Homework 2

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^∞ map. Then f is not injective.

   Consider two cases. First, if $D_{(x,y)}f = (0,0)$ for all $(x,y) \in \mathbb{R}^2$, then by the fundamental theorem of calculus, f is constant.

   Otherwise, without loss of generality we can find a point (x_0, y_0) such that $D_{y_0}f \neq 0$. Let $z_0 := f(x_0, y_0)$. Then it follows from the implicit function theorem that there is a smooth function $g : U \to V$, where U and V are open neighborhoods in $\mathbb{R}$ of x_0 and y_0 respectively, such that we have $f(x, g(x)) = z_0$ for all $x \in U$. In particular, this implies f is not injective.

2. (a.) Let π denote the stereographic projection $S^n \setminus \{N\} \to \mathbb{R}^n$, given by $\pi(x_1, \dots, x_{n+1}) = \frac{(x_1,\dots,x_n)}{1-x_{n+1}}$.

   We prove that $u = \pi(x)$ is the point at which the line through x and N intersects the linear subspace defined by $x_{n+1} = 0$.

   A point y on this line is given by a sum of the form $y = t(x) + (1-t)(N)$ for $t \in \mathbb{R}$.. We calculate the value of t such that $y_{n+1} = 0$. We see that

   $$y_{n+1} = tx_{n+1} + (1-t),$$

   and the value $t = \frac{1}{1-x_{n+1}}$ gives $y_{n+1} = 0$. Then subsituting this into our equation, we see that the rest of the coordinates of y are given by $y_i = \frac{x_i}{1-x_{n+1}}$ for $i = 1, \dots, n$, and of course we can regard y as an $n-tuple$ within the subspace $\mathbb{R}^{n+1}|_{x_{n+1}=0} \simeq \mathbb{R}^n$.

   We also see that the projection with respect to the south pole given by $\tilde{\pi}(x) = -\pi(-x)$ is the map obtained from first changing coordinates to be in the first situation, projecting, and returning to the original coordinates. Then this map has the same property.

   (b.) We can derive the inverse $\pi^{-1} : \mathbb{R}^n \to S^n \setminus \{N\}$ in a similar way. Namely, for $u \in \mathbb{R}^n$, consider the line between $(u, 0)$ and N given by $y = t(u,0) + (1-t)N$ for $t \in \mathbb{R}$. In this case, we need to solve for $\|y\| = 1$, or equivalently $\|y\|^2 = 1$. Since the vectors $(u, 0)$ and N are orthogonal, we simply have

   $$\|y\|^2 = t^2\|u\|^2 + (1-t)^2,$$

   which we can use to solve for t by setting this equal to 1. We obtain $t = \frac{2}{\|u\|^2+1}$. Define $\varphi(u)$ to be the value of the above equation at this given t. Then

   $$\varphi(u) = \frac{(2u, \|u\|^2 - 1)}{\|u\|^2 + 1}.$$

   Then it is clear that the maps φ and π are inverses of each other, so $\varphi = \pi^{-1}$ and π is bijective.

   (c.) We compute the transition map $\tilde{\pi} \circ \pi^{-1}$ for the charts (U, π), $(V, \tilde{\pi})$, with $U = S^n \setminus \{N\}$ and $V = S^n \setminus \{S\}$. To show this defines a smooth structure, we show $\tilde{\pi} \circ \pi^{-1} : \pi(U \cap V) \to \tilde{\pi}(U \cap V)$ is smooth, where $U \cap V = S^n \setminus \{N, S\}$, $\pi(U \cap V) = \tilde{\pi}(U \cap V) = \mathbb{R}^n \setminus \{0\}$.

   Both maps π and $\tilde{\pi}$ only involve scalings of the first n coordinates of the input variable, and similarly π^{-1}, in its first n coordinates, involves only a scaling of the input variable. Then we can expect $\tilde{\pi} \circ \pi^{-1}$ to be a function of $\|u\|$ only. So it suffices to consider the special case of $n = 1$.

