STA 1013 : Statistics through Examples

Lecture 23: Binomial distribution

Hwiyoung Lee

November 1, 2019

Department of Statistics, Florida State University

Bernoulli distribution

Bernoulli distribution

- The Bernoulli distribution is the discrete probability distribution of a random variable which takes the value $\mathbf{1}$ with probability $\mathbf{p}$ and the value $\mathbf{0}$ with probability $\mathbf{q = 1 - p}$

$$P(X = x) = \begin{cases} p & , \text{ if } x = 1 \\ 1 - p & , \text{ if } x = 0 \end{cases}$$

$$= p^x(1-p)^{1-x}$$

- 1 : Success , p : the probability of Success
- 0 : Failure , q = (1-p) : the probability of Failure
- These two outcomes are **mutually exclusive**

Examples of Bernoulli distribution

A. Head when we toss a fair coin

- 1 : Head (Success) $\rightarrow$ $p = \frac{1}{2}$
- 0 : Tail (Failure) $\rightarrow$ $q = (1 - p) = \frac{1}{2}$

B. 5 when we roll a fair die

- 1 : Getting 5 (Success) $\rightarrow$ $p = \frac{1}{6}$
- 0 : Getting 1, 2, 3, 4, 6 (Failure) $\rightarrow$ $q = (1 - p) = \frac{5}{6}$

C. Less than 3 when we roll a fair die

- 1 : Getting 1,2 (Success) $\rightarrow$ $p = \frac{1}{3}$
- 0 : Getting 3,4,5,6 (Failure) $\rightarrow$ $q = (1 - p) = \frac{2}{3}$

Expectation and Variance of Bernoulli distribution

- $\mathbf{E(X) = p}$

$$E(X) = \sum_x x \cdot P(X = x)$$
$$= 0 \cdot P(X = 0) + 1 \cdot P(X = 1) = 0 \cdot (1 - p) + 1 \cdot p = p$$

- $\mathbf{Var(X) = p(1 - p)}$

$$\begin{aligned} Var(X) &= E(X^2) - \{E(X)\}^2 \\ &= 0^2 \cdot P(X = 0) + 1^2 \cdot P(X = 1) - \{E(X)\}^2 \\ &= 0^2 \cdot (1 - p) + 1^2 \cdot p - p^2 = p - p^2 \\ &= p(1 - p) \end{aligned}$$

Expectation and Variance of Bernoulli distribution

A. Head when we toss a fair coin

- 1 : Head (Success) $\quad \rightarrow \quad p = \frac{1}{2}$
- 0 : Tail (Failure) $\quad \rightarrow \quad q = (1-p) = \frac{1}{2}$
- $E(X) = \dfrac{1}{2}$
- $Var(X) = \dfrac{1}{2} \times \dfrac{1}{2} = \dfrac{1}{4}$

C. Less than 3 when we roll a fair die

- 1 : Getting 1,2 (Success) $\quad \rightarrow \quad p = \frac{1}{3}$
- 0 : Getting 3,4,5,6 (Failure) $\quad \rightarrow \quad q = (1-p) = \frac{2}{3}$
- $E(X) = \dfrac{1}{3}$
- $Var(X) = \dfrac{1}{3} \times \dfrac{2}{3} = \dfrac{2}{9}$

Binomial distribution

Binomial distribution

- The binomial distribution can be used to compute the probability of getting a particular **number of successes** in a specifed **number of trials**
- How many heads when you toss 5 coins ?

$$P(X = 0|n = 5, p = 0.5), \cdots, P(X = 5|n = 5, p = 0.5)$$

- How many ⚂ when you roll a die 10 times ?

$$P(X = 0|n = 10, p = \frac{1}{6}), \cdots, P(X = 10|n = 10, p = \frac{1}{6})$$

How many success (x) in n trials ?

- Repeat bernoulli trial n times
- Summation of n independent bernoulli trial

Binomial distribution

Notations :

- n : Number of Trial
- x : Number of Successes
- $n - x$: Number of Failures
- p : Probability of Success (From Bernoulli)
- $1 - p$: Probability of Failure (From Bernoulli)

Binomial distribution

Probability distribution

$$P(X = x|n, p) = \binom{n}{x} p^x (1-p)^{n-x}, \text{ for } x = 0, 1, \cdots, n$$

Note : $\binom{n}{x} = {}_n\mathrm{C}_x = \dfrac{n!}{x!(n-x)!}$

- Do not calculate $P(X = x|n, p)$ by hand !!
- Probability of 50 successes (H) out of 80 trials (Toss a coin)

$$P(X = 50|n = 80, p = 1/2) = \binom{80}{50} \frac{1}{2}^{50} \cdot \frac{1}{2}^{30}$$

Expectation and Variance of Binomial distribution

Let $X_1, X_2, \cdots X_n$ be independent bernoulli trials with success probability p.

