655 Notes

Product Topology

Let Γ be a set and $(X_\gamma, \tau_\gamma)_{\gamma\in\Gamma}$ a collection of topological spaces. The Product topology on $\prod_{\gamma\in\Gamma} X_\gamma$ is defined as the weakest topology on $\prod_{\gamma\in\Gamma} X_\gamma$ which makes the projection maps $\pi_\gamma : \prod_{\gamma\in\Gamma} X_\gamma$ continuous.

Example. On $\mathbb{R}^\Gamma$, the product topology is given by the following neighborhood basis:

$$\{U(x; \gamma_1, \dots, \gamma_n; \varepsilon) : \gamma_1, \dots, \gamma_n \in \Gamma, \varepsilon > 0, n \geq 1, x \in \mathbb{R}^\gamma\},$$

where $U(x; \gamma_1, \dots, \gamma_n; \varepsilon) := \{z \in \mathbb{R}^\Gamma : |z_{\gamma_i} - x_{\gamma_i}| < \varepsilon, 1 \leq i \leq n\}$.

$\mathbb{R}^\Gamma$ with the product topology is hausdorff.

Locally Convex Topological Vector Spaces

Definition. A **topological vector space** is a vector space X equipped with a topology τ such that the maps

$$\begin{aligned} A :&X \times X \to X & \Omega :&\mathbb{R} \times X \to X \\ &(x_1, x_2) \mapsto x_1 + x_2 & &(a, x) \mapsto ax \end{aligned}$$

are both continuous.

A TVS is locally convex if every point has a local base consisting of convex sets.

Example. An arbitrary product of LCTVS's is an LCTVS with the product Topology. A vector subspace of an LCTVS is an LCTVS when given the relative topology.

Dual Pairs

Let E be a vector space and let $E^\# := \{f : E \to \mathbb{R} : f \text{ is linear}\}$ be the algebraic dual space.

Let E and F be vector spaces. Then a bilinear form $\langle \cdot, \cdot \rangle : E \times F \to \mathbb{R}$ induces two maps:

$$\begin{aligned} \varphi :&E \to F^\# & \psi :&F \to E^\# \\ &e \mapsto f \mapsto \langle e, f\rangle & &f \mapsto e \mapsto \langle e, f\rangle. \end{aligned}$$

Definition. A dual pair is a pair of vector spaces E, F and a bilinear map $\langle \cdot, \cdot \rangle : E \times F \to \mathbb{R}$ such that

a.) E separates points in F, meaning for all $f_1, f_2 \in E$, $f_1 \neq f_2$, there is an $e \in E$ such that $\langle e, f_1\rangle \neq \langle e, f_2\rangle$.

b.) F separates points in E.

We write $\langle E, F\rangle$ is a dual pair.

Remark. The statement that E separates points in F is equivalent to the statement that for $f \in F$, if for all $e \in E$, $\langle e, f\rangle = 0$, then $f = 0$. Then ψ is an injection, and we can identify F with its image in under ψ in $E^\#$

The dual statement is that if F separates points in E, we can identify E with its image under φ in $F^\#$.

Example. Given a vector space E, $\langle E, E^\#\rangle$ is a dual pair for $\langle \cdot, \cdot \rangle : E \times E^\# \to \mathbb{R}$ given by $(e, e^\#) \mapsto e^\#(e)$.

Example. Given a normed vector space X, $\langle X, X^*\rangle$ is a dual pair for $\langle \cdot, \cdot, \rangle : X \times X^* \to \mathbb{R}$ given by $(x, x^*) \mapsto x^*(x)$.

Definition. Let $\langle E, F\rangle$ be a dual pair. The weak topology associated to the dual pair, denoted by $\sigma(E, F)$, is defined as the restriction to E of the product topology on $\mathbb{R}^F$.

Remark. We showed that we can view E as a subset of $F^{\#}$ by the injection φ. $F^{\#}$ is a subset of $\mathbb{R}^F$, the space of all maps $F \to \mathbb{R}$, consisting of those maps which are linear. Then we can view E as a subset of $\mathbb{R}^F$.

Example. Let X be a normed vector space and consider the dual pair $\langle X, X^* \rangle$, with $\langle e, e^* \rangle = e^*(e)$. The topology $\sigma(X, X^*)$ on X is called the weak topology. The topology $\sigma(X^*, X)$ on X^* is called the weak* topology.

We now give some equivalent definitions for the weak topology in the case that X is a normed vector space and $\langle X, X^* \rangle$ is our dual pair.

Weak Topology

The weak topology on X is given by:

- The topology generated by the sets

$$\begin{aligned} U(x_0; x_1^*, \ldots, x_n^*; \varepsilon) &= \{x \in X : |\langle x_0, x_i^* \rangle - \langle x, x_i^* \rangle| < \varepsilon, 1 \leq i \leq n\} \\ &= \{x \in X : |x_i^*(x_0) - x_i^*(x)| < \varepsilon, 1 \leq i \leq n\} \end{aligned}$$

- If $\{x_\alpha\}_\alpha$ is a net in X and $x \in X$, then $x_\alpha \to x$ weakly if and only if for all $x^* \in X^*$, $x^*(x_\alpha) \to x^*(x)$
- the weakest topology on X which makes all of the bounded linear functionals on X continuous.

Weak* Topology

The weak* topology on X^* is given by

- the topology generated by sets

$$U(x_0^*; x_1, \ldots, x_n; \varepsilon) = \{x^* \in X^* : |x_0^*(x_i) - x^*(x_i)| < \varepsilon, 1 \leq i \leq n\}$$

- $x_\alpha^* \to x^*$ in the weak* topology if and only if $x_\alpha^*(x) \to x^*(x)$ for all $x \in X$
- the weakest topology on X^* for which the maps $x^* \to x^*(x)$ are continuous for every $x \in X$.

Remark. The map $i : (X^*, \sigma(X^*, X)) \to \mathbb{R}^X, x^* \mapsto (x^*(x))_{x \in X}$ is a homeomorphism from $(X^*, \sigma(X^*, X))$ onto its image in $\mathbb{R}^X$ with the product topology.

