Algebra Autumn 2025
Frank Sottile
11 September 2025 **Fourth Homework**

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Some problems were partially done in class. This is an opportunity to polish your skills at writing mathematics in a clear and organized manner.

I am also hoping that this assignment has some easier problems. The last homework was a bit more than I probably should assign.

Hand in at the start of class, Thursday 18 September:

21. Let G be a group with subgroups H and K.
    (a) Prove that $H \cap K$ is a subgroup of G.
    (b) If both H and K are normal subgroups of G, prove that $H \cap K$ is a normal subgroup of G.
    (c) If the set $H \cap K$ is replaced by $H \cup K$, do the answers to parts (a) and (b) change? Justify your answer.

22. Let G be a group. Show that its commutator subgroup, $[G, G]$, is a normal subgroup of G.

    Show that $G/[G, G]$ is an abelian group. This is the *abelianization* of G.

    Suppose that $\varphi\colon G \to H$ is a homomorphism of groups with H abelian. Show that $[G, G] \subset \ker(\varphi)$. Conclude that every homomorphism from G to an abelian group factors through its abelianization.

23. You will prove **Gorsat's Lemma**: Let G_1 and G_2 be groups and suppose that H is a subgroup of $G_1 \times G_2$ whose image under each coordinate projection $p_i\colon G_1 \times G_2 \to G_i$ for $i \in \{1, 2\}$ is surjective. Let $N_1 \subset H$ be the kernel of p_2 and $N_2 \subset H$ be the kernel of p_1 and show that we may identify N_i as a normal subgroup of G_i, for $i \in \{1, 2\}$.

    Finally, show that the image of H in $G_1/N_1 \times G_2/N_2$ is the graph of an isomorphism $G_1/N_1 \xrightarrow{\sim} G_2/N_2$.

24. Let $\mathbb{K}$ be a field. Recall that $B(n, \mathbb{K})$ is the group of invertible upper triangular matrices with entries in $\mathbb{K}$.

    Show that $B(2, \mathbb{K})$ and $B(3, \mathbb{K})$ are solvable.

    (A more elegant route than taken by Lang is to let $U(n, \mathbb{K})$ be the subgroup of $B(n, \mathbb{K})$ consisting of matrices with 1s along the diagonal, and explore the commutator subgroup $[U(n, \mathbb{K}), U(n, \mathbb{K})]$.)

25. Give two different composition series for the quaternion group Q_8 and verify that they are equivalent (à la Jordan-Hölder). Also, verify that Q_8 is solvable.

26. Repeat the previous question, but for the dihedral group of order 8.

27. Please give a complete proof of Theorem 3.2 in Lang:

    Let G be a group with normal subgroup H. Then G is solvable if and only if both H and G/H are solvable.