Strong convergence of the gradients for p-Laplacian problems as p → ∞

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STRONG CONVERGENCE OF THE GRADIENTS FOR $p-$LAPLACIAN PROBLEMS AS $p \to \infty$

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ABSTRACT. In this paper we prove that the gradients of solutions to the Dirichlet problem for $-\Delta_p u_p = f$, with $f > 0$, converge as $p \to \infty$ strongly in every L^q with $1 \leq q < \infty$ to the gradient of the limit function. This convergence is sharp since a simple example in 1-d shows that there is no convergence in L^∞. For a nonnegative f we obtain the same strong convergence inside the support of f. The same kind of result also holds true for the eigenvalue problem for a suitable class of domains (as balls or stadiums).

1. INTRODUCTION

When one deals with elliptic problems that involve the well-known $p-$Laplacian operator, $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$, there are two extreme special cases: $p = 1$ and $p = \infty$ (understood as the limit as $p \to \infty$, we refer to [5, 14] among many other references). Also the limit as $p \to 2$ with $p \neq 2$ is relevant for regularity issues, see [20].

Concerning limits as $p \to \infty$, one of the first papers that studies this kind of problems is [5], where it is proved the following result: let u_p denote the solution to

$$\begin{cases} -\Delta_p u_p = f & \text{in} \quad \Omega \\ u_p = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

with Ω a bounded smooth domain in $\mathbb{R}^N$, $f \in C(\bar{\Omega})$ and $f > 0$. Let us mention at this point that we will understand solutions to (1.1) both as weak solutions in the Sobolev space $W_0^{1,p}(\Omega)$ and as viscosity solutions (it is known that for the $p-$Laplacian both notions of solution coincide, see [17]). Then, from [5], we have that u_p converges uniformly as $p \to \infty$ to the distance to the boundary, i.e.,

$$\lim_{p\to\infty} u_p(x) = u_\infty(x) = d(x, \partial\Omega), \tag{1.2}$$

for $x \in \overline{\Omega}$. This limit u_∞ is characterized as the unique viscosity solution (see [9] for the precise definition) of the eikonal equation

$$\begin{cases} |\nabla u_\infty| = 1 & \text{in} \quad \Omega \\ u_\infty = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

When f is nonnegative and nontrivial (but not necessarily strictly positive), we still have uniform convergence, but now we can only assert that (1.2) holds in the set $\{f > 0\}$. Indeed the limit u_∞ is a solution to the eikonal equation in $\{f > 0\}$ and is $\infty-$harmonic (i.e. a viscosity solution to $-\Delta_\infty u_\infty = -\langle D^2 u_\infty \nabla u_\infty, u_\infty \rangle = 0$) in $\Omega \setminus \overline{\{f > 0\}}$.

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As far as the gradients are concerned (that is the main issue here), when $f \geq 0$, it was also proved in [5] that the sequence $\{\nabla u_p\}_p$ is uniformly bounded in $L^q(\Omega)$ for any $1 \leq q < \infty$ for p large (by a constant independent of p), and therefore we can obtain *weak* convergence in $L^q(\Omega)$ of ∇u_p to ∇u_∞ as $p \to \infty$. Moreover, one can obtain that the limit u_∞ is Lipschitz with $\|\nabla u_\infty\|_{L^\infty(\Omega)} = 1$.

Our main goal is to show that one can improve such a weak convergence: we start by showing the almost-everywhere convergence of the gradients. We will present two proofs of this result. The first one is based on arguments from [5] (and in fact shows strong convergence in L^2 directly) while the second one contains new ideas (and in fact is the main contribution of this note).

Theorem 1.1. *Assume that $f > 0$ in Ω. Let u_p be the sequence of solutions of (1.1). Then, ∇u_p converges to ∇u_∞ as $p \to \infty$ almost-everywhere in Ω.*

As a consequence we can conclude *strong* convergence of ∇u_p to ∇u_∞ in $L^q(\Omega)$ for any $1 \leq q < \infty$.

Corollary 1.2. *Assume that $f > 0$ in Ω. Let u_p be the sequence of solutions of (1.1). Then, ∇u_p converges to ∇u_∞ as $p \to \infty$ strongly in $L^q(\Omega)$ for every $1 \leq q < \infty$.*

Remark 1.3. The result is optimal, since there is no uniform convergence of the gradients as the simple one dimensional case shows, see Section 2.

In the case $f \geq 0$, we can also obtain convergence of the gradients a.e. (and then strong convergence in L^q for $1 \leq q < \infty$) but only in the set where the norm of the gradient of the limit u_∞ attains its maximum, that is, in the set $\{x \in \Omega \, : \, |\nabla u_\infty(x)| = 1\}$, that contains the set $\overline{\{x : f(x) > 0\}}$ and therefore has positive measure.