   I claim that in general, $\tilde{\pi} \circ \pi^{-1}$ is the map $u \mapsto \frac{u}{\|u\|^2}$. In one dimension we will obtain $\tilde{\pi} \circ \pi^{-1}(u) = \frac{u}{|u|^2}$ and the result will extend to arbitrary dimension by the discussion above.

   Consider a point $U \in \mathbb{R} \setminus \{0\}$. Let UN be the line segment from U to N. Denote point at the intersection of UN with S^1 by C, and denote by CS the line segment from C to S. Let O be the origin. Let B denote the point at the intersection of CS and the x-axis. By Thale's theorem from basic geometry, the line CS meets UN at a right angle. Thus the traingle $\triangle UCB$ is a right triangle, and clearly so is $\triangle BOS$. Moreover we see that the angles $\angle UBC$ and $\angle OBS$ are equal. Thus $\triangle UCB \sim \triangle BOS$, and we have the relation

$$\frac{|UO|}{|NO|} = \frac{|OS|}{|OB|}.$$

In other words, $B = \frac{1}{U}$, now considering these as real numbers. Then indeed the map is given by $U \mapsto \frac{U}{|U|^2}$ and extends to the case $\mathbb{R}^n$ in the obvious way. This map is smooth $\mathbb{R}^n \to \mathbb{R}^n$. The other transition map is defined in the same way. Then we have verified that these charts define a smooth structure.

(d.) Let $(U_i^\pm, \varphi_i^\pm)$ be the given atlas for S^n, and let $(U, \pi), (V, \tilde{\pi})$ be the atlas from above. To check that these two atlases are smoothly compatible, it will suffice to check that the charts $(\varphi_{n+1}^\pm, U_{n+1}^\pm)$ are both compatible with (U, π), and that charts $(\varphi_i^\pm, U_i^\pm)$ are both compatible with (U, π) for some $i < n+1$. The other cases are proven similarly.

First, $U_{n+1}^+ \cap U$ is the "upper" half $(x_{n+1} > 1)$ of the sphere minus the north pole. Then $\varphi_{n+1}^+ \circ \pi^{-1}$ has domain $\{x \in \mathbb{R}^n : \|x\| > 1$, and we can calculate that

$$\varphi_{n+1}^+ \circ \pi^{-1}(u) = \frac{2u}{\|u\|^2 + 1},$$

so this map is clearly smooth. Similarly we have

$$\pi \circ (\varphi_{n+1}^+)^{-1}(u) = \frac{u}{1 - \sqrt{1 - \|u\|^2}}$$

which is also smooth, noting that $0 < \|u\| < 1$ in this case.

The opposite chart $(U_{n+1}^-, \varphi_{n+1}^-)$ can be seen to be compatible with (U, π) as well.

Now for $0 \leq i < n+1$, we have that $\varphi_i^+ \circ \pi^{-1}$ maps $\{x \in \mathbb{R}^n : x_i \geq 0\}$ to $\{x \in \mathbb{R}^n : \|x\| \leq 1 \text{ and } x_i \geq 0\}$, and that the composition is smooth is due to the fact that π^{-1} is smooth on the given domain, and φ is smooth on the image of this domain under π^{-1}. The rest of the charts are similar.

3. Let $\tilde{\mathbb{R}}$ be the smooth manifold with a single chart $(\tilde{\mathbb{R}}, \varphi)$ where $\varphi(x) = x^3$. Let $f : \mathbb{R} \to \mathbb{R}$ be smooth in the usual sense.

(a.) To show that f is smooth as a map $\mathbb{R} \to \tilde{\mathbb{R}}$, we must show that its coordinate representation $\varphi \circ f \circ \mathrm{id}^{-1} = \varphi \circ f = f^3$ is a smooth map $\mathbb{R} \to \mathbb{R}$. It is, since it is a composition of smooth maps $\mathbb{R} \to \mathbb{R}$.

