Homework 4

1. Let $(x_n)_1^\infty \in \ell^\infty$. We will show that
$$d((x_n)_1^\infty, c_0) = \limsup_n |x_n|.$$
By definition,
$$d((x_n)_1^\infty, c_0) = \inf_{(y_n)\in\ c_0} d((x_n),(y_n)),$$
where $d\left((x_n),(y_n)\right) = \sup_n |x_n - y_n|$.

To show that the distance $d((x_n), c_0)$ is at most $\liminf_n |x_n|$, define a family of sequences in c_0 as follows: Let
$$(y_n^k) := \begin{cases} x_n & n < k \\ 0 & n \geq k \end{cases},$$
so that
$$d((x_n),(y_n^k)) = \sup_n |x_n - y_n^k| = \sup_{n\geq k} |x_n|.$$
Then we have
$$d((x_n), c_0) \leq \inf_k d((x_n),(y_n^k)) = \inf_k \sup_{n\geq k} |x_n| = \liminf_n |x_n|,$$
which proves one direction. On the other hand, for any $(y_n) \in c_0$,
$$d((x_n),(y_n)) = \sup_n |x_n - y_n| \geq \sup_n |x_n| - |y_n| = \sup_n |x_n| \geq \limsup_n |x_n|,$$
and thus $d((x_n), c_0) = \inf_{(y_n)\in c_0} d((x_n),(y_n)) \geq \limsup_n |x_n|$. Then we have shown $d((x_n), c_0) = \limsup_n |x_n|$.

2. Let $\mathcal{U}$ be a non-principal ultrafilter on a set I, $(X_i)_{i\in I}$ a collection of Banach spaces, and $(\prod_{i\in I} X_i)^{\mathcal{U}}$ its ultraproduct with respect to $\mathcal{U}$.

We will show that for some $(x_i)_{i\in I}$,
$$\|(x_i)\|_{\mathcal{U}} = \lim_{i,\mathcal{U}} \|x_i\|_{X_i}.$$
Let $\lim_{i,\mathcal{U}} \|x_i\|_{X_i} = a$. By the definition of the ultrafilter limit, if we choose some $\varepsilon > 0$ we get a set $U \in \mathcal{U}$ such that for $i \in U$, $|\|x_i\|_{X_i} - a| < \varepsilon$. Then we know $\|x_i\| < a + \varepsilon$ for all $i \in U$.

Define a sequence
$$y_i = \begin{cases} x_i & i \notin U \\ 0 & i \in U \end{cases}$$
so that
$$d((x_n),(y_n)) = \sup_i \|x_i - y_i\| = \sup_{i\in U} \|x_i\| < a + \varepsilon.$$
We know that $(y_n)_{i\in I}$ is an element of $N_{\mathcal{U}}$ since for any $\varepsilon > 0$, the set of $i \in I$ for which $\|y_i\|_{X_i} < \varepsilon$ contains the set U and thus is an element of $\mathcal{U}$.

Moreover,
$$\|(x_i)\|_{\mathcal{U}} = \inf_{(z_n)\in N_{\mathcal{U}}} d((x_n),(z_n)) \leq d((x_n),(y_n)) < a + \varepsilon.$$
Then this holds for any $\varepsilon > 0$, so in fact $\|(x_i)\|_{\mathcal{U}} \leq a = \lim_{i,\mathcal{U}} \|x_i\|_{X_i}$.

In the other direction, let $(y_n) \in N_{\mathcal{U}}$ and choose $\varepsilon > 0$. Then there is a set $U \in \mathcal{U}$ such that for $i \in U$, $\|y_i\| < \varepsilon$, and a set $V \in \mathcal{U}$ such that for $i \in V$, $\|x_i\| \geq a - \varepsilon$. So

$$d((x_n),(y_n)) = \inf_i \|x_i - y_i\| \geq \sup_{i \in U \cap V} \|x_i\| - \|y_i\| \geq a - 2\varepsilon.$$

Then $d((x_n),(y_n)) \geq \lim\limits_{i,\mathcal{U}} \|x_i\|$ for any $(y_n) \in N_{\mathcal{U}}$, and thus

$$\|(x_i)\|_{\mathcal{U}} = \inf_{(y_n) \in N_{\mathcal{U}}} d((x_i),(y_n)) \geq \lim_{i,\mathcal{U}} \|x_i\|$$

and we are done.

3. We will show that $\mathbb{R} \simeq \mathbb{R}^{\mathcal{U}}$, the ultrapower of $\mathbb{R}$ with respect to $\mathcal{U}$.

   Define a map $\Phi : \mathbb{R}^{\mathcal{U}} \to \mathbb{R}$ by $(x_i) \mapsto \lim\limits_{i,\mathcal{U}} |x_i|$. The map is linear because of the linearity of the limit. It is an isometry since

$$\|(x_i)\|_{\mathcal{U}} = \lim_{i,\mathcal{U}} |x_i| = |\lim_{i,\mathcal{U}} x_i|$$

   as $\lim\limits_{i,\mathcal{U}} x_i$ always exists. It is clearly surjective, since for any $x \in \mathbb{R}$ we can take $(x_n) = x$ for all n. Then the two spaces are isometrically isomorphic.