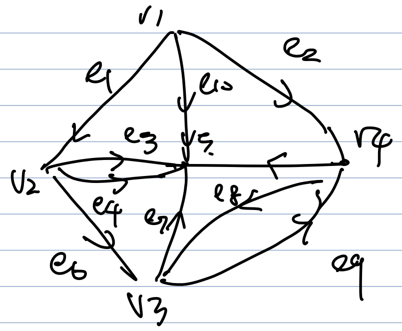
4. (a)

对G的每条边任给一方向



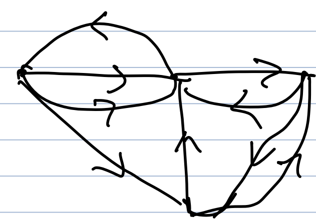
得 $B_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$

$\det(B_3B_3^T) = \begin{vmatrix} 3 & -1 & -1 & -1 \\ -1 & 4 & 0 & -2 \\ -1 & 0 & 4 & -1 \\ -1 & 2 & -1 & 5 \end{vmatrix} = 161$

(b) ①将结点 v_1, v_5 收缩为 $v_{1,5}$ 得图G'

在G'中,

$B = \begin{bmatrix} 1 & 1 & -1 & -1 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \end{bmatrix}$



$\det(B_3B_3^T) = \begin{vmatrix} 6 & -3 & -2 \\ -3 & 4 & 0 \\ -2 & 0 & 4 \end{vmatrix} = 44$

② 令 $G' = G - e_{10}$，得

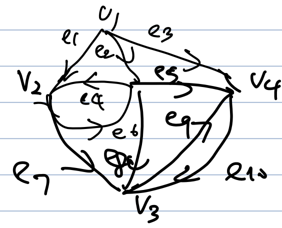
$B_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & 0 \end{bmatrix}$ $\det(B_3B_3^T) = \begin{vmatrix} 2 & -1 & -1 & 0 \\ -1 & 4 & 0 & -2 \\ -1 & 0 & 4 & -1 \\ 0 & -2 & -1 & 4 \end{vmatrix} = 57$

∴ 必含 (v_1, v_5) 的树数为 $161 - 57 = 44$

(c) 令 $G' = G - (v_4, v_5)$ 得

$B_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$, $\det(B_3B_3^T) = \begin{vmatrix} 3 & -1 & -1 & -1 \\ -1 & 4 & 0 & -2 \\ -1 & 0 & 3 & 0 \\ -1 & -2 & 0 & 4 \end{vmatrix} = 60$

5. 给图3.5边编号



得 G

(a) 由 $B_1 = \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 1 & -1 & 0 & 1 & 0 & 0 \end{bmatrix}$

得 $\vec{B_1} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\det(\vec{B_1} B_1^T) = \begin{vmatrix} 2 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 2 \end{vmatrix} = 24$

(b) 令 $G' = G - e_2$，在 G' 中，

$B_1 = \begin{bmatrix} -1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & -1 \\ 0 & -1 & 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 & 0 \end{bmatrix}$ $\vec{B_1} = \begin{bmatrix} -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\det(\vec{B_1} B_1^T) = \begin{vmatrix} 2 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 1 \end{vmatrix} = 8$

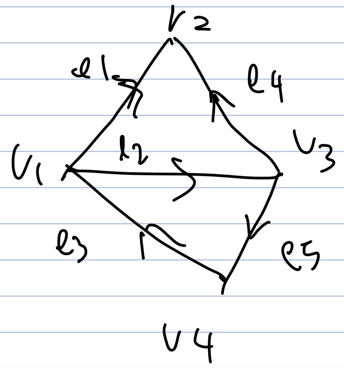
(c)

令 $G' = G - e_8 - e_{10}$，在 G' 中

$B_1 = \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 & 1 & -1 & 0 & 0 \end{bmatrix}$ $\vec{B_1} = \begin{bmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}$

$\det(\vec{B_1} B_1^T) = \begin{vmatrix} 2 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 2 \end{vmatrix} = 7$

9. 举例说明，$\det(\vec{B}_k\vec{B}_k^T)$ 不是以 v_k 为根的根树数目。



G

在G中，$B_1 = \begin{array}{c} \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \\ \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 \end{matrix} \end{array}$

$$\vec{B}_1 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

以 v_1 为根的根树有 $\{e_1, e_2, e_5\}$

而 $\det(\vec{B}_1\vec{B}_1^T) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 0$，显然不是以 v_1 为根的根树数目。

11.

