Homework 7

37. Let G be a group of order 3825 with a normal subgroup H of order 17. Because H is normal in G, G acts on H by conjugation inducing a map
$$\varphi : G \to \text{aut}(H).$$
Because H is cyclic, we know that $|\text{aut}(H)| = \phi(17) = 16$. Moreover, the kernel of this map will be the centralizer. We know that $\text{Im}(\varphi)$ is a subgroup of $\text{aut}(H)$, so its order must be $1, 2, 4, 8$, or 16. From lagrange's theorem, we know that $|G| = |\text{Im}(\varphi)||\ker(\varphi)|$, and thus the order of $|\text{Im}(\varphi)|$ divides 3825. Then $|\text{Im}(\varphi)| = 1$ and $\ker(\varphi) = G$. Then $ghg^{-1} = h$ for all $g \in G, h \in H$, and thus $H \subset Z(G)$.

38. First we clearly have the trivial semi-direct product $(S_2)^2 \times S_2$.

Aside from this one, we must consider homomorphisms $S_2 \to \text{aut}((S_2)^2)$. We note that $(S_2)^2 \simeq Z_2 \oplus Z_2$, and $S_2 \simeq Z_2$. I claim that $\text{aut}(Z_2 \oplus Z_2) = S_3$. There are clearly 6 possible permutations of the nonzero elements of $Z_2 \oplus Z_2$.

Moreover, each of these permutations defines an automorphism of $Z_2 \oplus Z_2$. It is helpful to regard $Z_2 \oplus Z_2$ with the presentation $\{a, b, c : a^2 = b^2 = c^2 = e, ab = c\}$. Here, we will use $a = (1, 0), b = (0, 1), c = (1, 1)$. As long as $\phi : Z_2 \oplus Z_2 \to Z_2 \oplus Z_2$ is injective and maps $e \mapsto e$, we will have $\phi(ab) = \phi(c) = \phi(a)\phi(b)$.

The only nontrivial homomorphisms $Z_2 \to S_3$ are the ones sending $1 \mapsto (ij)$ for $i, j \in \{a, b, c\}$. Without loss of generality suppose $\phi : 1 \mapsto (ab)$. Construct a semi-direct product $(Z_2 \oplus Z_2) \rtimes_\phi Z_2$. The element $(a, 1)$ is order 4:
$$(a, 1)(a, 1) = (c, 0) \tag{1}$$
$$(c, 0)(a, 1) = (b, 1) \tag{2}$$
$$(b, 1)(a, 1) = (e, 0). \tag{3}$$
The element $(c, 1)$ can similarly be verified to be of order 2. Moreover, we have
$$(c, 1)(a, 1)(c, 1) = (c, 1)(b, 0) = (b, 1) = (a, 1)^{-1}$$
and thus $(Z_2 \oplus Z_2) \rtimes_\phi Z_2 \simeq D_4$. The other two homeomorphisms create the same group.

39. Let $N = Z_2 \oplus Z_2$. We have already determined that $\text{aut}(N) = S_3$. Then to identify the group $N \rtimes \text{aut}(N)$, we need to find all possible homomorphisms $\varphi : S_3 \to S_3$. Clearly we have the trivial map which induces the ordinary product. We know the kernel of φ will be a normal subgroup of S_3. Then the only other options are for φ to have kernel $\langle(123)\rangle$ or to be injective.

We will again use the presentation $Z_2 \oplus Z_2 = \{a, b, c : a^2 = b^2 = c^2 = e, ab = c\}$.

(φ not injective) First let $\varphi : (ij) \mapsto (12)$ for all $ij \in [3]$, and let all 3-cycles be mapped to e. Then notice that this group contains three copies of D_4 due to problem 38, by taking $(a, (ij))$ instead of $(a, 1)$. This group is not S_4 because the element $(a, (abc))$ is order 6.

(φ injective) In this case φ will be some permutation of the 2-cycles. We can take φ to be the identity map, and the other cases will generate the same group.

We again get 3 copies of D_4 and some other elements, but none of order greater than 4. We can show that this group must be S_4.

40. (a.) Let A and A' be free on a set S. By the universal property of free groups, for every map $S \to B$ into an abelian group B, we get a unique homomorphism $A \to B$ of which the restriction to S is equal to the first map.

Since A and A' are free on S, there is a map $f : S \to A'$ such that $f(S)$ is a basis of A', and a map $g : S \to A$ so $g(S)$ is a basis for A. By the universal property we get a unique map $\phi : A \to A'$ such that $\phi(x) = f(x)$ for any $x \in S$. This map is clearly an ismorphism since $g(S)$ is a basis for A.

Moreover if any other map $\psi : A \to A'$ is an isomorphism, we know that $\psi(S) = f(S) \subset A'$. Then because the map ϕ above is the unique map with this property, we have $\psi = \phi$. Then ϕ is the unique isomorphism $A \to A'$.

(b.) Let M be a commutative monoid and $K(M)$ its grothendieck group. Suppose G is a group with a map $\gamma : M \to G$ such that for any abelian group B, the pullback map

$$\mathrm{Hom}_{\text{ab-gp}}(G, B) \to \mathrm{Hom}_{\text{monoid}}(M, B)$$

is a bijection.

Let $\phi : M \to K(M)$ be the universal homomorphism into its grothendieck group.

By the universal property of $K(M)$, the map γ induces a map $f : K(M) \to G$ such that $\gamma = f \circ \phi$.

Moreover, the map ϕ induces a map $g : G \to K(M)$ such that $\phi = g \circ \gamma$. Together, we have $\gamma = (f \circ g) \circ \gamma$.

