On the Birkhoff Conjecture for Convex Billiards

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Outline

Outline

Birkhoff Billiard

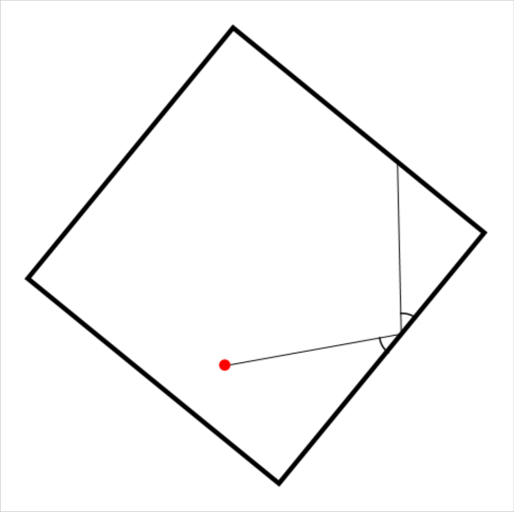


Figure: Square Billiard where angle of incidence equals the angle of reflection

Birkhoff Billiard

Billiard Map

$$\begin{aligned} f : [0, 2\pi) \times (0, \pi) &\to [0, 2\pi) \times (0, \pi) \\ (\theta_n, \alpha_n) &\mapsto (\theta_{n+1}, \alpha_{n+1}) \end{aligned}$$

Birkhoff Billiard

What Billiard Shapes are allowed?

- Convex

Birkhoff Billiard

What Billiard Shapes are allowed?

- Convex
- Smooth

Discs

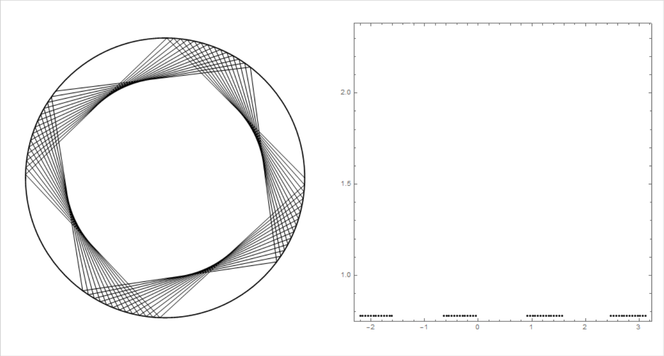


Figure: Disc Billiard with Phase Space Diagram

Ellipse

Definition

A curve $\Gamma \subset \Omega$ is called a caustic of a billiard Ω if any billiard orbit having one segment tangent to Γ has all its sements tangent to Γ.

Ellipse

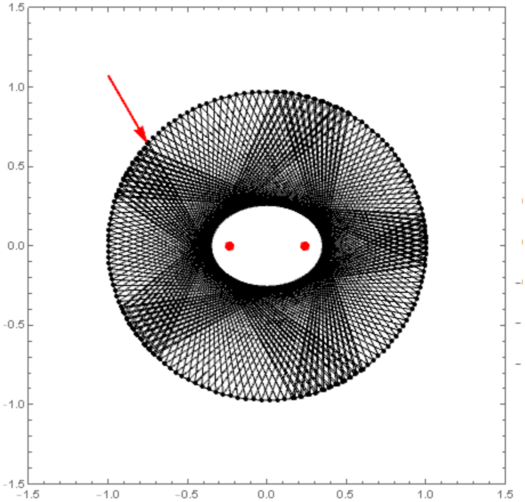


Figure: Elliptical Billiard with caustic confocal ellipse with foci shown in red

Ellipse

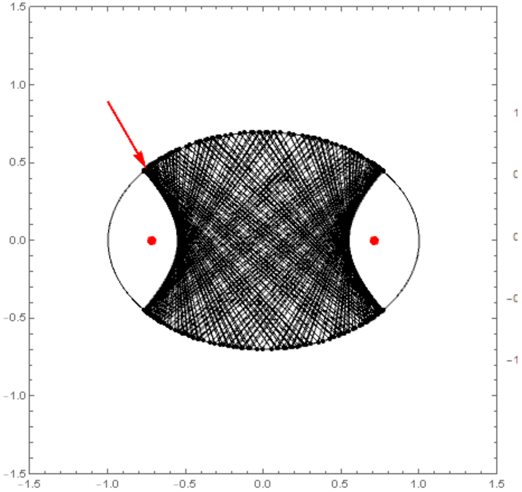


Figure: Elliptical Billiard with caustic confocal hyperbola with foci shown in red

Ellipse

- A billiard trajectory inside Ω stays tangent (caustic) to a fixed confocal ellipse or fixed confocal hyperbola.

