Homework 1

Let X be a finite dimensional normed vector space. Show that the weak topology and the norm topology coincide.

Let τ be the norm topology on X and τ_w the weak topology. We know that $\tau_w \subset \tau$ by definition.

To prove the other direction, we will show that a weakly convegent net is norm-convergent. Let $\{x_1, \dots, x_n\}$ be a basis for x and let $\varphi_j : (a_1 x_1 + \cdots + a_n x_n) \mapsto a_j$ be linear functionals on X. Let $x_\alpha = a_{1,\alpha} x_{1,\alpha} + \cdots + a_{n,\alpha} x_{n,\alpha}$ be a net and suppose WLOG that $x_\alpha \to 0$ weakly. Then $\varphi_j(x_\alpha) = a_{j,\alpha} \to 0$ for $j = 1, \dots, n$.

Now give X the norm $\| \cdot \|' : a_1 x_1 + \cdots + a_n x_n \mapsto \sum_1^n |a_j|$ so that $\|x_\alpha\|' \to 0$. Since X is finite dimensional, all norms are equivalent and $x_\alpha \to 0$ in τ. Then $\tau \subset \tau_w$.