Projective Shape Space

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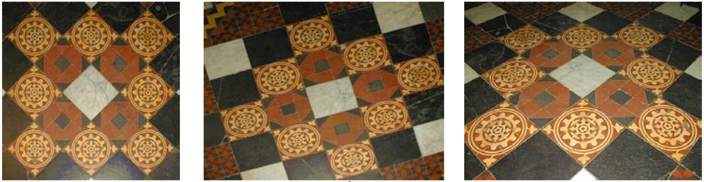
Projective Shape

Motivation (Scene's projective shape)



a b c

- m-dimensional images of a scene comprising of k landmarks
- Images differ only by a projective transformation between themselves and from the original scene
- Even if the images are taken with different camera angles

Mathematical Representation of the Projective Shape

Elements of $\mathbb{R}P^m$

$$[\mathbf{x}] = [\mathbf{x}^1 : \cdots : \mathbf{x}^{m+1}] = \{\lambda\mathbf{x} : \lambda \neq 0\}$$

- $\mathbf{x} = (\mathbf{x}^1, \cdots, \mathbf{x}^{m+1})^\top \in \mathbb{R}^{m+1} \backslash \{\mathbf{0}\}$
- lines through the origin in $\mathbb{R}^{m+1}$

A projective transformation α on $\mathbb{R}P^m$

$$\alpha([\mathbf{x}]) = [\mathbf{A}\mathbf{x}] \text{ , where}$$

- $\mathbf{A} \in \mathbf{GL}(m+1, \mathbb{R})$
- $\mathbf{PGL}(m)$: Group of all projective transformation on $\mathbb{R}P^m$

Mathematical Representation of the Projective Shape

k-**ad** (a scene or object described by configuration $\mathbf{y}$) :

$$\mathbf{y} = (\mathbf{y}_1, \cdots, \mathbf{y}_k) \in (\mathbb{R}P^m)^k \text{ , where}$$

- A finite, ordered set of landmarks $\mathbf{y}_i \in \mathbb{R}P^m$

Projective shape of $\mathbf{y}$:

$$[\mathbf{y}] = \{\alpha\mathbf{y} = (\alpha\mathbf{y}_1, \cdots, \alpha\mathbf{y}_k) : \alpha \in \mathbf{PGL}(m)\}$$

- Orbit of $\mathbf{y}$ under the component-wise action of $\mathbf{PGL}(m)$

The projective shape space

$$P_0\Sigma_m^k = (\mathbb{R}P^m)^k / \mathbf{PGL}(m)$$

Mathematical Representation of the Projective Shape

Alternatively, a configuration is given as a matrix form

$$\mathbf{Y} = \begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_k \end{pmatrix} \in \mathbb{R}^{k \times (m+1)}$$

Note :

1. Left-multiplication of $\mathbf{Y}$ with $\mathbf{D} \in \mathrm{Diag}(\lambda_1, \cdots, \lambda_k)$, where $\lambda_i \neq 0$, corresponds to same configuration in $\mathbb{R}P^m$
2. Projective transformations act as right multiplication with $\mathbf{B} \in \mathbf{GL}(m+1, \mathbb{R})$

Projective shape of a configuration matrix $\mathbf{Y}$

$$[\mathbf{Y}] = \{\mathbf{DYB} : \mathbf{D} \in \mathrm{Diag}(\lambda_1, \cdots, \lambda_k), \mathbf{B} \in \mathbf{GL}(m+1, \mathbb{R})\}$$

Mathematical Representation of the Projective Shape

Question : $P_0\Sigma_m^k$ is a manifold ?

Answer : No ! $P_0\Sigma_m^k$ is **neither metrizable nor a manifold**

- ✗ Definig a metric for comparison of shapes
- ✗ Performing statistical Inference

Solution : Find a topological subspace of $P_0\Sigma_m^k$

- ✔ **Mardia and Patrangenaru (2005)** used **projective frames** to define a topological subspace of projective shape space

Projective Shape with Projective frame

Let $\mathbf{V}$ be a vector space, and $\mathbf{0_v}$ be the zero vector in $\mathbf{V}$

Projective Space of $\mathbf{V}$

$$P(\mathbf{V}) = \{[\mathbf{x}] = \{\lambda\mathbf{x} : \lambda \neq \mathbf{0}\} : \mathbf{x} \in \mathbf{V}\backslash\{\mathbf{0_v}\}\}$$

Example : $\mathbb{R}P^m = P(\mathbb{R}^{m+1})$

d-dimensional Projective Subspace of $\mathbb{R}P^m$

$P(\mathbf{V})$, where $\mathbf{V}$ is a $d+1$ dimensional vector subspace of $\mathbb{R}^{m+1}$

