\alpha \to Z_\alpha$.

But then $g_\alpha \circ f_\alpha$ is proper, hence $g \circ f$ is ind-proper by the other direction of Proposition 4.8. The other classes of morphisms are treated the same way. □

Proposition 4.10. *Let $f : X \to Y$ be a morphism of geometric stacks. Then f is ind-proper (resp. an ind-closed immersion, of ind-finite cohomological dimension) if and only if it is proper (resp. a closed immersion, of finite cohomological dimension). If X and Y are truncated, then f is ind-tamely presented (resp. almost ind-finitely presented) if and only if it is tamely presented (resp. almost finitely presented).*

Proof. Recall that $X \cong \operatorname{colim} \tau_{\leq n} X$ and $X \cong \operatorname{colim} \tau_{\leq n} Y$ are ind-geometric presentations. Properness of f is equivalent to properness of $\tau_{\leq 0} f$ [Lur18, Rem. 5.1.2.2], hence to properness of each $\tau_{\leq n} f$, hence to ind-properness (Proposition 4.8). The corresponding claim for closedness is immediate, while for finiteness of cohomological dimension it follows from [HLP14, Lem. A.1.6] and the fact that $\mathrm{QCoh}(X)^\heartsuit \cong \mathrm{QCoh}(\tau_{\leq 0} X)^\heartsuit$. The last claim follows from Proposition 4.8, X and Y being reasonable presentations of themselves. □

Ind-closed immersions have the following closure property. Here $\mathrm{indGStk}_{k,\,cl} \subset \mathrm{indGStk}_k$ (resp. $\mathrm{indGStk}^{reas}_{k,\,cl,\,afp} \subset \mathrm{indGStk}^{reas}_k$) denotes the 1-full subcategory which only includes ind-closed immersions (resp. almost finitely presented ind-closed immersions), similarly for $\mathrm{GStk}^+_{k,\,cl} \subset \mathrm{GStk}^+_k$ (resp. $\mathrm{GStk}^+_{k,\,cl,\,afp} \subset \mathrm{GStk}^+_k$). Recall that a subcategory is 1-full if for $n > 1$ it includes all n-simplices whose edges belong to the indicated class of morphisms.

Proposition 4.11. *The canonical continuous functor $\mathrm{Ind}(\mathrm{GStk}^+_{k,\,cl}) \to \widehat{\mathrm{Stk}}_k$ factors through an equivalence $\mathrm{Ind}(\mathrm{GStk}^+_{k,\,cl}) \cong \mathrm{indGStk}_{k,\,cl}$, and likewise $\mathrm{Ind}(\mathrm{GStk}^+_{k,\,cl,\,afp}) \to \widehat{\mathrm{Stk}}_k$ factors through an equivalence $\mathrm{Ind}(\mathrm{GStk}^+_{k,\,cl,\,afp}) \cong \mathrm{indGStk}^{reas}_{k,\,cl,\,afp}$. In particular, $\mathrm{indGStk}_k$ (resp. $\mathrm{indGStk}^{reas}_k$) is closed in $\widehat{\mathrm{Stk}}_k$ under filtered colimits along ind-closed immersions (resp. almost ind-finitely presented ind-closed immersions).*

Proof. By definition $\mathrm{indGStk}_k$ is the essential image of $\mathrm{Ind}(\mathrm{GStk}^+_{k,\,cl})$. Let $X \cong \operatorname{colim} X_\alpha$, $Y \cong \operatorname{colim} Y_\beta$ be ind-geometric presentations. By abuse we denote the corresponding objects of $\mathrm{Ind}(\mathrm{GStk}^+_{k,\,cl})$ by X and Y as well, so that

$$\mathrm{Map}_{\mathrm{Ind}(\mathrm{GStk}^+_{k,\,cl})}(X, Y) \cong \lim_\alpha \operatorname*{colim}_\beta \mathrm{Map}_{\mathrm{GStk}^+_{k,\,cl}}(X_\alpha, Y_\beta).$$

Now the natural map

$$\lim_\alpha \operatorname*{colim}_\beta \mathrm{Map}_{\mathrm{GStk}^+_{k,\,cl}}(X_\alpha, Y_\beta) \to \lim_\alpha \operatorname*{colim}_\beta \mathrm{Map}_{\mathrm{GStk}^+_{k}}(X_\alpha, Y_\beta) \cong \mathrm{Map}_{\mathrm{indGStk}_k}(X, Y)$$

is a monomorphism since monomorphisms are stable under limits and filtered colimits (note that the isomorphism on the right follows from Proposition 3.27). It thus suffices to show its image is exactly the subspace of ind-closed immersions, but this follows from the definitions and Proposition 4.8. The other case is proved the same way. □

Remark 4.12. Note that a closed immersion of non-truncated geometric stacks is also an ind-closed morphism of ind-geometric stacks. It follows from Proposition 4.11 that $\mathrm{indGStk}_k$ is the essential image of the (not fully faithful) functor $\mathrm{Ind}(\mathrm{GStk}_{k,cl}) \to \widehat{\mathrm{Stk}}_k$, where $\mathrm{GStk}_{k,cl} \subset \mathrm{GStk}_k$ is the 1-full subcategory which only includes closed immersions. In other words, we obtain the same class of objects if in Definition 4.2 we do not require the X_α to be truncated.

By the following result the notion almost ind-finitely presented morphism is redundant, but the formulation of Definition 4.7 is often still convenient.

Proposition 4.13. *A morphism $f : X \to Y$ of reasonable ind-geometric stacks is almost of ind-finite presentation in the sense of Definition 4.7 if and only if it is almost of finite presentation in the sense of (4.1).*

Proof. The if direction follows from Propositions 3.5 and 4.6. Now let $X \cong \mathrm{colim}_\alpha X_\alpha$ be a reasonable presentation and $A \cong \mathrm{colim}\, A_\beta$ a filtered colimit in $\tau_{\leq n}\mathrm{CAlg}_k$ for some n. Then we have

$$\operatorname*{colim}_\beta X(A_\beta) \cong \operatorname*{colim}_{\alpha,\beta} X_\alpha(A_\beta) \cong \operatorname*{colim}_\alpha \left(X_\alpha(A) \times_{Y(A)} \operatorname*{colim}_\beta Y(A_\beta) \right),$$

the first isomorphism using Proposition 4.4 and the second Proposition 4.6. But the last expression is then isomorphic to $X(A) \times_{Y(A)} \mathrm{colim}_\beta Y(A_\beta)$ by the left exactness of filtered colimits of spaces. □

4.3. **Coherent pullback.** Recall that a morphism $f : X \to Y$ of stacks is geometric if for any morphism $Y' \to Y$ with Y' an affine scheme, $Y' \times_Y X$ is a geometric stack. By Proposition 3.2 this implies more generally that $Y' \times_Y X$ is a geometric stack whenever Y' is.

Definition 4.14. Let $f : X \to Y$ be a morphism of reasonable ind-geometric stacks. We say f has semi-universal coherent pullback if it is geometric and for every truncated geometric stack Y' and every ind-tamely presented morphism $Y' \to Y$, the base change $f' : Y' \times_Y X \to Y'$ has coherent pullback.

Likewise we will say a morphism of arbitrary ind-geometric stacks is of Tor-dimension $\leq n$ (resp. of finite Tor-dimension) if it is geometric and its base change to any truncated geometric stack is of Tor-dimension $\leq n$ (resp. of finite Tor-dimension) in the sense of Section 3.3. The following criterion for semi-universal coherent pullback entails in particular that the above definition is consistent with Definition 3.10 when Y is a truncated geometric stack.

Proposition 4.15. *Let $f : X \to Y$ be a geometric morphism of reasonable (resp. arbitrary) ind-geometric stacks and $Y \cong \mathrm{colim}_\alpha Y_\alpha$ a reasonable (resp. ind-geometric) presentation of Y. Then f has semi-universal coherent pullback (resp. is of finite Tor-dimension) if and only if its base change to every Y_α has semi-universal coherent pullback (resp. is of finite Tor-dimension).*

Proof. Propositions 4.6 and 4.13 imply the only if direction. Let Y' be a truncated geometric stack and $j : Y' \to Y$ a tamely presented morphism. By Proposition 4.6 and the definitions we can factor j through some $i_\alpha : Y_\alpha \to Y$ via a tamely presented morphism $j_\alpha : Y' \to Y_\alpha$, hence the base change of f to Y' has coherent pullback since its base change to Y_α has semi-universal coherent pullback. The finite Tor-dimension case is proved the same way. □

Proposition 4.16. *Morphisms with semi-universal coherent pullback (resp. of finite Tor-dimension) are stable under composition in* $\mathrm{indGStk}_k^{reas}$ *(resp.* $\mathrm{indGStk}_k$*).*

Proof. Let X, Y, and Z be reasonable ind-geometric stacks, $f : X \to Y$ and $g : Y \to Z$ morphisms with semi-universal coherent pullback, and $Z \cong \operatorname{colim} Z_\alpha$ a reasonable presentation. Define $g_\alpha : Y_\alpha \to Z_\alpha$ and $f_\alpha : X_\alpha \to Y_\alpha$ by base change. Then each Y_α is a truncated geometric stack since g_α has coherent pullback, the maps $Y_\alpha \to Y_\beta$ are almost finitely presented closed immersions by base change, and $Y \cong \operatorname{colim} Y_\alpha$ by left exactness of filtered colimits in $\widehat{\mathrm{Stk}}_k$. In particular each f_α has semi-universal coherent pullback, hence each $g_\alpha \circ f_\alpha$ does by Proposition 3.11, hence $g \circ f$ does by Proposition 4.15. The finite Tor-dimension case is proved the same way. □

4.4. **Fiber Products.** Now we consider fiber products of ind-geometric stacks, and the base change properties of the classes of morphisms considered above.

Proposition 4.17. *Ind-geometric stacks are closed under finite limits in* $\widehat{\mathrm{Stk}}_k$ *(and* Stk_k*).*

Proof. Note that $\widehat{\mathrm{Stk}}_k$ is closed under limits in Stk_k, so the two claims are equivalent. Since $\mathrm{indGStk}_k$ contains the terminal object $\operatorname{Spec} k$, it suffices to show closure under fiber products [Lur09, Prop. 4.4.2.6].

Let $f : X \to Y$ and $h : Y' \to Y$ be morphisms of ind-geometric stacks, and let $X' := X \times_Y Y'$. Suppose first that X and Y' are truncated geometric stacks, and let $Y \cong \operatorname{colim} Y_\alpha$ be an ind-geometric presentation. By Proposition 4.4 we can factor f and h through Y_α for some α. We have $X' \cong \operatorname{colim}_{\beta \geq \alpha} X'_\beta$, where $X'_\beta := X \times_{Y_\beta} Y'$, by left exactness of filtered colimits in $\widehat{\mathrm{Stk}}_k$ [Lur09, Ex. 7.3.4.7]. The transition maps are closed immersions of not necessarily truncated geometric stacks by Proposition 3.30. It follows they are ind-closed as morphisms of ind-geometric stacks, hence X' is ind-geometric by Proposition 4.11. Now suppose $X \cong \operatorname{colim} X_\alpha$ and $Y' \cong \operatorname{colim} Y'_\beta$ are ind-geometric presentations. Then as above $X' \cong \operatorname{colim} X_\alpha \times_Y Y'_\beta$ expresses X' as a filtered colimit in $\widehat{\mathrm{Stk}}_k$ of ind-geometric stacks along ind-closed immersions, so again X' is ind-geometric by Proposition 4.11. □

Already the self-intersection of the origin in $\mathbb{A}^\infty$ illustrates that reasonable ind-geometric stacks are not closed under arbitrary fiber products. Instead we have the following more limited result.

Proposition 4.18. *Let the following be a Cartesian diagram of ind-geometric stacks.*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Suppose that X, Y, and Y' are reasonable. Suppose also that h has semi-universal coherent pullback and f is ind-tamely presented (resp. that h is of finite Tor-dimension and f is arbitrary). Then X' is reasonable. Moreover, h' has semi-universal coherent pullback and f' is ind-tamely presented (resp. h' is of finite Tor-dimension), and f' is almost ind-finitely presented if f is.

Proof. Let $X \cong \operatorname{colim} X_\alpha$ be a reasonable presentation. For any α, there is by hypothesis a reasonable geometric substack $Y_\alpha \to Y$ and a commutative cube

$$\begin{tikzcd} & X'_\alpha \arrow[rr, "\phi'"] \arrow[dl] \arrow[dd, "\psi'"', near end] & & Y'_\alpha \arrow[dl] \arrow[dd, "\psi"] \\ X' \arrow[rr, "f'", near end, crossing over] \arrow[dd] & & Y' & \\ & X_\alpha \arrow[rr, "\phi", near start] \arrow[dl] & & Y_\alpha \arrow[dl] \\ X \arrow[rr, "f"] & & Y \arrow[from=uu, "h", near start, crossing over] & \end{tikzcd}$$

in $\mathrm{indGStk}_k$ such that all but the top and bottom faces are Cartesian, and such that ϕ and ϕ' are tamely presented.

By hypothesis ψ has semi-universal coherent pullback, and by Proposition 3.11 so does ψ'. In particular X'_α is a truncated geometric stack. We have $X' \cong \operatorname{colim} X'_\alpha$ in $\widehat{\mathrm{Stk}}_k$ since filtered colimits are left exact [Lur09, Ex. 7.3.4.7], and this is a reasonable presentation since almost finitely presented closed immersions are stable under base change. The rest of the claim now follows from Propositions 4.15 and 4.8. The other claims follow the same way. $\square$

Proposition 4.19. *Ind-proper morphisms (resp. ind-closed immersions, morphisms of ind-finite cohomological dimension) are stable under base change in* $\mathrm{indGStk}_k$.

Proof. Let $f : X \to Y$ and $h : Y' \to Y$ be morphisms in $\mathrm{indGStk}_k$ such that f is ind-proper. If $X \cong \operatorname{colim} X_\alpha$ is an ind-geometric presentation, we have for all α a commutative cube

$$\begin{tikzcd}
 & X'_\alpha \arrow[rr, "\phi'"] \arrow[dl] \arrow[dd, "\psi'"'] & & Y'_\alpha \arrow[dl] \arrow[dd, "\psi"] \\
X' \arrow[rr, "f'"] \arrow[dd] & & Y' \arrow[dd, "h"] & \\
 & X_\alpha \arrow[rr, "\phi"] \arrow[dl] & & Y_\alpha \arrow[dl] \\
X \arrow[rr, "f"] & & Y &
\end{tikzcd}$$

in $\mathrm{indGStk}_k$ such that all but the top and bottom faces are Cartesian, $Y_\alpha \to Y$ is a truncated geometric substack, and ϕ is proper. Let $Y'_\alpha \cong \operatorname{colim}_\beta Y'_{\alpha\beta}$ be an ind-geometric presentation. Then, letting $X'_{\alpha\beta} := X'_\alpha \times_{Y'_\alpha} Y'_{\alpha\beta}$, we have $X'_\alpha \cong \operatorname{colim}_\beta X'_{\alpha\beta}$ by left exactness of filtered colimits in $\widehat{\mathrm{Stk}}_k$. Note that for all β the morphisms $X'_{\alpha\beta} \to X'$ and $Y'_{\alpha\beta} \to Y'$ are closed immersions since $Y'_{\alpha\beta} \to Y'_\alpha$ and $Y_\alpha \to Y$ are, and in particular $Y'_{\alpha\beta}$ is a truncated geometric substack of Y'.

Now let $X' \cong \operatorname{colim} X'_\gamma$ be an ind-geometric presentation and fix some γ. By Proposition 4.4 we can choose α so that $X'_\gamma \to X$ factors through X_α, hence so that $X'_\gamma \to X'$ factors through X'_α. Proposition 3.27 then implies that $X'_\gamma \to X'$ factors through $X'_{\alpha\beta}$ for some β. This map $X'_\gamma \to X'_{\alpha\beta}$ is a closed immersion since $X'_\gamma \to X'$ and $X'_{\alpha\beta} \to X'$ are, while $X'_{\alpha\beta} \to Y'_{\alpha\beta}$ is proper since it is a base change of ϕ. Thus the composition $X'_\gamma \to X'_{\alpha\beta} \to Y'_{\alpha\beta}$ is proper, hence f' is ind-proper by Proposition 4.8. The other classes of morphisms are treated the same way. □

4.5. **Coherent sheaves and pushforward.** We now define the category $\mathrm{Coh}(X)$ of coherent sheaves on a reasonable ind-geometric stack X. Explicitly, if $X \cong \operatorname{colim} X_\alpha$ is a reasonable presentation we will have $\mathrm{Coh}(X) \cong \operatorname{colim} \mathrm{Coh}(X_\alpha)$, where the colimit is taken in Cat_∞ along the pushforward functors. Such expressions will follow from a more canonical definition of $\mathrm{Coh}(X)$, which will also make manifest its functoriality under proper pushforward.

Let $\mathrm{GStk}^+_{k,prop} \subset \mathrm{GStk}^+_k$ denote the 1-full subcategory which only includes proper, almost finitely presented morphisms, similarly for $\mathrm{indGStk}^{reas}_{k,prop} \subset \mathrm{indGStk}^{reas}_k$. Note that $\mathrm{GStk}^+_{k,prop}$ is a full subcategory of $\mathrm{indGStk}^{reas}_{k,prop}$ by Proposition 4.10. We have a canonical functor

$$\mathrm{Coh} : \mathrm{GStk}^+_{k,prop} \to \mathrm{Cat}_\infty \tag{4.20}$$

which takes X to $\mathrm{Coh}(X)$ and $f : X \to Y$ to $f_* : \mathrm{Coh}(X) \to \mathrm{Coh}(Y)$. This is obtained by restriction from the corresponding functor $\mathrm{QCoh} : \mathrm{GStk}_k \to \widehat{\mathrm{Cat}}_\infty$, given that proper, almost finitely presented pushforward preserves coherence (Proposition 3.17).

Definition 4.21. We write

$$\mathrm{Coh} : \mathrm{indGStk}^{reas}_{k,prop} \to \mathrm{Cat}_\infty \tag{4.22}$$

for the left Kan extension of (4.20) along the inclusion $\mathrm{GStk}^+_{k,\,prop} \subset \mathrm{indGStk}^{reas}_{k,\,prop}$. We write $f_* : \mathrm{Coh}(X) \to \mathrm{Coh}(Y)$ for the functor assigned to an ind-proper, almost ind-finitely presented morphism $f : X \to Y$.

Existence of the indicated left Kan extension follows from [Lur09, Lem. 4.3.2.13]. The following result implies that $\mathrm{Coh}(X)$ can be expressed in terms of reasonable presentations as stated earlier.

Proposition 4.23. *The functor* $\mathrm{Coh} : \mathrm{indGStk}^{reas}_{k,\,prop} \to \mathrm{Cat}_\infty$ *preserves filtered colimits along almost ind-finitely presented ind-closed immersions.*

Proof. For the proof we distinguish (4.20) and (4.22) by writing them as Coh_{geom} and Coh_{ind}, respectively. The proof of Proposition 4.11 adapts to show that the canonical continuous functor $\mathrm{Ind}(\mathrm{GStk}^+_{k,\,prop}) \to \widehat{\mathrm{Stk}}_k$ is faithful and that $\mathrm{indGStk}^{reas}_{k,\,prop}$ is the intersection of its image with $\mathrm{indGStk}^{reas}_k$. In particular, the inclusion $\mathrm{indGStk}^{reas}_{k,\,prop} \subset \widehat{\mathrm{Stk}}_k$ factors through a fully faithful functor $\mathrm{indGStk}^{reas}_{k,\,prop} \to \mathrm{Ind}(\mathrm{GStk}^+_{k,\,prop})$. By the transitivity of left Kan extensions [Lur09, Prop. 4.3.2.8], Coh_{ind} is the restriction to $\mathrm{indGStk}^{reas}_{k,\,prop}$ of Coh_{Ind}, the left Kan extension of Coh_{geom} to $\mathrm{Ind}(\mathrm{GStk}^+_{k,\,prop})$.

Recall from Proposition 4.11 that $\mathrm{indGStk}^{reas}_{k,\,cl,\,afp}$ admits filtered colimits and its inclusion into $\widehat{\mathrm{Stk}}_k$ continuous. By the previous paragraph this inclusion factors through a faithful functor $\mathrm{indGStk}^{reas}_{k,\,cl,\,afp} \to \mathrm{Ind}(\mathrm{GStk}^+_{k,\,prop})$, which is then also continuous. But Coh_{Ind} is continuous [Lur09, Lem. 5.3.5.8], hence so is its restriction to $\mathrm{indGStk}^{reas}_{k,\,cl,\,afp}$. $\square$

Suppose $f : X \to Y$ is an ind-proper, almost ind-finitely presented morphism and $X \cong \operatorname{colim} X_\alpha$ a reasonable presentation. Proposition 4.23 implies in particular that any $\mathcal{F} \in \mathrm{Coh}(X)$ can be written as $\mathcal{F} \cong i_{\alpha *}(\mathcal{F}_\alpha)$ for some α and some $\mathcal{F}_\alpha \in \mathrm{Coh}(X_\alpha)$. The behavior of ind-proper pushforward on objects is thus determined from the geometric setting by the following compatibility: we can factor $f \circ i_\alpha$ through a proper, almost finitely presented morphism $f_\alpha : X_\alpha \to Y_\alpha$ to a reasonable geometric substack $j_\alpha : Y_\alpha \to Y$, and we then have $f_*(\mathcal{F}) \cong j_{\alpha *} f_{\alpha *}(\mathcal{F}_\alpha)$.

4.6. **Pullback and base change.** Next we consider the pullback of coherent sheaves along suitable morphisms of reasonable ind-geometric stacks. For explicitness we first define this directly in terms of presentations, and then give a more global description that will make its compatibilities more manifest.

Let $f : X \to Y$ be a morphism with semi-universal coherent pullback, which we emphasize includes any morphism of finite Tor-dimension. Let $Y \cong \operatorname{colim} Y_\alpha$ a reasonable presentation, $f_\alpha : X_\alpha \to Y_\alpha$ the base change of f, and $i_{\alpha\beta} : X_\alpha \to X_\beta$, $j_{\alpha\beta} : Y_\alpha \to Y_\beta$ the induced maps. By proper base change we have isomorphisms $f^*_\beta j_{\alpha\beta *} \cong i_{\alpha\beta *} f^*_\alpha$ of functors $\mathrm{Coh}(Y_\alpha) \to \mathrm{Coh}(X_\beta)$. The identity $\mathrm{Coh}(Y) \cong \operatorname{colim} \mathrm{Coh}(Y_\alpha)$ implies there is an essentially unique functor $f^* : \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$ equipped with coherent isomorphisms $f^* j_{\alpha *} \cong i_{\alpha *} f^*_\alpha$ of functors $\mathrm{Coh}(Y_\alpha) \to \mathrm{Coh}(X)$ for all α. More formally, we can construct this functor as follows.

Write A for the index category of the presentation $Y \cong \operatorname{colim} Y_\alpha$, so that Y and the Y_α together define a diagram $A^\triangleright \to \mathrm{indGStk}_k^{reas}$. Taking Cartesian products with X we obtain a diagram $A^\triangleright \times \Delta^1 \to \mathrm{indGStk}_k^{reas}$ and by restriction a diagram $A \times \Delta^1 \to \mathrm{GStk}_k^+$. We then obtain a diagram $A \times \Delta^1 \to \mathrm{Cat}_\infty$ taking f_α to $f_\alpha^* : \mathrm{Coh}(Y_\alpha) \to \mathrm{Coh}(X_\alpha)$ and $i_{\alpha\beta}$ to $i_{\alpha\beta *} : \mathrm{Coh}(X_\alpha) \to \mathrm{Coh}(X_\beta)$. More precisely, this is constructed from the more primitive functor $A \times \Delta^1 \to \widehat{\mathrm{Cat}}_\infty$ that takes each map to the associated pullback of quasi-coherent sheaves, passing to adjoints along morphisms in A via [Lur17, Cor. 4.7.5.18] and then passing to coherent subcategories. This is equivalent to the data of the associated functor $A \to \mathrm{Cat}_\infty^{\Delta^1} := \mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$ taking α to $f_\alpha^* : \mathrm{Coh}(Y_\alpha) \to \mathrm{Coh}(X_\alpha)$. By [Lur09, Cor. 5.1.2.3] and Proposition 4.23 the colimit of this diagram is a functor $f^* : \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$ with the desired compatibilities.

To make this manifestly canonical, observe that $(\mathrm{GStk}^+_{k,cl,afp})_{/Y}$, the category of reasonable geometric substacks of Y, provides a canonical reasonable presentation $Y \cong \operatorname{colim}(\mathrm{GStk}^+_{k,cl,afp})_{/Y}$.

Definition 4.24. Let $f : X \to Y$ be a morphism with semi-universal pullback between reasonable ind-geometric stacks. Then we define $f^* : \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$ by applying the above construction to the canonical reasonable presentation $Y \cong \operatorname{colim}(\mathrm{GStk}^+_{k,cl,afp})_{/Y}$. That is, f^* is the unique functor which admits a coherent family of isomorphisms $f^* i_* \cong i'_* f'^*$ for any reasonable geometric substack $i : Y' \to Y$.

That this agrees with the above construction applied to an arbitrary reasonable presentation follows since the diagram $A \to \mathrm{GStk}_k^+$ associated to any other presentation factors through $(\mathrm{GStk}^+_{k,cl,afp})_{/Y}$.

Proposition 4.25. *Let the following be a Cartesian diagram of ind-geometric stacks.*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Suppose that X, Y, and Y' are reasonable, that h has semi-universal coherent pullback, and that f is ind-proper and almost ind-finitely presented. Then X' is reasonable, h' and f' have the same properties as h and f, and there is a canonical isomorphism $h^* f_* \cong f'_* h'^*$ of functors $\mathrm{Coh}(X) \to \mathrm{Coh}(Y')$.

Proof. The conclusions about X', h', and f' are Propositions 4.18 and 4.19. It follows from the proof of Proposition 4.23 and a straightforward variant of [Lur09, 5.3.5.15] that we can write $f \in (\mathrm{indGStk}^{reas}_{k,prop})^{\Delta^1}$ as the colimit of a filtered diagram $\{f_\alpha : X_\alpha \to Y_\alpha\}$ whose restrictions to the vertices of Δ^1 define reasonable presentations $X \cong \operatorname{colim} X_\alpha$, $Y \cong \operatorname{colim} Y_\alpha$. Writing A for the index category of this diagram, we repeat the above construction to obtain a functor $A \times \Delta^1 \times \Delta^1$... $\square$

Note that if $f : X \to Y$ and $g : Y \to Z$ are morphisms with semi-universal coherent pullback, we obtain a canonical isomorphism $(g \circ f)^* \cong f^* g^*$ by repeating the construction of each individual functor with Δ^2 in place of Δ^1. Similarly, one shows in this way that the base change isomorphisms of Proposition 4.25 are compatible with composition. We can encode such compatibilities more systematically using the formalism of correspondences.

Recall that given a category $\mathcal{C}$ with Cartesian products, we have a category $\mathrm{Corr}(\mathcal{C})$ with the same objects as $\mathcal{C}$, and in which a morphism from X to Z is a diagram $X \xleftarrow{f} Y \xrightarrow{h} Z$ [GR17a, Sec. 7.1.2.5]. Composition of correspondences is given by taking Cartesian products in the standard way. We note that we will never refer to $(\infty, 2)$-categories of correspondences in this text.

Let $\mathrm{Corr}(\mathrm{indGStk}_k^{reas})_{prop;coh}$ denote the 1-full subcategory of $\mathrm{Corr}(\mathrm{indGStk}_k)$ which only includes correspondences $X \xleftarrow{f} Y \xrightarrow{h} Z$ such that X, Y, and Z are reasonable, h has semi-universal coherent pullback, and f is ind-proper and almost of ind-finite presentation (these are stable under composition of correspondences by Propositions 4.9, 4.16, 4.19, and 4.18). We define $\mathrm{Corr}(\mathrm{GStk}_k^+)_{prop;coh}$ similarly, noting that it is a full subcategory of $\mathrm{Corr}(\mathrm{indGStk}_k)_{prop;coh}$ by Proposition 4.10.

Note now that the functor $\mathrm{Coh} : \mathrm{GStk}^+_{k,\,prop} \to \mathrm{Cat}_\infty$ of (4.20) extends to a functor

$$\mathrm{Coh} : \mathrm{Corr}(\mathrm{GStk}_k^+)_{prop;coh} \to \mathrm{Cat}_\infty. \tag{4.26}$$

To construct it, consider the 1-full subcategory $\mathrm{Corr}(\mathrm{GStk}_k)_{fcd;all} \subset \mathrm{Corr}(\mathrm{GStk}_k)$ in which we only include correspondences $X \xleftarrow{f} Y \xrightarrow{h} Z$ such that f is of finite cohomological dimension. By Proposition 3.15 and [GR17a, Thm. 3.2.2] there exists a canonical functor $\mathrm{QCoh} : \mathrm{Corr}(\mathrm{GStk}_k)_{fcd;all} \to \widehat{\mathrm{Cat}}_\infty$ which takes a correspondence $X \xleftarrow{f} Y \xrightarrow{h} Z$ to the functor $f_* h^* : \mathrm{QCoh}(Z) \to \mathrm{QCoh}(X)$. We obtain the desired functor (4.26) by restricting along the inclusion $\mathrm{Corr}(\mathrm{GStk}_k^+)_{prop;coh} \subset \mathrm{Corr}(\mathrm{GStk}_k)_{fcd;all}$, and observing that the associated functors $f_* h^*$ preserve coherence.