Ellipse

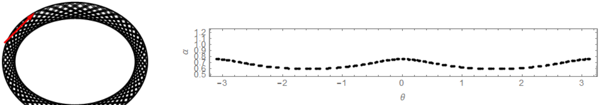
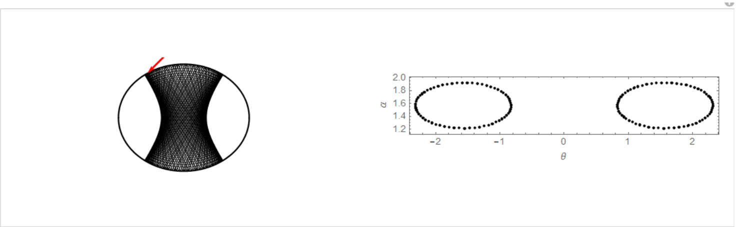


Figure: Elliptical Billiard with Phase Space Diagram $a = 1, b = 1.5$

Ellipse



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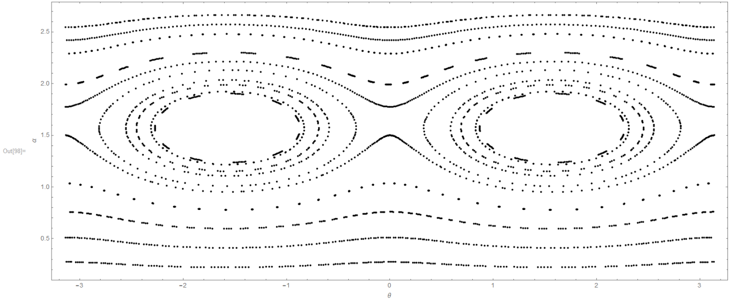

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Ellipse

- An f-invariant curve γ in the phase space correspond to a caustic Γ.

Near Elliptical Billiards

Consider billiards bounded by the curves of the form

$$ax^2 + by^2 + \varepsilon x^4 = 1$$

Near Elliptical Billiards

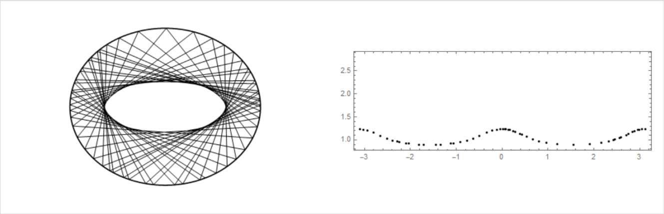

Figure: Near elliptical Billiard $a = 1, b = 1.5, \varepsilon = .01$ with Phase Space Diagram

Near Elliptical Billiards

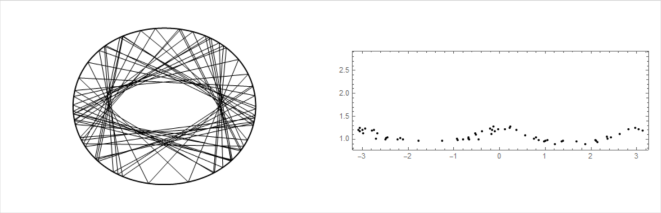

Figure: Near elliptical Billiard $a = 1, b = 1.5, \varepsilon = .1$ with Phase Space Diagram

Near Elliptical Billiards

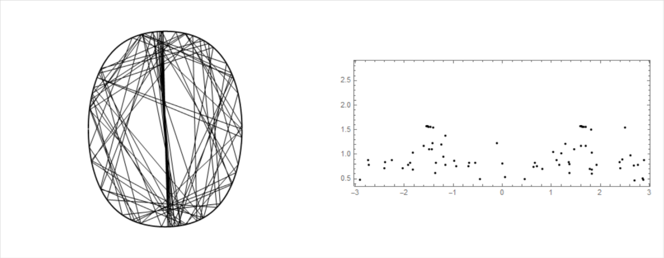

Figure: Near elliptical Billiard $a = b = \varepsilon = 1$ with Phase Space Diagram

Near Elliptical Billiards

Definition

A billiard is said to be integrable if the union of all smooth, convex caustics, has nonempty interior.

