Homework 7

1. Suppose $T : X \to Y$ is a compact operator. To show that T is bounded, we use the fact that a compact set is bounded. Then clearly $T(B_X)$ is bounded as well, so T is a bounded operator.

   Now suppose T is not strictly singular. Then there is a subspace $A \subset X$ which is infinite dimensional, such that T is an isomorphism of A onto its image $T(A)$. Then $T(B_A) \subset T(B_X)$, so $\overline{T(B_A)}$ is a compact subset of Y and therefore of $T(A)$. But then
$$T\big|_A^{-1}(\overline{T(B_A)})$$
   is compact, since bounded operators take compact sets to compat sets. Finally,
$$B_A = T\big|_A^{-1}(T(B_A)) \subset T\big|_A^{-1}(\overline{T(B_A)})$$
   implying B_A is a compact subset of A, since it is a closed subset of a compact set. But this is a contradiction, since the unit ball in any infinite dimensional space is not compact. Then T must be strictly singular.

2. Let $T : X \to Y$ be a compact operator. We will show that T is completely continuous, meaning it sends weak convergent sequences to strongly convergent sequences. Let $\{x_n\}$ be a weakly convergent sequence in X, and suppose $x_n \to_w x$ weakly. We know that $\{x_n\}$ is norm bounded by the uniform boundedness principle. Then $\{T(x_n)\}$ is contained in a relatively compact set. Consider some arbitrary subsequence $\{T(x_{n_k})\}$. Then there is a further subsequence $\{T(x_{n_{k_j}})\}$ which converges to some y.

   But bounded operators take weakly convergent sequences to weakly convergent sequences: For any $f^* \in Y^*$, the functional $f^*(T(\cdot))$ is in X^*. Then $T(x_n)$ converges weakly to $T(x)$. Then since strong convergence implies weak convergence, the above subsequence must also converge to $T(x)$ by uniqueness of weak limits. Then all subsequences have a subsequence which converges to $T(x)$, and thus the entire sequence $\{T(x_n)\}$ converges strongly to $T(x)$.

   For the converse, take the identity map $\ell_1 \to \ell_1$. It is clearly bounded, and since ℓ_1 has the schur property, it is completely continuous. But it is not compact since the unit ball of an infinite dimensional space is not compact.

3. Let $T : X \to Y$ be a bounded and completely continuous operator. We will show that $T(B_x)$ is relatively compact by showing every sequence in $T(B_X)$ has a convergent subsequence, converging in Y. This is equivalent to showing every sequence in the closure has a subsequence which converges in the closure, which is equivalent to relative compactness in a metric space.

   Let $\{x_n\}$ be a sequence in $T(B_X)$. Then choose $\{y_n\} \subset B_X$ such that $T(y_n) = x_n$. Since x is reflexive, the unit ball B_X is weakly sequentially compact, so there is a subsequence $\{y_{n_k}\}$ which converges weakly. Then $\{x_{n_k}\} = \{T(x_{n_k})$ converges strongly since T is completely continuous, and thus every sequence in $T(B_X)$ has a convergent subsequence, and $T(B_X)$ is relatively compact.

4. Let $1 \le p < q < \infty$ and let X be a closed subspace of ℓ_q. Suppose $T : X \to \ell_p$ is bounded. Since X is a closed subspace of a reflexive space, X is reflexive. Then to show T is compact, by problem 3 it is enough to show that T is completely continuous.

   Suppose T is not completely continuous, and take a weakly null sequence $\{x_n\}$ in the unit sphere of X such that $T(x_n)$ does not converge to 0 in norm. By passing to a subsequence, we can assume that $\{x_n\}$ is equivalent to the canonical basis of ℓ_q. By passing to a further subsequence, we can assume that $\|T(x_n)\| > a > 0$ for all n, and we also know that $\{T(x_n)\}$ is weakly null since T is bounded. By passing to a further subsequence we can assume that $\{T(x_n)\}$ is equivalent to a basis of ℓ_p. Then T is a bounded map from ℓ_q to ℓ_p which is impossible, hence T must be completely continuous.