Proposition 4.27. *Consider the left Kan extension*

$$\mathrm{Coh} : \mathrm{Corr}(\mathrm{indGStk}_k^{reas})_{prop;coh} \to \mathrm{Cat}_\infty \tag{4.28}$$

of (4.26) along the inclusion $\mathrm{Corr}(\mathrm{GStk}^+)_{prop;coh} \subset \mathrm{Corr}(\mathrm{indGStk}_k^{reas})_{prop;coh}$*. The restriction of this extension to* $\mathrm{indGStk}^{reas}_{k,\,prop}$ *is the functor of Definition 4.21, and its value on a morphism* $h : X \to Y$ *with semi-universal coherent pullback is the functor* $h^* : \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$ *of Definition 4.24.*

Proof. The first claim follows from [GR17a, Thm. 6.1.5] (in its opposite form for left Kan extensions). Now let $Y \cong \operatorname{colim} Y_\alpha$ be a reasonable presentation with index set A, so that as before we have a diagram $A \times \Delta^1 \to \mathrm{Cat}_\infty$ taking $\{\alpha\} \times \Delta^1$ to h_α^*, where $h_\alpha : X_\alpha \to Y_\alpha$ is the base change of h. It follows from the first claim, from Proposition **??**, and from the fact that $X \cong \operatorname{colim} X_\alpha$ is also a reasonable presentation that the restriction of (4.28) along the induced functor $A^{\triangleright} \times \Delta^1 \to \mathrm{Corr}(\mathrm{indGStk}_k^{reas})_{prop;coh}$ is a left Kan extension of its restriction to $A \times \Delta^1$. The second claim follows since this is the definition of h^*. $\square$

4.7. **External products.** When k is an ordinary ring of finite global dimension, the closure of GStk_k^+ under products implies the same for $\mathrm{indGStk}_k^{reas}$ (again using left exactness of filtered colimits in $\widehat{\mathrm{Stk}}_k$). Proposition 3.22 then extends as follows.

Proposition 4.29. *If k is an ordinary ring of finite global dimension, then the functor*

$$\mathrm{Coh} : \mathrm{Corr}(\mathrm{indGStk}_k^{reas})_{prop;coh} \to \mathrm{Cat}_\infty$$

of Definition 4.21 is canonically lax symmetric monoidal.

Proof. Applying [Lur17, Prop. 3.1.1.20, Thm. 3.1.2.3] we may consider the lax symmetric monoidal functor $\mathrm{Corr}(\mathrm{indGStk})_{prop;coh} \to \widehat{\mathrm{Cat}}_\infty$ defined by operadic left Kan extension (more precisely, in the notation of [Lur17, Rem. 3.1.3.15] we take $\mathcal{A} = \mathrm{Corr}(\mathrm{GStk}^+)_{prop;coh}$, $\mathcal{B} = \mathrm{Corr}(\mathrm{indGStk})_{prop;coh}$, $\mathcal{C} = \widehat{\mathrm{Cat}}_\infty$, and $\mathcal{O}^\otimes = \mathrm{Fin}_*$). That its underlying functor is obtained by (non-operadic) left Kan extension follows from comparing the diagrams in [Lur17, Def. 3.1.1.2] and [Lur09, Def. 4.3.2.2] after pulling back along $\mathcal{D}' = \{\langle 1 \rangle\} \hookrightarrow \mathcal{O}^\otimes = \mathrm{Fin}_*$ (trivial Kan fibrations being stable under pullback). In particular, this extension a posteriori takes values in Cat_∞. $\square$

4.8. **Truncated ind-geometric stacks.** We say an ind-geometric stack is n-truncated if it admits an ind-geometric presentation $X \cong \mathrm{colim}\, X_\alpha$ in which each X_α is an n-truncated geometric stack. Recall that $(-)_{\leq n} : \widehat{\mathrm{Stk}}_k \to \mathrm{Stk}_{k,\leq n}$ identifies the category of n-truncated geometric stacks with a full subcategory of $\mathrm{Stk}_{k,\leq n}$, the inverse equivalence being given by the left adjoint $\widehat{i}_{\leq n} : \mathrm{Stk}_{k,\leq n} \to \widehat{\mathrm{Stk}}_k$. Note that $(-)_{\leq n}$ takes filtered colimits in $1\text{-}\widehat{\mathrm{Stk}}_k$ to filtered colimits in $\tau_{\leq n+1}\mathrm{Stk}_{k,\leq n}$, since it restricts from a continuous functor $\widehat{\mathrm{PStk}}_k \to \mathrm{PStk}_{k,\leq n}$, and since as in Proposition 3.27 and its proof $1\text{-}\widehat{\mathrm{Stk}}_k \subset \widehat{\mathrm{PStk}}_k$ and $\tau_{\leq n+1}\mathrm{Stk}_{k,\leq n} \subset \mathrm{PStk}_{k,\leq n}$ are closed under filtered colimits. Since $\widehat{i}_{\leq n}$ is continuous it follows that $(-)_{\leq n}$ identifies the category of n-truncated ind-geometric stacks with the obvious subcategory of $\mathrm{Stk}_{k,\leq n}$. Again letting $\widehat{\tau}_{\leq n} : \widehat{\mathrm{Stk}}_k \to \widehat{\mathrm{Stk}}_k$ denote the composition of $(-)_{\leq n}$ and $\widehat{i}_{\leq n}$, the following result states in particular that Definition 4.2 is indeed the obvious extension of [GR14, Def. 1.4.2] from schemes to geometric stacks (it is stated differently so that reasonableness may be introduced more easily).

Proposition 4.30. *A convergent stack X is ind-geometric if and only if $\widehat{\tau}_{\leq n}X$ is an n-truncated ind-geometric stack for all n.*

Proof. The only if direction follows since $\widehat{\tau}_{\leq n}$ preserves closed immersions of geometric stacks, and since by the above discussion its restriction to $1\text{-}\widehat{\mathrm{Stk}}_k$ is continuous. The if direction follows from Proposition 4.11, since each $\widehat{\tau}_{\leq n}X \to \widehat{\tau}_{\leq n+1}X$ is classically an isomorphism, hence is an ind-closed immersion. $\square$

Example 4.31. A typical example of a classical (i.e. zero-truncated) ind-geometric stack is the following. Suppose $X \cong \operatorname{colim} X_\alpha$ is a presentation of a classical ind-scheme as a filtered colimit of quasi-compact, semi-separated ordinary schemes (regarded as objects of $\widehat{\mathrm{Stk}}_k$) along closed immersions, and that G is a classical affine group scheme acting on X. For each α the induced map $X_\alpha \times G \to X$ factors through some X_β. The closure $X'_\alpha \subset X_\beta$ of its image is a G-invariant closed subscheme of X. We obtain a presentation $X \cong \operatorname{colim} X'_\alpha$ by closed G-invariant subschemes, and it follows that the quotient X/G (in $\widehat{\mathrm{Stk}}_k$) is a classical ind-geometric stack with ind-geometric presentation $X/G \cong \operatorname{colim} X'_\alpha/G$. Note that the quotients X'_α/G taken in Stk_k are again geometric [Lur18, Prop. 9.3.1.3], hence convergent (Proposition 3.24), hence they coincide with the quotients X'_α/G taken in $\widehat{\mathrm{Stk}}_k$.

4.9. **Formal geometric stacks.** If X is geometric, it is generally not the case that a morphism $f : X \to Y$ in $\mathrm{indGStk}_k$ is geometric. However, the failure is mild in so far as the statement does hold at the level of the underlying classical stacks. We formalize the situation with the following notion, which will reappear in Section 7.2.

Definition 4.32. A formal geometric stack is an ind-geometric stack X such that X^{cl} is geometric. An ind-geometric presentation $X \cong \operatorname{colim} X_\alpha$ is formal if the natural map $X_\alpha^{\mathrm{cl}} \to X^{\mathrm{cl}}$ is an equivalence for all α. A morphism $f : X \to Y$ is formally geometric if $X \times_Y Y'$ is a formal geometric stack for any morphism $Y' \to Y$ whose source Y' is geometric.

Proposition 4.33. *Every formal geometric stack admits a formal ind-geometric presentation.*

Proof. □

Proposition 4.34. *Let $f : X \to Y$ be a morphism of ind-geometric stacks. If X is geometric, then f is formally geometric.*

Proof. Suppose first that X is truncated. Let $Y' \to Y$ be a morphism whose source Y' is truncated and geometric, let $X' := X \times_Y Y'$, and let $Y \cong \operatorname{colim} Y_\alpha$ be an ind-geometric presentation. Then $X' \cong \operatorname{colim} X \times_{Y_\alpha} Y'$ by left exactness of filtered colimits in $\widehat{\mathrm{Stk}}_k$, and this is a formal ind-geometric presentation by Proposition 3.30.

If X and Y' are not necessarily truncated, we now have $X' \cong \operatorname{colim}_{m,n} \tau_{\leq n} X \times_Y \tau_{\leq m} Y'$ by Proposition 3.26. But the classical stacks $(\tau_{\leq n} X \times_Y \tau_{\leq m} Y')^{\mathrm{cl}}$ are isomorphic for all m, n, hence isomorphic to X'^{cl}, and the claim follows (recalling from the proof of Proposition 3.27 that $Y \mapsto Y^{\mathrm{cl}}$ preserves filtered colimits in $1\text{-}\widehat{\mathrm{Stk}}_k$). □

4.10. **Semi-reasonable ind-geometric stacks.** If $X' := X \times_Y Y'$ is a fiber product of ind-geometric stacks, it is not true in general that X' is reasonable if X, Y, and Y' are. However, we can make precise a sense in which X' is not too far from being reasonable, at least if either X or Y' is almost ind-finitely presented over Y.

Definition 4.35. An ind-geometric stack X is semi-reasonable if it can be written as a filtered colimit $X \cong \operatorname{colim} X_\alpha$ in $\widehat{\mathrm{Stk}}_k$ of geometric stacks X_α along almost finitely presented closed immersions. We call such an expression a semi-reasonable presentation of X.

We caution that a semi-reasonable presentation need not be an ind-geometric presentation, as we do not assume the X_α are truncated. Clearly if an ind-geometric stack is either reasonable or geometric then it is also semi-reasonable. We will see in later sections that semi-reasonableness of X gives us much of the same control over ind-coherent sheaves on X that reasonableness does. On the other hand, the failure of non-truncated geometric stacks to be compact in 1-$\widehat{\mathrm{Stk}}_k$ means that semi-reasonable presentations provide much less control over maps out of X, hence they are a secondary notion.

Proposition 4.36. *Let the following be a Cartesian diagram of ind-geoemtric stacks.*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Suppose that X, Y, and Y' are reasonable and f is almost of ind-finite presentation. Then X' is semi-reasonable.

Lemma 4.37. *Let I be a finite poset and $X_\bullet : I \to \mathrm{indGStk}_k$ a diagram. Then we can write $X_\bullet$ as the colimit in $\mathrm{indGStk}_k^I$ of a filtered diagram $X_{\bullet,\bullet} : A \to (\mathrm{GStk}_k^+)^I \subset \mathrm{indGStk}_k^I$ with the following property: for all $i \in I$ and all $\beta \geq \alpha \in A$, the map $X_{i,\alpha} \to X_{i,\beta}$ is a closed immersion which is almost of finite presentation if X_i is reasonable.*

5. Digression: ind-coherent sheaves

We now expand our attention from coherent to ind-coherent sheaves. These are necessary for the discussion of !-pullback and sheaf Hom in Sections 7 and 8.

If X is a geometric stack we define $\mathrm{IndCoh}(X)$ as the left anticompletion of $\mathrm{QCoh}(X)$, following a construction of [Lur18, App. C]. This characterizes $\mathrm{IndCoh}(X)$ in terms of a universal property satisfied by bounded colimit-preserving functors out of it (see (5.4)). The definition is then extended to the ind-geometric setting so that $\mathrm{IndCoh}(X) \cong \operatorname{colim} \mathrm{IndCoh}(X_\alpha)$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ for any ind-geometric presentation $X \cong \operatorname{colim} X_\alpha$ (Proposition 5.11), following the case of ind-schemes [GR14, Sec. 2.4], [Ras19, Sec. 6.10]. In particular $\mathrm{Coh}(X)$ is naturally a subcategory of $\mathrm{IndCoh}(X)$(Proposition 5.24), though unless X is geometric $\mathrm{Coh}(X)$ is not generally a subcategory of $\mathrm{QCoh}(X)$ in any natural way.

In most cases of interest one can be more explicit. If X is a classical geometric stack then $\mathrm{IndCoh}(X)$ is (the dg nerve of) the category of injective complexes in $\mathrm{QCoh}(X)^\heartsuit$, introduced in [Kra05]. If X is coherent (for example, if it is a Noetherian scheme as in [Gai13]) then $\mathrm{IndCoh}(X)$ is the ind-completion of $\mathrm{Coh}(X)$. This is the primary case of interest (as our terminology abusively reflects) and is considered in detail in Section 6. As the focal results of Sections 7 and 8 are about this case, the reader concerned only with these may safely define $\mathrm{IndCoh}(X)$ via ind-completion and bypass the current section. But it is often convenient to have $\mathrm{IndCoh}(X)$ defined in greater generality, since for example the class of coherent ind-geometric stacks (or even of coherent affine schemes) is not closed under fiber products.

5.1. **Anticompletion.** We begin by recalling the notion of anticompleteness from [Lur18, App. C] in slightly adapted form. A t-structure on a stable ∞-category $\mathcal{C}$ is left complete if the natural functor $\mathcal{C} \to \lim_n \mathcal{C}^{\geq n}$ is an equivalence, and right complete if $\mathcal{C} \to \lim_n \mathcal{C}^{\leq n}$ is an equivalence. The category $\widehat{\mathcal{C}} := \lim_n \mathcal{C}^{\geq n}$ is called the left completion of $\mathcal{C}$. It has a canonical t-structure such that $\mathcal{C} \to \widehat{\mathcal{C}}$ is t-exact and restricts to an equivalence $\mathcal{C}^{\geq 0} \xrightarrow{\sim} \widehat{\mathcal{C}}^{\geq 0}$ [Lur17, Prop. 1.2.1.17]. If $\mathcal{D}$ is another stable ∞-category with a t-structure, an exact functor $F : \mathcal{C} \to \mathcal{D}$ is bounded if there exist m, n such that $F(\mathcal{C}^{\geq 0}) \subset \mathcal{D}^{\geq m}$ and $F(\mathcal{C}^{\leq 0}) \subset \mathcal{D}^{\leq n}$. If $\mathcal{C}$ and $\mathcal{D}$ are presentable, we write $\mathrm{LFun}^b(\mathcal{C}, \mathcal{D}) \subset \mathrm{LFun}(\mathcal{C}, \mathcal{D})$ for the full subcategory of bounded colimit-preserving functors. In the presentable case, a t-structure on $\mathcal{C}$ is accessible if $\mathcal{C}^{\geq 0}$ is also presentable, and is compatible with filtered colimits if $\mathcal{C}^{\geq 0}$ is closed under filtered colimits in $\mathcal{C}$.

Definition 5.1. Let $\mathcal{C}$ be a presentable stable ∞-category equipped with an accessible t-structure which is right complete and compatible with filtered colimits. We say $\mathcal{C}$ is left anticomplete if composition with $\mathcal{D} \to \widehat{\mathcal{D}}$ induces an equivalence

$$\mathrm{LFun}^b(\mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \mathrm{LFun}^b(\mathcal{C}, \widehat{\mathcal{D}})$$

for any other presentable stable ∞-category $\mathcal{D}$ equipped with an accessible t-structure which is right complete and compatible with filtered colimits.

Now let $\mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{rcpl}}$ denote the ∞-category whose objects are presentable stable ∞-categories equipped with accessible t-structures which are right complete and compatible with filtered colimits, and whose morphisms are bounded colimit-preserving functors. Explicitly, given $\mathcal{C} \in \mathcal{P}\mathrm{r}^{\mathrm{St}}$ we consider the set of cores $\mathcal{C}^{\leq 0}$ (subcategories closed under small colimits and extensions), partially ordered by $\mathcal{C}_1^{\leq 0} < \mathcal{C}_2^{\leq 0}$ if $\mathcal{C}_1^{\leq 0} \subset \mathcal{C}_2^{\leq 0}[n]$ for some n. These posets are contravariantly functorial under taking preimages along exact functors, and $\mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{rcpl}}$ is a full subcategory of the associated Cartesian fibration over $\mathcal{P}\mathrm{r}^{\mathrm{St}}$.

We further let $\mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{acpl}}$ and $\mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{cpl}}$ denote the full subcategories of $\mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{rcpl}}$ defined by only including t-structures which are respectively left anticomplete and left complete. We then have the following variant of [Lur18, Cor. C.3.6.4, Cor. C.5.5.11, Prop. C.5.9.2].

Proposition 5.2. *The inclusion $\mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{cpl}} \hookrightarrow \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{rcpl}}$ admits a left adjoint, which acts on objects by $\mathcal{C} \mapsto \widehat{\mathcal{C}}$. The inclusion $\mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{acpl}} \hookrightarrow \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{rcpl}}$ admits a right adjoint, which we denote by $\mathcal{C} \mapsto \check{\mathcal{C}}$. These restrictions of these adjoints define inverse equivalences between $\mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{cpl}}$ and $\mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{acpl}}$.*

Proof. Given $\mathcal{C}, \mathcal{D} \in \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{rcpl}}$, write $\mathrm{LFun}^{b,\leq n}(\mathcal{C}, \mathcal{D}) \subset \mathrm{LFun}^{b}(\mathcal{C}, \mathcal{D})$ for the full subcategory of functors which take $\mathcal{C}^{\leq 0}$ to $\mathcal{D}^{\leq n}$. Since $\mathcal{C} \to \widehat{\mathcal{C}}$ is t-exact, composition with it induces a functor $\mathrm{LFun}^{b,\leq n}(\widehat{\mathcal{C}}, \mathcal{D}) \to \mathrm{LFun}^{b,\leq n}(\mathcal{C}, \mathcal{D})$. The existence of the desired adjoint follows if this is an equivalence for all n and for all $\mathcal{D} \in \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{cpl}}$ [Lur09, Prop. 5.2.4.2]. By shifting we can reduce to $n = 0$. That composition with $\mathcal{C} \to \widehat{\mathcal{C}}$ induces an equivalence between right t-exact functors follows from [Lur18, Prop. C.3.1.1., Prop. C.3.6.3] (noting that $\mathcal{D} \cong \mathrm{Sp}(\mathcal{D}^{\leq 0})$ [Lur18, Rem. C.3.1.5]). But this further identifies right t-exact functors which are bounded since $\mathcal{C}^{\geq 0} \xrightarrow{\sim} \widehat{\mathcal{C}}^{\geq 0}$.

Now let $\check{\mathcal{C}} := \mathrm{Sp}(\check{\mathcal{C}}^{\leq 0}) \in \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{rcpl}}$, where $\check{\mathcal{C}}^{\leq 0}$ is the anticompletion of $\mathcal{C}^{\leq 0}$ in the sense of [Lur18, Prop. C.5.5.9]. We claim $\check{\mathcal{C}}$ is left anticomplete in the sense of Definition 5.1. It suffices to show that for any $\mathcal{D} \in \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{rcpl}}$, the functor $\mathrm{LFun}^{b,\leq n}(\check{\mathcal{C}}, \mathcal{D}) \to \mathrm{LFun}^{b,\leq n}(\check{\mathcal{C}}, \widehat{\mathcal{D}})$ given by composition with $\mathcal{D} \to \widehat{\mathcal{D}}$ is an equivalence for all n. By shifting we can reduce to the case $n = 0$, which follows from [Lur18, Prop. C.3.1.1, Def. C.5.5.4].

The left-exact functor $\check{\mathcal{C}}^{\leq 0} \to \mathcal{C}^{\leq 0}$ of [Lur18, Prop. C.5.5.9] induces a t-exact functor $\check{\mathcal{C}} \to \mathcal{C}$ [Lur18, Prop. C.3.1.1, Prop. C.3.2.1]. The existence of the desired adjoint follows if the induced functor $\mathrm{LFun}^{b}(\mathcal{D}, \check{\mathcal{C}}) \to \mathrm{LFun}^{b}(\mathcal{D}, \mathcal{C})$ is an equivalence for all $\mathcal{D} \in \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{acpl}}$ [Lur09, Prop. 5.2.4.2]. But this follows from Definition 5.1, given that the left completions of $\check{\mathcal{C}}$ and $\mathcal{C}$ are equivalent [Lur18, Prop. C.5.5.9].

Finally, it follows by adjunction and Definition 5.1 that $\mathcal{C} \mapsto \widehat{\mathcal{C}}$ has fully faithful restriction to $\mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{acpl}}$, likewise for $\mathcal{C} \mapsto \check{\mathcal{C}}$ and $\mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{cpl}}$. That these restrictions are inverse equivalences now follows since $\mathcal{C} \in \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{cpl}}$ implies $\mathcal{C}$ is the left completion of $\check{\mathcal{C}}$ [Lur18, Prop. C.5.5.9]. □

5.2. **Ind-coherent sheaves.** Recall that if X is a geometric stack, the standard t-structure on $\mathrm{QCoh}(X)$ is accessible, left and right complete, and compatible with filtered colimits [Lur18, Cor. 9.1.3.2].

Definition 5.3. The category of ind-coherent sheaves on a geometric stack X is

$$\mathrm{IndCoh}(X) := \widecheck{\mathrm{QCoh}(X)},$$

the left anticompletion of its category of quasicoherent sheaves.

Conversely, $\mathrm{QCoh}(X)$ is the left completion of $\mathrm{IndCoh}(X)$, and in particular the natural functor $\mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$ induces an equivalence

$$\mathrm{IndCoh}(X)^{+} \xrightarrow{\sim} \mathrm{QCoh}(X)^{+}.$$

Unwinding the definitions, we see that $\mathrm{IndCoh}(X)$ is uniquely characterized by the following universal property: for all $\mathcal{C} \in \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{rcpl}}$ we have

$$\mathrm{LFun}^{b}(\mathrm{IndCoh}(X), \mathcal{C}) \cong \mathrm{LFun}^{b}(\mathrm{QCoh}(X), \widehat{\mathcal{C}}). \tag{5.4}$$

By Proposition 5.2 ind-coherent sheaves inherit all bounded, colimit-preserving functorialities of quasicoherent sheaves. Explicitly, recall from the proof of Proposition 3.18 the functor $\mathrm{QCoh} : \mathrm{Corr}(\mathrm{GStk})_{fcd;all} \to \mathcal{P}\mathrm{r}^{\mathrm{St}}$, which takes a correspondence $X \xleftarrow{f} Y \xrightarrow{h} Z$ such that f is of finite cohomological dimension to the functor $f_* h^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Z)$. By construction its restriction to $\mathrm{Corr}(\mathrm{GStk}_k)_{fcd;ftd}$, the subcategory which only includes correspondences in which h is of finite Tor-dimension, lifts to a functor

$$\mathrm{QCoh} : \mathrm{Corr}(\mathrm{GStk}_k)_{fcd;ftd} \to \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{cpl}}. \tag{5.5}$$

Definition 5.6. We define a functor

$$\mathrm{IndCoh} : \mathrm{Corr}(\mathrm{GStk}_k)_{fcd;ftd} \to \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{acpl}}. \tag{5.7}$$

by composing (5.5) with the equivalence $\mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{cpl}} \xrightarrow{\sim} \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{acpl}}$ of Proposition 5.2.

If $f : X \to Y$ is a morphism of finite cohomological dimension in GStk_k, we denote the associated functor $\mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$ simply by f_* when the context is clear. If we need to distinguish it from the associated functor $\mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ we denote them respectively by f_{IC*} and f_{QC*}. The two are related by a commutative diagram

$$\begin{tikzcd} \mathrm{IndCoh}(X) \arrow[r] \arrow[d, "f_{IC*}"'] & \mathrm{QCoh}(X) \arrow[d, "f_{QC*}"] \\ \mathrm{IndCoh}(Y) \arrow[r] & \mathrm{QCoh}(Y), \end{tikzcd}$$

where the horizontal functors are the canonical ones. A similar discussion applies to the $*$-pullback functor associated to a morphism of finite Tor-dimension.

To extend the definitions to ind-geometric stacks, first consider the functor

$$\mathrm{IndCoh} : \mathrm{Corr}(\mathrm{GStk}^+_k)_{fcd;ftd} \to \mathcal{P}\mathrm{r}^{\mathrm{L}} \tag{5.8}$$

obtained by restricting (5.7) to correspondences of truncated geometric stacks and composing with the forgetful functor $\mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{acpl}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$. Now let $\mathrm{Corr}(\mathrm{indGStk}_k)_{fcd;ftd}$ denote the subcategory of $\mathrm{Corr}(\mathrm{indGStk}_k)$ which only includes correspondences $X \xleftarrow{f} Y \xrightarrow{h} Z$ such that f is of ind-finite cohomological dimension and h is geometric and of finite Tor-dimension (these are stable under composition of correspondences by Propositions 4.9 and 4.19). By Proposition 4.10 $\mathrm{Corr}(\mathrm{GStk}^+_k)_{fcd;ftd}$ is a full subcategory of $\mathrm{Corr}(\mathrm{indGStk}_k)_{fcd;ftd}$.

Definition 5.9. We define a functor

$$\mathrm{IndCoh} : \mathrm{Corr}(\mathrm{indGStk}_k)_{fcd;ftd} \to \mathcal{P}\mathrm{r}^{\mathrm{L}} \tag{5.10}$$

by left Kan extending (5.8) along $\mathrm{Corr}(\mathrm{GStk}^+_k)_{fcd;ftd} \subset \mathrm{Corr}(\mathrm{indGStk}_k)_{fcd;ftd}$.

If $f : X \to Y$ is a morphism of ind-geometric stacks which is of ind-finite cohomological dimension (resp. geometric and of finite Tor dimension), we write $f_* : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$ (resp. $f^* : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$) for the functor defined by Definition 5.9. We have the following variant of Proposition 4.23, which is proved the same way. Here we write $\mathrm{indGStk}_{k,fcd} \subset \mathrm{indGStk}_k$ for the subcategory which only includes morphisms of ind-finite cohomological dimension, indentifying it with a subcategory $\mathrm{Corr}(\mathrm{indGStk}_k)_{fcd;ftd}$ as before.

Proposition 5.11. *The restriction of* IndCoh *to* $\mathrm{indGStk}_{k,fcd}$ *preserves filtered colimits along ind-closed immersions.*

In particular, if $X \cong \operatorname{colim} X_\alpha$ is an ind-geometric presentation, then the functors $i_{\alpha*}$: $\mathrm{IndCoh}(X_\alpha) \to \mathrm{IndCoh}(X)$ induce an isomorphism

$$\mathrm{IndCoh}(X) \cong \operatorname{colim} \mathrm{IndCoh}(X_\alpha) \tag{5.12}$$

in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.

5.3. **!-pullback and t-structures.** If a morphism $f : X \to Y$ ind-geometric stacks is ind-proper (hence of ind-finite cohomological dimension by Proposition 3.15), we write $f^! : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$ for the right adjoint of f_*. For ind-geometric X we define a standard t-structure on $\mathrm{IndCoh}(X)$ in terms of these adjoints, following [GR17b, Sec. I.3.1.2].

Proposition 5.13. *If X is an ind-geometric stack,* $\mathrm{IndCoh}(X)$ *has a t-structure defined by*

$$\mathrm{IndCoh}(X)^{\geq 0} := \left\{ \begin{matrix} \mathcal{F} \in \mathrm{IndCoh}(X) \textit{ such that } i^!(\mathcal{F}) \in \mathrm{IndCoh}(X')^{\geq 0} \textit{ for any} \\ \textit{truncated geometric substack } i : X' \to X \end{matrix} \right\}.$$

This t-structure is accessible, compatible with filtered colimits, right complete, and left anticomplete. If $X \cong \operatorname{colim}_\alpha X_\alpha$ is an ind-geometric presentation, the functors $i_{\alpha*}$ are t-exact and induce equivalences

$$\mathrm{IndCoh}(X)^{\leq 0} \cong \operatorname*{colim}_\alpha \mathrm{IndCoh}(X_\alpha)^{\leq 0}, \quad \mathrm{IndCoh}(X)^{\geq 0} \cong \operatorname*{colim}_\alpha \mathrm{IndCoh}(X_\alpha)^{\geq 0} \tag{5.14}$$

in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$.

Lemma 5.15. *Let $\mathcal{P}\mathrm{r}^{\mathrm{St,t\text{-}ex}}_{\mathrm{rcpl}} \subset \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{rcpl}}$ denote the subcategory which only includes t-exact functors. Then $\mathcal{P}\mathrm{r}^{\mathrm{St,t\text{-}ex}}_{\mathrm{rcpl}}$ admits filtered colimits, and these are preserved by the functors to $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ given by $\mathcal{C} \mapsto \mathcal{C}$, $\mathcal{C} \mapsto \mathcal{C}^{\leq 0}$, and $\mathcal{C} \mapsto \mathcal{C}^{\geq 0}$. Moreover, the subcategory $\mathcal{P}\mathrm{r}^{\mathrm{St,t\text{-}ex}}_{\mathrm{acpl}} \subset \mathcal{P}\mathrm{r}^{\mathrm{St,t\text{-}ex}}_{\mathrm{rcpl}}$ which only includes left anticomplete t-structures is closed under filtered colimits.*

Proof. Let $\mathrm{Groth}^{\mathrm{lex}}_\infty$ denote the category of Grothendieck prestable ∞-categories and left-exact functors. By [Lur18, Rem. C.3.1.5, Prop. C.3.2.1] $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$ and $\mathcal{C} \mapsto \mathcal{C}^{\leq 0}$ induce inverse equivalences of $\mathrm{Groth}^{\mathrm{lex}}_\infty$ and $\mathcal{P}\mathrm{r}^{\mathrm{St,t\text{-}ex}}_{\mathrm{rcpl}}$. The claims about $\mathcal{C} \mapsto \mathcal{C}$ and $\mathcal{C} \mapsto \mathcal{C}^{\leq 0}$ now follow since $\mathrm{Groth}^{\mathrm{lex}}_\infty$ is closed under filtered colimits in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ [Lur18, Prop. C.3.3.5], and since $\mathcal{C} \mapsto \mathrm{Sp}(\mathcal{C})$ preserves small colimits in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ [Lur17, Ex. 4.8.1.22].