We have $x_\alpha^* \to x^*$ in the weak* topology if and only if for all $x \in X$, $x_\alpha^*(x) \to x^*(x)$, if and only if $i(x_\alpha^*) \to i(x^*)$ in the product topology.

Remark. The map $j : (X, \sigma(X, X^*)) \to X^{**} \subset \mathbb{R}^{X^*}, x \mapsto (x^*(x))_{x^* \in X^*}$ is a homeomorphism from $(X, \sigma(X, X^*))$ onto its image in $(X^{**}, \sigma(X^{**}, X^*))$.

We have $x_\alpha \to x$ weakly if and only if for all $x^* \in X^*$, $x^*(x_\alpha) \to x^*(x)$ if and only if $j(x_\alpha) \to j(x)$ in the weak* topology on X^{**}.

Proposition. *Let X be a normed space.*

1. $(X, \sigma(X, X^*))^* = X^*$
2. $(X^*, \sigma(X^*, x))^* = j(X)$

Proof. (1.) We have $(X, \sigma(X, X^*))^* \subset X^*$ because $\sigma(X, X^*)$ is weaker than the norm topology, thus every functional which is weak-continuous is also norm-continuous. That $X^* \subset (X, \sigma(X, X^*))$ follows by construction, since $\sigma(X, X^*)$ ensures that each functional which is norm-continuous is also $\sigma(X, X^*)$ continuous.

(2.) We have $j(X) \subset (X^*, \sigma(X^*, X))^*$ by construction, since $\sigma(X^*, X)$ is a topology such that the maps $j(x)$ are continuous.

To show the other direction, let $\varphi : (X^*, \sigma(X^*, X)) \to \mathbb{R}$ be a weak* continuous functional on X^*. Since φ is continuous, there is a weak* neighborhood $U \ni 0$ in X^* such that $U \subset \varphi^{-1}(-1, 1)$.

From one of the above characterizations of the weak* topology, we know that there must be elements $x_1, \ldots, x_n$ such that $U = \{x^* : |x^*(x_i)| < \varepsilon \text{ for} 1 \le i \le n\}$. Now suppose $f^* \in \bigcap_1^n \ker x_i$. In particular, we have $|f^*(x_i)| = 0 < \varepsilon$ for $i = 1, \ldots, n$, thus $f^* \in U$. Then for any $\lambda > 0$, $|\lambda f(x_i)| = \lambda 0 = 0 < \varepsilon$ for $i = 1, \ldots, n$, thus $\lambda f^* \in U$, and we have $|\varphi(\lambda f^*)| < 1$ and thus $|\varphi(f^*)| < 1/\lambda$.

Since this holds for all $\lambda > 0$, it must be that $\varphi(f^*) = 0$ and $f^* \in \ker \varphi$. We have therefore shown that $\ker \varphi \subset \bigcap_1^n \ker x_i$. Then linear algebra tells us that φ must be a linear combination of the functionals x_i, $\varphi = \sum_1^n a_i x_i := x$. Then $j(x) = \varphi$ □

Theorem (Banach-Alaoglu Theorem)**.** *Let X be a normed vector space. Then $(B_{X^*}, \sigma(X^*, X))$ is a compact topological space.*

Proof (outline) Observe that for all $x \in X, x^* \in X^*$, $\|x^*(x)\| \le \|x^*\|\|x\|$. Then B_{X^*} embeds in $\mathbb{R}^X$ by the map

$$\begin{aligned} i : & B_{X^*} \to \prod_{x \in X} [-\|x\|, \|x\|] \subset \mathbb{R}^X \\ & x^* \mapsto (x^*)_{x \in X}. \end{aligned}$$

$K := \prod_{x \in X} [-\|x\|, \|x\|]$ is compact by Tychonoff's theorem. $i(B_{X^*})$ consists of only the elements of K that are linear. To finish show nets in $i(B_{X^*})$ converge to linear elements of K. □

Theorem. *If X is reflexive, then $(B_X, \sigma(X, X^*))$ is compact.*

Hahn-Banach Theorems

Definition. Let E be a vector space over $\mathbb{R}$. A subset $A \subset E$ is called absorbing if for all $x \in E$, there exists $\lambda > 0$ such that $x \in \lambda A$.

A neighborhood of 0 in a topological vector space is absorbing: For all $x \in E$, the map $\mu_\lambda : \mathbb{R} \to E$ sending λ to λx is continuous and sends 0 to 0. Then if V is a neighborhood of 0 in E, there exists $r > 0$ such that $(-r, r) \subset \mu_x^{-1}(V)$, and thus for all $|\lambda| < r$, $\mu_x(\lambda) = \lambda x \in V$.

Definition. Let A be an absorbing set in a topological vector space E. We define the gauge, or Minkowski Functional, of A, denoted μ_A, as follows:

$$\begin{aligned} \mu_A : & X \to [0, \infty) \\ & x \mapsto \inf\{\lambda > 0 : x \in \lambda A\} \end{aligned}$$

Notice that $\mu_A(0) = 0$.

Lemma. *If C is a convex absorbing subset, then*

i. μ_C is a sublinear functional

ii. $\{x \in E : \mu_C(x) < 1\} \subset C \subset \{x \in E : \mu_C(x) \le 1\}$.

iii. If E is an LCTVS and $0 \in C^\circ$, then μ_C is continuous at 0.

Proof. (i.) Let $x, y \in E$ and $\varepsilon > 0$. By definition, there are $\lambda, \mu > 0$ such that $\lambda < \mu_C(x) + \varepsilon$, $\mu < \mu_C(y) + \varepsilon$ and $x \in \lambda C$, $y \in \mu C$. Then

$$\frac{x + y}{\lambda + \mu} = \frac{\lambda}{\lambda + \mu} \frac{x}{\lambda} + \frac{\mu}{\lambda + \mu} \frac{y}{\mu} \in C,$$

so that $x + y \in (\lambda + \mu)C$ and $\mu_C(x + y) \leq \lambda + \mu \leq \mu_C(x) + \mu_C(y) + 2\varepsilon$. This shows subadditivity. Positive homogeneity is obvious after expanding the definition of μ_C.

