Uniqueness and Nonuniqueness of Viscosity Solutions to Aronsson's Equation

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Uniqueness and nonuniqueness of viscosity solutions to Aronsson's equation

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ABSTRACT. For a bounded domain $\Omega \subset \mathbf{R}^n$ and $g \in C(\partial\Omega)$, assume that $H \in C^2(\mathbf{R}^n)$ is convex, coercive, and $\{p \in \mathbf{R}^n : H(p) = \min_{\mathbf{R}^n} H\}$ has no interior points. Then we establish the uniqueness for viscosity solutions to the Dirichlet problem of Aronsson's equation:

$$\mathcal{A}[u] := H_p(\nabla u) \otimes H_p(\nabla u) : \nabla^2 u = 0, \text{ in } \Omega$$
$$u = g, \text{ on } \partial\Omega.$$

For $H = H(p, x)$ depending on x, we illustrate the connection between uniqueness & nonuniqueness of Aronsson's equation and that of Hamilton-Jacobi equation $H(\nabla u, x) = c$.

§1 Introduction

Calculus of variations in L^∞ was initiated by Aronsson [A1,2,3] in 1960's. Because of both analytic difficulty and many potential applications, there have been great interests by many people to investigate this subject in the last few years. The basic problem is to study the minimization of the L^∞-functional:

$$F(u, \Omega) \equiv \text{esssup}_{x\in\Omega} H(\nabla u(x), u(x), x), \ \Omega \subset \mathbf{R}^n, \quad u \in W^{1,\infty}(\Omega). \tag{1.1}$$

As evident by studies on the model case $H(p) = |p|^2$ by [A1,2,3], Jensen [J], and Crandall-Evans-Gariepy [CEG] and many others, the correct notation of minimizers for

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L^∞-functional is *absolute minimizer* (there is an excellent survey article by Aronsson-Crandall-Juutinen [ACJ] on its recent development). More precisely, $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ is an absolute minimizer of H, if

$$F(u,U) \le F(v,U),\ \forall\, U \subset\subset \Omega,\ v \in W^{1,\infty}(U) \text{ with } v|_{\partial U} = u|_{\partial U}. \tag{1.2}$$

Through Barron-Jensen-Wang [BJW], it is known that the natural condition for the existence of absolute minimizer is quasiconvexity (or level set convexity) of $H(\cdot, z, x)$: $\{p \in \mathbf{R}^n : H(p,z,x) \le c\}$ *is convex, for any* $c \in \mathbf{R}$ *and* $(z,x) \in \mathbf{R} \times \Omega$. It is shown by [BJW] and Crandall [C] that if $H(p,z,x) \in C^2(\mathbf{R}^n \times \mathbf{R} \times \Omega)$ is quasiconvex in p-variable, then any absolute minimizer $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ is a viscosity solution to Aronsson's equation:

$$\mathcal{A}[u] := \nabla_x(H(\nabla u(x), u(x), x)) \cdot H_p(\nabla u(x), u(x), x) = 0, \ \text{ in } \Omega. \tag{1.3}$$

The readers can refer to Crandall-Ishii-Lions [CIL] for the background on the theory of viscosity solutions. Very recently, Crandall-Wang-Yu [CWY] extend the above result of [BJW] and [C] to $H \in C^1(\mathbf{R}^n \times \Omega)$. Moreover, Yu [Y] has showed that if $H(p,z,x) \in C^2(\mathbf{R}^n \times \mathbf{R} \times \Omega)$ is convex in p-variable, then any viscosity solution of Aronsson's equation (1.3) is an absolute minimizer of H. For a quasiconvex Hamiltonian $H = H(p) \in C^2(\mathbf{R}^n)$, inspired by [CEG], Gariepy-Wang-Yu [GWY] have developed *the comparison principle of generalized cones* $C_k^H(\cdot)$ for both absolute minimizers of H and viscosity solutions of Aronsson's equation (see §2 below).

Among many outstanding issues on Aronsson's equation (1.3), the uniqueness and regularity issue of its viscosity solutions are most challenging problems. Notice that for the model case $H(p) = |p|^2$, Aronsson's equation (1.3) reduces to the infinity Laplace equation:

$$\Delta_\infty u = \nabla u \otimes \nabla u : \nabla^2 u = 0 \ \text{ in } \Omega, \quad u = g \ \text{ on } \partial\Omega. \tag{1.4}$$

In a seminal paper [J], Jensen proved the uniqueness of viscosity solutions to (1.4). See Juutinen [Jp] for some extensions of [J]. Later, Barles-Busca [BB] found an alternative proof of Jensen's uniqueness theorem. Very recently, Crandall-Gunnarsson-Wang [CGW] rediscovered another proof based on the new observation of an approximation of infinity subharmonic functions that allows one to assume that gradients are bounded away from zero. Moreover, this observation enables [CGW] to extend Jensen's uniqueness of infinity harmonic functions to some unbounded domains and with respect to other norms s well. We would like to point out that a similar observation has also been employed by Barron-Jensen [BJ] on absolute minimizing functions with an Dirichlet energy constraint. For the regularity issue, Savin [S] and Evans-Savin [ES] have proved that any infinity harmonic function in $\mathbf{R}^2$ is $C^{1,\alpha}$ for some $0 < \alpha < 1$. Wang-Yu [WY], based on some extensions of ideas by [S] and the technique of discrete gradient flow for Aronsson's equations, proved the C^1-regularity for viscosity solutions to Aronsson's equation in $\mathbf{R}^2$ for any uniformly convex H.

In this paper, we are mainly interested in the uniqueness issue of Aronsson's equations. First, let's recall the definition of viscosity solutions to Aronsson's equation.

Definition 1.1. A function $u \in C(\Omega)$ is a viscosity sub(super)solution of Aronsson's equation (1.3), if for any $(x_0, \phi) \in \Omega \times C^2(\Omega)$, with

$$0 = (u - \phi)(x_0) \geq (\leq)(u - \phi)(x), \ \forall x \in \Omega, \tag{1.5}$$

then

$$\mathcal{A}[\phi](x_0) := \nabla_x(H(\nabla\phi, \phi, x)) \cdot H_p(\nabla\phi, \phi, x)|_{x=x_0} \geq (\leq)0. \tag{1.6}$$

A $u \in C(\Omega)$ is a viscosity solution of Aronsson's equation (1.3), if it is both viscosity subsolution and supersolution of (1.3). Notice that it is easy to check that u is a viscosity supersolution of Aronsson's equation (1.3), if $-u$ is a viscosity subsolution of (1.3) with H replaced by $\hat{H}(p, z, x) = H(-p, -z, x)$.

Now we are ready to state the uniqueness theorem.

Theorem 1.2. *For any bounded domain $\Omega \subset \mathbf{R}^n$ and $g \in C(\partial\Omega)$, assume that $H \in C^2(\mathbf{R}^n)$ satisfies*

(A1) H is convex, i.e., $\nabla^2 H(p) \geq 0$ for any $p \in \mathbf{R}^n$,

(A2) H is coercive:

$$\lim_{|p| \to +\infty} H(p) = +\infty, \tag{1.7}$$

(A3) the minimal level set, $\{p \in \mathbf{R}^n : H(p) = \min H\}$, has no interior points.

Then there exists a unique, viscosity solution $u \in C(\overline{\Omega})$ to:

$$\mathcal{A}[u] := \sum_{i,j=1}^{n} H_{p_i}(\nabla u) H_{p_j}(\nabla u) u_{ij} = 0, \quad \text{in } \Omega, \tag{1.8}$$

$$u = g, \quad \text{on } \partial\Omega. \tag{1.9}$$

We would like to point out that by suitably modifying H near infinity, one can always assume that H satisfies (A2). Notice that both the arguments by [J] and [BB] don't seem to be applicable for the proof of theorem 1.2. In fact, the proof of [J] relies heavily on the homogenity of $H(p) = |p|^2$, while the proof of [BB] depends on that a strong type maximum principle for viscosity solutions to the PDE under consideration. This is not true for our case here. For example, let $H(p_1, p_2) = |p_1|^2 : \mathbf{R}^2 \to \mathbf{R}$. Then it is easy to see that $u(x_1, x_2) = -x_2^2$ is a smooth solution of the corresponding Aronsson equation, which is not locally constant but has a local maximum point.

From [GWY], we know that for any viscosity subsolution u of (1.8), a upper semicontinuous representation of $H(\nabla u(x))$, $S^+(H, u, x)$, exists for any $x \in \Omega$. There are several crucial ingredients in the proof of theorem 1.2. First, for any $\epsilon > 0$, we apply the Perron's method to solve the Hamilton-Jacobi equation in the open set $\Omega_\epsilon = \{x \in \Omega : S^+(H, u, x) < \epsilon\}$:

$$\epsilon - H(\nabla v_\epsilon) = 0, \text{ in } \Omega_\epsilon, \quad v_\epsilon = u, \text{ on } \partial\Omega_\epsilon. \tag{1.10}$$

Second, we approximate any viscosity subsolution u of (1.8) by $u_\epsilon : \Omega \to \mathbf{R}$:

$$u_\epsilon = u \text{ in } \Omega \setminus \Omega_\epsilon, \; u_\epsilon = v_\epsilon \text{ in } \Omega_\epsilon. \tag{1.11}$$

Third, we show that u_ϵ is also a viscosity subsolution of (1.8), which is highly nontrivial to achieve for $x_0 \in \partial\Omega_\epsilon \cap \Omega$. To do it, we modify the argument by [CWY] to show that for any upper test function $\phi \in C^2(\Omega)$ of u_ϵ at x_0, there exists $x_r \in \partial B_r(x_0) \cap (\Omega \setminus \Omega_\epsilon)$ such that

$$\phi(x_r) - \phi(x_0) \geq C^H_{H(\nabla\phi(x_0))}(x_r - x_0). \tag{1.12}$$

This turns out to be sufficient.