Let $\mathcal{C} \cong \operatorname{colim} \mathcal{C}_\alpha$ be a filtered colimit in $\mathcal{P}\mathrm{r}^{\mathrm{St,t\text{-}ex}}_{\mathrm{rcpl}}$. Since the structure functors $F_{\alpha\beta} : \mathcal{C}_\alpha \to \mathcal{C}_\beta$, $F_\alpha : \mathcal{C}_\alpha \to \mathcal{C}$ are t-exact their right adjoints are left t-exact. Then since $\mathcal{C} \cong \lim \mathcal{C}_\alpha$ in $\mathcal{P}\mathrm{r}^{\mathrm{R}}$ and since the inclusions $\mathcal{C}^{\geq 0}_\alpha \subset \mathcal{C}_\alpha$ are morphisms in $\mathcal{P}\mathrm{r}^{\mathrm{R}}$, they identify the limit of the $\mathcal{C}^{\geq 0}_\alpha$ in $\mathcal{P}\mathrm{r}^{\mathrm{R}}$ as the full subcategory of $X \in \mathcal{C}$ such that $F^R_\alpha(X) \in \mathcal{C}^{\geq 0}_\alpha$ for all α. This is equivalent to $\mathrm{Map}_{\mathcal{C}}(F_\alpha(Y_\alpha), X) \cong 0$ for all α and all $Y_\alpha \in \mathcal{C}^{<0}_\alpha$. But this is equivalent to

$\mathrm{Map}_{\mathcal{C}}(Y, X) \cong 0$ for all $Y \in \mathcal{C}^{<0}$, since by Lemma A.2 and the previous paragraph $\mathcal{C}^{<0}$ is generated under small colimits by objects of the form $F_\alpha(Y_\alpha)$ with $Y_\alpha \in \mathcal{C}_\alpha^{<0}$. It follows that $\mathcal{C}^{\geq 0} \cong \operatorname{colim} \mathcal{C}_\alpha^{\geq 0}$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Finally, it follows from [Lur18, Cor. C.5.5.10] and the discussion above that $\mathcal{P}\mathrm{r}^{\mathrm{St,t\text{-}ex}}_{\mathrm{acpl}}$ is closed under all colimits that exist in $\mathcal{P}\mathrm{r}^{\mathrm{St,t\text{-}ex}}_{\mathrm{rcpl}}$. □

Proof of Proposition 5.13. Let $\mathrm{indGStk}_{k,\,cl} \subset \mathrm{indGStk}_k$ denote the subcategory which only includes ind-closed immersions, similarly for $\mathrm{GStk}^+_{k,\,cl} \subset \mathrm{GStk}^+_k$. By construction the restriction of (5.7) to $\mathrm{GStk}^+_{k,\,cl}$ factors through $\mathcal{P}\mathrm{r}^{\mathrm{St,t\text{-}ex}}_{\mathrm{acpl}}$, and we write $\mathrm{IndCoh}^t : \mathrm{indGStk}_{k,\,cl} \to \mathcal{P}\mathrm{r}^{\mathrm{St,t\text{-}ex}}_{\mathrm{acpl}}$ for its left Kan extension. Adapting again the proof of Proposition 4.23, we find that $\mathrm{IndCoh}^t(X) \cong \operatorname{colim} \mathrm{IndCoh}^t(X_\alpha)$ in $\mathcal{P}\mathrm{r}^{\mathrm{St,t\text{-}ex}}_{\mathrm{acpl}}$. But by Proposition 5.11 $\mathrm{IndCoh}(X)$ is the underlying category of $\mathrm{IndCoh}^t(X)$. The claims now follow from Lemma 5.15 and Proposition 4.6 (the Lemma ensures each $i_\alpha^!$ is left t-exact, the Proposition ensures $i^!$ is left t-exact for all $i : X' \to X$). □

Remark 5.16. Since $\mathrm{IndCoh}(X)$ is left anticomplete, it can be recovered functorially from $\mathrm{IndCoh}(X)^+$. By contrast, let $\mathrm{QCoh}'(X)$ denote the colimit of the categories $\mathrm{QCoh}(X_\alpha)$ in $\mathcal{P}\mathrm{r}^{\mathrm{St,t\text{-}ex}}_{\mathrm{rcpl}}$. Left completeness is stable under limits rather than colimits in $\mathcal{P}\mathrm{r}^{\mathrm{St,t\text{-}ex}}_{\mathrm{rcpl}}$, so $\mathrm{QCoh}'(X)$ will in general be neither left complete nor left anticomplete. In particular, $\mathrm{QCoh}'(X)$ is wild in so far as it cannot be recovered functorially from its bounded below objects.

Remark 5.17. Let X be a classical ind-geometric stack and $X \cong \operatorname{colim}_\alpha X_\alpha$ an ind-geometric presentation by classical geometric stacks. By construction we have $\mathrm{IndCoh}(X_\alpha)^\heartsuit \cong \mathrm{QCoh}(X_\alpha)^\heartsuit$ for all α, and by Proposition 5.13 we have $\mathrm{IndCoh}(X)^\heartsuit \cong \operatorname{colim}_\alpha \mathrm{IndCoh}(X_\alpha)^\heartsuit$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. It follows from [Lur18, Cor. 10.4.6.8, Prop. C.5.5.20, Thm. C.5.8.8] that $\mathrm{IndCoh}(X)$ is the dg nerve of the category of injective complexes in $\mathrm{IndCoh}(X)^\heartsuit$, a construction first considered in [Kra05].

Using t-structures we can address the potential ambiguity in the definition of $\mathrm{IndCoh}(X)$ when X is a non-truncated geometric stack.

Proposition 5.18. *Given a non-truncated geometric stack X, the categories* $\mathrm{IndCoh}(X)$ *defined by Definitions 5.3 and 5.9 are canonically equivalent, and this equivalence identifies the t-structure of the former with that of Proposition 5.13.*

Proof. Temporarily denote the two categories by $\mathrm{IndCoh}_{geom}(X)$ and $\mathrm{IndCoh}_{ind}(X)$. For each n the morphism $i_n : \tau_{\leq n} X \to X$ is affine, hence yields a t-exact functor $i_{n*} : \mathrm{IndCoh}(\tau_{\leq n} X) \to \mathrm{IndCoh}_{geom}(X)$. By Propositions 5.11 and 5.13 it suffices to show the induced t-exact functor

$$\mathrm{IndCoh}_{ind}(X) \cong \operatorname{colim} \mathrm{IndCoh}(\tau_{\leq n} X) \to \mathrm{IndCoh}_{geom}(X) \tag{5.19}$$

is an equivalence. Now for all $a \leq b$ (5.19) restricts to an equivalence $\mathrm{IndCoh}_{ind}(X)^{[a,b]} \cong \mathrm{IndCoh}_{geom}(X)^{[a,b]}$, since for all $n \geq b - a$ we have

$$\mathrm{IndCoh}(\tau_{\leq n} X)^{[a,b]} \cong \mathrm{QCoh}(\tau_{\leq n} X)^{[a,b]} \cong \mathrm{QCoh}(X)^{[a,b]} \cong \mathrm{IndCoh}(X)^{[a,b]}_{geom}.$$

By right completeness of the two t-structures it follows that (5.19) restricts to an equivalence $\mathrm{IndCoh}_{ind}(X)^{\geq a} \cong \mathrm{IndCoh}_{geom}(X)^{\geq a}$ for any a. But then (5.19) induces an equivalence of left completions, hence by Proposition 5.2 is itself an equivalence since its source and target are left anticomplete. □

If $f : X \to Y$ is ind-proper, the functor $f^!$ is not obviously continuous in general, but its failure to be so can be controlled by t-structures.

Definition 5.20. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable stable ∞-categories equipped with t-structures that are compatible with filtered colimits. Then F is almost continuous if its restriction to $\mathcal{C}^{\geq n}$ is continuous for all n.

Proposition 5.21. *Let $f : X \to Y$ be an ind-proper morphism of semi-reasonable ind-geometric stacks. Suppose either that f is geometric and almost of finite presentation, or that X is reasonable and f is of finite cohomological dimension and almost of ind-finite presentation. Then $f^!$ is almost continuous.*

The claim results from the following variant of [Lur18, Prop. 6.4.1.4], which we generalize further in Section 7.1. Here the Beck-Chevalley transformation $h'^* f^! \to f'^! h^*$ is the result of passing to adjoints from the natural isomorphism $f'_* h'^* \cong h^* f_*$.

Lemma 5.22. *Let the following be a Cartesian diagram of geometric stacks.*

(5.23)
$$
\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "h'"'] & \operatorname{Spec} A \arrow[d, "h"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
$$

Suppose h is a flat cover and that f is proper and almost of finite presentation. Then for any $\mathcal{F} \in \mathrm{IndCoh}(Y)^+$ the Beck-Chevalley map $h'^* f^!(\mathcal{F}) \to f'^! h^*(\mathcal{F})$ is an isomorphism.

Proof. Write $\operatorname{Spec} A^\bullet$ for the Cech nerve of h. Let $h_k : \operatorname{Spec} A^k \to Y$ denote the natural map, $h'_k : X_k \to X$ its base change along f, and $f_k : X_k \to \operatorname{Spec} A^k$ the base change of f along h_k (so $h_0 = h$, $h'_0 = h'$, and $f_0 = f'$). Finally, let $h_p : \operatorname{Spec} A^j \to \operatorname{Spec} A^i$ denote the map associated to a morphism $p : i \to j$ in Δ_s, and $h'_p : X_j \to X_i$ its base change along f_i.

Choose m so that f_* takes $\mathrm{IndCoh}(X)^{\leq 0}$ to $\mathrm{IndCoh}(Y)^{\leq m}$. The same holds for each f_{j*} since h_{j*} is t-exact and its restriction to $\mathrm{IndCoh}(X)^+$ is conservative, hence $f_j^!$ takes $\mathrm{IndCoh}(\operatorname{Spec} A^j)^{\geq 0}$ to $\mathrm{IndCoh}(X_j)^{\geq -m}$. Choosing n so that $\mathcal{F} \in \mathrm{IndCoh}(Y)^{\geq n}$, the Beck-Chevalley map $h'^*_p f^!_i \to f^!_j h^*_p$ in $\mathrm{Fun}(\mathrm{IndCoh}(\operatorname{Spec} A^i)^{\geq n}, \mathrm{IndCoh}(X_j)^{\geq n-m})$ is an isomorphism for any p, since h_p is a map of affine schemes [Lur18, Prop. 6.4.1.4]. Since h is faithfully flat, we have $\mathrm{IndCoh}(Y)^{\geq n} \cong \lim_{\Delta_s} \mathrm{IndCoh}(\operatorname{Spec} A^i)^{\geq n}$ and $\mathrm{IndCoh}(X)^{\geq n-m} \cong \lim_{\Delta_s} \mathrm{IndCoh}(X_i)^{\geq n-m}$. The claim now follows from [Lur17, Cor. 4.7.5.18]. □

Proof of Proposition 5.21. First suppose X and Y are geometric and f is almost of finite presentation, and fix a Cartesian square of the form (5.23). Since $f^!$ is left bounded, and since h'^* is conservative on $\mathrm{IndCoh}(X)^+$ in addition to being continuous, it suffices to show $h'^* f^!$ is almost continuous. It then suffices to show $f'^! h^*$ is almost continuous, since by Lemma 5.22 their restrictions to $\mathrm{IndCoh}(Y)^+$ are the same. But this follows since h^* is continuous and left t-exact, and since $f'^!$ is almost continuous by [Lur18, Lem. 6.4.1.5].

Now let $Y \cong \operatorname{colim} Y_\alpha$ be a semi-reasonable presenation. By the proof of [Lur09, Prop. 5.5.3.13] categories admitting filtered colimits and continuous functors among them form a subcategory of $\widehat{\mathrm{Cat}}_\infty$ which is closed under limits. By the previous paragraph $i^!_{\alpha\beta}$ restricts to a continuous functor $\mathrm{IndCoh}(Y_\beta)^{\geq n} \to \mathrm{IndCoh}(Y_\alpha)^{\geq n}$ for all $\beta \geq \alpha$ and any n. But the proof of Proposition 5.11 extends to show $\mathrm{IndCoh}(Y)^{\geq n} \cong \lim \mathrm{IndCoh}(Y_\alpha)^{\geq n}$, hence $i^!_\alpha$ is almost continuous for any α.

If f is geometric and almost of finite presentation, then letting $X_\alpha := X \times_Y Y_\alpha$ we have a semi-reasonable presentation $X \cong \operatorname{colim} X_\alpha$ by left exactness of filtered colimits in $\widehat{\mathrm{Stk}}_k$. Let $\mathcal{F} \cong \operatorname{colim} \mathcal{F}_\beta$ be a filtered colimit in $\mathrm{IndCoh}(Y)^{\geq n}$ for some n. Since $\mathrm{IndCoh}(X) \cong \lim \mathrm{IndCoh}(X_\alpha)$ it suffices to show the second factor of

$$\operatorname*{colim}_\beta i'^!_\alpha f^!(\mathcal{F}_\beta) \to i'^!_\alpha \operatorname*{colim}_\beta f^!(\mathcal{F}_\beta) \to i'^!_\alpha f^!(\operatorname*{colim}_\beta \mathcal{F}_\beta)$$

is an isomorphism for all α, where $i'_\alpha : X_\alpha \to X$ is the natural map. Since $i'^!_\alpha$ is almost continuous by the previous paragraph, and since $f^!$ is left bounded, the first factor is an isomorphism. It now suffices to show $i'^!_\alpha f^!$ is almost continuous, since then the composition is an isomorphism. But $i'^!_\alpha f^! \cong f^!_\alpha i^!_\alpha$, where $f_\alpha : X_\alpha \to Y_\alpha$ is the base change of f, and $f^!_\alpha i^!_\alpha$ is almost continuous by the previous paragraph and left t-exactness of $i^!_\alpha$.

Finally, suppose $X \cong \operatorname{colim} X_\gamma$ is a reasonable presentation and f is of finite cohomological dimension and almost of ind-finite presentation. Then for any γ we can factor $f \circ i_\gamma$ through a proper, almost finitely presented map $f_{\gamma\alpha} : X_\gamma \to Y_\alpha$ for some α, and the claim follows by the same argument as above. □

5.4. **Relation to coherent sheaves.** We now turn to the relationship between $\mathrm{Coh}(X)$ and $\mathrm{IndCoh}(X)$ when X is reasonable. We recall from [Lur18, Prop. 9.1.5.1] that when X is geometric, $\mathrm{Coh}(X)$ can characterized as the full subcategory of bounded, almost compact objects in $\mathrm{QCoh}(X)$ (i.e. of $\mathcal{F} \in \mathrm{QCoh}(X)^+$ such that $\tau^{\geq n}\mathcal{F}$ is compact in $\mathrm{QCoh}(X)^{\geq n}$ for all n). The equivalence $\mathrm{IndCoh}(X)^+ \cong \mathrm{QCoh}(X)^+$ thus also identifies $\mathrm{Coh}(X)$ with the full subcategory of bounded, almost compact objects in $\mathrm{IndCoh}(X)$.

Proposition 5.24. *For any reasonable ind-geometric stack X there is a canonical fully faithful functor $\mathrm{Coh}(X) \hookrightarrow \mathrm{IndCoh}(X)$. Its essential image consists of bounded, almost compact objects. It is induced from a canonical natural transformation between the functors $\mathrm{Corr}(\mathrm{indGStk}_k^{reas})_{prop;ftd} \to \widehat{\mathrm{Cat}}_\infty$ given by restricting the domains of (4.28) and (5.10) and* *composing them with the natural functors to* $\widehat{\mathrm{Cat}}_\infty$*. In particular, we have a diagram*

$$\begin{tikzcd} \mathrm{Coh}(X') \arrow[r, hook] \arrow[d, "i_*"'] & \mathrm{IndCoh}(X') \arrow[d, "i_*"] \\ \mathrm{Coh}(X) \arrow[r, hook] & \mathrm{IndCoh}(X), \end{tikzcd}$$

for any reasonable geometric substack $i : X' \to X$*.*

Proof. By construction the inclusion $\mathrm{Coh}(X) \subset \mathrm{IndCoh}(X)$ for geometric X enhances to a natural transformation of functors $\mathrm{Corr}(\mathrm{GStk}_k^+)_{prop;ftd} \to \widehat{\mathrm{Cat}}_\infty$. The variant of (4.28) appearing in the statement is the left Kan extension of its restriction to $\mathrm{Corr}(\mathrm{GStk}_k^+)_{prop;ftd}$, since following the proof of Proposition 4.23 the restrictions of both to $\mathrm{indGStk}_{k,prop}$ are left Kan extended from $\mathrm{GStk}^+_{k,prop}$. The desired natural transformation and the pictured diagram then follow from the characteristic adjunction of left Kan extensions [Lur09, Prop. 4.3.2.17].

Fix a reasonable presentation $X \cong \mathrm{colim}_\alpha X_\alpha$. Given $\mathcal{F} \in \mathrm{Coh}(X)$ we can write $\mathcal{F} \cong i_{\alpha*}(\mathcal{F}_\alpha)$ for some α and some $\mathcal{F}_\alpha \in \mathrm{Coh}(X_\alpha)$. The t-exactness of $i_{\alpha*}$ implies $\mathcal{F}$ is bounded in $\mathrm{IndCoh}(X)$, while the almost continuity of $i_\alpha^!$ (Proposition 5.21) implies $\mathcal{F}$ is almost compact.

Given $\mathcal{G} \in \mathrm{Coh}(X)$ we can write $\mathcal{G} \cong i_{\alpha*}(\mathcal{G}_\alpha)$ for some $\mathcal{G}_\alpha \in \mathrm{Coh}(X_\alpha)$, increasing α if needed. Identifying $\mathcal{F}$, $\mathcal{G}$ with their images in $\mathrm{IndCoh}(X)$ we then have

$$\begin{aligned} \mathrm{Map}_{IC(X)}(\mathcal{F},\mathcal{G}) &\cong \mathrm{Map}_{IC(X_\alpha)}(\mathcal{F}_\alpha, i_\alpha^! i_{\alpha*}(\mathcal{G}_\alpha)) \\ &\cong \mathrm{Map}_{IC(X_\alpha)}(\mathcal{F}_\alpha, \operatorname*{colim}_{\beta\geq\alpha} i_{\alpha\beta}^! i_{\alpha\beta*}(\mathcal{G}_\alpha)) \\ &\cong \operatorname*{colim}_{\beta\geq\alpha} \mathrm{Map}_{IC(X_\alpha)}(\mathcal{F}_\alpha, i_{\alpha\beta}^! i_{\alpha\beta*}(\mathcal{G}_\alpha)) \\ &\cong \operatorname*{colim}_{\beta\geq\alpha} \mathrm{Map}_{IC(X_\beta)}(i_{\alpha\beta*}(\mathcal{F}_\alpha), i_{\alpha\beta*}(\mathcal{G}_\alpha)). \end{aligned}$$

Here the second isomorphism follows from Proposition 5.11 and [GR17b, Lemma 1.1.10], and the third follows since $\mathcal{F}_\alpha$ and $\mathcal{G}_\alpha$ are coherent and each $i_{\alpha\beta}^! i_{\alpha\beta*}$ is left t-exact. But since $\mathrm{Coh}(X_\beta)$ is a full subcategory of $\mathrm{IndCoh}(X_\beta)$ for all $\beta \geq \alpha$, the last expression is equivalent to $\mathrm{Map}_{\mathrm{Coh}(X)}(\mathcal{F},\mathcal{G})$ [Roz, Lem. 0.2.1]. □

5.5. **The pushforward/$*$-pullback adjunction.** Suppose $f : X \to Y$ is both of finite Tor-dimension and ind-finite cohomological dimension. Then we have separately defined functors f_{IC*} and f^*_{IC}, but the definition does not explicitly entail any direct relationship between them. Nonetheless, the two functors are adjoint in the expect way.

Proposition 5.25. *Let X and Y be ind-geometric stacks and $f : X \to Y$ a morphism which is both of finite Tor-dimension and of ind-finite cohomological dimension. Then f_{IC*} is right adjoint to f^*_{IC}.*

Lemma 5.26. *Let $\widehat{\mathcal{C}}, \widehat{\mathcal{D}}$ be the left completions of $\check{\mathcal{C}}, \check{\mathcal{D}} \in \mathcal{P}r^{\mathrm{St,b}}_{\mathrm{acpl}}$. Let $\check{F} : \check{\mathcal{C}} \to \check{\mathcal{D}}$ and $\check{G} : \check{\mathcal{D}} \to \check{\mathcal{C}}$ be bounded colimit-preserving functors, and let $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ and $\widehat{G} : \widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$ be their* *images under the equivalences* $\mathrm{LFun}^b(\check{\mathcal{C}}, \check{\mathcal{D}}) \cong \mathrm{LFun}^b(\widehat{\mathcal{C}}, \widehat{\mathcal{D}})$ *and* $\mathrm{LFun}^b(\check{\mathcal{D}}, \check{\mathcal{C}}) \cong \mathrm{LFun}^b(\widehat{\mathcal{D}}, \widehat{\mathcal{C}})$*. Then* $\check{G}$ *is right adjoint to* $\check{F}$ *if and only if* $\widehat{G}$ *is right adjoint to* $\widehat{F}$*. If this is the case, the equivalences* $\mathrm{LFun}^b(\check{\mathcal{C}}, \check{\mathcal{C}}) \cong \mathrm{LFun}^b(\widehat{\mathcal{C}}, \widehat{\mathcal{C}})$ *and* $\mathrm{LFun}^b(\check{\mathcal{D}}, \check{\mathcal{D}}) \cong \mathrm{LFun}^b(\widehat{\mathcal{D}}, \widehat{\mathcal{D}})$ *together identify pairs of a compatible unit and counit for the two adjunctions.*

Proof. We consider the if direction, the other being symmetric. Recall that $\widehat{G}$ being right adjoint to $\widehat{F}$ is equivalent to the existence of unit and counit transformations $\widehat{u} : id_{\widehat{\mathcal{D}}} \to \widehat{G}\widehat{F}$, $\widehat{\epsilon} : \widehat{F}\widehat{G} \to id_{\widehat{\mathcal{C}}}$ and diagrams

$$\begin{tikzcd} & \widehat{F}\widehat{G}\widehat{F} \arrow[dr, "\widehat{\epsilon}\cdot id_{\widehat{F}}"] & \\ \widehat{F} \arrow[ur, "id_{\widehat{F}}\cdot\widehat{u}"] \arrow[rr, "id_{\widehat{F}}"'] & & \widehat{F} \end{tikzcd} \qquad \begin{tikzcd} & \widehat{G}\widehat{F}\widehat{G} \arrow[dr, "id_{\widehat{G}}\cdot\widehat{\epsilon}"] & \\ \widehat{G} \arrow[ur, "\widehat{u}\cdot id_{\widehat{G}}"] \arrow[rr, "id_{\widehat{G}}"'] & & \widehat{G}. \end{tikzcd}$$

This is a reformulation of [RV22, Def. 2.1.1], which is equivalent to [Lur09, Def. 5.2.2.1] by [RV22, Prop. F.5.6].

By hypotheses $\widehat{u}$ and $\widehat{\epsilon}$ are morphisms in $\mathrm{LFun}^b(\widehat{\mathcal{D}}, \widehat{\mathcal{D}})$ and $\mathrm{LFun}^b(\widehat{\mathcal{C}}, \widehat{\mathcal{C}})$, and we write $\check{u}$ and $\check{\epsilon}$ for the corresponding morphisms in $\mathrm{LFun}^b(\check{\mathcal{D}}, \check{\mathcal{D}})$ and $\mathrm{LFun}^b(\check{\mathcal{C}}, \check{\mathcal{C}})$. The above diagrams are respectively in $\mathrm{LFun}^b(\widehat{\mathcal{C}}, \widehat{\mathcal{D}})$ and $\mathrm{LFun}^b(\widehat{\mathcal{D}}, \widehat{\mathcal{C}})$, and we claim the corresponding diagrams in $\mathrm{LFun}^b(\check{\mathcal{C}}, \check{\mathcal{D}})$ and $\mathrm{LFun}^b(\check{\mathcal{D}}, \check{\mathcal{C}})$ witness $\check{u}$ and $\check{\epsilon}$ as the unit and counit of an adjunction between $\check{F}$ and $\check{G}$. In other words, we claim the equivalence $\mathrm{LFun}^b(\widehat{\mathcal{C}}, \widehat{\mathcal{D}}) \cong \mathrm{LFun}^b(\check{\mathcal{C}}, \check{\mathcal{D}})$ takes $\widehat{F}\widehat{G}\widehat{F}$, $id_{\widehat{F}} \cdot \widehat{u}$, and $\widehat{\epsilon} \cdot id_{\widehat{F}}$ respectively to $\check{F}\check{G}\check{F}$, $id_{\check{F}} \cdot \check{u}$, and $\check{\epsilon} \cdot id_{\check{F}}$, similarly for the right diagram.

Write $\Psi_{\mathcal{C}} : \check{\mathcal{C}} \to \widehat{\mathcal{C}}$ and $\Psi_{\mathcal{D}} : \check{\mathcal{D}} \to \widehat{\mathcal{D}}$ for the canonical functors. Then $\check{F}$ is characterized in $\mathrm{LFun}^b(\check{\mathcal{C}}, \check{\mathcal{D}})$ by the condition $\Psi_{\mathcal{D}}\check{F} \cong \widehat{F}\Psi_{\mathcal{C}}$, similarly for $\check{G}$. It follows that $\Psi_{\mathcal{D}}\check{F}\check{G}\check{F} \cong \widehat{F}\widehat{G}\widehat{F}\Psi_{\mathcal{C}}$, which likewise characterizes $\check{F}\check{G}\check{F}$ as the functor corresponding to $\widehat{F}\widehat{G}\widehat{F}$. Now consider the following diagram, in which all horizontal arrows equivalences.

$$\begin{tikzcd} \mathrm{LFun}^b(\widehat{\mathcal{C}}, \widehat{\mathcal{C}}) \arrow[r, "-\Psi_{\mathcal{C}}"] \arrow[d, "\widehat{F}-"'] & \mathrm{LFun}^b(\check{\mathcal{C}}, \widehat{\mathcal{C}}) \arrow[d, "\widehat{F}-"] & \mathrm{LFun}^b(\check{\mathcal{C}}, \check{\mathcal{C}}) \arrow[l, "\Psi_{\mathcal{C}}-"'] \arrow[d, "\check{F}-"] \\ \mathrm{LFun}^b(\widehat{\mathcal{C}}, \widehat{\mathcal{D}}) \arrow[r, "-\Psi_{\mathcal{C}}"] & \mathrm{LFun}^b(\check{\mathcal{C}}, \widehat{\mathcal{D}}) & \mathrm{LFun}^b(\check{\mathcal{C}}, \check{\mathcal{D}}) \arrow[l, "\Psi_{\mathcal{D}}-"'] \end{tikzcd}$$

Here the compositions around the left square are evidently isomorphic, while those around the right square are because of the isomorphism $\Psi_{\mathcal{D}}\check{F} \cong \widehat{F}\Psi_{\mathcal{C}}$. The morphism $id_{\widehat{F}} \cdot \widehat{u}$ is the image of $\widehat{u}$ under the left vertical map, while $id_{\check{F}} \cdot \check{u}$ is the image of $\check{u}$ under the right. But by definition $\check{u}$ is the image of $\widehat{u}$ under the overall top equivalence, hence $id_{\check{F}} \cdot \check{u}$ is the image of $id_{\widehat{F}} \cdot \widehat{u}$ under the overall bottom equivalence. The remaining conditions are checked the same way. $\square$

Proof of Proposition 5.25. We let IC subscripts be implicit in the proof, f_* always meaning f_{IC*}, etc. If X and Y are truncated and geometric, the claim follows immediately from

Lemma 5.26. In general, let $Y \cong \operatorname{colim} Y_\alpha$ be an ind-geometric presentation and $f_\alpha : X_\alpha \to Y_\alpha$ the base change of f. We then have an ind-geometric presentation $X \cong \operatorname{colim} X_\alpha$ since f is of finite Tor-dimension and filtered colimits in $\widehat{\mathrm{Stk}}_k$ are left exact. Let A denote the index category and $\Delta^1_{\alpha\beta} \subset A$ the morphism associated to $\alpha \leq \beta$.

By construction we have a functor $\Delta^1 \times A \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ taking $\Delta^1 \times \Delta^1_{\alpha\beta}$ to the diagram witnessing the isomorphism $i_{\alpha\beta*} f_{\alpha*} \cong f_{\beta*} i'_{\alpha\beta*}$, and a similar functor $(\Delta^1)^{\mathrm{op}} \times A \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ packaging the isomorphisms $f^*_\beta i_{\alpha\beta*} \cong i'_{\alpha\beta*} f^*_\alpha$. Passing to right adjoints, these are equivalent to the data of a functor $A^{\mathrm{op}} \to \mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \widehat{\mathrm{Cat}}_\infty)$ taking α to $f^R_{\alpha*}$ and a functor $A^{\mathrm{op}} \to \mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$ taking α to f^{*R}_α.

Note that the unit/counit compatibility of Lemma 5.26 implies more precisely that the isomorphism $f^*_\beta i_{\alpha\beta*} \cong i'_{\alpha\beta*} f^*_\alpha$ is the Beck-Chevalley transformation associated to the isomorphism $i_{\alpha\beta*} f_{\alpha*} \cong f_{\beta*} i'_{\alpha\beta*}$. In the notation of [Lur17, Def. 4.7.5.16], the above functors thus take values in $\mathrm{Fun}^{\mathrm{RAd}}((\Delta^1)^{\mathrm{op}}, \widehat{\mathrm{Cat}}_\infty)$ and $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$, respectively, and correspond under the equivalence $\mathrm{Fun}^{\mathrm{RAd}}((\Delta^1)^{\mathrm{op}}, \widehat{\mathrm{Cat}}_\infty) \cong \mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$ of [Lur17, Cor. 4.7.5.18]. By the same result these subcategories are closed under limits in $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \widehat{\mathrm{Cat}}_\infty)$ and $\mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$. But by Proposition 5.11 and [Lur09, Cor. 5.1.2.3] we have $f^R_* \cong \lim f^R_{\alpha*}$ and $f^{*R} \cong \lim f^{*R}_\alpha$, hence f^R_* is right adjoint to f^{*R}, hence f_* is right adjoint to f^*. □

The definition of ind-coherent pushforward and $*$-pullback also defines base change isomorphisms for suitable Cartesian squares. These isomorphisms are compatible with the adjunction of Proposition 5.25 in the following sense.

Proposition 5.27. *Let the following be a Cartesian diagram of ind-geometric stacks in which f is both of finite Tor-dimension and of ind-finite cohomological dimension.*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

If h is of ind-finite cohomological dimension (resp. of finite Tor-dimension), the isomorphism $f^* h_* \cong h'_* f'^*$ of functors $\mathrm{IndCoh}(Y') \to \mathrm{IndCoh}(X)$ is the Beck-Chevalley transformation associated to the isomorphism $f_* h'_* \cong h_* f'_*$ of functors $\mathrm{IndCoh}(X') \to \mathrm{IndCoh}(Y)$ (resp. the isomorphism $h'^* f^* \cong f'^* h^*$ of functors $\mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X')$).

Proof. Consider the case with h of ind-finite cohomological dimension, the finite Tor-dimension case following by a variation of the same argument. If X, Y, and Y' are truncated and geometric the claim follows immediately from Lemma 5.26. If h is the immersion of an ind-geometric substack the claim was established during the proof of Proposition 5.25.