(ii.) If $x \in C$, then $x = \frac{x}{1}$, which proves the second inclusion. For the first, if $\mu_C(x) < 1$, then for some $\lambda < 1$, we have $x \in \lambda C$. Since C is convex, writing $x = \lambda \frac{x}{\lambda} + (1 - \lambda)0$ shows that $x \in C$.

(iii.) Since $x \in C^\circ$, there is a convex open neighborhood $U \ni 0$ in C. Let $\varepsilon > 0$, then εU is also an open neighborhood of 0, and if x_α is a net in E converging to 0, then there exists α_0 such that $x_\alpha \in \varepsilon U$ for all $\alpha > \alpha_0$. Then $\mu_C(x_\alpha) \leq \mu_U(x_\alpha) \leq \varepsilon$.

Geometric Hahn-Banach Separation Theorem for LCTVS

Theorem. *Let (X, τ) be an LCTVS, C a nonempty closed convex subset, and $x_0 \in X \setminus C$. Then there exists $x^* \in (X, \tau)^*$ such that*

$$x^*(x_0) > \sup_{x \in C} x^*(x)$$

Proof. WLOG, suppose $0 \in C$. Since C is closed, $X \setminus C$ is open and there exists a convex neighborhood U of 0 such that $x_0 + U \subset X \setminus C$. Then take a convex neighborhood V of 0 such that $V - V \subset U$ by continuity of operations in a TVS.

Let $D = C + V$ and observe that $(x_0 + V) \cap D = \varnothing$, and D is convex and $0 \in D^\circ$. Need to write this step out to see how $V - V \subset U$ is used.

Let μ_D be the gauge of D. Then for all $z \in x_0 + V$, $\mu_D(z) \geq 1$. Since V is open, there is a $\lambda > 1$ such that $\lambda x_0 \in x_0 + V$ and in fact $\mu_D(x_0) > 1$.

Now define

$$\begin{aligned} f :& \mathbb{R}x_0 \to \mathbb{R} \\ & \alpha x_0 \mapsto \alpha \mu_D(x_0) \end{aligned}$$

and observe that f is linear. Then for any $\alpha \geq 0$, we have

$$f(\alpha x_0) = \alpha \mu_D(x_0) = \mu_D(\alpha x_0).$$

Likewise if $\alpha < 0$ we have

$$f(\alpha x_0) = \alpha \mu_D(x_0) \leq \mu_D(\alpha x_0),$$

so that $f \leq \mu_D$ on $\mathbb{R}x_0$. By the algebraic Hahn-Banach theorem, we can extend f to a function $F : X \to \mathbb{R}$ such that F equals f on the subspace $\mathbb{R}x_0$, and $F \leq \mu_D$ on X. In particular, $x \in D$ implies $\mu_D(x) \leq 1$ and thus $F(x) \leq 1$ on D and $F(x) \geq -1$ on $-D$. Then we have $|F(x)| \leq 1$ on $D \cap (-D)$ and F is continuous at 0.

The inequality holds since $F(x_0) \geq 1$ but $F(x) < 1$ for all $x \in D$.

Applications

Theorem (Goldstine's Theorem)**.** *Let X be a normed space. Then*

$$\overline{j(B_x)}^{\sigma(X^{**}, X^*)} = B_{X^{**}}.$$

In particular,

$$\overline{j(X)}^{\sigma(X^{**}, X^*)} = X^{**}.$$

Proof. First notice that

$$\overline{j(B_x)}^{\sigma(X^{**}, X^*)} \subset B_{X^{**}}$$

since $B_{X^{**}}$ is weak* compact and hence closed.

Next suppose $x_0 \in B_{X^{**}} \setminus \overline{j(B_x)}^{\sigma(X^{**},X^*)}$. $\sigma(X^{**}, X^*)$ is a hausdorff LCVT, so we can apply geometric Hahn-Banach theorem to obtain $\varphi \in (X, \sigma(X^{**}, X^*)) = j(X^*)$ such that $\varphi(x_0) > \sup_{x \in \overline{j(B_x)}^{\sigma(X^{**},X^*)}} \varphi(x)$.

Then since $\varphi = j(x_0^*)$ for some $x_0^* \in X^*$, we have

$$\varphi(x_0) > \sup_{x \in \overline{j(B_x)}^{\sigma(X^{**},X^*)}} x(x_0^*) \geq \sup_{x \in j(B_X)} x(x_0) = \sup_{x \in B_x} x_0^*(x) = \|x_0^*\|.$$

However, $j(x_0^*)(x_0) = x_0(x_0^*) \leq \|x_0\|_{\sigma(X^{**},X^*)} \|x_0^*\|_{X^*} \leq \|x_0\|_{X^*}$, which shows $\|x_0^*\| < \|x_0^*\|$, a contradiction.

Theorem (Mazur's Theorem)**.** *Let C be a convex subset of a normed space X. Then $\overline{C}^{\|\cdot\|} = \overline{C}^{w}$.*

Proof. We have $\overline{C}^{\|\cdot\|} \subset \overline{C}^{w}$ by definition. The intuition is that since the weak topology is less restrictive, it allows more into the closure.

Then suppose $x_0 \in \overline{C}^{w} \setminus \overline{C}^{\|\cdot\|}$.

By the Geometric Hahn-Banach theorem, there exists $x_0^* \in (X, \|\cdot\|)^*$ such that $x_0^*(x_0) > \sup_{x \in C} x_0^*(x)$. Now let x_α be a net in C converging weakly to x_0. Then for all $x^* \in X^*$, $x^*(x_\alpha) \to x^*(x_0)$. In particular, $x_0^*(x_\alpha) \to x_0^*(x_0)$. However, we have $x_0^*(x_\alpha) \leq \sup_{x \in C} x_0^*(x)$, implying $x_0^*(x_0) \leq \sup_{x \in C} x_0^*(x) < x_0^*(x_0)$, a contradiction.

