Homework 2

Section 19

2.

Section 20

2. Let $R : \ell^2 \to \ell^2$ be the map $(a_0, a_1, \dots) \mapsto (0, a_0, a_1, \dots)$. First, it is clear that zero is not an eigenvalue, since R is invertible. With $x = (a_0, a_1, \dots)$, suppose $Rx = \lambda x$ for $\lambda \in \mathbb{C}$. Then

$$(0, a_0, a_1, \dots) = (\lambda a_0, \lambda a_1, \dots),$$

so $\lambda a_0 = 0, \lambda a_1 = a_0$, etc. Then $a_0 = 0$, implying $a_i = 0$ for all i. Then λ is not an eigenvalue.

3. A similar argument from the $p = 2$ case applies. Fix $1 \le p < \infty$. It is still clear that the ℓ^p norm of $\mathbf{L}$ and $\mathbf{R}$ is 1, and the same for any integer power. Then the spectral radius will be 1 of both operators. We will show that all λ in the open disk are eigenvalues of L. Suppose $\mathbf{L}x = \lambda x$. This is equivalent to

$$(a_1, a_2, \dots) = \lambda(a_0, a_1, \dots).$$

So $a_n = \lambda^n a_0$. Then since $x \in \ell^p$, we have $\sum_1^\infty |a_n|^p < \infty$. This is satisfied iff $|\lambda| < 1$. Then every λ in the open unit disk is an eigenvalue and thus in the spectrum. Since the spectrum is closed, the spectrum is exactly the closed unit disk. If p and q are conjugate exponents, then $\mathbf{R} : \ell^q \to \ell^q$ is the adjoint to $\mathbf{L}$, and thus has the same spectrum. Then for $1 < p < \infty$, the spectrum of $\mathbf{R}$ and $\mathbf{L}$ is the closed unit disk in $\mathbb{C}$. The case $\mathbf{R} : \ell^\infty \to \ell^\infty$ is handled as well by taking the adjoint of $\mathbf{L} : \ell^1 \to \ell^1$. The final case is $\mathbf{L} : \ell^\infty \to \ell^\infty$, which handles $\mathbf{R} : \ell^1 \to \ell^1$ as well.

The spectral radius of $\mathbf{L}$ for $p = \infty$ is still 1 by the same argument. We need $a_n = \lambda^n a_0$ to be bounded, which is true if and only if $|\lambda| < 1$. Then taking the closure we see that the spectrum is again the closed unit disk.

4. Let $\{\lambda_n\}$ be a bounded sequence of complex numbers, $X = \ell^2$. Define $M : X \to X$ by

$$(a_0, a_1, \dots) \mapsto (\lambda_0 a_0, \lambda_1 a_1, \dots)$$

First, it is clear that each λ_i is in the spectrum, since $Me_i = \lambda_i e_i$.

Next, take λ not a limit point of $\{\lambda_n\}$. Then $|\lambda - \lambda_i| > \varepsilon$ for all $i = 0, 1, 2, \dots$. Then we can construct an inverse N of $(\lambda - M)$ by defining

$$N^{-1}(a_0, a_1, \dots) = \left(\frac{1}{\lambda - \lambda_0} a_0, \frac{1}{\lambda - \lambda_1} a_1, \dots \right)$$

and this is a bounded operator as well, since $\frac{1}{\lambda - \lambda_i} < \varepsilon^{-1}$ for all i. Then λ is not an eigenvalue. Since the spectrum is closed and thus contains all limit points of $\{\lambda_n\}$, it is exactly the closure of $\{\lambda_n\}$.

6. Let $K(s,t)$ be a continuous function on $[0,1] \times [0,1]$, $t \le s$. Define

$$Kf(s) = \int_0^s K(s,t) f(t) dt.$$

We may choose M such that $|K(s,t)| < M$ everywhere. Since K is continuous on a compact domain, we can choose δ so that $|K(a,t) - K(b,t)| < \varepsilon$ for all $t \in [0,1]$, whenever $|a-b| < \delta$. Then for $a < b$, $|a-b| < \delta$, we have

$$|Kf(a) - Kf(b)| = \left|\int_0^a K(a,t)f(t)dt - \int_0^b K(b,t)f(t)dt\right| \leq \int_0^a |K(a,t) - K(b,t)||f(t)|dt + \int_a^b |K(b,t)||f(t)|dt$$
$$\leq \varepsilon M +$$