Finally, since $\gamma : M \to G$, there is a unique homomorphism $p : G \to G$ such that $\gamma = p \circ \gamma$. But $\mathrm{id} : G \to G$ satisfies this property, so $p = \mathrm{id}$. Since also $(f \circ g)$ satisfies this property, we must have $f \circ g = \mathrm{id}$.

41. a. **Groups of order 3** There is only one, the cyclic group $\mathbb{Z}_3$. Its automorphism group is $\mathbb{Z}_2$, since we must either fix the generators $\{1, 2\}$ or send one to another.

   Groups of order 4 There are 2 such groups, either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

   We have shown that $\mathrm{aut}(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = S_3$.

   To calculate $\mathrm{aut}(\mathbb{Z}_4)$, we know that there will be $\phi(4) = 2$ automorphisms, and thus this group must be $\mathbb{Z}_2$.

   b. Let G be a group of order 12 and let N_3 and N_2 be 3 and 2-sylow subgroups. We have shown that either N_2 or N_3 is normal in G, and they clearly have trivial intersection. Then $G = N_2N_3 = N_3N_2$. Then one of N_2 or N_3 act on the other by conjugation, say N_3 is normal and N_2 acts on it by conjugation. Then

$$xyx'y' = x\phi(y)(x')yy'$$

   where $\phi(y)(x')$ is conjugation of x' by y. So this is a semidirect product with the automorphism given by conjugation of one subgroup by another. This is true in both cases, so we have that G is a semidirect product of N_3 with N_2 or vice versa.

   c. **($N_2 = \mathbb{Z}_4$)** If we take the action of N_3 on N_2 to be trivial, we get the ordinary product $\mathbb{Z}_4 \times Z_3 \simeq \mathbb{Z}_{12}$.

   Otherwise we first consider homomorphisms $Z_3 \to \mathrm{aut}(\mathbb{Z}_4) = Z_2$. There are none except the trivial one.

Alternatively what are the possible homomorphisms $Z_4 \to \text{aut}(\mathbb{Z}_3) = \mathbb{Z}_2$? There is the trivial one which we have already covered. There is also the one sending $0, 2 \mapsto 0$ and $1, 3 \mapsto 1$.

We can calculate that $(0, 1)$ is an element of order 4, and that $(1, 2)$ is an element of order 6. Then this is not A_4, $\mathbb{Z}_3 \oplus \mathbb{Z}_4$, $\mathbb{Z}_2 \oplus \mathbb{Z}_6$, $\mathbb{Z}_{12}$, or D_6. Then it must be another group.

($N_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$) First we consider homomorphisms $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \text{aut}(Z_3) = Z_2$. We have either the map $(a, b) \mapsto a$, or $(a, b) \mapsto b$, or $(a, b) \mapsto a + b$. I claim these all generate the same semi-direct product. Consider $(a, b) \mapsto a$. We get many elements of order 2, such as $(0, (1, 1)), (0, (0, 1))$, etc. Moreover the element $(1, (0, 1))$ is order 6, and we have

$$(1, (1, 1))(1, (0, 1))(1, (1, 1)) = (2, (0, 1)) = (1, (1, 1))^{-1}$$

so we have the group D_6. The other cases also yeild this group.

If we have the trivial homomorphism, this is the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 = \mathbb{Z}_2 \oplus \mathbb{Z}_6$.

For homomorphisms $\mathbb{Z}_3 \to \text{aut}(\mathbb{Z}_2 \oplus \mathbb{Z}_2) = S_3$, we have the trivial one covered above, and the ones sending elements into the 3-cycle. These both give the same semi-direct product. We can see that there are 6 elements of order 2, four element of order 3, the identity, and one element of order 4. This is clearly A_4.

d. We can calculate that $\mathbb{Z}_2 \oplus S_3 \simeq D_6$ based on the orders of the elements. Then we have created each of the listed groups, plus one additional group which contains elements of order 4 and 6.

42. (a.) Suppose X is linearly independent. Then if

$$x = \sum a_i x_i = \sum b_i x_i$$

we have

$$\sum (a_i - b_i) x_i = 0,$$

and thus $a_i = b_i$ and there is a unique representation of the elements in the group it generates.

Now suppose every element in $\langle X \rangle$ has a unique representation. Suppose $\sum_1^n a_i x_i = 0$. Then $\sum_1^{n-1} a_i x_i = a_n x_n = x$, and unless every $a_i = 0$, we have written the element x as two different linear combination of elements of X.

(b.) Let F be a free abelian group of rank n and let B be linearly independent. Let $B = \{b_i\}_1^n$ and let $\{x_i\}_1^n$ be a basis for F. Construct a map $\phi : F \to \langle B \rangle$ which extends the map $x_i \mapsto b_i$ to all of F by linearity. Let $x \in F$ with $x = \sum a_i x_i$, and suppose $\phi(x) = 0$. Then $\sum a_i \phi(x_i) = \sum a_i b_i = 0$, and thus $a_i = 0$ for all i since b_i is a basis for the set it generates. Then $x = 0$ and ϕ is injective. Thus $F \sim \langle B \rangle$ and B generates F.

(c.)

(d.) Let V be a generating set of F. Place a partial order on the set of linearly independent subsets of V by inclusion. Note that every totally ordered subset contains an upper bound, namely its union. Then by Zorn's lemma, there is a maximal element $B = \{x_i\}$. Suppose B does not generate F. Then there is another element $x \in F$ which is not in the span of B. Then we cannot write $\sum a_i x_i + \alpha x = 0$ unless every coefficient is zero. Then $x \cup B$ is a larger linearly independent set than B, which contradicts the maximality of B. So B is a basis of F.