Homework 1

1. Let $F : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathbb{R}$ be given by $F(A) := \det A$.

   (a.) F is a polynomial in the n^2 entries of the matrix. It is thus differentiable as a map $\mathbb{R}^{n^2} \to \mathbb{R}$. The space $\mathbb{R}^{n^2}$ is isomorphic to the space of $n \times n$ matrices as a finite dimensional normed vector space.

   (b.) Suppose A is invertible. We are looking for a linear map $D_AF : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathrm{Mat}_{n\times n}(\mathbb{R})$ such that

$$F(A+H) = F(A) + D_AF(H) + r(H)$$

   where $r(H) = o(\|H\|)$. We have

$$F(A+H) = \det(A+H) = \det(A)\det(I + A^{-1}H),$$

   and we can write

$$\det(I + A^{-1}H) = 1 + \mathrm{Tr}(A^{-1}H) + r(H),$$

   where $r(H)$ consists of terms in the determinate of at least order 2 in the components of $A^{-1}H$.

   We can bound $|r(H)|$ by a sum of absolute values of monics in elements of $A^{-1}H$ of orders at least 2. Then use $(A^{-1}H)_{ij} \le \|A^{-1}H\| \le \|A^{-1}\|\|H\|$, so that

$$\lim_{\|H\|\to 0} \frac{|r(H)|}{\|H\|} = 0.$$

   Then we have

$$\det(A+H) = \det(A) + \det(A)\mathrm{Tr}(A^{-1}H) + \det(A)r(H),$$

   where of course the last term is still $o(\|H\|)$, and where $D_AF(H) = \det(A)\mathrm{Tr}(A^{-1}H)$ is a linear map.

   (c.) Using the formula for the inverse of an invertible matrix A, we have

$$D_AF(H) = \det(A)\mathrm{Tr}(A^{-1}H) = \det(A)\frac{1}{\det(A)}\mathrm{Tr}(\mathrm{adj}(A)H) = \mathrm{Tr}(\mathrm{adj}(A)H).$$

   Since the set of invertible matrices is dense in $\mathrm{Mat}_{n\times n}(\mathbb{R})$, we can approach a non-invertible matrix B by a sequence of invertible matrices $\{A_j\}_{j=1}^{\infty}$ so that $A_j \to B$. Since F is continuously differentiable, $\lim_{j\to\infty} D_{A_j}F(H) = D_BF(H)$. Then $\lim_{j\to\infty} D_{A_j}F(H) = \lim_{j\to\infty}\mathrm{Tr}(\mathrm{adj}(A_j)H) = \mathrm{Tr}(\mathrm{adj}B)$, as trace and adjoint are both continuous functions.

2. Suppose $\Gamma : \mathbb{R} \to GL_n(\mathbb{R})$ is a smooth map such that $\Gamma(t)$ is orthogonal for every $t \in \mathbb{R}$. By the product rule, we have

$$\frac{d}{dt}(\Gamma(t)^T\Gamma(t)) = \frac{d}{dt}(\Gamma(t)^T)\Gamma(t) + \Gamma(t)^T\frac{d}{dt}\Gamma(t) = 0.$$

   And we also see that

$$\frac{d}{dt}(\Gamma(t)^T) = \lim_{h\to 0}\frac{\Gamma(t+h)^T - \Gamma(t)^T}{h} = \lim_{h\to 0}\frac{(\Gamma(t+h)-\Gamma(t))^T}{h} = \left(\frac{d}{dt}\Gamma(t)\right)^T.$$

   Then

$$\left(\Gamma(t)^{-1}\frac{d}{dt}\Gamma(t)\right)^T = \left(\Gamma(t)^T\frac{d}{dt}\Gamma(t)\right)^T = \left(\frac{d}{dt}\Gamma(t)\right)^T\Gamma(t) = -\Gamma(t)^T\frac{d}{dt}\Gamma(t) = -\Gamma(t)^{-1}\frac{d}{dt}\Gamma(t)$$

