Homework 6

Section 26

4. We must show that every compact subset of a metric space is closed and bounded.

   Let X be a metric space and A a compact subspace of X. Then choose any point $a \in A$. The collection of balls centered at a with radius $r = 1, 2, \dots$ is an open cover of A, since these balls cover X. Then there is a finite subcover, and we can choose the largest ball out of this subcover, say of radius r. Then A is contained in a ball of radius r and thus each of its points are within $2r$ distance of one another.

   Since every metric space is hausdorff, A is closed, since a compact subset of a hausdorff space is closed.

   Now we give an example of a metric space in which not every closed bounded subpace is compact. Consider the space $C([0,1])$ of continuous functions mapping $[0,1]$ into $\mathbb{R}$, equipped with the metric

   $$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

   Define $A := \{f \in C([0,1]) : |f(x)| \leq 1 \text{ for all } x \in [0,1]\}$. A is bounded in the sup metric since the distance between any two of its elements is bounded by 2. Is is closed by the sequential characterization, since if $f_n \to f$ for some $\{f_n\} \subset A$, then for $x \in [0,1]$, $f_n(x) \to f(x)$ and thus $|f(x)| \leq 1$.

   However, it is not compact, since the sequence

   $$f_n(x) = \begin{cases} -nx + 1 & 0 \leq x \leq \frac{1}{n} \\ 0 & \frac{1}{n} < x \leq 1 \end{cases}$$

   does not converge to any element of A. This is because for any n, we can choose $m > n$ such that the distance between f_n and f_m is greater than $1/2$. Then suppose there was some f such that for any $\varepsilon > 0$ we could choose N such that $d(f_n, f) < \varepsilon/2$ for all $n > N$. For $n > N$, we could choose $m > n$ such that

   $$1/2 \leq d(f_n, f_m) \leq d(f_n, f) + d(f_m, f) \leq \varepsilon$$

   which is a contradiction for $\varepsilon < 1/2$.

6. Let $f : X \to Y$ for X compact and Y hausdorff. We will show f takes closed sets to closed sets.

   Let $A \subset X$ be closed. Then A is compact because a closed subset of a compact space is compact. Then $f(A)$ is compact in Y because continuous functions map compact sets to compact sets. Then $f(A)$ is closed in Y because compact subsets of hausdorff spaces are closed.

Section 27

2. Let X be a metric space with the metric d, and $A \subset X$ a nonempty subset.

   a. Show that $d(x, A) = 0$ if and only if $x \in \overline{A}$.

      First suppose $d(x,A) = 0$. Then by the definition of infimum, for every $n = 1, 2, \dots$, there is a point $x_n \in A$ such that $d(x, x_n) < 1/n$. Then $x_n \to x$ and thus $x \in \overline{A}$ since X is a metric space.

      Now suppose $x \in \overline{A}$. Then X is a limit point of A, so there must be a sequence x_n converging to x, since X is first countable. Then is is clear that $d(x, A) = 0$, as we can get arbitrarily close to x by points in A.

b. Now suppose A is compact, and we will show there is a point a such that $d(x, A) = d(x, a)$. Choose a sequence $x_n \in A$ such that $d(x_n, x) \to d(A, x)$ by the argument above. Since X is a metric space, A is sequentially compact and there is a subsequence x_{n_k} converging to some $a \in A$. By the continuity of the metric, we have $d(x_{n_k}, x) \to d(a, x)$, and we already know that $d(x_{n_k}, x) \to d(A, x)$ because subsequences of convergent sequences converge to the same limit. Then by uniqueness of limits in $\mathbb{R}$, we have $d(x, a) = d(x, A)$.

c. We will show the ε-neighborhood of A is the union of balls of radius ε with centers in A.

First suppose $x \notin \bigcup_{a \in A} B(a, \varepsilon)$. Then the distance between x and every $a \in A$ is at least ε, so $d(x, A) \geq \varepsilon$. Then $x \notin U(A, \varepsilon)$.

Conversely suppose $x \in \bigcup_{a \in A} B(a, \varepsilon)$. Then x is in some ε-ball with its center in A, and thus the distance to A is less than epsilon. So $x \in U(A, \varepsilon)$.

Then $\bigcup_{a \in A} B(a, \varepsilon) = U(A, \varepsilon)$.

d. Suppose A is compact and U is an open set containing A. We will show that some ε-neigborhood of A is contained in U.

