Homework 9

Section 51

1. Let $h, h' : X \to Y$ and $k, k' : Y \to Z$ each be homotopic pairs of maps. Let $F : X \times I \to Y$ and $G : Y \times I \to Z$ be homotopies from h to h' and k to k' respectively.

   Then define a function

$$H : X \times I \to Z \tag{1}$$
$$(x,t) \mapsto G(H(x,t),t). \tag{2}$$

   H is continuous since it is a composition of continuous functions. It is a path homotopy from $k \circ h$ to $k' \circ h'$ since we have $H(x,0) = G(H(x,0),0) = G(h(x),0) = (k \circ h)(x)$, and $H(x,1) = G(H(x,1),1) = k' \circ h'(x)$.

2. (a.) Let $[X,I]$ be the set of homotopy classes of maps $X \to I$. Let $f : X \to I$ be a continuous map. Then define

$$F(x,t) = (1-t)f(x).$$

   This is a path homotopy from f to the constant path at 0: We have $F(x,0) = f(x)$, $F(x,1) = 0$.

   Then the only element of $[X,I]$ is the class $[e]$, where $e(x) = 0$.

   (b.) Let f and g be two paths in Y, a path connected space. Then connect $f(1)$ with $g(0)$ by a path h. Then $f * h * g$ is a path containing f and g, and there is a clear homotopy between the two by moving the time interval $[0,1]$ along the path from f to g.

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2. Let α be a path from x_0 to x_1 and let β be a path from x_1 to x_2. Let $\gamma = \alpha * \beta$. Then we have

$$\hat{\gamma}([f]) = [\overline{\gamma}] * [f] * [\gamma] = [\overline{\beta}] * [\overline{\alpha}] * [f] * [\alpha] * [\beta] = \hat{\beta}([\overline{\alpha}] * [f] * [\alpha]) = \hat{\beta} \circ \hat{\alpha}([f]),$$

   verifying that $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$

3. Let X be a path connected space with $x_0, x_1 \in X$. We will show that $\pi_1(X, x_0)$ is abelian if and only if for every pair of paths α, β from x_0 to x_1, we have $\hat{\alpha} = \hat{\beta}$.

   First suppose the second condition holds. Then

$$\hat{\alpha}([\beta]) = [\hat{\alpha}] * [\beta] * [\alpha] = \hat{\beta}([\beta]) = \beta,$$

   and hence $[\alpha] * [\beta] = [\beta] * [\alpha]$.

   Conversely suppose $\pi_1(X, x_0)$ is abelian. Then

$$\hat{\alpha}([f]) = [\overline{\alpha}] * [f] * [\alpha] = [\overline{\alpha}] * [f] * [\alpha] * [\overline{\beta}] * [\beta] = [\overline{\alpha}] * [\alpha] * [\overline{\beta}] * [f] * [\beta] = [\overline{\beta}] * [f] * [\beta] = \hat{\beta}([f])$$

   since $[\alpha] * [\overline{\beta}]$ is a loop at x_0 and thus commutes with $[f]$.

4. Let r be a retraction of X onto a subspace A. Let $a_0 \in A$. Then the map

$$r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$$

   is a surjection.

   Take a loop $\tilde{f}$ in $\pi_1(A, a_0)$. Then consider this as a map f from I to X. Then we have $r \circ f = f$, since $f(t)$ is in A for all $t \in [0,1]$.

5. Let A be a subspace of $\mathbb{R}^n$ and let $h : (A, a_0) \to (Y, y_0)$. Suppose h can be extended to a map $\tilde{h}$ with domain all of $\mathbb{R}^n$.

   Let $f \in \pi_1(A, a_0)$. Then the function

$$\tilde{h}((1-t)f(s) + ta_0)$$

   is a path homotopy from $h \circ f$ to $h(a_0) = y_0$, and hence the image of every element of $\pi_1(A, a_0)$ under h is homotopic to the identity element, and h is a trivial homomorphism.