Homework 1

1. Let G be a group, $H \subset G$ be a subgroup, and $g \in G$. We will show $gHg^{-1} = \{ghg^{-1} : h \in H\}$ is a subgroup.

   It contains the identity since $geg^{-1} = gg^{-1} = e$.

   It contains inverses: let gag^{-1} be some element of gHg^{-1}, with $a \in H$. Then since H is a subgroup, $a^{-1} \in H$, and $ga^{-1}g^{-1} \in gHg^{-1}$ is the inverse of gag^{-1} since $gag^{-1}ga^{-1}g^{-1} = gaa^{-1}g^{-1} = e$.

   Finally, gHg^{-1} is closed under multiplication since for $gag^{-1}, gbg^{-1} \in gHg^{-1}$, $gag^{-1}gbg^{-1} = gabg^{-1}$. As H is a subgroup, we have $ab \in H$ and thus $gabg^{-1} \in gHg^{-1}$.

   Now we show that gHg^{-1} is isomorphic to H. I claim the map $\varphi : H \to gHg^{-1}$ defined by $a \mapsto gag^{-1}$ is an isomorphism of subgroups. It is a group homomorphism since $\varphi(ab) = gabg^{-1} = gag^{-1}gbg^{-1} = \varphi(a)\varphi(b)$. It is injective: suppose $\varphi(a) = e$. Then $gag^{-1} = e$, and thus $a = g^{-1}eg = e$. It is surjective: choose any $gag^{-1} \in gHg^{-1}$. Then $\varphi(a) = gag^{-1}$. Thus φ is an isomorphism.