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A priori error estimates for optimal control problems with pointwise constraints on the gradient of the state

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Abstract We analyze a finite element approximation of an elliptic optimal control problem with pointwise bounds on the gradient of the state variable. We derive convergence rates if the control space is discretized implicitly by the state equation. In contrast to prior work we obtain these results directly from classical results for the $W^{1,\infty}$-error of the finite element projection, without using adjoint information. If the control space is discretized directly, we first prove a regularity result for the optimal control to control the approximation error, based on which we then obtain analogous convergence rates.

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1 Introduction

Many physical processes modeled by partial differential equations require bounds on the gradient of the state variable. For example, large temperature gradients during cooling or heating of an object may lead to its destruction, or, in solid mechanics, the deformation gradient determines the change between elastic and plastic material behavior. Therefore, optimization of such processes may require pointwise constraints on the gradient of the state.

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Little attention has been given, to date, to optimal control problems with pointwise constraints on the gradient. For semilinear elliptic and parabolic state equations existence of solutions to such optimal control problems, as well as first order optimality conditions, are derived in [4–7]. In [13] semi-smooth Newton methods and regularized active set methods are discussed for the solution of an elliptic PDE with gradient constraints. An analysis for a barrier method for optimization with constraints on the gradient of the state can be found in [21], and corresponding a posteriori error estimates for the barrier parameter and the mesh size in [23,24].

The focus of this work is the derivation of a priori error estimates for a model optimization problem subject to pointwise constraints on the gradient of the state variable. The convergence rate obtained will be analogous to those in [9,10,15,17]. However, our proof is far more elementary, since we will not make use of a discrete adjoint equation. We consider this an advantage for several reasons: First, since the adjoint variables possess low regularity [6], we cannot expect more than qualitative convergence for these variables. Second, if one uses "black-box" optimization software, one might be unable to influence the discretization of the adjoint variables. In particular, our techniques are essentially independent of the choices of discretization spaces for the state and control variables. Our technique is similar to the one used in [11,19] for state constraints, however, in our situation the analysis is complicated by the missing regularity of the control variable and the fact that it is not always feasible to choose a Hilbert space as the control space.

Our presentation is structured as follows: In Sect. 2 we will define the model problem, followed by a description of the discretization in Sect. 3. The a priori error estimates will be established in two steps. In Sect. 4.1 we consider the case when only the state equation is discretized, e.g., the discretization of the control space is given implicitly by the necessary optimality conditions. This will lead to the same results as in [9,10]. In Sect. 4.2 we will then analyze the case of a given discretization of the control space. Here, we require a regularity result on the optimal control, which can be circumvented in the case of piecewise constant approximations as is used in [15]. While we do use information about the continuous adjoint variables to establish this result (cf. Appendix A) we stress that our technique requires no special features of the approximation space. Finally we will derive the additional regularity required for our proof in Appendix A.

2 Problem formulation

Let Ω be a convex polygonal (or polyhedral) domain in $\mathbb{R}^d$, $d \leq 3$. We consider the linear elliptic PDE

$$-\Delta u = Bq \quad \text{in } \Omega, \tag{2.1a}$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{2.1b}$$

where $B : Q \to L^r(\Omega), r \in (d, +\infty)$, is a linear continuous operator, and Q is a reflexive Banach space which we specify below.

It will be crucial for our analysis that there exists $d < t \leq r$ such that (2.1) is $W^{2,t}$-regular (the weak solution $u \in H_0^1(\Omega)$ belongs in fact to $W^{2,t}(\Omega)$). If $d = 2$ then this result follows from [14, Thm. 4.4.3.7]. If $d = 3$ then one needs to assume, in addition, that the angle between any two faces of Ω is bounded strictly above by $\frac{3}{4}\pi$ [8, Cor. 3.7]. We assume throughout that this is satisfied, that is, if $Bq \in L^t(\Omega)$ then

$$u \in W^{2,t}(\Omega) \cap W_0^{1,t}(\Omega) \subset C^{1,1-d/t}(\overline{\Omega}),$$

and that there exist constants c, c' such that

$$\|u\|_{C^{1,1-d/t}} \leq c\|u\|_{2,t} \leq c'\|Bq\|_{L^t}. \tag{2.2}$$

We pose the following optimal control problem:

$$\underset{Q\times V}{\text{Minimize}}\ J(q,u) := \tfrac{1}{2}\|u - u^d\|_{L^2}^2 + R(q), \tag{2.3a}$$

$$\text{such that}(u, q) \text{ satisfies (2.1)}, \tag{2.3b}$$

$$\text{and such that } |\nabla u| \leq 1 \text{ in } \overline{\Omega}, \tag{2.3c}$$

where $u^d \in L^2(\Omega)$ is fixed, and the regularization function R depends on Q. We consider two possible situations:

(Q.1) $Q = \mathbb{R}^n$, for some $n \in \mathbb{N}$, $R(q) = \frac{1}{2}\|q\|_{\ell^2}^2$, and $B : \mathbb{R}^n \to L^r(\Omega)$ is an arbitrary linear continuous operator.
(Q.2) $Q = L^r(\Omega)$, $R(q) = \frac{1}{r}\|q\|_{L^r}^r$, and $B = \mathrm{Id}$.