Then Binomial distribution $X = X_1 + X_2 + \cdots + X_n$

- $\mathbf{E(X) = np}$

$$E(X_1 + X_2 + \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n) = np$$

- $\mathbf{Var(X) = np(1 - p)}$

$$\begin{aligned} Var(X) &= Var(X_1 + \cdots + X_n) = Var(X_1) + \cdots + Var(X_n) \\ &= np(1-p) \end{aligned}$$

Examples : Expectation of Binomial distribution

Expected value of Head (Tossing coin)

- Suppose you toss a fair coin 12 times
- Expected value : 6 ($\because 12 \times \frac{1}{2}$)
- You expect to get 6 heads when you toss 12 coins.
- The most likely outcome in 12 tosses is 6 heads.

Expected value of Defects

- Suppose a defective rate of a process is 0.002, and you examine 2,000 products
- Expected value : 4 ($\because 2,000 \times \frac{2}{1,000}$)
- On average 4 products are expected to have defects when you examine 2,000 products

Calculator (binompdf function)

- $P(X = x|n, p)$: point probability
- **distr → binompdf**
- Enter the required data : **binompdf**(n, p, x)
  - n : Number of trials
  - p : success probability
  - x : Number of successes
- Example :
  - The probability to get exactly two heads in 4 coin flips

$$P\left(X = 2|4, \frac{1}{2}\right) = \textbf{binompdf}\left(4, \frac{1}{2}, 2\right)$$

  - The probability to get exactly 5 ⚂ in 7 dice rollings

$$P\left(X = 5|7, \frac{1}{6}\right) = \textbf{binompdf}\left(7, \frac{1}{6}, 5\right)$$

Binomial probability (Binompdf)

- 55% of the students at FSU are female. What is the probability that a randomly selected group of 25, there will be exactly 15 females?

- Suppose a biased coin is tossed 22 times and that the probability it lands on tails is 0.6. Find the probability that the coin lands on tails 9 times.

Binomial probability (Binompdf)

- An air conditioner made by a certain company is made of 20 distinct parts. Each part has a .004 probability of being defective. What is the probability that a randomly selected air conditioner will not work perfectly? [**Note that it will not work perfectly if even one of these parts is defective**.]

Calculator (binomcdf function)

- $P(X \leqslant x|n,p)$: cumulative probability

$$P(X \leqslant x) = P(X=0) + P(X=1) + \cdots + P(X=x-1) + P(X=x)$$

- **distr → binomcdf**
- Enter the required data : **binomcdf**(n,p,x)
- Example :
  - The probability to get less than or equal to two heads in 4 coin flips

$$P\left(X \leqslant 2|4,\frac{1}{2}\right) = \textbf{binomcdf}\left(4,\frac{1}{2},2\right)$$

  - The probability to get less than six ⚂ in 7 dice rollings

$$P\left(X \leqslant 5|7,\frac{1}{6}\right) = \textbf{binomcdf}\left(7,\frac{1}{6},5\right)$$

Binomial probability (Binomcdf)

- 55% of the students at FSU are female. What is the probability that a randomly selected group of 25, there will be less than or equal to 15 females?

- An air conditioner made by a certain company is made of 20 distinct parts. Each part has a .004 probability of being defective. What is the probability that a randomly selected air conditioner will have at least 3 defective parts ?

Exercie 1

The coins used at a casino are biased; the probability a coin lands on heads is 0.4. You play a game where **you must flip a coin ten times.** You win \$2 every time the coin lands on heads and you win nothing when the coin lands on tails.

1. What is the probability that you win \$20 ?
2. What is the probability that you win less than \$6 ?
3. What is the probability that you win at least \$4 ?
4. If you pay \$10 to play this game, should you play ?

Exercie 2

An exam consists of 10 multiple choice questions. For each question, one must choose from 4 possible answers. Assume the Albert has not studied and decides to answer each question at random.

1. What is the probability that Albert obtains 100% ?
2. What is the probability that Albert obtains 0% ?
3. When 6 correct answers needed to pass a test, what is the probability that Albert fails ?