The uniqueness issue is much more complicated for Aronsson's equation (1.3) when $H(p, x)$ depends on x. In general, it may fail. Here is an example pointed out by [Y]. For $H(p, x) = |p|^2 + \sin^2 x : \mathbf{R} \times [0, 2\pi] \to \mathbf{R}$, then $u_1 = 0$ and $u_2(x) = \sin x$ are smooth solutions to Aronsson's equation:

$$u'(u'u'' + \sin x \cos x) = 0 \text{ in } [0, 2\pi], \; u(0) = u(2\pi) = 0.$$

However, we believe that the following should be true.

Conjecture 1.3. *For any bounded domain $\Omega \subset \mathbf{R}^n$ and $g \in C(\partial\Omega)$, let $H(p, x) \in C^2(\mathbf{R}^n \times \Omega) \cap C(\mathbf{R}^n \times \overline{\Omega})$ satisfy (1) $H(\cdot, x)$ is convex for any $x \in \overline{\Omega}$, and (2) $0 = H(0, x) < H(p, x)$ for any $0 \neq p \in \mathbf{R}^n$ and $x \in \overline{\Omega}$. Then there exists a unique viscosity solution $u \in C(\overline{\Omega})$ to Aronsson's equation:*

$$\mathcal{A}[u] := \sum_{i,j=1}^n H_{p_i}(\nabla u, x) H_{p_j}(\nabla u, x) u_{ij} + \sum_{i=1}^n H_{p_i}(\nabla u, x) H_{x_i}(\nabla u, x) = 0 \text{ in } \Omega, \tag{1.13}$$

$$u = g \text{ on } \partial\Omega. \tag{1.14}$$

A plausible approach to attack the conjecture is to first modify the proof of theorem 1.2 to obtain a viscosity subsolution u_ϵ of (1.13) such that $H_p(\nabla u_\epsilon, x)\cdot\nabla u_\epsilon \geq c(\epsilon) > 0$, and then use the super-convolution to deform such a subsolution to a *strict* subsolution. Here speical cares need to be taken, due to the presence of x-variable. We plan to investigate Conjecture 1.3 in the future.

For $H(p, x)$ that may not satisfy the condition (2), we show in §5 that there are interesting connections between the nonuniqueness of Aronsson's equation and that of viscosity solution to Hamilton-Jacobi equation:

$$H(\nabla u(x), x) = c_0 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega, \tag{1.15}$$

where

$$c_0 = \inf_{\phi \in C(\overline{\Omega}) \cap C^1(\Omega)} \sup_{x \in \Omega} H(\nabla\phi, x). \tag{1.16}$$

More precisely, we show in §5

Theorem 1.4. *If $u \in C(\overline{\Omega})$ is a viscosity solution of (1.15), then there exists a viscosity solution $U = T(u) \in C(\overline{\Omega})$ of (1.13)-(1.14). Moreover, if $u_1 \neq u_2 \in C(\overline{\Omega})$ are two different solutions of (1.15), then $U_1 = T(u_1) \neq U_2 = T(u_2)$.*

In §5, utilizing theorem 1.4, we provide three typical examples in which Aronsson's equation has infinitely many solutions.

The paper is organized as follows. In section 2, we review and collect some preliminary results on Aronsson's equations. In section 3, we provide the construction of Hamilton-Jacobi equation as a nondegenerate approximation of subsolutions of Aronsson's equation. In section 4, we prove theorem A. In section 5, we discuss nonuniqueness of Aronsson's equation for Hamilton functions depending on x-variables, and its connection to Hamilton-Jacobi equations.

§2 Some preliminary properties of Aronsson's equations

This section is devoted to some basic properties of viscosity solutions to Aronsson's equations that are needed in this paper. We refer the readers to [GWY] for more details.

Throughout this section and §3, the Hamiltonian H is assumed to satisfy:

(**H1**) $H \in C^2(\mathbf{R}^n)$ is nonnegative, and $H(0) = 0$.

(**H2**) H is quasiconvex:

$$H(tp + (1-t)q) \leq \max\{H(p), H(q)\}, \ \forall p, q \in \mathbf{R}^n, \ 0 \leq t \leq 1. \tag{2.1}$$

(**H3**) H is coercive:

$$\lim_{|p|\to+\infty} H(p) = +\infty. \tag{2.2}$$

(**H4**) For any $c \geq 0$, $\{p \in \mathbf{R}^n \mid H(p) = c\}$ has no interior points.

First we would like to point out that for any $H \in C^2(\mathbf{R}^n)$ satisfying **H2-H3**, one can check that if $p_0 \in \mathbf{R}^n$ satisfies $H(p_0) = \min_{p\in\mathbf{R}^n} H(p)$, then $\hat{H}(p) = H(p_0 + p) - H(p_0)$, $p \in \mathbf{R}^n$, satisfies **H1-H3**. Therefore, without loss of generality, we may always assume throughout this paper that H satisfies **H1**.

Definition 2.1. For $k \geq 0$, a generalized cone centered at 0 with slope k, $C_k^H(x)$, is defined by

$$C_k^H(x) = \max_{\{H(p)=k\}} \langle p, x\rangle,\ x \in \mathbf{R}^n. \tag{2.3}$$

It is proved by [GWY] that if H satisfies (**H1-H3**), then $C_k^H(x)$ is Lipschitz continuous, convex, positively homogeneous of degree 1, and satisfies the triangle inequality:

$$C_k^H(x+y) \leq C_k^H(x) + C_k^H(y),\ \forall x, y \in \mathbf{R}^n. \tag{2.4}$$

Definition 2.2. A upper semicontinuous function $u \in \mathrm{USC}(\Omega)$ enjoys comparison with generalized cones from above in Ω (or abbreviated $u \in CGCA(\Omega)$) if, for any $V \subset\subset \Omega$, $x_0 \in \Omega$, and $k \geq 0$,

$$u(x) \leq u(x_0) + C_k^H(x - x_0),\ \forall x \in \partial V, \tag{2.5}$$

then

$$u(x) \leq u(x_0) + C_k^H(x - x_0),\ \forall x \in V. \tag{2.6}$$

The following two theorems by [GWY] will be needed throughout this paper. The first the upper semicontinuous representation of $H(\nabla u)$ for viscosity subsolutions u of Aronsson's equation.

Theorem 2.3. *Under the assumptions (***H1-H4***). If $u \in C(\Omega) \cap CGCA(\Omega)$, then $u \in W^{1,\infty}_{loc}(\Omega)$, and*
(i) For any $x_0 \in \Omega$, $0 < r < d(x_0, \partial\Omega)$,

$$S_r^+(H, u, x_0) = \inf\{k \geq 0 \mid u(x) \leq u(x_0) + C_k^H(x - x_0),\ \forall x \in \partial B_r(x_0)\} \tag{2.7}$$

exists and is monotonically nondecreasing with respect to r. In particular

$$S^+(H, u, x_0) = \lim_{r\to 0} S_r^+(H, u, x_0) \tag{2.8}$$

exists and is upper semicontinuous.

(ii) If $x_0 \in \Omega$ is a differentiable point of u, then

$$S^+(H,u,x_0) = H(\nabla u(x_0)). \tag{2.9}$$

In particular,

$$\|H(\nabla u)\|_{L^\infty(U)} = \max_{x\in U} S^+(H,u,x),\ \forall U \subset \Omega. \tag{2.10}$$

(iii) For $x_0 \in \Omega$, $0 < r < d(x_0, \partial\Omega)$, if $x_r \in \partial B_r(x_0)$ satisfies

$$u(x_r) - u(x_0) = C^H_{S^+(H,u,x_0)}(x_r - x_0)$$

then

$$S^+(H,u,x_r) \geq S^+_r(H,u,x_0). \tag{2.11}$$

Proof. See [GWY] Proposition 3.1, Proposition 3.3, and Proposition 3.4. ∎

The property (iii) in Theorem 2.3 is called an *end point estimate.* The next is the equivalence between viscosity subsolutions of Aronsson's equation and CGCA(Ω), namely,

Theorem 2.4. *Under the assumptions (**H1-H4**). Then the following statements are equivalent:*

(a) $u \in C(\Omega)$ is a viscosity subsolution of Aronsson's equation (1.8).

(b) $u \in C(\Omega)$ is in CGCA(Ω).

