# Homework 10

1. (a.) Take $(y_i^k)_{i=1}^\infty = \begin{cases} 1 & i \le k \\ 0 & i > k \end{cases}$. Then for any $x^* \in \ell_1$, we have

$$x^*(y_i^k) = \sum_{i=1}^{k} x_i^* \le \sum_{1}^{k} |x_i^*| \to_k \|x^*\|_{\ell_1}$$

meaning $(y_i^k)$ is weakly cauchy. However, it is clear by taking the limit against unit vectors $e_i$ that if we did have $y_i^k \to_k (y_i)$ in norm, we must have $y_i = 1$ for all $i$. But this is not an element of $c_0$.

(b.) Suppose $X$ has the schur property. Take a weakly cauchy sequence $x_n$. Then for every $x^* \in X^*$, we must have

$$x^*(x_n - x_m) \to 0$$

in $n$ and $m$. Then for every pair of increasing sequences $n_k$ and $m_k$, $x_{n_k} - x_{m_k}$ is weakly convergent to zero and thus strongly convergent, by the schur property. Then this is equivalent to $x_n$ being norm cauchy. To see this we can suppose it is not, and this would allow us to find a pair of sequences such that the limit does not converge to zero in norm.

Then since $X$ is complete, $x_n$ converges to some element in norm and thus weakly.

2. Suppose every closed subspace of $\ell_1$ is complemented. We know that $\ell_p$ is isomorphic to the quotient $\ell_1/M$ for some closed subspace $M$. Then $M$ is complemented, and we can decompose $\ell_1 = M \oplus N$, and $N$ will be isomorphic to $\ell_p$. But then $\ell_1$ contains a subspace $N$ isomorphic to $\ell_p$, which is impossible.

Moreover, since $\ell_p$ is not isomorphic to $\ell_q$ for $p \ne q$, there are uncountably many non-isomorphic separable Banach spaces. Each must correspond to a unique uncomplemented closed subspace $M_p$ of $\ell_1$ such that $\ell_p = \ell_1/M_p$. Then there must be uncountably many such subspaces.

3. Let $T : X \to \ell_1$ be the composition of the projection onto $X/M$ with the isomorphism with $\ell_1$. Let $(e_n)$ be the canonical basis of $\ell_1$ and choose a bounded sequence $(x_n) \in X$ such that $T(x_n) = e_n$ using the open mapping theorem. Then $(e_n)$ is equivalent to $(x_n)$: We first have

$$\left\| \sum_{1}^{\infty} a_n x_n \right\| \le \sum_{1}^{\infty} |a_n| \|x_n\| \le C \sum_{1}^{\infty} |a_n|$$

since $x_n$ is bounded. We also have

$$\|T\| \left\| \sum_{1}^{\infty} a_n x_n \right\| \ge \| \sum_{1}^{\infty} a_n e_n \| = \sum_{1}^{\infty} |a_n|.$$

Now let $S : \ell_1 \to X$ be the map sending $e_n$ to $x_n$ and extend to $X$ by linearity. Then $S$ is an isomorphism of $[x_n]$ with $\ell_1$. Also, the subspace is complemented with $P = ST$ since $STST = S(TS)T$, and $TS$ is the identity on $\ell_1$.

This solution is not my original work.