Theorem (Eberlein-Smulian)**.** *Let $(X, \|\cdot\|)$ be a normed vector space. Then $A \subset X$ is (relatively) weakly compact if and only if A is (relatively) weakly sequentially compact.*

Remark.
1. The weak topology on X is metrizable iff X is finite dimensional
2. The weak topology on X is not $1st$ countable
3. $(B_X, \sigma(X, X^*))$ is metrizable iff X^* is separable
4. $(B_{X^*}, \sigma(X^*, X))$ is metrizable iff X is separable.

Lemma. *Let $(X, \|\cdot\|)$ be a normed space. If X is separable, then there exists a norm on X that induces a topology that is weaker than the weak topology on the unit ball.*

Proof of Lemma. Let $\{x_n\}$ be a dense sequence in B_X. Choose $x_n^* \in B_X$ such that $x_n^*(x_n) = \|x_n\|$ using algebraic Hahn-Banach theorem. Let $p(x) = \sum_1^\infty \frac{1}{2^n}|x_n^*(x)|$, taking values in $[0, \infty)$. Check that p is a sublinear functional. Assume that $p(x) = 0$ and $\|x\| \leq 1$. Let $i \geq 1$ such that $\|x - x_i\| < \varepsilon$. Then

$$\|x_i\| = |x_i^*(x_i)| = |x_i^*(x - x_i)| \leq \|x_i - x\| < \varepsilon$$

Now let $r > 0$ and consier $\{x \in B_X : p(x) < r\}$. Let $V = \{x \in B_X : |x_i^*(x)| < \varepsilon, 1 \leq i \leq N\}$. We can choose ε small so that the first N terms of $p(x)$ sum to less than $r/2$ and N large so that the remaining terms sum to less than $r/2$. □

Proof of Eberlein-Smulian ($\Rightarrow$) Since X is a normed vector space and A relatively weakly compact, every sequence in A has a subsequence which is convergent in X.

Let $K = \overline{A}^{\sigma(X,X^*)}$. Then K is weakly compact. Let $a_n \in A$ and define $Z := \overline{\text{span}\{a_n\}}^{\|\cdot\|} \subset X$. Z is a separable subspace of X.

Let $K_0 = \overline{\{a_n\}}^{\sigma(X,X^*)}$. Note that $K_0 \subset Z$, since Z is a convex set and is thus also weakly closed by Mazur's theorem. Also K_0 is a wealky closed subset of K, which is compact, thus K_0 is weakly compact.

In fact K_0 is $\sigma(Z, Z^*)$ compact by Hahn-Banach extension theorem, since every linear functional on Z extends to one on X.

Note that K_0 is weakly compact and hence bounded in Z. By the previous lemma, there is a norm ρ on Z which induces a topology on K_0 which is weaker than the weak topology.

The ρ topology actually coincides with $\sigma(Z, Z^*)$ on K_0. This is because if $\tau_1 \subset \tau_2$ are both topologies, with τ_1 hausdorff and τ_2 compact, then $\tau_1 = \tau_2$. Then K_0 is metrizable, so a_n has a subsequence which is weakly convergent in Z. □

Definition. Let $A \subset (X, \|\cdot\|)$. We say that A is weakly bounded if for all $x^* \in X^*$, the set $x^*(A) \subset \mathbb{R}$ is bounded.

Remark. Every originally bounded subset is also weakly bounded.

Lemma. *If A is weakly bounded, then A is norm bounded.*

Proof. Consider linear maps $T_a : X^* \to \mathbb{R}$, where $x^* \mapsto x^*(a)$ for $a \in A$. Then $\|T_a\| = \|a\|$. Since A is weakly bounded, for each $x^* \in X*$ we have $\sup_{a\in A} |T_a(x^*)| < \infty$. Then the Uniform Boundedness Principle implies that $\sup_{a\in A} \|T_a\| < \infty$ and thus $\sup_{a\in A} \|a\| < \infty$. $\square$

Corollary. *If $A \subset (X, \|\cdot\|)$ is (relatively) weakly compact OR (relatively) weakly sequentially compact, then A is norm-bounded.*

Proof. Prof. only sketched. Prove by contradiction.

Lemma. *Let $(X, \|\cdot\|)$ be a normed space and $E \subset X^*$ a finite dimensional subspace. Then there exists a finite subset $F \subset X$ such that for all $x^* \in E$, we have*

$$\frac{\|x^*\|}{2} \leq \max_{x\in F} |x^*(x)| \leq \|x^*\|$$

Proof. Since E is finite dimensional, the unit sphere S_E is compact. Then we can choose a finite η-net $\{x_1^*, \dots, x_N^*\}$ such that for all $x^* \in S_E$, there is some $i \in \{1, \dots, N\}$ such that $\|x^* - x_i^*\| < \eta$. For each i, choose $x_i \in B_X$ such that $|x_i^*(x_i)| > 1 - \eta$.

Then for any $x^* \in E$, choose $i \in \{1, \dots, N\}$ such that $\left\|\frac{x^*}{\|x^*\|} - x_i^*\right\| < \eta$. Then we have

$$\left|\frac{x^*}{\|x^*\|}(x_i)\right| = \left|\left(\frac{x^*}{\|x^*\|} - x_i^*\right)(x_i) + x_i^*(x_i)\right| \geq |x_i^*(x_i)| - \left|\left(\frac{x^*}{\|x^*\|} - x_i^*\right)(x_i)\right| \geq 1 - \eta - \eta,$$

using the reverse triangle inequality. Then take $\eta = 1/4$.

Proof of Eberlein-Smulian ($\Leftarrow$) Our first observation is that A is bounded by the above corollary. The second and main observation is that if $A \subset X$ is bounded, then $\overline{A}^{\sigma(X,X^*)}$ is compact if and only if $\overline{J(A)}^{\sigma(X^{**},X^*)} \subset J(X)$.

