Homework 4

Section 20

5. Let $\mathbb{R}^\infty$ be the subset of $\mathbb{R}^\omega$ consisting of sequences which are eventually 0. Equip $\mathbb{R}^\omega$ with the uniform topology induced by the uniform metric D.

   Claim: The uniform closure of $\mathbb{R}^\infty$ is c_0, the space of convergent sequences in $\mathbb{R}$.

   Because the closure of $\mathbb{R}^\infty$ is the smallest closed set containing $\mathbb{R}^\infty$, to show that $\overline{\mathbb{R}^\infty} \subset c_0$, we just need to show that c_0 is closed in the uniform topology. To do this, we will show c_0 contains all of its limit points.

   Let (x_n) be a limit point of c_0 and suppose $x_n \to a > 0$. Then choose $\varepsilon = a/2 > 0$, and we are guaranteed some $(y_n) \in c_0$ such that $D((x_n),(y_n)) < a/2$. But this implies $|y_i| > a/2 > 0$ for all i, contradicting that $|y_i| \to 0$. Then we must have $(x_n) \in c_0$ and c_0 is closed.

   Conversely, in order to show $c_0 \subset \overline{\mathbb{R}^\infty}$, it is enough to show that every point in c_0 is a limit point of $\mathbb{R}^\infty$. Take any $(y_n) \in c_0$, and let $\varepsilon > 0$. Choose $N > 0$ such that $|y_i| < \varepsilon$ whenever $i \geq N$. Define a sequence in $\mathbb{R}^\infty$ by

$$(x_n) = \begin{cases} y_i & n < N \\ 0 & n \geq N \end{cases}.$$

   Clearly (x_n) is eventually zero. Then we have

$$D(x_n, y_n) \leq \sup_i |x_i - y_i| = \sup_{i \geq N} |y_i| \leq \varepsilon,$$

   and thus every ε-ball about (x_n) intersects $\mathbb{R}^\infty$ at a point other than (x_n).

   Then $\overline{\mathbb{R}^\infty} = c_0$.

7. Let $(a_n)_1^\infty$ and $(b_n)_1^\infty$ be sequences of real numbers. Consider the map

$$\begin{aligned} h :&\mathbb{R}^\omega \to \mathbb{R}^\omega \\ &(x_n) \mapsto (a_1x_1 + b_1, a_2x_2 + b_2, \dots) \end{aligned}$$

   Let f be the map $(x_n) \mapsto (a_1x_1, a_2x_2, \dots)$, and g the map $(x_n) \mapsto (x_1 + b_1, x_2 + b_2, \dots)$. Then $h = g \circ f$, and in order to show h is continuous we may consider f and g separately.

   For g, i claim there is no restriction of the values of b_i in order for g to be continuous. This is because for any $g(x)$ in the image of g, the preimage of any open ball about $f(x)$ is just the open ball of the same radius about x. Then it is clear that g is continuous.

   In order for f to be continuous, I claim that the sequence of $|a_i|$'s must be bounded. First suppose this is the case, and we will show that f is indeed continuous. Let $a = \sup_i |a_i|$ and choose $\varepsilon > 0$. Let $(y_n) \in \mathbb{R}^\omega$ such that $d((x_n),(y_n)) < \frac{\varepsilon}{a}$. Then

$$|x_i - y_i| < \frac{\varepsilon}{a}$$

   assuming $\frac{\varepsilon}{a} < 1$, and thus

$$|f(x_i) - f(y_i)| = |a_i||x_i - y_i| \leq a|x_i - y_i| < \varepsilon.$$

   Then f is continuous, with $\delta = \frac{\varepsilon}{a}$.

   Now suppose $|a_i|$ is unbounded. For any $\delta > 0$, choose some $(y_n) \in \mathbb{R}^\omega$ such that $|x_i - y_i| = \frac{\delta}{2}$. We see that

$$|f(x_i) - f(y_i)| = |a_i||x_i - y_i| = |a_i|\delta,$$

and for any $N > 0$ there is some i such that $|a_i| > N$ and $|f(x_i) - f(y_i)|$ is unbounded. Then this implies f cannot be continuous if we choose $0 < \varepsilon < 1$.

