STA 1013 : Statistics through Examples

Lecture 15: Standard Normal distribution

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October 7, 2019

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Transformation

Fact : If $X \sim N(\mu, \sigma)$, then $aX + b \sim N(a\mu + b, |a|\sigma)$

- $X + b$: Changes in mean / location cause shifts in the density curve along the x axis
  - $b > 0$: shifts to right
  - $b < 0$: shifts to left

- aX : Changes in spread / standard deviation cause changes in both the location and the shape of the density curve
  - If $\mu = 0$,then aX cause changes only in the shape
  - $|a| < 1$: more peaky curve ($\because$ small standard dev)
  - $|a| > 1$: more flatter curve ($\because$ large standard dev)

Example : Transformation

Example : $X \sim N(0,1)$

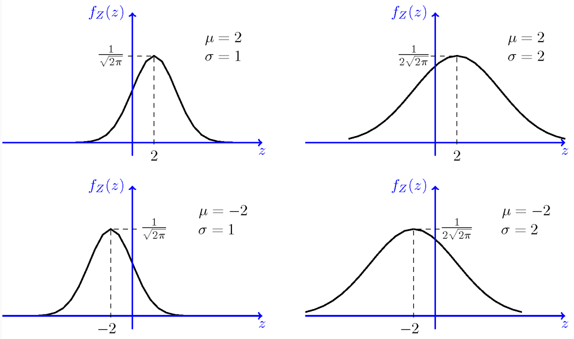
- $X + 3 \sim N(3,1)$
- $0.5X \sim N(0,0.5)$
- $2X \sim N(0,2)$
- $2X + 2 \sim N(2,2)$

Example : $X \sim N(2,1)$

- $X + 3 \sim N(5,1)$
- $0.5X \sim N(1,0.5)$
- $2X \sim N(4,2)$
- $2X + 2 \sim N(6,2)$

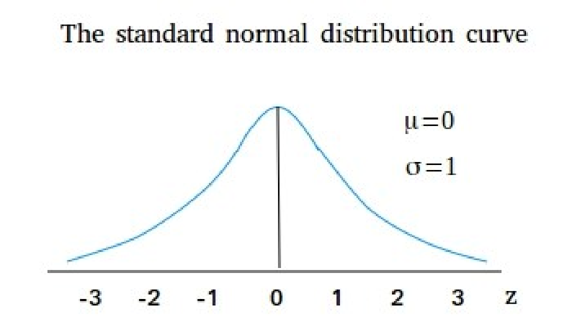
Transformation

Example : $X \sim N(2,1)$ (upper left panel) Identify a, and b



Standard Normal distribution

Standard Normal distribution



1. Bell-shaped
2. A mean equal to 0 ($\mu = 0$)
3. A standard deviation equal to 1 ($\sigma = 1$)

Why do we need the standard normal

- Probability between any two points on this distribution is well documented in the statistical tables.
- We can look into the **standard normal distribution table** to calculate the probabilities.
- It also makes life easier because we only need one table, rather than doing calculations individually for each value of mean and standard deviation.

Standard Normal table

Standard Normal Probabilities

Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

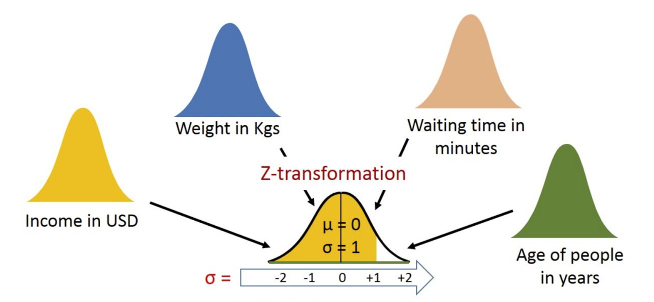
Standard Normal Probabilities

Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

How can we get the standard normal

In order to use the standard normal distribution, we need to convert the parent dataset (irrespective of the unit of measurement) into standard normal distribution using Z-transformation.

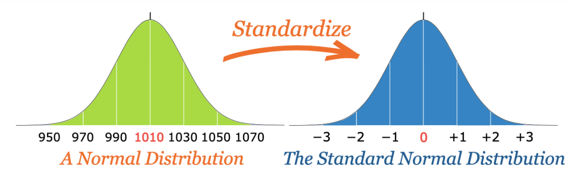


Z - Transformation

So to convert a value to a Standard Score (or z-score):

- first subtract the mean,
- then divide by the Standard Deviation

And doing that is called "Standardizing":



We can take any Normal Distribution and convert it to The Standard Normal Distribution.

