Algebra Autumn 2025
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16 October 2025 **Eighth Homework**

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

The last two problems were partially done in class. This is an opportunity to polish your skills at writing mathematics in a clear and organized manner.

Hand in at the start of class, Thursday 23 October:

43. How many elements of order seven are there in a simple group of order 168?

44. Free abelian groups and the rational numbers.
    (a) Show that the additive group of the rational numbers is not a free abelian group.
    (b) Show that the multiplicative group of the positive rational numbers is a free abelian group of countable rank.

45. Show that $\mathbb{Q}/\mathbb{Z}$ has a unique subgroup of order n, for each integer $n > 1$, and that this subgroup is cyclic.

46. Let $\mathbb{C}^\times$ be the group of nonzero complex numbers under multiplication. Write $\mathbb{T}$ for the set of unit complex numbers, those $z \in \mathbb{C}^\times$ such that $\overline{z} = z^{-1}$.

    Find the torsion subgroup $\mathbb{T}_{\rm tor}$ of this circle group $\mathbb{T}$.

47. Prove that the group $\mathbb{Q}/\mathbb{Z}$ is *divisible*. That is, if $x \in \mathbb{Q}/\mathbb{Z}$, then for every $n \in \mathbb{N} \smallsetminus \{0\}$ there is an element $y \in \mathbb{Q}/\mathbb{Z}$ with $ny = x$.

    Let A be an abelian group with a subgroup B and suppose that $\varphi\colon B \to \mathbb{Q}/\mathbb{Z}$ is group homomorphism. Show that there exists a group homomorphism $\psi\colon A \to \mathbb{Q}/\mathbb{Z}$ such that $\psi|_B = \varphi$.

    If ψ unique? Why or why not?

48. Show that a finite abelian p-group is generated by its elements of maximal order.

49. Let p be a prime.
    (a) How many subgroups of order p does $\mathbb{Z}_p \oplus \mathbb{Z}_p$ have?
    (b) How many subgroups H isomorphic to $\mathbb{Z}_{p^2}$ does $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$ have?
    (c) If $H \simeq \mathbb{Z}_{p^2}$ is a subgroup of $G := \mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^2}$, what type does G/H have?

50. Let p be a prime. Recall that the Tate group $T_p(\mathbb{Q}/\mathbb{Z})$ of $\mathbb{Q}/\mathbb{Z}$ is the inverse limit of the p^n-torsion subgroups of $\mathbb{Q}/\mathbb{Z}$. What are the possible orders of elements of $T_p(\mathbb{Q}/\mathbb{Z})$?

    Compare $T_p(\mathbb{Q}/\mathbb{Z})$ to the Tate subgroup $T_p(\mathbb{T})$ of the circle group. What about $T_p(\mathbb{C}^\times)$?