Together, as long as $|a_i|$ is bounded, the function h will be continuous in the uniform metric.

8. (a.) Let $X \subset \mathbb{R}^\omega$ be the space of square summable sequences. We will show that on X there are inclusions

$$\text{box topology} \supset \ell^2 \text{ topology} \supset \text{uniform topology}.$$

Let D be the uniform metric and d be the ℓ^2 metric.

We will show the second inclusion first. Choose $\varepsilon > 0$. Without loss of generality, consider a uniform ε ball $B_{u,\varepsilon}$ about 0, and say $\varepsilon < 1$. This works because for any uniform open set U and a point $x \in U$, we can translate x to the origin. If $(y_n) \in X$ with $d((x_n), (y_n)) < \varepsilon$, then

$$\sum |x_i|^2 < \varepsilon^2, \text{ so that } |x_i|^2 < \varepsilon^2 \text{ and } |x_i| < \varepsilon \text{ for all } i.$$

Then $D((x_n), (y_n)) < \varepsilon$ and the ℓ^2 epsilon ball of radius ε is contained in $B_{u,\varepsilon}$. Then the uniform topology is contained in the ℓ^2 topology.

Now consider an ℓ^2 ball $B_{2,\varepsilon}$ of radius ε for some $\varepsilon > 0$. Define an open set U containing 0 in the box topology by

$$U = \prod_{i=1}^{\infty} (-\varepsilon 2^{i-1}, \varepsilon 2^{i-1}).$$

Then if $(x_i) \in U$, we have

$$d((x_i), 0)^2 = \sum_{1}^{\infty} |x_i|^2 \leq \sum_{1}^{\infty} \varepsilon^2 2^i = \varepsilon^2,$$

and thus $d((x_i), 0) < \varepsilon$ and $(x_n) \in B_{2,\varepsilon}$. Then the ℓ^2 topology is contained in the box topology.

(b.) We will show that the uniform, box, product, and ℓ^2 topologies are all different on $\mathbb{R}^\infty$ as a subspace of X.

Box topology is distinct: First, for $\varepsilon > 0$,

$$\prod_{i=1}^{\infty} (\varepsilon, \varepsilon)$$

is an open set in the box topology which is not open in the product or ℓ^2 topologies.

The set

$$\prod_{i=1}^{\infty} U_i$$

where $U_i = X_i$ for all i except some j, where $U_j = (-1, 1)$ is not open in the uniform topology.

Product topology $\neq \ell^2$ topology $\neq$ uniform topology:

The set $\{x \in \mathbb{R}^\infty : d(x, 0) < \varepsilon\}$ is open in the ℓ^2 topology but not in the product topology, since an open set in the product topology cannot have infinitely many of its projections not all of $\mathbb{R}$, but for every i, we must have $d(x_i) < \varepsilon$.

Likewise it is not open in the uniform topology, since every uniform open ball contains sequences of arbitrarily large ℓ^2 norm.

Product topology $\neq$ uniform topology: The set $\{x \in \mathbb{R}^\infty : D(x, 0) < \varepsilon\}$ is open in the uniform topology but is not open in the product topology, for the same reason as above.

Then all four topologies are distinct.

Section 21

1. Let d be a metric on X and let $A \subset X$ be a subspace. We will show that the restriction of d to $A \times A$ induces the subspace topology that A inherets from X. Let d' be the restricted metric on A.

   First note that the collection of intersection of open balls in X with A forms a basis for the subspace topology on A. Then if U is a basic open set in A, we have $U = B_\varepsilon(x) \cap A$ for some epsilon ball $B_\varepsilon(x)$ centered at $x \in X$. For any $y \in U$, we have $d(x, y) < \varepsilon$, and thus the d' ball about y of radius $\varepsilon - d(x, y)$ is contained in U. Then every set from the subspace topology is open in the d' metric topology.

   Conversely let $B'_\varepsilon(x)$ be a d' ball of radius ε at some $x \in A$. Then $B'_\varepsilon(x) = B_\varepsilon(x) \cap A$, where $B_\varepsilon(x)$ is a ball in the original metric. Thus d' balls are open in the subspace topology and we are done.

