Microlocal Notes

These notes are following “Microlocal Analysis for Pseudodifferential Operators” by Grigis and Sjostrand. This will be an informal collection of notes on my reading.

Chapter 3: Pseudodifferential Operators

A Pseudodifferential operator is a Fourier integral operator $A : C_0^\infty(X) \to \mathscr{D}'(X)$ of the form

$$Au(x) = \frac{1}{(2\pi)^n} \int\int e^{i(x-y)\theta} a(x,y,\theta)u(y)dyd\theta, \quad u \in C_0^\infty(X), \tag{1}$$

where $a \in S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n)$. The space of such operators is denoted by $L^m_{\rho,\delta}$, and we say $a \in L^m_{\rho,\delta}$ is of type (ρ,δ) and order m.

Example 1. Every ordinary differential operator is a pseudodifferential operator. In fact, let $A = \sum_{|\alpha|\le m} a_\alpha(x)D_x^\alpha$ be a differential operator with $a_\alpha \in C^\infty(X)$. Using the Fourier Inversion formula, we have

$$\begin{aligned} Au(x) &= \sum_{|\alpha|\le m} a_\alpha(x) \int e^{ix\xi} D^\alpha \hat{u}(\xi)d\xi = \sum_{|\alpha|\le m} a_\alpha(x) \int e^{ix\xi} \xi^\alpha \hat{u}(\xi)d\xi \\ &= \int e^{ix\xi} \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha \hat{u}(\xi)d\xi = \int e^{ix\xi} a(x,\xi)\hat{u}(x)d\xi = \int\int e^{i(x-y)\xi} a(x,\xi)u(y)dy\frac{d\xi}{(2\pi)^n}, \end{aligned}$$

so that $A \in L^m_{1,0}(X)$.

Moreover we see that the distribution kernel is given by

$$K_A(x,y) = \int e^{i(x-y)\xi} a(x,\xi)\frac{d\xi}{(2\pi)^n}.$$

We now list some important facts about FIO’s:

1. If $K_A \in \mathscr{D}'(X \times X)$ is the distribution kernel of $A \in L^m_{\rho,\delta}(X)$, then sing supp$(K_A) \subset \Delta(X \times X)$. This is due to the fact that the phase $\varphi(x,y,\theta) = (x-y)\theta$ has vanishing differential in θ when $x = y$.

2. Since $(x-y)\theta$ is a phase function in either variable x or y for $\theta \neq 0$, the pseudodifferential operators in $L^m_{\rho,\delta}$ are continuous $C_0^\infty(X) \to C^\infty(X)$ and have continuous extensions $\mathscr{E}'(X) \to \mathscr{D}'(X)$. Both (1.) and (2.) use results from the end of chapter 1.

3. sing supp$Au \subset$ sing supp$u$ for all $u \in \mathscr{E}'(X)$. To see this, let $u \in \mathscr{E}'(X)$, and choose some $x_0 \in X \setminus$ sing supp(u). We choose disjointly supported $\varphi, \psi \in C_0^\infty(X)$ such that $\varphi = 1$ in a neighborhood of x_0 and $\psi = 1$ in a neighborhood of sing supp(u). Then $Au \equiv A\psi u \mod C^\infty(X)$, since $(u - \psi u) \in C_0^\infty(X)$. Moreover, $\varphi A \psi$ is a smoothing operator, with $\varphi(x)\psi(y)K_A(x,y) \in C^\infty(X)$ by (1.), as φ and ψ have disjoint supports. Then $\varphi A \psi u \in C^\infty$.

   So we have proven that Au is C^∞ at any point not in the singular support of u, and thus sing supp$Au \subset$ sing supp$u$.

0.1 Properly Supported Operators

If C is a closed subset of $X \times Y$, we say that C is **proper** if both of its projections are proper, meaning the inverse image of any compact set is compact. Often we will view C as the graph of a relation $Y \to X$, so that $C(K) = \{x \in X : \exists y \in K \text{ s.t. } (x,y) \in C\} = \Pi_y^{-1}(K)$ for example.

An operator $A : C_0^\infty(Y) \to \mathscr{D}'(X)$ is called **properly supported** if the set supp$K_A \subset X \times Y$ is proper. Letting $C =$ suppK_A, we notice that supp$Au \subset C($supp$u)$, and thus if A is properly supported, A is continuous $C_0^\infty(Y) \to \mathscr{E}'(X)$.

By the other projection on C, we get a unique continuous extension $\tilde{A} : C^\infty(Y) \to \mathscr{D}'(X)$. We put $\tilde{A}u = A\chi_{\tilde{X}}u$, where $\chi_{\tilde{X}} \in C_0^\infty(Y)$ is equal to 1 near $C^{-1}(\overline{\tilde{X}})$.

If $A \in L^m_{\rho,\delta}(X)$ is properly supported, then A is continuous

$$C_0^\infty(X) \to C_0^\infty,\, C^\infty(X) \to C^\infty(X),$$
$$\mathscr{E}'(X) \to \mathscr{E}'(X),\, \mathscr{D}'(X) \to \mathscr{D}'(X)$$

Thus we can compose finitely many pseudodifferental operators as long as all but one of them is properly supported.

We also have the existence of a function $\chi(x,y)$ for which suppχ is proper and $\chi = 1$ in a neighborhood of the diagonal $\Delta(X \times X)$.

Using this new function, we discover that every $A \in L^m_{\rho,\delta}(X)$ has a decomposition $A = A' + A''$, where $A' \in L^m_{\rho,\delta}(X)$ is properly supported and $L'' \in L^{-\infty}$. This is seen by using the functions $\chi(x,y)$ and $(1-\chi(x,y))$ in the integral 1. One term will have a distribution kernel with proper support, and the other will be a smoothing operator.

The next theorem is about expressing the symbol without dependence on y.

Theorem. *Let $A \in L^m_{\rho,\delta}(X)$ be properly supported. Assume $\rho > \delta$. Then $b(x,\xi) := e^{-ix\xi}A(e^{i(\cdot)\xi})$ belongs to $S^m_{\rho,\delta}(X \times \mathbb{R}^n)$ and has the asymptotic development*

$$b(x,\xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{i^{-|\alpha|}}{\alpha!} (\partial_\xi^\alpha \partial_y^\alpha a(x,y,\xi))|_{y=x}.$$

Moreover, $Au(x) = \int e^{ix\xi} b(x,\xi)\hat{u}(\xi)d\xi, u \in C_0^\infty(X)$.

We call $b(x,\xi)$ the **complete symbol** of A.