Homework 3

Section 18

4. We will show the function f is an embedding, and g will follow in the same way. f is clearly a bijection onto its image. We will show f is open and continuous.

   Let U be open in X. Then $f(U) = U \times \{y_0\}$ is open in the subspace topology. Also let W be open in the subspace topology on $X \times \{y_0\}$ so that $W = V \cap (X \times \{y_0\})$ for some V open in $X \times Y$. Then $f^{-1}(W) = \pi_X(V)$ is open in X, where π_X is the projection onto X.

5. We will show that $(a, b) \subset \mathbb{R}$ is homeomorphic with $(0, 1)$, and the same is true for the corresponding closed intervals.

   Let $f : \mathbb{R} \to \mathbb{R}$ be the map $x \mapsto \frac{x-a}{b-a}$ which maps (a, b) to $[0, 1]$ and $[a, b]$ to $[0, 1]$. f is clearly a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$, and the restriction of f to (a, b) ($[a, b]$) is continuous. Moreover the same is true when restricting the inverse function to $(0, 1)$ ($[0, 1]$) and thus f is a homeomorphism in both cases.

7a. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous from the right. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$ when $x > a$.

   Choose some open set V in $\mathbb{R}$ and choose some $x \in \mathbb{R}$ such that $f(x) \in V$. Without loss of generality suppose $V = (f(x) - \varepsilon, f(x) + \varepsilon)$, otherwise choose a basic open set in V containing $f(x)$. Then using the δ from above, let $U \subset \mathbb{R}$ be the set $[x, x + \delta)$. Then $a \in U$ implies $|x - a| < \delta$ which implies $|f(x) - f(a)| < \varepsilon$ and $f(a) \in V$. Then $f(U) \subset V$ and we are done.

8. Let Y be an ordered set equipped with its order topology. Let $f, g : X \to Y$ be continuous functions from a topological space X.

   (a.) We will show the set $\{x : f(x) \leq g(x)\}$ is closed in X. Define a function $h(x) = f(x) - g(x)$ which is also continuous mapping $X \to Y$. Then the set $h^{-1}(\{y \leq 0\})$ is closed in X and is equal to the set $\{x : f(x) \leq g(x)\}$.

   (b.) Let $h(x)$ be the function $h(x) = \min\{f(x), g(x)\}$. We show h is continuous. Let $A = \{x : f(x) \leq g(x)\}$ and $B = \{x : f(x) \geq g(x)\}$. Both are closed by the above result. Moreover, $f = g$ on $A \cap B$. Thus by the pasting lemma, the function

$$p(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

   is continuous and equal to $h(x)$.

13. Let $A \subset X$ and $f : A \to Y$ be continuous with Y hausdorff. Suppose g_1 and g_2 are two continuous extensions of f to the domain $\overline{A}$.

   Choose any $x_0 \in \overline{A} \setminus A$ and choose open sets U_1 and U_2 containing $g_1(x_0)$ and $g_2(x_0)$ respectively. Then $g_1^{-1}(U_1)$ and $g_2^{-1}(U_2)$ are two open sets containing x_0 in X, and thus their intersection $g_1^{-1}(U_1) \cap g_2^{-1}(U_2) = V$ is an open set containing x_0 as well, and must intersect A as $x \in \overline{A} \setminus A$. Choose any point $y \in V \cap A$, so that $g_1(y) = g_2(y)$ and thus $U_1 \cap U_2 \neq \varnothing$.

   Then we have shown that there are no disjoint sets containing $g_1(x_0)$ and $g_2(x_0)$, which together with the hausdorffness of Y implies that $g_1(x_0) = g_2(x_0)$, and thus this holds for any point in $\overline{A}$. Then the extension of f is uniquely determined by its values on A.

Section 19

6. Let $\{\mathbf{x}_i\}_1^\infty$ be a sequence in $\prod X_\alpha$ which converges to $\mathbf{x}$. We will show convergence in the product topology is equivalent to convergence pointwise in each coordinate.

   First suppose $\mathbf{x}_n$ converges in the product topology. Then for any U_α open in X_α, the set $\pi_\alpha^{-1}(U_\alpha)$ is open in the product space, and we can find some $N > 0$ such that $\mathbf{x}_n \in \pi_\alpha^{-1}(U_\alpha)$ for $n > N$, implying $\pi_\alpha(\mathbf{x}_n) \in U_\alpha$ for $n > N$.

   Conversely suppose $\pi_\alpha(\mathbf{x}_n)$ converges in each X_α. Choose any open set $U = \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})$ in the product topology. Choose $N = \min\{N_1, \dots, N_n\}$ such that $\pi_{\alpha_i}(\mathbf{x}_n) \in U_{\alpha_i}$ for $n > N_i$. Then $\mathbf{x}_n \in U$ for $n > N$.

8. Let $(a_1, a_2, \dots)$ and $(b_1, b_2, \dots)$ be sequences with $a_i > 0$. Let $h : \mathbb{R}^\omega \to \mathbb{R}^\omega$ be the function

$$h((x_1, x_2, \dots)) = (a_1 x_1 + b_1, a_2 x_2 + b_2, \dots).$$

   h is clearly a bijection. Then we will show it is both open and continuous. Let $U = \prod_1^\infty U_n$ be an open set in either the box topology or the product topology, in which case only finitely many $U_n \neq \mathbb{R}$. We have $h(U) = (a_1 U_1 + b_1, a_2 U_2 + b_2, \dots)$, and in either topology, for each n, these sets are open.
   Moreover $h^{-1}(U) = (\frac{U_1 - b_1}{a_1}, \frac{U_2 - b_2}{a_2}, \dots)$, and again for each n the sets are open. Then h is a homeomorphism in either topology.

Section 20

3. Let X be a metric space with metric d. (a.) We will show the metric is continuous as a function $d : X \times X \to \mathbb{R}$. We can use the sequence criterion for continuity since $X \times X$ is a metric space. Let $(x_n, y_n) \to (x, y)$ in $X \times X$, so that $d(x_n, x) \to 0$ and $d(y_n, y) \to 0$. Then

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n),$$

   implying $\lim_{n\to\infty} d(x_n, y_n) \leq d(x, y)$. Moreover,

$$d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y, y_n),$$

   implying $d(x, y) \leq \lim_{n\to\infty} d(x_n, y_n)$. Then $d(x_n, y_n) \to d(x, y)$ and d is continuous.