We pass to right adjoints and identify the isomorphisms $f^{*R} h^R_* \cong h'^R_* f'^{*R}$ and $f^R_* h'^R_* \cong h^R_* f'^R_*$ respectively with a morphism $f^{*R} \to f'^{*R}$ in $\mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$ and a morphism $f^R_* \to f'^R_*$ in $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \widehat{\mathrm{Cat}}_\infty)$, performing such identifications without comment in the rest of the proof.

In the notation of [Lur17, Def. 4.7.5.16], we want to show these belong to $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$ and $\mathrm{Fun}^{\mathrm{RAd}}((\Delta^1)^{\mathrm{op}}, \widehat{\mathrm{Cat}}_\infty)$, respectively, and correspond to each other under the equivalence $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty) \cong \mathrm{Fun}^{\mathrm{RAd}}((\Delta^1)^{\mathrm{op}}, \widehat{\mathrm{Cat}}_\infty)$ of [Lur17, Cor. 4.7.5.18].

First suppose X and Y are truncated and geometric, let $Y' \cong \operatorname{colim} Y'_\alpha$ be an ind-geometric presentation, and write $f'_\alpha : X'_\alpha \to Y'_\alpha$ for the base change of f'. As in the proof of Proposition 5.25, we have $f'^{*R} \cong \lim f'^{*R}_\alpha$ in both $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$ and $\mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$, hence $f^{*R} \to f'^{*R}$ is in $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$ since each $f^{*R} \to f'^{*R}_\alpha$ is. Similarly, $f^R_* \to f'^R_* \cong \lim f'^R_{\alpha *}$ is in $\mathrm{Fun}^{\mathrm{RAd}}((\Delta^1)^{\mathrm{op}}, \widehat{\mathrm{Cat}}_\infty)$ since each $f^R_* \to f'^R_{\alpha *}$ is, and it corresponds to $f^{*R} \to f'^{*R}$ under [Lur17, Cor. 4.7.5.18] since each $f^R_* \to f'^R_{\alpha *}$ corresponds to $f^{*R} \to f'^{*R}_\alpha$.

Now let $Y \cong \operatorname{colim} Y_\alpha$ be an ind-geometric presentation. For any α we have a diagram

$$\begin{tikzcd}
& X'_\alpha \arrow[rr, "f'_\alpha"] \arrow[dl, "j'_\alpha"'] \arrow[dd, "h'_\alpha"] & & Y'_\alpha \arrow[dl, "j_\alpha"] \arrow[dd, "h_\alpha"] \\
X' \arrow[rr] \arrow[dd] & & Y' \arrow[dd] & \\
& X_\alpha \arrow[rr, "f_\alpha"] \arrow[dl, "i'_\alpha"'] & & Y_\alpha \arrow[dl, "i_\alpha"] \\
X \arrow[rr] & & Y &
\end{tikzcd}$$

with Cartesian faces. We have already shown that the compositions $f^{*R} \to f^{*R}_\alpha \to f'^{*R}_\alpha$ and $f^R_* \to f^R_{\alpha *} \to f'^R_{\alpha *}$ belong to $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$ and $\mathrm{Fun}^{\mathrm{RAd}}((\Delta^1)^{\mathrm{op}}, \widehat{\mathrm{Cat}}_\infty)$, and correspond under [Lur17, Cor. 4.7.5.18]. By closure of these subcategories under limits, and by Proposition 5.11, it suffices to show $f'^{*R}_\beta \to f'^{*R}_\alpha$ and $f'^R_{\beta *} \to f'^R_{\alpha *}$ belong to $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$ and $\mathrm{Fun}^{\mathrm{RAd}}((\Delta^1)^{\mathrm{op}}, \widehat{\mathrm{Cat}}_\infty)$ and correspond under [Lur17, Cor. 4.7.5.18].

Consider the following diagram in $\mathrm{Fun}(\mathrm{IndCoh}(X_\beta), \mathrm{IndCoh}(Y'_\alpha))$.

$$\begin{tikzcd}
j^R_{\alpha\beta *} h^R_{\beta *} f^{*R}_\beta \arrow[r, "\sim"] \arrow[d, "\wr"] & j^R_{\alpha\beta *} f'^{*R}_\beta h'^R_{\beta *} \arrow[r, "\sim"] & f'^{*R}_\alpha j'^R_{\alpha\beta *} h'^R_{\beta *} \arrow[d, "\wr"] \\
h^R_{\alpha *} i^R_{\alpha\beta *} f^{*R}_\beta \arrow[r, "\sim"] & h^R_{\alpha *} f^{*R}_\alpha i'^R_{\alpha\beta *} \arrow[r, "\sim"] & f'^{*R}_\alpha h'^R_{\alpha *} i'^R_{\alpha\beta *}
\end{tikzcd}$$

We have already shown f'^{*R}_α and $f'^R_{\alpha *}$, etc., are adjoint, and the claim at hand is equivalent to the top right arrow being the Beck-Chevalley transformation associated to the isomorphism $j'^R_{\alpha\beta *} f'^R_{\beta *} \cong f'^R_{\alpha *} j^R_{\alpha\beta *}$. But this follows since we have shown the corresponding claim for the top left and bottom arrows, and since all arrows in the diagram are isomorphisms. □

6. Coherent ind-geometric stacks

Given an arbitrary ind-geometric stack X, we have noted that in general the category $\mathrm{IndCoh}(X)$ defined in the previous section is not necessarily compactly generated by $\mathrm{Coh}(X)$ (despite the notation). This is true, however, in the main cases of interest. This includes, of course, Noetherian schemes as considered in [Gai13], as well as locally Noetherian geometric stacks (in particular geometric stacks which are almost of finite type and QCA as in [DG13, Thm. 3.3.5]) and reasonable inductive limits of these. More generally it includes the class of coherent ind-geometric stacks considered in this section (Proposition 6.5), which also includes examples such as the quotient $\mathcal{R}_{G,N}/G_{\mathcal{O}}$ studied in [BFN18, CW23].

6.1. **Definitions and Key Properties.** Recall that a geometric stack X is locally coherent if there exists a flat cover $\operatorname{Spec} A \to X$ such that A is coherent.

Definition 6.1. A geometric stack X is coherent if it is locally coherent and $\operatorname{QCoh}(X)^\heartsuit$ is compactly generated. An ind-geometric stack X is coherent if it is reasonable and every reasonable geometric substack is coherent.

If X is a coherent geometric stack, it follows from [Lur18, Prop. 9.1.5.1] that more specifically $\operatorname{QCoh}(X)^\heartsuit$ is compactly generated by $\operatorname{Coh}(X)^\heartsuit$. As a basic example, recall that a geometric stack X is locally Noetherian if it admits a flat cover $\operatorname{Spec} A \to X$ such that A is Noetherian. In this case $\operatorname{QCoh}(X)^\heartsuit$ is compactly generated, hence X is coherent, by [Lur18, Prop. 9.5.2.3]. More generally, the following result implies $\operatorname{QCoh}(X)^\heartsuit$ is compactly generated if X is affine over a locally Noetherian geometric stack.

Proposition 6.2. *Let $f : X \to Y$ be an affine morphism of geometric stacks. If $\operatorname{QCoh}(Y)^\heartsuit$ is compactly generated, then so is $\operatorname{QCoh}(X)^\heartsuit$.*

Proof. Since f is affine $f_* : \operatorname{QCoh}(X) \to \operatorname{QCoh}(Y)$ is t-exact and conservative, hence restricts to a conservative functor $\operatorname{QCoh}(X)^\heartsuit \to \operatorname{QCoh}(Y)^\heartsuit$. This restriction is continuous and has a left adjoint, the restriction of $\tau^{\geq 0} \circ f^*$. Thus compact generation of $\operatorname{QCoh}(Y)^\heartsuit$ implies that of $\operatorname{QCoh}(X)^\heartsuit$ by [Lur17, Prop. 7.1.4.12] (whose proof applies to compact generation, not just compact projective generation). □

Together with Proposition 4.3, the following result ensures the definition of coherence is consistent when we regard a geometric stack as an ind-geometric stack.

Proposition 6.3. *Let $f : X \to Y$ be an almost finitely presented closed immersion of geometric stacks. If Y is locally coherent (resp. coherent), then so is X.*

Proof. Let $\operatorname{Spec} A \to Y$ be a flat cover with A coherent, and $f' : \operatorname{Spec} B \to \operatorname{Spec} A$ the base change of f (recall that f is affine by Proposition 3.29). By [Lur18, Cor. 5.2.2.2] B is almost perfect as an A-module, hence $H^n(B)$ is finitely presented over $H^0(A)$ for all $n \leq 0$. Moreover $H^0(B)$ is a quotient of $H^0(A)$ by a finitely generated ideal, so $H^0(B)$ is coherent and the $H^n(B)$ are finitely presented over $H^0(B)$ [Gla89, Thm. 2.4.1]. Moreover, if $\operatorname{QCoh}(Y)^\heartsuit$ is compactly generated then so is $\operatorname{QCoh}(X)^\heartsuit$ by Proposition 6.2. □

Proposition 6.4. *Coherent ind-geometric stacks are closed under filtered colimits along almost ind-finitely presented closed immersions in $\widehat{\operatorname{Stk}}_k$. In particular, an ind-geometric stack is coherent if and only if it has a reasonable presentation whose terms are coherent.*

Proof. If $X \cong \operatorname{colim} X_\alpha$ is a filtered colimit as in the statement, then X is a reasonable ind-geometric stack by Proposition 4.11. If $i : X' \to X$ is a reasonable geometric substack, then i factors through some X_α by Proposition 3.27. Moreovoer, if $X_\alpha \cong \operatorname{colim}_\beta X_{\alpha\beta}$ is a reasonable presentation i then factors through an almost finitely presented closed immersion to some $X_{\alpha\beta}$ by Proposition 4.6. But $X_{\alpha\beta}$ is coherent by hypothesis, hence X' is coherent by Proposition 6.3. □

We have the following key property of coherent ind-geometric stacks.

Proposition 6.5. *If X is a coherent ind-geometric stack then the canonical functor* $\operatorname{Ind}(\operatorname{Coh}(X)) \to \operatorname{IndCoh}(X)$ *is an equivalence, and induces equivalences* $\operatorname{Ind}(\operatorname{Coh}(X)^{\leq 0}) \cong \operatorname{IndCoh}(X)^{\leq 0}$, $\operatorname{Ind}(\operatorname{Coh}(X)^{\geq 0}) \cong \operatorname{IndCoh}(X)^{\geq 0}$.

Proof. For a coherent geometric stack this follows from [Lur18, Thm. C.6.7.1]. The ind-geometric case then follows from Propositions 4.23, 5.11, and 5.24, since ind-completion of idempotent-complete categories admitting finite colimits commutes with filtered colimits [Lur17, Lem. 7.3.5.11]. □

Remark 6.6. In light of Proposition 6.5, the notion of coherent ind-geometric stack is in a sense formally dual to the notion of perfect stack considered in [BZFN10].

6.2. **Fiber Products.** Coherent ind-geometric stacks are not closed under general fiber products (e.g. [Gla89, Sec. 7.3.13]), but we record a few cases of interest where they are (beyond more elementary ones with Noetherian hypotheses).

Proposition 6.7. *Let the following be a Cartesian diagram of ind-geometric stacks.*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Suppose that X and Y are reasonable, that Y' is coherent, and that f is an almost ind-finitely presented ind-closed immersion. Then X' is coherent and f' is an almost ind-finitely presented ind-closed immersion.

Proof. Suppose first that X and Y' are truncated geometric stacks, and let $Y \cong \operatorname{colim} Y_\alpha$ be a reasonable presentation. We may assume f and h factor through maps $f_\alpha : X \to Y_\alpha$, $h_\alpha : Y' \to Y_\alpha$ for all α. Letting $X'_\alpha := X \times_{Y_\alpha} Y'$, each $f'_\alpha : X'_\alpha \to Y'$ is an almost finitely presented closed immersion by base change and Proposition 4.13, hence X'_α is coherent by Proposition 6.3. For any $\beta \geq \alpha$ the induced map $i'_{\alpha\beta} : X'_\alpha \to X'_\beta$ is an almost finitely presented closed immersion since $f'_\beta \circ i'_{\alpha\beta} \cong f'_\alpha$ (Proposition 3.5). Since $X' \cong \operatorname{colim} X \times_{Y_\alpha} Y'$ in $\widehat{\operatorname{Stk}}_k$ by left exactness of filtered colimits [Lur09, Ex. 7.3.4.7], it follows that X' is coherent by Proposition 6.4.

In general, fix reasonable presentations $X \cong \operatorname{colim} X_\alpha$ and $Y' \cong \operatorname{colim} Y'_\beta$. Then as above $X' \cong \operatorname{colim} X_\alpha \times_Y Y'_\beta$ presents X' as a filtered colimit of coherent ind-geometric stacks along almost ind-finitely presented ind-closed immersions, hence X' is coherent by Proposition 6.4. That f' is an almost ind-finitely presented ind-closed immersion follows from Propositions 4.13 and 4.19. □

Proposition 6.8. *Let the following be a Cartesian diagram of ind-geometric stacks.*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Suppose that X is coherent, Y is reasonable, and Y' is ind-tamely presented. Suppose also that f is ind-tamely presented and that h is affine and has coherent pullback. Then X' is coherent.

Proof. If $X \cong \operatorname{colim} X_\alpha$ is a reasonable presentation, then as in the proof of Proposition 4.18 $X' \cong \operatorname{colim} X'_\alpha$ is a reasonable presentation, where $X'_\alpha := X_\alpha \times_Y Y'$. By Propositions 4.9 and 4.10 each X'_α is tamely presented, hence locally coherent by Proposition 2.8. But X_α is coherent, hence by Proposition 6.2 so is X'_α since h is affine. □

We say an ind-geometric stack is ind-locally Noetherian if it is reasonable and every reasonable geometric substack is locally Noetherian.

Proposition 6.9. *Suppose $X \to X'$ and $Y \to Y'$ are tamely presented affine morphisms of ind-geometric stacks such that X' and Y' are ind-locally Noetherian. Then X, Y, and $X \times Y$ are coherent.*

Proof. Let $X' \cong \operatorname{colim} X'_\alpha$ be a reasonable presentation. Then $X \cong \operatorname{colim} X_\alpha$, where $X_\alpha := X \times_{X'} X'_\alpha$, by left exactness of filtered colimits in $\widehat{\mathrm{Stk}}_k$. Each X'_α is locally Noetherian, hence each X_α is coherent since $X_\alpha \to X'_\alpha$ is tamely presented and affine (Propositions 2.8 and 6.2). Coherence of X follows from Proposition 6.4. The other claims are the same, noting that $X' \times Y'$ is ind-locally Noetherian and that $X \times Y$ is tamely presented and affine over it. □

6.3. **Pushforward and $*$-pullback.** In the context of coherent ind-geometric stacks, Proposition 6.5 lets us extend both of the basic functorialities of IndCoh.

Definition 6.10. Let X and Y be reasonable ind-geometric stacks such that Y is coherent, and let $f : X \to Y$ be a morphism with coherent pullback. We write $f^* : \operatorname{IndCoh}(Y) \to \operatorname{IndCoh}(X)$ for the unique continuous functor whose restriction to $\operatorname{Coh}(Y)$ factors through the functor $f^* : \operatorname{Coh}(Y) \to \operatorname{Coh}(X)$ of Definition 4.21.

When f is of finite Tor-dimension this is indeed consistent with Definition 5.9, since the previously defined f^* is continuous and preserves coherence. To describe the pushforward counterpart of Definition 6.10 first note that if $f : X \to Y$ is any morphism of ind-geometric stacks such that X is reasonable, there is a canonical functor $f_* : \mathrm{Coh}(X) \to \mathrm{IndCoh}(Y)$ defined as follows. Write $\mathrm{IndCoh}^+_{naive} : \mathrm{indGStk}_k \to \widehat{\mathrm{Cat}}_\infty$ for the left Kan extension of the evident functor $\mathrm{IndCoh}^+ : \mathrm{GStk}^+_k \to \widehat{\mathrm{Cat}}_\infty$. Explicitly, $\mathrm{IndCoh}^+_{naive}(X)$ is the full subcategory of $\mathcal{F} \in \mathrm{IndCoh}^+(X)$ which are pushed forward from some ind-geometric substack of X. By construction we have a functor $f_* : \mathrm{IndCoh}(X)^+_{naive} \to \mathrm{IndCoh}(Y)^+_{naive}$, while by the universal property of left Kan extensions we have canonical functors $\mathrm{Coh}(X) \to \mathrm{IndCoh}(X)^+_{naive}$ and $\mathrm{IndCoh}(Y)^+_{naive} \to \mathrm{IndCoh}(Y)^+$, and we let f_* be the composition of these.

Definition 6.11. Let $f : X \to Y$ be a morphism of ind-geometric stacks, and suppose that X is coherent. Then we write $f_* : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$ for the unique continuous functor whose restriction to $\mathrm{Coh}(X)$ is the functor above.

Suppose $f : X \to Y$ and $g : Y \to Z$ are morphisms of ind-geometric stacks such that X is coherent, and such that either Y is coherent or g is of ind-finite cohomological dimension. Then have an isomorphism $g_* f_* \cong (g \circ f)_*$ of functors $\mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Z)$, since both are continuous and have their restrictions to $\mathrm{Coh}(X)$ are isomorphic by construction.

If the source and target of $f : X \to Y$ are coherent, we have the following extension of Proposition 5.25.

Proposition 6.12. *Let X and Y be coherent ind-geometric stacks and $f : X \to Y$ a morphism with coherent pullback. Then $f_* : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$ is right adjoint to $f^* : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$. In particular, suppose f sits in a Cartesian diagram of the following form, where h and h' have coherent pullback and X' and Y' are coherent.*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{6.13}$$

Then we obtain a Beck-Chevalley transformation $h^* f_* \to f'_* h'^*$ from the the isomorphism $h_* f'_* \cong f_* h'_*$ of functors $\mathrm{IndCoh}(X') \to \mathrm{IndCoh}(Y)$, and the former is itself an isomorphism if f is proper and almost of finite presentation.

Lemma 6.14. *Let $\widehat{\mathcal{C}}, \widehat{\mathcal{D}}$ be the left completions of $\check{\mathcal{C}}, \check{\mathcal{D}} \in \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{acpl}}$, and let $\Psi_{\mathcal{C}} : \check{\mathcal{C}} \to \widehat{\mathcal{C}}$ and $\Psi_{\mathcal{D}} : \check{\mathcal{D}} \to \widehat{\mathcal{D}}$ be the canonical functors. Let $\check{F} : \check{\mathcal{C}} \to \check{\mathcal{D}}$, $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ be colimit-preserving functors such that $\widehat{F}$ is right bounded and $\Psi_{\mathcal{D}} \check{F} \cong \widehat{F} \Psi_{\mathcal{C}}$, and let $\check{G} : \check{\mathcal{D}} \to \check{\mathcal{C}}$, $\widehat{G} : \widehat{\mathcal{D}} \to \widehat{\mathcal{C}}$ be their right adjoints. Then the Beck-Chevalley map $\Psi_{\mathcal{C}} \check{G}(X) \to \widehat{G} \Psi_{\mathcal{D}}(X)$ is an isomorphism for all $X \in \check{\mathcal{D}}^+$.*

Proof. By definition the Beck-Chevalley map is the composition

$$\Psi_{\mathcal{C}} \check{G}(X) \to \Psi_{\mathcal{C}} \check{G} \Psi^R_{\mathcal{D}} \Psi_{\mathcal{D}}(X) \cong \Psi_{\mathcal{C}} \Psi^R_{\mathcal{C}} \widehat{G} \Psi_{\mathcal{D}}(X) \to \widehat{G} \Psi_{\mathcal{D}}(X) \tag{6.15}$$

of unit and counit maps. Since $\Psi_{\mathcal{C}}$ is t-exact and restricts to an equivalence $\check{\mathcal{C}}^+ \xrightarrow{\sim} \widehat{\mathcal{C}}^+$, its right adjoint $\Psi_{\mathcal{C}}^R$ is left t-exact and restricts to the inverse equivalence $\widehat{\mathcal{C}}^+ \xrightarrow{\sim} \check{\mathcal{C}}^+$, likewise for $\Psi_{\mathcal{D}}^R$. In particular, the first map in (6.15) is an isomorphism since $X \in \check{\mathcal{D}}^+$. But $\widehat{G}$ is left bounded since $\widehat{F}$ is right bounded, hence $\widehat{G}\Psi_{\mathcal{D}}(X) \in \widehat{\mathcal{C}}^+$, hence the last map in (6.15) is an isomorphism. □

Proof. First suppose X and Y are truncated and geometric. Since $\mathrm{IndCoh}(X)$ is compactly generated and f^* preserves compactness, the right adjoint f^{*R} is continuous. But f_* and f^{*R} have isomorphic restrictions to $\mathrm{Coh}(X)$ by Lemma 6.14 and the definitions, hence by continuity they are themselves isomorphic. When X' and Y' are truncated and geometric, the final claim follows immediately since $h^* f_*$ and $f'_* h'^*$ are continuous and the transformation restricts to an isomorphism of functors $\mathrm{Coh}(X) \to \mathrm{Coh}(Y')$.

Now let $Y \cong \operatorname{colim} Y_\alpha$ be a reasonable presentation, let $f_\alpha : X_\alpha \to Y_\alpha$ be the base change of f, and let $i'_{\alpha\beta} : X_\alpha \to X_\beta$ the base change of $i_{\alpha\beta} : Y_\alpha \to Y_\beta$. Unwinding the definition of f_*, it follows from Proposition 5.11 and [Lur09, Cor. 5.1.2.3] that $f_* \cong \operatorname{colim} f_{\alpha *}$ in $\mathrm{Fun}(\Delta^1, \mathcal{P}\mathrm{r}^{\mathrm{L}})$, the structure maps being given by the isomorphisms $i'_{\alpha\beta *} f_{\beta *} \cong f_{\alpha *} i_{\alpha\beta *}$. Likewise, we have $f_* \cong \operatorname{colim} f_{\alpha *}$ in $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \mathcal{P}\mathrm{r}^{\mathrm{L}})$, the structure maps being given by the Beck-Chevalley isomorphisms $f^*_\beta i_{\alpha\beta *} \cong i'_{\alpha\beta *} f^*_\alpha$. Passing to right adjoints we have $f^R_* \cong \lim f^R_{\alpha *}$ in $\mathrm{Fun}((\Delta^1)^{\mathrm{op}}, \widehat{\mathrm{Cat}}_\infty)$ and $f^{*R} \cong \lim f^{*R}_\alpha$ in $\mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$. In the notation of [Lur17, Def. 4.7.5.16], it follows from [Lur17, Cor. 4.7.5.18] and the previous paragraph that $f^{*R} \in \mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$, $f^R_* \in \mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$, and that f^{*R} and f^R_* correspond under the equivalence $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty) \cong \mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$ of [Lur17, Cor. 4.7.5.18]. But then f_* is right adjoint to f^* since f^R_* is to f^{*R}. The general case of the final claim now follows as in the geometric case. □

7. Exceptional inverse image

Let $f : X \to Y$ be a morphism of ind-geometric stacks. In this section we study the right adjoint $f^! : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$ of the functor f_* when X and Y are reasonable and f is almost of ind-finite presentation. The main result (Proposition 7.5) studies conditions under which !-pullback commutes with $*$-pullback inside Cartesian diagrams. We also consider criteria for the commutation of !-pullback and pushforward (Propositions 7.11 and 7.13).

7.1. **!-pullback and $*$-pullback.** We first consider morphisms of finite Tor-dimension between geometric stacks, the proofs closely following similar results with different finiteness conditions [Lur18, Prop. 6.4.1.4], [Gai13, Prop. 7.1.6], [Ras19, Lem. 6.14.2]. Having the ind-geometric case in mind we formulate the statement in terms of ind-coherent sheaves, but this is essentially a notational choice since $\mathrm{IndCoh}(X)^+ \cong \mathrm{QCoh}(X)^+$.

Proposition 7.1. *Let the following be a Cartesian diagram of geometric stacks.*

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Suppose h is of finite Tor dimension and f is proper and almost of finite presentation. Then for any $\mathcal{F} \in \mathrm{IndCoh}(Y)^+$ the Beck-Chevalley map $h'^* f^!(\mathcal{F}) \to f'^! h^*(\mathcal{F})$ is an isomorphism.

Proof. First note that we may assume that Y' is affine. Otherwise choose a flat cover $u : \operatorname{Spec} B \to Y'$ and consider the following diagram of Cartesian squares.

$$\begin{tikzcd}
X'' \arrow[r, "u'"] \arrow[d, "f''"'] & X' \arrow[r, "h'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
\operatorname{Spec} B \arrow[r, "u"] & Y' \arrow[r, "h"] & Y
\end{tikzcd}$$

We have $h'^* f^!(\mathcal{F}), f'^! h^*(\mathcal{F}) \in \mathrm{IndCoh}(X')^+$ since h and h' are of finite Tor dimension, and since f is proper and almost of finite presentation. By faithful flatness the restriction of u'^* to $\mathrm{IndCoh}(X')^+$ is conservative, so it suffices to show the first factor of

$$u'^* h'^* f^!(\mathcal{F}) \to u'^* f'^! h^*(\mathcal{F}) \to f''^! u^* h^*(\mathcal{F}),$$

is an isomorphism. But the second factor is an isomorphism by Lemma 5.22, hence it suffices to show the composition is.

Now fix a flat cover $g : \operatorname{Spec} A \to Y$. We have a commutative cube

$$\begin{tikzcd}
& Z' \arrow[rr, "\phi'"] \arrow[dl, "\theta'"'] \arrow[dd, "\psi'"] & & \operatorname{Spec} A' \arrow[dl, "\theta"] \arrow[dd, "\psi"] \\
X' \arrow[rr, crossing over] \arrow[dd] & & Y' & \\
& Z \arrow[rr, "\phi"] \arrow[dl, "g'"'] & & \operatorname{Spec} A \arrow[dl, "g"] \\
X \arrow[rr] & & Y \arrow[from=uu, crossing over] &
\end{tikzcd} \tag{7.2}$$

with Cartesian faces (here ψ is affine since Y' is). Again by faithful flatness the restriction of θ'^* to $\mathrm{IndCoh}(X')^+$ is conservative, hence it suffices to show $\theta'^* h'^* f^!(\mathcal{F}) \to \theta'^* f'^! h^*(\mathcal{F})$ is an isomorphism. Since $\theta'^* f'^! h^*(\mathcal{F}) \to \phi'^! \theta^* h^*(\mathcal{F})$ is an isomorphism by Lemma 5.22, it suffices to show their composition is an isomorphism. But this can be refactored as

$$\theta'^* h'^* f^!(\mathcal{F}) \cong \psi'^* g'^* f^!(\mathcal{F}) \to \psi'^* \phi^! g^*(\mathcal{F}) \to \phi'^! \psi^* g^*(\mathcal{F}) \cong \phi'^! \theta^* h^*(\mathcal{F}), \tag{7.3}$$

whose first factor is an isomorphism by Lemma 5.22, and whose second factor is an isomorphism since ψ is a map of affine schemes [Lur18, Prop. 6.4.1.4]. $\square$

We now turn to morphisms with coherent pullback, first in the setting of geometric stacks. Note that Proposition 7.5 will show the truncatedness hypothesis in the following statement is ultimately unnecessary.

Proposition 7.4. *Let the following be a Cartesian diagram of geometric stacks.*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Suppose that all stacks in the diagram are coherent and that Y and Y' are truncated and tamely presented. Suppose also that h has semi-universal coherent pullback and that f is proper and almost of finite presentation. Then for any $\mathcal{F} \in \mathrm{IndCoh}(Y)$ the Beck-Chevalley map $h'^* f^!(\mathcal{F}) \to f'^! h^*(\mathcal{F})$ is an isomorphism.

Proof. Note that f_* and f'_* preserve compact objects and have compactly generated source by Proposition 6.5, hence $f^!$ and $f'^!$ are continuous [Lur09, Prop. 5.5.7.2]. By compact generation of $\mathrm{IndCoh}(Y)$ it thus suffices to assume $\mathcal{F} \in \mathrm{Coh}(Y)$. The proof of Proposition 7.1 then carries over except we must argue differently at two points. First, it is no longer immediate that $h'^* f^!(\mathcal{F})$ is bounded below, since $f^!(\mathcal{F})$ need not be coherent and h'^* need not be left bounded. Second, to show (7.3) is an isomorphism we can no longer directly apply [Lur18, Prop. 6.4.1.4], since ψ need not be of finite Tor dimension.

By hypothesis we may assume the flat cover $g : \mathrm{Spec}\, A \to Y$ is strictly tamely presented and that A is strictly tamely presented over k. Since Y is truncated, there exists for some n a strictly tame presentation $A \cong \mathrm{colim}\, A_\alpha$ of order n over k. Using Noetherian approximation [Lur18, Prop. 4.2.1.5, Thm. 4.4.2.2, Prop. 5.5.4.1] and faithful flatness of the $u_\alpha : \mathrm{Spec}\, A \to \mathrm{Spec}\, A_\alpha$ we can, for some α, extend (8.4) to a diagram

$$\begin{tikzcd} & Z' \arrow[ld, "\theta'"'] \arrow[rr, "\phi'"] \arrow[dd, "\psi'"] & & W \arrow[ld, "\theta"] \arrow[dd, "\psi"] \\ X' \arrow[rr] \arrow[dd] & & Y' \arrow[dd] & \\ & Z \arrow[ld, "g'"'] \arrow[rr, "\phi"] \arrow[dd, "u'_\alpha"] & & \mathrm{Spec}\, A \arrow[ld, "g"] \arrow[dd, "u_\alpha"] \\ X \arrow[rr] & & Y & \\ & Z_\alpha \arrow[rr, "\phi_\alpha"] & & \mathrm{Spec}\, A_\alpha \end{tikzcd}$$

in which all squares are Cartesian and $\phi_\alpha : Z_\alpha \to \mathrm{Spec}\, A_\alpha$ is proper and almost of finite presentation. Moreover, increasing α if needed, we may assume by [Lur18, Cor. 4.5.1.10], flatness of the u_α, and coherence of the A_α that there exists $\mathcal{F}_\alpha \in \mathrm{Coh}(\mathrm{Spec}\, A_\alpha)$ such that $g^*(\mathcal{F}) \cong u_\alpha^*(\mathcal{F}_\alpha)$.