Proof. See [GWY] §5. ∎

§3. Hamilton-Jacobi equation and viscosity subsolutions of Aronsson's equation

The major difficulty to show uniqueness for viscosity solutions to Aronsson's equation is that $S^+(H,u,\cdot)$, the upper semicontinuous representive of $H(\nabla u)$, may vanish somewhere. To overcome this difficulty for the infinity Laplace equation, Jensen [J] constructed subsolution approximations that have nondegenerate gradients by employing a L^p approximation scheme, namely,

$$\max\{-\Delta_\infty v, \epsilon - |\nabla v|\} = 0, \text{ in } \Omega.$$

Very recently, Crandall-Gunnarsson-Wang [CGW] (cf. also Crandall [C]) and Barron-Jensen [BJ] discovered another approximation scheme by solving the Eikonal equation in the domain where the solution has degenerate gradients. Here we undergo a similar surgery procedure for Aronsson's equations.

Throughout this section, H satisfies the assumptions (**H1-H4**), and $u \in C(\overline{\Omega}) \cap W^{1,\infty}_{\text{loc}}(\Omega)$ is a viscosity subsolution to Aronsson's equation (1.8).

For any $\epsilon > 0$, define

$$\Omega_\epsilon := \{x \in \Omega : S^+(H, u, x) < \epsilon\}, \quad T_\epsilon := \Omega \setminus \Omega_\epsilon = \{x \in \Omega : S^+(H, u, x) \geq \epsilon\}.$$

Theorem 2.3 implies that T_ϵ is a closed subset of Ω and hence Ω_ϵ is an open subset of Ω.

For $\epsilon > 0$, assume $\Omega_\epsilon \neq \emptyset$. Define

$$\Gamma_\epsilon = \{v \in W^{1,\infty}(\Omega_\epsilon) \cap C(\overline{\Omega}_\epsilon) \mid \|H(\nabla v)\|_{L^\infty(\Omega_\epsilon)} \leq \epsilon, \quad v|_{\partial\Omega_\epsilon} = u|_{\partial\Omega_\epsilon}\}. \tag{3.1}$$

It is clear that $u \in \Gamma_\epsilon$. We define $v_\epsilon : \overline{\Omega}_\epsilon \to \mathbf{R}$ by

$$v_\epsilon(x) = \inf\{v(x) \mid v \in \Gamma_\epsilon\}, \ x \in \overline{\Omega}_\epsilon. \tag{3.2}$$

It is clear that $v_\epsilon = u$ on $\partial\Omega_\epsilon$. Moreover, since $u \in \Gamma_\epsilon$, we have $v_\epsilon \leq u$ on $\overline{\Omega}_\epsilon$.

Now we have

Proposition 3.1. *Under the assumptions (***H1-H4***), assume that $u \in C(\overline{\Omega}) \cap W^{1,\infty}_{loc}(\Omega)$ is a viscosity subsolution of Aronsson's equation (1.8). For $\epsilon > 0$, assume that $\Omega_\epsilon \neq \emptyset$. Then*

(1) $v_\epsilon \in \Gamma_\epsilon$.

(2) v_ϵ is a viscosity solution of the Hamiltonian-Jacobi equation:

$$\epsilon - H(\nabla\psi) = 0 \quad \textit{in } \Omega_\epsilon, \ \psi = u \quad \textit{on } \partial\Omega_\epsilon. \tag{3.3}$$

(3) v_ϵ is a viscosity subsolution of Aronsson's equation (1.8) in Ω_ϵ.

Proof. It follows from **H3** that there is a constant $K_\epsilon > 0$ such that

$$\|\nabla v\|_{L^\infty(\Omega_\epsilon)} \leq K_\epsilon, \ \forall v \in \Gamma_\epsilon. \tag{3.4}$$

By the definition (3.2), this implies

$$\|\nabla v_\epsilon\|_{L^\infty(\Omega_\epsilon)} \leq K_\epsilon.$$

To show $v_\epsilon \in C(\overline{\Omega}_\epsilon)$. Assume $x \in \Omega_\epsilon$ and $y \in \partial\Omega_\epsilon$. Let $\bar{x} \in \partial\Omega_\epsilon$ be such that $|x - \bar{x}| = \text{dist}(x, \partial\Omega_\epsilon)$. Then we have

$$\begin{aligned} |v_\epsilon(x) - v_\epsilon(y)| &\leq |v_\epsilon(x) - v_\epsilon(\bar{x})| + |v_\epsilon(\bar{x}) - v_\epsilon(y)| \\ &\leq K_\epsilon|x - \bar{x}| + |u(\bar{x}) - u(y)| \\ &\leq K_\epsilon|x - y| + |u(\bar{x}) - u(y)|, \end{aligned} \tag{3.5}$$

where we have used the fact that $|x-\bar{x}| \le |x-y|$ in the last step. Hence we have, for any $y \in \partial\Omega_\epsilon$,

$$\lim_{x\in\Omega_\epsilon, x\to y} |v_\epsilon(x) - v_\epsilon(y)| = 0.$$

$\|H(\nabla v_\epsilon)\|_{L^\infty(\Omega_\epsilon)} \le \epsilon$ follows from **H2**. In fact, it is not hard to show that there exist $\{v_i\} \subset \Gamma_\epsilon$ such that $v_i \to v_\epsilon$ uniformly on $\overline{\Omega}_\epsilon$, and $\nabla v_i \to \nabla v_\epsilon$ weak* in $L^\infty(\Omega_\epsilon)$. Hence, by the lower semicontinuity, we then have

$$\|H(\nabla v_\epsilon)\|_{L^\infty(\Omega_\epsilon)} \le \liminf_i \|H(\nabla v_i)\|_{L^\infty(\Omega_\epsilon)} \le \epsilon. \tag{3.6}$$

Hence $v_\epsilon \in \Gamma_\epsilon$.

(2) is obvious. Here, for the completeness, we sketch it. It suffices to prove v_ϵ is a viscosity subsolution of (3.3). For this, let $(x_0, \phi) \in \Omega_\epsilon \times C^1(\Omega_\epsilon)$ be such that

$$0 = (v_\epsilon - \phi)(x_0) < (v_\epsilon - \phi)(x),\ \forall x \in \Omega_\epsilon \setminus \{x_0\}. \tag{3.7}$$

Suppose that $H(\nabla\phi)(x_0) < \epsilon$. Then there exists a $\delta_0 > 0$ such that $H(\nabla\phi)(x) < \epsilon$ for any $x \in B_{\delta_0}(x_0)$. Observe that if $\delta_1 > 0$ is sufficiently small, then there exists an open set $V_1 \subset B_{\delta_0}(x_0)$ such that $v_\epsilon > \phi - \delta_1$ in V_1 and $v_\epsilon = \phi - \delta_1$ on V_1. Define $w_\epsilon : \overline{\Omega}_\epsilon \to R$ by

$$w_\epsilon(x) = (\phi - \epsilon)(x),\ x \in V_1 \tag{3.8}$$

$$= v_\epsilon(x),\ x \in \overline{\Omega}_\epsilon \setminus V_1. \tag{3.9}$$

Then we have $w_\epsilon \in \Gamma_\epsilon$. Hence we have

$$v_\epsilon(x_0) \le w_\epsilon(x_0) = \phi(x_0) - \delta_1 < v_\epsilon(x_0). \tag{3.10}$$

This yields a desired contradiction. Hence $\epsilon - H(\nabla\phi)(x_0) \le 0$ and v_ϵ is a viscosity subsolution of (3.3).

To prove (3), first recall the super-convolution of v_ϵ (cf. [CIL]). For any $\delta > 0$, let

$$v_\epsilon^\delta(x) = \sup_{y\in\overline{\Omega}_\epsilon} \{v_\epsilon(y) - \frac{1}{2\delta}|x-y|^2\},\ \forall x \in \overline{\Omega}_\epsilon. \tag{3.11}$$

For any $\tau > 0$, define $\Omega_\epsilon^\tau = \{x \in \Omega_\epsilon : \mathrm{d}(x, \partial\Omega_\epsilon) > \tau\}$. We have

Lemma 3.2. *Under the same conditions as in Proposition 3.1, for any $\tau > 0$ there exists $\delta_0 = \delta_0(\tau) > 0$ such that for any $0 < \delta < \delta_0$, if v_ϵ^δ is differentiable at $x_0 \in \Omega_\epsilon^\tau$, then*

$$H(\nabla v_\epsilon^\delta)(x_0) = \epsilon. \tag{3.12}$$

In particular, if v_ϵ^δ is twice differentiable at $x_0 \in \Omega_\epsilon^\tau$, then

$$\mathcal{A}[v_\epsilon^\delta](x_0) = 0. \tag{3.13}$$

Proof. The proof of (3.12) is similar to [GWY] Lemma 2.2. For convenience of readers, we outline it as follows. For $x_0 \in \Omega_\epsilon^\tau$, denote $p = \nabla v_\epsilon^\delta(x_0)$. For any $\alpha > 0$, there is $r_0 = r_0(\tau) > 0$ such that $B_{r_0}(x_0) \subset \Omega_\epsilon^{\frac{\tau}{2}}$ and

$$v_\epsilon^\delta(x) < v_\epsilon^\delta(x_0) + \langle p, x - x_0\rangle + \alpha(\sqrt{2} - 1)r_0,\ \forall x \in B_{r_0}(x_0). \tag{3.14}$$