To prove the only if, first we have that $j(\overline{A}^{\sigma(X,X^*)})$ is $\sigma(X^{**}, X^*)$ compact since J is weak to weak* continuous. Then $j(\overline{A}^{\sigma(X,X^*)})$ is closed since the weak* topology is hausdorff. Then since $A \subset \overline{A}^{\sigma(X,X^*)}$, we have $\overline{J(A)}^{\sigma(X^{**},X^*)} \subset j(\overline{A}^{\sigma(X,X^*)})$.

For the other direction, A bounded implies $j(A)$ bounded, so $\overline{j(A)}^{\sigma(X^{**},X^*)}$ is $\sigma(X^{**}, X^*)$-compact by Banach-Alaoglu. Now if $\overline{j(A)}^{\sigma(X^{**},X^*)} \subset J(X)$, the $\sigma(X^{**}, X^*)$ topology restricted to $J(X)$ coincides with the weak topology on X and thus $\overline{A}^{\sigma(X,X^*)}$ is weakly compact.

Now we begin the proof. Let $x_0^{**} \in \overline{J(A)}^{\sigma(X^{**},X^*)}$. Our goal will be to show that there is some $x_0 \in X$ such that $x_0^{**} = J(x)$. We will construct a sequence $\{a_n\} \subset A$ and $\{x_n^*\} \subset B_{X^*}$ inductively.

Begin by taking $x_1^* \in S_{X^*}$ and consider the $\sigma(X^{**}, X^*)$ neighborhood $V = \{x^{**} \subset X^{**} : |x^{**}(x_1^*) - x_0^{**}(x_1^*)| < 1\}$ of x_0^{**}. Since $x_0^{**} \in \overline{J(A)}^{\sigma(X^{**},X^*)}$, there is $a_1 \in A$ such that $J(a_1) \in V$ and hence $|J(a_1)(x_1^*) - x_0^{**}(x_1^*)| < 1$.

Now $E_1 := \operatorname{span}\{x_0^{**}, x_0^{**} - J(a_1)\}$ is a finite dimensional subspace of X^*, so by the lemma there is a finite sequence $x_2^*, \dots, x_{n_2}^* \in B_{X^*}$ such that for all $x^{**} \in E_1$,

$$\frac{\|x^{**}}{2} \leq \max_{2\leq i\leq n_2} |x^{**}(x_i^*)| \leq \|x^{**}\|$$

Then in a similar fashion to above, there is some $a_2 \in A$ such that

$$|J(a_2)(x_i^*) - x_0^{**}(x_i^*)| < \frac{1}{2}$$

for all $1 \le i \le n_2$. By the lemma there exist $x^*_{n_2+1}, \dots, x^*_{n_3} \in B_{X^*}$ such that for all $x^{**} \in \text{span}\{x_0^{**}, x_0^{**} - j(a_1), x_0^{**} - j(a_2)\}$, we have

$$\frac{\|x^{**}\|}{2} \le \max_{n_2+1 \le i \le n_3} |x^{**}(x_i^*)| \le \|x^{**}\|.$$

Continue inductively to obtain sequences $\{a_n\} \subset A$ and $\{x_n^*\} \subset B_{X^*}$, such that

1. for all $x^{**} \in \text{span}\{x_0^{**}, x_0^{**} - J(a_1), x_0^{**} - J(a_2), \dots\}$,

$$\frac{\|x^{**}\|}{2} \le \sup_{i \ge 1} |x^{**}(x_i)| \le \|x^{**}\|$$

2. $|J(a_k)(x_i^*) - x_0^{**}(x_i^*)| < \frac{1}{k}$ for all $1 \le i \le n_k$.

Since A is relatively weakly sequentially compact, there is some $x \in X$ and a subsequence $\{a_{n_k}\}$ converging to x in the $\sigma(X, X^*)$ topology.

Note that by Mazur's theorem, $x \in \overline{\text{span}\{a_n : n \ge 1\}}$. Hence $x_0^{**} - j(x) \in \overline{\text{span}\{x_0^{**} - J(a_n) : n \ge 1\}} =: Z$. This needs to be verified. Then for any $z^{**} \in Z$, we have

$$\frac{\|z^{**}\|}{2} \le \sup_{i \ge 1} |z^{**}(x_i^*)|$$

by a continuity argument.

In particular,

$$\frac{\|x_0^{**} - J(x)\|}{2} \le \sup_{i \ge 1} |(x_0^{**} - J(x))(x_i^*)|.$$

Finally we will show this last term must be zero. Let $i \ge 1$. Then

$$|(x_0^{**} - J(x))(x_i^*)| \le |(x_0^{**} - J(a_k))(x_i^*)| + |(J(a_k) - J(x))(x_i^*)| \le \varepsilon/2 + \varepsilon/2$$

by choosing k large enough that the second term is small by weak convergence, and the first is small by (2.) above, such that $a_k > i$.

Reflexive Spaces

Definition. A normed space is called reflexive if the canonical map

$$\begin{aligned} J :& X \to X^{**} \\ & x \mapsto (x^* \mapsto x^*(x)) =: \langle J(x), x^* \rangle \end{aligned}$$

Remark. A reflexive space is always a Banach space.

The obvious examples are the spaces ℓ_p and $L_p([0,1])$ for $1 < p < \infty$.

Topological Characterization of Reflexivity

Theorem. *Let X be a Banach space. X is reflexive if and only if B_X is $\sigma(X, X^*)$ compact.*

Proof. The forward direction is immediate by Banach-Alaoglu theorem.

For the other direction, if $(B_X, \sigma(X, X^*))$ is compact, then $J(B_X)$ is $\sigma(X^{**}, X^*)$ compact. Then $J(B_X)$ is closed since $\sigma(X^{**}, X^*)$ is a hausdorff topology. But by Goldstine's theorem, $J(B_X) = \overline{J(B_X)}^{\sigma(X^{**}, X^*)} = B_{X^{**}}$. Then $J(B_X) = B_{X^{**}}$, implying that $J(X) = X^{**}$.