The case (Q.1) corresponds to the, possibly more realistic case for applications of optimal control, of a finite dimensional control that has distributed influence on the solution variable. The second case is important in inverse problems, e.g, when the 'control' is in fact an unknown volume force that one tries to recover from a set of measurements.

We also note that, in many applications bound constraints on the control variable are imposed. This introduces some additional technicalities, and hence we decided to not include this case in the present work.

Either of the assumptions (Q.1) or (Q.2) guarantee coercivity and strong convexity of the optimal control problem, that is, the following Clarkson-type inequality holds for arbitrary $w_1 = (q_1, u_1)$, $w_2 = (q_2, u_2) \in Q \times V$

$$\tfrac{1}{2}\|\tfrac{1}{2}(u_1 - u_2)\|_{L^2}^2 + R(\tfrac{1}{2}(q_1 - q_2)) + J\left(\tfrac{1}{2}(w_1 + w_2)\right) \leq \tfrac{1}{2}J(w_1) + \tfrac{1}{2}J(w_2). \tag{2.4}$$

The inequality for the q-variable is simply Clarkson's inequality. The inequality for the u-variable is obtained by applying Clarkson's inequality to $u_1 - u^d$ and $u_2 - u^d$. Using (2.4) and standard arguments [18], one can show that there exists a unique solution $(\overline{q}, \overline{u}) \in Q \times V$ to (2.3).

We will later use the fact that both $R : Q \to \mathbb{R}$ and $J : Q \times V \to \mathbb{R}$ are twice differentiable, and denote the first derivatives, respectively, by R' and J', for example,

$$\langle \delta q, R'(q) \rangle_{Q,Q^*} := \lim_{t\to 0} \frac{R(\|q + t\delta q\|_Q) - R(\|q\|_Q)}{t}.$$

We note that, in the case (Q.1), $R'(q) = q$, while in the case (Q.2), the first derivative takes the form $R'(q) = |q|^{r-2}q$.

3 Discretization

For the discretization of (2.3), we assume that we are given a family $(\mathcal{T}_h)_{h\in(0,1]}$ of triangulations, consisting of triangles or quadrilaterals in 2d, and of tetrahedrons or hexahedrons in 3d, which are *affine-equivalent* to their respective reference elements, such that $\operatorname{diam}(T) \le h$ for all $T \in \mathcal{T}_h$, $h \in (0, 1]$. We assume throughout that the family is quasi-uniform in the sense of [3, Def. 4.4.13], that is, there exists $\rho > 0$ such that, for each $T \in \mathcal{T}_h$ and $h \in (0, 1]$ there exists a ball $B_T \subset T$ such that $\operatorname{diam}(B_T) \ge \rho h$.

We define the discrete state space V_h as the space of continuous piecewise linear (or bi-, or tri-linear) functions with respect to the mesh $\mathcal{T}_h$. For fixed $q \in Q$, the semi-discretized state equation then reads

$$(\nabla u_h, \nabla \phi_h) = (Bq, \phi_h) \quad \forall\, \phi_h \in V_h. \tag{3.1}$$

The corresponding semi-discretized optimal control problem becomes

$$\underset{Q\times V_h}{\text{Minimize}}\ J(q_h, u_h) := \tfrac{1}{2}\|u_h - u^d\|_{L^2}^2 + R(q_h), \tag{3.2a}$$
$$\text{such that } (u_h, q_h) \text{ satisfies (3.1)}, \tag{3.2b}$$
$$\text{and such that } |\nabla u_h| \le 1 \text{ a.e. in } \overline{\Omega}. \tag{3.2c}$$

In the case (Q.2), i.e., $Q = L^r(\Omega)$, we also need to discretize the control space Q. We consider two different discretizations: either $Q^h = Q^h_{(0)}$ or $Q^h = Q^h_{(1)}$, where

$$Q^h_{(0)} = \{q_h \in Q : q_h \text{ is p.w. constant w.r.t. } \mathcal{T}_h\}, \quad \text{and}$$
$$Q^h_{(1)} = \left\{q_h \in C(\bar{\Omega}) : q_h \text{ is p.w. (bi-/tri-)linear w.r.t. } \mathcal{T}_h\right\}.$$

Our analysis applies to both choices, however, we will see that for the choice $Q^h = Q^h_{(0)}$ it yields a better convergence rate. The choice $Q^h = Q^h_{(1)}$ was not previously considered in the literature.

The fully discretized optimal control problem reads

$$\underset{Q^h\times V_h}{\text{Minimize}}\ J\big(q_h^h,u_h^h\big) := \tfrac{1}{2}\left\|u_h^h-u^d\right\|_{L^2}^2+R\big(q_h^h\big), \tag{3.3a}$$

$$\text{such that } \big(u_h^h,q_h^h\big) \text{ satisfies (3.1)}, \tag{3.3b}$$

$$\text{and such that } \left|\nabla u_h^h\right| \le 1 \text{ a.e. in } \overline{\Omega}. \tag{3.3c}$$

We remark that the restrictions we imposed on the family $(\mathcal{T}_h)_{h\in(0,1]}$ ensure that the usual interpolation error results, best approximation results, and inverse estimates hold [2, Sec. 4 and 5]. In particular, it follows that the Ritz projection is stable in $W^{1,\infty}(\Omega)$, that is, if $u\in W_0^{1,\infty}(\Omega)$, and if $u_h\in V_h$ satisfies

$$(\nabla u_h,\nabla\varphi)=(\nabla u,\nabla\varphi)\quad\forall\varphi\in V_h,\quad\text{then}\quad\|\nabla u_h\|_\infty\le c\|\nabla u\|_\infty; \tag{3.4}$$

see [20] for simplicial meshes and [2, Thm. 8.1.11 and Ex. 8.x.6] for the general case.

