1. $\int \frac{A}{ax+b}dx \quad (a\neq 0) = \frac{A}{a}\int \frac{1}{x+\frac{b}{a}}dx = \frac{A}{a}\ln\left|x+\frac{b}{a}\right| + C$

2. $\int \frac{A}{(ax+b)^n}dx$ ($a\neq 0$, $n\geq 2$的正整数) $= \frac{A}{a^n}\int \frac{1}{(x+\frac{b}{a})^n}dx = \frac{A}{a^n}\cdot\frac{1}{1-n}\left(x+\frac{b}{a}\right)^{-n+1}$

3. $\int \frac{Bx+D}{px^2+qx+r}dx \quad (p\neq 0,\ B\neq 0,\ q^2-4pr<0)$

换元

$$= B\int \frac{x+\frac{D}{B}}{px^2+qx+r} = \frac{B}{2p}\int \frac{2px+q-q+\frac{D}{B}2p}{px^2+qx+r}dx$$

$$= \frac{B}{2p}\int \frac{d(px^2+qx+r)}{px^2+qx+r} + \frac{B}{2p}\left(-q+\frac{D}{B}2p\right)\int \frac{1}{px^2+qx+r}dx$$

$$= \frac{B}{2p}\ln|px^2+qx+r| + \frac{B}{2p}\left(-q+\frac{D}{B}2p\right)\int \frac{1}{p\left[\left(x+\frac{q}{2p}\right)^2+\frac{4pr-q^2}{4p^2}\right]}dx$$

注: $4p-q^2>0$

$(\arctan x)' = \frac{1}{x^2+1}$ $\Rightarrow \int \frac{1}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$

$$= \frac{B}{2p}\ln|px^2+qx+r| + \frac{B}{2p}\left(-q+\frac{D}{B}2p\right)\frac{1}{p}\sqrt{\frac{4p^2}{4p-q^2}}\arctan\left(\frac{2p\left(x+\frac{q}{2p}\right)}{\sqrt{4p-q^2}}\right) + C$$

递推公式

$\boxed{令\ I_n = \int \frac{dx}{(x^2+a^2)^n} \quad (n\ 正整数)}$

$n=1 \Rightarrow I_1 = \int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C$

$\left[(x^2+a^2)^{-n+1}\right]' = \frac{2x}{1-n}(x^2+a^2)^{-n}$

$$I_n = \int \frac{dx}{(x^2+a^2)^n} = \int \underbrace{\frac{1}{(x^2+a^2)^n}}_{u(x)}\underbrace{dx}_{v(x)} = \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2}{(x^2+a^2)^{n+1}}dx$$

$$= \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2+a^2-a^2}{(x^2+a^2)^{n+1}}dx$$

$$= \frac{x}{(x^2+a^2)^n} + 2n\int \frac{1}{(x^2+a^2)^n}dx - 2na^2\int \frac{1}{(x^2+a^2)^{n+1}}dx$$

$$I_n = \frac{x}{(x^2+a^2)^n} + 2nI_n - 2na^2 I_{n+1}$$

$1 \qquad (x^2+a^2)^{-n}$

$\downarrow \quad + \quad \downarrow$

$x \qquad -2nx(x^2+a^2)^{-n-1}$

$\frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2}{(x^2+a^2)^{n+1}}dx$

$$\Rightarrow \quad I_{n+1} = \frac{1}{2na^2}\left((2n-1)I_n + \frac{x}{(x^2+a^2)^n} \right) \quad n = 1, 2, \cdots, \cdots$$

4. $\displaystyle \int \frac{Bx+D}{(px^2+qx+r)^n}dx$ $\quad (n > 1$, 正整数$)$

$$= B\int \frac{x+\frac{D}{B}}{(px^2+qx+r)^n}dx = \frac{B}{2p}\int \frac{2px+q-q+\frac{D}{B}2p}{(px^2+qx+r)^n}dx$$

$$= \frac{B}{2p}\int \frac{d(px^2+qx+r)}{(px^2+qx+r)^n} + \frac{B}{2p}\left(-q+\frac{D}{B}2p\right)\frac{1}{p^n}\int \frac{dx}{\left[(x+\frac{q}{2p})^2+\frac{4pr-q^2}{4p^2}\right]^n}$$

$$u = x + \frac{q}{2p}$$

$$= \frac{B}{2p}\cdot\frac{1}{1-n}(px^2+qx+r)^{-n+1} + \frac{B}{2p}\left(-q+\frac{D}{B}2p\right)\frac{1}{p^n}\int \frac{du}{(u^2+a^2)^n}$$