Corollary. *Let X be a Banach space. If X is reflexive, then*

1. *X^* is reflexive*

2. *Every closed subspace of X is reflexive*

3. *Every $x^* \in X^*$ attains its norm*

4. *Y is reflexive whenever Y is isomorphic to X*

5. *Every bounded sequence in X has a weakly convergent subsequence.*

Proof. (1.) Assume X is reflexive. Then $(B_{X^*}, \sigma(X^*, X^{**})) \simeq (B_{X^*}, \sigma(X^*, X))$, and since the second space is compact, the unit ball in X^* is weakly compact and thus X^* is reflexive.

(2.) Let X be reflexive and Y be a closed subspace. By assumption, $(B_X, \sigma(X, X^*))$ is compact. The restriction of $\sigma(X, X^*)$ to Y is $\sigma(Y, Y^*)$. Therefore, $(B_Y, \sigma(Y, Y^*))$ is compact because it is a $\sigma(X, X^*)$ closed subset of B_X.

(3.) Compactness argument.

(4.) Assume there exists $T : X \to Y$ such that $1/C\|x\| \leq \|Tx\| \leq C\|x\|$ for some $C > 0$. Then $\frac{1}{C}B_Y \subset T(B_X) \subset CB_Y$. We have that $(B_X, \sigma(X, X^*))$ is compact. Since T is weak to weak continuous, $T(B_X)$ is $\sigma(Y, Y^*)$ compact. Finally, since $\frac{1}{C}B_Y$ is a $\sigma(Y, Y^*)$ closed subset of a $\sigma(Y, Y^*)$ compact set, it is also $\sigma(Y, Y^*)$ compact.

(5.) x_n bounded implies $x_n \subset cB_X$ for some c. Since the unit ball is weakly compact and thus weakly sequentially compact by Eberlein-Smulian, there is a weakly convergent subsequence. □

Proposition. *If X^* is reflexive, then X is reflexive.*

Proof. The above corollary implies that X^{**} is reflexive in this case. Then $J(X)$ is a closed subspace of X^{**} and thus $J(X)$ and X are reflexive.

Sequential/Geometric Characterization of Reflexivity

Theorem. *Let X be a Banach space. The following are equivalent:*

1. *X is not reflexive.*

2. *For all $\theta \in (0, 1)$, there exists a sequence $\{x_n\} \subset B_X$ and $\{x_n^*\} \subset B_{X^*}$ such that $x_n^*(x_k) = 0$ if $k < n$ and θ if $k \geq n$.*

3. *For all $\theta \in (0, 1)$, there exists a sequence $\{x_n\} \subset B_X$ such that for all $k > 1$,*

$$d(\text{conv}\{x_1, \dots, x_k\}, \text{conv}\{x_{n+1}, \dots\}) \geq \theta$$

Moment Problem

Let $(X, \|\cdot\|)$ be a normed vector space. Let $x_1^*, \dots, x_n^* \in X^*$ and $c_1, \dots, c_n \in \mathbb{R}$. Does there exist $x \in X$ such that $x_i^*(x) = c_i$ for all $1 \leq i \leq n$.

Theorem (Helly's Theorem)**.** *Let $x_1^*, \dots, x_n^* \in X^*$, $c_1, \dots, c_n \in \mathbb{R}$, and $k > 0$. Then the following are equivalent:*

1. *For all $\varepsilon > 0$, there exists $x_\varepsilon \in X$ such that $\|x_\varepsilon\| \leq k + \varepsilon$ and $x_i^*(x_\varepsilon) = c_i$ for $1 \leq i \leq n$.*

2. *For all $a_1, \dots, a_n \in \mathbb{R}$,*

$$\left| \sum_1^N a_i c_i \right| < k \left\| \sum_1^N a_i x_i^* \right\|$$

Proof. $(1 \Rightarrow 2)$

$$\left|\sum a_i c_i\right| = \left|\sum a_i x_i^*(x_\varepsilon)\right| = \|x_\varepsilon\| \left\|\sum_1^N a_i x_i^*\right\| \leq (x+\varepsilon)\left\|\sum a_i x_i^*\right\|$$

$(2 \Rightarrow 1)$ Without loss of generality, suppose not all $c_i = 0$. Say $c_{i_0} \neq 0$. Also suppose not all x_i^* are zero.

Therefore we can assume $x_1^*, \ldots, x_k^*$ are linearly independent for $k \leq n$. Thus for all $1 \leq i \leq n$, $x_i^* = \sum_1^k \alpha_j^{(i)} x_j^*$.

Given this assumption, if we show 2 holds for the linearly independent elements, there is an argument to show that it holds for the rest of the elements. See notes.

Then we can assume $x_1, \ldots, x_n$ are linearly independent.

Consider $T : X \to \mathbb{R}^n$, $x \mapsto (x_1^*(x), \ldots, x_n^*(x))$. T is linear. Because the x_i^* are linearly independent, for all $1 \leq k \leq n$ we have $\bigcap_{i\neq k} \ker x_i^* \subset \ker(x_k^*)$.

For all $1 \leq k \leq n$, there exists $y_k \in \bigcap \ker x_i^* \setminus \ker x_k^*$ such that $x_k^*(y_k) = 1$, and $x_j^*(y_k) = 0$ for all $j \neq k$. Let $y = \sum_1^n c_j y_j$. Then $x_i^*(y) = \sum_1^n c_j x_j^*(y_j) = c_i$.

Let $Z := \bigcap_1^n \ker x_i^*$, a set closed in X. By Hahn-Banach, there exists $x^* \in X^*$ such that $x^*(y) = d(y, z)$. Since $\ker(x^*) \supset Z$, we have $x^* = \sum_1^n \alpha_i x_i^*$.

$d(y, Z) = x^*(y) = \sum_1^n \alpha_i x_i^*(y) = \sum \alpha_i c_i \leq k \left\|\sum_1^n \alpha_i x_i^*\right\|$.

Fix $\varepsilon > 0$. There exists $z \in Z$ such that $\|y - z\| \leq (k+\varepsilon)\left\|\sum_1^n \alpha_i x_i^*\right\| = \|x^*\| = 1$.

