Homework 1

17.1. Let P be a nonzero projection, so that $P^2 = P \neq 0$. Then by norm properties, we have

$$\|P\| = \|P^2\| \leq \|P\| \cdot \|P\|.$$

Since $\|P\| \neq 0$, we have $\|P\| \geq 1$

17.2. Fix some $\varepsilon > 0$. Suppose the inequality does not hold. Then for every $N > 0$, there exists $n \geq N$ and λ_n such that $\lambda_n \in \sigma(M_n)$ and $|\lambda_n| \geq |\sigma(M)| + \varepsilon$. Obtain an increasing sequence n_k such that $\lambda_{n_k} \in \sigma(M_{n_k})$ by induction, taking n_{k+1} to be the next n greater than n_k with the desired property.

Since M_n is a convergent sequence, it is bounded in norm, and thus $|\sigma(M_n)|$ is bounded uniformly in n. Choose a convergent subsequence of λ_{n_k} which converges to some λ, which also must satisfy $|\lambda| \geq |\sigma(M)| + \varepsilon$. Denote this also by λ_{n_k} for simplicity. Then $(M - \lambda_{n_k}) \to (M - \lambda)$. Since $(M - \lambda_{n_k})$ is not invertible, and the set of non-invertible elements is closed, we have that $(M - \lambda)$ is not invertible as well, implying $\lambda \in \sigma(M)$. But this is impossible as it lies outside the spectral radius of M. Then we are done.

17.4. We know from complex analysis that we can define a branch of the logarithm on any open set in $\mathbb{C}$ which does not contain zero. Let Ω be an open set not containing 0 which contains $\sigma(M)$. This is possible as $\sigma(M)$ is closed. Define a branch of the complex logarithm, which is analytic on Ω. By the hypothesis that 0 can be connected to ∞ by a path in $\rho(M)$, it is possible to enclose $\sigma(M)$ by a path γ contained within Ω which does not encircle 0. Using this path, we define

$$\log(M) = \oint_\gamma (\zeta - M)^{-1} f(\zeta) d\zeta.$$

Moreover, exp is analytic on $\mathbb{C}$ so that $\exp(\log(M))$ is well defined, and equals the identity by theorem 5, since $\exp(\log(\zeta)) = \zeta$ as functions on $\mathbb{C}$.

17.5. Let $\overline{\mathcal{L}_M}$ be the closure of the algebra generated by elements M and $(\lambda - M)^{-1}$ for $\lambda \in \rho(M)$ within some larger Banach algebra $\mathcal{M}$.

Since $(\lambda - M)^{-1}$ is analytic on $\rho(M)$, it can be expressed as a power series. Then it is clear that the elements M and $(\lambda - M)^{-1}$ commute for all $\lambda \in \rho(M)$. Every element of $\mathcal{L}_M$ is a polynomial in M and elements of the form $(\lambda - M)^{-1}$. Then it is clear that $\mathcal{L}$ is a commutative algebra.

Moreover, take any $A, B \in \overline{\mathcal{L}_M}$ and choose $A_n, B_n \in \mathcal{M}_L$ converging to A and B respectively. Then by the continuity of the multiplication operation in an algebra,

$$A \cdot B = \lim_n A_n \cdot B_n = \lim_n B_n \cdot A_n = B \cdot A,$$

which proves $\overline{\mathcal{M}_L}$ is a commutative subalgebra of $\mathcal{M}$

19.1. Let M be a maximal ideal in $C(S)$. If M contains a nonzero function f, then $\frac{1}{f} \cdot f = 1 \in M$, and thus $M = C(S)$, a contradiction. Then every element of M is zero at some point. Moreover, there must exist at least one point at which all elements of M vanish. If not, choose a collection of functions $\{f_n\}_{n \in I}$ which do not uniquely vanish at any $x \in S$. Without loss of generality we can assume each f_n is positive in some neighborhood. By urysohn's lemma, we can assume the f_n are positive and supported within this neighborhood. Then by taking a finite subcover and summing over these indices, we obtain an invertible element in M, giving a contradiction.

Choosing any x_0 at which every element of M vanishes, we have that $M \subset \{f \in C(S) : f(x_0) = 0\}$ implying M is of the desired form since M is maximal.