Homework 3

1. Let (X, τ) be a compact topological space and $\varphi : (X, \tau) \to \mathbb{R}$ a lower semicontinuous function.

   Let $a = \inf_{x \in X} \varphi(x)$. For each $\lambda > a$, define $U_\lambda = \varphi^{-1}((-\infty, \lambda])$. Then the sets U_λ are non-increasing as λ decreases to a, and each of them are closed sets by the lower semicontinuity of φ. Moreover, for any finite set $\lambda_1 < \cdots < \lambda_n$, we have

   $$\bigcap_{i=1}^{n} U_{\lambda_i} = U_{\lambda_1} \neq \varnothing,$$

   since $\lambda_1 > a$ and by the definition of infimum. Then since X is compact, we have

   $$\bigcap_{\lambda > a} U_\lambda \neq \varnothing.$$

   Choosing some $x_0 \in \bigcap_{\lambda > a} U_\lambda = \varphi^{-1}((-\infty, a])$, we have $\varphi(x_0) = a$ and thus φ achieves its infimum at x_0.

2. Let $(X, \|\cdot\|)$ be a normed space and let $\varphi : X \to \mathbb{R}$ be a convex, lower semicontinuous function. We will show that φ is lower semicontinuous in the $\sigma(X, X^*)$ topology on X. Since we have that $\varphi^{-1}((-\infty, a])$ is norm-closed for any $a \in \mathbb{R}$, we only need to show that these sets are convex, and then we can apply Mazur's theorem to deduce that they are $\sigma(X, X^*)$-closed as well.

   Suppose $\varphi^{-1}((-\infty, a]) = A$ is not convex. Then there exist points $x_1, x_2 \in A$ and $t \in (0, 1)$ such that $x_t = tx_1 + (1-t)x_2 \notin A$. In particular, we have $\varphi(x_1), \varphi(x_2) \leq a < \varphi(x_t)$. Then since φ is convex,

   $$a < \varphi(x_t) \leq t\varphi(x_1) + (1-t)\varphi(x_2) \leq a,$$

   a contradiction. Then the sets $\varphi^{-1}((-\infty, a])$ are convex and thus weakly closed, and φ is lower semicontinuous in the $\sigma(X, X^*)$ topology.

3. Let X be a reflexive Banach space, and $C \subset X$ a closed, convex subset. Let $\varphi : C \to \mathbb{R}$ be a convex, lower semicontinuous function such that

   $$\lim_{x \in C, \|x\| \to \infty} \varphi(x) = +\infty.$$

   Then we show that $\varphi(x)$ achieves its infimum a on X.

   By (2.), we know that φ is lower semicontinuous in $\sigma(X, X^*)$ as well. Since X is reflexive, the unit ball is compact in $\sigma(X, X^*)$. Then in order to apply the result of (1.), we just need to show that for some $M > a$, the preimages $\varphi^{-1}((-\infty, \lambda])$ are contained in some closed ball about $0 \in X$.

   But by the limit condition for φ, we have that for all $M > 0$, there exists $\varepsilon > 0$ such that $\|x\| > \varepsilon$ implies $\varphi(x) > M$. Thus if $\varphi(x) < M$, it must be that $\|x\| < \varepsilon$. Fix some $M > a$. Then for every $a < \lambda < M$, we have $\varphi^{-1}((-\infty, \lambda]) \subset B(0, \varepsilon)$. We note that $B(0, \varepsilon)$ is compact by reflexivity. Then similar to (1.), there is some point $x_0 \in \varphi^{-1}((-\infty, a])$ and $\varphi(x_0) = a$.