The function $a \mapsto d(a, U^c)$ for $a \in A$ is a continuous function, using the definition of $d(a, U^c)$ and the triangle inequality. It is defined on a compact set, and thus achieves a minimum. Suppose this minimum is 0. Then there is a point $a \in A$ such that $d(a, U^c) = 0$. But this means that $a \in U^c$ which is a contradiction since $A \subset U$. Then there is some $\varepsilon > 0$ such that $d(a, U^c) \geq \varepsilon$ for all $a \in A$, and thus the ε neighborhood of A is entirely contained in U.

e. We will construct a closed set A and an open set $U \supset A$ such that no ε neighborhood of A is contained in U.

Let
$$A = \bigcup_{1}^{\infty} [n - 0.5, n - \frac{1}{3n}]$$
and let
$$U = \bigcup_{1}^{\infty} = (n - 1, n).$$
Then A is closed, U is open, $A \subset U$, but points in A can be arbitrarily close to points in U.

Section 28

4. Suppose X is a T1 space. Then we show that countable compactness is equivalent to limit point compactness.

First suppose X is countably compact, and let A be a subset. We will prove the contrapositive, that if A has no limit points then A is finite. Then suppose A has no limit points. We know A is closed vacuously. Choose some $x_1 \in A$; there exists an open set U_1 containing x_1 such that U_1 is disjoint from A except at x_1 since x_1 is not a limit point. Then continue for $n = 2, 3, \ldots$ to obtain a sequence x_n and a countable collection of open sets U_n. Then the U_n's, together with the open set $X \setminus A$, form a countable open cover of X. Then take a finite subcover which may contain $X \setminus A$ along with $U_{n_1}, \ldots, U_{n_k}$ open sets which cover X. But $\{x_n\} \not\subset X \setminus A$, and each U_{n_i} may contain only one x_n. Then the set $\{x_n\}$ must be finite, and it follows that there can be no countable subset of A, hence A is finite.

Conversely, suppose X is limit point compact, but that there is a countable open cover $\{U_n\}$ which has no finite open cover. For each $n = 1, 2, \ldots$, choose a point $x_n \in X \setminus (\cap_1^n U_n)$. Then $\{x_n\}$ is an infinite set, and thus has a limit point x which must be contained in some U_N. But since X is T1, every neighborhood of x must intersect $\{x_n\}$ at infinitely many points, which contradicts the construction of the sequence $\{x_n\}$, as only the first N terms may be contained in U_N.

Section 29

5. Let $f : X_1 \to X_2$ be a homeomorphism of locally compact spaces. We will show that f extends to a homeomorphism $F : Y_1 \to Y_2$, the one point compactifications of X_1 and X_2 respectively.

   Extend f to Y_1 by mapping ∞_1 to ∞_2. It is clear that f is still a bijection. To show it is continuous, first consider some open $U \subset Y_2$ which does not contain ∞ and is open in X_2. Since f is a homeomorphism $X_1 \to X_2$, we know $F^{-1}(U) = f^{-1}(U)$ is open. Now suppose $V = Y_2 \setminus C$ is an open set in Y_2 with $C \subset X_2$. Then $F^{-1}(V) = F^{-1}(Y_2) \setminus F^{-1}(C) = Y_1 \setminus f^{-1}(C)$. Since f^{-1} is continuous and takes compact sets to compact sets, $Y_1 \setminus f^{-1}(C)$ is open in Y_1.

   The proof that F^{-1} is continuous is the same.

10. Let X be locally compact at a point $x \in X$ and let U be an open set containing x. We will show that there is an open set V containing x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

    Since X is locally compact at x, there is a compact neighborhood C which contains an open neighborhood E of x. Then $Q := U \cap E$ is an open set within C, so that $Q^c \cap C$ is a closed set contained in a compact set and is compact. Then there exist disjoint open sets $V \ni x$ and $W \supset Q^c \cap C$. Now the set $W^c \cap C$ is a closed set contained in Q and thus contained in U, and V we can assume V is contained in $W^c \cap C$ by intersecting it with Q. Thus $\overline{V}$ is contained in $W^c \cap C$ since this is a closed set, and thus $\overline{V} \subset U$. Also since V is contained in C, we know that $\overline{V}$ is compact.