Homework 5

Topological Groups

4. Since G is a topological group, the multiplication operation is a continuous map $G \times G \to G$. Then the map sending $x \mapsto \alpha \cdot x$ is just the composition of the projection in the first coordinate onto α with the multiplication map. Both are continuous, so the map $x \mapsto \alpha \cdot x$ is also. The same is true for right multiplication by α.

   Also, it is well known that left (right) multiplication by an element is a bijection $G \to G$.

   The inverse map $f_{\alpha^{-1}}$ is also continuous, thus f_α is a homeomorphism.

   Then for any pair $x, y \in G$, the map $f_{yx^{-1}}$ is a homeomorphism sending x to y.

5. Let H be a subgroup of G and give G/H the quotient topology.

   (a) Let f_α be the map from question 4 and let f'_α be the induced map on G/H the set of cosets of H in G. Specifically, f'_α sends an element $x \in G/H$ corresponding to a coset xH into the coset αxH, and returns the corresponding element αx in G/H. It is well known that this map is a bijection on G/H the set of cosets of H.

   Let U be an open set in G/H. We will show $f'^{-1}_\alpha(U)$ is open. U corresponds to a set of cosets we will denote by $U \cdot H$. This set is open in G by the definition of the quotient topology on G/H. The map f'^{-1}_α sends $U \cdot H$ to $\alpha^{-1} \cdot (U \cdot H)$. Of course, $\alpha^{-1} \cdot (U \cdot H)$ is a subset of G which is the image of the set $U \cdot H$ under the map f^{-1}_α. Thus $\alpha^{-1} \cdot U \cdot H$ is open in G by problem (4.). On the other hand, we know that f^{-1}_α sends entire cosets to entire cosets, so that $\alpha^{-1} \cdot U \cdot H$ is saturated in the projection map onto G/H. Then by the definition of the quotient map, $\alpha^{-1}U$ is in fact an open set in G/H.

   The inverse map $f'^{-1}_\alpha = f'_{\alpha^{-1}}$ is continuous by the same reasoning, and we have already mentioned that f'_α is a bijection. Thus it is a homemomorphism of G/H. If xH and yH are two cosets in G/H, the homeomorphism $f'_{yx^{-1}}$ sends xH to yH.

   (b) Let H be a closed set in G. Then a single point set $\{xH\} \in G/H$ is closed if and only if the set xH is closed as an element of G by the definition of a quotient map. By problem (4.) we know that this is true.

   (c) Let $U \subset G$ be an open set. We cannot say whether $p(U)$ is open in G because U may not be saturated. However, the image of U under p is the same as the image of its "saturation", $p^{-1}(p(U))$, which shows $p(U)$ is open in the quotient topology on G/H.

   (d) We will use the map p from the previous part. If H is closed and normal, G/H forms a group with multiplication $(xH) \cdot (yH) = (xy)H$. It satisfies the T1 property by part (c.). Let $m : G \times G \to G$ be the multiplication map, and m' the multiplication on G/H.

   If $UH \subset G/H$ is open, then $m'^{-1}(U) = \{(xH, yH) \in G/H \times G/H : xyH \in U\}$. This is equivalent to the set $A = \{(x, y) \in G \times G : xy \in UH\}$, since $xyH = zH$ for $zH \in U$ if and only if $xy \in zH$. But A is the preimage $m^{-1}(UH)$, and we know UH is open in G. Then A is open in G as well.

   Then $m'^{-1}(U) = (p \times p)(A)$. We know p is an open map, and for a basic open set $E \times F$ in $G/H \times G/H$, $(p \times p)(E \times F) = (p(E), p(F))$, an open set in $G/H \times G/H$. Since open sets are unions of basic open sets and $(p \times p)(A \cup B) = (p \times p)(A) \cup (p \times p)(B)$, we know that $(p \times p)$ is an open map, and thus $m'^{-1}(U) = (p \times p)(A)$ is open in $G/H \times G/H$.

   To show the map $j : xH \to x^{-1}H$ is continuous is the same as part (a.), where we need to show that for some open $U \in G/H$, $U^{-1} \cdot H$ is open and saturated in G. This is again true by the well-definedness of the group operation on G/H when H is normal.

