

Extra Problem

Let D be a division ring. Let S be the ring of $n \times n$ matrices, $n > 1$.

1. S has no proper ideals besides the zero ideal.

We proved on homework 10 that the only ideals in such a matrix ring are of the form $M_n(I)$ for $I \subset D$ an ideal. Since D is a division ring, the only ideals are D and the zero ideal. Then the only ideals of S are S and the zero ideal.

2. S has zero divisors: as long as $n > 1$, we have

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \ddots & \dots & 0 \\ \vdots & & \vdots & \\ 0 & & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & \ddots & & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & & 0 & & 0 \end{bmatrix} = 0.$$

3. Since S is non-commutative, we must use the more general version of prime ideals. Suppose $AB \subset \langle 0 \rangle$. Since D has no proper ideals, the only possibilities for A and B are 0 and S itself. If both A and B are S , clearly their product is not contained in the trivial ideal. Then one or both of A and B must be equal to 0 and thus the ideal 0 is prime.