Homework 2

1. Suppose X is $\sigma(X, X^*)$ separable. Let $\{x_n\}$ be a countable $\sigma(X, X^*)$ dense set in X. Let $A = \overline{\text{span}_{\mathbb{Q}}\{a_n\}}^{\|\cdot\|}$ be the norm closure of the rational span of the a_n's. Then A is a norm closed, convex set, and thus $A = \overline{A}^{\sigma(X,X^*)}$ by Mazur's theorem. But A contains $\{a_n\}$ whose weak closure is X, so we must have $A = X$. Then $\text{span}_{\mathbb{Q}}\{a_n\}$ is a countable set whose norm closure is X, and thus X is separable.

2. Suppose $(B_{X^*}, \sigma(X^*, X))$ is metrizable. Then B_{X^*} is a $\sigma(X^*, X)$ compact, metrizable space. Thus the space $C(B_{X^*})$ of $\sigma(X^*, X)$-continuous functions $B_{X^*} \to \mathbb{R}$ is separable in the supremum norm. Also note that the subspace of $C(B_{X^*})$ consisting of functions which are linear is exactly the image of X under the canonical surjection $J : X \to X^*$.

   Then it follows that $J(X)$ is separable as a subspace of $C(B_{X^*})$, and thus X is separable since the map J is an isometry.

3.