Homework 10

Section 53

3. Let $p : E \to B$ be a covering map and let E be simply connected. Suppose there is a point $b_0 \in B$ such that $p^{-1}(b_0)$ has k elements. We will prove for every $b \in B$, $p^{-1}(b)$ has k elements.

   Let A be the set of points in B which are covered "k-fold". Then I claim A is an open set. Choose any $b \in A$. There is an open set $U \ni b$ which is evenly covered by p. Choose any $a \in A$. Since p maps each disjoint set in $p^{-1}(U)$ onto U surjectively, each must contain some point which maps onto a. Moreover, since p maps these sets injectively onto U, there can be at most one point in each which is mapped to a. So the number of disjoint open sets in $p^{-1}(U)$ is equal to the number of elements in $p^{-1}(a)$ for any $a \in A$, namely b. Then $p^{-1}(U)$ has k disjoint open sets, and $p^{-1}(a)$ has k elements for every $a \in A$, and thus $U \subset A$ and A is open.

   The same is true for the points b of B such that $p^{-1}(b)$ has any other cardinality. Each of these sets are disjoint from the others. Since A is nonempty and B is connected, it must be that $A = B$.

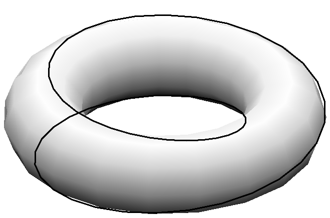
5. Consider the unit circle S^1 in the complex plane. We will prove the map $\varphi : x \mapsto x^n$ is a covering map from S^1 to itself.

   The initial segment $\{e^{i\theta} \in S^1 : \theta \in \left[0, \frac{2\pi}{n}\right)$ is mapped onto all of S^1, along with the other $n-1$ segments of equal width. The map from each half open segment is a bijection. Then for every $z \in S^1$, $\varphi^{-1}(z)$ will consist of precisely n points, each from a different segment. These points are equidistant from one another, since each must occupy the same position in its slice. Then for any $b \in S^1$, we can choose the open set U containing b to be the open half of S^1 that b occupies, and the sets mapped onto U are each arcs of radius π/n centered at each of the points in the primage of b. φ is clearly a bijection from each of these sets onto U, and φ is a homeomorphism since it merely dialates the open sets by a factor of n.

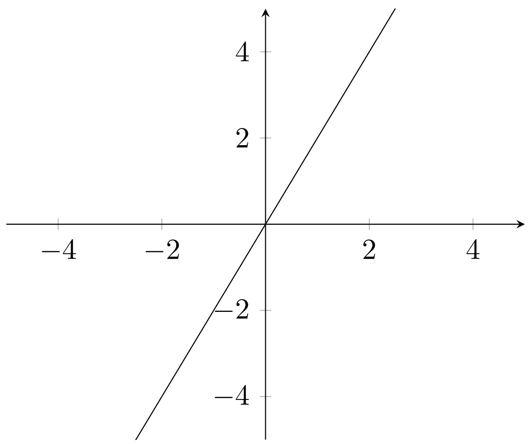
Section 54

3. Let $p : E \to B$ be a covering map, and let α and β be paths in B such that $\alpha(1) = \beta(0)$. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be their liftings, such that $\tilde{\alpha}(1) = \tilde{\beta}(0)$. Then $\tilde{\alpha} * \tilde{\beta}$ is a well defined path. We also know that $p \circ (\tilde{\alpha} * \tilde{\beta}) = p \circ \tilde{\alpha} * p \circ \tilde{\beta} = \alpha \circ \beta$. Then $\tilde{\alpha} * \tilde{\beta}$ is a lifting of $\alpha * \beta$.

5. This is a picture of the torus with the path drawn on it.

   and here is a drawing of the lift to $\mathbb{R}^2$



8. Suppose $p : E \to B$ is a covering map, with E path connected and B simply connected. Then for any $b \in B$, $\pi_1(B, b) = \{e\}$ the trivial group. Theorem 54.4 says that the map $\varphi : \pi_1(B, b) \to p^{-1}(b)$ is surjective, so we must have that $p^{-1}(b)$ contains one element for all $b \in B$. Then p must be injective. Moreover, since p is a covering map, we know that it is continuous, open, and surjective. Then p is a homeomorphism.