Z - Transformation

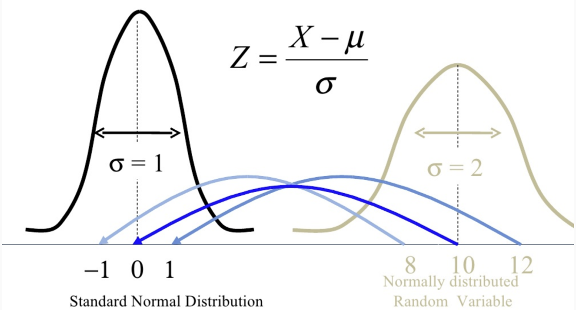
Here is the formula for z-score that we have been using:

Z - Transformation (Z - scores)

$$z = \frac{x - \mu}{\sigma}$$

- z is the "z-score" (Standard Score)
- x is the value to be standardized
- μ is the mean
- σ is the standard deviation

Z - Transformation



- The numerator $(X-\mu)$ shited the distribution so it is centered on 0
- By dividing by the standard deviation of the original distribution, it compressed the width of the distribution so it has a standard deviation equal to 1

Z - Transformation (Proof)

Recall : If $X \sim N(\mu, \sigma)$, then $aX + b \sim N(a\mu + b, |a|\sigma)$

- z transformation can be written as

$$Z = \frac{1}{\sigma}X - \frac{\mu}{\sigma}$$

- Use the theorem given above, $a = \frac{1}{\sigma}, b = -\frac{\mu}{\sigma}$
- Thus,

$$\begin{aligned} Z = \frac{1}{\sigma}X - \frac{\mu}{\sigma} &\sim N\left(\frac{1}{\sigma}\mu - \frac{\mu}{\sigma}, \frac{1}{\sigma}\sigma\right) \\ &\sim N(0,1) \end{aligned}$$

Z distribution

The standard normal distribution is also called the Z **distribution.**

$$Z \sim N(0, 1)$$

Note : Any normal distribution $X \sim N(\mu, \sigma)$ can be expressed in terms of the Z distribution

$$X = \sigma Z + \mu$$

Exmaple of z score

IQ test

The Stanford-Binet IQ test is scaled so that scores have a mean of 100 and a standard deviation of 16. Find the standard scores for IQs of 85, 100, and 125.

1. convert 85 into the z-score
   - first subtract the mean : $85 - 100$
   - then divide by the standard deviation : $\frac{85-100}{16} = -0.94$
2. convert 100 into the z-score

$$\frac{100-100}{16} = 0$$

3. convert 125 into the z-score

$$\frac{125-100}{16} = 1.56$$

Exmaple of z score

Women in the Army

The heights of American women ages 18 to 24 are normally distributed with a mean of 65 inches and a standard deviation of 2.5 inches. In order to serve in the U.S. Army, women must be between 58 inches and 80 inches tall.

Find the z scores for the army's minimum (58 inches) and maximum (80 inches) heights

z scores

- 58 :

- 80 :

Meaning of the z score

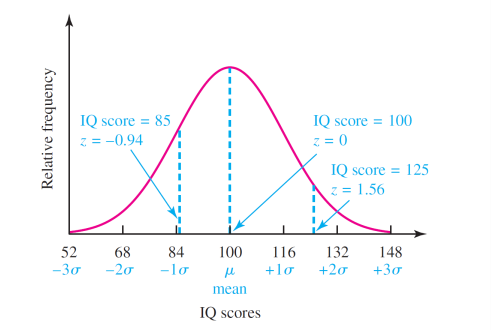
The number of standard deviations a data value lies above or below the mean is called its standard score (or z-score), often abbreviated by the letter z.

For example:

- The standard score of the mean is $z = 0$, because it is 0 standard deviations from the mean.
- The standard score of a data value **1.5 standard deviations above the mean** is $z = 1.5$.
- The standard score of a data value **2.4 standard deviations below the mean** is $z = -2.4$.

Example (IQ data): Meaning of the z score

The IQ test data follows Normal distribution with mean 100 and a standard deviation of 16. $N(\mu = 100, \sigma = 16)$

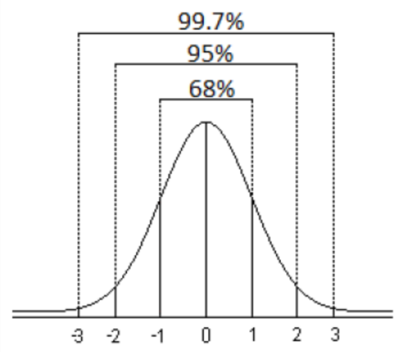


- 85 (z = -0.94) is 0.94 standard deviation below the mean
- 100 (z score = 0) is equal to the mean
- 125 (z = 1.56) is 1.56 standard deviations above the mean

Empirical rule for z distribution

$Z \sim N(0,1)$

- $[-1,1]$: 68% of data
- $[-2,2]$: 95% of data
- $[-3,3]$: 99.7% of data



Empirical rule for z distribution