

Homework 11

1. Define the binomial coefficient $\binom{a}{b}$ to be

$$\frac{n!}{k!(n-k)!}$$

for integers $0 \leq k \leq n$. Let R be a commutative ring. We will prove the binomial theorem:

$$\forall a, b \in R \forall n \in \mathbb{N}, \quad (a+b)^n = \sum_0^k \binom{n}{k} a^k b^{n-k}$$

We will prove by induction. First, clearly the formula holds for $n = 1$. Then assume the formula holds for $(a+b)^n$, and we will show it is valid for $(a+b)^{n+1}$ as well.

We have

$$\begin{aligned} (a+b)^{n+1} &= (a+b)^n(a+b) = \left(\sum_0^n \binom{n}{k} a^k b^{n-k} \right) (a+b) \\ &= \sum_0^n \binom{n}{k} a^{k+1} b^{n-k} + \sum_0^n \binom{n}{k} a^k b^{n-k+1} \\ &= \sum_1^{n+1} \binom{n}{k-1} a^k b^{n-k+1} + \sum_0^n \binom{n}{k} a^k b^{n-k+1} \\ &= b^{n+1} + \sum_1^n \left(\binom{n}{k-1} + \binom{n}{k} \right) a^k b^{n+1-k} + a^{n+1} \\ &= b^{n+1} + \sum_1^n \binom{n+1}{k} a^k b^{n+1-k} + a^{n+1} \\ &= \sum_0^{n+1} \binom{n+1}{k} a^k b^{(n+1)-k}. \end{aligned}$$

as desired. The fact that $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ comes from the following calculation:

$$\begin{aligned} \binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\ &= \frac{n!k}{k!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k+1)!} \\ &= \frac{n!k + n \cdot n! - n!k + n!}{k!(n-k+1)!} \\ &= \frac{n!(n+1)}{k!(n-k+1)!} = \frac{(n+1)!}{k!((n+1)-k)!} = \binom{n+1}{k} \end{aligned}$$

2. Let R be a commutative ring of characteristic p for p prime. Consider the map $a \mapsto a^p$. We will prove this is a ring homomorphism.

Since R is commutative, we have $(ab)^p = a^p b^p$.

For addition, note that when k is not zero or p , the binomial coefficient $\binom{p}{k}$ is divisible by p . Then since R is characteristic p , we have

$$(a+b)^p = \sum_{k=0}^p \binom{p}{k} a^k b^{p-k} = a^p + b^p$$

verifying that the map is a ring homomorphism.

3. Let R be a commutative ring and let Z be the set consisting of all zero divisors and zero.

The set of ideals contained in Z is ordered by set containment. Any linearly ordered subset has an upper bound, the union of the subset. Then by Zorn's lemma, there is an ideal M which is maximal with respect to containment amongst the set of all ideals contained in Z . We must show M is prime.

Suppose not. Then there exist $a, b \in R$ such that $ab \in M$ but neither a or b is contained in M . Then the ideal $M + (a)$ is strictly larger than M , and the same for $M + (b)$. By maximality of M , there must exist elements $x \in M + (a)$ and $y \in M + (b)$ such that $x, y \notin Z$. Then we can write

$$x = m_1 + r_1a, \quad y = m_2 + r_2b,$$

and their product

$$xy = m_1m_2 + r_1am_2 + r_2bm_1 + r_1r_2ab.$$

Then $xy \in M$ since m_1, m_2, ab are each in M and M is an ideal. But then xy is a zero divisor, implying either x or y must be a zero divisor: if $c(ab) = 0$, then either ca is zero and a is a zero divisor, or ca is nonzero and b is a zero divisor. This is a contradiction. Thus M is a prime ideal contained in Z .

4. Let G be a finite group. We will show the center $Z(\mathbb{C}[G])$ of the group algebra $\mathbb{C}[G]$ has dimension equal to the number of conjugacy classes in G .

To do this, we will give an explicit basis. For $g \in G$, define

$$e_{[g]} = \sum_{h \in G} hgh^{-1}.$$

Note that for all $h \in [g]$, $e_{[h]} = e_{[g]}$. Then the number of elements $e_{[g]}$ is equal to the number of conjugacy classes in G . Let $N \subset G$ be a set containing one representative from each conjugacy class.

We first show $e_{[g]} \in Z(\mathbb{C}[G])$ for all $g \in G$. Fix $g \in G$ and take any $a \in \mathbb{C}[G]$, with

$$a = \sum_{g \in G} a_g \cdot g$$

for some $a_g \in \mathbb{C}$. We will show these elements commute. We have

$$e_{[g]}a = \sum_{h,k \in G} a_k \cdot hgh^{-1}k = \sum_{k \in G} \left(\sum_{h \in G} a_k hgh^{-1}k \right) = \sum_{k \in G} \left(\sum_{h \in G} a_k khgh^{-1}k^{-1}k \right) = \sum_{h,k \in G} a_k khgh^{-1},$$

since the map $h \mapsto kh$ is a bijection of G , allowing us to interchange h with kh for a fixed k .

Since the conjugacy classes of G are disjoint, if we have

$$\sum_{g \in N} a_g e_{[g]} = 0,$$

it must be that $a_g = 0$ for all $g \in N$. Then $\{e_{[g]} : g \in N\}$ is a linearly independent set.

Finally we will show these elements span $Z(\mathbb{C}[G])$. Suppose there is an element f which cannot be written as a linear combination of the $e_{[g]}$'s. Then f is not constant on the conjugacy classes of G . Writing

$$f = \sum_{g \in G} f(g) \cdot g,$$

this means there exist $h, g \in G$ such that $f(g) \neq f(hgh^{-1})$. Then consider the element $h = 1 \cdot h \in \mathbb{C}[G]$. We have

$$fh(hg) = f(hgh^{-1}),$$

but

$$hf(hg) = f(g).$$

Then $fh \neq hf$, meaning $f \notin Z(\mathbb{C}[G])$. By the contrapositive, every element in the center $Z(\mathbb{C}[G])$ is constant on the conjugacy classes of G . Then $Z(\mathbb{C}[G])$ is spanned by $\{e_{[g]} : g \in N\}$ and its dimension is equal to the size of this set, the number of conjugacy classes in G .

5. For an arbitrary ring R and an infinite cyclic group G with generator ξ , the group ring $R[G]$ is not necessarily isomorphic to the polynomial ring in one variable $R[x]$.

To see this, consider $R = \mathbb{Z}$. We will show the group of units of $\mathbb{Z}[G]$ is not isomorphic to the group of units of $\mathbb{Z}[x]$. The units of $\mathbb{Z}[x]$ are only the constant polynomials 1 and -1 , as multiplication of any two polynomials in $\mathbb{Z}[x]$ results in a polynomial of equal or higher degree. In particular, you will never get 1. However, the units of $\mathbb{Z}[G]$ include the constant polynomials 1 and -1 , as well as the elements ξ^n for $n \in \mathbb{Z}$. Then the group of units of $\mathbb{Z}[G]$ is infinite, and thus cannot be isomorphic to the two element group of units of $\mathbb{Z}[x]$. Then the two rings cannot be isomorphic.