Algebra Autumn 2025
Frank Sottile
9 October 2025 Seventh Homework

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in at the start of class, Thursday 16 October:

37. Let G be a group of order 3825. Prove that if H is a normal sugroup of G of order 17, then H is a subgroup of the center of G.

38. Which group of order 8 is $(S_2)^2 \rtimes S_2$?

39. Let $N := \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Determine $\mathrm{Aut}(N)$ and identify the group $N \rtimes \mathrm{Aut}(N)$.

40. Universal constructions are uniquely isomorphic.

    (a) Let S be a set and suppose that the abelian group A is free on S. If A' is another abelian group that is free on S, show that there is a unique isomorphism from A to A', as abelian groups that are free on S.

    (b) Let M be a commutative monoid with Grothendieck group $K(M)$ and suppose that G is a group with a map $M \to G$ such that for any abelian group B, the pullback map
    $$\mathrm{Hom}_{\mathrm{ab-gp}}(G, B) \longrightarrow \mathrm{Hom}_{\mathrm{monoid}}(M, B)$$
    is a bijection. Show that there is a unique isomorphism $K(M) \to G$ that preserves this universal property.

41. **Groups of Order Twelve.** Recall: if $p \neq q$ prime numbers, all groups of order p^2q have a normal Sylow subgroup.

    (a) Determine all groups of orders 3 and 4, along with their groups of automorphisms.

    (b) Let G be a group of order twelve, and set N_2 and N_3 to be 2- and 3-Sylow subgroups. Show that $G = N_2N_3$ and deduce that G is a semi-direct product (in some order, $N_2 \rtimes N_3$ or $N_3 \rtimes N_2$) of its Sylow subgroups.

    (c) Using the previous two questions, determine all possible groups of order twelve, up to isomorphism. That is, consider all possible semi-direct products and identify those which are isomnorphic

    (d) You have seen a few groups of order twelve, namely $\mathbb{Z}_{12}$, $\mathbb{Z}_4 \oplus \mathbb{Z}_3$, $\mathbb{Z}_2 \oplus \mathbb{Z}_6$, D_6, $\mathbb{Z}_2 \times S_3$, and A_4. Identify these with groups you constructed as semi-direct products. Does one get all them?

42. A subset X of an abelian group F is *linearly independent* if $n_1x_1 + n_2x_2 + \cdots + n_kx_k = 0$ implies that $n_i = 0$ for all i, where $n_i \in \mathbb{Z}$ and $x_1, \ldots, x_k$ are distinct elements of X.

    (a) Show that X is linearly independent if and only if every nonzero element of the subgroup $\langle X \rangle$ it generates may be written uniquely in the form $n_1x_1 + \cdots + n_kx_k$, where $n_i \in \mathbb{Z}$ and $x_1, \ldots, x_k$ are distinct elements of X.

    (b) Prove or give a counter example to the following statement:
    If F is free abelian of finite rank n, then every linearly independent subset of n elements is a basis.

    (c) Prove or give a counter example to the following statement:
    If F is free abelian, then every linearly independent subset of F may be extended to a basis of F.

    (d) Prove or give a counter example to the following statement:
    If F is free abelian, then every generating set of F contains a basis of F.