By the definition, there exists $y_0 \in \overline{\Omega}_\epsilon$ such that

$$v_\epsilon^\delta(x_0) = v_\epsilon(y_0) - \frac{1}{2\delta}|x_0 - y_0|^2 (\geq v_\epsilon(x_0)). \tag{3.15}$$

It is easy to see that $|x_0 - y_0| \leq 2\delta\|\nabla v_\epsilon\|_{L^\infty(\overline{\Omega}_\epsilon)} \leq C\delta$. Therefore there is a $0 < \delta_0 \leq \frac{\tau}{2C}$ such that $|y_0 - x_0| \leq \frac{\tau}{2}$ and $y_0 \in \Omega_\epsilon^{\frac{\tau}{2}}$. It follows from (3.14) and (3.15) that for any $x \in B_{r_0}(x_0)$ and $y \in \Omega_\epsilon^\tau$,

$$v_\epsilon(y) + \frac{1}{2\delta}|x - y|^2 < v_\epsilon(y_0) - \frac{1}{2\delta}|x_0 - y_0|^2 + \langle p, x - x_0\rangle + \alpha(\sqrt{2} - 1)r_0. \tag{3.16}$$

Notice that $y \in \partial B_{r_0}(y_0)$ implies $x = x_0 - y_0 + y \in \partial B_{r_0}(x_0)$. Hence (3.16) gives

$$v_\epsilon(y) < \phi_\epsilon(y) \equiv v_\epsilon(y_0) + \langle p, y - y_0\rangle + \alpha(\sqrt{r_0^2 + |y - y_0|^2} - r_0),\ \forall y \in \partial B_{r_0}(y_0). \tag{3.17}$$

Since $v_\epsilon(y_0) = \phi_\epsilon(y_0)$, it follows that there exists $y_* \in B_{r_0}(y_0)$ such that

$$(v_\epsilon - \phi_\epsilon)(y_*) = \max_{y \in B_{r_0}(y_0)} (v_\epsilon(y) - \phi_\epsilon(y)). \tag{3.18}$$

Since v_ϵ is a viscosity subsolution of (3.3), we have

$$H(\nabla \phi_\epsilon(y_*)) \geq \epsilon. \tag{3.19}$$

By direct calculations, we have

$$\nabla \phi_\epsilon(y_*) = p + \frac{\alpha(y_* - y_0)}{\sqrt{|y_* - y_0|^2 + r_0^2}}, \tag{3.20}$$

and hence

$$H(p + \frac{\alpha(y_* - y_0)}{\sqrt{|y_* - y_0|^2 + r_0^2}}) \geq \epsilon. \tag{3.21}$$

Taking α to zero, this implies $H(\nabla v_\epsilon^\delta(x_0)) \geq \epsilon$.

To prove another direction, notice that (3.18) imply that for $v \in \mathbf{R}^n$ with $|v| = 1$ and $0 < t \leq \frac{r_0}{2}$,

$$\phi_\epsilon(y_*) - \phi_\epsilon(y_* - tv) \leq v_\epsilon(y_*) - v_\epsilon(y_* - tv). \tag{3.22}$$

Since ∇v_ϵ exists a.e. in Ω_ϵ^τ, $H(\nabla v_\epsilon) = \epsilon$ a.e. in Ω_ϵ^τ. Hence $\|H(\nabla v_\epsilon)\|_{L^\infty(\Omega_\epsilon^\tau)} = \epsilon$. This, combined with **H2** and standard approximations, implies

$$v_\epsilon(y_*) - v_\epsilon(y_* - tv) \leq tC_\epsilon^H(v). \tag{3.23}$$

Therefore we have

$$\frac{\phi_\epsilon(y_*) - \phi_\epsilon(y_* - tv)}{t} \leq C_\epsilon^H(v). \tag{3.24}$$

By taking t into zero, this implies

$$\langle \nabla\phi_\epsilon(y_*), v\rangle \leq C_\epsilon^H(v), \ \forall v \in \mathbf{R}^n \text{ with } |v| = 1.$$

Hence $H(\nabla\phi_\epsilon(y_*)) \leq \epsilon$. Taking α into zero, this yields $H(\nabla v_\epsilon^\delta(x_0)) \leq \epsilon$. This proves (3.12).

If v_ϵ^δ is twice differentiable at $x_0 \in \Omega_\epsilon^\tau$, then v_ϵ is differentiable for any x near x_0. Hence, by (3.12) we have $H(\nabla v_\epsilon^\delta)(x) = \epsilon$ for x near x_0. Taking one derivative of this equation, we have

$$H_p(\nabla v_\epsilon^\delta) \cdot \nabla^2 v_\epsilon^\delta|_{x=x_0} = 0.$$

This clearly implies (3.13). The proof of Lemma 3.2 is complete. ■

Remark 3.3. It is well-known (cf. [CIL]) that v_ϵ^δ is semiconvex and hence is twice differentiable for a.e. $x \in \Omega_\epsilon^\tau$. Therefore (4.13) holds for a.e. $x \in \Omega_\epsilon^\tau$.

To conclude that v_ϵ^δ is a viscosity subsolution of Aronsson's equation (1.8) in Ω_ϵ, we need the following useful lemma (see also [ACJ] Remark 6.7).

Lemma 3.4. *Let $u \in C(\Omega_\epsilon)$ be semiconvex and satisfy that if u is twice differentiable at $x_0 \in \Omega_\epsilon$, then*

$$\mathcal{A}[u](x_0) \geq 0. \tag{3.25}$$

Then u is a viscosity subsolution of Aronsson's equation (1.8) in Ω_ϵ.

Proof. Let $(x_0, \phi) \in \Omega_\epsilon \times C^2(\Omega_\epsilon)$ be such that

$$0 = (u - \phi)(x_0) \geq (u - \phi)(x), \ \forall x \in \Omega_\epsilon.$$

Since $u-\phi$ is semiconvex, Jensen's maximum principle ([J]) on semiconvex functions implies that there exist $\{x_i\}\subset\Omega_\epsilon$, $\{p_i\}\subset\mathbf{R}^n$, with $\lim_{i\to\infty}x_i=x_0$ and $\lim_{i\to\infty}p_i=0$, such that $(\nabla u+p_i-\nabla\phi)(x_i)=0$, and $\nabla^2(u+p_ix-\phi)(x_i)$ exist and

$$\nabla^2(u+p_ix-\phi)(x_i)\le 0.$$

Hence

$$\nabla u(x_i)=\nabla\phi(x_i)+p_i,\quad \nabla^2u(x_i)\le\nabla^2\phi(x_i). \tag{3.26}$$

Since u is twice differentiable at x_i, (3.25) implies

$$\mathcal{A}[u](x_i)\ge 0.$$

Since

$$\mathcal{A}[u](x_i)\le\mathcal{A}[\phi+p_ix](x_i)=H_p(\nabla\phi(x_i)+p_i)\otimes H_p(\nabla\phi(x_i)+p_i):\nabla^2\phi(x_i),$$

we have, by taking i into infinity,

$$\mathcal{A}[\phi](x_0)\ge 0.$$

This implies that u is a viscosity subsolution of Aronsson's equation (1.8) in Ω_ϵ. ■

§4 Nondegenerate approximation of Aronsson's equation and proof of Theorem 1.2

In this section, we first use the viscosity solution to the Hamilton-Jacobi equation constructed by Proposition 3.1 to approximate viscosity subsolutions to Aronsson's equation (1.8). Then we prove theorem 1.2.

For any $\epsilon>0$, construct an approximation u_ϵ for a viscosity subsolution u of (1.8) as follows. Let $v_\epsilon:\overline{\Omega}_\epsilon\to\mathbf{R}$ be defined by (3.2). Define $u_\epsilon:\overline{\Omega}\to\mathbf{R}$ by

$$u_\epsilon(x)=u(x),\quad x\in T_\epsilon, \tag{4.1}$$

$$=v_\epsilon(x),\ x\in\Omega_\epsilon. \tag{4.2}$$

The first result of this section is the following theorem on u_ϵ.

Theorem 4.1. *Under the assumptions (***H1-H4***), and assume that $u\in C(\overline{\Omega})$ is a viscosity subsolution of Aronsson's equation (1.8) and $u_\epsilon\in C(\overline{\Omega})$ is defined by (4.1)-(4.2). Then it holds*

(1) $u_\epsilon\le u$ in Ω, and $u_\epsilon=u$ on $\partial\Omega$.

