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One can formulate the two-dimensional Navier-Stokes equations not only in the so-called primitive variables (u, v, p, see assignment 2), but also in using the stream function ψ and the vorticity ζ. We do not want to deduce this formulation here explicit, but we want to define these sizes and give an discriptive interpretation, to have further possibilities to analyze the calculated flow.

1 The stream function ψ and the vorticity ζ

Definition 1 *If u and v are the velocities of the flow field, the stream function $\psi(x, y)$ is defined by:*

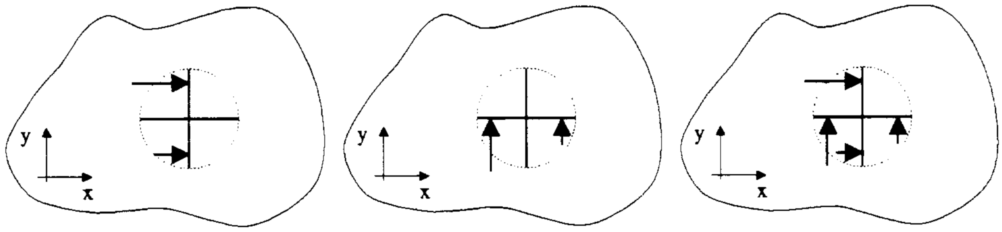
$$\frac{\partial \psi(x,y)}{\partial x} := -v, \quad \frac{\partial \psi(x,y)}{\partial y} := u, \tag{1}$$

and the vorticity $\zeta(x, y)$ by:

$$\zeta(x,y) := \frac{\partial u}{\partial y}(x,y) - \frac{\partial v}{\partial x}(x,y)\,. \tag{2}$$

The stream function ψ and the vorticity ζ can be interpreted as follows:

1. **Vorticity ζ:** ζ measures the vorticity in the velocity field. Take a look at figure 1:



(a) (b) (c)

Figure 1: *Thought experiment to the vorticity* ζ

In a thought experiment a wheel is inserted into a velocity field. In case (a) the velocity u rises in y-direction:

$$\frac{\partial u}{\partial y} > 0,$$

and a turn of the wheel in the clockwise direction is caused. In case (b) a turn in the clockwise direction is likewise caused, however here the velocity v falls in x-direction:

$$\frac{\partial v}{\partial x} < 0.$$

In order to realize the total dimension of the turning effect, the cases (a) and (b) are collected in case (c), so that the voticity

$$\zeta = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$$

results.

2. **Stream function ψ:** At first we want to show that the stream function ψ is well defined according to the definition (1). Thus a function in two independent variables x and y can meaningfully being defined as in (1) by means of its partial derivatives if a sufficient condition is fulfilled, the so-called "Integrabilitaetsbedingung" (permutability of the derivatives):

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}\,. \tag{3}$$

One can show easily with the help of the continuity equation that in our case the Integrabilitaetsbedingung (3) is fulfilled. For a descriptive interpretation of the stream function we first introduce the term *streamline*.

Definition 2 *A streamline is a curve, which has a parallel tangent to the appropriate velocity vector* $\begin{pmatrix} u & v \end{pmatrix}$ *in each of its points* (x, y) *at a fixed time* t.

The streamlines can be visualized as contour lines $\psi(x, y) = const$ of the stream function. Contour lines of a function $f(x, y)$ in two independed variables are defined by the sets:

$$N_c := \{(x, y) \in I\!R^2 : f(x, y) = c\}, \quad c \in I\!R.$$

2 Tasks

Calculate the discrete sizes $\psi_{i,j}$ and $\zeta_{i,j}$ of the stream function ψ and the vorticity ζ. In order to avoid unnecessarily averaging in the „staggered grid“ these sizes are to be calculated not in the respective cell focal points, but in the respective right upper corners of the cell (i, j) $fori = 1, \ldots, imax$, $j = 1, \ldots, jmax$. Therefore one should implement the following:

1. `[Psi, Zeta] = psi_zeta(U, V, imax, jmax, delx, dely):`
   This function calculates the discrete stream function $\psi_{i,j}$ with the use of the discretisation
   $$\left[\frac{\partial\psi}{\partial y}\right]_{i,j} = \frac{\psi_{i,j} - \psi_{i,j-1}}{\delta y}$$
   of the definition equation
   $$\frac{\partial\psi}{\partial y} = u$$
   and the predefinition $\psi_{i,1} = 0$. Also the discrete vorticity $\zeta_{i,j}$ is computed by an appropriate discretisation of the formula
   $$\zeta := \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\,.$$

2. Expand the function `visual` with an visualization of the fields `Psi` and `Zeta`. You should also visualize the contour lines (Matlab instruction `contour`). Visualize these new sizes particulary with the velocity field of the flow.