We now claim that $\tau^{\leq n} f^!(\mathcal{F})$ is coherent for all n. It follows from Lemma 5.22 and [Lur18, Prop. 6.4.1.4] that $g'^* f^!(\mathcal{F}) \cong u'^*_\alpha \phi^!_\alpha(\mathcal{F}_\alpha)$. By flatness of g' and u'_α we then have

$$g'^* \tau^{\leq n} f^!(\mathcal{F}) \cong \tau^{\leq n} g'^* f^!(\mathcal{F}) \cong \tau^{\leq n} u'^*_\alpha \phi^!_\alpha(\mathcal{F}_\alpha) \cong u'^*_\alpha \tau^{\leq n} \phi^!_\alpha(\mathcal{F}_\alpha).$$

Thus $\tau^{\leq n} f^!(\mathcal{F})$ is coherent if $\tau^{\leq n} \phi^!_\alpha(\mathcal{F}_\alpha)$ is, since g' is faithfully flat and $\tau^{\leq n} f^!(\mathcal{F})$ is bounded below. But $\tau^{\leq n} \phi^!_\alpha(\mathcal{F}_\alpha)$ is coherent for all n by [Lur18, Prop. 6.4.3.4].

Since the standard t-structure is right complete we have $f^!(\mathcal{F}) \cong \operatorname{colim} \tau^{\leq n} f^!(\mathcal{F})$ by Lemma A.2. Since it is compatible with filtered colimits, and since h'^* is continuous, $h'^* f^!(\mathcal{F})$ is then bounded below if the sheaves $h'^* \tau^{\leq n} f^!(\mathcal{F})$ are uniformly bounded below. Since θ' is faithfully flat, hence θ'^* conservative on $\operatorname{IndCoh}(X')^+$, it suffices to show this for the sheaves $\theta'^* h'^* \tau^{\leq n} f^!(\mathcal{F})$. But since Y and Y' are truncated and tamely presented, $u_\alpha \circ \psi$ is of finite Tor-dimension by Proposition 3.13, hence so is $u'_\alpha \circ \psi'$. The claim then follows since

$$\theta'^* h'^* \tau^{\leq n} f^!(\mathcal{F}) \cong \psi'^* g'^* \tau^{\leq n} f^!(\mathcal{F}) \cong \psi'^* u'^*_\alpha \tau^{\leq n} \phi^!_\alpha(\mathcal{F}_\alpha).$$

Now we show (7.3) is an isomorphism, which as before reduces to showing $\psi'^* \phi^! g^*(\mathcal{F}) \to \phi'^! \psi^* g^*(\mathcal{F})$ is. The composition

$$\psi'^* u'^*_\alpha \phi^!_\alpha(\mathcal{F}_\alpha) \to \psi'^* \phi^! u^*_\alpha(\mathcal{F}_\alpha) \to \phi'^! \psi^* u^*_\alpha(\mathcal{F}_\alpha),$$

and its first factor are now isomorphisms by Proposition 7.1, since again $u_\alpha \circ \psi$ is of finite Tor-dimension by Proposition 3.13 (note that Z_α is geometric since Z is and since u'_α is faithfully flat [Lur18, Prop. 9.3.1.3]). But then the second factor, which is a rewriting of $\psi'^* \phi^! g^*(\mathcal{F}) \to \phi'^! \psi^* g^*(\mathcal{F})$, is also an isomorphism. ☐

The extension to ind-geometric stacks now follows the corresponding result for flat morphisms among ind-schemes [Ras19, Lem. 6.17.2].

Proposition 7.5. *Let the following be a Cartesian diagram of ind-geometric stacks.*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Suppose that all stacks in the diagram are coherent and that Y and Y' are ind-tamely presented. Suppose also that h has semi-universal coherent pullback and that f is ind-proper and almost of ind-finite presentation. Then for any $\mathcal{F} \in \operatorname{IndCoh}(Y)$ the Beck-Chevalley map $h'^* f^!(\mathcal{F}) \to f'^! h^*(\mathcal{F})$ is an isomorphism.

Proof. Suppose first that X is truncated and geometric. Let $Y \cong \operatorname{colim} Y_\alpha$ be a reasonable presentation, and define $h_\alpha : Y'_\alpha \to Y_\alpha$, $i'_\alpha : Y'_\alpha \to Y'$ by base change, similarly $i'_{\alpha\beta} : Y'_\alpha \to Y'_\beta$ for $\beta \geq \alpha$. Note that since h has semi-universal coherent pullback each Y'_α is a reasonable geometric substack of Y', and in particular is tamely presented. By Proposition 7.4 we have $h'^*_\alpha i^!_{\alpha\beta} \cong i'^!_{\alpha\beta} h^*_\beta$ in $\operatorname{Fun}(\operatorname{IndCoh}(Y_\beta), \operatorname{IndCoh}(Y'_\alpha))$. For any α we then have $h'^*_\alpha i^!_\alpha \cong i'^!_\alpha h^*$ in

$\mathrm{Fun}(\mathrm{IndCoh}(Y), \mathrm{IndCoh}(Y'_\alpha))$ by (5.12) and [Lur17, Prop. 4.7.5.19]. By Proposition 4.4 we can factor f as $X \xrightarrow{f_\alpha} Y_\alpha \xrightarrow{i_\alpha} Y$ for some α. Consider the diagram

$$\begin{tikzcd} X' \arrow[r, "f'_\alpha"] \arrow[d, "h'"'] & Y'_\alpha \arrow[r, "i'_\alpha"] \arrow[d, "h_\alpha"] & Y' \arrow[d, "h"] \\ X \arrow[r, "f_\alpha"] & Y_\alpha \arrow[r, "i_\alpha"] & Y \end{tikzcd}$$

of Cartesian squares. The map in the statement now factors as

$$h'^* f^!(\mathcal{F}) \cong h^* f_\alpha^! i_\alpha^!(\mathcal{F}) \to f'^!_\alpha h^*_\alpha i^!_\alpha(\mathcal{F}) \to f'^!_\alpha i'^!_\alpha h^*(\mathcal{F}) \cong f'^! h^*(\mathcal{F}),$$

and we have shown both factors are isomorphisms (again using Proposition 7.4).

Now suppose $X \cong \operatorname{colim} X_\alpha$ is a reasonable presentation. For any α we have a diagram

$$\begin{tikzcd} X'_\alpha \arrow[r, "i'_\alpha"] \arrow[d, "h'_\alpha"'] & X' \arrow[r, "f'"] \arrow[d, "h'"] & Y' \arrow[d, "h"] \\ X_\alpha \arrow[r, "i_\alpha"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

of Cartesian squares. By Proposition 4.18 each h'_α has semi-universal coherent pullback, hence we have a reasonable presentation $X' \cong \operatorname{colim} X'_\alpha$ since filtered colimits are left exact in $\widehat{\mathrm{Stk}}_k$. The functors $i'^!_\alpha$ thus determine an isomorphism $\mathrm{IndCoh}(X') \cong \lim_\alpha \mathrm{IndCoh}(X'_\alpha)$ in $\widehat{\mathrm{Cat}}_\infty$, hence it suffices to show that $i'^!_\alpha h'^* f^!(\mathcal{F}) \to i'^!_\alpha f'^! h^*(\mathcal{F})$ is an isomorphism for all α. But by the previous paragraph both the composition

$$h'^*_\alpha i^!_\alpha f^!(\mathcal{F}) \to i'^!_\alpha h'^* f^!(\mathcal{F}) \to i'^!_\alpha f'^! h^*(\mathcal{F}) \tag{7.6}$$

and its first factor are isomorphisms, hence the second is as well. $\square$

We can relax the coherence hypotheses in Proposition 7.5 if we assume h is of finite Tor-dimension and impose suitable boundedness conditions on $\mathcal{F}$. However, the proof does not quite work with the hypothesis $\mathcal{F} \in \mathrm{IndCoh}(Y)^+$, for the following reason: ind-properness of f does not imply that $f^!$ takes $\mathrm{IndCoh}(Y)^+$ to $\mathrm{IndCoh}(X)^+$.

Instead, consider the full subcategory $\mathrm{IndCoh}(Y)^+_{lim} \subset \mathrm{IndCoh}(Y)$ of $\mathcal{F}$ such that $i^!(\mathcal{F}) \in \mathrm{IndCoh}(Y')^+$ for every truncated geometric substack $i : Y' \to Y$. By Proposition 4.6 this is equivalent to $i^!_\alpha(\mathcal{F}) \in \mathrm{IndCoh}(Y_\alpha)^+$ for all α, where $Y \cong \operatorname{colim} Y_\alpha$ is any ind-geometric presentation. In particular, the equivalance $\mathrm{IndCoh}(Y) \cong \lim \mathrm{IndCoh}(Y_\alpha)$ in $\widehat{\mathrm{Cat}}_\infty$ restricts to an equivalence

$$\mathrm{IndCoh}(Y)^+_{lim} \cong \lim \mathrm{IndCoh}(Y_\alpha)^+. \tag{7.7}$$

We have $\mathrm{IndCoh}(Y)^+ \subset \mathrm{IndCoh}(Y)^+_{lim}$ since each $i^!$ is left t-exact, and it follows from the definitions that $f^!$ takes $\mathrm{IndCoh}(Y)^+_{lim}$ to $\mathrm{IndCoh}(X)^+_{lim}$. We now have the following variant of Proposition 7.5, proved the same way with Proposition 7.1 in place of Proposition 7.4 and

(7.7) in place of (5.12) (and [Lur17, Cor. 4.7.5.18] in place of [Lur17, Prop. 4.7.5.19], since bounded below categories are not presentable).

Proposition 7.8. *Let the following be a Cartesian diagram of ind-geometric stacks.*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Suppose that X and Y are reasonable. Suppose also that h is geometric and of finite Tor-dimension, and that f is ind-proper and almost of ind-finite presentation. Then for any $\mathcal{F} \in \operatorname{IndCoh}(Y)^+_{lim}$ the Beck-Chevalley map $h'^* f^!(\mathcal{F}) \to f'^! h^*(\mathcal{F})$ is an isomorphism.

7.2. **!-pullback and pushforward.** We turn next to the commutation of !-pullback and pushforward. The proofs below largely follow those of corresponding results for ind-schemes [Gai13, Prop. 3.4.2], [GR14, Prop. 2.9.2], [Ras19, Lem. 6.17.2].

Proposition 7.9. *Let the following be a Cartesian diagram of ind-geometric stacks.*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Suppose that X and Y are geometric stacks, that h is of ind-finite cohomological dimension, and that f is proper and almost of finite presentation. Then for any $\mathcal{F} \in \operatorname{IndCoh}(Y')^+$ the Beck-Chevalley map $h'_* f'^!(\mathcal{F}) \to f^! h_*(\mathcal{F})$ is an isomorphism.

Proof. Suppose first that X' and Y' are geometric. Passing through the equivalences $\operatorname{IndCoh}(-)^+ \cong \operatorname{QCoh}(-)^+$ it suffices to show the natural transformation $h'_{QC*} f'^!_{QC} \to f^!_{QC} h_{QC*}$ in $\operatorname{Fun}(\operatorname{QCoh}(Y'), \operatorname{QCoh}(X))$ is an isomorphism, $f^!_{QC}$ and $f'^!_{QC}$ denoting the right adjoints of f_{QC*} and f_{QC*}. But this follows since it is obtained from the isomorphism $f'_{QC*} h'^*_{QC} \xrightarrow{\sim} h^*_{QC} f_{QC*}$ in $\operatorname{Fun}(\operatorname{QCoh}(X), \operatorname{QCoh}(Y'))$ by passing to right adjoints.

Now let $Y' \cong \operatorname{colim} Y'_\alpha$ be an ind-geometric presentation. For every α we have a diagram

$$\begin{tikzcd} Y'_\alpha \arrow[r, "i_\alpha"] & Y' \arrow[r, "h"] & Y \\ X'_\alpha \arrow[r, "i'_\alpha"] \arrow[u, "f_\alpha"] & X' \arrow[r, "h'"] \arrow[u, "f'"'] & X \arrow[u, "f"'] \end{tikzcd}$$

of Cartesian squares. Expanding the Beck-Chevalley maps in terms of units and counits, it follows from the basic properties of these that the counits $i_{\alpha*} i^!_\alpha \to id_{IC(Y')}$ and $i'_{\alpha*} i'^!_\alpha \to id_{IC(X')}$

fit into diagrams

$$\begin{tikzcd} h'_* i'_{\alpha*} i'^!_\alpha f^!(\mathcal{F}) \arrow[r, "\sim"] \arrow[d] & h'_* i'_{\alpha*} f^!_\alpha i^!_\alpha(\mathcal{F}) \arrow[r] & f^! h_* i_{\alpha*} i^!_\alpha(\mathcal{F}) \arrow[d] \\ h'_* f'^!(\mathcal{F}) \arrow[rr] & & f^! h_*(\mathcal{F}), \end{tikzcd} \tag{7.10}$$

compatibly for $\beta \geq \alpha$. That is, the counits $i_{\alpha*} i^!_\alpha \cong i_{\beta*} i_{\alpha\beta*} i^!_{\alpha\beta} i^!_\beta \to i_{\beta*} i^!_\beta$ and $i'_{\alpha*} i'^!_\alpha \cong i'_{\beta*} i'_{\alpha\beta*} i'^!_{\alpha\beta} i'^!_\beta \to i'_{\beta*} i'^!_\beta$ intertwine the top compositions in (7.10) for $\beta \geq \alpha$. Passing to colimits we obtain a diagram

$$\begin{tikzcd} \operatorname{colim} h'_* i'_{\alpha*} i'^!_\alpha f^!(\mathcal{F}) \arrow[r] \arrow[d] & \operatorname{colim} f^! h_* i_{\alpha*} i^!_\alpha(\mathcal{F}) \arrow[d] \\ h'_* f'^!(\mathcal{F}) \arrow[r] & f^! h_*(\mathcal{F}). \end{tikzcd}$$

The vertical arrows are isomorphisms by Lemma A.2, the continuity of h_* and h'_*, and the left t-exactness of $h_* i_{\alpha*} i^!_\alpha$ together with the almost continuity of $f^!$ (Proposition 5.21). But the top arrow is an isomorphism since the top right arrow in (7.10) is by the first paragraph, hence so is the bottom arrow. □

Note that when the stacks appearing in Proposition 7.9 are coherent, it immediately follows that its conclusion holds for all $\mathcal{F} \in \operatorname{IndCoh}(Y')$, as $f^!$ and $f'^!$ are continuous and $\operatorname{IndCoh}(Y')$ is compactly generated by $\operatorname{Coh}(Y')$. When X and Y are ind-geometric, the coherent case differs from the general case more substantially in terms of what other hypotheses are needed, so we isolate these into separate results below.

Proposition 7.11. *Let the following be a Cartesian diagram of ind-geometric stacks.*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Suppose that all stacks in the diagram are coherent, that f is ind-proper and almost of ind-finite presentation, and that h is of ind-finite cohomological dimension. Then for any $\mathcal{F} \in \operatorname{IndCoh}(Y')$ the Beck-Chevalley map $h'_* f'^!(\mathcal{F}) \to f^! h_*(\mathcal{F})$ is an isomorphism.

Proof. First note that the coherence hypotheses imply that $f^!$ and $f'^!$ are continuous. If X and Y are geometric, the claim now follows immediately from Proposition 7.9 since $\operatorname{IndCoh}(Y')$ is compactly generated by $\operatorname{Coh}(Y')$.

Now suppose that X is truncated and geometric, and let $Y \cong \operatorname{colim} Y_\alpha$ be a reasonable presentation. Define $h_\alpha : Y'_\alpha \to Y_\alpha$, $i'_\alpha : Y'_\alpha \to Y'$ by base change, similarly $i'_{\alpha\beta} : Y'_\alpha \to Y'_\beta$ for $\beta \geq \alpha$. Each Y'_α is coherent by Proposition 6.7, so by the previous paragraph we have $h_{\alpha*} i'^!_{\alpha\beta} \cong i^!_{\alpha\beta} h_{\beta*}$ in $\operatorname{Fun}(\operatorname{IndCoh}(Y'_\beta), \operatorname{IndCoh}(Y_\alpha))$. For any α we then have $h_{\alpha*} i'^!_\alpha \cong i^!_\alpha h_*$ in

$\mathrm{Fun}(\mathrm{IndCoh}(Y'), \mathrm{IndCoh}(Y_\alpha))$ by [Lur17, Prop. 4.7.5.19]. By Proposition 4.4 we can factor f as $X \xrightarrow{f_\alpha} Y_\alpha \xrightarrow{i_\alpha} Y$ for some α. Consider the diagram

$$\begin{tikzcd} X' \arrow[r, "f'_\alpha"] \arrow[d, "h'"'] & Y'_\alpha \arrow[r, "i'_\alpha"] \arrow[d, "h_\alpha"] & Y' \arrow[d, "h"] \\ X \arrow[r, "f_\alpha"] & Y_\alpha \arrow[r, "i_\alpha"] & Y \end{tikzcd}$$

of Cartesian squares. The map in the statement now factors as

$$h'_* f'^!(\mathcal{F}) \cong h'_* f'^!_\alpha i'^!_\alpha(\mathcal{F}) \to f^!_\alpha h_{\alpha *} i'^!_\alpha(\mathcal{F}) \to f^!_\alpha i^!_\alpha h_*(\mathcal{F}) \cong f^! h_*(\mathcal{F}),$$

and we have already shown both factors are isomorphisms.

Now suppose $X \cong \mathrm{colim}\, X_\alpha$ is a reasonable presentation. For any α we have a diagram

$$\begin{tikzcd} X'_\alpha \arrow[r, "i'_\alpha"] \arrow[d, "h'_\alpha"'] & X' \arrow[r, "f'"] \arrow[d, "h'"] & Y' \arrow[d, "h"] \\ X_\alpha \arrow[r, "i_\alpha"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

of Cartesian squares. Each X'_α is coherent by Proposition 6.7, and $X' \cong \mathrm{colim}\, X'_\alpha$ since filtered colimits are left exact in $\widehat{\mathrm{Stk}}_k$. Since the functors $i'^!_\alpha$ determine an isomorphism $\mathrm{IndCoh}(X') \cong \lim_\alpha \mathrm{IndCoh}(X'_\alpha)$ in $\widehat{\mathrm{Cat}}_\infty$, it suffices to show that $i^!_\alpha h'_* f'^!(\mathcal{F}) \to i^!_\alpha f^! h_*(\mathcal{F})$ is an isomorphism for all α. But by the previous paragraph both the composition

$$h'_{\alpha *} i'^!_\alpha f'^!(\mathcal{F}) \to i^!_\alpha h'_* f'^!(\mathcal{F}) \to i^!_\alpha f^! h_*(\mathcal{F})$$

and its first factor are isomorphisms, hence the second is as well. □

As with Propositions 7.5 and 7.8, the coherence hypotheses in Proposition 7.11 can be relaxed given boundedness hypotheses on $\mathcal{F}$. Now, however, we encounter the following problem: it is not the case in general that h_* takes $\mathrm{IndCoh}(Y')^+_{lim}$ to $\mathrm{IndCoh}(Y)^+_{lim}$. To circumvent this, recall from Definition 4.32 that a formal geometric stack is an ind-geometric stack X such that X^{cl} is geometric, and a morphism $Y' \to Y$ is formally geometric if $X \times_Y Y'$ is formally geometric whenever X is geometric.

Proposition 7.12. *If X is a formal geometric stack then* $\mathrm{IndCoh}(X)^+ \cong \mathrm{IndCoh}(X)^+_{lim}$. *If $h : Y' \to Y$ is a formally geometric morphism of ind-geometric stacks and Y is reasonable, then h_* takes* $\mathrm{IndCoh}(Y')^+_{lim}$ *to* $\mathrm{IndCoh}(Y)^+_{lim}$.

Proof. Let $X \cong \mathrm{colim}\, X_\alpha$ be a formal ind-geometric presentation (Definition 4.32). Given $\mathcal{F} \in \mathrm{IndCoh}(X)^+_{lim}$, fix α arbitrarily and choose n so that $i^!_\alpha(\mathcal{F}) \in \mathrm{IndCoh}(X_\alpha)^{\geq n}$. We claim $\mathcal{F} \in \mathrm{IndCoh}(X)^{\geq n}$. By Proposition 5.13 it suffices to show $i^!_\beta(\mathcal{F}) \in \mathrm{IndCoh}(X_\beta)^{\geq n}$ for any $\beta \geq \alpha$. By hypothesis $i^!_\beta(\mathcal{F}) \in \mathrm{IndCoh}(X_\beta)^{\geq m}$ for some m. By induction it suffices to show that $i^!_\beta(\mathcal{F}) \in \mathrm{IndCoh}(X_\beta)^{\geq m+1}$ if $m < n$.

Since $i_{\alpha\beta*}$ is t-exact it restricts to a functor $\mathrm{IndCoh}(X_\alpha)^\heartsuit \to \mathrm{IndCoh}(X_\beta)^\heartsuit$ whose right adjoint is the restriction of $\tau^{\leq 0} i^!_{\alpha\beta}$. But in fact these are inverse equivalences, since $\tau_{\leq 0} i_{\alpha\beta} : \tau_{\leq 0} X_\alpha \to \tau_{\leq 0} X_\beta$ is an isomorphism. It then follows that $\tau^{\leq m} i^!_\beta(\mathcal{F}) \cong i_{\alpha\beta*}\tau^{\leq m} i^!_\alpha(\mathcal{F}) \cong 0$.

Let $Y \cong \operatorname{colim} Y_\alpha$ be a reasonable presentation. Define $h_\alpha : Y'_\alpha \to Y_\alpha$, $i'_\alpha : Y'_\alpha \to Y'$ by base change, similarly $i'_{\alpha\beta} : Y'_\alpha \to Y'_\beta$ for $\beta \geq \alpha$. Since i_α is an almost finitely presented closed immersion, any truncated geometric substack of Y'_α is a truncated geometric substack of Y'. It follows that h_α is of ind-finite cohomological dimension since h is. In particular, the right adjoint h^+_α of $h_{\alpha*}$ takes $\mathrm{IndCoh}(Y_\alpha)^+$ to $\mathrm{IndCoh}(Y'_\alpha)^+_{lim}$, hence to $\mathrm{IndCoh}(Y'_\alpha)^+$ since Y'_α is formally geometric.

On the other hand, $h_{\alpha*}$ is left t-exact, so by restriction we obtain an adjunction between $\mathrm{IndCoh}(Y_\alpha)^+$ and $\mathrm{IndCoh}(Y'_\alpha)^+$. Passing to adjoints, the isomorphisms $h_{\beta*} i'_{\alpha\beta*} \cong i_{\alpha\beta*} h_{\alpha*}$ yield isomorphisms $i'^!_{\alpha\beta} h^+_\beta \cong h^+_\alpha i^!_{\alpha\beta}$ in $\mathrm{Fun}(\mathrm{IndCoh}(Y_\beta)^+, \mathrm{IndCoh}(Y'_\alpha)^+)$ for $\beta \geq \alpha$. But by the previous paragraph and Proposition 7.9 we have isomorphisms $h_{\alpha*} i'^!_{\alpha\beta} \cong i^!_{\alpha\beta} h_{\beta*}$ in $\mathrm{Fun}(\mathrm{IndCoh}(Y'_\beta)^+, \mathrm{IndCoh}(Y_\alpha)^+)$. Note that $\mathrm{IndCoh}(Y')^+_{lim} \cong \lim \mathrm{IndCoh}(Y'_\beta)^+ \cong \lim \mathrm{IndCoh}(Y'_\beta)^+_{lim}$ by the same proof as Proposition 5.11 (that is, the functor $\mathrm{indGStk}_{k,prop} \to (\widehat{\mathrm{Cat}}_\infty)^{\mathrm{op}}$ that takes $f : X \to Y$ to $f^! : \mathrm{IndCoh}(Y)^+_{lim} \to \mathrm{IndCoh}(X)^+_{lim}$ preserves filtered colimits along ind-closed immersions). For any α we then have $h_{\alpha*} i'^!_\alpha \cong i^!_\alpha h_*$ in $\mathrm{Fun}(\mathrm{IndCoh}(Y')^+_{lim}, \mathrm{IndCoh}(Y_\alpha)^+)$ by [Lur17, Cor. 4.7.5.18], and the claim follows. $\square$

We now have the following variant of Proposition 7.11, proven the same way but using Proposition 7.12 (and, as in the proof of the latter, using [Lur17, Cor. 4.7.5.18] in place of [Lur17, Prop. 4.7.5.19] since bounded below categories are not presentable).

Proposition 7.13. *Let the following be a Cartesian diagram of ind-geometric stacks.*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"'] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Suppose that X and Y are reasonable, that f is ind-proper and almost of ind-finite presentation, and that h is of ind-finite cohomological dimension. Suppose also that h is formally geometric (resp. that f is of finite cohomological dimension). Then for any $\mathcal{F} \in \mathrm{IndCoh}(Y')^+_{lim}$ (resp. $\mathcal{F} \in \mathrm{IndCoh}(Y')^+$) the Beck-Chevalley map $h'_* f'^!(\mathcal{F}) \to f^! h_*(\mathcal{F})$ is an isomorphism.

8. External products and sheaf Hom

Given a geometric stack Y, the sheaf Hom out of $\mathcal{F} \in \mathrm{QCoh}(Y)$ is defined by the adjunction

$$- \otimes \mathcal{F} : \mathrm{QCoh}(Y) \leftrightarrows \mathrm{QCoh}(Y) : \mathcal{H}om(\mathcal{F}, -).$$

For any $f : X \to Y$, the isomorphism $f^*(- \otimes \mathcal{F}) \cong - \otimes f^*(\mathcal{F})$ then gives rise to a map

$$f^* \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(f^*(\mathcal{F}), f^*(\mathcal{G})) \tag{8.1}$$

which is natural in $\mathcal{G} \in \mathrm{QCoh}(Y)$. This is not an isomorphism in general, but is when f is of finite Tor-dimension under certain hypotheses on $\mathcal{F}$ and $\mathcal{G}$. The basic goal of this section is to generalize this and related results, in particular allowing f to have coherent pullback, X and Y to be ind-geometric, and $\mathcal{F}$ and $\mathcal{G}$ to be ind-coherent.

Since $\mathrm{IndCoh}(X)$ does not generally have a tensor product, external products instead play the primary role. That is, for suitable $\mathcal{F} \in \mathrm{IndCoh}(Y)$ we have an adjunction

$$- \boxtimes \mathcal{F} : \mathrm{IndCoh}(X) \leftrightarrows \mathrm{IndCoh}(X \times Y) : (- \boxtimes \mathcal{F})^R.$$

To make explicit their dependence on X we will denote these functors by $e_{\mathcal{F},X}$ and $e^R_{\mathcal{F},X}$. When X and Y are geometric, we have an isomorphism $\mathcal{H}om(\mathcal{F}, -) \cong e^R_{\mathcal{F},Y} \Delta_{Y*}$, letting us subsume results about $\mathcal{H}om(\mathcal{F}, -)$ in corresponding results about $e^R_{\mathcal{F},Y}$. This formula moreover provides a useful definition of sheaf Hom in the ind-geometric setting.

On a technical level, there is a close analogy between $e^R_{\mathcal{F},X}$ for coherent $\mathcal{F}$ and the functor $g^!$ associated to a morphism $g : X' \to Y'$ which is ind-proper and almost of ind-finite presentation — the two functors have similar formal properties for similar reasons. In particular, many proofs about $e^R_{\mathcal{F},X}$ in this section closely follow corresponding proofs in Section 7. The main difference is that the role of the map g is now played by the projection $X \times Y \to X$, so a contravariant functoriality has been replaced with covariant one.

8.1. **Ind-coherent external products.** We begin by defining the the external product of a pair of ind-coherent sheaves, at least provided one of them is bounded. *We assume that k is an ordinary ring of finite global dimension for the rest of the paper.* Suppose first that X and Y are geometric stacks and $\mathcal{F} \in \mathrm{IndCoh}(Y)^b$. By Lemma 3.21 the assignment $\mathcal{G} \mapsto \mathcal{G} \boxtimes \Psi_Y(\mathcal{F})$ defines a bounded colimit-preserving functor $e_{\Psi_Y(\mathcal{F}),X} : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X \times Y)$. The universal property of $\mathrm{IndCoh}(-)$ guarantees that this functor has a unique ind-coherent lift in the following sense.

Definition 8.2. If X and Y are geometric stacks and $\mathcal{F} \in \mathrm{IndCoh}(Y)^b$, we let $e_{\mathcal{F},X}$ denote the unique bounded colimit-preserving functor fitting into a diagram of the following form.

$$\begin{tikzcd} \mathrm{IndCoh}(X) \arrow[r, "e_{\mathcal{F},X}"] \arrow[d, "\Psi_X"'] & \mathrm{IndCoh}(X \times Y) \arrow[d, "\Psi_{X \times Y}"] \\ \mathrm{QCoh}(X) \arrow[r, "e_{\Psi_Y(\mathcal{F}),X}"] & \mathrm{QCoh}(X \times Y) \end{tikzcd} \tag{8.3}$$

When X and Y are truncated this is compatible with our earlier definition of external products of coherent sheaves (Proposition 3.22) in the obvious way, given the identification of $\mathrm{Coh}(X)$ and $\mathrm{Coh}(X \times Y)$ with full subcategories of $\mathrm{IndCoh}(X)$ and $\mathrm{IndCoh}(X \times Y)$. When X is coherent Definition 8.2 is determined by this compatibility, as $e_{\mathcal{F},X}$ is then the left Kan extension of its restriction to $\mathrm{Coh}(X)$ (Proposition 6.5). The functoriality of Definition 8.2 in X, Y, and $\mathcal{F}$ is described by the following result.

Proposition 8.4. *There exists a diagram*

$$\begin{tikzcd} \mathrm{IndCoh}(-) \times \mathrm{IndCoh}(-)^b \arrow[r] \arrow[d, "\Psi_{(-)} \times \Psi_{(-)}"'] & \mathrm{IndCoh}(- \times -) \arrow[d, "\Psi_{(- \times -)}"] \\ \mathrm{QCoh}(-) \times \mathrm{QCoh}(-)^b \arrow[r] & \mathrm{QCoh}(- \times -) \end{tikzcd} \tag{8.5}$$

of functors $\mathrm{Corr}(\mathrm{GStk}_k)^{\times 2}_{fcd;ftd} \to \widehat{\mathrm{Cat}}_\infty$ *which specializes to the diagram (8.3) when evaluated on any* $X, Y \in \mathrm{GStk}_k$ *and any* $\mathcal{F} \in \mathrm{IndCoh}(Y)^b$*.*

We postpone the proof of Proposition 8.4 while we extend Definition 8.2 to ind-geometric stacks. To simplify the needed constructions we restrict our attention to the case where Y is reasonable and $\mathcal{F}$ is coherent. This is not strictly essential, but most good properties of $e^R_{\mathcal{F},X}$ (e.g. almost continuity) will require $\mathcal{F}$ to be coherent anyway.