3. (a.) First we write down the action of $\widehat{V_1} \circ \widehat{V_2} - \widehat{V_2} \circ \widehat{V_1}$ on a function $f \in C^\infty(U)$:

$$
\begin{aligned}
&\widehat{V_1} \circ \widehat{V_2} - \widehat{V_2} \circ \widehat{V_1}(f) = \widehat{V_1}\left(\sum_{i=1}^{n} V_2^i \frac{\partial f}{\partial x_i}\right) - \widehat{V_2}\left(\sum_{i=1}^{n} V_1^i \frac{\partial f}{\partial x_i}\right) \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n}\left[V_1^j \frac{\partial V_2^i}{\partial x_j}\frac{\partial f}{\partial x_i} + V_1^j V_2^i \frac{\partial^2 f}{\partial x_j \partial x_i}\right] - \sum_{j=1}^{n}\sum_{i=1}^{n} V_2^j \left[\frac{\partial V_1^i}{\partial x_j}\frac{\partial f}{\partial x_i} + V_2^j V_1^i \frac{\partial^2 f}{\partial x_j \partial x_i}\right] \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n} V_1^j \frac{\partial V_2^i}{\partial x_j}\frac{\partial f}{\partial x_i} - \sum_{j=1}^{n}\sum_{i=1}^{n} V_2^j \frac{\partial V_1^i}{\partial x_j}\frac{\partial f}{\partial x_i}. \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} V_1^j \frac{\partial V_2^i}{\partial x_j}\frac{\partial f}{\partial x_i} - \sum_{i=1}^{n}\sum_{j=1}^{n} V_2^j \frac{\partial V_1^i}{\partial x_j}\frac{\partial f}{\partial x_i}.
\end{aligned}
$$

Then we see that $\widehat{V_1} \circ \widehat{V_2} - \widehat{V_2} \circ \widehat{V_1}$ corresponds to a vector field W with components

$$
W_i = \sum_{j=1}^{n}\left[V_1^j \frac{\partial V_2^i}{\partial x_j} - V_2^j \frac{\partial V_1^i}{\partial x_j}\right].
$$

(b.) We calculate the Lie bracket of the vector fields $V_1 = \frac{\partial}{\partial x^1}$ and $V_2 = \frac{\partial}{\partial x^2} + (x^1)^2 \frac{\partial}{\partial x^3}$ to find

$$
[V_1, V_2] = (2x^1)\frac{\partial}{\partial x^3}.
$$

In order for $[V_1, V_2](x) \in \operatorname{span} V_1(x), V_2(x)$, we need $a, b \in \mathbb{R}$ such that

$$
(0, 0, 2x^1) = (a, b, b(x^1)^2),
$$

which is only possible if $x = (0, 0, 0)$ and $a = b = 0$.

4. Assume the Implicit Function Theorem holds. Let $F : X \to Y$ be a C^1 map, where $X, Y \in \mathbb{R}^n$, and let $x_0 \in X, y_0 \in Y$ such that $y_0 = F(x_0)$, with $D_{x_0}F$ non-singular.

   Define a new function $\tilde{F} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by $\tilde{F}(x, y) := F(x) - y$. Then $D_{x_0}\tilde{F}_1 = D_{x_0}F$, thus $D_{x_0}\tilde{F}_1$ is non-singular. By the Implicit Function Theorem in the first variable instead of the second, with $\tilde{F}(x_0, y_0) = 0 = z_0$, there exist neighborhoods $U \ni x_0$ and $V \ni y_0$ and a C^1 function $G : V \to U$ such that an element $(x, y) \in U \times V$ satisfies $\tilde{F}(x, y) = 0$ if and only if $x = G(y)$, and thus $F(x) = y$ if and only if $x = G(y)$. Then

$$
F(G(y)) = y \qquad G(F(x)) = x
$$

   for all $x \in U, y \in V$. Then it is clear that $G = F^{-1}$, and F is a C^1 diffeomorphism $U \to V$.