Algebra Autumn 2025
Frank Sottile
4 September 2025 **Third Homework**

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

You will note that some problems were partially done in class. This is an opportunity to polish your skills at writing mathematics in a clear and organized manner.

Hand in **to Frank, on a separeate sheet** at the start of class, Thursday 11 September:

13. Let H, K be subgroups of a group G. Show that any left coset of $H \cap K$ is the intersection of a left coset of H with a left coset of K. Use this to prove Poincaré's Theorem that if H and K have finite index, then so does $H \cap K$.

Hand in at the start of class, Thursday 11 September:

14. Let G be a group. For $g \in G$, show that conjugation by g, $G \ni a \mapsto gag^{-1} =: {}^g a$ is an automorphism of G. This is called an *inner automorphism*.

    Show that the set of inner automorphisms of G forms a normal subgroup of $\mathrm{Aut}(G)$, the group of all automorphisms of G.

15. Suppose that $\varphi\colon G \to H$ is a group homomorphism. If φ is a bijection, then the inverse function $\varphi^{-1}\colon H \to G$ is also a homomorphism. (Do this by checking that it preserves the identity, sends products to products, and sends inverses to inverses.)

16. Let $S \subset G$ be a subset of a group G and define the relation $\sim$ by $a \sim b$ if and only if $ab^{-1} \in S$. Show that $\sim$ is an equivalence relation if and only if S is a subgroup of G.

17. Let H be a subgroup of a group G. Define the *normalizer* of H in G to be

    $$N_G(H) \ := \ \{g \in G \mid gHg^{-1} = H\}\,.$$

    (a) Prove that $N_G(H)$ is a subgroup of G that contains H.

    (b) Let K be a subgroup of G that contains H. Prove that if H is normal in K, then $K \subset N_G(H)$.

    (c) Let K be a subgroup of $N_G(H)$. Prove that KH is a group and H is normal in KH. Recall from class that $KH := \{kh \mid k \in K,\ h \in H\}$.

18. Let p be the smallest prime number dividing the order $|G|$ of a finite group G and suppose that G has a subgroup H of index p, $[G : H] = p$. Prove that H is normal in G.

19. Prove that the absolute value map $|\ |\colon \mathbb{C}^\times \to \mathbb{R}^\times$ that sends a nonzero complex number to its absolute value is a group homomorphism, and determine its image and kernel.

20. Suppose that $\varphi\colon G \twoheadrightarrow H$ is a surjective homomorphism of groups with kernel N. Show that $K \mapsto \varphi(K)$ and $L \mapsto \varphi^{-1}(L)$ are inverse bijections between,

    (a) $\{$subgroups K of G that contain $N\}$ and $\{$subgroups L of $H\}$.

    (b) $\{$normal subgroups K of G that contain $N\}$ and $\{$normal subgroups L of $H\}$.