Basic Projective Geometry

Hwiyoung Lee

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Department of Statistics, Florida State University

Homogeneous Coordinate

Homogeneous Coordinate

- Represents coordinates in 2D with a 3-vector

$$\begin{bmatrix} X \\ Y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

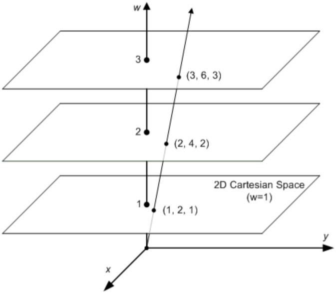
- Homogeneous coordinate to Cartesian coordinate

$$\underbrace{(x, y, w)}_{\text{Homogeneous}} \Leftrightarrow \underbrace{\left(\frac{x}{w}, \frac{y}{w}\right)}_{\text{Cartesian}}$$

Why is it called Homogeneous ?

Example : Homogeneous Coordinate $\Rightarrow$ Cartesian coordinate

$$(1,2,1) \Rightarrow (1,2)$$
$$(2,4,2) \Rightarrow (1,2)$$
$$\vdots$$
$$(1w,2w,1w) \Rightarrow (1,2)$$



The homogeneous coordinate is **Scale invariant**

Consider points $(1w, 2w, 1w)$ w.r.t Homogeneous coordinate

- Correspond to point $(1, 2)$ in Cartesian coordinate
- These points are homogeneous

Homogeneous Coordinate

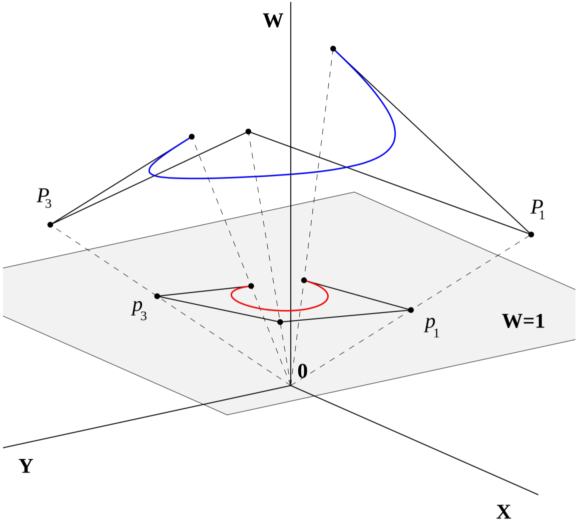


Figure 1: Graphical Illustration of the Homogeneous Coordinates

1. Why do we use the Homogeneous Coordinate ?

Question : Two parallel lines can intersect ?

- Railroad becomes narrower while it moves far away from eyes
- Finally they meet at the horizon (**the point at infinity**)

Difference between Euclidean and Projective geometry

Euclidean Geometry

- Euclidean geometry describes shapes **“as they are”**
- Two parallel lines on the plane **cannot** intersect

Projective Geometry

- Projective geometry describes objects **“as they appear”**
- Study geometric properties related to projective trans
- Angles, Parallelism : **“distorted”** when we look at objects

Two Parallel lines in 2D

Euclidean Geometry : Cartesian Coordinate

$$\begin{cases} Ax + By + C = 0 \\ Ax + By + D = 0 \end{cases}$$

- No Solution ($\because C \neq D$)

Projective Geometry : Homogeneous Coordinate

$$\begin{cases} A\frac{x}{w} + B\frac{y}{w} + C = 0 \\ A\frac{x}{w} + B\frac{y}{w} + D = 0 \end{cases} \Rightarrow \begin{cases} Ax + By + Cw = 0 \\ Ax + By + Dw = 0 \end{cases}$$

- Solution : $(x, y, 0) \quad \because (C - D)w = 0$

Represent Infinity point

Q. What if the point $(1, 2)$ moves toward to infinity?

- **Cartesian Coordinate** : (∞, ∞)
  - Meaningless
  - Cannot treat infinity like a regular number

- **Homogeneous Coordinate** : $(1, 2, 0)$
  - Can represent infinity without using ∞
  - $w = 0$: also called ideal point

Note : Many geometric concepts and computations can be greatly simplified if the concept of infinity is used.

