STA 1013 : Statistics through Examples

Lecture 22: Expectation and Variance of Probability distributions

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Discrete distributions

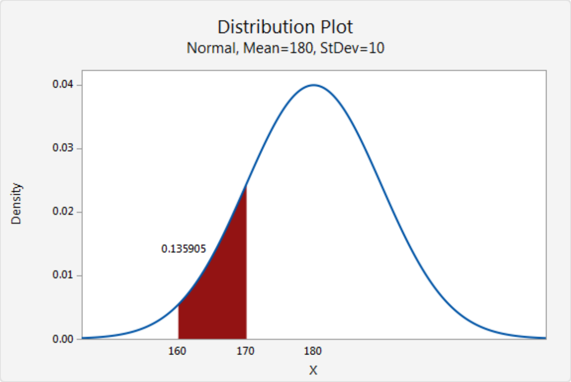
1. X : Number of Head (When tossing a fair coin twice)

x	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

2. X : Number (When rolling a die)

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Exmaple : Normal distributoin



Expectation

Expectation

The expected value, or expectation of a random variable X is written as $E(X)$

- X : discrete random variable

$$E(X) = \sum_x x \cdot P(X = x)$$

- X : continuous random variable

$$E(X) = \int_x x \cdot f_X(x) dx$$

Expectation

- Sometimes written as μ_X
- If we observe n random values of X, then the mean of the n values will be approximately equal to $E(X)$ for large n

Example

1. X : Number of Head (When tossing a fair coin twice)

x	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

2. X : Number (When rolling a die)

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Example

1. X : Number of Head (When tossing a fair coin twice)

x	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$$E(X) = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1$$

2. X : Number (When rolling a die)

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\begin{aligned} E(X) &= 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} \\ &= 3.5 \end{aligned}$$

Properties

1. $E(aX + b) = aE(X) + b$
2. Let X and Y be ANY random variables, Then
$$E(X + Y) = E(X) + E(Y)$$
3. Let X and Y be **independent** random variables, Then
$$E(XY) = E(X)E(Y)$$

Example

The U.S. Powerball Expected Value at $384 Million Jackpot

U.S. Powerball Expected Value at $384 Million Jackpot				
Matches	**Prize**	**Prize-cost**	**Probability**	**Prize-cost x probability**
5 White and Powerball	$384,000,000.00	$383,999,998.00	0.00000000342	$1.31
5 White	$1,000,000.00	$999,998.00	0.00000008556	$0.09
4 White and the Powerball	$50,000.00	$49,998.00	0.00000109514	$0.05
4 White	$100.00	$98.00	0.00002737838	$0.00
3 White and the Powerball	$100.00	$98.00	0.00006899352	$0.01
3 White	$7.00	$5.00	0.00172483810	$0.01
2 White and the Powerball	$7.00	$5.00	0.00142586616	$0.01
1 White and the Powerball	$4.00	$2.00	0.01087222948	$0.02
The red Powerball	$4.00	$2.00	0.02609335074	$0.05
Nothing	0	-$2.00	0.95978615950	-$1.92
			Expected Value	-$0.37

- Ticket price : 2$
- E(Prize − Ticket price) = E(Prize) − Ticket price
  (From the first formula : a=1, b= −Ticket price)

Example

- We roll **two dice** (X, Y are 1st and 2nd roll, respectively)
- Let $Z = X + Y$
- Possible outcomes and probabilities

$Z = X + Y$							Prob
2	(1,1)						1/36
3	(1,2)	(2,1)					2/36
4	(1,3)	(2,2)	(3,1)				3/36
5	(1,4)	(2,3)	(3,2)	(4,1)			4/36
6	(1,5)	(2,4)	(3,3)	(4,2)	(5,1)		5/36
7	(1,6)	(2,5)	(3,4)	(4,3)	(5,2)	(6,1)	6/36
8	(2,6)	(3,5)	(4,4)	(5,3)	(6,2)		5/36
9	(3,6)	(4,5)	(5,4)	(6,3)			4/36
10	(4,6)	(5,5)	(6,4)				3/36
11	(5,6)	(6,5)					2/36
12	(6,6)						1/36

Example

- Direct calculation of $E(Z)$

$$\begin{aligned} E(Z) &= 2\frac{1}{36} + 3\frac{2}{36} + 4\frac{3}{36} + 5\frac{4}{36} + 6\frac{5}{36} \\ &+ 7\frac{6}{36} + 8\frac{5}{36} + 9\frac{4}{36} + 10\frac{3}{36} + 11\frac{2}{36} + 12\frac{1}{36} \\ &= 7 \end{aligned}$$

- Use formula : $E(X + Y) = E(X) + E(Y)$

$$E(Z) = E(X + Y) = E(X) + E(Y) = 3.5 + 3.5 = 7$$

Example

- X : Number of Head (When tossing a fair coin twice)
- Y : Number (When rolling a die)
- $E(XY) = E(X)E(Y) = 1 \times 3.5 = 3.5$

Variance

Variance

- The variance measures how far the values of X are from their Expectation (Mean).
- Formula :

$$Var(X) = E((X - E(X))^2) = \mathbf{E(X^2) - \{E(X)\}^2}$$

- where $E(X^2) = \sum_x x^2 P(X = x)$
- Thus,

$$Var(X) = \sum_x x^2 \cdot P(X = x) - \left\{ \sum_x x \cdot P(X = x) \right\}^2$$

Example

- X : Number of Head (When tossing a fair coin twice)

x	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

- $$E(X^2) = 0^2 \times \frac{1}{4} + 1^2 \times \frac{1}{2} + 2^2 \times \frac{1}{4} = 1.5$$
- $\{E(X)\}^2 = 1^2$
- $Var(X) = E(X^2) - \{E(X)\}^2 = 0.5$

Example

- X : Number (When rolling a die)

x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

- $$\begin{aligned} E(X^2) &= 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 5^2 \times \frac{1}{6} + 6^2 \times \frac{1}{6} \\ &= \frac{91}{6} = 15.167 \end{aligned}$$

- $\{E(X)\}^2 = 3.5^2 = 12.25$

- $Var(X) = E(X^2) - \{E(X)\}^2 = 2.917$

Properties

- $Var(aX + b) = a^2 Var(X)$
- Let X and Y be independent random variables. Then

$$Var(X + Y) = Var(X) + Var(Y)$$

- (General version) : Let X and Y be independent random variables. Then

$$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y)$$

Examples

Let X, Y be independent

- $\text{Var}(X) = 2$
- $\text{Var}(Y) = 4$

Find the variance

1. Var(-X)
2. Var(-2X)
3. $\text{Var}(X+Y)$
4. Var(X-Y)
5. Var(2X - 3Y)

Notice

3rd Quiz :

- **November 8 (Fri)**
- You can use your calculator
- Bring one piece of hand written cheat sheet (both side allowed)