Note first that the top arrow of (8.5) can be encoded as a functor $\mathrm{Corr}(\mathrm{GStk}_k)^{\times 2}_{fcd;ftd} \to \widehat{\mathrm{Cat}}^{\Delta^1}_\infty$. Restricting its domain and values we then obtain a functor $\mathrm{Corr}(\mathrm{GStk}^+_k)_{fcd;ftd} \times \mathrm{Corr}(\mathrm{GStk}^+_k)_{prop;ftd} \to \widehat{\mathrm{Cat}}^{\Delta^1}_\infty$ of the form

$$\mathrm{IndCoh}(-) \times \mathrm{Coh}(-) \to \mathrm{IndCoh}(- \times -).$$

For any X, $Y \in \mathrm{GStk}_k$ the specialization of this expression preserves small colimits in $\mathrm{IndCoh}(X)$, hence there exists a unique extension to a functor $\mathrm{Corr}(\mathrm{GStk}^+_k)_{fcd;ftd} \times \mathrm{Corr}(\mathrm{GStk}^+_k)_{prop;ftd} \to (\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\Delta^1}$ of the form

$$\mathrm{IndCoh}(-) \otimes \mathrm{Ind}(\mathrm{Coh}(-)) \to \mathrm{IndCoh}(- \times -). \tag{8.6}$$

We now define a functor $\mathrm{Corr}(\mathrm{indGStk}_k)_{fcd;ftd} \times \mathrm{Corr}(\mathrm{indGStk}^{reas}_k)_{prop;ftd} \to (\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\Delta^1}$ via left Kan extension. This extension exists, and moreover is of the same form (8.6), since $(\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\Delta^1}$ admits small colimits and $(\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\Delta^1} \to (\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\times 2}$ preserves them [Lur09, Cor. 5.1.2.3], since the tensor product in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ preserves small colimits in each variable [Lur17, Rem. 4.8.1.23], and since ind-completion of idempotent-complete categories admitting finite colimits commutes with filtered colimits [Lur17, Lem. 7.3.5.11].

Definition 8.7. We define a functor $\mathrm{Corr}(\mathrm{indGStk}_k)_{fcd;ftd} \times \mathrm{Corr}(\mathrm{indGStk}^{reas}_k)_{prop;ftd} \to \widehat{\mathrm{Cat}}^{\Delta^1}_\infty$ of the form

$$\mathrm{IndCoh}(-) \times \mathrm{Coh}(-) \to \mathrm{IndCoh}(- \times -)$$

by taking the functor $\mathrm{Corr}(\mathrm{indGStk}_k)_{fcd;ftd} \times \mathrm{Corr}(\mathrm{indGStk}^{reas}_k)_{prop;ftd} \to (\mathcal{P}\mathrm{r}^{\mathrm{L}})^{\Delta^1}$ defined above, passing to its underlying $\widehat{\mathrm{Cat}}^{\Delta^1}_\infty$-valued functor, and then composing with the canonical natural transformation $\mathrm{IndCoh}(-) \times \mathrm{Coh}(-) \to \mathrm{IndCoh}(-) \otimes \mathrm{Ind}(\mathrm{Coh}(-))$. Given ind-geometric stacks X and Z such that Z is reasonable, and given $\mathcal{F} \in \mathrm{Coh}(Z)$, we write

$$e_{\mathcal{F},X} : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(X \times Y)$$

for the associated functor.

Explicitly, Definition 8.7 says that if $\mathcal{F} \cong i_*(\mathcal{F}')$ for some reasonable geometric substack $i : Z' \to Z$, and if $X \cong \operatorname{colim} X_\alpha$ is an ind-geometric presentation, then we have a diagram

$$\begin{tikzcd} \operatorname{IndCoh}(X_\alpha) \arrow[r, "e_{\mathcal{F}',X_\alpha}"] \arrow[d, "i_{\alpha*}"'] & \operatorname{IndCoh}(X_\alpha \times Z') \arrow[d, "(i_\alpha \times i)_*"] \\ \operatorname{IndCoh}(X) \arrow[r, "e_{\mathcal{F},X}"] & \operatorname{IndCoh}(X \times Z) \end{tikzcd}$$

for all α. The functor $e_{\mathcal{F},X}$ is determined by these diagrams together with the fact that by construction it preserves small colimits.

If X and $X \times Z$ are reasonable and $\mathcal{F} \in \operatorname{Coh}(X)$, it follows from the definitions that $e_{\mathcal{F},X}$ takes $\operatorname{Coh}(X)$ to $\operatorname{Coh}(X \times Z)$. In particular, suppose X and $X \times Z$ are coherent, X' and Z' are reasonable, and $h : X' \to X$, $\phi : Z' \to Z$ are morphisms with coherent pullback. Then there is a canonical isomorphism

$$(h \times \phi)^* e_{\mathcal{F},X} \cong e_{\phi^*(\mathcal{F}),X'} h^* \tag{8.8}$$

since both sides are continuous and have canonically isomorphic restrictions to $\operatorname{Coh}(X)$.

Proposition 8.9. *Let X and Z be ind-geometric stacks such that Z reasonable, and let $\mathcal{F} \in \operatorname{Coh}(Z)$. Then $e_{\mathcal{F},X}$ is bounded.*

Proof. Fix an ind-geometric presentation $X \cong \operatorname{colim} X_\alpha$ and write $\mathcal{F} \cong i_*(\mathcal{F}')$, where $i : Z' \to Z$ is a reasonable geometric substack and $\mathcal{F}' \in \operatorname{Coh}(Z')^{[m,n]}$. If $\mathcal{G}_\alpha \in \operatorname{IndCoh}(X_\alpha)^{\geq 0}$ for some α, then by t-exactness of $(i_\alpha \times i)_*$ and the proof of Lemma 3.21 we have $e_{\mathcal{F},X} i_{\alpha*}(\mathcal{G}_\alpha) \cong (i_\alpha \times i)_* e_{\mathcal{F}',X_\alpha}(\mathcal{G}_\alpha) \in \operatorname{IndCoh}(X)^{\geq m'}$, where m' is m minus the global dimension of k. Similarly, if $\mathcal{G}_\alpha \in \operatorname{IndCoh}(X_\alpha)^{\leq 0}$ then $e_{\mathcal{F},X} i_{\alpha*}(\mathcal{G}_\alpha) \in \operatorname{IndCoh}(X)^{\leq n}$. Given that $\mathcal{G} \cong \operatorname{colim} i_{\alpha*} i_\alpha^!(\mathcal{G})$ for any $\mathcal{G} \in \operatorname{IndCoh}(X)$ (by Proposition 5.11 and Lemma A.2), it follows that $e_{\mathcal{F},X}$ takes $\operatorname{IndCoh}(X)^{\geq 0}$ to $\operatorname{IndCoh}(X \times Z)^{\geq m'}$ since $i_{\alpha*} i_\alpha^!$ is left t-exact and $\operatorname{IndCoh}(X \times Z)^{\geq m'}$ is closed under filtered colimits. If $\mathcal{G} \in \operatorname{IndCoh}(X)^{\leq 0}$, then we additionally have $\mathcal{G} \cong \operatorname{colim} i_{\alpha*} \tau^{\leq 0} i_\alpha^!(\mathcal{G})$ (by Proposition 5.13 and Lemma A.2, given that $\operatorname{IndCoh}(X)^{\leq 0}$ is closed under colimits and $\tau^{\leq 0} i_\alpha^!$ is right adjoint to the restriction $i_{\alpha*} : \operatorname{IndCoh}(X_\alpha)^{\leq 0} \to \operatorname{IndCoh}(X)^{\leq 0}$). It now follows that $e_{\mathcal{F},X}$ takes $\operatorname{IndCoh}(X)^{\leq 0}$ to $\operatorname{IndCoh}(X \times Z)^{\leq n}$. □

We note the following consistency between Definitions 8.2 and 8.7, which is true by construction when X and Z are truncated, and whose statement implicitly uses Proposition 5.18.

Proposition 8.10. *Let X and Z be geometric stacks such that Z is reasonable, and let $\mathcal{F} \in \operatorname{Coh}(Z)$. Then the functors $e_{\mathcal{F},X}$ of Definitions 8.2 and 8.7 are canonically isomorphic.*

Proof. Since the functor $e_{\mathcal{F},X}$ of Definition 8.7 is bounded (Proposition 8.9), it suffices to show it fits into a diagram of the form (8.3). Let $X \cong \operatorname{colim} X_\alpha$, $Z \cong \operatorname{colim} Z_\beta$ be respectively an ind-geometric and a reasonable presentation, and write $\mathcal{F} \cong i_{\beta*}(\mathcal{F}_\beta)$ for some β and $\mathcal{F}_\beta \in \operatorname{Coh}(Z_\beta)$. By Proposition 8.4 the functors $e_{i_{\beta\gamma*}(\mathcal{F}_\beta),X_\alpha}$ form a filtered system in $(\mathcal{P}\mathrm{r}^L)^{\Delta^1}$

which lifts to a filtered system in $(\mathcal{P}\mathrm{r}^L)^{\Delta^1}_{/e_{\Psi_Z(\mathcal{F}),X}}$ (i.e. given termwise by taking $e_{i_{\beta\gamma*}(\mathcal{F}_\beta),X_\alpha}$ to the diagram realizing the isomorphism $(i_\alpha \times i_\gamma)_{*,QC}\Psi_{X_\alpha \times Z_\gamma} e_{i_{\beta\gamma*}(\mathcal{F}_\beta),X_\alpha} \cong e_{\Psi_Z(\mathcal{F}),X} i_{\alpha*,QC}\Psi_{X_\alpha})$. By Proposition 5.11 and [Lur09, Prop. 1.2.13.8] its colimit in $(\mathcal{P}\mathrm{r}^L)^{\Delta^1}_{/e_{\Psi_Z(\mathcal{F}),X}}$ is a diagram whose top and bottom arrows are $e_{\mathcal{F},X}$ and $e_{\Psi_Z(\mathcal{F}),X}$. But the vertical arrows in this diagram are t-exact and induce equivalences of left completions by Proposition 5.13 and t-exactness of the $\Psi_{(-)}$ functors and the pushforward functors in the filtered system, hence they are isomorphic to Ψ_X and $\Psi_{X\times Z}$. □

We now return to the (tedious but ultimately straightforward) proof of Proposition 8.4.

Proof of Proposition 8.4. We can regard the natural transformation on the bottom of (8.5) as a functor $\mathrm{Corr}(\mathrm{GStk}_k)^{\times 2}_{fcd;ftd} \to \widehat{\mathrm{Cat}}^{\Delta^1}_\infty$ taking (X,Y) to $\mathrm{QCoh}(X)\times\mathrm{QCoh}(Y)^b \to \mathrm{QCoh}(X\times Y)$. Here we write $\widehat{\mathrm{Cat}}^{\Delta^1}_\infty := \mathrm{Fun}(\Delta^1, \widehat{\mathrm{Cat}}_\infty)$, similarly in other cases below. By construction this functor factors through the category $\widehat{\mathrm{Cat}}^{b,\Delta^1}_{cc}$ defined as follows. We set $\mathcal{P}\mathrm{r}_{cc} := \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{cpl}} \times \widehat{\mathrm{Cat}}_\infty \times \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{cpl}}$, regarding it as a category over $\widehat{\mathrm{Cat}}^{\times 2}_\infty$ via $(\widehat{\mathcal{C}}, \mathcal{D}, \widehat{\mathcal{E}}) \mapsto (\widehat{\mathcal{C}} \times \mathcal{D}, \widehat{\mathcal{E}})$. We then write $\widehat{\mathrm{Cat}}^{\Delta^1}_{cc} := \widehat{\mathrm{Cat}}^{\Delta^1}_\infty \times_{\widehat{\mathrm{Cat}}^2_\infty} \mathcal{P}\mathrm{r}_{cc}$ for the category of tuples $(\widehat{\mathcal{C}}, \mathcal{D}, \widehat{\mathcal{E}}, F)$, where $\widehat{\mathcal{C}}, \widehat{\mathcal{E}} \in \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{cpl}}$, $\mathcal{D} \in \widehat{\mathrm{Cat}}_\infty$, and $F : \widehat{\mathcal{C}} \times \mathcal{D} \to \widehat{\mathcal{E}}$. Finally, we write $\widehat{\mathrm{Cat}}^{b,\Delta^1}_{cc}$ for the full subcategory of such tuples whose associated functor $\mathcal{D} \to \mathrm{Fun}(\widehat{\mathcal{C}}, \widehat{\mathcal{E}})$ takes values in $\mathrm{LFun}^b(\widehat{\mathcal{C}}, \widehat{\mathcal{E}})$.

We will prove the claim by constructing a functor $\widehat{\mathrm{Cat}}^{b,\Delta^1}_{cc} \to \widehat{\mathrm{Cat}}^{\Delta^1\times\Delta^1}_\infty$ which takes the bottom arrow in (8.5), evaluated on any (X,Y), to the entire diagram. Let us set $\mathcal{P}\mathrm{r}_{aa} := \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{acpl}} \times \widehat{\mathrm{Cat}}_\infty \times \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{acpl}}$ and $\mathcal{P}\mathrm{r}_{ac} := \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{acpl}} \times \widehat{\mathrm{Cat}}_\infty \times \mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{cpl}}$, defining $\widehat{\mathrm{Cat}}^{\Delta^1}_{aa}$, etc., as above. The main step will be to first construct a diagram

$$\begin{tikzcd} \widehat{\mathrm{Cat}}^{\Delta^1}_{aa} \arrow[r] & \widehat{\mathrm{Cat}}^{\Delta^1}_{ac} & \widehat{\mathrm{Cat}}^{\Delta^1}_{cc} \arrow[l] \\ \widehat{\mathrm{Cat}}^{b,\Delta^1}_{aa} \arrow[u, hook] \arrow[r, "\sim"] & \widehat{\mathrm{Cat}}^{b,\Delta^1}_{ac} \arrow[u, hook] & \widehat{\mathrm{Cat}}^{b,\Delta^1}_{cc} \arrow[u, hook] \arrow[l, "\sim"'] \end{tikzcd} \tag{8.11}$$

in which the bottom functors are equivalences, and such that under these equivalences the bottom arrow in (8.5) (as a $\widehat{\mathrm{Cat}}^{b,\Delta^1}_{cc}$-valued functor) corresponds to the top arrow and overall composition of (8.5) (respectively as a $\widehat{\mathrm{Cat}}^{b,\Delta^1}_{aa}$-valued and a $\widehat{\mathrm{Cat}}^{b,\Delta^1}_{ac}$-valued functor).

Let us explicitly construct the top left functor in (8.11) and show that it restricts to the equivalence on the bottom left; the construction of the right square is parallel. To do this we introduce the following pair of diagrams.

$$\begin{tikzcd} \widehat{\mathrm{Cat}}^{\Delta^2}_\infty \arrow[r] \arrow[d] & \widehat{\mathrm{Cat}}^{\Lambda^2_2}_\infty \arrow[r] \arrow[d] & \widehat{\mathrm{Cat}}^{\Delta^1_{02}}_\infty \arrow[d] \\ \widehat{\mathrm{Cat}}^{\Lambda^2_1}_\infty \arrow[r] \arrow[d] & \widehat{\mathrm{Cat}}^{\Delta^0_0\cup\Delta^1_{12}}_\infty \arrow[r] \arrow[d] & \widehat{\mathrm{Cat}}^{\Delta^0_0\cup\Delta^0_2}_\infty \\ \widehat{\mathrm{Cat}}^{\Delta^1_{01}}_\infty \arrow[r] & \widehat{\mathrm{Cat}}^{\Delta^0_0\cup\Delta^0_1}_\infty & \mathcal{P}\mathrm{r}_{aa} \arrow[l] \arrow[u] \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} \widehat{\mathrm{Cat}}^{\Delta^2}_{aac} \arrow[r] \arrow[d] & \widehat{\mathrm{Cat}}^{\Lambda^2_2}_{aac} \arrow[r] & \widehat{\mathrm{Cat}}^{\Delta^1_{02}}_{ac} \arrow[dd] \\ \widehat{\mathrm{Cat}}^{\Lambda^2_1}_{aac} \arrow[d] & & \\ \widehat{\mathrm{Cat}}^{\Delta^1_{01}}_{aa} \arrow[rr] & & \mathcal{P}\mathrm{r}_{aa} \end{tikzcd} \tag{8.12}$$

Here the subscripts in e.g. Δ^1_{02} indicate a particular 1-simplex of Δ^2, and the arrows in the left diagram not involving $\mathcal{P}\mathrm{r}_{aa}$ are induced by restriction. The unit of the localization $\mathcal{C} \mapsto \widehat{\mathcal{C}}$ on $\mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{rcpl}}$ induces a functor $\mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{acpl}} \to \widehat{\mathrm{Cat}}^{\Delta^1}_\infty$ taking $\check{\mathcal{C}}$ to $\check{\mathcal{C}} \to \widehat{\mathcal{C}}$, and the diagonal arrow out of $\mathcal{P}\mathrm{r}_{aa}$ is the induced functor $(\check{\mathcal{C}}, \mathcal{D}, \check{\mathcal{E}}) \mapsto (\check{\mathcal{C}} \times \mathcal{D}, \check{\mathcal{E}} \to \widehat{\mathcal{E}})$. The horizontal and vertical arrows out of $\mathcal{P}\mathrm{r}_{aa}$ thus take $(\check{\mathcal{C}}, \mathcal{D}, \check{\mathcal{E}})$ to $(\check{\mathcal{C}} \times \mathcal{D}, \check{\mathcal{E}})$ and $(\check{\mathcal{C}} \times \mathcal{D}, \widehat{\mathcal{E}})$, respectively.

In the right diagram, $\widehat{\mathrm{Cat}}^{\Delta^1_{01}}_{aa}$ and $\widehat{\mathrm{Cat}}^{\Delta^1_{02}}_{ac}$ are respectively the fiber products of the bottom row and right column of the left diagram (which is consistent with our existing notation after forgetting subscripts). The remaining three categories are the fiber products of their counterparts on the left with $\mathcal{P}\mathrm{r}_{aa}$ over $\widehat{\mathrm{Cat}}^{\Delta^0_0 \cup \Delta^1_{12}}_\infty$. Note that their natural maps to $\mathcal{P}\mathrm{r}_{aa}$ indeed factor through those of $\widehat{\mathrm{Cat}}^{\Delta^1_{01}}_{aa}$ and $\widehat{\mathrm{Cat}}^{\Delta^1_{02}}_{ac}$ as indicated.

We claim the leftmost vertical functors in the right diagram are equivalences. For the top, this follows since it is base changed from its counterpart on the left, which is an equivalence by [Lur09, Cor. 2.3.2.2]. For the bottom, this follows from the bottom left square of the left diagram being Cartesian. Composing the inverse equivalences with the top arrows we obtain a functor $\widehat{\mathrm{Cat}}^{\Delta^1_{01}}_{aa} \to \widehat{\mathrm{Cat}}^{\Delta^1_{01}}_{ac}$ as desired.

The fiber of this functor over a particular $(\check{\mathcal{C}}, \mathcal{D}, \check{\mathcal{E}}) \in \mathcal{P}\mathrm{r}_{aa}$ is the map $\mathrm{Fun}(\check{\mathcal{C}} \times \mathcal{D}, \check{\mathcal{E}})^{\cong} \to \mathrm{Fun}(\check{\mathcal{C}} \times \mathcal{D}, \widehat{\mathcal{E}})^{\cong}$ given by composition with $\check{\mathcal{E}} \to \widehat{\mathcal{E}}$ (the superscripts indicate that non-invertible natural transformations are excluded). Since the corresponding map $\mathrm{LFun}^b(\check{\mathcal{C}}, \check{\mathcal{E}}) \to \mathrm{LFun}^b(\check{\mathcal{C}}, \widehat{\mathcal{E}})$ is an equivalence by definition, it follows that $\widehat{\mathrm{Cat}}^{\Delta^1_{01}}_{aa} \to \widehat{\mathrm{Cat}}^{\Delta^1_{01}}_{ac}$ restricts to a functor $\widehat{\mathrm{Cat}}^{b,\Delta^1_{01}}_{aa} \to \widehat{\mathrm{Cat}}^{b,\Delta^1_{01}}_{ac}$ which in turns restricts to an isomorphism of fibers over $\mathcal{P}\mathrm{r}_{aa}$.

We recall that $\widehat{\mathrm{Cat}}^{\Delta^1}_\infty$ is a bifibration over $\widehat{\mathrm{Cat}}^2_\infty$ [Lur09, Cor. 2.4.7.11]. It follows from the definitions that bifbrations are stable under pullback along products of maps and under restriction to full subcategories. In particular, $\widehat{\mathrm{Cat}}^{b,\Delta^1_{01}}_{aa}$ and $\widehat{\mathrm{Cat}}^{b,\Delta^1_{01}}_{ac}$ are bifibrations over $\mathcal{P}\mathrm{r}_{aa}$, factored as the product of $\mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{acpl}} \times \widehat{\mathrm{Cat}}_\infty$ and $\mathcal{P}\mathrm{r}^{\mathrm{St,b}}_{\mathrm{acpl}}$. It now follows from [Lur09, Prop. 2.4.7.6] and the previous paragraph that $\widehat{\mathrm{Cat}}^{b,\Delta^1_{01}}_{aa} \to \widehat{\mathrm{Cat}}^{b,\Delta^1_{01}}_{ac}$ is an equivalence.

To complete the proof, note that by construction the bottom row of (8.11) factors as

$$\widehat{\mathrm{Cat}}^{b,\Delta^1_{01}}_{aa} \xleftarrow{\sim} \widehat{\mathrm{Cat}}^{b,\Delta^2}_{aac} \xrightarrow{\sim} \widehat{\mathrm{Cat}}^{b,\Delta^1_{02}}_{ac} \xleftarrow{\sim} \widehat{\mathrm{Cat}}^{b,\Delta^2}_{acc} \xrightarrow{\sim} \widehat{\mathrm{Cat}}^{b,\Delta^1_{12}}_{cc}.$$

Here we again use subscripts to indicate edges in Δ^2, $\widehat{\mathrm{Cat}}^{\Delta^2}_{acc}$ is the evident counterpart of $\widehat{\mathrm{Cat}}^{\Delta^2}_{aac}$, and $\widehat{\mathrm{Cat}}^{\Delta^2}_{acc}$, $\widehat{\mathrm{Cat}}^{\Delta^2}_{aac}$ are the full subcategories corresponding to $\widehat{\mathrm{Cat}}^{b,\Delta^1}_{aa}$. The middle terms in this factorization map to $\widehat{\mathrm{Cat}}^{\Delta^2}_\infty$, $\widehat{\mathrm{Cat}}^{\Delta^1}_\infty$, and $\widehat{\mathrm{Cat}}^{\Delta^2}_\infty$ compatibly with the relevant maps, hence we obtain a functor

$$\widehat{\mathrm{Cat}}^{b,\Delta^1}_{cc} \to \widehat{\mathrm{Cat}}^{\Delta^2}_\infty \times_{\widehat{\mathrm{Cat}}^{\Delta^1_{02}}_\infty} \widehat{\mathrm{Cat}}^{\Delta^2}_\infty \cong \widehat{\mathrm{Cat}}^{\Delta^1 \times \Delta^1}_\infty. \quad \square$$

8.2. **External and internal adjoints.** Let X and Z be ind-geometric stacks such that Z is reasonable, and let $\mathcal{F} \in \mathrm{Coh}(Z)$. We denote the right adjoint of $e_{\mathcal{F},X}$ by

$$e^R_{\mathcal{F},X} : \mathrm{IndCoh}(X \times Z) \to \mathrm{IndCoh}(X).$$

When X and Z are geometric and $\mathcal{F} \in \mathrm{QCoh}(X)$ we define $e^R_{\mathcal{F},X} : \mathrm{QCoh}(X \times Z) \to \mathrm{QCoh}(X)$ similarly. These are external counterparts of the internal sheaf Hom, which we define in the setting of ind-coherent sheaves on a reasonable ind-geometric stack X as

$$\mathcal{H}om(\mathcal{F}, -) := e^R_{\mathcal{F},X} \Delta_{X*} : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(X).$$

Here we again assume $\mathcal{F} \in \mathrm{Coh}(X)$. This definition is justified in part by the following result, and will be more fully justified by Corollary 8.43.

Proposition 8.13. *Let X and Z be ind-geometric stacks such that Z is reasonable, and let $\mathcal{F} \in \mathrm{Coh}(Z)$. Then $e^R_{\mathcal{F},X}$ is left bounded. If X and Z are geometric, the Beck-Chevalley map $\Psi_X e^R_{\mathcal{F},X}(\mathcal{G}) \to e^R_{\Psi_Z(\mathcal{F}),X} \Psi_{X \times Z}(\mathcal{G})$ is an isomorphism for all $\mathcal{G} \in \mathrm{IndCoh}(X \times Z)^+$, and the induced map $\Psi_X \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(\Psi_X(\mathcal{F}), \Psi_X(\mathcal{G}))$ is an isomorphism for all $\mathcal{G} \in \mathrm{IndCoh}(X)^+$.*

Proof. Since $e_{\mathcal{F},X}$ is bounded (Proposition 8.9), $e^R_{\mathcal{F},X}$ is left bounded and the two functors restrict to an adjunction between $\mathrm{IndCoh}(X)^+$ and $\mathrm{IndCoh}(X \times Z)^+$. The analogous statement holds for $e_{\Psi_Z(\mathcal{F}),X}$ and $e^R_{\Psi_Z(\mathcal{F}),X}$, and the second claim follows and since $\Psi_{(-)}$ restricts to an equivalence $\mathrm{IndCoh}(-)^+ \xrightarrow{\sim} \mathrm{QCoh}(-)^+$ and since $\Psi_{X \times Z} e_{\mathcal{F},X} \cong e_{\Psi_Z(\mathcal{F}),X} \Psi_X$ (Proposition 8.10). The third follows since Δ_{X*} is also compatible with the $\Psi_{(-)}$ functors, and since we have an isomorphism $- \otimes \Psi_Z(\mathcal{F}) \cong \Delta^*_X e_{\Psi_Z(\mathcal{F}),X}$ of functors $\mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$. □

Suppose that $f : X' \to X$, $g : Z' \to Z$ are morphisms of ind-finite cohomological dimension between ind-geometric stacks, and that Z' and Z are reasonable. Suppose also that either f and g are of finite Tor-dimension, or that they have coherent pullback and X, X', $X \times Z$, and $X' \times Z'$ are coherent. Then for $\mathcal{F} \in \mathrm{Coh}(Z)$ the isomorphism $(f \times g)^* e_{\mathcal{F},X} \cong e_{g^*(\mathcal{F}),X'} f^*$ of functors $\mathrm{IndCoh}(X) \to \mathrm{IndCoh}(X' \times Z')$ yields an isomorphism

$$e^R_{\mathcal{F},X}(f \times g)_* \cong f_* e^R_{g^*(\mathcal{F}),X'} \tag{8.14}$$

of right adjoints.

Similarly, suppose instead that f and g are ind-proper and that g is almost of ind-finite presentation. Then for $\mathcal{F} \in \mathrm{Coh}(Z')$ the isomorphism $(f \times g)_* e_{\mathcal{F},X'} \cong e_{g_*(\mathcal{F}),X} f_*$ of functors $\mathrm{IndCoh}(X') \to \mathrm{IndCoh}(X \times Z)$ yields an isomorphism

$$e^R_{\mathcal{F},X'}(f \times g)^! \cong f^! e^R_{g_*(\mathcal{F}),X} \tag{8.15}$$

of right adjoints.

Proposition 8.16. *Let X and Z be ind-geometric stacks such that Z is reasonable and X is semi-reasonable, and let $\mathcal{F} \in \mathrm{Coh}(Z)$. Then $e^R_{\mathcal{F},X}$ is almost continuous.*

Lemma 8.17. *Proposition 8.16 is true when X is affine and Z is geometric.*

Proof. In this case $\mathrm{QCoh}(X)$ is compactly generated by perfect sheaves. Thus $e^R_{\Psi_Z(\mathcal{F}),X}$ is almost continuous since $e_{\Psi_Z(\mathcal{F}),X}$ takes compact objects to almost compact objects by the proof of Lemma 3.21 (if X is not truncated and $\mathcal{G} \in \mathrm{QCoh}(X)$ is perfect, $e_{\Psi_Z(\mathcal{F}),X}(\mathcal{G})$ may be unbounded but is still almost perfect, hence almost compact [Lur18, Prop. 9.1.5.1]). It follows that $e^R_{\mathcal{F},X}$ is almost continuous since it and $e^R_{\Psi_Z(\mathcal{F}),X}$ are left bounded and intertwined by the equivalences $\Psi^+_{(-)}$ (Proposition 8.13). $\square$

Lemma 8.18. *Let X and Y be affine schemes, Z a reasonable geometric stack, and $\mathcal{F} \in \mathrm{Coh}(Z)$. If $h : X \to Y$ is a morphism of finite Tor-dimension and $h' = h \times id_Z$, then for any $\mathcal{G} \in \mathrm{IndCoh}(Y \times Z)^+$ the Beck-Chevalley map $h^* e^R_{\mathcal{F},Y}(\mathcal{G}) \to e^R_{\mathcal{F},X} h'^*(\mathcal{G})$ is an isomorphism.*

Proof. Write $X \cong \operatorname{Spec} A$ and $Y \cong \operatorname{Spec} B$, and assume first that Z is truncated and geometric. Since h is affine and $h^* e^R_{\mathcal{F},Y}(\mathcal{G})$, $e^R_{\mathcal{F},X} h'^*(\mathcal{G}) \in \mathrm{IndCoh}(X)^+$, it suffices to show the given map is an isomorphism after composing with $\Psi_Y h_*$. Since the second factor of

$$\Psi_Y h_* h^* e^R_{\mathcal{F},Y}(\mathcal{G}) \to \Psi_Y h_* e^R_{\mathcal{F},X} h'^*(\mathcal{G}) \to \Psi_Y e^R_{\mathcal{F},Y} h'_* h'^*$$

is an isomorphism by (8.14), it suffices to show the composition is. After commuting the given functors with Ψ_Y (hence using Proposition 8.13) and applying the projection formula, the composition becomes identified with the Beck-Chevalley map

$$\theta_M : e^R_{\Psi_Z(\mathcal{F}),Y} \Psi_{Y\times Z}(\mathcal{G}) \otimes M \to e^R_{\Psi_Z(\mathcal{F}),Y}(\Psi_{Y\times Z}(\mathcal{G}) \otimes p^*_Y(M)), \tag{8.19}$$

where $p_Y : Y \times Z \to Y$ is the projection and we substitute A for $M \in \mathrm{Mod}_B \cong \mathrm{QCoh}(Y)$.

