13. 14 21 25 27 30

13. 证明：设 $a = x^{-1}h_1 x$. $b = x^{-1}h_2 x$. $h_1, h_2 \in H$.

则 $b^{-1} = (x^{-1}h_2 x)^{-1} = x^{-1}h_2^{-1}x$.

$ab^{-1} = x^{-1}h_1 x x^{-1}h_2^{-1}x = x^{-1}h_1 h_2^{-1}x$.

$\because$ $h_1, h_2^{-1} \in H$，H是群 $\therefore$ $h_1 h_2^{-1} \in H$.

$ab^{-1} \in H_1$ $\therefore$ H_1 是G子群

14. 证明：只需证明 G中任两个子群的交是G的子群即可.

设 H_1, H_2是G子群.

则 $H_1 \cap H_2 = \phi$时. 命题成立

$H_1 \cap H_2 \neq \phi$时，设 $a, b \in H_1 \cap H_2$.

则 $\because$ $a, b \in H_1$ $\therefore$ $ab^{-1} \in H_1$

$a, b \in H_2$, $\therefore$ $ab^{-1} \in H_2$

$\therefore$ $ab^{-1} \in H_1 \cap H_2$，$H_1 \cap H_2$是G的子群

命题得证

21.

$$\sigma\tau = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 5 & 6 \end{bmatrix} = (3\,1)(2\,4)$$

$$\tau\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 3 & 4 & 2 & 1 \end{bmatrix} = (1\,6)(2\,5)$$

$$\sigma^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 2 & 1 & 3 & 4 \end{bmatrix} = (53264)$$

$$\sigma\tau\sigma^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 4 & 3 & 1 & 2 \end{bmatrix} = (15)(26)(34)$$

25. 设 G 为偶数阶的有限群

且 x 为其生成元之一，则 $o\langle x\rangle = 2n$, $n \in Z^+$, $|G| = 2n$

假设 $a = x^k$, $a \neq e$

则 $a^{2n} = (x^k)^{2n} = (x^{2n})^k = e^k = e$.　　$\therefore a^n$ 为所求的元素

27. 设 $H = \{(1324), (12)(34), (1423), e\}$

e : $eH = H$

(13): $(13)H = \{(142), e, (13), (243)\}$,

(12): $(12)H = \{(13)(24), (12), (34), (14)(23)\}$

$(2\ 4)$: $(2\ 4)H = \{(1\ 3\ 4), (1\ 4\ 3\ 2), (2\ 4)(1\ 2\ 3)\}$

$(2\ 3)$: $(2\ 3)H = \{(1\ 2\ 4), (1\ 3\ 4\ 2), (2\ 3), (1\ 4\ 3)\}$

$(1\ 3\ 2)$: $(1\ 3\ 2)H = \{(1\ 2\ 4\ 3), (2\ 3\ 4), (1\ 3\ 2), (1\ 4)(2\ 3)\}$

$(1\ 3\ 4)$: $(1\ 3\ 4)H = \{(1\ 4\ 3\ 2), (1\ 2\ 3), (1\ 3\ 4), (2\ 4)\}$

$(1\ 2\ 3\ 4)$: $(1\ 2\ 3\ 4)H = \{(1\ 4\ 2), (1\ 3), (1\ 2\ 3\ 4), (2\ 4\ 3)\}$

20. 设 $Aa_1, Aa_2, \dots, Aa_n$ 是 G 关于 A 的陪集 ($Aa_i \cap Ab_j = \phi$, $i \neq j$ 时)

$Bb_1, Bb_2, \dots, Bb_m$ 是 A 关于 B 的陪集 ($Bb_i \cap Bb_j = \phi$)

再设 $S = \{B_{i,j} : 1 \le i \le m, 1 \le j \le n, B_{i,j} := Bb_i a_j\}$

首先证 $G \subseteq S$.

$\forall x \in G$, $\exists j$ s.t. $x \in Aa_j$. 设 $x = ha_j$, $h \in A$

then $\because h \in A$. $\exists i$ s.t. $h \in Bb_i$. $\Rightarrow h = h' b_i$. $h' \in B$

$\therefore x = ha_j = h' b_i a_j \in B(b_i a_j) = B_{i,j}$ $\therefore G \subseteq S$

再证 $|S| = mn$. 若 $B_{p,q} = B_{r,s}$.

$\because Bb_p, Bb_r \subset A$, $\therefore B_{p,q} \subset Aa_q$, $B_{r,s} \subset Ba_s$

$\because q \neq s$ 时 $Aa_q \cap Aa_s = \phi$ $\therefore q = s$, $\Rightarrow a_q = a_s = a$

$\Rightarrow \quad Bb_p = (Bb_p)(a_q a^{-1}) = (B_{p,q})a^{-1} = (B_{r,s})a^{-1}$

$= (Bb_r)(a_s a^{-1}) = Bb_r$ 又∵ $p \neq r$ 时 $Bb_p \cap Bb_r = \emptyset$ ∴ $p = r$

因此 $[G:B] = |S| = mn = [G:A][A:B]$