Finally, we define $\Pi_h : L^1(\Omega)\to Q^h$ to be the natural extension of the L^2-projection operator, that is, for $u\in L^1(\Omega)$, we define $\Pi_h u\in Q^h$ via

$$(\Pi_h q,\varphi)=(q,\varphi)\qquad\forall\varphi\in Q^h. \tag{3.5}$$

It is shown in [12] that Π_h is stable as an operator from $L^p(\Omega)$ to $L^p(\Omega)$, for any $p\in[1,\infty]$, that is, there exist constants c_p, independent of h, such that

$$\|\Pi_h f\|_{L^p}\le c_p\|f\|_{L^p}\qquad\forall f\in L^p(\Omega). \tag{3.6}$$

4 A priori estimates

4.1 State discretization

First we consider the case, where only the state space is discretized. This is reasonable if the control space is finite dimensional (i.e., case (Q.1)), or if we use the 'variational discretization' concept discussed in [16]. In general this is an intermediate step that gives us preliminary insights into the convergence behavior of the discretization. The results that we will obtain are essentially the same as those in [9,10,17], however, we do not require bounds on the discrete adjoint variables in our analysis.

Theorem 1 *Let $(\overline{q},\overline{u})\in Q\times V$ be the solution to (2.3), and $(\overline{q}_h,\overline{u}_h)\in Q\times V_h$ be the solution to the semi-discretized problem (3.2). Then there exists a constant C, independent of h, such that*

$$\left|J(\overline{q},\overline{u})-J(\overline{q}_h,\overline{u}_h)\right|\le Ch^{1-d/t}. \tag{4.1}$$

Proof Instead of using the criticality conditions for solutions, the idea of the proof is to construct discrete an continuous competitors for which the energy error can be estimated immediately.

We begin by investigating the solutions $\overline{u}$ of (2.1) and its Ritz projection u_h which are, respectively, given by

$$\begin{aligned}(\nabla\overline{u}, \nabla\phi) &= (B\overline{q}, \phi) \qquad \forall\, \phi \in V, \qquad \text{and}\\ (\nabla u_h, \nabla\phi) &= (B\overline{q}, \phi) \qquad \forall\, \phi \in V_h.\end{aligned}$$

The difficulty is that, possibly, $|\nabla u_h| \not\leq 1$. However, using the stability of the Ritz projection in $W^{1,\infty}(\Omega)$ [20], we can see that the constraint on the gradient is *almost* satisfied. Namely, in view of the regularity estimate (2.2), it follows from [20, Eq. (1.7)] that

$$\|\nabla\overline{u} - \nabla u_h\|_{L^\infty} \leq ch^{1-d/t}\|\overline{u}\|_{C^{1,1-d/t}} \leq ch^{1-d/t}\|B\overline{q}\|_{L^t}.$$

From this, we derive the bound

$$|\nabla u_h(x)| \leq |\nabla\overline{u}(x)| + ch^{1-d/t}\|B\overline{q}\|_{L^t} \qquad \text{for a.e. } x \in \Omega.$$

Setting $\beta = 1 - d/t$ and $\tilde{c} \geq c\|B\overline{q}\|_{L^t}$, it follows that

$$\left(1 - \tilde{c}h^\beta\right)|\nabla u_h| \leq \left(1 - \tilde{c}h^\beta\right) + \left(1 - \tilde{c}h^\beta\right)ch^\beta\|B\overline{q}\|_{L^t} < 1 \quad \text{a.e. in } \overline{\Omega} \quad \forall h \in (0, 1].$$

Thus, we find that the sequence

$$(\tilde{q}_h, \tilde{u}_h) := \left(\left(1 - \tilde{c}h^\beta\right)\overline{q}, \left(1 - \tilde{c}h^\beta\right)u_h\right) \tag{4.2}$$

is feasible for (3.2) and that the following estimates hold:

$$\begin{aligned}\|B\overline{q} - B\tilde{q}_h\|_{L^r} &\leq c\|\overline{q} - \tilde{q}_h\|_Q \leq ch^\beta\|\overline{q}\|_Q, \quad \text{and}\\ \|\overline{u} - \tilde{u}_h\|_{1,\infty} &\leq \|\overline{u} - u_h\|_{1,\infty} + \|u_h - \tilde{u}_h\|_{1,\infty} \leq ch^\beta\|B\overline{q}\|_{L^r} + ch^\beta\|u_h\|_{1,\infty}.\end{aligned}$$