Linear span of $\mathbf{D} \subseteq \mathbb{R}\mathbf{P^m}$

The smallest projective subspace of $\mathbb{R}P^m$ containing $\mathbf{D}$

General Position

k-ad $\mathbf{y} = (\mathbf{y}_1, \cdots, \mathbf{y}_k) \in (\mathbb{R}P^m)^k$ is said to be **General position** if

The linear span of k-ad is $\mathbb{R}P^m$

$\Leftrightarrow$ The smallest projective subspace of k-ad is $\mathbb{R}P^m$

$\Leftrightarrow$ Any $m+1$ points in k-ad spans $\mathbb{R}P^m$

In matrix notation

$$\mathbf{Y} = \begin{pmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_k \end{pmatrix} \text{ is general position if}$$

any subset of $m+1$ rows of $\mathbf{Y}$ is linearly independent

Projective Frame

Projective Frame in $\mathbb{R}P^m$

An ordered set of $\mathbf{m+2}$ points in **general position**.

Example : (Standard Projective Frame)

Let $(\mathbf{e_1}, \cdots, \mathbf{e_{m+1}})$ be the standard basis of $\mathbb{R}^{m+1}$

Then the standard projective frame is given by

$$\boldsymbol{\pi}_0 = ([\mathbf{e_1}], \cdots, [\mathbf{e_{m+1}}], [\mathbf{e_1} + \cdots \mathbf{e_{m+1}}])$$

Proposition

Let $(\mathbf{p_1}, \cdots \mathbf{p_{m+2}}), (\mathbf{q_1}, \cdots \mathbf{q_{m+2}})$ be two projective frames of $\mathbb{R}P^m$, then there exists a unique $\alpha \in \mathbf{PGL}(m)$, s.t.

$$\mathbf{p}_i = \alpha(\mathbf{q}_i)\ ,\ i = 1, \cdots, m+2$$

Projective Coordinates

Given a projective frame $\boldsymbol{\pi} = (\boldsymbol{\pi}_1, \cdots \boldsymbol{\pi}_{m+2})$.

The projective coordinates

The projective coordinates $\mathbf{p}^{\boldsymbol{\pi}}$ of $\mathbf{p} \in \mathbb{R}P^m$ w.r.t $\boldsymbol{\pi}$

$$\mathbf{p}^{\boldsymbol{\pi}} = \alpha^{-1}(\mathbf{p}) \ , \ \text{where } \alpha \in \mathbf{PGL}(m) \ , \ \text{s.t.,}$$

$$\begin{cases} \boldsymbol{\pi}_i = \alpha([\mathbf{e}_i]) & i = 1, \cdots, m+1 \\ \boldsymbol{\pi}_{m+2} = \alpha([\mathbf{e}_1 + \cdots + \mathbf{e}_{m+1}]) & i = m+2 \end{cases}$$

- $\mathbf{p}^{\boldsymbol{\pi}}$ is the preimage of $\mathbf{p}$ under the projective transformation α.
- The projective shape space represented w.r.t the projective coordinates has a manifold structure

Projective Shape Space (Mardia and Patrangenaru, 2005)

Idea : For $k > m + 2$,

1. Fix the first $m + 2$ landmarks as a projective frame
2. Then the projective shape is uniquely determined by the projective coordinates of the remaining $k - m - 2$ landmarks

Then the Projective shape space of k-ad in $\mathbb{R}P^m$

$$P\Sigma_m^k = \mathcal{G}_m^k / \mathbf{PGL}(m)$$

- $\mathcal{G}_m^k$ denotes the set of k-ad $(\underbrace{\mathbf{p}_1, \cdots, \mathbf{p}_{m+2}}_{\boldsymbol{\pi}}, \mathbf{p}_{m+3}, \cdots, \mathbf{p}_k)$
- $\mathbf{PGL}(m)$ acts on $\mathcal{G}_m^k$ by $\alpha(\mathbf{p}_1, \cdots, \mathbf{p}_k) = (\alpha\mathbf{p}_1, \cdots, \alpha\mathbf{p}_k)$

How do we get the Projective Coordinates ?

Given a projective frame $\boldsymbol{\pi} = (\boldsymbol{\pi}_1, \cdots \boldsymbol{\pi}_{m+2})$.

