# Homework 3

14. We first show the conjugation map is a bijection. If $gag^{-1} = gbg^{-1}$, multiplication on the left and right by $g$ gives $a = b$, which proves injectivity. For surjectivity, the element $g^{-1}ag$ is mapped to $a \in G$.

    The map is a homomorphism: $geg^{-1} = e$, and $gag^{-1}gbg^{-1} = g(ab)g^{-1}$.

    Now we show that the set $A$ of inner automorphisms form a normal subgroup of $\mathrm{Aut}(G)$. Let $C_g$ be conjugation by some $g \in G$ and let $\varphi \in \mathrm{Aut}(G)$. Then for $x \in G$, we have

    $$(\varphi \circ C_g \circ \varphi^{-1})(x) = \varphi(C_g(\varphi^{-1}(x))) = \varphi(g)x\varphi(g)^{-1} = C_{\varphi(g)}(x).$$

    Moreover, $C_g = \varphi \circ C_{\varphi^{-1}(g)} \circ \varphi^{-1}$, and so $\varphi A \varphi^{-1} = A$ for any $\varphi \in \mathrm{Aut}(G)$ and $A$ is normal.

15. Let $\varphi : G \to H$ be a bijective group homomorphism. Then

    $$\varphi^{-1}(ab) = \varphi^{-1}(\varphi(\varphi^{-1}(a))\varphi(\varphi^{-1}(b))) = \varphi^{-1}(\varphi(\varphi^{-1}(a)\varphi^{-1}(b))) = \varphi^{-1}(a)\varphi^{-1}(b),$$

    and $\varphi^{-1}(e_H) = e_G$ since $\varphi(e_G) = e_H$ and $\varphi$ is a bijection. Then $\varphi^{-1}$ is a homomorphism.

16. Let $S$ be a subset of a group $G$, and define $a \sim b$ if and only if $ab^{-1} \in S$. We will start by supposing $S$ is a subgroup and showing $\sim$ is an equivalence relation. Because $e \in S$, we have $e = aa^{-1} \in S$ and thus $a \sim a$. If $a \sim b$, then $ab^{-1} \in S$, and so $S$ contains the inverse $ba^{-1}$ and we have $b \sim a$ as well. Finally if $a \sim b$ and $b \sim c$, this means $ab^{-1}, bc^{-1} \in S$. Then the product $ab^{-1}bc^{-1} = ac^{-1}$ is contained in $S$, so that $a \sim c$ and $\sim$ is indeed an equivalence relation on $G$.

    Conversely suppose $\sim$ is an equivalence relation. Since $a \sim a$ for $a \in G$, this means $aa^{-1} = e \in S$. Next let $a \in S$. Then $ae^{-1} \in S$ so that $a \sim e$ and $e \sim a$, which gives $ea^{-1} = a^{-1} \in S$. Finally let $a, b \in S$. Then $a \sim e$ and $e \sim b^{-1}$, so $a \sim b^{-1}$ and $ab \in S$. Then $S$ is a subgroup.

17. (a.) Clearly $N_G(H)$ contains $H$ by problem 14. If $g, k \in N_G(H)$, then $(gk)H(gk)^{-1} = g(kHk^{-1})g^{-1} = gHg^{-1} = H$, so $gk \in N_G(H)$. We obviously have $e \in N_G(H)$, as well as inverses, since $aHa^{-1} = H$ implies $H = a^{-1}Ha$.

    (b.) If $H$ is normal in $K$, we have $kHk^{-1} = H$ for any $k \in K$, and thus $K \subset N_G(H)$.

    (c.) Since both $H$ and $K$ contain identity, we have $e \in HK$. For inverses, if $kh \in KH$, we have $h^{-1}k^{-1} = k^{-1}h' \in KH$ for some $h' \in H$ since $K \subset N_G(H)$ and $K$ is a subgroup. Finally if $kh, k'h' \in KH$, then $khk'h' = kk'h^*h' \subset KH$ for some $h^* \in H$.

    To show $H$ is normal in $KH$, let $kh \in KH$. We have $khH(kh)^{-1} = khHh^{-1}k^{-1} = kHk^{-1} = H$.

