Homework 8

43. Let G be a simple group of order 168. Then the number of 7-sylow subgroups, n_7, is equal to $1 \mod 7$, and divides 24. Then we must have $n_7 = 1, 8$. But G is simple, so we cannot have $n_7 = 1$. Then $n_7 = 8$. Since a group of order 7 is cyclic, each 7-sylow subgroup is generated by each of its non-identity elements. Then the 8 subgroups must have trivial intersection. They each contain 6 elements of order 7, and G can have no other elements of order 7 as they would generate other 7-sylow subgroups. Then G has 48 elements of order 7.

44. (a.) Let $\mathbb{Q}$ be the additive group of the rational numbers. Suppose there is a basis B such that every element of $\mathbb{Q}$ is a unique finite sum of elements of B. Take some $a \in B$. Then we can write

$$\frac{a}{2} = \sum_1^n a_i b_i$$

for $a_i \in \mathbb{Z}$ and $b_i \in B$. But then

$$\sum_1^n 2a_i b_i = a,$$

which is only possible if some $b_j = a$ and $b_i = 0$ for $i \neq j$, and if $2a_i = 1$. But this is a contradiction as $\frac{1}{2} \notin \mathbb{Z}$.

(b.) Let G be the group of nonzero rational numbers under multiplication. I claim that G has a basis given by the set of prime numbers. Choose any $a \in G$. We can uniquely write

$$a = \frac{p_1^{a_1} \cdots p_n^{a_n}}{q_1^{b_1} \cdots q_m^{b_m}} = p_1^{a_1} \cdots p_n^{a_n} \cdot q_1^{-b_1} \cdots q_m^{-b_m},$$

which is a countable product for any $a \in G$. Our basis is a countable and thus G is a free group with countable rank.

45. The element $1/n \in \mathbb{Q}/\mathbb{Z}$ generates a subgroup of order n. Moreover, if A is a subgroup in $QQ/\mathbb{Z}$ of order n,