Homework 3

1. Define $Z := \{(x,a,b,c) \in \mathbb{R}^4 : ax^2+bx+c=0, a \neq 0\}$.

   (a.) We show that Z is a smooth submanifold of $\mathbb{R}^4$ by showing it is the level set of a map $\mathbb{R}^4 \to \mathbb{R}$ such that the differential at each point is onto. Consider the smooth map $F: \mathbb{R}^4 \to \mathbb{R}$ given by $F(x,a,b,c) = ax^2+bx+c$. Its differential is given by
$$D_{(x,a,b,c)}F = (2ax+b, x^2, x, 1).$$
   In particular, it is always onto since the last coordinate is constant.

   (b.) Let $\pi : Z \to \mathbb{R}^3$ be the projection onto (a,b,c). We will find the critical values of the map π.

   If the polynomial ax^2+bx+c does not intersect the x-axis, the level set will be empty, and in particular (x,a,b,c) is a regular value of F.

   Suppose instead the polynomial does have at least one root, and let $(x,a,b,c) \in F^{-1}(0)$. By the implicit function theorem, since $\partial_c F$ is always nonzero, there is a neighborhood of (x,a,b,c) in which $F^{-1}(0)$ is the graph of a function in the variables (x,a,b). We let the coordinate map φ be the projection onto these coordinates. We can see that $\pi \circ \varphi^{-1}$ is the map $(x,a,b) \mapsto (a,b,\varphi(x,a,b))$. Then the coordinate representation of the differential of π at $\varphi(x,a,b,c)$ is given by the matrix
$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial a} & \frac{\partial\varphi}{\partial b} \end{bmatrix}.$$
   The differential at (x,a,b,c) of $F^{-1}(0)$ is of full rank if and only if D is. Moreover, by differentiating $F(x,a,b,\varphi(x,a,b)) = 0$, we see that $\partial_x F = -\partial_x \varphi$ at the points (x,a,b,c) and (x,a,b) respectively, using the chain rule. Then $\partial_x \varphi$ vanishes whenever $\partial_x F$ does. Thus, if ax^2+bx+c has a single root, the derivative will vanish at this x value and thus $\partial_x F$ will vanish at (x,a,b,c). In this case, the differential will not be onto, making (x,a,b,c) a critical value.

   In the case that ax^2+bx+c has two roots, the differnetial will be onto at either of these values as the partial derivative in x will not vanish.

   The critical case will happen when the vertex of the polynomial has y-value zero. We can solve $2ax+b=0$ to get the x-value of $\frac{-b}{2a}$. Then plugging this into our polynomial, we get
$$\frac{b^2-2b^2+4ac}{4a} = 0.$$
   Thus whenever $-b^2+4ac=0$, (x,a,b,c) will be a critical value, and all other values are regular.

   One can relate this to the notion of the discriminant, the distance from the middle of a polynomial to its roots. When this is zero, the critical case occurs.

2. Let $F: \mathbb{R}^4 \to \mathbb{R}^2$ be defined by $F(x,y,s,t) = (x^2+y, x^2+y^2+s^2+t^2+y)$. (a.) We show that $(0,1)$ is a regular value of F. We have
$$D_{(x,y,s,t)}F = \begin{bmatrix} 2x & 1 & 0 & 0 \\ 2x & 2y+1 & 2s+2t \end{bmatrix}.$$
   By a calculation, the first 2×2 square matrix is of full rank whenever x is nonzero. In the case that $x=0$, we also have $y=0$ since $x^2+y=0$. Then either s or t must be nonzero, and it is clear that the differential is onto as well.

   (b.) Since $x^2+y=0$, the value of y is completely determined by x. Then the projection $(x,y,s,t) \mapsto (x,s,t)$ is a diffeomorphism, as we are modifying y by the smooth function x^2 to map it to zero. We see that the coordinates of the projection satisfy $x^4+y^2+z^2=1$, which is easily seen to be diffeomorphic to the unit sphere S^2 by the coordinate transform $x^2 \mapsto x$. Then the composition of diffeomorphisms is a diffeomorphism $F^{-1}(0,1) \mapsto S^2$.