$\because m=8,\ n=5$ $\therefore$ G的完全回路 C_e 的

秩是 $8-5+1=4$. $\therefore$ 通过行变换

$$C' \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & -1 & -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ & & 1 & 0 & 0 & 1 & -1 & 0 \\ & & & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = C''$$

由 C'' 的前4行构成 G的完全回路 $C_e = (\ I \quad C_{f12}\ )$

$\because$ 在上述过程中没有进行列变换，$\therefore$ C_e 也是关于 $\{e_5, e_6, e_7, e_8\}$ 的基本回路矩阵

令 S_f 为 G的关于 $\{e_5, e_6, e_7, e_8\}$ 的基本割集矩阵。

并设 $S_f = (\bar{S}_{f11}, I)$.

则由 $S_f C_e' = 0$. $(S_{f11}, I)\begin{pmatrix} I \\ C_{f12}^T \end{pmatrix} = 0$ $\quad S_{f11} + C_{f12}^T = 0$,

$$S_{f11} = -C_{f12}^T = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & 0 & -1 \end{bmatrix},$$

$$\therefore S_f = \begin{array}{c} \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & & & \\ 1 & 0 & -1 & 1 & & 1 & & \\ 0 & -1 & 1 & -1 & & & 1 & \\ 0 & -1 & 0 & -1 & & & & 1 \end{bmatrix} \\ \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \end{array}$$

13. 设 G 是无向图，对任意结点 $v \in V(G)$，$G-v$ 仍是连通图，而且 G 的基本割集矩阵 S_f 的每一行都有偶数个 1 元素。证明 G 中有欧拉回路。

$\Leftrightarrow$ G中各结点度都是偶数

设T为G的一棵树，

设树T中度为1的所有点$(v_{11}, v_{12}, \cdots v_{1i})$关联的T中边各为$(e_{11}, e_{12} \cdots e_{1i})$，则

∵ G的基本割集矩阵S_f的每一行都有偶数个1，故 在G-T中，跟$v_{11}, v_{12} \cdots v_{1i}$

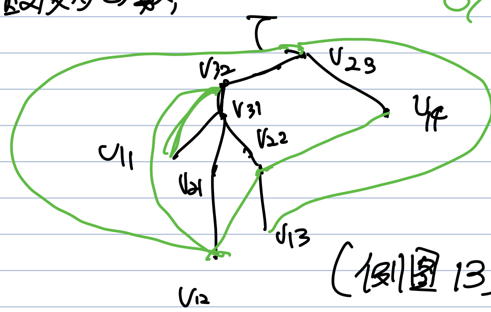
关联的边数各为奇数，$\Rightarrow$ 在G中，$v_{11}, v_{12}, \cdots v_{1i}$的度为偶数（1+奇=偶）

设这些点集为V_1，则 $T' = T - V_1$中，度为1的所有点$(v_{21}, v_{22}, \cdots, v_{2k})$

在G中度依然为偶数，这是因为 任结$(v_{21}, v_{22} \cdots v_{2k})$中一个结点，

与之对应的V_1中结点在G-T中的度为奇数，

∴ 由已知条件，$(v_{21}, v_{22}, \cdots v_{2k})$这些点在G中度也为偶数。∵ G为有完图，这样依次类推，得到T中所有结点在G中度为偶数的结论



（例图13）

$\Rightarrow$ G中所有结点度为偶数 $\Leftrightarrow$ G存在欧拉回路.