Using again the $W^{1,\infty}$-stability of the Ritz projection, we obtain

$$\|\overline{u} - \tilde{u}_h\|_{1,\infty} \leq ch^\beta\|B\overline{q}\|_{L^r} \leq ch^\beta\|\overline{q}\|_Q.$$

Differentiability of the cost functional implies local Lipschitz continuity, and therefore we can deduce that

$$|J(\overline{q}, \overline{u}) - J(\tilde{q}_h, \tilde{u}_h)| \leq ch^\beta.$$

Since $(\tilde{u}_h, \tilde{q}_h)$ is an admissible pair for (3.2), the relation $J(\overline{q}_h, \overline{u}_h) \leq J(\tilde{q}_h, \tilde{u}_h)$ is satisfied, and hence

$$J(\overline{q}_h, \overline{u}_h) - J(\overline{q}, \overline{u}) \leq J(\tilde{q}_h, \tilde{u}_h) - J(\overline{q}, \overline{u}) \leq ch^\beta.$$

It should be mentioned that the last inequality already implies that $R(\overline{q}_h)$ is uniformly bounded for $h \in (0, 1]$, and hence there exists c independent of h such that $\|\overline{q}_h\|_Q \leq c$.

To obtain the reverse inequality, we start from $(\overline{q}_h, \overline{u}_h)$ and, using precisely the same arguments, construct $(\hat{q}, \hat{u})$ that are feasible for the exact problem (2.3) (note though, that $\hat{q}, \hat{u}$ do depend on h) and satisfy

$$|J(\overline{q}_h, \overline{u}_h) - J(\hat{q}, \hat{u})| \leq ch^{\beta}.$$

In summary, we obtain

$$-ch^{\beta} \leq J(\overline{q}, \overline{u}) - J(\tilde{q}_h, \tilde{u}_h) \leq J(\overline{q}, \overline{u}) - J(\overline{q}_h, \overline{u}_h) \leq J(\hat{q}, \hat{u}) - J(\overline{q}_h, \overline{u}_h) \leq ch^{\beta},$$

which concludes the proof of the theorem. □

From the error estimate on the objective, we can derive an estimate for the control and for the state.

Corollary 1 *Let $(\overline{q}, \overline{u}) \in Q \times V$ be the solution of (2.3), and let $(\overline{q}_h, \overline{u}_h) \in Q \times V_h$ be the solution of (3.2), then*

$$\|\overline{q} - \overline{q}_h\|_Q \leq ch^{\frac{1-d/t}{s}} \quad \textit{and} \quad \|\overline{u} - \overline{u}_h\|_{L^2} \leq ch^{\frac{1-d/t}{2}},$$

where $s = 2$ in the case (Q.1) and $s = r$ in the case (Q.2).

Proof Let $(\tilde{q}_h, \tilde{u}_h)$ be defined by (4.2), then it follows that

$$\begin{aligned} \|\overline{q} - \overline{q}_h\|_Q &\leq \|\overline{q} - \tilde{q}_h\|_Q + \|\tilde{q}_h - \overline{q}_h\|_Q \leq ch^{1-d/t} + \|\tilde{q}_h - \overline{q}_h\|_Q, \\ \|\overline{u} - \overline{u}_h\|_{L^2} &\leq \|\overline{u} - \tilde{u}_h\|_{L^2} + \|\tilde{u}_h - \overline{u}_h\|_{L^2} \leq ch^{1-d/t} + \|\tilde{u}_h - \overline{u}_h\|_{L^2}. \end{aligned} \tag{4.3}$$

To bound the remaining terms on the right hand side, we apply Clarkson's inequality (2.4), which gives

$$\begin{aligned} &\frac{1}{2}\left\|\frac{1}{2}\left(\tilde{u}_h - \overline{u}_h\right)\right\|_{L^2}^2 + R\left(\frac{1}{2}\left(\tilde{q}_h - \overline{q}_h\right)\right) \\ &\qquad \leq \frac{1}{2}J\left(\tilde{q}_h, \tilde{u}_h\right) + \frac{1}{2}J\left(\overline{q}_h, \overline{u}_h\right) - J\left(\frac{1}{2}\left(\tilde{q}_h + \overline{q}_h\right), \frac{1}{2}\left(\tilde{u}_h + \overline{u}_h\right)\right). \end{aligned}$$

Since $(\tilde{q}_h, \tilde{u}_h)$ is feasible for (3.2) it follows that

$$J\left(\overline{q}_h, \overline{u}_h\right) \leq J\left(\frac{1}{2}\left(\tilde{q}_h + \overline{q}_h\right), \frac{1}{2}\left(\tilde{u}_h + \overline{u}_h\right)\right),$$

and hence, using Theorem 1,

$$\frac{1}{2}\left\|\frac{1}{2}\left(\tilde{u}_h - \overline{u}_h\right)\right\|_{L^2}^2 + R\left(\frac{1}{2}\left(\tilde{q}_h - \overline{q}_h\right)\right) \leq \frac{1}{2}J\left(\tilde{q}_h, \tilde{u}_h\right) - \frac{1}{2}J\left(\overline{q}_h, \overline{u}_h\right) \leq ch^{1-d/t}.$$

This establishes the assertion. □

4.2 Control discretization

We are now concerned with the error introduced by a discretization of the control space Q. We assume, from now on, that (Q.2) holds, that is, $B = \mathrm{Id}$ and $R(q) = \frac{1}{r}\|q\|_{L^r}^r$, and hence $R'(q) = |q|^{r-2}q$. Note that in the other case there is no point in discretizing Q.

