Homework 10

1. (a.) Take $(y_i^k)_{i=1}^\infty = \begin{cases} 1 & i \le k \\ 0 & i > k \end{cases}$. Then for any $x^* \in \ell_1$, we have

$$x^*(y_i^k) = \sum_{i=1}^k x_i^* \le \sum_1^k |x_i^*| \to_k \|x^*\|_{\ell_1}$$

meaning (y_i^k) is weakly cauchy. However, it is clear by taking the limit against unit vectors e_i that if we did have $y_i^k \to_k (y_i)$ in norm, we must have $y_i = 1$ for all i. But this is not an element of c_0.

(b.) Suppose X has the schur property. Take a weakly cauchy sequence x_n. Then for every $x^* \in X^*$, we must have

$$x^*(x_n - x_m) \to 0$$

in n and m. Then for every pair of increasing sequences n_k and m_k, $x_{n_k} - x_{m_k}$ is weakly convergent to zero and thus strongly convergent, by the schur property. Then this is equivalent to x_n being norm cauchy. To see this we can suppose it is not, and this would allow us to find a pair of sequences such that the limit does not converge to zero in norm.

Then since X is complete, x_n converges to some element in norm and thus weakly.

2. Suppose every closed subspace of ℓ_1 is complemented. We know that ℓ_p is isomorphic to the quotient ℓ_1/M for some closed subspace M. Then M is complemented, and we can decompose $\ell_1 = M \oplus N$, and N will be isomorphic to ℓ_p. But then ℓ_1 contains a subspace N isomorphic to ℓ_p, which is impossible.

Moreover, since ℓ_p is not isomorphic to ℓ_q for $p \ne q$, there are uncountably many non-isomorphic separable Banach spaces. Each must correspond to a unique uncomplemented closed subspace M_p of ℓ_1 such that $\ell_p = \ell_1/M_p$. Then there must be uncountably many such subspaces.

3. Let $T : X \to \ell_1$ be the composition of the projection onto X/M with the isomorphism with ℓ_1. Let (e_n) be the canonical basis of ℓ_1 and choose a bounded sequence $(x_n) \in X$ such that $T(x_n) = e_n$ using the open mapping theorem. Then (e_n) is equivalent to (x_n): We first have

$$\left\|\sum_1^\infty a_n x_n\right\| \le \sum_1^\infty |a_n| \|x_n\| \le C \sum_1^\infty |a_n|$$

since x_n is bounded. We also have

$$\|T\| \left\|\sum_1^\infty a_n x_n\right\| \ge \| \sum_1^\infty a_n e_n\| = \sum_1^\infty |a_n|.$$

Now let $S : \ell_1 \to X$ be the map sending e_n to x_n and extend to X by linearity. Then S is an isomorphism of $[x_n]$ with ℓ_1. Also, the subspace is complemented with $P = ST$ since $STST = S(TS)T$, and TS is the identity on ℓ_1.