For $\mathbf{x} \in \mathbb{R}^m$, and $\mathbf{p} = (\mathbf{x}^\top, 1)^\top$, Define the followings

1. $\mathbf{\Pi} = [\boldsymbol{\pi}_1, \cdots, \boldsymbol{\pi}_{m+1}] \in \mathbb{R}^{(m+1)\times(m+1)}$
2. $\mathbf{v}(\mathbf{p}) = \left(\mathbf{v}^1(\mathbf{p}), \cdots, \mathbf{v}^{m+1}(\mathbf{p})\right)^\top = \mathbf{\Pi}^{-1}\mathbf{p}$
3. $\mathbf{v}(\boldsymbol{\pi}_{m+2}) = \left(\mathbf{v}^1(\boldsymbol{\pi}_{m+2}), \cdots, \mathbf{v}^{m+1}(\boldsymbol{\pi}_{m+2})\right)^\top = \mathbf{\Pi}^{-1}\boldsymbol{\pi}_{m+2}$

Then the projective coordinate of $\mathbf{p}$ is given by

$$\mathbf{p}^{\boldsymbol{\pi}} = \left[\frac{\mathbf{v}^1(\mathbf{p})}{\mathbf{v}^1(\boldsymbol{\pi}_{m+2})} : \cdots : \frac{\mathbf{v}^{m+1}(\mathbf{p})}{\mathbf{v}^{m+1}(\boldsymbol{\pi}_{m+2})}\right]$$

Properties of $P\Sigma_m^k$

Note : Since $\mathbf{v}(\boldsymbol{\pi}_i) = \mathbf{e}_i \in \mathbb{R}^{m+1}$, for $i = 1, \cdots, m+1$

$$\because \boldsymbol{\Pi}\mathbf{e}_i = \boldsymbol{\pi}_i \ \rightarrow \ \mathbf{e}_i = \boldsymbol{\Pi}^{-1}\boldsymbol{\pi}_i = \mathbf{v}(\boldsymbol{\pi}_i)$$

Thus, we have

$$\boldsymbol{\pi}_i^{\boldsymbol{\pi}} = \mathbf{e}_i \ , \ \text{for } i = 1, \cdots, m+1$$
$$\boldsymbol{\pi}_{m+2}^{\boldsymbol{\pi}} = \mathbf{e}_1 + \cdots \mathbf{e}_{m+1}$$

1. Projective frame allows us to register the shape

$$\mathbf{P} = \begin{pmatrix} 1 & & & 1 & & & \\ & \ddots & & \vdots & \mathbf{p}_{m+3}^{\boldsymbol{\pi}} & \cdots & \mathbf{p}_k^{\boldsymbol{\pi}} \\ & & 1 & 1 & & & \end{pmatrix}^\top$$

Projective Coordinates

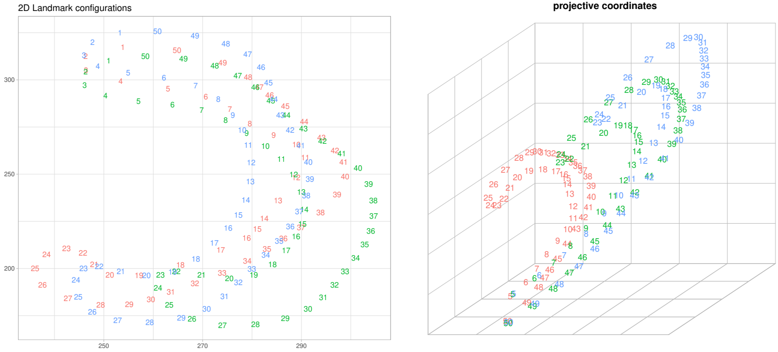


Figure 1: Projective Coordinates of the C.C shapes

Properties of $P\Sigma_m^k$

2. $P\Sigma_m^k$ is an open dense subset of $P_0\Sigma_m^k$

3. $P\Sigma_m^k$ is manifold diffeomorphic with $(\mathbb{R}P^m)^{k-m-2}$

   Define $\phi : P\Sigma_m^k \to (\mathbb{R}P^m)^q$

$$\phi\left((\mathbf{p}_1, \cdots, \mathbf{p}_k) \bmod \mathbf{PGL}(m)\right) = \left(\frac{\mathbf{p}_{m+3}^{\boldsymbol{\pi}}}{\|\mathbf{p}_{m+3}^{\boldsymbol{\pi}}\|}, \cdots, \frac{\mathbf{p}_k^{\boldsymbol{\pi}}}{\|\mathbf{p}_k^{\boldsymbol{\pi}}\|}\right)$$

   Then ϕ is a well-defined diffeomorphism

   Proof : Mardia and Patrangenaru (2005)

References

Mardia, K. V. and Patrangenaru, V. (2005). Directions and projective shapes. *Annals of Statistics*, 33:1666–1699.