2. First we show that f is continuous. Choose $\varepsilon > 0$. Then if $d_X(x, y) < \varepsilon$, we have $d_Y(f(x), f(y)) = d_X(x, y) < \varepsilon$, and we are done.

   Now we show f^{-1} is continuous as a map $f(X) \to X$. Suppose $d_Y(f(x), f(y)) < \varepsilon$. Then $d_X(x, y) = d_Y(f(x), f(y)) < \varepsilon$ and we are done.

   Finally, we show f is injective. This follows from f being an isometry. If $f(x) = f(y)$, then

   $$0 = d_Y(f(x), f(y)) = d_X(x, y),$$

   so $x = y$ since d_X is a metric.

7. Let X be a set and $f_n : X \to \mathbb{R}$ a sequence of functions. We will show that uniform convergence of f_n to f is equivalent to convergence of f_n to f as elements of $\mathbb{R}^X$ in the uniform topology. We can suppose f_n converges to 0 without loss of generality, otherwise take $f_n - f$.

   First suppose $f_n \to 0$ uniformly on X. Then for all $\varepsilon > 0$ there exists $N > 0$ such that $|f_n(x)| < \varepsilon$ for all $x \in X$ and $n > N$. Then with d the uniform metric,

   $$d(f_n, 0) \leq \sup_{x \in X} |f_n(x)| < \varepsilon$$

   and thus f_n converges to 0 in the uniform topology as an element of $\mathbb{R}^X$.

   Conversely consider f_n as an element of $\mathbb{R}^X$ and suppose it converges in the uniform topology to 0. Choose $0 < \varepsilon < 1$ and $N > 0$ such that $d(f_n(x), 0) < \varepsilon$ for $n > N$. Then

   $$|f_n(x)| < \varepsilon < 1$$

   for all $x \in X$ and $n > N$, so f_n converges uniformly as well.

Section 22

2. (a.) Let $p : X \to Y$ be a continuous map and suppose there is a continuous map $f : Y \to X$ such that $p \circ f$ is the identity on Y. We will show p is a quotient map.

   For surjectivity of p, for any $y \in Y$, since $p(f(y)) = y$, the point $f(y)$ maps to y under p.

   Let $U \subset Y$. We already have that if U is open in Y, $p^{-1}(U)$ must be open in X, since p is continuous. Now suppose $p^{-1}(U)$ is open in X. Then $f^{-1}(p^{-1}(U))$ is open in Y. But this is just U, since the inverse of $p \circ f$, $f^{-1} \circ p^{-1}$, must also be the identity (as maps of sets).

   (b.) Let $A \subset X$ and let $r : X \to A$ be a retraction. r is clearly surjective since it is the identity on A.

   Let $U \subset A$. We only need to show $r^{-1}(U)$ open implies U open since r is continuous. In this case, we have $U = r^{-1}(U) \cap A$, and thus U is open in the subspace topology on A.

4. (a.) Define an equivalence relation on $\mathbb{R}^2$ by

$$x_0 \times y_0 \sim x_1 \times y_1 \text{ if } x_0 + y_0^2 = x_1 + y_1^2,$$

and let X^* be the quotient space. Then X^* is homeomorphic to the real line $\mathbb{R}$. To see this define a map

$$\begin{aligned} g :& \mathbb{R}^2 \to \mathbb{R} \\ & x \times y \mapsto x + y^2. \end{aligned}$$

Then the fibers of g are exactly the equivalence classes under the relation above. Then by 22.3, X^* is homeomorphic to $\mathbb{R}$ if g is a quotient map. We can already see that g is continuous and surjective. Also, g is an open map because is it the composition of the map $x \times y \mapsto x - y^2 \times y$ and projection onto the y axis. Then it is a quotient map.

(b.) Define $g : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ by $x \times y \mapsto x^2 + y^2$. Then g is continuous and surjective, and is an open map as it maps open sets to the segment of $\mathbb{R}_{\geq 0}$ obtained by collecting all the points intersected by sweeping a ray from 0 around 360 degrees and then squaring.