2. Why do we use the Homogeneous Coordinate ?

$(x, y)^\top \in \mathbb{R}^2$, represent a transformation by matrix multiplication

1. Rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2. Scaling

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

3. Translations : Can't be represented as a matrix multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

In computer Vision, need to do a series of translations.

Why do we use the Homogeneous Coordinate ?

Example :Transformation

1. **Scale, Translate**
2. **then Rotate and Scale**
3. **then Translate again**

In Cartesian coordinates system : $\mathbf{x} = (x, y)^\top \in \mathbb{R}^2$

- The above transformation

$$\mathbf{x}' = \mathbf{SR}(\mathbf{Sx} + \mathbf{T}) + \mathbf{T}$$

- The inverse of the transformation ? $(\mathbf{x}' \rightarrow \mathbf{x})$

$$\mathbf{S}^{-1}\left(\mathbf{R}^{-1}\mathbf{S}^{-1}(\mathbf{x}' - \mathbf{T}) - \mathbf{T}\right) = \mathbf{x}$$

Why do we use the Homogeneous Coordinate ?

In Homogeneous coordinates system : $\mathbf{p} = (x, y, 1)^\top \in \mathbb{R}^3$

- The same transformation

$$\mathbf{p}' = \mathbf{T}_H \mathbf{S}_H \mathbf{R}_H \mathbf{T}_H \mathbf{S}_H \mathbf{x} = \mathbf{H}\mathbf{p}$$

- The inverse of the transformation ? $(\mathbf{x}' \rightarrow \mathbf{x})$

$$\mathbf{H}^{-1}\mathbf{p}' = \mathbf{p}$$

Note : Using the homogeneous coordinates is **More convenient !!**

A Projective Transformation

Basic 2D Transformations as 3×3 Matrices

1. **Translation**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2. **Scale**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

3. **Rotation**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D Transformations Matrices

4. **Isometric Transformation** : preserve Euclidean distance

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \epsilon\cos\theta & -\sin\theta & t_x \\ \epsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{H}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}, \text{ where } \epsilon = \pm 1$$

- Invariant : length, area, angle between lines
- $\epsilon = 1$: Planar Euclidean transformation

$$\mathbf{H}_E = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

- $\epsilon = -1$: Reverses orientation (composition of a reflection)

$$\mathbf{H}_R = \text{Diag}(-1, 1, 1) \circ \mathbf{H}_E$$

- D.o.f : 3 (θ, t_x, t_y)

5. **Similarity Transformation**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{H_S}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Block form :

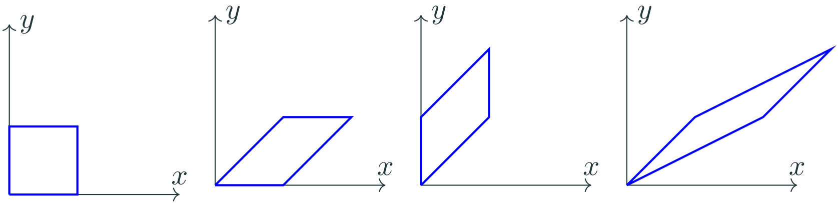
$$\mathbf{H_S} = \begin{bmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}, \text{ where}$$

  s denotes the isotropic scaling
- Composition of an isometry and an isotropic scaling
- Preserves the shape, angle between lines
- D.o.f : 4 (s, θ, t_x, t_y)

2D Transformations as 3×3 Matrices

6. **Shear** : slants the shape of object

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & sh_x & 0 \\ sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$