(b.) Let $f : \tilde{\mathbb{R}} \to \mathbb{R}$ be a map. First suppose it is smooth. By taylor's theorem, we can write

$$f(x) = f(0) + f'(0)x + f''(0)x^2 + f'''(0)x^3 + r(x)$$

where $r(x) = 0(x^3)$, and thus we have

$$f(x^{1/3}) = f(0) + f'(0)x^{1/3} + f''(0)x^{2/3} + f'''(0)x + r(x^{1/3}).$$

That f is a smooth map between manifolds implies $f(x^{1/3})$ is a smooth map $\mathbb{R} \to \mathbb{R}$. But we see by taking one or two derivatives on the right side that unless $f'(0) = f''(0) = 0$, this cannot be true. We may have $f^{(3n)}(0)$ nonzero, but for all other derivatives the value at zero must vanish, or else we can derive a similar contradiction.

conversely suppose all derivatives of f order not a multiple of 3 vanish at 0. Again we can write

$$f(x) = f(0) + f'''(0)x^3 + r(x)$$

where $r(x) = o(x^3)$. Then substituting $x^{1/3}$, we have

$$g(x) = f(x^{1/3}) = f(0) + f'''(0)x + r(x^{1/3}),$$

and thus g is a smooth function since $r(x^{1/3}) = o(x)$. Then f is a smooth map $\tilde{\mathbb{R}} \to \mathbb{R}$.

4. Let $P = \text{span}\{e_1, \dots, e_k\}$, $Q = \text{span}\{e_{k+1}, \dots, e_n\}$ be subspaces of $\mathbb{R}^n$. Let S be a k-dimensional subspace which intersects Q trivially.

   The map φ assigns to S a $(n-k) \times k$ matrix B such that

   $$S = \{v + Bv : v \in P\} = \left\{ \begin{pmatrix} I_k \\ B \end{pmatrix} v : v \in P \right\}.$$

   Then since $P = \text{span}\{e_1, \dots, e_k\}$, we have

   $$S = \text{span} \left\{ \begin{pmatrix} I_k \\ B \end{pmatrix} e_i \right\}_{i=1}^{k}.$$

   In other words, the columns of $\begin{pmatrix} I_k \\ B \end{pmatrix}$ span S.

   For uniqueness, suppose there is another such matrix K. Then in other words, S is the graph of the linear map $K : P \to Q$. But by the discussion from the book, there is a unique map whose graph is equal to S. Then we must have $B = K$, and B is unique.

5. Let M be a smooth n dimensional manifold which is also a group, such that the group operation is a smooth map $M \times M \to M$. Fix some $x \in M$. Choose charts $(U, \varphi), (U', \varphi)$ with $x \in U$ and $x^{-1} \in U'$. Also let (V, ψ) be a chart with $e \in V$.

   Let $F : \varphi(U) \times \varphi'(U') \to \psi(V)$ be the map $F(s,t) = \psi\left(\varphi^{-1}(s) \cdot \varphi'^{-1}(t)\right)$. Fix $s_0 = \varphi(x)$. Let $F_{s_0}(t) = F(s_0, t)$. Without carefully defining the charts, we can construct a map G which maps a neighborhood of $\psi(e)$ in $\mathbb{R}^n$ to a neighborhood of $\varphi'(x^{-1})$ such that $F_{s_0} \circ G$ is the identity on its domain. Similarly construct a map H which maps a neighborhood of $\varphi(s)$ in $\mathbb{R}^n$ to a neighborhood of $\psi(e)$ in $\mathbb{R}^n$ such that $H \circ F_{s_0}$ is the identity on its domain. Both G and H should map from $\mathbb{R}^n$ into M, multiply by x^{-1}, and map back into $\mathbb{R}^n$.

   Then by the chain rule, we have

   $$D_{\varphi'(x^{-1})}F \circ D_e G = D_e H \circ D_{\varphi'(x^{-1})}F = I$$

   which verifies that the map F has nonsingular differential in the second coordinate at the point $(\varphi(x), \varphi'(x^{-1}))$. By the implicit function theorem, there exists $U \ni \varphi(x)$ and $V \ni \varphi'(x^{-1})$ and a smooth function $g : U \to V$. This is the coordinate representation of the map $x \mapsto x^{-1}$, and thus the map $x \mapsto x^{-1}$ is a smooth map $M \to M$.