Homework 1

1. Let $F : \text{Mat}_{n \times n}(\mathbb{R}) \to \mathbb{R}$ be given by $F(A) := \det A$.

   (a.) F is a polynomial in the n^2 entries of the matrix. It is thus differentiable as a map $\mathbb{R}^{n^2} \to \mathbb{R}$. The space $\mathbb{R}^{n^2}$ is isomorphic to the space of $n \times n$ matrices as a finite dimensional normed vector space.

   (b.) Suppose A is invertible. We are looking for a linear map $D_A F : \text{Mat}_{n \times n}(\mathbb{R}) \to \text{Mat}_{n \times n}(\mathbb{R})$ such that

   $$F(A + H) = F(A) + D_A F(H) + r(H)$$

   where $r(H) = o(\|H\|)$. We have

   $$F(A + H) = \det(A + H) = \det(A) \det(I + A^{-1}H),$$

   and we can write

   $$\det(I + A^{-1}H) = 1 + \text{Tr}(A^{-1}H) + r(H),$$

   where $r(H)$ consists of terms in the determinate of at least order 2 in the components of $A^{-1}H$.

   We can bound $|r(H)|$ by a sum of absolute values of monics in elements of $A^{-1}H$ of orders at least 2. Then use $(A^{-1}H)_{ij} \leq \|A^{-1}H\| \leq \|A^{-1}\|\|H\|$, so that

   $$\lim_{\|H\| \to 0} \frac{|r(H)|}{\|H\|} = 0.$$

   Then we have

   $$\det(A + H) = \det(A) + \det(A)\text{Tr}(A^{-1}H) + \det(A)r(H),$$

   where of course the last term is still $o(\|H\|)$, and where $D_A F(H) = \det(A)\text{Tr}(A^{-1}H)$ is a linear map.

   (c.) Using the formula for the inverse of an invertible matrix A, we have

   $$D_A F(H) = \det(A)\text{Tr}(A^{-1}H) = \det(A)\frac{1}{\det(A)}\text{Tr}(\text{adj}(A)H) = \text{Tr}(\text{adj}(A)H).$$

   Since the set of invertible matrices is dense in $\text{Mat}_{n \times n}(\mathbb{R})$, we can approach a non-invertible matrix B by a sequence of invertible matrices $\{A_j\}_{j=1}^{\infty}$ so that $A_j \to B$. Since F is continuously differentiable, $\lim_{j \to \infty} D_{A_j} F(H) = D_B F(H)$. Then $\lim_{j \to infty} D_{A_j} F(H) = \text{Tr}(\text{adj}(A_j)H) = \text{Tr}(\text{adj}B)$, as trace and adjoint are continuous functions.

2. Suppose $\Gamma : \mathbb{R} \to GL_n(\mathbb{R})$ is a smooth map such that $\Gamma(t)$ is orthogonal for every $t \in \mathbb{R}$. By the product rule, we have

   $$\frac{d}{dt}(\Gamma(t)^T \Gamma(t)) = \frac{d}{dt}(\Gamma(t)^T)\Gamma(t) + \Gamma(t)^T \frac{d}{dt}\Gamma(t) = 0.$$

   And we also see that

   $$\frac{d}{dt}(\Gamma(t)^T) = \lim_{h \to 0} \frac{\Gamma(t+h)^T - \Gamma(t)^T}{h} = \lim_{h \to 0} \frac{(\Gamma(t+h) - \Gamma(t))^T}{h} = \left(\frac{d}{dt}\Gamma(t)\right)^T.$$

   Then

   $$\left(\Gamma(t)^{-1}\frac{d}{dt}\Gamma(t)\right)^T = \left(\Gamma(t)^T \frac{d}{dt}\Gamma(t)\right)^T = \left(\frac{d}{dt}\Gamma(t)\right)^T \Gamma(t) = -\Gamma(t)^T \frac{d}{dt}\Gamma(t) = -\Gamma(t)^{-1}\frac{d}{dt}\Gamma(t)$$