Our analysis is based on the following regularity result for the optimal control the proof of which is postponed to the appendix.

Theorem 2 *Let $(\bar{u}, \bar{q})$ be the solution of (2.3), $r' = \frac{r}{r-1}$, then there exist constants $\gamma, \gamma' > 0$ such that $\gamma + \gamma' \geq 1 - d/t$, with*

$$\overline{q} \in W_0^{\gamma,r}\Omega \qquad and \qquad R'(\overline{q}) \in W_0^{\gamma',r'}(\Omega). \tag{4.4}$$

Although this regularity result is somewhat technical, and our proof uses information about the adjoint system, we note that Theorem 3 only requires the regularity result itself which could, alternatively, be formulated as an assumption.

Before we state our main result, we first deduce an approximation property from the regularity result 2.

Corollary 2 *There exists a constant c, independent of h such that*

$$\|\overline{q} - \Pi_h\overline{q}\|_{L^r} \leq ch^{\gamma} \quad and \quad \left\|R'(\overline{q}) - \Pi_h R'(\overline{q})\right\|_{L^{r'}} \leq ch^{\gamma'}. \tag{4.5}$$

Proof Stability of Π_h in L^p (cf. (3.6)) provides, for example focusing on $\Pi_h\overline{q}$,

$$\|\overline{q} - \Pi_h\overline{q}\|_{L^r} \leq \left(1 + \|\Pi_h\|_{L(L^r,L^r)}\right) \inf_{q_h\in Q^h} \|\overline{q} - q_h\|_{L^r} .$$

Choosing a suitable quasi-interpolation operator for q_h, for example the Clément operator, gives the desired result. □

We are now ready to prove our main result.

Theorem 3 *Let $(\overline{q}, \overline{u}) \in Q \times V$ be the solution of (2.3) and $(\overline{q}_h^h, \overline{u}_h^h) \in Q^h \times V_h$ be the solution of (3.3), then*

$$\left|J\left(\overline{q}, \overline{u}\right) - J\left(\overline{q}_h^h, \overline{u}_h^h\right)\right| \leq \begin{cases} Ch^{\min(2\gamma,1-d/t)}, & \text{if } Q^h = Q_{(1)}^h, \\ Ch^{1-d/t}, & \text{if } Q^h = Q_{(0)}^h. \end{cases} \tag{4.6}$$

Proof As above, we set $\beta = 1 - d/t$ throughout this proof. Let $(\overline{q}^h, \overline{u}^h) \in Q^h \times V$ be the solution of the following auxiliary problem where only the control variable is discretized:

$$\underset{Q^h\times V}{\text{Minimize}}\ J\left(q^h, u^h\right) := \tfrac{1}{2}\left\|u^h - u^d\right\|_{L^2}^2 + R\left(q^h\right), \tag{4.7a}$$

$$\text{such that } \left(u^h, q^h\right) \text{ satisfies (2.1)}, \tag{4.7b}$$

$$\text{and such that } \left|\nabla u^h\right| \leq 1 \text{ in } \overline{\Omega}. \tag{4.7c}$$

We will first show that

$$\left|J\left(\overline{q},\overline{u}\right)-J\left(\overline{q}^h,\overline{u}^h\right)\right|\leq Ch^{\min(2\gamma,\beta)}. \tag{4.8}$$

Once this is established, we can repeat the proof of Theorem 1 verbatim to show that

$$\left|J\left(\overline{q}^h,\overline{u}^h\right)-J\left(\overline{q}_h^h,\overline{u}_h^h\right)\right|\leq Ch^{\beta}.$$

This is possible since all constants in this proof would only depend on the regularity of the triangulation and on $\|\overline{q}^h\|_{L^r}$. The fact that $\|\overline{q}^h\|_{L^r}$ remains bounded, as $h\to 0$, is immediately deduced from the fact that $J(\overline{q}^h,\overline{u}^h)$ converges and is therefore bounded itself. Combining the two estimates gives the desired result.

To establish (4.8) we proceed along the lines of the proof of Theorem 1 as well. Let $q^h=\Pi_h\overline{q}$ and let $u^h\in V$ solve the state equation (2.1) with right-hand side $q=q^h$, then, using our regularity assumptions on the state equation, Corollary 2, and the continuous embedding $W^{1,\infty}(\Omega)\subset W^{1+d/t+\epsilon,t}(\Omega)$ for all $\epsilon>0$, we can estimate

$$\left\|u^h-\overline{u}\right\|_{1,\infty}\leq c\left\|u^h-\overline{u}\right\|_{1+d/t+\epsilon,t}\leq c\left\|q^h-\overline{q}\right\|_{-1+d/t+\epsilon,t}\leq ch^{1-d/t-\epsilon+\gamma}. \tag{4.9}$$

