Homework 1

1. Let $S = \{1, 2\}$, a set with 2 elements. We write the elements of Func(S) as

$$e := \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \qquad \alpha := \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \beta := \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \qquad \gamma := \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}.$$

Then the multiplication table for the monoid Func(S) under composition is as follows:

$\circ$	e	α	β	γ
e	e	α	β	γ
α	α	e	γ	β
β	β	β	β	β
γ	γ	γ	γ	γ

This monoid is not commutative since $\gamma \circ \beta = \beta$ but $\beta \circ \gamma = \gamma$.

2. Let $\mathbb{R}_{\geq 0}$ be the monoid of nonnegative real numbers under addition, and let $\mathbb{R}^+$ denote the monoid of positive real numbers under multiplication.

   (a.) We know that any submonoid of $\mathbb{R}_{\geq 0}$ must contain 0. Also, any submonoid containing $\sqrt{3}$ must contain $\sqrt{3}\mathbb{N}$, where $\mathbb{N}$ is the monoid of nonnegative integers under addition. $\sqrt{3}\mathbb{N}$ is in fact the smallest submonoid containing $\sqrt{3}$, since no smaller subset is a monoid, and any submonoid containing $\sqrt{3}$ must contain $\sqrt{3}\mathbb{N}$.

   (b.) A submonoid of $\mathbb{R}^+$ containing $\sqrt{3}$ necessarily contains $\sqrt{3}^n$ for $n = 0, 1, 2, \ldots$. The set $\{\sqrt{3}^n : n = 0, 1, 2, \ldots\}$ is a submonoid of $\mathbb{R}^+$ under multiplication, and removing any element makes it not a monoid. This is the smallest monoid containing $\sqrt{3}$.

   (c.) $\mathbb{R}_{\geq 0}$ is not a group because none of the positive reals have additive inverses.
   $\mathbb{R}^+$ is a group because it is a monoid where every $r \in \mathbb{R}^+$ has an inverse, namely $1/r$ since $r > 0$.
   If we want to find the smallest subgroup of $\mathbb{R}^+$ containing $\sqrt{3}$, we need all integer powers of $\sqrt{3}$, not just the nonnegative ones. It is trivial to check that this also defines a submonoid. Also, every element $(\sqrt{3})^n$ has inverse $(\sqrt{3})^{-n}$, and we still have identity $1 = (\sqrt{3})^0$.

3. The identity in GL$(2, \mathbb{R})$ is the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. This is indeed contained in $B_2(\mathbb{R}) := \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) : a, b, c \in \mathbb{R},\ ac \neq 0\}$.

   We check that $B_2(\mathbb{R})$ is closed under multiplication: for $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right), \left(\begin{smallmatrix} d & e \\ 0 & f \end{smallmatrix}\right) \in B_2(\mathbb{R})$, we have

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} ad & ea + bf \\ 0 & cf \end{pmatrix} \in B_2(\mathbb{R}),$$

   since $adcf = (ac)(df) \neq 0$.

   Also, if

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} d & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

   Then $ad = cf = 1$, and $ea + bf = 0$. Then we can solve for d, e, f to get $d = a^{-1}$, $f = c^{-1}$, and $e = -cab^{-1}$, giving us the inverse of $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$.

4. To see that f is injective, we note that a complex number is uniquely determined by its real and imaginary parts.

   We have

$$f((a + bi)(c + di)) = f((ac - bd) + (bc + ad)i) = \begin{pmatrix} ac - bd & bc + ad \\ -(bc + ad) & ac - bd \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = f(a+bi)f(c+di),$$

so that f is a monoid homomorphism and thus a group homomorphism.

5. We have $e \in Z(G)$ since for any $g \in G$, $eg = ge = g$.

   If $a \in Z(G)$, then $a^{-1} \in Z(G)$, since for any $g \in G$,

   $$a^{-1}g = a^{-1}gaa^{-1} = a^{-1}aga^{-1} = ga^{-1}.$$

   If $a, b \in Z(G)$, then the product $ab \in Z(G)$ since for any $g \in G$,

   $$gab = agb = abg.$$

   Then $Z(G)$ is a subgroup since it contains identity, is closed, and contains inverses.

   $C_G(g)$ is a subgroup for the same reasons and the proof is similar.