(2) $S^+(H, u_\epsilon, x)$ *exists for any* $x \in \Omega$*, and*

$$S^+(H, u_\epsilon, x) = S^+(H, u, x), \ \forall x \in T_\epsilon, \tag{4.3}$$

$$= \epsilon, \ \forall x \in \Omega_\epsilon. \tag{4.4}$$

In particular, $S^+(H, u_\epsilon, x) \geq \epsilon$ *for any* $x \in \Omega$*.*

(3) u_ϵ *is a viscosity subsolution of Aronsson's equation (1.8) on* Ω*.*

Proof. (1) follows directly from (1) of Proposition 3.1. For (2), we will show

$$S_r^+(H, u_\epsilon, x) = S_r^+(H, u, x), \ \forall x \in T_\epsilon, \ 0 < r < \mathrm{d}(x, \partial\Omega). \tag{4.5}$$

In fact, it is easy to see from the definition of S^+ and (1) that

$$S_r^+(H, u_\epsilon, x) \leq S_r^+(H, u, x), \ \forall x \in T_\epsilon, \ 0 < r < \mathrm{dist}(x, \partial\Omega).$$

On the other hand, if $x \in T_\epsilon$ and $0 < r < \mathrm{dist}(x, \partial\Omega)$, then there exists $x_r \in \partial B_r(x)$ such that

$$u(x_r) = u(x) + C^H_{S_r^+(H,u,x)}(x_r - x). \tag{4.6}$$

By theorem 2.3, we have

$$S^+(H, u, x_r) \geq S_r^+(H, u, x) \geq S^+(H, u, x) \geq \epsilon. \tag{4.7}$$

Therefore $x_r \in T_\epsilon$, and $u_\epsilon(x_r) = u(x_r)$. Since $u_\epsilon(x) = u(x)$, (4.7) implies

$$u_\epsilon(x_r) = u_\epsilon(x) + C^H_{S_r^+(H,u,x)}(x_r - x). \tag{4.8}$$

This clearly implies that

$$S_r^+(H, u_\epsilon, x) \geq S_r^+(H, u, x)$$

and hence (4.5) follows. Taking r into zero, (4.5) implies (4.3). For (4.4), notice that Proposition 3.1 implies that u_ϵ is a viscosity subsolution of Aronsson's equation on Ω_ϵ. Hence, by theorem 2.3, we have $u_\epsilon \in W^{1,\infty}_{\mathrm{loc}}(\Omega_\epsilon)$ and $S^+(H, u_\epsilon, x)$ exists for any $x \in \Omega_\epsilon$. Since u_ϵ is differentiable for a.e. $x \in \Omega_\epsilon$, (3.3) yields that for a.e. $x \in \Omega_\epsilon$, $H(\nabla u_\epsilon(x)) = \epsilon$. Therefore, by (2.9), we have

$$S^+(H, u_\epsilon, x) = \epsilon, \ \text{for a. e. } x \in \Omega_\epsilon.$$

This, combined with the definition of v_ϵ and the upper semicontinuity of $S^+(H, u_\epsilon, \cdot)$, implies $S^+(H, u_\epsilon, x) = \epsilon$ for any $x \in \Omega_\epsilon$.

For (3), since by Proposition 3.1, v_ϵ is a viscosity subsolution of Aronsson's equation (1.8) on Ω_ϵ, and u is a viscosity subsolution of Aronsson's equation (1.8) on the interior of T_ϵ, it suffices to prove that u_ϵ is a viscosity subsolution of Aronsson's equation (1.8) on $\partial T_\epsilon \cap \Omega = \partial\Omega_\epsilon \cap \Omega$. For any $x_0 \in \partial T_\epsilon \cap \Omega$, let $\phi \in C^2(\Omega)$ satisfy

$$0 = (\phi - u_\epsilon)(x_0) \leq (\phi - u_\epsilon)(x) \quad \forall x \in \Omega.$$

If $H_p(\nabla\phi(x_0)) = 0$, then $\mathcal{A}[\phi](x_0) = 0$ and we are done. Hence we may assume $H_p(\nabla\phi(x_0)) \neq 0$. For $r > 0$, it follows from the definition of $S_r^+(H, u, x_0)$ that there exists $x_r \in \partial B_r(x_0)$ such that

$$u(x_r) = u(x_0) + C^H_{S_r^+(H,u,x_0)}(x_r - x_0). \tag{4.9}$$

Since $S^+(H, u, x_0) = \epsilon$, Theorem 2.3 implies

$$S^+(H, u, x_r) \geq S^+(H, u, x_0) = \epsilon$$

and hence $x_r \in T_\epsilon$ and $u_\epsilon(x_r) = u(x_r)$. Notice also that $u_\epsilon(x_0) = u(x_0) = \phi(x_0)$. Therefore we have

$$\begin{aligned}\phi(x_r) - \phi(x_0) &\geq u_\epsilon(x_r) - u_\epsilon(x_0) = u(x_r) - u(x_0)\\ &= C^H_{S_r^+(H,u,x_0)}(x_r - x_0) \geq C^H_{S^+(H,u,x_0)}(x_r - x_0)\\ &= C^H_\epsilon(x_r - x_0). \end{aligned}\tag{4.10}$$

Now we claim

$$S^+(H, u, x_0) \geq H(\nabla\phi(x_0)). \tag{4.11}$$

In fact, by [CWY] Proposition 2.3, we have

$$\begin{aligned} H(\nabla\phi(x_0)) &\leq \lim_{\delta\downarrow 0} \|H(\nabla u_\epsilon)\|_{L^\infty(B_\delta(x_0))}\\ &= \lim_{\delta\downarrow 0} \max\{\|H(\nabla u)\|_{L^\infty(B_\delta(x_0)\cap T_\epsilon)}, \|H(\nabla v_\epsilon)\|_{L^\infty(B_\delta(x_0)\cap\Omega_\epsilon)}\}\\ &= \lim_{\delta\downarrow 0} \max\{\|H(\nabla u)\|_{L^\infty(B_\delta(x_0)\cap T_\epsilon)}, \epsilon\}\\ &= \lim_{\delta\downarrow 0} \|H(\nabla u)\|_{L^\infty(B_\delta(x_0)\cap T_\epsilon)}\\ &= \lim_{\delta\downarrow 0} \sup_{x\in B_\delta(x_0)\cap T_\epsilon} S^+(H, u, x). \end{aligned}\tag{4.12}$$

Since $S^+(H, u, x)$ is upper semicontinuous, we have

$$\lim_{\delta\downarrow 0} \max_{x\in B_\delta(x_0)\cap T_\epsilon} S^+(H, u, x) \leq S^+(H, u, x_0),$$

this, combined with (4.12), implies (4.11).

It follows from (4.10) and (4.11) that

$$\phi(x_r) - \phi(x_0) \geq C^H_{H(\nabla\phi(x_0))}(x_r - x_0). \tag{4.13}$$

Since $\phi \in C^2(\Omega)$, we have

$$\phi(x_r) - \phi(x_0) = \nabla\phi(x_0) \cdot (x_r - x_0) + O(r^2). \tag{4.14}$$

We may assume that there is a $q \in \mathbf{R}^n$ with $|q| = 1$ such that $\lim_{r\to 0} \frac{x_r - x_0}{|x_r - x_0|} = q$. Hence, by taking r to 0, (4.13) and (4.14) imply

$$\nabla\phi(x_0) \cdot q \geq C^H_{H(\nabla\phi(x_0))}(q). \tag{4.15}$$

In particular,

$$\max_{H(p)=H(\nabla\phi(x_0))} p \cdot q = \nabla\phi(x_0) \cdot q.$$

Hence, by the Lagrange multiple theorem, we have

$$q = \frac{H_p(\nabla\phi(x_0))}{|H_p(\nabla\phi(x_0))|}. \tag{4.16}$$

It also from from (4.13) that there exists $\theta = \theta_r \in (0,1)$ such that

$$\begin{aligned} C^H_{H(\nabla\phi(x_0))}(x_r - x_0) &\leq \nabla\phi(\theta x_r + (1-\theta)x_0) \cdot (x_r - x_0) \\ &\leq C^H_{H(\nabla\phi(\theta x_r + (1-\theta)x_0))}(x_r - x_0). \end{aligned} \tag{4.17}$$

This implies that there exists $\theta_r \in (0,1)$ such that

$$H(\nabla\phi(\theta_r x_r + (1-\theta_r)x_0)) \geq H(\nabla\phi(x_0)). \tag{4.18}$$

Hence

$$\begin{aligned} 0 &\leq \lim_{r\downarrow 0} \frac{H(\nabla\phi(\theta_r x_r + (1-\theta_r)x_0)) - H(\nabla\phi(x_0))}{\theta_r r} \\ &= H_p(\nabla\phi(x_0)) \otimes q : \nabla^2\phi(x_0) \\ &= \frac{1}{|H_p(\nabla\phi(x_0))|} H_p(\nabla\phi) \otimes H_p(\nabla\phi) : \nabla^2\phi|_{x=x_0}. \end{aligned}$$

This implies $\mathcal{A}[\phi](x_0) \geq 0$ and u_ϵ is a viscosity subsolution of Aronsson's equation (1.8) on Ω. ■

We are ready to prove theorem 1.2. First, as mentioned in §2, we can assume that $H(0) = 0$ and $H(p) \geq 0$ for any $p \in \mathbf{R}^n$. Second, observe that the conditions **H1-H4** are satisfied if H satisfies the conditions (A1-A3) of theorem 1.2. The idea is to first use (A3) to show that u_ϵ, constructed as above, converges uniformly to u, and then use (4.3)-(4.4) to show that u_ϵ can be deformed into a *strict* viscosity subsolution of Aronsson's equation (1.8).

