Algebra Autumn 2025

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29 August 2025 **Second Homework**

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

The last two problems were partially done in class. This is an opportunity to polish your skills at writing mathematics in a clear and organized manner.

Hand in at the start of class, Thursday 4 September:

1. Let G be a group, H be a subgroup of G, and $g \in G$. Prove that gHg^{-1} is a subgroup of G and that it is isomorphic to H.

2. Show that a group G cannot be the union of two proper subgroups.

   Show that the additive group of ordered pairs of integers $\mathbb{Z} \oplus \mathbb{Z}$ is the union of three proper subgroups, by exhibiting three such subgroups.

3. Let A, B be groups with elements $a \in A$ and $b \in B$. What is the order of the element $(a, b) \in A \times B$?

4. Assume that $G = \{e, a, b, c\}$ is a group with four elements and identity e. Suppose that G has no element of order four. Prove that there is a unique group structure for G and deduce that G is abelian. What is the case if G has an element of order 4?

5. Prove that every finitely generated subgroup of the rational numbers $\mathbb{Q}$ is cyclic. Give an example (with proof) of a subgroup of $\mathbb{Q}$ that is not finitely generated.

6. Let D_4 be the group under matrix multiplication generated by the real 2×2 matrices $S := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $R := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Show that D_4 is a nonabelian group of order 8.

   Let $\square$ be the square with vertices $(\pm 1, \pm 1)$ in $\mathbb{R}^2$. Show that D_4 acts on $\square$, and is its group of symmetries, called the *dihedral group* of order 8.

7. Let Q_8 be the group generated by the complex 2×2 matrices $\mathbf{i} := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\mathbf{j} := \left(\begin{smallmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{smallmatrix}\right)$. Show that Q_8 is a nonabelian group of order 8. Hint: Observe that $\mathbf{ij} = \mathbf{ji}^3$, so that every element of Q_8 has the form $\mathbf{i}^a\mathbf{j}^b$. Note further that $\mathbf{i}^4 = \mathbf{j}^4 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, the identity. This is the *quaternion* group.

   Comparing subgroups, or the number of elements of different orders, show that Q_8 is not isomorphic to D_4.