Write $\mathcal{C}$ for the full subcategory of $M \in \mathrm{Mod}_B$ such that θ_M is an isomorphism. The assignment $M \mapsto \theta_M$ extends to a functor $\mathrm{Mod}_B \to \mathrm{Mod}_B^{\Delta^1}$, which is exact since both terms in (8.19) are exact in M. It follows that $\mathcal{C}$ is stable and closed under retracts, as isomorphisms form a stable subcategory closed under retracts in $\mathrm{Mod}_B^{\Delta^1}$. Clearly $B \in \mathcal{C}$, hence $\mathcal{C}$ contains all perfect B-modules. If A is of Tor-dimension $\leq n$ over B, then we can write it as a filtered colimit $A \cong \operatorname{colim}_\alpha M_\alpha$ of perfect B-modules of Tor-dimension $\leq n$ [Lur18, Prop. 9.6.7.1]. The claim now follows since tensoring is continuous, since the $\mathcal{G} \otimes p^*_Y(M_\alpha)$ are uniformly bounded below, and since $e^R_{\mathcal{F},Y}$ is almost continuous by Lemma 8.17. $\square$

Lemma 8.20. *Let X and Z be geometric stacks such that Z is reasonable, and let $\mathcal{F} \in \mathrm{Coh}(Z)$. If U is an affine scheme, $h : U \to X$ a flat cover, and $h' = h \times id_Z$, then for any $\mathcal{G} \in \mathrm{IndCoh}(X \times Z)^+$ the Beck-Chevalley map $h^* e^R_{\mathcal{F},X}(\mathcal{G}) \to e^R_{\mathcal{F},U} h'^*(\mathcal{G})$ is an isomorphism.*

Proof. Write $U_\bullet$ for the Cech nerve of h. Let $h_k : U_k \to X$ denote the natural map and let $h'_k := h_k \times id_Z$. Given a morphism $p : i \to j$ in Δ_s, let $h_p : U_j \to U_i$ denote the associated map and let $h'_p := h_p \times id_Z$. By construction we have compatible isomorphisms $e_{\mathcal{F},U_j} h^*_p(\mathcal{G}') \cong h'^*_p e_{\mathcal{F},U_i}(\mathcal{G}')$ for any p and any $\mathcal{G}' \in \mathrm{IndCoh}(U_i)$. By Lemma 8.18 the Beck-Chevalley map $h^*_p e^R_{\mathcal{F},U_i}(\mathcal{G}') \to e^R_{\mathcal{F},U_j} h'^*_p(\mathcal{G}')$ is an isomorphism for any p and any $\mathcal{G}' \in \mathrm{IndCoh}(U_i \times Z)^+$. Since h is faithfully flat, we have $\mathrm{IndCoh}(X)^+ \cong \lim_{\Delta_s} \mathrm{IndCoh}(U_i)^+$ and $\mathrm{IndCoh}(X \times Z)^+ \cong \lim_{\Delta_s} \mathrm{IndCoh}(U_i \times Z)^+$. The claim now follows from [Lur17, Cor. 4.7.5.18]. $\square$

Proof of Proposition 8.16. Suppose first that X and Z are geometric, and let $h : U \cong \operatorname{Spec} A \to X$ be a flat cover. Since $e^R_{\mathcal{F},X}$ is left bounded, and since h^* is conservative on $\operatorname{IndCoh}(X)^+$ in addition to being continuous, it suffices to show $h^* e^R_{\mathcal{F},X}$ is almost continuous. It then suffices to show $e^R_{\mathcal{F},U} h'^*$ is almost continuous, since by Lemma 8.20 it has the same restriction to $\operatorname{IndCoh}(X \times Z)^+$. But this follows since h'^* is continuous and left bounded, and since $e^R_{\mathcal{F},U}$ is almost continuous by 8.17.

Still assuming Z is geometric, let $X \cong \operatorname{colim} X_\alpha$ be a semi-reasonable presentation and $\mathcal{G} \cong \operatorname{colim} \mathcal{G}_\beta$ a filtered colimit in $\operatorname{IndCoh}(X \times Z)^{\geq n}$ for some n. Since $\operatorname{IndCoh}(X) \cong \lim \operatorname{IndCoh}(X_\alpha)$ in $\widehat{\mathrm{Cat}}_\infty$, it suffices to show the second factor in

$$\operatorname*{colim}_\beta i^!_\alpha e^R_{\mathcal{F},X}(\mathcal{G}_\beta) \to i^!_\alpha \operatorname*{colim}_\beta e^R_{\mathcal{F},X}(\mathcal{G}_\beta) \to i^!_\alpha e^R_{\mathcal{F},X}(\operatorname*{colim}_\beta \mathcal{G}_\beta)$$

is an isomorphism for all α. The first factor is an isomorphism since $e^R_{\mathcal{F},X}$ is left bounded and since $i^!_\alpha$ is almost continuous by the proof of Proposition 5.21. But $i^!_\alpha e^R_{\mathcal{F},X} \cong e^R_{\mathcal{F},X_\alpha}(i_\alpha \times id_Z)^!$ by (8.15), so the composition is an isomorphism by the first paragraph and the almost continuity and left t-exactness of $(i_\alpha \times id_Z)^!$.

Finally, suppose $Z \cong \operatorname{colim} Z_\alpha$ is a reasonable presentation, and write $\mathcal{F} \cong i_{\alpha*}(\mathcal{F}_\alpha)$ for some α and $\mathcal{F}_\alpha \in \operatorname{Coh}(Z_\alpha)$. By (8.15) we have $e^R_{\mathcal{F},X} \cong e^R_{\mathcal{F}_\alpha,X} i^!_\alpha$, and the claim follows since $i^!_\alpha$ is left t-exact and since $e^R_{\mathcal{F}_\alpha,X}$ and $i^!_\alpha$ are almost continuous. □

8.3. **External products and $*$-pullback.** Suppose that X, Y, and Z are ind-geometric stacks, that Z is reasonable, that $h : X \to Y$ is a morphism of finite Tor-dimension, and that $\mathcal{F} \in \operatorname{Coh}(Z)$. We have an isomorphism $e_{\mathcal{F},X} h^* \cong (h \times id_Z)^* e_{\mathcal{F},Y}$, hence an associated Beck-Chevalley map

$$h^* e^R_{\mathcal{F},Y}(\mathcal{G}) \to e^R_{\mathcal{F},X}(h \times id_Z)^*(\mathcal{G}) \tag{8.21}$$

for $\mathcal{G} \in \operatorname{IndCoh}(Y \times Z)$. We also have a corresponding map when h has coherent pullback and suitable coherence hypotheses are satisfied, and this section studies when (8.21) and related maps are isomorphisms.

In particular, results about these maps immediately imply more familiar statements about sheaf Hom. Suppose X and Y are reasonable ind-geometric stacks and $h : X \to Y$ is a morphism of finite Tor-dimension and ind-finite cohomological dimension. Then for any $\mathcal{F} \in \operatorname{Coh}(Y)$ and $\mathcal{G} \in \operatorname{IndCoh}(Y)$ there is a natural map

$$h^* \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(h^*(\mathcal{F}), h^*(\mathcal{G})) \tag{8.22}$$

given by the composition

$$h^* e^R_{\mathcal{F},Y} \Delta_{Y*}(\mathcal{G}) \to e^R_{\mathcal{F},X}(h \times id_Y)^* \Delta_{Y*}(\mathcal{G}) \to e^R_{\mathcal{F},X} \Delta'_{Y*} h^*(\mathcal{G}) \cong e^R_{h^*(\mathcal{F}),X} \Delta_{X*} h^*(\mathcal{G}). \tag{8.23}$$

Here $\Delta'_Y : Y \to X \times Y$ is the base change of Δ_Y, and the last isomorphism follows from (8.14) and $\Delta'_{Y*} \cong (id_X \times h)_* \Delta_{X*}$. When X and Y are geometric and $\mathcal{G} \in \operatorname{IndCoh}(Y)^+$, one can check using Proposition 8.13 that the equivalences $\Psi^+_{(-)} : \operatorname{IndCoh}(-)^+ \cong \operatorname{QCoh}(-)^+$ identify

(8.22) with the Beck-Chevalley map (8.1) associated to the isomorphism $h^*(- \otimes \Psi_Y(\mathcal{F})) \cong - \otimes h^*\Psi_Y(\mathcal{F})$.

If h has coherent pullback, we have a map (8.22) given by the same formula provided X, Y, $X \times X$, $X \times Y$, and $Y \times Y$ are coherent. In our cases of interest X and Y are coherent because they are tamely presented and affine over an ind-locally Noetherian geometric stack, and in this case $X \times X$, $X \times Y$, and $Y \times Y$ are automatically coherent (Proposition 6.9).

Remark 8.24. The finite cohomological dimension hypotheses above and throughout this section are satisfied in our applications, but can mostly be relaxed. Note that (8.14) remains true with such hypothesis, replacing $(f \times g)_*$ and f_* with the (not necessarily continuous) right adjoints $(f \times g)^{*R}$ and f^{*R}. However, even when these are continuous because of coherence hypotheses, this relaxation raises subtleties it is convenient to avoid.

Proposition 8.25. *Let X, Y, and Z be geometric stacks such that Z is reasonable, and let $\mathcal{F} \in \mathrm{Coh}(Z)$. If $h : X \to Y$ is a morphism of finite Tor-dimension, then for any $\mathcal{G} \in \mathrm{IndCoh}(Y \times Z)^+$ the Beck-Chevalley map $h^* e^R_{\mathcal{F},Y}(\mathcal{G}) \to e^R_{\mathcal{F},X}(h \times id_Z)^*(\mathcal{G})$ is an isomorphism.*

Proof. Set $h' = h \times id_Z$, and note first that we may assume X is affine. Otherwise choose a flat cover $g : U = \operatorname{Spec} A \to X$ and consider the following diagram of Cartesian squares.

$$\begin{tikzcd} U \times Z \arrow[r, "g'"] \arrow[d] & X \times Z \arrow[r, "h'"] \arrow[d] & Y \times Z \arrow[d] \\ U \arrow[r, "g"] & X \arrow[r, "h"] & Y \end{tikzcd}$$

We have $h^* e^R_{\mathcal{F},Y}(\mathcal{G}), e^R_{\mathcal{F},X} h'^*(\mathcal{G}) \in \mathrm{IndCoh}(X)^+$ since $h^* e^R_{\mathcal{F},Y}$ and $e^R_{\mathcal{F},X} h'^*$ are left bounded (Proposition 8.13). By faithful flatness the restriction of g^* to $\mathrm{IndCoh}(X)^+$ is conservative, so it suffices to show the first factor of

$$g^* h^* e^R_{\mathcal{F},Y}(\mathcal{G}) \to g^* e^R_{\mathcal{F},X} h'^*(\mathcal{G}) \to e^R_{\mathcal{F},U} g'^* h'^*(\mathcal{G}),$$

is an isomorphism. But the second factor is an isomorphism by Lemma 8.20, hence it suffices to show the composition is.

Now fix a flat cover $\phi : V = \operatorname{Spec} B \to Y$. We have a diagram

$$\begin{tikzcd} & U \arrow[dl, "\theta"'] \arrow[dd, "\psi"] & & U \times Z \arrow[ll] \arrow[dl, "\theta'"] \arrow[dd, "\psi'"] \\ X \arrow[dd, "h"'] & & X \times Z \arrow[ll] \arrow[dd, "h'"] & \\ & V \arrow[dl, "\phi"'] & & V \times Z \arrow[ll] \arrow[dl, "\phi'"] \\ Y & & Y \times Z \arrow[ll] & \end{tikzcd} \tag{8.26}$$

with Cartesian faces, and an associated diagram

$$\begin{tikzcd} \theta^* h^* e^R_{\mathcal{F},Y}(\mathcal{G}) \arrow[r] \arrow[d, "\wr"] & \theta^* e^R_{\mathcal{F},X} h'^*(\mathcal{G}) \arrow[r] & e^R_{\mathcal{F},U} \theta'^* h'^*(\mathcal{G}) \arrow[d, "\wr"] \\ \psi^* \phi^* e^R_{\mathcal{F},Y}(\mathcal{G}) \arrow[r] & \psi^* e^R_{\mathcal{F},V} \phi'^*(\mathcal{G}) \arrow[r] & e^R_{\mathcal{F},U} \psi'^* \phi'^*(\mathcal{G}) \end{tikzcd}$$

in $\mathrm{IndCoh}(U)$. Again by faithful flatness the restriction of θ^* to $\mathrm{IndCoh}(X)^+$ is conservative, hence it suffices to show the top left map is an isomorphism. Noting that U is affine since X and V are, the bottom right map is an isomorphism by Lemma 8.18 and flatness of ϕ'. But the top right and bottom left maps are isomorphisms by Lemma 8.20 and flatness of h', so the top left map must be as well. □

Corollary 8.27. *Let X and Y be reasonable geometric stacks and $h : X \to Y$ a morphism of finite Tor-dimension and ind-finite cohomological dimension. Then for any $\mathcal{F} \in \mathrm{Coh}(Y)$, $\mathcal{G} \in \mathrm{IndCoh}(Y)^+$ the natural map $h^* \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(h^*(\mathcal{F}), h^*(\mathcal{G}))$ is an isomorphism.*

Proof. Follows from Proposition 8.25 and the definition (8.23), as Δ_{Y*} is left t-exact. □

In relaxing the finite Tor-dimension hypothesis of Proposition 8.25 we will need to replace $e^R_{\mathcal{F},X}$ by its composition with a suitable generalized diagonal map. The example to keep in mind in the following statement is $Z = W = Y$, in which case Y' is Y and δ_Y is Δ_Y.

Proposition 8.28. *Let the following be a diagram of geometric stacks in which both squares are Cartesian, X and Y are truncated and coherent, h and g are of finite cohomological dimension and have semi-universal coherent pullback, and f is proper and almost of finite presentation.*

$$\begin{tikzcd} X' \arrow[r, "h'"] \arrow[d, "f''"'] & Y' \arrow[r, "g'"] \arrow[d, "f'"] & Z \arrow[d, "f"] \\ X \arrow[r, "h"] & Y \arrow[r, "g"] & W \end{tikzcd}$$

Suppose that $X \times Z$ and $Y \times Z$ are coherent and that X, Y, and Z are tamely presented. Then X' and Y' are coherent, and for any $\mathcal{F} \in \mathrm{Coh}(Z)$ and $\mathcal{G} \in \mathrm{IndCoh}(Y')$ the composition

$$h^* e^R_{\mathcal{F},Y} \delta_{Y*}(\mathcal{G}) \to e^R_{\mathcal{F},X}(h \times id_Z)^* \delta_{Y*}(\mathcal{G}) \to e^R_{\mathcal{F},X} \delta_{X*} h'^*(\mathcal{G}) \tag{8.29}$$

is an isomorphism, where $\delta_X : X' \to X \times Z$ and $\delta_Y : Y' \to Y \times Z$ are the natural maps.

Lemma 8.30. *Let X be a locally coherent geometric stack and $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(X)$. Then $\tau^{\leq n} \mathcal{H}om(\mathcal{F}, \mathcal{G})$ is coherent for all n.*

Proof. First suppose $A \in \mathrm{CAlg}_k$ is coherent and $M, N \in \mathrm{Coh}_A$. For any m there exists an exact triangle $P \to M \to Q$ such that P is perfect and $Q \in \mathrm{Mod}_A^{\leq m}$ [Lur18, Cor. 2.7.2.2], yielding an exact triangle

$$\mathcal{H}om(Q, N) \to \mathcal{H}om(M, N) \to \mathcal{H}om(P, N).$$

If $N \in \mathrm{Mod}^{\geq i}$ then $\mathcal{H}om(Q,N) \in \mathrm{Mod}^{\geq i-m}$, hence it follows from the associated long exact sequence that $H^j\,\mathcal{H}om(M,N) \cong H^j\,\mathcal{H}om(P,N)$ for $j < i-m-1$. Since P is perfect $\mathcal{H}om(P,N) \cong P^\vee \otimes N$, which is coherent since N is and since $P^\vee$ is perfect ([Lur17, Prop. 7.2.4.11, Prop. 7.2.4.23]). Since A is coherent $H^j\,\mathcal{H}om(M,N)$ is then finitely presented over $H^0(A)$ for $j < i-m-1$. But since m was arbitrary and $\mathcal{H}om(M,N)$ is bounded below, coherence of A then also implies $\tau^{\leq n}\,\mathcal{H}om(M,N)$ is coherent for all n.

Now let $f : \operatorname{Spec} A \to X$ be a flat cover such that A is coherent. By Proposition 8.13 we can conflate $\mathcal{F}$ and $\mathcal{G}$ with their images in $\mathrm{QCoh}(X)$. It suffices to show $f^*\tau^{\leq n}\,\mathcal{H}om(\mathcal{F},\mathcal{G})$ is coherent. We have $f^*\tau^{\leq n}\,\mathcal{H}om(\mathcal{F},\mathcal{G}) \cong \tau^{\leq n}\,\mathcal{H}om(f^*(\mathcal{F}), f^*(\mathcal{G}))$ by flatness and Corollary 8.27, hence the claim follows from the previous paragraph. ☐

Proof of Proposition 8.28. First note that X' and Y' are tamely presented by our hypotheses on X, Y, and f. The maps δ_X and δ_Y are affine since they are base changes of Δ_W (along $gh \times f$ and $g \times f$), hence X' and Y' are coherent by Proposition 6.2. In particular, the second factor of (8.29) is well-defined by Proposition 6.12. Moreover, $\mathrm{IndCoh}(Y')$ is compactly generated and all functors in the statement are continuous, so it suffices to assume $\mathcal{G} \in \mathrm{Coh}(Y')$.

We first claim that $h^* e^R_{\mathcal{F},Y}\delta_{Y*}(\mathcal{G})$ and $e^R_{\mathcal{F},X}\delta_{X*}h'^*(\mathcal{G})$ are bounded below. This is immediate for the latter but not the former, since $e^R_{\mathcal{F},Y}\delta_{Y*}(\mathcal{G})$ need not be coherent and h^* need not be left bounded. Fix a strictly tamely presented flat cover $\phi : V = \operatorname{Spec} A \to Y$ such that A is strictly tamely presented over k. Since Y is truncated there exists for some n a strictly tame presentation $A \cong \operatorname{colim} A_\alpha$ of order n over k. Set $V_\alpha = \operatorname{Spec} A_\alpha$. Using Noetherian approximation [Lur18, Prop. 4.2.1.5, Thm. 4.4.2.2, Prop. 5.5.4.1] and faithful flatness of the $u_\alpha : V \to V_\alpha$ we have, for some α, a diagram

$$\tag{8.31}
\begin{tikzcd}
 & W \arrow[dl, "\theta"'] \arrow[dd, "\psi"] & & W' \arrow[ll, "\xi'"'] \arrow[dl, "\theta'"] \arrow[dd, "\psi'"] \\
X \arrow[dd] & & X' \arrow[ll] \arrow[dd] & \\
 & V \arrow[dl, "\phi"'] \arrow[dd, "u_\alpha"] & & V' \arrow[ll, "\xi"'] \arrow[dl, "\phi'"] \arrow[dd, "u'_\alpha"] \\
Y & & Y' \arrow[ll] & \\
 & V_\alpha & & V'_\alpha \arrow[ll, "\xi_\alpha"']
\end{tikzcd}$$

in which all faces are Cartesian and ξ_α is proper and almost of finite presentation. Since f is representable and almost of finite presentation g' has semi-universal coherent pullback and $g'^*(\mathcal{F}) \in \mathrm{Coh}(Y')$. Increasing α, we may then assume by [Lur18, Thm. 4.5.12.3], flatness of the u_α, and coherence of the A_α that there exist $\mathcal{F}_\alpha, \mathcal{G}_\alpha \in \mathrm{Coh}(Y'_\alpha)$ such that $\phi'^* g'^*(\mathcal{F}) \cong u'^*_\alpha(\mathcal{F}_\alpha)$ and $\phi'^*(\mathcal{G}) \cong u'^*_\alpha(\mathcal{G}_\alpha)$.

Letting $\delta'_Y : Y' \to Y \times Y'$ denote the natural map, we have isomorphisms

$$e^R_{\mathcal{F},Y}\delta_{Y*}(\mathcal{G}) \cong e^R_{g'^*(\mathcal{F}),Y}\delta'_{Y*}(\mathcal{G}) \cong f'_* e^R_{g'^*(\mathcal{F}),Y'}\Delta_{Y'*}(\mathcal{G}) \cong f'_* \mathcal{H}om(g'^*(\mathcal{F}), \mathcal{G}).$$

Here the first uses (8.14) and the fact that $\delta_Y \cong (id_Y \times g') \circ \delta'_Y$, the second uses Proposition 8.40 (which does not depend on the current Proposition) and the fact that $\delta'_Y \cong (f' \times id_{Y'}) \circ \Delta_{Y'}$, and the last is by definition. Since $\operatorname{IndCoh}(Y')$ is right complete and $h^* f'_*$ is continuous we then have $h^* e^R_{\mathcal{F},Y}\delta_{Y*}(\mathcal{G}) \cong \operatorname{colim} h^* f'_* \tau^{\leq n} \mathcal{H}om(g'^*(\mathcal{F}), \mathcal{G})$.

Now $h^* f'_*$ preserves coherence, hence $h^* f'_* \tau^{\leq n} \mathcal{H}om(g'^*(\mathcal{F}), \mathcal{G})$ is coherent for all n by Lemma 8.30. Since the t-structure on $\operatorname{IndCoh}(X)$ is compatible with filtered colimits, $h^* e^R_{\mathcal{F},Y}\delta_{Y*}(\mathcal{G})$ is then bounded below if $h^* f'_* \tau^{\leq n} \mathcal{H}om(g'^*(\mathcal{F}), \mathcal{G})$ is uniformly bounded below in n. Since θ is faithfully flat, hence θ^* t-exact and conservative on $\operatorname{IndCoh}(X)^+$, it suffices to show $\theta^* h^* f'_* \tau^{\leq n} \mathcal{H}om(g'^*(\mathcal{F}), \mathcal{G})$ is uniformly bounded below in n. But we have

$$\begin{aligned}
\theta^* h^* f'_* \tau^{\leq n} \mathcal{H}om(g'^*(\mathcal{F}), \mathcal{G}) &\cong \psi^* \xi_* \phi'^* \tau^{\leq n} \mathcal{H}om(g'^*(\mathcal{F}), \mathcal{G}) \\
&\cong \psi^* \xi_* \tau^{\leq n} \mathcal{H}om(\phi'^*(g'^*(\mathcal{F})), \phi'^*(\mathcal{G})) \\
&\cong \psi^* \xi_* \tau^{\leq n} \mathcal{H}om(u'^*_\alpha(\mathcal{F}_\alpha), u'^*_\alpha(\mathcal{G}_\alpha)) \\
&\cong \psi^* \xi_* u'^*_\alpha \tau^{\leq n} \mathcal{H}om(\mathcal{F}_\alpha, \mathcal{G}_\alpha) \\
&\cong \psi^* u^*_\alpha \xi_{\alpha*} \tau^{\leq n} \mathcal{H}om(\mathcal{F}_\alpha, \mathcal{G}_\alpha),
\end{aligned}$$

where the second and fourth isomorphisms use Corollary 8.27 and flatness of ϕ' and u'_α. The claim now follows since $u_\alpha \circ \psi$ is of finite Tor-dimension by Proposition 3.13.

Note that we may now assume X is affine and strictly tamely presented over $\operatorname{Spec} k$. Otherwise choose a flat cover $u : U = \operatorname{Spec} B \to X$ by such an affine, letting $u' : U' \to X'$ denote its base change and $\delta_U : U' \to U \times Z$ the base change of δ_Y. By faithful flatness the restriction of u^* to $\operatorname{IndCoh}(X)^+$ is conservative, so it suffices to show the first factor of

$$u^* h^* e^R_{\mathcal{F},Y}\delta_{Y*}(\mathcal{G}) \to u^* e^R_{\mathcal{F},X}\delta_{X*}h'^*(\mathcal{G}) \to e^R_{\mathcal{F},U}\delta_{U*}u'^* h'^*(\mathcal{G}),$$

is an isomorphism. But the second factor is an isomorphism by Lemma 8.20, hence it suffices to show the composition is.

Returing to (8.31), it now follows that W is affine and strictly tamely presented over $\operatorname{Spec} k$, and that $W \times Z$, $V \times Z$, W', and V' are tamely presented. Proposition 6.2 now implies $W \times Z$, $V \times Z$, W', and V' are coherent since $X \times Z$, $Y \times Z$, X', and Y' are, and since ϕ and its base changes are affine. We then have a diagram

$$\begin{tikzcd}
\theta^* h^* e^R_{\mathcal{F},Y}\delta_{Y*}(\mathcal{G}) \arrow[r] \arrow[d, "\wr"] & \theta^* e^R_{\mathcal{F},X}\delta_{X*}h'^*(\mathcal{G}) \arrow[r] & e^R_{\mathcal{F},W}\delta_{W*}\theta'^* h'^*(\mathcal{G}) \arrow[d, "\wr"] \\
\psi^* \phi^* e^R_{\mathcal{F},Y}\delta_{Y*}(\mathcal{G}) \arrow[r] & \psi^* e^R_{\mathcal{F},V}\delta_{V*}\phi'^*(\mathcal{G}) \arrow[r] & e^R_{\mathcal{F},W}\delta_{W*}\psi'^*\phi'^*(\mathcal{G}),
\end{tikzcd} \tag{8.32}$$

where $\delta_W : W' \to W \times Z$ and $\delta_V : V' \to V \times Z$ are the base changes of δ_Y, and where the bottom right map is well-defined by Proposition 6.12. As in the previous paragraph, it suffices by faithful flatness of θ to show the top left map is an isomorphism. The top right and bottom left maps are by Lemma 8.20, so it further suffices to show the bottom right is.

Let $\delta'_V : V' \to V \times V'$ and $\delta'_W : W' \to W \times V'$ denote the natural maps, noting that $\delta_V \cong (id_V \times g'\phi') \circ \delta'_V$ and $\delta_W \cong (id_W \times g'\phi') \circ \delta'_W$. Since $V \times V'$ is an algebraic space, $\mathrm{QCoh}(V \times V')$ is compactly generated by perfect sheaves [Lur18, Prop. 9.6.1.1]. But $V \times V'$ is also tamely presented, hence locally coherent. Thus $\mathcal{H}^0 : \mathrm{QCoh}(V \times V') \to \mathrm{QCoh}(V \times V')^\heartsuit$ preserves compactness, hence $\mathrm{QCoh}(V \times V')^\heartsuit$ is compactly generated, hence $V \times V'$ is coherent. Similarly $W \times V'$ is coherent. We then have a diagram

$$\begin{tikzcd}
\psi^* e^R_{\mathcal{F},V} \delta_{V*} \arrow[r, "\sim"] \arrow[d] & \psi^* e^R_{\mathcal{F},V} (id_V \times g'\phi')_* \delta'_{V*} \arrow[r, "\sim"] \arrow[d] & \psi^* e^R_{\phi'^* g'^*(\mathcal{F}),V} \delta'_{V*} \arrow[d] \\
e^R_{\mathcal{F},W} \delta_{W*} \psi'^* \arrow[r, "\sim"] & e^R_{\mathcal{F},W} (id_W \times g'\phi')_* \delta'_{W*} \psi'^* \arrow[r, "\sim"] & e^R_{\phi'^* g'^*(\mathcal{F}),W} \delta'_{W*} \psi'^*,
\end{tikzcd}$$

where the right square is well-defined by (8.14) and the left by Proposition 6.12.

In particular, the bottom right map in (8.32) is an isomorphism if and only if the map $\psi^* e^R_{\phi'^* g'^*(\mathcal{F}),V} \delta'_{V*} u'^*_\alpha(\mathcal{G}_\alpha) \to e^R_{\phi'^* g'^*(\mathcal{F}),W} \delta'_{W*} \psi'^* u'^*_\alpha(\mathcal{G}_\alpha)$ is an isomorphism (recall that $\phi'^*(\mathcal{G}) \cong u'^*_\alpha(\mathcal{G}_\alpha)$). Varying the above argument with $V \times V'_\alpha$ in place of $V \times Z$, we see that this is the case if and only if the second factor of

$$\psi^* u^*_\alpha e^R_{\mathcal{F}_\alpha, V_\alpha} \delta''_{V_\alpha *}(\mathcal{G}_\alpha) \to \psi^* e^R_{\mathcal{F}_\alpha, V} \delta''_{V*} u'^*_\alpha(\mathcal{G}_\alpha) \to e^R_{\mathcal{F}_\alpha, W} \delta''_{W*} \psi'^* u'^*_\alpha(\mathcal{G}_\alpha)$$

is an isomorphism, where $\delta''_W : W' \to W \times V'_\alpha$, $\delta''_V : V' \to V \times V'_\alpha$ are the natural maps. But the first factor and the composition are isomorphisms by Proposition 8.25, since u_α is flat and $u_\alpha \circ \psi$ is of finite Tor-dimension. □

In generalizing Proposition 8.28 to ind-geometric stacks we will use the following terminology. We say a (not necessarily geometric) morphism $f : X \to Y$ of ind-geometric stacks is ind-flat (resp. of ind-finite Tor-dimension) if X can be written as a filtered colimit $X \cong \operatorname{colim} X_\alpha$ of ind-geometric stacks along ind-closed immersions $i_{\alpha\beta} : X_\alpha \to X_\beta$ such that $f \circ i_\alpha : X_\alpha \to Y$ is flat (resp. of finite Tor-dimension) for all α. For example, the condition that k is an ordinary ring of finite global dimension implies that any ind-geometric stack X is of ind-finite Tor-dimension over $\operatorname{Spec} k$. Similarly, if X and Y are reasonable we say f has inductively semi-universal coherent pullback if X can be written as a filtered colimit $X \cong \operatorname{colim} X_\alpha$ of reasonable ind-geometric stacks along almost ind-finitely presented closed immersions such that $f \circ i_\alpha : X_\alpha \to Y$ has semi-universal coherent pullback for all α.