(a) Original

(b) Shear in X by 1 $(sh_x = 1, sh_y = 0)$

(c) Shear in Y by 1 $(sh_x = 0, sh_y = 1)$

(d) Shear X, Y by $(2, 1)$ $(sh_x = 2, sh_y = 1)$

2D Transformations Matrices

7. **Affine Transformation**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{H_A}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Block form:

$$\mathbf{H_A} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix}, \text{ where } \mathbf{A} : \text{Nonsingular matrix}$$

- Nonsingular linear transformation followed by a translation
- **Preserves the parallel lines**
- D.o.f : 6

2D Transformations Matrices

- $\mathbf{A}$ can be decomposed by

$$\begin{aligned}\mathbf{A} &= \mathbf{U}\mathbf{D}\mathbf{V}^\top \\ &= (\mathbf{U}\mathbf{V}^\top)(\mathbf{V}\mathbf{D}\mathbf{V}^\top) \\ &= \mathbf{R}(\theta)\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi)\ , \text{ where}\end{aligned}$$

  - $\mathbf{R}$ denotes rotation matrix
  - $\mathbf{U}, \mathbf{V}$: orthogonal matrices
  - $\mathbf{D} = \mathrm{Diag}(\lambda_{\mathbf{1}}, \lambda_{\mathbf{2}})$

- **Meaning** : Concatenation of

  i. Rotation by ϕ
  ii. Scaling x, y directions by λ_1, λ_2, respectively
  iii. Rotation by $-\phi$

  Deformation

  iv. Another rotation by θ

Graphical Illustration of the Affine Transformation

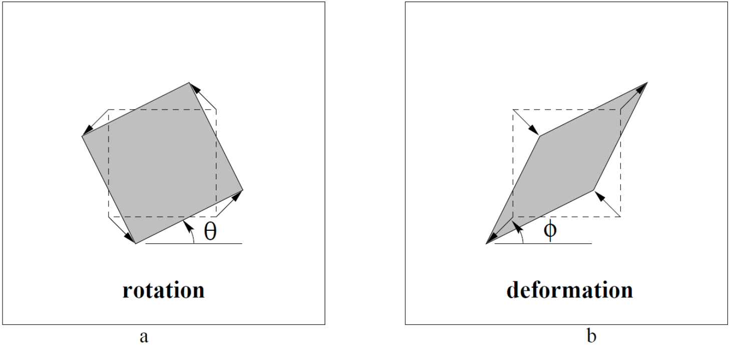


Figure 3: Graphical Illustration of the Affine transformation : (a) Rotation by $\mathbf{R}(\theta)$. (b) A deformation $\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi)$ Note, the scaling directions in the deformation are orthogonal.

8. **Projective Transformation**

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ v_1 & v_2 & \nu \end{bmatrix}}_{\mathbf{H_P}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Block form :

$$\mathbf{H_P} = \begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\top & \nu \end{bmatrix}, \text{ where } \mathbf{A} : \text{Nonsingular matrix}$$

- General nonsingular linear transformation :

Perspective + Scale + Rotation + Translation

- D.o.f : 8 (The ratio of elements are significant)

Let $\mathbf{H} = \begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\top & \nu \end{pmatrix}$ be a general projective transformation matrix.

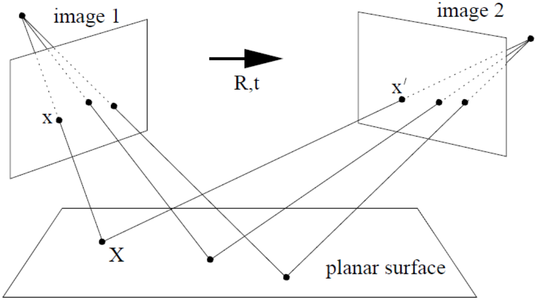
Decomposition of the projective transformation

$$\begin{aligned} \mathbf{H} &= \mathbf{H_S} \mathbf{H_A} \mathbf{H_P} \\ &= \begin{pmatrix} s\mathbf{R} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{pmatrix} \begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0}^\top & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{v}^\top & \nu \end{pmatrix} \end{aligned}$$

