655 Notes

Product Topology

Let Γ be a set and $(X_\gamma, \tau_\gamma)_{\gamma\in\Gamma}$ a collection of topological spaces. The Product topology on $\prod_{\gamma\in\Gamma} X_\gamma$ is defined as the weakest topology on $\prod_{\gamma\in\Gamma} X_\gamma$ which makes the projection maps $\pi_\gamma : \prod_{\gamma\in\Gamma} X_\gamma$ continuous.

Example. On $\mathbb{R}^\Gamma$, the product topology is given by the following neighborhood basis:

$$\{U(x;\gamma_1,\ldots,\gamma_n;\varepsilon) : \gamma_1,\ldots,\gamma_n \in \Gamma, \varepsilon > 0, n \geq 1, x \in \mathbb{R}^\gamma\},$$

where $U(x;\gamma_1,\ldots,\gamma_n;\varepsilon) := \{z \in \mathbb{R}^\Gamma : |z_{\gamma_i} - x_{\gamma_i}| < \varepsilon, 1 \leq i \leq n\}$.

$\mathbb{R}^\Gamma$ with the product topology is hausdorff.

Locally Convex Topological Vector Spaces

Definition. A **topological vector space** is a vector space X equipped with a topology τ such that the maps

$$\begin{aligned} A :& X \times X \to X & \Omega :& \mathbb{R} \times X \to X \\ & (x_1, x_2) \mapsto x_1 + x_2 & & (a, x) \mapsto ax \end{aligned}$$

are both continuous.

A TVS is locally convex if every point has a local base consisting of convex sets.

Example. An arbitrary product of LCTVS's is an LCTVS with the product Topology. A vector subspace of an LCTVS is an LCTVS when given the relative topology.

Dual Pairs

Let E be a vector space and let $E^\# := \{f : E \to \mathbb{R} : f \text{ is linear}\}$ be the algebraic dual space.

Let E and F be vector spaces. Then a bilinear form $\langle\cdot,\cdot\rangle : E \times F \to \mathbb{R}$ induces two maps:

$$\begin{aligned} \varphi :& E \to F^\# & \psi :& F \to E^\# \\ & e \mapsto f \mapsto \langle e, f\rangle & & f \mapsto e \mapsto \langle e, f\rangle. \end{aligned}$$

Definition. A dual pair is a pair of vector spaces E, F and a bilinear map $\langle\cdot,\cdot\rangle : E \times F \to \mathbb{R}$ such that

a.) E separates points in F, meaning for all $f_1, f_2 \in E$, $f_1 \neq f_2$, there is an $e \in E$ such that $\langle e, f_1\rangle \neq \langle e, f_2\rangle$.

b.) F separates points in E.

We write $\langle E, F\rangle$ is a dual pair.

Remark. The statement that E separates points in F is equivalent to the statement that for $f \in F$, if for all $e \in E$, $\langle e, f\rangle = 0$, then $f = 0$. Then ψ is an injection, and we can identify F with its image in under ψ in $E^\#$

The dual statement is that if F separates points in E, we can identify E with its image under φ in $F^\#$.

Example. Given a vector space E, $\langle E, E^{\#}\rangle$ is a dual pair for $\langle\cdot,\cdot\rangle : E \times E^{\#} \to \mathbb{R}$ given by $(e, e^{\#}) \mapsto e^{\#}(e)$.

Example. Given a normed vector space X, $\langle X, X^*\rangle$ is a dual pair for $\langle\cdot,\cdot,\rangle : X \times X^* \to \mathbb{R}$ given by $(x, x^*) \mapsto x^*(x)$.

Definition. Let $\langle E, F\rangle$ be a dual pair. The weak topology associated to the dual pair, denoted by $\sigma(E, F)$, is defined as the restriction to E of the product topology on $\mathbb{R}^F$.

Remark. We showed that we can view E as a subset of $F^{\#}$ by the injection φ. $F^{\#}$ is a subset of $\mathbb{R}^F$, the space of all maps $F \to \mathbb{R}$, consisting of those maps which are linear. Then we can view E as a subset of $\mathbb{R}^F$.

Example. Let X be a normed vector space and consider the dual pair $\langle X, X^*\rangle$, with $\langle e, e^*\rangle = e^*(e)$. The topology $\sigma(X, X^*)$ on X is called the weak topology. The topology $\sigma(X^*, X)$ on X^* is called the weak* topology.

We now give some equivalent definitions for the weak topology in the case that X is a normed vector space and $\langle X, X^*\rangle$ is our dual pair.

Weak Topology

The weak topology on X is given by:

- The topology generated by the sets

$$\begin{aligned} U(x_0; x_1^*, \ldots, x_n^*; \varepsilon) &= \{x \in X : |\langle x_0, x_i^*\rangle - \langle x, x_i^*\rangle| < \varepsilon, 1 \le i \le n\} \\ &= \{x \in X : |x_i^*(x_0) - x_i^*(x)| < \varepsilon, 1 \le i \le n\} \end{aligned}$$

- If $\{x_\alpha\}_\alpha$ is a net in X and $x \in X$, then $x_\alpha \to x$ weakly if and only if for all $x^* \in X^*$, $x^*(x_\alpha) \to x^*(x)$
- the weakest topology on X which makes all of the bounded linear functionals on X continuous.

