Homework 8

1. Suppose ℓ_1 does not have the Schur property: then without loss of generality, there is a sequence $\{x_n\}$ which converges weakly to 0 but not in norm. Then by the Bessage-pelczynski selection principle, there is a subsequence $\{x_{n_k}\}$ which is equivalent to a block-basic sequence of the standard basis of ℓ_1, which in turn is equivalent to the standard basis of ℓ_1. Then $[\{x_{n_k}\}]$ is isomorphic to ℓ_1.

   We want to show there is a bounded linear functional φ given by

   $$x = \sum_1^\infty a_k x_{n_k} \mapsto_\varphi \sum_1^\infty a_k.$$

   Then we need to show this map is well defined, i.e. that the series converges.

   Note that
   $$\frac{1}{C}\left\|\sum_1^\infty a_k e_k\right\| \leq \left\|\sum_1^\infty a_k x_{n_k}\right\| \leq C\left\|\sum_1^\infty a_k e_k\right\|$$

   where $\{e_k\}$ is the standard basis of ℓ_1. Then

   $$\left|\varphi\left(\sum_1^\infty a_k x_{n_k}\right)\right| \leq \sum_1^\infty |a_k| = \left\|\sum_1^\infty a_k e_k\right\| \leq C\left\|\sum_1^\infty a_k x_{n_k}\right\|,$$

   which shows φ is well defined and bounded, and linearity is obvious.

   Then we can extend this to a functional $\varphi \in \ell^\infty$ by Hahn-Banach. But then we have

   $$\varphi(x_{n_k}) = 1$$

   for all k, contradicting the weak convergence of $\{x_{n_k}\}$ to zero. So ℓ_1 must have the schur property.

2. Let X be a Banach space with the Schur property and suppose A is a weakly compact subset. By Eberlein-Smulian, A is also sequentially weakly compact, and thus norm sequentially compact, since every weak convergent subsequence is also norm convergent. Then since X is a metric space, sequential norm compactness is equivalent to norm compactness, and thus A is compact.

   Conversely suppose A is a norm compact subset of X. Since every weak open cover is also a norm open cover, we can always reduce to a finite subcover. Thus A is weakly compact.

3. Let X be a reflexive Banach space with the Schur property. By the topological characterization of reflexivity, the unit ball in X is weakly compact, and thus norm compact by problem 2. Then X must be a finite dimensional space, since the unit ball in a normed vector space is compact if and only if the space is finite dimensional.

4. Let X be a reflexive space and suppose $T : X \to \ell_1$ is a bounded operator. Since X is reflexive, the unit ball is weakly compact. Note that T is also weakly bounded. Then the image of the unit ball $T(B_X)$ is weakly compact in ℓ_1, and thus norm compact by problem 2. Since X is hausdorff, this is a stronger condition than the closure $\overline{T(B_X)}$ being compact, and thus T is a compact operator.