Proposition 8.33. *Let the following be a diagram of ind-geometric stacks in which both squares are Cartesian, X and Y are coherent, W and Z are reasonable, h and g are of ind-finite cohomological dimension, h has semi-universal coherent pullback, g has inductively* *semi-universal coherent pullback, and f is ind-proper and almost of ind-finite presentation.*

$$\begin{tikzcd} X' \arrow[r, "h'"] \arrow[d, "f''"'] & Y' \arrow[r, "g'"] \arrow[d, "f'"] & Z \arrow[d, "f"] \\ X \arrow[r, "h"] & Y \arrow[r, "g"] & W \end{tikzcd}$$

Suppose that $X \times Z$ and $Y \times Z$ are coherent and that X, Y, and Z are ind-tamely presented. Then X' and Y' are coherent, and for any $\mathcal{F} \in \mathrm{Coh}(Z)$ and $\mathcal{G} \in \mathrm{IndCoh}(Y')$ the composition

$$h^* e^R_{\mathcal{F},Y} \delta_{Y*}(\mathcal{G}) \to e^R_{\mathcal{F},X}(h \times id_Z)^* \delta_{Y*}(\mathcal{G}) \to e^R_{\mathcal{F},X} \delta_{X*} h'^*(\mathcal{G})$$

is an isomorphism, where $\delta_X : X' \to X \times Z$ and $\delta_Y : Y' \to Y \times Z$ are the natural maps.

Proof. Suppose first that Z is truncated and geometric, and that g has semi-universal coherent pullback (not just inductively so). If $W \cong \operatorname{colim} W_\alpha$ is a reasonable presentation, then for some α we can refine the given diagram to a diagram

$$\begin{tikzcd} X' \arrow[r, "h'"] \arrow[d, "f''_\alpha"'] & Y' \arrow[r, "g'"] \arrow[d, "f'_\alpha"] & Z \arrow[d, "f_\alpha"] \\ X_\alpha \arrow[r, "h_\alpha"] \arrow[d, "i''_\alpha"'] & Y_\alpha \arrow[r, "g_\alpha"] \arrow[d, "i'_\alpha"] & W_\alpha \arrow[d, "i_\alpha"] \\ X \arrow[r, "h"] & Y \arrow[r, "g"] & W \end{tikzcd}$$

of Cartesian squares in which f_α is proper and almost of finite presentation. Proposition 4.18, our hypotheses on k, and coherence of X, Y, $X \times Z$, and $Y \times Z$ imply that X_α, Y_α, $X_\alpha \times Z$, and $Y_\alpha \times Z$ are truncated and coherent, and that h_α and g_α have coherent pullback. Thus X' and Y' are coherent by Proposition 8.28. Letting $\delta_{X_\alpha} : X' \to X_\alpha \times Z$, $\delta_{Y_\alpha} : Y' \to Y_\alpha \times Z$ denote the natural maps, we have an associated diagram

$$\begin{tikzcd} h^* e^R_{\mathcal{F},Y} \delta_{Y*}(\mathcal{G}) \arrow[rr] \arrow[d, "\wr"] & & e^R_{\mathcal{F},X} \delta_{X*} h'^*(\mathcal{G}) \arrow[d, "\wr"] \\ h^* i'_{\alpha*} e^R_{\mathcal{F},Y_\alpha} \delta_{Y_\alpha *}(\mathcal{G}) \arrow[r] & i''_{\alpha*} h^*_\alpha e^R_{\mathcal{F},Y_\alpha} \delta_{Y_\alpha *}(\mathcal{G}) \arrow[r] & i''_{\alpha*} e^R_{\mathcal{F},X_\alpha} \delta_{X_\alpha *} h'^*(\mathcal{G}). \end{tikzcd}$$

Here the vertical isomorphisms are given by Proposition 8.40 (which does not depend on the current Proposition) and the identities $\delta_X \cong (i''_\alpha \times id_Z) \circ \delta_{X_\alpha}$, $\delta_Y \cong (i'_\alpha \times id_Z) \circ \delta_{Y_\alpha}$. But the bottom maps are isomorphisms by Propositions 6.12 and 8.28, hence so is the top map.

Now suppose only that g has semi-universal coherent pullback, and let $Z \cong \operatorname{colim} Z_\alpha$ be a reasonable presentation. For each α the given diagram extends to a diagram

$$\begin{tikzcd}
X_\alpha \arrow[r, "h_\alpha"] \arrow[d, "i''_\alpha"'] & Y_\alpha \arrow[r, "g_\alpha"] \arrow[d, "i'_\alpha"] & Z_\alpha \arrow[d, "i_\alpha"] \\
X' \arrow[r, "h'"] \arrow[d, "f''"'] & Y' \arrow[r, "g'"] \arrow[d, "f'"] & Z \arrow[d, "f"] \\
X \arrow[r, "h"] & Y \arrow[r, "g"] & W
\end{tikzcd}$$

of Cartesian squares. We have reasonable presentations $X' \cong \operatorname{colim} X_\alpha$ and $Y' \cong \operatorname{colim} Y_\alpha$ by left exactness of filtered colimits in $\widehat{\mathrm{Stk}}_k$ and since the X_α and Y_α are truncated (as g_α and $g_\alpha \circ h_\alpha$ have coherent pullback). Since $X \times Z$ and $Y \times Z$ are coherent so are $X \times Z_\alpha$ and $Y \times Z_\alpha$ by Proposition 6.7. By the previous paragraph X_α and Y_α are then coherent for every α, hence X' and Y' are coherent by Proposition 6.4.

Write $\mathcal{F} \cong i_{\alpha *}(\mathcal{F}_\alpha)$ for some α and some $\mathcal{F}_\alpha \in \mathrm{Coh}(Z_\alpha)$. Letting $\delta_{X_\alpha} : X_\alpha \to X \times Z_\alpha$, $\delta_{Y_\alpha} : Y_\alpha \to Y \times Z_\alpha$ denote the natural maps, we have a diagram

$$\begin{tikzcd}
h^* e^R_{\mathcal{F},Y} \delta_{Y*}(\mathcal{G}) \arrow[rr] \arrow[d, "\wr"] & & e^R_{\mathcal{F},X} \delta_{X*} h'^*(\mathcal{G}) \arrow[d, "\wr"] \\
h^* e^R_{\mathcal{F}_\alpha,Y} \delta_{Y_\alpha *} i'^!_\alpha(\mathcal{G}) \arrow[r] & e^R_{\mathcal{F}_\alpha,X} \delta_{X_\alpha *} h^*_\alpha i'^!_\alpha(\mathcal{G}) \arrow[r] & e^R_{\mathcal{F}_\alpha,X} \delta_{X_\alpha *} i''^!_\alpha h'^*(\mathcal{G}).
\end{tikzcd}$$

Here the vertical isomorphisms are given by (8.15) and Proposition 7.11. But the bottom maps are isomorphisms by the first paragraph and Proposition 7.5, hence so is the top map.

Finally, suppose that $Y \cong \operatorname{colim} Y_\alpha$ expresses Y as a filtered colimit of reasonable ind-geometric stacks along almost ind-finitely presented ind-closed immerions such that $g \circ i_\alpha : Y_\alpha \to W$ has coherent pullback for all α. For any α we have a diagram

$$\begin{tikzcd}
& X_\alpha \arrow[dl, "j_\alpha"'] \arrow[dd, "h_\alpha"] & & X_\alpha \times Z \arrow[ll] \arrow[dl] \arrow[dd] & & X'_\alpha \arrow[ll, "\delta_{X_\alpha}"'] \arrow[dl, "j'_\alpha"] \arrow[dd, "h'_\alpha"] \\
X \arrow[dd, "h"'] & & X \times Z \arrow[ll] \arrow[dd] & & X' \arrow[ll, "\delta_X"'] \arrow[dd, "h'"] & \\
& Y_\alpha \arrow[dl, "i_\alpha"'] & & Y_\alpha \times Z \arrow[ll] \arrow[dl] & & Y'_\alpha \arrow[ll, "\delta_{Y_\alpha}"'] \arrow[dl, "i'_\alpha"] \\
Y & & Y \times Z \arrow[ll] & & Y' \arrow[ll, "\delta_Y"'] &
\end{tikzcd}$$

with Cartesian faces. Proposition 4.11 implies that i_α is an almost ind-finitely presented ind-closed immersion, and Proposition 6.7 implies that X_α, Y_α, $X_\alpha \times Z$, and $Y_\alpha \times Z$ are coherent and j_α, $(i_\alpha \times id_Z)$, and $(i_\alpha \times id_Z)$ are almost ind-finitely presented ind-closed immersions. It then follows from the previous paragraphs that X'_α and Y'_α are coherent for all α. We have $X' \cong \operatorname{colim} X'_\alpha$ and $Y' \cong \operatorname{colim} Y'_\alpha$ by left exactness of filtered colimits in $\widehat{\mathrm{Stk}}_k$, and since the maps $j'_{\alpha\beta} : X_\alpha \to X_\beta$, $i'_{\alpha\beta} : Y_\alpha \to Y_\beta$ are almost ind-finitely presented ind-closed immersions by Proposition 6.7, X' and Y' are then coherent by Proposition 6.4.

For every α we now have a diagram

$$\begin{tikzcd} h_\alpha^* i_\alpha^! e^R_{\mathcal{F},Y} \delta_{Y*}(\mathcal{G}) \arrow[r] \arrow[d, "\wr"] & j_\alpha^! h^* e^R_{\mathcal{F},Y} \delta_{Y*}(\mathcal{G}) \arrow[r] & j_\alpha^! e^R_{\mathcal{F},X} \delta_{X*} h'^*(\mathcal{G}) \arrow[d, "\wr"] \\ h_\alpha^* e^R_{\mathcal{F},Y_\alpha} \delta_{Y_\alpha *} i'^!_\alpha(\mathcal{G}) \arrow[r] & e^R_{\mathcal{F},X_\alpha} \delta_{X_\alpha *} h'^*_\alpha i'^!_\alpha(\mathcal{G}) \arrow[r] & e^R_{\mathcal{F},X_\alpha} \delta_{X_\alpha *} j'^!_\alpha h'^*(\mathcal{G}) \end{tikzcd}$$

in $\operatorname{IndCoh}(X_\alpha)$. Here the vertical isomorphisms are given by (8.15) and Proposition 7.11. Since the functors $j_\alpha^!$ determine an isomorphism $\operatorname{IndCoh}(X) \cong \lim \operatorname{IndCoh}(X_\alpha)$ in $\widehat{\mathrm{Cat}}_\infty$, it suffices to show that the top right map is an isomorphism for all α. But the top left and bottom right maps are isomorphisms by Proposition 7.5 and the bottom left is by the previous paragraphs, hence the top right is as well. □

Corollary 8.34. *Let X and Y be coherent, ind-tamely presented ind-geometric stacks such that $X \times Y$ and $Y \times Y$ are coherent. Let $h : X \to Y$ be a morphism of ind-finite finite cohomological dimension with semi-universal coherent pullback. Then for any $\mathcal{F} \in \operatorname{Coh}(Y)$, $\mathcal{G} \in \operatorname{IndCoh}(Y)$ the natural map $h^* \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(h^*(\mathcal{F}), h^*(\mathcal{G}))$ is an isomorphism.*

Proof. Follows from Proposition 8.33, taking $Z = W = Y$. □

8.4. **External products and pushforward.** If X, Y, and Z are geometric stacks and $\mathcal{F} \in \operatorname{QCoh}(Z)$, then for any $f : X \to Y$ the isomorphism $(f \times id_Z)^* e_{\mathcal{F},Y} \cong e_{\mathcal{F},X} f^*$ of functors $\operatorname{QCoh}(Y) \to \operatorname{QCoh}(X \times Z)$ yields an isomorphism

$$f_* e^R_{X,\mathcal{F}} \cong e^R_{Y,\mathcal{F}} (f \times id_Z)_* \tag{8.35}$$

of right adjoints $\operatorname{QCoh}(X \times Z) \to \operatorname{QCoh}(Y)$. This is an external counterpart of the isomorphism

$$\mathcal{H}om(\mathcal{F}, f_*(-)) \cong f_* \mathcal{H}om(f^*(\mathcal{F}), -)$$

obtained for $\mathcal{F} \in \operatorname{QCoh}(X)$ by taking right adjoints of the isomorphism $f^*(- \otimes \mathcal{F}) \cong f^*(-) \otimes f^*(\mathcal{F})$. Similarly, if f is proper the projection isomorphism $f_*(\mathcal{F} \otimes f^*(-)) \cong f_*(\mathcal{F}) \otimes -$ yields an isomorphism

$$f_* \mathcal{H}om(\mathcal{F}, f^!(-)) \cong \mathcal{H}om(f_*(\mathcal{F}), -) \tag{8.36}$$

of right adjoints.

This section generalizes these isomorphisms to ind-coherent sheaves under suitable hypotheses, letting X, Y, and Z be ind-geometric and $f : X \to Y$ a morphism of ind-finite cohomological dimension. In this setting $f_* : \operatorname{IndCoh}(X) \to \operatorname{IndCoh}(Y)$ typically does not have a left adjoint. Instead, if Z is reasonable and $\mathcal{F} \in \operatorname{Coh}(Z)$, we may take the isomorphism

$(f \times id_Z)_* e_{\mathcal{F},X} \cong e_{\mathcal{F},Y} f_*$ of functors $\mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y \times Z)$ and consider the associated Beck-Chevalley transformation

$$f_* e^R_{\mathcal{F},X} \to e^R_{\mathcal{F},Y}(f \times id_Z)_* \tag{8.37}$$

of functors $\mathrm{IndCoh}(X \times Z) \to \mathrm{IndCoh}(Y)$.

Suppose in addition that X and Y are reasonable and that f is ind-proper and almost of ind-finite presentation. Then for $\mathcal{F} \in \mathrm{Coh}(Y)$ and $\mathcal{G} \in \mathrm{IndCoh}(Y)$ we have a transformation

$$f_* \mathcal{H}om(\mathcal{F}, f^!(-)) \to \mathcal{H}om(f_*(\mathcal{F}), -) \tag{8.38}$$

of functors $\mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(Y)$ given by the composition

$$f_* e^R_{\mathcal{F},X} \Delta_{X*} f^! \to e^R_{\mathcal{F},Y}(f \times id_X)_* \Delta_{X*} f^! \to e^R_{\mathcal{F},Y}(id_Y \times f)^! \Delta_{Y*} \cong e^R_{f_*(\mathcal{F}),Y} \Delta_{Y*}. \tag{8.39}$$

Here the last isomorphism is given by (8.15), and we have used the fact that $\Delta_X \circ (f \times id_X)$ is the base change of Δ_Y along $id_Y \times f$. In the geometric case one can check that if we restrict to bounded below subcategories, (8.38) and (8.37) are indeed identified with the isomorphisms (8.35) and (8.36) under the equivalences $\mathrm{IndCoh}(-)^+ \cong \mathrm{QCoh}(-)^+$.

Proposition 8.40. *Let X, Y, and Z be ind-geometric stacks such that Z is reasonable and Y is semi-reasonable. Let $f : X \to Y$ be a morphism of ind-finite cohomological dimension and $f' = f \times id_Z$. Then for any $\mathcal{F} \in \mathrm{Coh}(Z)$ and $\mathcal{G} \in \mathrm{IndCoh}(X \times Z)^+$ the Beck-Chevalley map $f_* e^R_{\mathcal{F},X}(\mathcal{G}) \to e^R_{\mathcal{F},Y} f'_*(\mathcal{G})$ is an isomorphism.*

Proof. First assume X, Y, and Z are geometric and Z is truncated. By Proposition 8.13 all functors involved are left bounded and compatible with the equivalences $\mathrm{IndCoh}(-)^+ \cong \mathrm{QCoh}(-)^+$, hence the claim follows from (8.35).

Now let $X \cong \operatorname{colim} X_\alpha$ be an ind-geometric presentation, still supposing Y and Z are geometric and Z is truncated. For every α we have a diagram

$$\begin{tikzcd} X_\alpha \times Z \arrow[r, "i'_\alpha"] \arrow[d] & X \times Z \arrow[r, "f'"] \arrow[d] & Y \times Z \arrow[d] \\ X_\alpha \arrow[r, "i_\alpha"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

of Cartesian squares. Expanding the Beck-Chevalley maps in terms of units and counits, it follows from the basic properties of these that the counits $i_{\alpha*} i^!_\alpha \to id_{IC(X)}$ and $i'_{\alpha*} i'^!_\alpha \to id_{IC(X \times Z)}$ fit into diagrams

$$\begin{tikzcd} f_* i_{\alpha*} i^!_\alpha e^R_{\mathcal{F},X}(\mathcal{G}) \arrow[r] \arrow[d] & f_* i_{\alpha*} e^R_{\mathcal{F},X_\alpha} i'^!_\alpha(\mathcal{G}) \arrow[r] & e^R_{\mathcal{F},Y} f'_* i'_{\alpha*} i'^!_\alpha(\mathcal{G}) \arrow[d] \\ f_* e^R_{\mathcal{F},X}(\mathcal{G}) \arrow[rr] & & e^R_{\mathcal{F},Y} f'_*(\mathcal{G}) \end{tikzcd} \tag{8.41}$$

compatibly for $\beta \geq \alpha$. That is, the counits $i_{\alpha*}i_\alpha^! \cong i_{\beta*}i_{\alpha\beta*}i_{\alpha\beta}^!i_\beta^! \to i_{\beta*}i_\beta^!$ and $i'_{\alpha*}i'^!_\alpha \cong i'_{\beta*}i'_{\alpha\beta*}i'^!_{\alpha\beta}i'^!_\beta \to i'_{\beta*}i'^!_\beta$ intertwine the top compositions in (8.41) for $\beta \geq \alpha$. Passing to colimits we obtain a diagram

$$\begin{tikzcd}
\operatorname{colim} f_*i_{\alpha*}i_\alpha^! e^R_{\mathcal{F},X}(\mathcal{G}) \arrow[r] \arrow[d] & \operatorname{colim} e^R_{\mathcal{F},Y} f'_* i'_{\alpha*}i'^!_\alpha(\mathcal{G}) \arrow[d] \\
f_*e^R_{\mathcal{F},X}(\mathcal{G}) \arrow[r] & e^R_{\mathcal{F},Y} f'_*(\mathcal{G}).
\end{tikzcd}$$

The vertical arrows are isomorphisms by Lemma A.2, the continuity of f_* and f'_*, and the left t-exactness of $f'_*i'_{\alpha*}i'^!_\alpha$ together with the almost continuity of $e^R_{\mathcal{F},Y}$ (Proposition 8.16). But the top arrow is an isomorphism since the top arrows in (8.41) are by (8.15) and the previous paragraph, hence so is the bottom arrow.

Now let $Y \cong \operatorname{colim} Y_\alpha$ be a semi-reasonable presentation and write $\mathcal{F} \cong i_*(\mathcal{F}')$ for some truncated geometric substack $i : Z' \to Z$ and $\mathcal{F}' \in \operatorname{Coh}(Z')$. For any α we have a diagram

$$\begin{tikzcd}
& X_\alpha \arrow[dl, "j_\alpha"'] \arrow[dd, "f_\alpha" near start] & & X_\alpha \times Z \arrow[ll] \arrow[dl, "j'_\alpha"] \arrow[dd, "f'_\alpha"] \\
X \arrow[dd, "f"'] & & X \times Z \arrow[ll, crossing over] & \\
& Y_\alpha \arrow[dl, "i_\alpha"'] & & Y_\alpha \times Z \arrow[ll] \arrow[dl, "i'_\alpha"] \\
Y & & Y \times Z \arrow[ll] \arrow[from=uu, crossing over, "f'" near start] &
\end{tikzcd}$$

with all faces but the top and bottom Cartesian, and with $i'_\alpha = i_\alpha \times i$. We have a diagram

$$\begin{tikzcd}
f_{\alpha*}j_\alpha^! e^R_{\mathcal{F},X}(\mathcal{G}) \arrow[r] \arrow[d, "\wr"] & i_\alpha^! f_* e^R_{\mathcal{F},X}(\mathcal{G}) \arrow[r] & i_\alpha^! e^R_{\mathcal{F},Y} f'_*(\mathcal{G}) \arrow[d, "\wr"] \\
f_{\alpha*}e^R_{\mathcal{F},X_\alpha} j'^!_\alpha(\mathcal{G}) \arrow[r] & e^R_{\mathcal{F},Y_\alpha} f'_{\alpha*} j'^!_\alpha(\mathcal{G}) \arrow[r] & e^R_{\mathcal{F},Y_\alpha} i'^!_\alpha f'_*(\mathcal{G})
\end{tikzcd}$$

in $\operatorname{IndCoh}(Y_\alpha)$, where the vertical isomorphisms are given by (8.15). Since the functors $i_\alpha^!$ determine an isomorphism $\operatorname{IndCoh}(Y) \cong \lim \operatorname{IndCoh}(Y_\alpha)$ in $\widehat{\operatorname{Cat}}_\infty$, it suffices to show the top right map is an isomorphism for all α. The top left and bottom right maps are isomorphisms by Proposition 7.13, right t-exactness of $i_{\alpha*}$ and $i'_{\alpha*}$, and left boundedness of $e^R_{\mathcal{F},X}$. But the bottom left is an isomorphism by the previous paragraph and left t-exactness of $j'^!_\alpha$, hence the top right is as well. $\square$

Corollary 8.42. *Let X, Y, and Z be ind-geometric stacks such that Z is reasonable and X, Y, $X \times Z$, and $Y \times Z$ are coherent. Let $f : X \to Y$ be a morphism of ind-finite cohomological dimension and $f' = f \times id_Z$. Then for any $\mathcal{F} \in \operatorname{Coh}(Z)$ and $\mathcal{G} \in \operatorname{IndCoh}(X \times Z)$ the Beck-Chevalley map $f_*e^R_{\mathcal{F},X}(\mathcal{G}) \to e^R_{\mathcal{F},Y} f'_*(\mathcal{G})$ is an isomorphism.*

Proof. Under these hypotheses $f_* e^R_{\mathcal{F},X}(\mathcal{G})$ and $e^R_{\mathcal{F},Y} f'_*$ are continuous and $\mathrm{IndCoh}(X \times Z)$ is compactly generated by $\mathrm{Coh}(X \times Z) \subset \mathrm{IndCoh}(X \times Z)^+$, hence the claim follows from Proposition 8.40. ☐

Note that the extension of sheaf Hom from the geometric to the ind-geometric setting is uniquely determined by the following corollary, since we can always write $\mathcal{F} \in \mathrm{Coh}(X)$ as $i_*(\mathcal{F}')$ for some reasonable geometric substack $i : X' \to X$ and $\mathcal{F}' \in \mathrm{Coh}(X')$.

Corollary 8.43. *Let X and Y be reasonable ind-geometric stacks, $f : X \to Y$ an ind-proper, almost ind-finitely presented morphism of finite cohomological dimension, and $\mathcal{F} \in \mathrm{Coh}(X)$. Then for any $\mathcal{G} \in \mathrm{IndCoh}(Y)^+$ the natural map $f_* \mathcal{H}om(\mathcal{F}, f^!(\mathcal{G})) \to \mathcal{H}om(f_*(\mathcal{F}), \mathcal{G})$ is an isomorphism.*

Proof. Under these hypotheses $\Delta_{X*} f^!$ is left bounded, hence the claim follows from Propositions 7.13 and 8.40. ☐

Corollary 8.44. *Let X and Y be coherent ind-geometric stacks such that $X \times X$, $X \times Y$, and $Y \times Y$ are coherent, $f : X \to Y$ an ind-proper, almost of ind-finitely presented morphism, and $\mathcal{F} \in \mathrm{Coh}(X)$. Then for any $\mathcal{G} \in \mathrm{IndCoh}(Y)$ the natural map $f_* \mathcal{H}om(\mathcal{F}, f^!(\mathcal{G})) \to \mathcal{H}om(f_*(\mathcal{F}), \mathcal{G})$ is an isomorphism.*

Proof. Follows from Proposition 7.11 and Corollary 8.42. ☐

Appendix A. Conventions and notation

Our notation generally follows [Lur18] unless otherwise specified. We write Cat_∞ (resp. $\widehat{\mathrm{Cat}}_\infty$) for the ∞-category of small (resp. not necessarily small) ∞-categories, $\mathcal{S} \subset \mathrm{Cat}_\infty$ for the ∞-category of spaces, Sp for the ∞-category of spectra, and $\mathcal{P}r^{\mathrm{L}} \subset \widehat{\mathrm{Cat}}_\infty$ for the category of presentable ∞-categories and functors which preserve small colimits. We use the terms category and ∞-category interchangeably, and say ordinary category when we specifically mean a category in the traditional sense. Given morphisms $f : X \to Y$, $g : Y \to Z$ in an ∞-category $\mathcal{C}$, we often refer by abuse to the composition $g \circ f$, with the understanding that this is only well-defined up to homotopy.

Our most significant departure from [Lur18] is that we use cohomological indexing and notation for t-structures. Thus if $\mathcal{C}$ has a t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ with heart $\mathcal{C}^{\heartsuit}$ we write $\tau^{\leq n} : \mathcal{C} \to \mathcal{C}^{\leq n}$, $H^n : \mathcal{C} \to \mathcal{C}^{\heartsuit}$, etc., for the associated functors. This is of course more consistent with the general conventions in algebraic geometry, but it does create some awkwardness in that it remains most convenient to write $\tau_{\leq n}$ for the subcategory of n-truncated objects in an ∞-category $\mathcal{D}$. Thus, for example, $\tau_{\leq n}(\mathcal{C}^{\leq 0})$ and $\mathcal{C}^{[-n,0]}$ refer to the same subcategory of $\mathcal{C}$. The reader can remain oriented by distinguishing subscripts from superscripts, which should respectively be read homologically and cohomologically.

If $\mathcal{C} \in \widehat{\mathrm{Cat}}_\infty$ is monoidal we write $\mathrm{CAlg}(\mathcal{C})$ for the category of commutative algebra objects of $\mathcal{C}$. When $\mathcal{C} \cong \mathrm{Sp}^{\leq 0}$ is the category of connective spectra we omit it from the notation, so that CAlg denotes the category of connective $\mathbb{E}_\infty$-rings (we omit the superscript used in [Lur18] since we never consider nonconnective $\mathbb{E}_\infty$-rings). Its full subcategory $\tau_{\leq 0}\mathrm{CAlg}$ is equivalently the ordinary category of ordinary commutative rings.

Given $A \in \mathrm{CAlg}$, we write $\mathrm{CAlg}_A := \mathrm{CAlg}_{A/}$ for the category of commutative A-algebras, $\tau_{\leq n}\mathrm{CAlg}_A$ for the subcategory of n-truncated algebras, and $\tau_{<\infty}\mathrm{CAlg}_A := \cup_n \tau_{\leq n}\mathrm{CAlg}_A$ for its subcategory of truncated algebras (i.e. n-truncated for some n). If A is an ordinary ring containing $\mathbb{Q}$ then CAlg_A is equivalently the (enhanced homotopy) category of nonpositively graded commutative dg algebras. We write Mod_A for the category of A-modules (i.e. A-module objects in the category of spectra). If A is an ordinary ring this is the (enhanced) unbounded derived category of ordinary A-modules (i.e. of $\mathrm{Mod}_A^\heartsuit$).

Throughout the text we fix a Noetherian base $k \in \mathrm{CAlg}$. That is, $H^0(k)$ is an ordinary Noetherian ring and $H^n(k)$ is finitely generated over $H^0(k)$ for all $n < 0$.

We write PStk_k for the category of prestacks over $\operatorname{Spec} k$, i.e. functors $\mathrm{CAlg}_k \to \mathcal{S}$. Similarly we write $\widehat{\mathrm{PStk}}_k$, $\mathrm{PStk}_{k,\leq n}$ for the categories of functors from $\tau_{<\infty}\mathrm{CAlg}_k$, $\tau_{\leq n}\mathrm{CAlg}_k$ to $\mathcal{S}$. A stack will mean a prestack which is a sheaf for the fpqc topology on CAlg_k [Lur18, Prop. B.6.1.3], and we denote the category of stacks by $\mathrm{Stk}_k \subset \mathrm{PStk}_k$. Similarly we write $\widehat{\mathrm{Stk}}_k \subset \widehat{\mathrm{PStk}}_k$, $\mathrm{Stk}_{k,\leq n} \subset \mathrm{PStk}_{k,\leq n}$ for the subcategories of fpqc sheaves on $\tau_{<\infty}\mathrm{CAlg}_k$, $\tau_{\leq n}\mathrm{CAlg}_k$ (note that $\tau_{<\infty}\mathrm{CAlg}_k$ does not admit arbitrary pushouts, but for the application of [Lur18, Prop. A.3.2.1] in defining the fpqc topology closure under flat pushouts is sufficient). We write Spec for the Yoneda embedding $\mathrm{CAlg}_k \to \mathrm{Stk}_k$.

Given $k' \in \mathrm{CAlg}_k$, the natural functor $\mathrm{PStk}_{k'} \to (\mathrm{PStk}_k)_{/\operatorname{Spec} k'}$ is an equivalence by [Lur09, Cor. 5.1.6.12], and we record the following analogue for stacks.

Lemma A.1. *The equivalence* $\mathrm{PStk}_{k'} \to (\mathrm{PStk}_k)_{/\operatorname{Spec} k'}$ *restricts to an equivalence* $\mathrm{Stk}_{k'} \to (\mathrm{Stk}_k)_{/\operatorname{Spec} k'}$.

We will make use several times of the following standard fact (c.f. [Gai, Sec. 1.3.4]).

Lemma A.2. *Let* $\mathcal{C} \cong \operatorname{colim} \mathcal{C}_\alpha$ *be a small colimit in* $\mathcal{P}\mathrm{r}^{\mathrm{L}}$*, with* $F_\alpha : \mathcal{C}_\alpha \to \mathcal{C}$ *the canonical functors and* $G_\alpha : \mathcal{C} \to \mathcal{C}_\alpha$ *their right adjoints. Then any* $X \in \mathcal{C}$ *can be written as* $X \cong \operatorname{colim} F_\alpha G_\alpha(X)$.

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