Section 23

3. Let A_α and A be connected subspaces of X, and suppose $A \cap A_\alpha \neq \varnothing$ for all α. By Theorem 23.3, for each α, $A \cup A_\alpha$ is connected since they have nonempty intersection. Then

$$A \cup (\bigcup A_\alpha) = \bigcup (A \cup A_\alpha)$$

is a union of sets each sharing a common point, namely any point in A, and is connected by 23.3.

8. Give $\mathbb{R}^\omega$ the uniform topology. Let A be the set of bounded sequences in $\mathbb{R}^\omega$. I claim that A is both open and closed.

To show A is closed, let (x_n) be a limit point of A. Then by the definition of the uniform metric, for every $\varepsilon > 0$ there is some $(y_n) \in A$ which is not equal to (x_n), such that $|x_n - y_n| < \varepsilon$ for all n. Thus if $|y_n| < M$ for all n, then $|x_n| < M + \varepsilon$ for all n, and thus $(x_n) \in A$. Then A contains all of its limit points and is closed.

Now we show A is open. Let $(x_n) \in A$ with $|x_n| < M$ for all n. Then for any $0 < \varepsilon < 1$, the ε-ball centered at (x_n) contains only sequences which are bounded by $M + \varepsilon$, and thus is contained in A. Then A is open.

Since A is not the entire space, $\mathbb{R}^\omega$ is disconnected by the alternate formulation of connectedness.

11. Let $p : X \to Y$ be a quotient map, with Y connected and $p^{-1}(y)$ connected for each $y \in Y$. Suppose $X = A \cup B$ for open and disjoint sets A and B. Since $p^{-1}(y)$ is connected, we must have $p^{-1}(y)$ entirely contained in either A or B for all $y \in Y$. Then the sets A and B are saturated open sets, so $p(A)$ and $p(B)$ are open and disjoint, with $Y = p(A) \cup p(B)$. Since Y is connected, either set must be empty and thus one of A and B must be empty. Then X has no disconnection.

Section 24

4. Let X be an ordered set equipped with the order topology and suppose X is connected. To show that X is a linear continuum, we must show that for all $x < y$, there is some $z \in X$ with $x < z < y$, and that every set has a least upper bound.

The first condition is easy. If this is not true, then $(-\infty, y) \cup (x, \infty)$ is a separation of X which is a contradiction.

For the least upper bound property, let A be a set in X. Let

$$F := \bigcap_{y \in A} \{x \in X : x \geq y\}$$

be the intersection of all closed rays going to infinity starting from elements of A. Then F is closed since it is an intersection of closed sets in the order topology. Similarly let

$$E = \bigcup_{y \in X y > A} \{x \in X : x > y\}$$

be the union of open rays to infinity starting from points greater than every element in X. It is open.

It is clear that $E \subset F$. But if $E = F$, then F is a set which is both open and closed. Moreover, F is not X since it does not contain A. Then this contradicts that X is connected.

Then there must be some point $x \in F \setminus E$. Since $x \in F$, we know that x is an upper bound. Since $x \notin E$, we have $x \leq y$ for all upper bounds y of A. Then x is a least upper bound for A.

8. a. The product of path connected spaces is path connected. Let X and Y be path connected and choose points $(x_1, y_1), (x_2, y_2) \in X \times Y$. Let $f : \mathbb{R} \to X$ be a path from x_1 to x_2, and $g : \mathbb{R} \to Y$ be a path from y_1 to y_1. Then the function

$$g := \begin{cases} (f(2t), y_1) & 0 \leq t < \frac{1}{2} \\ (x_2, g(2t-1)) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a path from (x_1, y_1) to (x_2, y_2).

b. The closure of a path connected space is not path connected. The set $\{(x, sin(\frac{1}{x})) : x > 0\}$ is path connected, but its closure is not.

c. Let $f : X \to Y$ be continuous and suppose X is path connected. Let $f(x), f(y) \in f(X)$. If g is a path from x to y in X, then $f \circ g$ is a path from $f(x)$ to $f(y)$ in $f(X)$. Hence, $f(X)$ is path connected.

d. Let A_α each be path connected subspaces of X and suppose there is some $a \in \cap A_\alpha$. Then for any $x, y \in \bigcup A_\alpha$, we can make a path from x to y by joining the path from x to a with the path from a to y, which both exist by the connectedness of the A_αs. This process is the same as in part (a.).