Homework 4

21. Let G be a group with subgroups H and K.

   a. $H \cap K$ is a subgroup. It clearly contains identity. Since both H and K are subgroups, $H \cap K$ clearly contains inverses and is closed under the group operation.

   b. For any $g \in G$, we have $g(H \cap K)g^{-1} \subset H \cap K$ since $H \cap K$ is a subset of both H and K, and thus its conjugates must lie within H and K since both are normal. Then we are done, since $|g(H \cap K)g^{-1}| = |H \cap K|$ implies $g(H \cap K)g^{-1} = H \cap K$.

   c. If H and K are subgroups of G, $H \cup K$ is not always a subgroup. For example, take $G = \mathbb{Z} \times \mathbb{Z}$, and let $H = \langle (1,0) \rangle$ and $K = \langle (0,1) \rangle$. Then their union does not contain the element $(1,1)$, so it cannot be a group. Since H and K are both normal, this shows (b.) fails as well.

22. Let G be a group and $[G,G]$ the subgroup generated by the commutators. To show it is normal, consider one element $aba^{-1}b^{-1}$ and some $g \in G$. Then the conjugate

$$gaba^{-1}b^{-1}g^{-1} = gag^1gbg^{-1}ga^{-1}g^{-1}gb^{-1}g^{-1} = (gag^{-1})(gbg^{-1})(gag^{-1})^{-1}(gbg^{-1})^{-1}$$

is also a commutator. This also applies to arbitrary combinations of elements of $[G,G]$ and thus $g[G,G]g^{-1} \subset [G,G]$, which again shows that $g[G,G]g^{-1} = [G,G]$. Then $[G,G]$ is normal in G.

Now we show $G/[G,G]$ is abelian. Let $g' = g[G,G]$ and $h' = h[G,G]$ be elements of $G/[G,G]$. Then by taking their commutator, we see that

$$g'h'g'^{-1}h'^{-1} = (ghg^{-1}h^{-1})[G,G] = [G,G]$$

and thus $g'h' = h'g'$.

Let $\varphi : G \to H$ be a homomorphism of groups with H abelian. To show that $[G,G] \subset \ker \varphi$, take some $aba^{-1}b^{-1} \in [G,G]$. Then

$$\varphi(aba^{-1}b^{-1}) = \varphi(a)\varphi(b)\varphi(a^{-1})\varphi(b^{-1}) = \varphi(a)\varphi(a)^{-1}\varphi(b)\varphi(b)^{-1} = e_H,$$

since H is abelian. Then the same applies to arbitrary elements in $[G,G]$.

Then by one of the isomorphism theorems, there is a unique homomorphism $\psi : G/[G,G] \to H$ such that we have a commutative diagram

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\pi"] & H \\ G/[G,G] \arrow[ur, "\psi"] & \end{tikzcd}$$

23. Assume the hypothesis of the problem. Then $N_1 = \ker p_2 \cap H$, and $N_2 = \ker p_1 \cap H$. Thus N_1 contains all of the elements of the form $(a,0) \in H$, and we may identify N_1 with its projection N_1' G_1. Then let $g \in G_1$ and $a \in N_1'$. Using the surjectivity of H under P_1, we have

$$gag^{-1} = p_1(g,h)p_1(a,0)p_1(g^{-1},h^{-1}) = p_1((g,h)(a,0)(g^{-1},h^{-1}))$$

for some (g,h) in H such that $p_1(g,h) = g$. Then because N_1 is normal in G, as it is the kernel of p_2 restricted to H, we have that this equals $p_1((b,0)) = b$ for some $b \in N_1'$. Then by the same argument as above we know that N_1' is normal in G_1. The same argument applies to G_2.

Now we will construct an isomorphism $\varphi : G_1/N_1 \to G_2/N_2$. Since $p_1(H) = G_1$, for any $g \in G_1$ there is some $h \in G_2$ such that $(g, h) \in H$. Then let $\varphi(gN_1) = hN_2$.

We need to check that φ is well defined. Suppose we have elements $h_1, h_2 \in G_2$ such that both (g, h_1) and (g, h_2) are in H. Then $(0, h_1h_2^{-1}) \in H$, and we see that $h_1h_2^{-1} \in N_2$, identified as a subgroup of G_2. Thus $h_1N_2 = h_2N_2$ and the image of gN_1 is well defined.

Moreover, suppose $g_1N_1 = g_2N_1$ for $g_1, g_2 \in G_1$ and take $h_1, h_2 \in G_2$ such that (g_1, h_1) and (g_2, h_2) are elements of H. Then we have $g_1g_2^{-1} \in N_1$ and thus

$$(g_1g_2^{-1}, h_1h_2^{-1}) \in p_2^{-1}(g_1g_2^{-1}) = (g_1g_2^{-1}, 0) + \ker p_2.$$

I It follows that $h_1h_2^{-1} \in \ker p_2$ and thus $h_1N_2 = h_2N_2$.

Now we show that φ is a homomorphism. Suppose $a, b \in G_1$. Then there exist $h, k \in G_2$ such that $(a, h), (b, k) \in H$ and thus $(ab, hk) \in H$ as H is a subgroup. Then

$$\varphi(abN_1) = hkN_2 = (hN_2)(kN_2) = \varphi(aN_1)\varphi(bN_1).$$

Also $\varphi(eN_1) = eN_2$ because H is a subgroup.

Finally, we show φ is an isomorphism. It is clearly surjective since for every $h \in G_2$ there is some $g \in G_2$ such that $(g, h) \in H$. To show injectivity, suppose $\varphi(gN_1) = eN_2$. Then $(g, e) \in H$ and $g \in N_1$.

