Homework 3

13. Let $a \in G$, we will show $a(H \cap K) = (aH) \cap (aK)$. Let $x \in a(H \cap K)$. Then $x = ar$ for some $r \in H \cap K$, thus $x \in aH \cap aK$.

    If $x \in (aH) \cap (aK)$, we have $x = ah = ak$ for $h \in H$, $k \in K$, and thus $h = k$ and $x \in a(H \cap K)$.

    Now suppose H and K have finite index. Let $\varphi : G/H \cap K \to G/H \times G/K$ be the map $a(H \cap K) \mapsto (aH, aK)$. We will show φ is an injection, proving the theorem. First note that φ is well defined, since $a(H \cap K) = b(H \cap K)$ if and only if $(aH) \cap (aK) = (bH) \cap (bK)$, in which case $aH \cap bH \neq \varnothing$ and $aH = bH$. similarly $aK = bK$.

    Then if $\varphi(a(H \cap K)) = \varphi(b(H \cap K))$, we must have $(aH, aK) = (bH, bK)$. This gives $aH = bH$, $aK = bK$, meaning $(aH) \cap (aK) = (bH) \cap (bK)$ and finally $a(H \cap K) = b(H \cap K)$. This proves φ is injective. Then we have an injection into a set of finite cardinality, so $G/H \cap K$ must have finite cardinality.