Lemma 4.2. *Under the assumptions as in theorem 1.2. Let u_ϵ be defined by (4.1)-(4.2). Then we have*

$$\lim_{\epsilon \to 0} \|u_\epsilon - u\|_{C^0(\Omega)} = 0. \tag{4.19}$$

Proof. For $\epsilon \geq 0$, let $\mathcal{A}_\epsilon = \{p \in \mathbf{R}^n \mid H(p) \leq \epsilon\}$. Since $\mathcal{A}_0$ is a compact, convex set without interior points, we know that $\mathcal{A}_0$ is contained in a hyperplane of $\mathbf{R}^n$ and hence

$$\lim_{\epsilon \to 0} |\text{co}\{p - q \mid p, q \in \mathcal{A}_\epsilon\}| = 0, \tag{4.20}$$

where $\text{co}(E)$ denotes the convex hull of a set E, and $|\cdot|$ denotes the Lebesgue measure in $\mathbf{R}^n$.

Let $v = u - u_\epsilon$. Then, by Theorem 4.1, we have $v \geq 0$. By the Alexandroff-Bakelman-Pucci estimate (cf. [GT]), we have

$$\begin{aligned} \|u_\epsilon - u\|_{C^0(\Omega)} &= \max_{x \in \Omega_\epsilon} (u - u_\epsilon)(x) \\ &\leq C(n) \text{diam}(\Omega_\epsilon) |\text{co}\{\nabla v(x) \mid x \in \Omega_\epsilon\}|^{\frac{1}{n}} \\ &\leq C(n) \text{diam}(\Omega_\epsilon) |\text{co}\{p - q \mid p, q \in \mathcal{A}_\epsilon\}|^{\frac{1}{n}} \end{aligned} \tag{4.21}$$

where $\text{diam}(\Omega_\epsilon)$ is the diameter of Ω_ϵ and $C(n) > 0$ is a constant depending only on n. Taking ϵ into zero, (4.20) and (4.21) imply (4.19). ■

Proof of Theorem 1.2.

The conclusion of theorem 1.2 follows from the comparison principle. Let $u \in C(\overline{\Omega})$ ($v \in C(\overline{\Omega})$, resp.) be any viscosity subsolution (supersolution, resp.) of Aronsson's equation (1.8), we need to prove

$$\max_{x \in \overline{\Omega}} (u - v)(x) = \max_{x \in \partial\Omega} (u - v)(x). \tag{4.22}$$

For any $\alpha > 0$, let $\Omega_\alpha = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \alpha\}$. Notice that it suffices to prove (4.22) with Ω replaced by Ω_α. Since, by theorem 2.3, $u, v \in W^{1,\infty}_{\text{loc}}(\Omega)$, we have

$$\max\{\|\nabla u\|_{L^\infty(\Omega_\alpha)},\ \|\nabla v\|_{L^\infty(\Omega_\alpha)}\} \leq C_\alpha < +\infty.$$

Hence we assume for simplicity that $u, v \in W^{1,\infty}(\Omega)$.

Let $u_\epsilon \in C(\overline{\Omega})$ be given by theorem 4.1. Then we have $S^+(H, u_\epsilon, x) \geq \epsilon > 0$ for $x \in \Omega$ that u_ϵ is a viscosity subsolution of (1.8) on Ω. Since $u_\epsilon \to u$ uniformly on Ω, it suffices to prove (4.22) with u replaced by u_ϵ. Therefore we may assume that for sufficiently small $\epsilon > 0$, $S^+(H, u, x) \geq \epsilon$ for $x \in \Omega$.

For any $\delta > 0$, let $u^\delta \in C(\overline{\Omega})$ be the sup-convolution of u defined by (3.11), and

$$v_\delta(x) := \inf_{y \in \overline{\Omega}} (v(y) + \frac{1}{2\delta}|x - y|^2), \ x \in \overline{\Omega}$$

be the inf-convolution of v. It is well-known (cf. [J], [CIL], [ACJ]) that (i) $\|\nabla u_\epsilon^\delta\|_{L^\infty(\Omega)} \leq \|\nabla u\|_{L^\infty(\Omega)}$ and $\|\nabla v_\delta\|_{L^\infty(\Omega)} \leq \|\nabla v\|_{L^\infty(\Omega)}$, $u^\delta \to u$ and $v_\delta \to v$ uniformly on Ω as $\delta \to 0$, (ii) u^δ is semiconvex and v_δ is semiconcave, i.e.,

$$\nabla^2 u^\delta(x) \geq -8\delta^{-2} I_n, \quad \nabla^2 v_\delta(x) \leq 8\delta^{-2} I_n, \ x \in \Omega \tag{4.23}$$

in the sense of distributions, and (iii) u^δ (or v_δ, resp.) is a viscosity subsolution (or supersolution, resp.) of Aronsson's equation (1.8). Moreover, it is easy to check

$$S^+(H, u^\delta, x) \geq \epsilon, \quad x \in \Omega. \tag{4.24}$$

Since $u^\delta + c$ is also a viscosity subsolution of Aronsson's equation (1.8) for any $c \in \mathbf{R}$, we further assume $u^\delta(x) \geq 1$ for $x \in \Omega$.

For sufficiently large $m > 0$, define

$$\phi_m(t) = t - \frac{1}{m} e^{-mt}, \ \forall t > 0.$$

Then

$$\phi_m'(t) = 1 + e^{-mt}, \quad \phi_m''(t) = -me^{-mt}, \ t > 0.$$

This implies that there exists a unique $u_m^\delta : \Omega \to \mathbf{R}$ such that

$$u^\delta(x) = \phi_m(u_m^\delta(x)), \ x \in \Omega. \tag{4.25}$$

Moreover, direct calculations imply

$$\nabla u^\delta = \phi_m'(u_m^\delta) \nabla u_m^\delta,$$

$$\nabla^2 u^\delta = \phi_m'(u_m^\delta) \nabla^2 u_m^\delta + \phi_m''(u_m^\delta) \nabla u_m^\delta \otimes \nabla u_m^\delta \leq \phi_m'(u_m^\delta) \nabla^2 u_m^\delta. \tag{4.26}$$

Hence u_m^δ is also semiconvex on Ω.

We want to show (4.22) with u (v, resp.) replaced by u_m^δ (v_δ, resp.). To do it, we argue by contradiction. Suppose that this were false. Then there exists a $x_0 \in \Omega$ such that

$$(u_m^\delta - v_\delta)(x_0) = \max_{x\in\overline{\Omega}}(u_m^\delta - v_\delta)(x). \tag{4.27}$$

Since $u_m^\delta - v_\delta$ is semiconvex, Jensen's maximum principle for seminconvex functions (cf. [J] [CIL]) implies that $\nabla u_m^\delta(x_0)$ and $\nabla v_\delta(x_0)$ exist and

$$\nabla u_m^\delta(x_0) = \nabla v_\delta(x_0),$$

and there exist $\{p_k\} \subset \mathbf{R}^n$ with $|p_k| \leq k^{-1}$, and $x_k \in \Omega$ with $x_k \to x_0$ such that both $\nabla^2 u_m^\delta(x_k)$ and $\nabla^2 v_\delta(x_k)$ exist, and $u_m^\delta(x) - v_\delta(x) + p_k \cdot x$ attains its local maximum at x_k. Hence we have

$$\nabla u_m^\delta(x_k) + p_k = \nabla v_\delta(x_k),\ \nabla^2 u_m^\delta(x_k) \leq \nabla^2 v_\delta(x_k). \tag{4.28}$$

Since v_δ is a viscosity supersolution of (1.8), this implies

$$\begin{aligned}\mathcal{A}[u_m^\delta + p_k\cdot x](x_k) &= H_p(\nabla u_m^\delta + p_k)\otimes H_p(\nabla u_m^\delta + p_k) : \nabla^2 u_m^\delta|_{x=x_k}\\ &\leq H_p(\nabla v_\delta)\otimes H_p(\nabla v_\delta) : \nabla^2 v_\delta|_{x=x_k}\\ &= \mathcal{A}[v_\delta](x_k) \leq 0.\end{aligned} \tag{4.29}$$

On the other hand, since u^δ is a viscosity subsolution of (1.8), by using (4.26) we have, at x_k,

$$\begin{aligned}H_p(\nabla u^\delta)\otimes H_p(\nabla u^\delta) : \nabla^2 u_m^\delta &\geq -\frac{\phi_m''(u_m^\delta)}{(\phi_m'(u_m^\delta))^3}\langle H_p(\nabla u^\delta), \nabla u^\delta\rangle^2\\ &\geq \frac{me^{-mu_m^\delta}}{(1+me^{-mu_m^\delta})^3}\epsilon^2 \geq me^{-mu_m^\delta}\epsilon^2.\end{aligned} \tag{4.30}$$

Here we have used the condition (A1):

$$\langle H_p(\nabla u^\delta(x_k)), \nabla u^\delta(x_k)\rangle \geq H(\nabla u^\delta(x_k)) = S^+(H, u^\delta, x_k) \geq \epsilon. \tag{4.31}$$

Observe that (4.23) and (4.26) imply

$$-16\delta^{-2} I_n \leq \nabla^2 u_m^\delta(x_k) \leq 8\delta^{-2} I_n. \tag{4.32}$$

Moreover, at x_k, we have

$$
\begin{aligned}
|H_p(\nabla u^\delta) - H_p(\nabla u_m^\delta + p_k)| &\le C|\phi_m'(u_m^\delta) - 1)\nabla u_m^\delta - p_k| \\
&\le C(k^{-1} + e^{-mu_m^\delta})
\end{aligned} \tag{4.33}
$$

where $C = C(\delta, H, \|\nabla u\|_{L^\infty(\Omega)}) > 0$. Therefore, putting all these inequalities together, we have

$$
me^{-mu_m^\delta(x_k)}\epsilon^2 \le C(k^{-1} + e^{-mu_m^\delta(x_k)}).
$$

Taking k into infinity, this implies

$$
m\epsilon^2 \le C.
$$

This is impossible if m is sufficiently large. We get the desired contradiction. Hence (4.22) holds. The proof of theorem 1.2 is complete. ■

§5 Connection between Hamilton-Jacobi equations and Aronsson equations

In this section, we discuss the nonuniqueness of Aronsson's equation (1.13) and its connection with Hamilton-Jacobi equations.