Then $x_\varepsilon = y - z$ satisfies $\|x_\varepsilon\| \leq k + \varepsilon$.

See notes for proof of sequential characterization of Reflexivity.

Theorem. *Let X be a Banach space. The following are equivalent:*

1. *X is reflexive*
2. *Every bounded sequence in X has a weakly convergent subsequence*
3. *If (C_n) is a nonincreasing sequence of nonempty, bounded, closed, convex sets, then $\cap C_n \neq \varnothing$*

Proof. $(1 \to 2)$ follows by Eberlein-Smulian.

$(2 \to 3)$: Let $x_n \in C_n$ for all $n \geq 1$. (x_n) is bounded and hence there exists $x \in X$ such that $x_{n_k} \to x$ weakly for some n_k.

Claim: $x \in \cap_1^\infty C_n$. Assume otherwise. Then there is some n_0 such that $x \notin C_{n_0}$. By geoemtric Hahn-Banach, there exists $x^* \in S_{X^*}$ such that $x^*(x) > \sup_{C_{n_0}} x^*(z)$. Note that $x^*(x) = \lim x^*(x_{n_k})$. But there exists $K \geq 1$ such that for all $k \geq K$, $x_{n_k} \in C_{n_0}$ and hence $x^*(x_{n_k}) \leq \sup_{C_{n_0}} x^*(z)$ for all $k \geq K$. Then we have a contradiction.

$(3 \to 1)$: Assume X is not reflexive. Then apply the sequential characterization to obtain some $\theta \in (0,1), x_n \in B_X, and x_n^* \in B_{X^*}$. Consider $C_n = \overline{\text{conv}\{x_k : k \geq n\}}$. Then C_n is nonincreasing, nonempty, closed, and bounded. We claim $\cap C_n = \varnothing$. Suppose not. Then let $x \in \cap C_n$ and observe that for all $\varepsilon > 0$ and all $k \geq 1$, there exists $y \in C_n$ such that $\|x - y\| < \varepsilon$, where $y = \sum_1^{n_\varepsilon} \lambda_i x_i$ for $\lambda_1 > 0$ and where y is a convex combination.

Then for all $n > n_\varepsilon$, we have $|x_n^*(x-y)| = |x_n^*(x)| < \varepsilon$ and thus $\lim_{n\to\infty} x_n^*(x) = 0$. But because $x \in C_k$ for $k \geq 1$, we have $x_k^*(x) \geq \theta/2$,a contradiction.

Finite Representability

Definition. Let X, Y be Banach spaces and $\lambda \geq 1$.

1. Y is λ-finitely representable in X if for every finite dimensional subspace $E \subset Y$, there exists an isomorphism $T : E \to X$ such that $\|T\|\|T^{-1}\| \leq \lambda$. In other words, there exists $k \geq 0$ with $k^2 \leq \lambda$ such that for all $e \in E$,

$$\frac{\|e\|}{k} \leq \|Te\| \leq k\|e\|.$$

2. Y is finitely representable in X if it is $(1+\varepsilon)$-finitely representable in X for all $\varepsilon > 0$.

Example. 1. $L_p([0,1])$ is finitely representable in ℓ_p for $1 \leq p < \infty$.

2. Every Banach space is finitely representable in any Banach space which contains the ℓ_∞^n's, such as $c_0, \ell_\infty, C([0,1])$, etc.

Lemma. *1. If X_1 is λ_1 finite representable in X_2 and X_2 is λ_2 finite representable in X_3, then X_1 is $\lambda_1\lambda_2$ finite representable in X_3*

2. If X_1 is finitely representable in X_2 and X_2 is finitely representable in X_3, then X_1 is finitely representable in X_3

Definition. Let P be a property of Banach spaces. We say that a Banach space X has super-P if every Banach space that is finitely representable in X has P.

Remark. 1. Super-P implies P

2. Super-super-P is equal to super-P

Theorem. *If X is a Banach space, then X^{**} is finitely representable in X.*

The proof of this theorem is a consequence of the principle of local reflexivity.

Ultraproducts

Definition. Let I be a set, $\mathcal{U} \in \beta I$ an untrafilter on I, and $(X_i)_{i\in I}$ a collection of Banach spaces. The Ultraproduct of (X_i) with respect to $\mathcal{U}$ is $(\prod_{i\in I} X_i)_\mathcal{U} := \ell_\infty(I;(X_i)_i)/N_\mathcal{U}$,

where $\ell_\infty(I;(X_i)i) := \{(x_i)_{i\in I} : \forall i \in I, x_i \in X_i, \sup \|x_i\|_{X_i} < \infty\}$ equipped with the sup norm, and $N_\mathcal{U} := \{(x_i)_{i\in I} \subset \ell_\infty(I,(x_i)_i) : \lim_{i,\mathcal{U}} \|x_i\|_{X_i} = 0\}$.

One important notion used here is that of a limit along an ultrafilter. If $f : I \to (X, \tau)$, we say $\lim_{i,\mathcal{U}} f(i) = x$ iff for all neighborhoods V of x, we have $f^{-1}(V) \in \mathcal{U}$.

Lemma. *If $(x_i)_\mathcal{U} \in (\prod_{i\in I} X_i)_\mathcal{U}$, then*

$$\|(x_i)_\mathcal{U}\|_\mathcal{U} = \lim_{i,\mathcal{U}} \|x_i\|_{X_i}$$

where the first norm is the quotient norm.

Lemma. *If $\mathcal{U} \in \beta I$ is non-principle, then $\mathbb{R}^\mathcal{U}$ is linearly isomorphic to $\mathbb{R}$.*

Theorem. *Let X be a Banach space, $\mathcal{U} \in \beta I$ non-principle. Then $X^\mathcal{U}$ is finitely representable in X.*

See notes for proof.

