Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Show the set of points at which f is continuous is a borel set.

Proof: For each $n = 1, 2, \ldots$, Let U_n be the union of all open intervals V such that $|f(x) - f(y)| < 1/n$ for all $x, y \in V$.

Suppose f is continuous at $x \in \mathbb{R}$. Then for every $\varepsilon > 0$, there exists a δ such that $|f(y) - f(x)| < \varepsilon$ for all $y \in (y - \delta, y + \delta)$. For a given n, choose $\varepsilon < 1/n$ etc.

Then let $U = \cap_n U_n$. Then I claim the set of points at which f is continuous is precisely U.