Choosing $\epsilon\leq\gamma$, we obtain $\|u^h-\overline{u}\|_{1,\infty}\leq ch^{\beta}$. Thus, setting

$$\left(\tilde{q}^h,\tilde{u}^h\right)=\left(1-\tilde{c}h^{\beta}\right)\left(q^h,u^h\right),$$

for $\tilde{c}$ sufficiently large, gives an admissible pair and we obtain

$$0\leq J\left(\overline{q}^h,\overline{u}^h\right)-J\left(\overline{q},\overline{u}\right)\leq J\left(\tilde{q}^h,\tilde{u}^h\right)-J\left(\overline{q},\overline{u}\right).$$

Since J is differentiable (hence locally Lipschitz) it follows that

$$\left|J\left(\tilde{q}^h,\tilde{u}^h\right)-J\left(q^h,u^h\right)\right|\leq ch^{\beta},$$

hence, we only need to bound the term $J(q^h,u^h)-J(\overline{q},\overline{u})$ from above. Using convexity of J, we can estimate

$$\begin{aligned}J\left(\overline{q},\overline{u}\right)&\geq J\left(q^h,u^h\right)+\left\langle J'\left(q^h,u^h\right),\left(\overline{q}-q^h,\overline{u}-u^h\right)\right\rangle\\&=J\left(q^h,u^h\right)+\left\langle R'(q^h),\overline{q}-q^h\right\rangle+\left(u^h-u^d,\overline{u}-u^h\right)\end{aligned}$$

The term $(u^h-u^d,\overline{u}-u^h)$ is easily bounded by ch^{β}, using (4.9). In summary, we obtain

$$0\leq J\left(\overline{q}^h,\overline{u}^h\right)-J\left(\overline{q},\overline{u}\right)\leq J\left(\tilde{q}^h,\tilde{u}^h\right)-J\left(\overline{q},\overline{u}\right)\leq R'\left(q^h\right),q^h-\overline{q}>+ch^{\beta}. \tag{4.10}$$

Up to this point, the proof is entirely independent of the choice of control discretization space Q^h.

If $Q^h = Q^h_{(0)}$ is the space of piecewise constant functions then $R'(q^h) = |q^h|^{r-2}q^h$ also belongs to Q^h, and since $q^h = \Pi_h \overline{q}$ it follows that $< R'(q^h), q^h - \overline{q} >= 0$. This concludes the proof of (4.6) for the case $Q^h = Q^h_{(0)}$. We note that for precisely the same reason, namely that $R'(q^h) \in Q^h$, the analysis in [15] did not require regularity of the optimal control.

If $Q^h = Q^h_{(1)}$ is the space of linear (or bi- or tri-linear) functions then this argument is not available. Instead, we estimate

$$\begin{aligned}\left\langle R'(q^h), q^h - \overline{q}\right\rangle &= \left\langle R'(q^h) - R'(\overline{q}), q^h - \overline{q}\right\rangle + \left\langle R'(\overline{q}), q^h - \overline{q}\right\rangle \\ &= \left\langle R'(q^h) - R'(\overline{q}), q^h - \overline{q}\right\rangle + \left\langle R'(\overline{q}) - \Pi_h R'(\overline{q}), q^h - \overline{q}\right\rangle \\ &\le \left(\|R'(q^h) - R'(\overline{q})\|_{L^{r'}} + \|R'(\overline{q}) - \Pi_h R'(\overline{q})\|_{L^{r'}}\right)\|q^h - \overline{q}\|_{L^r}.\end{aligned}$$

Using the fact that R' is differentiable as a mapping from $L^r(\Omega) \to L^{r'}(\Omega)$ (hence locally Lipschitz continuous) as well as Corollary 2, we finally obtain

$$\left\langle R'(q^h), q^h - \overline{q}\right\rangle \le c\left(h^\gamma + h^{\gamma'}\right)h^\gamma.$$

Since $\gamma + \gamma' \ge \beta$, we obtain the convergence rate $O(h^{\min(2\gamma,\beta)})$. This concludes the proof of the theorem. □

As before, the error estimate on the objective provides an error estimate for the primal variables.

Corollary 3 *Let $(\overline{q}, \overline{u}) \in Q \times V$ be the solution of (2.3), and let $(\overline{q}^h_h, \overline{u}^h_h) \in Q^h \times V_h$ be the solution of (3.3), then*

$$\left\|\overline{q} - \overline{q}^h_h\right\|_Q \le ch^{\alpha/r} \quad \textit{and} \quad \left\|\overline{u} - \overline{u}^h_h\right\|_{L^2} \le ch^{\alpha/2},$$

where r is defined in (Q.2), and where $\alpha = 1 - d/t$ if $Q^h = Q^h_{(0)}$, and $\alpha = \min(2\gamma, 1 - d/t)$ if $Q^h = Q^h_{(1)}$.