Then we have also shown that the image of G in $G_1/N_1 \times G_2/N_2$ is the graph of this isomorphism, since an element $(g, h) \in H$ corresponds to a pair of cosets (gN_1, hN_2), such that gN_1 is mapped to hN_2 under φ.

24. Throughout we will denote $U(n, \mathbb{K})$ by U and $B(2, \mathbb{K})$ by B. We first show the case $n = 2$. We proved in class that there is an exact sequence

$$1 \longrightarrow U \xrightarrow{f} B \xrightarrow{g} \mathbb{T} \longrightarrow 1,$$

where $\mathbb{T}$ is the diagonal subgroup.

First, since U is a subgroup in B which is equal to the kernel of the homomorphism g, U is normal in B. Moreover, it is clear that U is abelian in the case $n = 2$.

This exact sequence implies that B/U is isomorphic to $\mathbb{T}$, an abelian group.

Then we have the tower of subgroups

$$B \supset U \supset \{e\}$$

where U is normal in B, $\{e\}$ is normal in U, and the quotient groups B/U and $U/\{e\} = U$ are both abelian. Then we have shown $B(2, \mathbb{K})$ is solvable.

For the case where $n = 3$, we must consider the commutator subgroup in U. Collapsing many steps of the calculation, we see that for matricies

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix},$$

their commutator is

$$ABA^{-1}B^{-1} = \begin{bmatrix} 1 & 0 & af - dc \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then it is easy to see that $[U, U]$ is the subgroup of U consisting of matrices with 1's on the diagonal and all other entries zero except for the top right corner, which is an arbitrary element of $\mathbb{K}$. Note that $[U, U]$ is abelian by a simple calculation.

By a previous excercise, $[U, U]$ is a normal subgroup of U, and $U/[U, U]$ is abelian. Then we have a normal tower of subgroups

$$B \supset U \supset [U, U] \supset \{e\}.$$

We have aleady shown that each quotient group is abelian as well, so it is an abelian tower and $B(3, \mathbb{K})$ is solvable.

25. Recall that the quaternion group Q_8 is generated by elements i, j, such that if $k = ij$ and $m = i^2$, we have $i^2 = j^2 = k^2 = m$, $m^2 = e$, and $ij = mji$.

    Consider the subgroup $\langle i \rangle$ generated by i. Since $i^4 = e$, $\langle i \rangle$ is order 4 and cyclic, and has index 2, meaning it is normal in Q_8. It has a cyclic subgroup generated by m of order 2 which is thus normal in $\langle i \rangle$. Then we have the normal tower

    $$Q_8 \supset \langle i \rangle \supset \langle m \rangle \supset \{e\}.$$

    Additionally, each of the quotient groups is order 2, so that they have no nontrivial subgroups and are simple. Then this is a composition series for Q_8.

    Another equivalent composition series is

    $$Q_8 \supset \langle j \rangle \supset \langle m \rangle \supset \{e\},$$

    since all possible quotients are the group of 2 elements.

26. Recall that the dihedral group of order 8, D_4, is the group generated by elements a, b, such that $a^4 = b^2 = e$, and $bab^{-1} = a^{-1}$.

    We have a subgroup of order 4 generated by a which is again normal in D_4. It has a normal subgroup generated by a^2. Then we have a composition series

    $$D_4 \supset \langle a \rangle \supset \langle a^2 \rangle \supset \{e\},$$

    since each of the quotients have order 2.

    Another subgroup of order 4 is generated by elements b and aba^{-1}, which is seen to be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ by considering both elements as reflections about perpendicular lines of symmetry of the square. Then we have another equivalent composition series

    $$D_4 \supset \langle b, aba^{-1} \rangle \supset \langle b \rangle \supset \{e\}$$

    as all quotients are isomorphic to the group of order 2.

27. Let G be a group with normal subgroup H. First suppose H and G/H are solvable. Then we have abelian towers

    $$H = H_0 \supset H_1 \supset \cdots \supset H_m = \{e\}$$
    $$G/H = K_0 \supset K_1 \supset \cdots \supset K_n = \{e\}.$$

    Let $\pi : G \to G/H$ be the canonical projection. Define $G_i = \pi^{-1}(K_i)$, so that there is a normal tower

    $$G_0 = G \supset G_1 \supset \cdots G_n = H,$$

    which is an abelian tower by one of the isomorphism theorems which states a quotient of 2 quotient groups with the same denominator is isomorphic to the quotient of the first numerator by the second.

    Then we can join this with the first abelian tower to get an abelian tower

    $$G_0 = G \supset G_1 \supset \cdot \supset G_n = H = H_0 \supset H_1 \supset \cdots \supset H_m = \{e\},$$

    proving that G is solvable

Conversely suppose that G is solvable, and we have an abelian tower

$$G = G_0 \supset G_1 \supset \cdots \supset G_n = \{e\}.$$

By intersecting with H and letting $H_i = G_i \cap H$, we obtain a new normal tower

$$H = H_0 \supset H_1 \supset \cdots \supset H_0 = \{e\},$$

after removing possible duplicates. In fact, it is an abelian tower since we have an embedding $H_i/H_{i+1} \to G_i/G_{i+1}$. Then H is solvable.

By taking quotients by H instead and letting $K_i = G_i/H$, we obtain a new normal tower

$$G/H = K_0 \supset K_1 \supset \cdots \supset K_0 = \{e\},$$

after removing duplicates. Then this is an abelian tower since K_i/K_{i+1} is isomorphic to G_i/G_{i+1} by the same isomorphism theorem. Then G/H is solvable.