Homework 5

Let $(h_n)_{n\geq 1}$ be the Haar system.

1. We will prove that this is a monotone basis for the space $L_p([0,1])$ for $p \in [1,\infty)$. By the Grunbaum criterion, it is enough to show there exists a C such that for all $(a_i)_{i\geq 1} \subset \mathbb{R}$ and $m > n$, we have

$$\|\sum_1^n a_i h_i\|_p \leq \|\sum_1^m a_i h_i\|_p,$$

and that the closure of the span of (h_n) is all of $L_p([0,1])$. Moreover, (h_i) is monotone iff $C = 1$.

It is enough to show that

$$\|\sum_1^n a_i h_i\|_p \leq C\|\sum_1^{n+1} a_i h_i\|_p \tag{1}$$

for any n. In this case, the function $\sum_1^n a_i h_i$ will be a constant value, say r, on the support of h_{i+1}. We can write

$$\left\|\sum_1^n a_i h_i\right\|_p^p = \int_{[0,1]} \left|\sum_1^n a_i h_i(t)\right|^p dt = \int_{\text{supp}(h_i)} |r|^p dt + \int_{[0,1]\setminus\text{supp}(h_i)} \left|\sum_1^n a_i h_i(t)\right|^p dt,$$

$$\left\|\sum_1^{n+1} a_i h_i\right\|_p^p = \int_{\text{supp}(h_i)} |r + h_{n+1}|^p dt + \int_{[0,1]\setminus\text{supp}(h_i)} \left|\sum_1^n a_i h_i(t)\right|^p dt,$$

and thus the inequality 1 holds if and only if we have

$$\int_{\text{supp}(h_i)} |r|^p dt \leq \int_{\text{supp}(h_i)} |r + h_{n+1}|^p dt.$$

Then it clearly suffices to show that for any a, b, we have

$$\int_0^1 |a|^p \leq \int_0^1 |a + bh_1|^p = \int_0^{1/2} |a+b|^p + \int_{1/2}^1 |a-b|^p = \frac{|a-b|^p + |a+b|^p}{2}.$$

But this is true by the convexity of the function $|x|^p$, since for any convex φ, we have

$$\varphi\left(\frac{x+y}{2}\right) \leq \frac{\varphi(x)+\varphi(y)}{2},$$

and thus

$$|a|^p = \left|\frac{a+b+a-b}{2}\right|^p \leq \frac{|a+b|^p + |a-b|^p}{2}.$$

Note that the span of (h_n) is dense in the set of indicator functions on diadic interals. For example, the function $\chi_[0,0.5] = 0.5h_1 + 0.5h_2$. These functions are dense in the space of simple functions, which are dense in $L_p[0,1]$. Then (h_n) is dense in $L_p[0,1]$.

Another way to solve this is by using biorthogonal functionals. Define

$$h^*_{2^k+r}(f) = \int_0^1 2^k h_{2^k+r} f.$$

These are clearly linear functionals on $L_p[0,1]$. Then we have

$$h_i^*(h_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Then to prove that (h_n) is a basis, we need to show that we can express any $x \in L_p[0,1]$ as a sum

$$x = \sum_1^\infty h_i^*(x) h_i$$

My idea was to first show this sum converges to some element in L_p.

After having done this, suppose that $h_i^*(f) = 0$ for all i for some $f \in L_p[0,1]$. Denote the average value of f on the interval $[a,b]$ by $A[a,b]$. From the functionals evaluating to zero on f, we know that we must have $A[0,0.5] = A[0.5,1]$. Moreover, since the average value $A[0,1]$ is the average of the two above, we know that $A[0,0.5] = A[0,1]$, and the same for $A[0.5,1]$. Then we can continue this process to show that f must have the same average value on every diadic interval. Moreover, $h_0^*(f) = \int_0^1 f = 0 = A[0,1]$. Then since the average value on an interval of radius r containing x converges to $f(x)$ almost everywhere for any locally integrable function, we must have that $f(x) = 0$ for almost every x.

Then note that $h_i(x - \sum_0^\infty h_i^*(x)h_i) = h_i^*(x) - h_i^*(x) = 0$, so that indeed $x = \sum_1^\infty h_i^*(x)h_i$.

2. Here we can use either approach as well. First note that for any N and $(a_i)_1^{N+1}$, the function $\sum_1^N a_i\varphi_i$ must achieve its maximum at either the endpoints of $[0,1]$ or at the center of one of the supports of φ_i, $1 \leq i \leq N$. Then adding φ_{N+1} to this sum cannot decrease the supremum, which shows that

$$\left\| \sum_1^N a_i \varphi_i \right\| \leq \left\| \sum_1^{N+1} a_i \varphi_i \right\|.$$

We can define linear functionals

$$\varphi_{2^k+1}^*(f) = \int_0^1 2^k h_{2^k+r-1} f'$$
$$\varphi_0^*(f) = f(0)$$

on the dense subspace C^1 of $C[0,1]$, and then extend these to $C[0,1]$ using Hahn-Banach. One can see that these functionals have the desired property evaluated at each φ_i.

Similar to above, we must somehow show that the sum

$$\sum_1^\infty \varphi_i^*(f)\varphi_i$$

converges.

Having done this, note that for $f \in C^1[0,1]$, if $\varphi_i^*(f) = 0$ for all i, we must have that $f(0) = 0$, and that the derivative of f is zero everywhere. Then $f = 0$. The same applies then for continuous functions by density. Thus

$$f = \sum_1^\infty \varphi_i^*(f)\varphi_i$$

and we are done.