Proof We set $\overline{w} = (\overline{q}, \overline{u})$, and so forth. We split the error

$$\overline{w}^h_h - \overline{w} = \left(\overline{w}^h_h - \overline{w}^h\right) + \left(\overline{w}^h - \overline{w}\right),$$

where $\overline{w}^h$ is the solution of the auxiliary problem (4.7). The first contribution, $(\overline{w}^h_h - \overline{w}^h)$, can be estimated precisely as in the proof of Corollary 1, yielding

$$\left\|\overline{u}^h_h - \overline{u}^h\right\|_{L^2} \le ch^{\frac{1-d/t}{2}} \quad \text{and} \quad \left\|\overline{q}^h_h - \overline{q}^h\right\|_{L^2} \le ch^{\frac{1-d/t}{r}}.$$

To estimate $(\overline{w}^h - \overline{w})$ we employ again Clarkson's inequality (2.4),

$$\frac{1}{2}\left\|\frac{1}{2}(\overline{u}^h - \overline{u})\right\|_{L^2}^2 + R\big(\tfrac{1}{2}(\overline{q}^h - \overline{q})\big) \le \tfrac{1}{2}J(\overline{w}^h) + \tfrac{1}{2}J(\overline{w}) - J\big(\tfrac{1}{2}(\overline{w}^h + \overline{w})\big).$$

Since $\overline{w}^h$ is admissible for the full problem (2.3), we have $J\left(\frac{1}{2}(\overline{w}^h + \overline{w})\right) \ge J(\overline{w})$ which gives

$$\frac{1}{2}\left\|\frac{1}{2}\left(\overline{u}^h - \overline{u}\right)\right\|_{L^2}^2 + R\left(\frac{1}{2}\left(\overline{q}^h - \overline{q}\right)\right) \le \frac{1}{2}\left(J\left(\overline{w}^h\right) - J(\overline{w})\right) \le ch^{\min(2\gamma, 1-d/t)},$$

where we also used (4.8). □

Remark 1 The analysis in the appendix shows that possible choices for the constants γ, γ' appearing in Theorem 2 and in the subsequence results are

$$\gamma = \frac{1 - d/t - \epsilon}{r - 1} \quad \text{and} \quad \gamma' = 1 - d/t,$$

for any $\epsilon > 0$. We have deliberately not included these explicit formulas in the convergence results above since we have no reason to believe that these estimates are optimal.

We note, however, that $2\gamma = \frac{2}{r-1}(1 - d/t - \epsilon)$. Thus, if $d = 2$ then choosing $r < 3$ allows us to recover the rate $1 - d/t$. If $d = 3$ then choosing $r = 3 + \epsilon$ gives $2\gamma = 1 - d/t - \epsilon'$ for some $\epsilon' > 0$ which tends to zero as $\epsilon \to 0$.

Appendix A: Proof of Theorem 2

We begin by deriving additional regularity for the optimal control $\overline{q}$, given as solution to (2.3), from the first order necessary optimality conditions (cf. [6]).

Theorem 4 *Let $(\overline{q}, \overline{u}) \in L^r(\Omega) \times H_0^1(\Omega) \cap W^{2,t}(\Omega)$ be the solution to (2.3), with $d < r < \infty, t \in (d, r]$, and $B = \mathrm{Id}$. Then there exist $\overline{z} \in L^{r'}(\Omega)$ and a measure μ with support contained in $\bar{\Omega}$, such that*

$$(\nabla\overline{u}, \nabla\phi) = (\overline{q}, \phi), \qquad \forall \phi \in H_0^1(\Omega), \tag{A.1}$$
$$(\overline{z}, -\Delta\phi) = \left(\overline{u} - u^d, \phi\right) + \langle \nabla\phi\nabla\overline{u}, \mu\rangle_{C,C^*}, \qquad \forall \phi \in H_0^1(\Omega) \cap W^{2,r}(\Omega), \tag{A.2}$$
$$\langle \phi\nabla\overline{u}, \mu\rangle_{C,C^*} \le \langle |\nabla\overline{u}|^2, \mu\rangle_{C,C^*}, \qquad \forall \phi \in C\left(\overline{\Omega}, \mathbb{R}^d\right),\ |\phi| \le 1, \tag{A.3}$$
$$\left(|\overline{q}|^{r-2}\overline{q}, q\right) \le \left(-\overline{z}, q\right), \qquad \forall q \in Q. \tag{A.4}$$

From these necessary optimality conditions we can, in a first step, derive additional regularity for the adjoint state $\overline{z}$. To do so we will employ the K-Method of interpolation (although any other method would do fine). Hence we define fractional-order Sobolev spaces $W^{s,p}$ by Besov-Spaces $B^s_{p,p}$. For details on this see, e.g., [22, Definition 4.2.1] or [1, Chap. 7].

Lemma 1 *The solution $\overline{z}$ of (A.2) belongs to $W^{1-d/t-\epsilon,t'}(\Omega) \subset W^{1-d/t-\epsilon,r'}(\Omega)$, for every $\epsilon > 0$.*

Proof Let $\epsilon > 0$ be given, then

$$\langle \nabla\phi\nabla\overline{u}, \mu\rangle_{C,C^*} \leq \|\overline{u}\|_{C^1}\|\mu\|_{C^*}\|\phi\|_{C^1} \leq C\|\phi\|_{W^{1+d/t+\epsilon,t}}$$

by standard embedding theorems [22, Theorem 4.6.1]. Hence, the right hand side of (A.2) is an element of $W^{-1-d/t-\epsilon,t'}(\Omega)$.

