Algebra Autumn 2025
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26 August 2025 **First Homework**

Write your answers neatly, in complete sentences. I highly recommend recopying your work before handing it in. Correct and crisp proofs are greatly appreciated; oftentimes your work can be shortened and made clearer.

Hand in at the start of class, Thursday 28 August:

0. Read Sections I.1 and I.2 of Lang's Algebra.

1. We explained that $\mathsf{Func}(S)$, the set of functions $f\colon S \to S$, where S is set, forms a monoid. Let $S := \{1,2\}$, a set with two elements. We write the elements of S in 2-line notation as

$$e := \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \qquad \alpha := \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \beta := \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \qquad \gamma := \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}.$$

Please give the composition (multiplication) table for this monoid $\mathsf{Func}(\{1,2\})$.

$\circ$	e	α	β	γ
e				
α			$\alpha \circ \beta$	
β				
γ				

(Evaluate the composition $\alpha \circ \beta$, place it in that cell, and do the same for the other 15 cells.)

Is this monoid commutative?

2. Let $\mathbb{R}_{\geq 0}$ be the monoid of nonnegative real numbers under addition, and let $\mathbb{R}_+$ denote the monoid of positive real numbers under multiplication.

   (a) Find the smallest submonoid of $\mathbb{R}_{\geq 0}$ that contains $\sqrt{3}$. (That is, describe it elements as a set. The term "smallest" here means, by defintion, the submonoid that is contained in any other submonoid that contains $\sqrt{3}$.)

   (b) Find the smallest submonoid of $\mathbb{R}_+$ that contains $\sqrt{3}$.

   (c) Are either of $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_+$ a group? If so, do the answers to the previous quesions change if "submonoid" is replaced by "subgroup"?

   (You need not justify your answers.)

3. Let $B_2(\mathbb{R}) := \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \mid a,b,c \in \mathbb{R},\ ac \neq 0\}$. Show that $B_2(\mathbb{R})$ is a subgroup of $\mathsf{GL}(2,\mathbb{R})$, under matrix multiplication. (This subgroup is called a *Borel subgroup*.)

4. Let $\mathbb{C}^\times$ be the group of nonzero complex numbers under multiplication. Define $f\colon \mathbb{C}^\times \to \mathsf{GL}(2,\mathbb{R})$ by $f(z) = \begin{pmatrix} a(z) & b(z) \\ -b(z) & a(z) \end{pmatrix}$, where $a(z) := \frac{z+\overline{z}}{2}$ and $b(z) := \frac{z-\overline{z}}{2\sqrt{-1}}$ are the real and imaginary parts of the complex number z.

   Show that f is an injective group homomorphism.

5. The *center* of a group G is the set $Z(G) := \{a \in G \mid ag = ga\ \forall g \in G\}$. For a fixed $g \in G$, the *centralizer* of g is the set $C_G(g) := \{a \in G \mid ag = ga\}$. Prove that both $Z(G)$ and $C_G(g)$ are subgroups of G.