It is well-known that if $H \in C(\mathbf{R}^n)$ is convex and $c > \min_{\mathbf{R}^n} H$, then the Hamilton-Jacobi equation

$$
H(\nabla u) = c
$$

has at most one viscosity solution with any given boundary data. See Ishii [I] for example. If $c = \min_{\mathbf{R}^n} H > -\infty$, by the proof of Lemma 5.2 below, we have

Theorem 5.1. *If $H \in C(\mathbf{R}^n)$ is convex and the set $\{p \in \mathbf{R}^n |\ H = \min_{\mathbf{R}^n} H\}$ has no interior points, then for any $g \in C(\partial\Omega)$, the Hamilton-Jacobi equation*

$$
H(\nabla u) = \min_{\mathbf{R}^n} H \ \text{ in } \Omega, \quad u = g \ \text{ on } \partial\Omega \tag{5.1}
$$

has at most one solution.

Next we want to discuss the case that $H = H(p,x)$ depends on x-variable. If $H(p,x) \in C^2(\mathbf{R}^n \times \Omega) \cap C(\mathbf{R}^n \times \overline{\Omega})$ satisfies (i) $H(\cdot, x)$ is convex for all $x \in \overline{\Omega}$, (ii) $0 = H(0,x) < H(p,x)$ for all $p \neq 0$ and $x \in \overline{\Omega}$, then, as stated Conjecture 1.3 in §1, we believe that suitable modifications of the proof of theorem 1.2 can lead that there exists at most one viscosity solution to the following Aronsson's equation

$$
\sum_{i,j=1}^n H_{p_i}(\nabla u, x) H_{p_j}(\nabla u, x) u_{x_i x_j} + H_{p_i}(\nabla u, x) H_{x_i}(\nabla u, x) = 0, \text{ in } \Omega \tag{5.2}
$$

$$
u = g \qquad \text{on } \partial\Omega.
$$

In general, the uniqueness of Aronsson's equation (5.2) is complicated and may fail in some interesting situations.

In this section, we indicate how to construct viscosity solutions of equation (5.2) from viscosity solutions of the Hamilton-Jacobi equation:

$$H(\nabla u(x), x) = c \ \text{ in } \Omega, \quad u = g \ \text{ on } \partial\Omega. \tag{5.3}$$

Henceforth we assume that $\partial\Omega$ is smooth, $H(p,x) \in C^2(\mathbf{R}^n \times \Omega) \cap C(\mathbf{R}^n \times \overline{\Omega})$, $H(\cdot, x)$ is convex for all $x \in \overline{\Omega}$, and

$$\lim_{|p| \to +\infty} H(p,x) = +\infty \text{ uniformly in } \overline{\Omega}. \tag{5.4}$$

Solutions to equations (5.2) and (5.4) are refered to be viscosity solutions. By the main theorem of [I], we have that *if*

$$c > c_0 = \inf_{\phi \in C(\overline{\Omega}) \cap C^1(\Omega)} \sup_{x \in \Omega} H(\nabla \phi(x), x), \tag{5.5}$$

then equation (5.3) has at most one solution. If $c = c_0$, then equation (5.3) may have infinitely many solutions even if H is strictly convex in p-variable. This permits us to construct infinitely many solutions of equation (5.2).

First we want to investigate nonuniqueness of solutions to the equation (5.3). Our method is motivated by Fathi-Siconolfi [FS]. We say that $x_0 \in \Omega$ is a *strict* point, if there exists a Lipschitz continuous function v in Ω such that

$$\text{esssup}_{x \in \Omega} H(\nabla v(x), x) = c_0,$$

and

$$\text{esssup}_{x \in U_{x_0}} H(\nabla v(x), x) < c_0$$

for some neighborhood U_{x_0} of x_0. Denote

$$\mathcal{O} = \{x \in \Omega : x \text{ is a strict point in } \Omega\}, \quad \Gamma = \overline{\Omega} \setminus \mathcal{O}. \tag{5.6}$$

It is readily seen that $\mathcal{O}$ is an open subset of Ω and Γ is a closed subset of Ω. The following theorem asserts that any solution of equation (5.3) is uniquely determined by its value on Γ.

Theorem 5.2. *Assume $\Gamma \neq \emptyset$. If both u and v are solutions of equation (5.3) with $c = c_0$, and*

$$u|_\Gamma = v|_\Gamma,$$

then $u \equiv v$ on Ω. Also, if w is a subsolution of equation (5.3), then there exists a unique viscosity solution u of equation (5.3) such that

$$u|_\Gamma = w|_\Gamma.$$

Proof. Let ω be an arbitrary open set, with $\omega \subset \bar{\omega} \subset \mathcal{O}$. For each $x \in \bar{\omega}$, it follows from the definition of strict points that there exists a Lipschitz continuous function v_x and a neighborhood U_x of x such that

$$\text{esssup}_{y \in U_x} H(\nabla v_x(y), y) < c_0, \ \text{esssup}_{y \in \Omega} H(\nabla v_x(y), y) = c_0. \tag{5.7}$$

Since $\overline{\omega}$ is compact, there exists finite many points $\{x_i\}_{i=1}^m \subset \overline{\omega}$ such that $\overline{\omega} \subset \cup_{i=1}^m U_{x_i}$. Define

$$v = \frac{1}{m} \sum_{i=1}^m v_{x_i}.$$

Then, since $H(\cdot, x)$ is convex, Jensen's inequality implies

$$\text{esssup}_{x \in \omega} H(\nabla v(x), x) \leq \frac{1}{m} \sum_{i=1}^m \text{esssup}_{y \in U_{x_i}} H(\nabla v_{x_i}(y), y) < c_0. \tag{5.8}$$

Hence, by the main theorem of [I], the equation

$$H(\nabla w(x), x) = c_0 \ \text{ in } \mathcal{O}, \ w|_{\partial \mathcal{O}} = u (= v).$$

has at most one solution w. This implies $u = v$ on $\mathcal{O}$.

Next we prove the second part of theorem 5.2. By (5.4) and Perron's method, there exists a solution u of the following equation

$$H(\nabla u(x), x) = c_0 \quad \text{in } \mathcal{O}, \quad u|_{\partial \mathcal{O}} = w. \tag{5.9}$$

In fact, u is given by

$$u = \sup_{v \in S} v, \tag{5.10}$$

where

$$S = \{v \in W^{1,\infty}(\Omega) \mid \text{esssup}_{x \in \Omega} H(\nabla v(x), x) = c_0 \text{ and } v|_\Gamma = w|_\Gamma\}. \tag{5.11}$$

By the first part of theorem 5.2, we know that such a u is unique. Now we claim that u is a viscosity solution of $H(\nabla u(x), x) = c_0$ in Ω. It is clear that u is a subsolution of

$H(\nabla u(x), x) = c_0$ in Ω. So we need to show that u is a viscosity supersolution on the set Γ, but this directly follows from the definition of strict points. ■

The following is an easy consequence of theorem 5.2.

Corollary 5.3. *For $\Gamma \neq \emptyset$, if equation (5.3) has more than one solution, then it has infinitely many solutions.*

Proof. Suppose that u_1 and u_2 are two distinct solutions. Then, by theorem 5.2, we have $u_1 \neq u_2$ on Γ. For each $t \in [0, 1]$, let

$$u_t = tu_1 + (1-t)u_2.$$

It is clear that u_t is a subsolution of equation (5.3). By theorem 5.2, we conclude that for any $t \in (0, 1)$, there exists a solution v_t to equation (5.3) on Ω such that

$$v_t = u_t \quad \text{on } \Gamma.$$

Since for any $t_1, t_2 \in (0, 1)$, $v_{t_1}(= u_{t_1}) \neq v_{t_2}(= u_{t_2})$ on Γ, we get infinitely many distinct solutions to (5.3). ■

Next we indicate that there is an injective map from the solutions of equation (5.3) to the solutions of equation (5.2). Under the above assumptions on H, it is known that u is a solution of equation (5.2) if and only if it is an absolute minimizer for H with the boundary value g (see [Y], [BJW], and [C]).