- $\mathbf{K}$: an upper triangular matrix (normalized as $\det \mathbf{K} = 1$)
- Valid for $\nu \neq 0$
- Unique if $s > 0$

Application of 2D Projective Transformation

1. The projective transformation between two images obtained from different camera angles



Application of 2D Projective Transformation

Removing perspective distortion (Perspective correction)

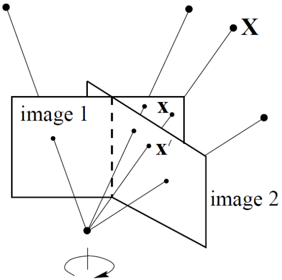
- The projective transformation can be recovered by the 4 points
- Then it is applied to the whole image

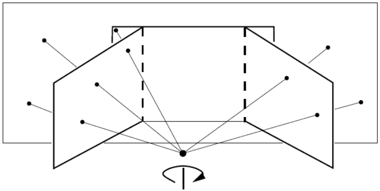
a b

Figure 4: a:The original image (the facade of the building) with perspective distortion. b:Rectified image (frontal orthogonal view of the building, perspective distortion is removed).

Application of 2D Projective Transformation

2. The projective transformation images with **the same camera centre (e.g. a camera rotating about its centre)**





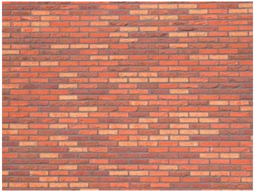
Application of 2D Projective Transformation

Figure 5: Planar panoramic mosaicing Eight images (out of thirty) acquired by rotating a camcorder about its centre. The thirty images are registered (automatically) using planar homographies and composed into the single panoramic mosaic shown. Note the characteristic "bow tie" shape resulting from registering to an image at the middle of the sequence.

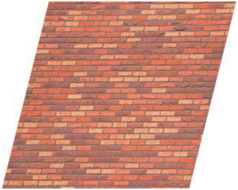
Affine vs Projective Transformations

Key Difference :

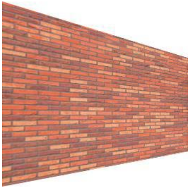
- Affine transformation : Preserve parallelism
- Projective transformation : Parallelism **may not** be kept



(a) the raw image



(b) Affine transformation



(c) Projective transformation

Affine vs Projective Transformations

Ideal point : The point at infinity

- Affine transform : ideal point remains ideal

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^\top & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix} \\ 0 \end{bmatrix}$$

- Projective transform : ideal point is mapped to a finite point

$$\begin{bmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{v}^\top & \nu \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix} \\ v_1 x + v_2 y \end{bmatrix}$$

Note : Projective transformation can model vanishing points

Summary of 2D Transformation

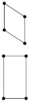
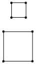
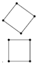
Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

Figure 7: Summary of 2D transformation

Summary of 3D Transformation

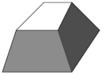
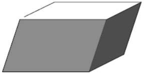
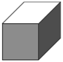
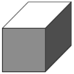
Group	Matrix	Distortion	Invariant properties
Projective 15 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{v}^\mathsf{T} & v \end{bmatrix}$		Intersection and tangency of surfaces in contact. Sign of Gaussian curvature.
Affine 12 dof	$\begin{bmatrix} \mathtt{A} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$		Parallelism of planes, volume ratios, centroids. The plane at infinity, $\boldsymbol{\pi}_\infty$, (see section 3.5).
Similarity 7 dof	$\begin{bmatrix} s\mathtt{R} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$		The absolute conic, Ω_∞, (see section 3.6).
Euclidean 6 dof	$\begin{bmatrix} \mathtt{R} & \mathbf{t} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$		Volume.

Figure 8: Summary of 3D transformation