Definition

A billiard is said to be integrable if there exists a (smooth) foliation of the whole phase space consisting of invariant curves of the billiard map.

*Note that a billiard inside an ellipse is integrable.

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- A small integrable perturbation of an ellipse of small eccentricity must be an ellipse. (Avila, Simoi, Kaloshin; 2016)
- A small integrable perturbation of an ellipse of arbitrary eccentricity must be an ellipse. (Kaloshin, Sorrentino; 2017)

Outline

Near Elliptical Billiards

Consider billiards bounded by the curves of the form

$$ax^2 + by^2 + \varepsilon x^4 = 1 \qquad (1)$$

The billiard mapping is definied as

$$\begin{cases} F(\theta_{n+1}) = 0 = Y(\theta_{n+1}) - Y(\theta_n) - \tan(\alpha_n + \phi_n)(X(\theta_{n+1}) - X(\theta_n)) \\ \alpha_{n+1} = \phi_{n+1} - (\alpha_n + \phi_n) \end{cases} \qquad (2)$$

Near Elliptical Billiards

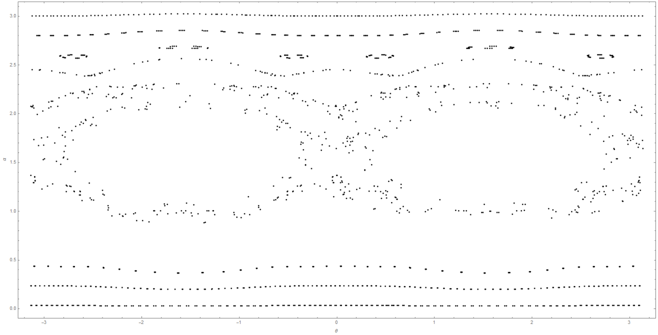

Figure: Near elliptical Billiard $a = 1, b = 1.5, \varepsilon = .1$ with Phase Space Diagram

Near Elliptical Billiards

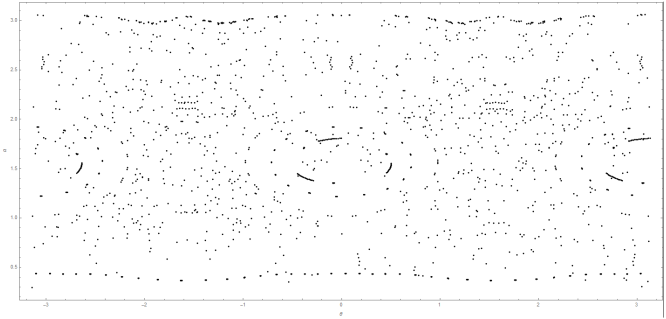

Figure: Near elliptical Billiard $a = 1, b = 1.5, \varepsilon = 10$ with Phase Space Diagram

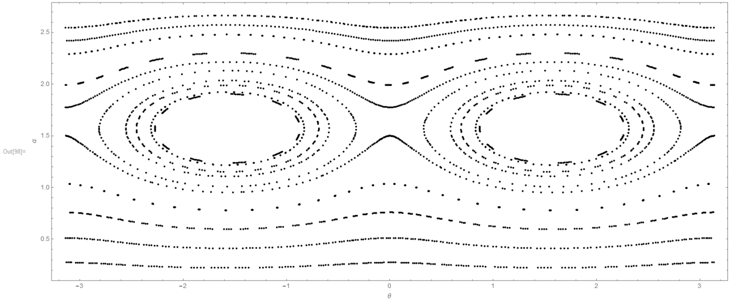

Figure: Elliptical Billiard with Phase Space Diagram
$a = 1, b = 1.5, \varepsilon = 0$

Near Elliptical Billiards

Theorem

Mather,1982

If the curvature of $d\Omega$ vanishes at some point, there exists no caustic of Ω and hence is non-integrable.

Summary

- Relationship between caustics, invariant curves, and integrability.
- For ε significantly large, all of the invariant curves are destroyed.

- Outlook
  - Sensitive dependence on initial conditions Lyapunov Exponents