Theorem. *Let X be a Banach space, $E \subset X^{**}$, $F \subset X^*$ both finite dimensional subspaces. For all $\varepsilon > 0$, there exists an injective operator $T : X \to X$ such that*

1. $Tx = x$ for all $x \in E \cap X$

2. $\|T\|\|T^{-1}\| \leq 1 + \varepsilon$

3. For all $x^{**} \in E$ and $x^* \in F$, $x^{**}(x^*) = x^*(Tx^{**})$.

Corollary. *X^{**} is finitely representable in X.*

Lemma (Helly's Lemma)**.** *Let X be a Banach space and $G \subset X^*$ finite dimensional. For all $x^{**} \in X^*$ and $\varepsilon > 0$, there exists $x \in X$ such that*

1. $\|x\| \leq (1+\varepsilon)\|x^{**}\|$

2. For all $x^* \in G$*,* $x^{**}(x^*) = x^*(x) = J(x)(x^{**})$*.*

Proof of Helly's lemma
Let $G = \text{span}\{x_i^* : 1 \leq i \leq n\}$ and let $c_i = x^{**}(x_i^*)$ for $1 \leq i \leq n$.
Choose any $(\alpha_i)_1^\infty \subset \mathbb{R}$. Then we have

$$\left|\sum_1^n \alpha_i c_i\right| = \left|\sum_1^n \alpha_i x^{**}(x_i)\right| \leq \|x^{**}\| \left\|\sum_1^n \alpha_i x^*\right\|$$

Then the conditions for Helly's theorem are satisfied, and there exists $x \in X$ such that $\|x\| \leq (1+\varepsilon)\|x^{**}\|$ and $x_i^*(x) = c_i = x^{**}(x_i^*)$ for $1 \leq i \leq n$. Then by linearity, we can extend this to

$$\text{for all } x^* \in G, x^*(x) = x^{**}(x^*)$$

□

Remark. Let X be a real vector space. Then there is a canonical identification of $L(\mathbb{R}, X)$ with X, where $f \mapsto f(1)$. Then we have $L(\mathbb{R}, X)^{**} \equiv X^{**} \equiv L(\mathbb{R}, X^{**})$.

Thus, we can restate Helly's Lemma as follows:

Lemma (Generalized Helly's Lemma)**.** *Let* X *be a Banach space,* $E \subset X$ *and* $F \subset X^*$ *finite dimensional subspaces. For all* $S \in L(E, X^{**})$ *and all* $\varepsilon > 0$*, there exists* $T \in L(E, X)$ *such that*

$$\|T\| \leq (1+\varepsilon)\|S\|$$

and for all $x^* \in F$ *and* $e \in E$*,* $(Se)(x^*) = x^*(Te)$*.*

Proof of GHL For all $x \in E$ and $x^* \in F$, define

$$\begin{aligned} x \otimes x^* : L(E, X) &\to \mathbb{R} \\ A &\to x^*(Ax) \end{aligned}$$

Then $x \otimes x^*$ is linear, and we have $\|x \otimes x^*\| \leq \|x\|\|x^*\|$. Then $x \otimes x^* \in L(E, X)^*$ for all $x \in E, x^* \in F$.

Define $G := \text{span}\{x \otimes x^* : x \in E, x^* \in F\}$, a finite dimensional subspace of $L(E, X)^*$. By Helly's lemma on $G \subset L(E, X)^*$, for all $S \in L(E, X)^{**}$ and $\varepsilon > 0$, there exists $T \in L(E, X)$ such that $\|T\| \leq (1+\varepsilon)\|S\|$, and

$$\text{for all } R \in G, S(R) = R(T),$$

i.e. for all $x \in E$ and $x^* \in F$, $S(x \otimes x^*) = (x \otimes x^*)(T) = x^*(Tx)$.

Then if for all $\tilde{S} \in L(E, X)^{**}$ we can find an $S \in L(E, X^{**})$ such that

$$(\tilde{S}e)(x^*) = S(e \otimes x^*) \text{ for all } e \in E, x^* \in F,$$

we will be done. We will save the proof of this fact until after proving PLR.
Proof of PLR
Let $E \subset X^{**}, F \subset X^*$ be finite dimensional subspaces. Consider

$$\begin{aligned} S : E &\to X^{**} \\ x^{**} &\mapsto x^{**}. \end{aligned}$$

By Generalized Helly's Lemma, for all $\varepsilon > 0$, there exists some bounded operator $T : E \to X$ such that

1. $\|T\| \leq (1+\varepsilon)\|S\| \leq (1+\varepsilon)$
2. For all $x^* \in F$ and $e \in E$, $(Se)(x^*) = x^*(Te)$.

The problem is that we do not know if T is injective or if $\|T\| \leq 1$.

We need to enlarge F in order to get these results. Let $\delta > 0$ and choose a $\delta/2$ net $\{x_1^{**}, \ldots, x_n^{**}\} \subset S_E$, and let $x_i^* \in S_{X^*}$ such that $x_i^{**}(x_i^*) \geq 1 - \delta$ for $1 \leq i \leq n$. Let $\tilde{F} = \operatorname{span}\{F \cup \{x_i^* : 1 \leq i \leq n\}\}$, and observe that $\tilde{F}$ is also a finite dimensional subspace of X^*. Now apply GHL to $\tilde{F}$. Then for all $\varepsilon > 0$, there exists $T \in L(E, X)$ such that

1. $\|T\| \leq (1 + \varepsilon)$
2. For all $x^* \in \tilde{F}$ and all $x^{**} \in E$, $x^{**}(x^*) = x^*(Tx^{**})$

Next we will prove that T is the identity on E. Let $x \in E$. Then for all $x^* \in F$, $x^*(Tx - x) = 0$. Assume by contradiction that $Tx \neq x$. Pick some $x_{i_0}^{**} \in S_E$ such that

$$\left\| \frac{Tx - x}{\|Tx - x\|} - x_{i_0}^{**} \right\| \leq \delta.$$

But then

$$1 - \delta \leq |x_{i_0}^{**}(x_{i_0}^*)| = \left| \left(x_{i_0}^* - \frac{Tx - x}{\|Tx - x\|} \right) (x_{i_0}^*) \right| \leq \left\| x_{i_0}^* - \frac{Tx - x}{\|Tx - x} \right\| \|x_{i_0}^*\| < \delta$$