From [8,14], and our assumptions on Ω in §2, we obtain that the mappings

$$\begin{aligned} A_1 &:= -\Delta \ : \ W_0^{1,t}(\Omega) \to W^{-1,t}(\Omega), \quad \text{and} \\ A_2 &:= -\Delta \ : \ W^{2,t}(\Omega) \cap W_0^{1,t}(\Omega) \to L^t(\Omega) \end{aligned}$$

are isomorphisms. Hence, the adjoint operators

$$\begin{aligned} A_1^* &: W_0^{1,t'}(\Omega) \to W^{-1,t'}(\Omega), \quad \text{and} \\ A_2^* &: L^{t'}(\Omega) \to W^{-2,t'}(\Omega) \end{aligned}$$

are isomorphisms as well. By interpolation we obtain [22, Theorems 4.6.1, 4.8.2], that

$$W^{-1-d/t-\epsilon,t'}(\Omega) = \left(W^{-1,t'}(\Omega), W^{-2,t'}(\Omega)\right)_{d/t+\epsilon,t'},$$

and hence that [22, Theorem 1.3.3]

$$\overline{z} \in \left(W_0^{1,t'}(\Omega), L^{t'}(\Omega)\right)_{d/t+\epsilon,t'} = W^{1-d/t-\epsilon,t'}(\Omega).$$

This concludes the proof. □

Lemma 1 and (A.4) together provide us with a regularity result for $R'(\overline{q}) = |\overline{q}|^{r-2}\overline{q}$, and therefore the following convergence rate for the projection in Corollary 2.

Corollary 4 *Given any $\epsilon > 0$, then $R'(\overline{q})$ belongs to $W^{\gamma',r'}(\Omega)$ where $\gamma' = 1 - d/t - \epsilon$.*

Proof The result follows from the fact that $R'(\overline{q}) = |\overline{q}|^{r-2}\overline{q} = -\overline{z}$. □

Setting $\psi(g) = \text{sign}(g)|g|^{1/(r-1)}$ it follows that $\psi(R'(q)) = q$, and hence we can deduce regularity of $\overline{q}$ from regularity of $R'(\overline{q})$.

Lemma 2 *Let $f \in W^{s,r'}(\Omega)$, where $s < 1$ and $r > d$, then*

$$\text{sign}(f)|f|^{\frac{1}{r-1}} \in W^{\frac{s}{r-1},r}(\Omega).$$

Proof The result follows from the fact that the function $\psi(g) = \operatorname{sign}(g)|g|^\alpha$ belongs to $C^{0,\alpha}(\mathbb{R})$, more precisely, that it satisfies the Hölder condition

$$\sup_{\substack{g_1,g_2\in\mathbb{R}\\ g_1\neq g_2}} \frac{|\psi(g_1)-\psi(g_2)|}{|g_1-g_2|^\alpha} \leq 2. \tag{A.5}$$

The stated result follows if we show, setting $\alpha = 1/(r-1)$ in the definition of ψ, that $\psi \circ f \in W^{\frac{s}{r-1},r}(\Omega)$. To this end, we need to show that $\psi \circ f \in L^r(\Omega)$ (this is easy to see), and that the semi-norm

$$|\psi \circ f|^r_{\frac{s}{r-1},r} = \int_\Omega \int_\Omega \frac{|\psi(f(x))-\psi(f(y))|^r}{|x-y|^{d+\frac{s}{r-1}r}}\, dx\, dy$$

is finite. Using the Hölder-condition (A.5) we estimate

$$|\psi \circ f|^r_{W^{\frac{s}{r-1},r}} \leq 2^r \int_\Omega \int_\Omega \frac{|f(x)-f(y)|^{\frac{r}{r-1}}}{|x-y|^{d+s\frac{r}{r-1}}} dxdy = 2^r |f|^{r'}_{s,r'},$$

which is finite due to our assumption that $f \in W^{s,r'}(\Omega)$. □

We can now formulate the desired regularity result for the optimal control $\overline{q}$.

Corollary 5 *For any* $\epsilon > 0$*, the optimal control* $\overline{q}$ *belongs to the space* $W^{\gamma,r}$*, where* $\gamma = (1-d/t-\epsilon)/(r-1)$.

Proof Recall from the proof of Corollary 4 that $|\overline{q}|^{r-2}\overline{q} \in W^{1-d/t-\epsilon,r'}(\Omega)$. Applying Lemma 2 with $f = |\overline{q}|^{r-2}\overline{q}$, and $s = 1-d/t-\epsilon$, we obtain that $\overline{q} \in W^{\frac{1}{r-1}(1-d/t-\epsilon),r}(\Omega)$ which establishes the stated regularity. □

Finally, we note that

$$\gamma + \gamma' = \left(1 + \frac{1}{r-1}\right)\left(1 - \frac{d}{t} - \epsilon\right) \geq 1 - \frac{d}{t}$$

if ϵ is chosen sufficiently small. This concludes the proof of Theorem 2.

Remark 2 We note that Corollary 5 shows that the convergence orders obtained in this paper, as well as in [9,15,17], namely $O(h^{\frac{1-d/t}{r}})$, are not of optimal order for the control variable.

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