Theorem 5.4 *Assume $\Gamma \neq \emptyset$, let u be a solution of equation (5.3). Then there exists an absolute minimizer v for H on Ω such that*

$$v|_\Gamma = u|_\Gamma.$$

Proof. Using the standard L^p-approximation scheme, we can find an absolute minimizer v for H on $\Omega \setminus \Gamma$ such that

$$v|_\Gamma = u|_\Gamma.$$

We want to show that such a v is an absolute minimizer for H on Ω. To see this, let $V \subset\subset \Omega$ be any open set and $w \in W^{1,\infty}(V)$ with $v|_{\partial V} = w|_{\partial V}$. If $V \cap \Gamma = \emptyset$, then $V \subset\subset \Omega \setminus \Gamma$ and hence

$$\text{esssup}_{x\in V} H(\nabla v(x), x) \leq \text{esssup}_{\in V} H(\nabla w(x), x).$$

If $V \cap \Gamma \neq \emptyset$, then, by the choice of v, we have

$$\text{esssup}_{x \in V} H(\nabla v(x), x) \leq \text{esssup}_{x \in \Omega} H(\nabla v(x), x) \leq \text{esssup}_{x \in \Omega} H(\nabla u(x), x) = c_0.$$

On the other hand, if we define

$$\begin{aligned} \tilde{u} &= u, \ \text{ in } \Omega \setminus V \\ &= w, \text{in } V, \end{aligned}$$

then, by the the definition of strict points, we have

$$\text{esssup}_{x \in V} H(\nabla w(x), x) = \text{esssup}_{x \in V} H(\nabla \tilde{u}(x), x) \geq c_0 \geq \text{esssup}_{x \in V} H(\nabla v(x), x).$$

This completes the proof. ■

In general, it is a difficult task to determine the set Γ. Here we investigate three typical cases. We will illustrate concrete examples where $\Gamma \cap \Omega$ is nonempty and equation (5.3) has infinitely many solutions. Hence by Theorem 5.4, equation (5.2) also has infinitely many solutions.

Case 1: $H(p, x) = H(p) + V(x)$.

In this case, it is easy to see

$$c_0 = \max_{\overline{\Omega}} V + \min_{p \in \mathbf{R}^n} H,$$

and

$$\Gamma = \partial\Omega \cup \{x \in \Omega \mid V(x) = \max_{\overline{\Omega}} V\}.$$

Assume that $\{x \in \Omega |\ V(x) = \max_{\overline{\Omega}} V\}$ is not empty and contains finitely many points. Then Proposition 5.4 in [L] implies that the Hamilton-Jacobi equation

$$H(\nabla u) + V(x) = c_0 \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

has infinitely many solutions. Hence Theorem 5.4 implies that there are infinitely many absolute minimizers for H with boundary value 0.

For example, let $H(p, x) = |p|^2 + \sin^2 x : \mathbf{R} \times [0, \pi] \to \mathbf{R}$. Then $c_0 = 1$ and $\Gamma = \{\frac{\pi}{2}\}$. It is easy to see that both $u_1 = \sin x$ and $u_2 = -\sin x$ are smooth solutions to the Hamilton-Jacobi equation

$$|u'|^2 + \sin^2 x = 1, \ \text{ in } (0, \pi), u(0) = u(\pi) = 0.$$

Hence u_1 and u_2 are solutions of Aronsson's equation for $H(p,x) = |p|^2 + \sin^2 x$, with the boundary value $u(0) = u(\pi) = 0$.

Case 2: $H(p,x) = a(x)H(p)$.

This is essentially a variant of Case 1. In fact, u is a solution of $a(x)H(\nabla u) = c$ if and only u is a solution of $H(\nabla u) - \frac{c}{a(x)} = 0$.

For example, let $H(p,x) = \frac{1}{2-\sin^2 x}(|p|^2+1) : \mathbf{R} \times [0,\pi]$. Then both $u_1 = \sin x$ and $u_2 = -\sin x$ are smooth solutions of the following Hamilton-Jacobi equation

$$\frac{1}{2-\sin^2 x}(|u'|^2+1) = 1 \quad \text{in } (0,\pi),\ u(0) = u(\pi) = 0.$$

Hence both u_1 and u_2 are solutions of Aronsson equation on $[0,\pi]$ for H, with boundary value $u(0) = u(\pi) = 0$.

Case 3: $H(p,x) = |p - B(x)|^2$.

Here we assume that $B(x)$ is a smooth vector field in $\mathbf{R}^n$. For $n = 1$, this case is equivalent to the case $H(p) = |p|^2$, since $u' - B(x) = (u - \int_0^x B(x)\,dx)'$. Hence the uniqueness holds. But for $n \geq 2$, these two cases are different since there may not exist a smooth function w such that $\nabla w = B$. For this case, it is usually not an easy task to determine the set Γ. Here we will employ some ideas from the weak KAM theory (cf. [FS]) to construct an example where Γ can be determined and the equation (5.3) has infinitely many solutions.

Example 5.5. Denote by $\mathbf{T}^2 = (-1,1)\times(-1,1)$ an open square in $\mathbf{R}^2$. Let $B : \overline{\mathbf{T}}^2 \to \mathbf{R}^2$ be a continuous vector field such that $\max_{\mathbf{T}^2}|B| = 1$,

$$\{x \in \overline{\mathbf{T}}^2 \mid |B(x)| = 1\} = \partial B_{\frac{1}{2}}(0) (= \{x \in \mathbf{R}^2 | \ |x| = \frac{1}{2}\}), \tag{5.12}$$

and $B(x) = 2(x_2, -x_1)$ for $x = (x_1,x_2) \in \partial B_{\frac{1}{2}}(0)$.

Claim I. $c_0 = 1$. Since $\max_{\mathbf{T}^2}|0 - B(x)| = 1$, $c_0 \leq 1$. Let $\phi \in C^1((-1,1)\times(-1,1))$. Denote $\xi(t) = (\frac{1}{2}\cos t, \frac{1}{2}\sin t)$ for $0 \leq t \leq 2\pi$. Since $\phi(\xi(0)) = \phi(\xi(2\pi))$, there exists $t_0 \in (0,2\pi)$ such that

$$0 = \frac{d\phi(\xi(t))}{dt}|_{t=t_0} = \nabla\phi(\xi(t_0)) \cdot \xi'(t_0).$$

Notice that $B(\xi(t_0)) = -2\xi'(t_0)$. Hence we have

$$(\nabla\phi(\xi(t_0)) - B(\xi(t_0))) \cdot B(\xi(t_0)) = -1.$$

This, combined with the Cauchy inequality, implies that

$$|\nabla \phi(\xi(t_0)) - B(\xi(t_0))| \geq 1.$$

Therefore

$$\sup_{\mathbf{T}^2} |\nabla \phi(x) - B(x)| \geq |D\phi(\xi(t_0)) - B(\xi(t_0))| \geq 1.$$

and $c_0 \geq 1$.

Claim II. $\Gamma \cap (-1,1) \times (-1,1) = \partial B_{\frac{1}{2}}(0)$.

By (5.12), it is clear $\Gamma \cap (-1,1) \times (-1,1) \subset \partial B_{\frac{1}{2}}(0)$. To show the equality, let $L(q,x)$ be the Lagrangian of $H = |p - B(x)|^2$. Then, by a simple calculation, we have

$$L(q,x) = \frac{1}{4}|q|^2 + B(x) \cdot q.$$

Suppose that the claim were false. Then there exists $v \in W^{1,\infty}(\mathbf{T}^2)$ such that

$$\mathrm{esssup}_{\mathbf{T}^2} H(\nabla v(x), x) = c_0 = 1,$$

but

$$\mathrm{esssup}_{U_{t_1}} H(\nabla v(x), x) < c_0 = 1 \tag{5.13}$$

for some $t_1 \in [0, 2\pi]$ and a neighborhood U_{t_1} of $\xi(t_1)$. Let $s(t) = \xi(4t)$ for $0 \leq t \leq \frac{\pi}{2}$. Then $2B(s(t)) = -s'(t)$. A standard argument then shows that

$$0 = v(s(\frac{\pi}{2})) - v(s(0)) < \int_0^{\frac{\pi}{2}} 1 + L(s'(t), s(t))\, dt = 0, \tag{5.14}$$

where the strict inequality is due to (5.13). This is a contradiction. Hence the claim holds.

Claim III. There are infinitely many solutions of the following equation:

$$|\nabla u - B(x)| = 1 \quad \text{in } \mathbf{T}^2,\ u|_{\partial \mathbf{T}^2} = 0. \tag{5.15}$$

In fact, by [L] and (5.12), there exists $\epsilon > 0$ such that for any $0 < \delta \leq \epsilon$, the following equation has a solution u_δ

$$|\nabla u_\delta - B(x)| = 1 \quad \text{in } T^2 \backslash \partial B_{\frac{1}{2}}(0) \tag{5.16}$$

$$u_\delta|_{\partial T^2} = 0 \tag{5.17}$$

$$u_\delta|_{\partial B_{\frac{1}{2}}(0)} = \delta. \tag{5.18}$$

Therefore, by Theorem 5.2, u_δ is also a solution of equation (5.2). ■

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