Weak* Topology

The weak* topology on X^* is given by

- the topology generated by sets

$$U(x_0^*; x_1, \ldots, x_n; \varepsilon) = \{x^* \in X^* : |x_0^*(x_i) - x^*(x_i)| < \varepsilon, 1 \le i \le n\}$$

- $x_\alpha^* \to x^*$ in the weak* topology if and only if $x_\alpha^*(x) \to x^*(x)$ for all $x \in X$
- the weakest topology on X^* for which the maps $x^* \to x^*(x)$ are continuous for every $x \in X$.

Remark. The map $i : (X^*, \sigma(X^*, X)) \to \mathbb{R}^X, x^* \mapsto (x^*(x))_{x\in X}$ is a homeomorphism from $(X^*, \sigma(X^*, X))$ onto its image in $\mathbb{R}^X$ with the product topology.

We have $x_\alpha^* \to x^*$ in the weak* topology if and only if for all $x \in X$, $x_\alpha^*(x) \to x^*(x)$, if and only if $i(x_\alpha^*) \to i(x^*)$ in the product topology.

Remark. The map $j : (X, \sigma(X, X^*)) \to X^{**} \subset \mathbb{R}^{X^*}, x \mapsto (x^*(x))_{x^* \in X^*}$ is a homeomorphism from $(X, \sigma(X, X^*))$ onto its image in $(X^{**}, \sigma(X^{**}, X^*))$.

We have $x_\alpha \to x$ weakly if and only if for all $x^* \in X^*$, $x^*(x_\alpha) \to x^*(x)$ if and only if $j(x_\alpha) \to j(x)$ in the weak* topology on X^{**}.

Proposition. *Let X be a normed space.*

1. $(X, \sigma(X, X^*)) = X^*$
2. $(X^*, \sigma(X^*, x))^* = j(X)$

Proof. (1.) We have $(X, \sigma(X, X^*))^* \subset X^*$ because $\sigma(X, X^*)$ is weaker than the norm topology, thus every functional which is weak-continuous is also norm-continuous. That $X^* \subset (X, \sigma(X, X^*))$ follows by construction, since $\sigma(X, X^*)$ ensures that each functional which is norm-continuous is also $\sigma(X, X^*)$ continuous.

(2.) We have $j(X) \subset (X^*, \sigma(X^*, X))^*$ by construction, since $\sigma(X^*, X)$ is a topology such that the maps $j(x)$ are continuous.

To show the other direction, let $\varphi : (X^*, \sigma(X^*, X)) \to \mathbb{R}$ be a weak* continuous functional on X^*. Since φ is continuous, there is a weak* neighborhood $U \ni 0$ in X^* such that $U \subset \varphi^{-1}(-1, 1)$.

From one of the above characterizations of the weak* topology, we know that there must be elements $x_1, \ldots, x_n$ such that $U = \{x^* : |x^*(x_i)| < \varepsilon \text{ for} 1 \leq i \leq n\}$. Now suppose $f^* \in \bigcap_1^n \ker x_i$. In particular, we have $|f^*(x_i)| = 0 < \varepsilon$ for $i = 1, \ldots, n$, thus $f^* \in U$. Then for any $\lambda > 0$, $|\lambda f(x_i)| = \lambda 0 = 0 < \varepsilon$ for $i = 1, \ldots, n$, thus $\lambda f^* \in U$, and we have $|\varphi(\lambda f^*)| < 1$ and thus $|\varphi(f^*)| < 1/\lambda$.

Since this holds for all $\lambda > 0$, it must be that $\varphi(f^*) = 0$ and $f^* \in \ker \varphi$. We have therefore shown that $\ker \varphi \subset \bigcap_1^n \ker x_i$. Then linear algebra tells us that φ must be a linear combination of the functionals x_i, $\varphi = \sum_1^n a_i x_i := x$. Then $j(x) = \varphi$ □

Theorem (Banach-Alaoglu Theorem)**.** *Let X be a normed vector space. Then $(B_{X^*}, \sigma(X^*, X))$ is a compact topological space.*

Proof (outline) Observe that for all $x \in X, x^* \in X^*$, $\|x^*(x)\| \leq \|x^*\| \|x\|$. Then B_{X^*} embeds in $\mathbb{R}^X$ by the map

$$\begin{aligned} i : & B_{X^*} \to \prod_{x \in X} [-\|x\|, \|x\|] \subset \mathbb{R}^X \\ & x^* \mapsto (x^*)_{x \in X}. \end{aligned}$$

$K := \prod_{x \in X} [-\|x\|, \|x\|]$ is compact by Tychonoff's theorem. $i(B_{X^*})$ consists of only the elements of K that are linear. To finish show nets in $i(B_{X^*})$ converge to linear elements of K. □

Theorem. *If X is reflexive, then $(B_X, \sigma(X, X^*))$ is compact.*