3. For each $a \in \mathbb{R}$, let M_a be the subset of $\mathbb{R}^2$ defined by

$$M_a = \{(x,y) : y^2 = x(x-1)(x-a)\}.$$

Let $F_a = x(x-1)(x-a) - y^2$. We see that $M_a = F_a^{-1}(0)$. By a result from class, the level set of a smooth map is an embedded submanifold if the differential is onto at every point. A critical point will only happen when $y = 0$. Then only when $x(x-1)(x-a)$ has a critical point on the x-axis will F have a critical point. This happens when $a = 0$ or 1. In the absence of these two cases, M_a will indeed be an embedded submanifold. If $a = 0$, both the x and y second derivatives of F will be negative, and thus there is an isolated point in M_a. In this case, we can still embed M_a. If $a = 1$, there will be a saddle point in F, and this creates a crossing in M_a. Thus M_a cannot be a submanifold of either type.

However if $a = 1$, M_a is the image of a non-injective immersion of the real line.

4. (a.) Let $F : \mathrm{Mat}_{2n\times 2n}\,\mathrm{Skew}_{2n\times 2n}(\mathbb{R})$ be the map $A \mapsto A^T JA$, where J is as defined in the homework. Then $\mathrm{Sp}_{2n}(\mathbb{R}) = F^{-1}(J)$. By the result from class, $\mathrm{Sp}_{2n}(\mathbb{R})$ is an embedded submanifold of $\mathrm{Mat}_{2n\times 2n}$ if the differential of F at every point in $F^{-1}(J)$ is of full rank, in this case onto. We can calculate that

$$F(A+H) = A^T JA^T + A^T JH + H^T JA + H^T JH.$$

Then $D_A F : H \mapsto A^T JH + H^T JA$, and $H^T JH$ is the higher order term. We see that indeed the image of $D_A F$ is the space of skew symmetric matrices. We just need to solve, for every $A \in F^{-1}(J)$, the equation $D_A FH = S$ for H, where $S \in \mathrm{Skew}_{2n\times 2n}(\mathbb{R})$.

Let $H = -\frac{1}{2}AJS$, so that indeed

$$-\frac{1}{2}A^T JAJS - \frac{1}{2}SJA^T JA = -\frac{1}{2}JJS - \frac{1}{2}SJJ = S,$$

and the differential is onto at every point in the level set, making this an embedded submanifold.

moreover, the dimension of $\mathrm{Sp}_{2n}(\mathbb{R})$ is given by $\dim \mathrm{Mat}_{2n\times 2n} - \dim \mathrm{Skew}_{2n\times 2n}(\mathbb{R}) = (2n)^2 - \frac{2n(2n-1)}{2} = \frac{2n(2n+1)}{2}$.

(b.) We showed in class that the tangent space of $\mathrm{Sp}_{2n}(\mathbb{R})$ at the identity is given by the kernel of the differential of F at I_{2n}. We have

$$D_{I_{2n}} FH = JH + H^T J.$$

Thus the kernel consists of matrices H such that $JH = -H^T J$, or $H = JH^T J$. With $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, we can calculate that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = H = JH^T J = J\begin{bmatrix} -c^T & A^T \\ -D^T & B^T \end{bmatrix} = \begin{bmatrix} -D^T & B^T \\ C^T & -A^T \end{bmatrix}.$$

We see that the tangent space $T_{I_{2n}}\mathrm{Sp}_{2n}$ is given by matrices $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ with $B = B^T$, $C = C^T$, and $A = -D^T$.

5. Let G be the manifold consisting of k element sets of vectors in $\mathbb{R}^n$ which are an orthonormal basis for their linear span. By arranging these k vectors in an $n \times k$ matrix, we can view this manifold as a subset of $\mathbb{R}^{n\times k}$. Let $F : \mathrm{Mat}_{n\times k}(\mathbb{R}) \to \mathrm{Sym}_{k\times k}$ be the map $A \mapsto A^T A$. We can calculate similar to problem 4 that the differential of F is given by $D_A FH = A^T H + H^T A$. Its image is indeed the symmetric $k \times k$ matrices. Moreover, it is surjective. For any $S \in \mathrm{Sym}_{k\times k}(\mathbb{R})$, we can solve $A^T H + H^T A = S$ by letting $H = \frac{1}{2}AS \in \mathrm{Mat}_{n\times k}$. Thus, G is an embedded submanifold of $\mathbb{R}^{n\times k}$ of dimension $nk - \frac{k(k-1)}{2}$.