Homework 11

Section 55

1. Suppose A is a retract of B^2. We prove every continuous map $f : A \to A$ has a fixed point.

   Let f be such a map. Then we have maps

   $$B^2 \xrightarrow{r} A \xrightarrow{f} A \xrightarrow{i} B^2$$

   where r is a the retraction $B^2 \to A$ and i is the inclusion map $A \to B^2$. Then the map $i \circ f \circ r$ has a fixed point x since it is a continuous map $B^2 \to B^2$. But the image of i is A, so we must have $x \in A$. Then since both r and i fix points in A, we must have $f(x) = i \circ f \circ r(x) = x$.

Section 57

2. Let $g : S^2 \to S^2$ be continuous with $g(x) \neq g(-x)$ for all $x \in S^2$. We will prove g is surjective.

   Suppose it is not surjective, say $p \notin \operatorname{im}(g)$. Then there is a homeomorphism $\varphi : S^2 \setminus \{p\} \to R^2$. Then $\varphi \circ g$ is a map $S^2 \to \mathbb{R}^2$, so there must be a point $x \in S^2$ such that $\varphi \circ g(x) = \varphi \circ g(-x)$. Then we have $g(x) = g(-x)$, a contradiction, since φ is injective. So we must have that g is a surjection.

3. Let $h : S^1 \to S^1$ be continuous and antipode preserving with $h(b_0) = b_0$. We prove that h_* sends a generator of $\pi_1(S^1, b_0)$ to an odd power of itself.

   To do this, we show that the map k_* from the proof of theorem 57.1 does the same. Let $\tilde{f}$ be the an injective path from b_0 to $-b_0$. Let $f = q \circ \tilde{f}$, a generator. Then $k_*[f] = [k \circ (q \circ f)] = [q \circ h \circ \tilde{f}]$. But $q \circ \tilde{f}$ is also a path beginning at b_0 and ending at $-b_0$. Then $k_*[f] = [q \circ h \circ \tilde{f}]$ cannot be an even power of a generator, since q is the map $z \mapsto z^2$, implying the preimage under q, $[h \circ \tilde{f}]$, iwould be a loop.

   Finally, since $q_* \circ h_* = k_* \circ q_*$, h_* cannot send a generator to an even power. Suppose it does. Let a be a generator. Then there would be integers r, s such that

   $$(a^{2r})^2 = (a^2)^{2s+1},$$

   which is impossible. Then h must send a generator to an odd power as well.

Section 58

2. (a.) Infinite cyclic

   (b.) Figure 8

   (c.) Infinite cyclic

   (d.) Infinite cyclic

   (e.) Figure 8

5. Suppose X is contractible. Let $i : X \to \{x\}$ be the constant map to some point $x \in X$, and let $j : \{x\} \to X$ be the inclusion map into X. Then $i \circ j$ is the unique map $\{x\} \to \{x\}$, the identity map, and $j \circ i : X \to X$ is the map sending all of X to the point x, which is homotopic to the identity by hypothesis. Then X has the same homotopy type as a point.

   Conversely suppose X has the same homotopy type as a point. Let $i : X \to \{x\}$ be the map sending all elements to a point $x \in X$, and let g be its homotopy inverse. Then $g \circ i$ is the constant map to some point in x, as the image of g must be a single point. Since g is a homotopy inverse of i, $g \circ i$ must be homotopic to the identity on X, and thus X is nullhomotopic.

7. Let $A \subset X$, let $j : A \to X$ be the inclusion map, and $f : X \to A$ be continuous. Suppose $H : X \times I \to X$ is a homotopy from $j \circ f$ to the idnetity on X.

   (a.) If f is a retraction, then $f \circ j$ is just the constant map on A, meaning j is a homotopy equivalence with inverse f and j_* is an isomorphism.

   (b.) If $H|_{A\times I}$ maps into A, then $H|_{A\times I}(x, 0) = j \circ f|_A(x)$ takes values in A, thus equals $f|_A(x) = f \circ j(x)$, and $H|_{A\times I}(x, 1) = \mathrm{id}_X|_A(x) = \mathrm{id}_A(x)$. Then $H|_{A\times I}$ is a homotopy from $f \circ j$ to the identity on A, so j is a homotopy equivalence and j_* is an isomorphism.

   (c.) Let $X = B^1$, $A = S^1$, and f be the map sending points in B^1 directly to the left to the boundary. Then f maps continuously into the left half of S^1. $j \circ f = f$ is homotopic to the identity, but j_* is the trivial homomorphism from S^1 into B^1.