Homework 8

Section 33

4. We will show that if X is a normal space with $A \subset X$, then A is a closed G_δ set if and only if there is a function f such that $f(x) = 0$ for $x \in A$ and $f(x) > 0$ for $x \notin A$.

   First suppose A is a closed G_δ set, and say $A = \cup_1^\infty A_n$. For each n, apply Urysohn's lemma to get a function $f_n : X \to [0, 2^{-n}]$ which is 0 on A and 2^{-n} on the closed set A_n^c. Define the function $f = \sum_1^\infty f_n$. Then f converges uniformly by the Weierstrass M-test, since each f_n is uniformly bounded by 2^{-n}, which sums to 1. Moreover f is continuous since each f_n is continuous, and by the uniform convergence theorem.

   Finally note that $f(x) = 0$ for all $x \in A$ by construction, and if $x \notin A$, then there must be some n with $x \in A^c$, and thus $f(x) > 0$.

   Conversely suppose A is a set such that there is a function f for which $f(x) = 0$ for $x \in A$ and $f(x) > 0$ for $x \notin A$. Then A is closed, as A is the preimage of the closed set $\{0\}$ under the continuous map f. For $n = 1, 2, \ldots$, define a set $A_n = f^{-1}([0, 1/n))$. The A_n's are open as preimages of open sets. Also if $x \in A_n$ for all n, we must have $f(x) < 1/n$ for all n, so $f(x) = 0$ and $x \in A$. If $x \in A$, then clearly $x \in A_n$ for all n. Then $A = \cap A_n$.

Section 34

3. We show that a compact Hausdorff space X is metrizable if and only if it has a countable basis.

   First suppose X is metrizable. Then X is a compact metric space, and we have previously shown that this implies X is second countable by taking finite subcovers of open covers of increasingly small balls and taking their centers.

   Conversely suppose X has a countable basis. Then X is normal (hence regular) and second countable, and so X is metrizable by the Urysohn metrization theorem.

6. Let X be a space in which one point sets are closed. Suppoose $\{f_\alpha\}_{\alpha \in J}$ is an indexed family of continuous functions $f_\alpha : X \to \mathbb{R}$ which separates points and closed sets. Then the function $F : X \to \mathbb{R}^J$ defined by

$$F(x) = (f_\alpha(x))_{\alpha \in J}$$

   is an embedding of X into $\mathbb{R}^J$. If f_α maps X into $[0, 1]$, then F embeds X into $[0, 1]^J$.

   Proof. F is injective, since for any $x \neq y$ there is some map f_{α_0} such that $f_{\alpha_0}(x) = 0$ and $f_{\alpha_0}(y) > 0$, and thus $F(x) \neq F(y)$.

   It is continuous in the product topology on $\mathbb{R}^J$ since each component is continuous.

   Then we must show it is open. Take some open $U \subset X$. Choose $z_0 \in F(U)$. Let x_0 be the point in U such that $F(x_0) = z_0$. Choose some α_0 where $f_{\alpha_0}(x_0) > 0$ and $f_\alpha(X \setminus U) = 0$. Take

$$V = \pi_{\alpha_0}^{-1}((0, \infty)),$$

   where π_{α_0} is the projection onto the α_0 component, and let $W = V \cap F(X)$. Then W is open in $F(X)$ in the subspace topology, and we just need to show that $z_0 \in W \subset F(U)$. First, we have $z_0 \in W$ since

$$\pi_{\alpha_0}(z_0) = \pi_{\alpha_0}(F(x_0)) = f_{\alpha_0}(x_0) > 0.$$

   We have $W \subset F(U)$ since if $z \in W$, then $z = F(x)$ for some $x \in X$ and $\pi_{\alpha_0}(x) > 0$. Then since $\pi_{\alpha_0}(z) = f_{\alpha_0}(x)$, and we know f_{α_0} vanishes outside U, we must have $x \in U$ and thus $z \in F(U)$. Then $z_0 \in W \subset F(U)$.

   The same proof works for the case where $f_\alpha : X \to [0, 1]$.

Section 35

1. Assume the conclusion of the Tietze Extension Theorem. Suppose X is normal, with closed disjoint subsets A and B. Then the function on $A \cup B$ defined by $f(x) = a$ if $x \in A$, and $f(x) = b$ if $x \in B$ is continuous. By Tietze extension theorem, f extends to a function $\tilde{f} : X \to [a, b]$, and thus we have proved Urysohn's lemma.

Categories

a. The identity morphism on $X \in \mathcal{C}$ is defined by the property that for $A, B \in \mathcal{C}$, and maps $f \in \mathrm{Mor}(X, A)$ and $g \in \mathrm{Mor}(B, X)$, we have

$$f \circ \mathrm{id}_X = f, \quad \mathrm{id}_X \circ g = g.$$

Then suppose we have two identity maps on X, id_X and id'_X. The above property tells us that

$$\mathrm{id}_X = \mathrm{id}_X \circ \mathrm{id}'_X = \mathrm{id}'_X$$

since both map $X \to X$. Then the two must be equal.

b. A functor $\mathcal{F}$ has the properties that $\mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{F}(X)}$, and that $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$.

Now let $A, B \in \mathcal{C}$ and suppose $f \in \mathrm{Mor}(A, B)$ is an isomorphism, so that there is a map $g \in \mathrm{Mor}(B, A)$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Then by the properties of $\mathcal{F}$, we have

$$\mathcal{F}(g) \circ \mathcal{F}(f) = \mathcal{F}(g \circ f) = \mathcal{F}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{F}(A)}$$

and similarly to show $\mathcal{F}(f) \circ \mathcal{F}(g) = \mathrm{id}_{\mathcal{F}(B)}$. Then $\mathcal{F}(f)$ is an isomorphism $\mathcal{F}(A) \to \mathcal{F}(B)$ with inverse $\mathcal{F}(g)$.