

# Extra Problem

Let  $D$  be a division ring. Let  $S$  be the ring of  $n \times n$  matrices,  $n > 1$ .

1.  $S$  has no proper ideals besides the zero ideal.

We proved on homework 10 that the only ideals in such a matrix ring are of the form  $M_n(I)$  for  $I \subset D$  an ideal. Since  $D$  is a division ring, the only ideals are  $D$  and the zero ideal. Then the only ideals of  $S$  are  $S$  and the zero ideal.

2.  $S$  has zero divisors: as long as  $n > 1$ , we have

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \ddots & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & \ddots & & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & & & & 0 \end{bmatrix} = 0.$$

3. Since  $S$  is non-commutative, we must use the more general version of prime ideals. Suppose  $AB \subset \langle 0 \rangle$ . Since  $D$  has no proper ideals, the only possibilities for  $A$  and  $B$  are 0 and  $S$  itself. If both  $A$  and  $B$  are  $S$ , clearly their product is not contained in the trivial ideal. Then one or both of  $A$  and  $B$  must be equal to 0 and thus the ideal 0 is prime.