Homework 2

Section 17

2. Let A be closed in Y and Y be closed in X. Then $A = C \cap Y$ for some C closed in X. Then A is an intersection of two closed sets and is closed.

6. (a.) If $A \subset B$, then $B \subset \overline{B}$ and thus $\overline{B}$ is a closed set containing A and $\overline{A} \subset \overline{B}$.

   (b.) We can write
   $$\overline{A \cup B} = \bigcap \{C \supset A \cup B : C \text{ closed}\} = \bigcap \{C \cup D : C \supset A, D \supset B;\ C, D \text{ closed}\}.$$
   To see these two sets are the same, write $C = C \cup \varnothing$ for one direction and in the other take C to be the union of both closed sets.

   Moreover, we have
   $$\bigcap \{C \cup D : C \supset A, D \supset B;\ C, D \text{ closed}\} \supset \left(\bigcap \{C \supset A : C \text{ closed}\}\right) \cup \left(\bigcap \{D \supset B : D \text{ closed}\}\right) = \overline{A} \cup \overline{B}.$$
   Then since $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$ and $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$, we must have $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

   (c.) The above proof generalizes to the case where $\bigcup A_\alpha$ is a union of arbitrarily many sets A_α, except for the last step, where $\bigcup \overline{A_\alpha}$ is potentially not open.

   An example of this failing is for the family of open sets $A_n = (\frac{1}{n+1}, \frac{1}{n})$ for $n \geq 1$. Their union is $(0, 1)$, and the union of closures is $(0, 1]$. However, the closure of the union is $[0, 1]$.

9. $(\overline{A \times B}) \subset \overline{A} \times \overline{B})$: Let $(a, b) \in \overline{A \times B}$. Let $U \subset X$ and $V \subset Y$ be open sets containing a and b respectively. Then $(U \times V) \cap (A \times B) \neq \varnothing$. But $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$, and thus both $U \cap A$ and $V \cap B$ are nonempty for all such U and V, and $(a, b) \in \overline{A} \times \overline{B}$.

   $(\overline{A} \times \overline{B} \subset \overline{A \times B})$: Conversely let $(a, b) \in \overline{A} \times \overline{B}$. Let $U \times V$ be a basic open set in $X \times Y$ containing (a, b). Then $U \cap A$ and $V \cap B$ are both nonempty, thus $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$ is nonempty, and $(a, b) \in \overline{A \times B}$.

12. Let X be hausdorff with subspace Y. Let $a, b \in Y$ with $a \neq b$. Choose U, V open disjoint sets in X containing a and b respectively. Then $Y \cap U$ and $Y \cap V$ are disjoint open sets in Y containing a and b respectively. Thus Y is hausdorff.

13. First suppose X is hausdorff. Choose any $(a, b) \notin \Delta$, so that $a \neq b$, and choose disjoint open sets U, V containing a and b respectively. Then $A \times B$ is an open set in $X \times X$ which is disjoint from δ, since no point in A is equal to any point in B. Then $X \times X \setminus \Delta$ is open and Δ is closed.

    Conversely if Δ is closed, for any $(a, b) \notin \Delta$, there is a basic open set $A \times B \ni (a, b)$ such that $A \times B$ is disjoint from Δ, and thus $A \cap B = \varnothing$ and X is hausdorff.

19. (a.) Suppose $x \in \operatorname{int} A$. Then there is some open $U \subset A$ with $x \in U$. Then $X \setminus U$ is a closed set containing $X \setminus A$. But $x \notin X \setminus U$, so x cannot be in $\overline{X \setminus A}$.

    Conversely let $x \in \overline{X \setminus A} \supset \operatorname{Bd} A$. Then if there is some open $U \subset A$ containing x, $X \setminus U$ is a closed set containing $X \setminus A$ which does not contain x, contradicting $x \in \overline{X \setminus A}$.Then $x \notin \operatorname{int} A$.

    Now suppose $x \in \overline{A}$. If $x \notin \operatorname{int} A$, then x is not contained in any open subsets of A. Choose any closed $B \supset X \setminus A$, so that $X \setminus B$ is an open subset of A. Then $x \notin X \setminus B$ implying $x \in B$ and $x \in \overline{X \setminus A} \cap \overline{A} = \operatorname{Bd} A$. Then $\overline{A} = \operatorname{int} A \cup \operatorname{Bd} A$.

(b.) If A is both open and closed, then $\operatorname{int} A = A = \overline{A}$. Then by (a.), $\operatorname{Bd} A = \overline{A} \setminus \operatorname{int} A = A \setminus A = \varnothing$.

If $\operatorname{Bd} A = \varnothing$, we have $\overline{A} = \operatorname{int} A$, implying A is equal to its interior and closure and therefore must be open and closed.

(c.) If A is open, by reasoning above we have $\operatorname{Bd} A = \overline{A} \setminus \operatorname{int} A = \overline{A} \setminus A$.

If $\operatorname{Bd} A = \overline{A} \setminus A$, we have $\operatorname{int} A = \overline{A} \setminus \operatorname{Bd} A = \overline{A} \setminus (\overline{A} \setminus A) = A$, and A must be open.

(d.) No. Let $U = (-1, 0) \cup (0, 1)$. Then $\overline{U} = [-1, 1]$, and so $\operatorname{int}(\overline{U}) = (-1, 1)$.

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2. This is not necessarily the case. Let $f : \mathbb{R} \to \mathbb{R}$ be the function mapping all $x \in \mathbb{R}$ to 1. Then if $A = [0, 1]$, the point 1 is a limit point of A, however $f(1) = 1$ is not a limit point of $f(A) = \{1\}$ since any neighborhood of 1 only intersects $f(A)$ at 1.