18. We will construct a homomorphism from $G$ into the permutation group on $p$ elements. First we will show that right multiplication by elements of $G$ induces a permutation on the set $G/H$. For $g \in G$, let $\pi_g : G \to G$ be the map $a \mapsto ga$. Then $\pi_g$ is a bijection $G \to G$, so we only must show that if $a, b \in G$ belong to the same coset, they will be mapped to the same coset under $\pi_g$. We let $aH = bH$, so that $b^{-1}a = h$ for $h \in H$, and thus $b^{-1}g^{-1}ga = h$. Then $ga$ and $gb$ differ by an element of $H$, so $gaH = gbH$. Then we have shown that $\pi_g$ permutes the cosets of $H$, of which there are $p$.

    Let $\varphi : g \mapsto \pi_g$ be the map above. It maps $G \to \mathrm{Sym}(G/H)$, the symmetric group on the $p$ cosets. Note $|\mathrm{Sym}(G/H)| = p!$, so that the subgroup $\mathrm{im}(\varphi) \subset \mathrm{Sym}(G/H)$ must have order dividing $p!$. However the order of the image of $\varphi$ must also divide the order of $G$ by lagrange's theorem. Since $p$ is the smallest prime dividing $|G|$, we must have $|\mathrm{im}(\varphi)| = p$. Then since every prime order group is cyclic, $\mathrm{im}(\varphi)$ must be a cycle of all $p$ cosets.

    Finally we want to show that $\ker \varphi = H$. We note that if $h \in H$, $\pi_h$ fixes the coset $H$, and since $\pi_h$ cycles each of the $p$ cosets, it must be the trivial permutation fixing each of them. Then $\varphi(H) = e \in \mathrm{Sym}(G/H)$, so $H \subset \ker \varphi$. Since $|G| = |\ker \varphi| \cdot [G : \ker \varphi] = |\ker \varphi| \cdot |\mathrm{im}(\varphi)| = |\ker \varphi| \cdot p$, we must have $|H| = |\ker \varphi|$, so that in fact $H = \ker \varphi$.

19. We will represent nonzero complex numbers as $ae^{i\theta}$ for $a > 0$ and $\theta \in [0, 2\pi)$. We have $|1| = |1e^0| = 1$, and $|ae^{i\theta}be^{i\varphi}| = |abe^{i(\theta+\varphi)}| = |ab| = |a||b|$, so $|\cdot|$ is a homomorphism.

    Its image is the multiplicative group of positive real numbers. The kernel contains all elements of unit norm, which are the elements of $S_1 \subset \mathbb{C}$.

20. By one of the isomorphism theorems from class, since $\varphi$ is a surjective homomorphism with kernel $N$, we have $G/N \simeq H$. Then every subgroup $L$ of $H$ is a subgroup of $G/N$. Then the problem amounts to showing there is a bijective correspondence between subgroup $K$ of $G$ which contain $N$ and subgroups $L$ of $G/N$, and the same for normal subgroups.

    Let $\pi$ be the canonical surjection $G \to G/N$. We will show $\pi$ is the desired bijection, first in the non-normal case. Let $K \neq K'$ be two subgroups of $G$ containing $N$. Then WLOG choose some $k \in K \setminus K'$. Suppose $KN = K'N$. Then $k = k'n$ for $k' \in K'$, $n \in N$. Since $N \subset K'$ and $K'$ is a subgroup, this implies $k \in K'$, a contradiction. Then $\pi(K) \neq \pi(K')$, and $\pi$ is injective.

    Let $L \subset G/N$ be a subgroup. Then $\pi^{-1}(A)$ is a subgroup in $G$, proving surjectivity. Thus we have the desired bijection between subgroups of $G$ containing $N$ and subgroups of $G/N$, and canonically from subgroups of $G/N$ to subgroups of $H$.

    For the normal case, note that $\pi$ preserves normality. The injectivity condition still holds. Finally if $A \subset G/N$ is a normal subgroup, we must show that $\pi^{-1}(A)$ is normal in $G$. For $g \in G$, we have

    $$g\pi^{-1}(A)g^{-1} = \pi^{-1}(\pi(g)A\pi(g)^{-1}) = \pi^{-1}(A),$$

    and we are done.