Homework 1

13.1 Let X be a topological space and $A \subset X$. Suppose for each $x \in A$ there is an open set U_x with $x \in U_x \subset A$.

Then I claim $U := \bigcup_{x\in A} U_x = A$. Since each $x \in A$ is in some U_x, clearly $A \subset U$. Conversely, each U_x is a subset of A, so their union must be. Then A is a union of open sets and is open.

13.3 Let X be any set. Let $\mathcal{T}_c$ be the collection of subsets U of X such that $X \setminus U$ is either countable or all of X.

$\mathcal{T}_c$ contains the empty set and X since $X \setminus \emptyset = X$ and $X \setminus X = \emptyset$, which is countable. Also, if $\{U_\alpha\}_{\alpha\in A}$ is any collection of open sets, we have

$$X \setminus \bigcup_{\alpha\in A} U_\alpha = X \cap \left(\bigcup_{\alpha\in A} U_\alpha\right)^c = X \cap \bigcap_{\alpha\in A} {U_\alpha}^c = \bigcap_{\alpha\in A} X \cap U_\alpha^c = \bigcap_{\alpha\in A} X \setminus U_\alpha$$

by De Morgan's laws, and thus $\bigcup_{\alpha\in A} U_\alpha \in \mathcal{T}_c$, since any intersection of countable sets is countable.

Similarly, if $i_1, \dots, i_n \in A$, we have

$$X \setminus \bigcap_1^n U_{i_n} = \bigcup A \setminus U_{i_n},$$

which is a finite union of countable sets and is countable. Then $\bigcap_1^n U_{i_n} \in \mathcal{T}_c$.

Then we have shown that $\mathcal{T}_c$ is a topology.

Now let $\mathcal{T}_\infty$ be the subsets U of X such that $X \setminus U$ is infinite, empty, or all of X. $\mathcal{T}_\infty$ is not a topology. For example, let $X = \mathbb{N}$. Then $\{x\}$ is open for $x \in X$, but $V := \bigcup_{x\notin\{0\}}\{x\}$ is a union of open sets that is not open, as $X \setminus V = \{0\}$.

13.6 Let $\mathbb{R}_\ell$ be the reals equipped with the lower limit topology, and let $\mathbb{R}_K$ be the reals equipped with the K-topology. Here $K := \{1/z : z \in \mathbb{Z}\}$ and the K topology is generated by the collection of open intervals (a, b) along with sets of the form $(a, b) \setminus K$.

To show that the topologies on $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable, we must show the existence of sets $U, V \in \mathbb{R}$ such that U is open in the K-topology and not in the lower limit topology, and V is open in the lower limit topology but not the K-topology.

Let $U = (-2, 2) \setminus K$. U is clearly open in the K-topology, but it is not open in the lower limit topology. For $0 \in U$, suppose there is an interval $[a, b)$ containing 0. Then $0 < b$, so there is some $n \in \mathbb{Z}$ such that $0 < 1/n < b$ and thus $[a, b) \not\subset U$. Then U is not open in the lower limit topology, which is generated by all such half-open intervals.

Let $V = [2025, 2026)$. Then V is not open in the K-topology since there is no open interval containing 2025 which is contained in V, and the only basic open sets from the K-topology contained in V are open intervals.

16.1 Let X be a topological space, Y a subspace, and A a subset of Y. Let $\mathcal{T}_Y$ be the topology A inherets as a subspace of Y and let $\mathcal{T}_X$ be the topology A inherets as a subspace of X.

If U is in $\mathcal{T}_Y$, then $U = W \cap A$ for some W open in Y. We can write $W = K \cap Y$ for some K open in X since Y is a subspace of X. Then we have $U = K \cap Y \cap A = K \cap A$, showing that U is open in $\mathcal{T}_X$.

Likewise if V is in $\mathcal{T}_X$, Then $V = W \cap A$ for some W open in X. Since $A \subset Y$, $V = W \cap Y \cap A = (W \cap Y) \cap A$. Then $W \cap Y$ is open in Y and V is in $\mathcal{T}_Y$.

16.4 Let X and Y be topological spaces, and give $X \times Y$ the product topology. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projection maps.

Let U be open in $X \times Y$ and let $x \in \pi_1(U) \subset X$. Then choose a point $z \in U$ such that $\pi_1(z) = x$ using the surjectivity of π_1. Since U is open, there is a basic open set $A \times B$ for open sets $A \subset X$ and $B \subset Y$, such that $z \in A \times B \subset U$.

Then $x = \pi_1(z) \in \pi_1(A \times B) = A \subset \pi_(U)$, showing that $\pi_1(U)$ is open.

A similar argument shows that π_2 is an open map.

16.9 Let $\mathcal{T}_d$ be the dictionary order topology on $\mathbb{R} \times \mathbb{R}$ and let $\mathcal{T}_p$ be the product topology on $\mathbb{R}_d \times \mathbb{R}$ where $\mathbb{R}_d$ is $\mathbb{R}$ with the discrete topology and $\mathbb{R}$ has the standard topology. Let $\mathcal{B}_d$ be the basis for $\mathcal{T}_d$ consisting of intervals of the form $(a \times b, c \times d)$ where either $a < c$ or $a = c$ and $b < d$. Let $\mathcal{B}_p$ be the basis for $\mathcal{T}_d$ consisting of intervals $(a \times b, a \times c)$ for $b < c$, possibly $\pm\infty$.

Since $\mathcal{B}_p \subset \mathcal{B}_d$, we must have $\mathcal{T}_p \subset \mathcal{B}_d$.

In the other direction, we can write $(a \times b, c \times d) = (a \times b, a \times \infty) \cup \bigcup_{x \in (a,c)} (x \times -\infty, x \times \infty) \cup (c \times -\infty, c \times c)$. Then $\mathcal{B}_d \subset \mathcal{T}_p$ and thus $\mathcal{T}_d \subset \mathcal{T}_p$.

I claim that $\mathcal{T}_d$ is strictly finer than the standard topology on $\mathbb{R} \times \mathbb{R}$. All products of intervals $A \times B$ are unions $\cup_{x \in A} x \times B$, but the interval $(a \times b, a \times c)$ cannot contain any product of intervals $A \times B$ since this would imply $B \subset \{a\}$, which is not possible.