Theorem 1.4. *Assume $f \geq 0$. Let u_p be the sequence of solutions of (1.1). Then, ∇u_p converges to ∇u_∞ as $p \to \infty$ almost-everywhere in $\{x \in \Omega \, : \, |\nabla u_\infty(x)| = 1\}$ and in turn this implies that ∇u_p converges to ∇u_∞ strongly in $L^q(\{|\nabla u_\infty| = 1\})$ for every $1 \leq q < \infty$.*

Finally, our ideas can also be used to deal with the limit as $p \to \infty$ of the eigenvalue problem

$$
\begin{cases} -\Delta_p u_p = \lambda |u_p|^{p-2} u_p & \text{in} \quad \Omega \\ u_p = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.3}
$$

The first eigenvalue of (1.3) is given by

$$
\lambda_{1,p} = \inf_{u \in W_0^{1,p}(\Omega)} \frac{\|\nabla u\|_{L^p(\Omega)}^p}{\|u\|_{L^p(\Omega)}^p}.
$$

In [16] it is proved that

$$
\lim_{p \to \infty} (\lambda_{1,p})^{\frac{1}{p}} = \lambda_{1,\infty} = \frac{1}{R}
$$

with R the maximum of the distance to the boundary, i.e. $R = \max_{x \in \Omega} d(x, \partial\Omega)$. Moreover, the eigenfunctions, normalized according to $\|\nabla u_p\|_{L^\infty(\Omega)} = 1$, converge (along a subsequence) to a limit u_∞ uniformly in $\overline{\Omega}$ and, as before, $\nabla u_p \rightharpoonup \nabla u_\infty$ weakly in $L^q(\Omega)$ for $1 \leq q < \infty$. This limit u_∞ verifies $\|\nabla u_\infty\|_{L^\infty(\Omega)} = 1$. Our last result says that we can improve the converge of the gradients also in this case on the set where the maximum of the gradient is attained.

Theorem 1.5. *Let u_p be the sequence of eigenfunction of* (1.3) *corresponding to the first eigenvalue $\lambda_{1,p}$ normalized according to $\|\nabla u_p\|_{L^\infty(\Omega)} = 1$. Then, up to a (not relabeled) subsequence, ∇u_p a.e. converges to ∇u_∞ as $p \to \infty$ in $\{x \in \Omega \ : \ |\nabla u_\infty(x)| = 1\}$ and hence strongly in $L^q(\{|\nabla u_\infty| = 1\})$ for every $1 \le q < \infty$.*

Remark 1.6. For some special domains (like the ball or the stadium) it is well-known (see [21]) that the limit of the normalized eigenfunctions u_p is unique and is given, again, by the distance to the boundary $u_\infty(x) = d(x, \partial\Omega)$. In this case our result implies strong convergence of the gradients in L^q in the whole Ω.

However, in general, the distance to the boundary is not the limit of u_p (for example, in a square, [16]). Hence, in general, it is not know that the set where the gradient attains its maximum norm has positive measure.

2. An explicit example in 1-d

In this section we include a simple example in 1-d where u_p can be computed explicitly. This example shows that our results are optimal and gives a flavor of what can be expected for the convergence of the gradients. The essence of this example is somehow related to the $C^{p'}-$regularity conjecture, see [1, 2].

Let $\Omega = (-1, 1)$ and $f \equiv 1$, then (1.1) reads as

$$\begin{cases} -(|u_p'|^{p-2}u_p')' = 1 & \text{in} \quad (-1,1) \\ u_p(\pm 1) = 0. \end{cases}$$

Then, from uniqueness we deduce that the solution u_p is an even function, $u_p(x) = u_p(-x)$ (and therefore we can concentrate on what happens in $(0, 1)$). Moreover, u_p is decreasing in $(0, 1)$ with $u_p'(0) = 0$. Therefore, we can integrate the equation in order to obtain

$$(-u_p'(x))^{p-1} = x, \qquad x \in (0, 1).$$

Therefore, we obtain, integrating one more time and using that $u_p(1) = 0$

$$u_p(x) = \int_0^1 s^{\frac{1}{p-1}}\, ds - \int_0^x s^{\frac{1}{p-1}}\, ds = \frac{p-1}{p}\Big(1 - x^{\frac{p}{p-1}}\Big), \qquad x \in (0, 1),$$

that implies that, in fact, $u_p(x) = \frac{p-1}{p}\Big(1 - |x|^{\frac{p}{p-1}}\Big)$.

From these explicit expressions we immediately obtain that

$$\lim_{p\to\infty} u_p(x) \quad = \lim_{p\to\infty} \frac{p-1}{p}\Big(1 - |x|^{\frac{p}{p-1}}\Big) = 1 - |x| = d(x, \partial\Omega)$$

and moreover

$$\lim_{p\to\infty} \sup_{x\in(-1,1)} \Big|\frac{p-1}{p}\Big(1 - |x|^{\frac{p}{p-1}}\Big) - (1 - |x|)\Big| = \lim_{p\to\infty} \frac{1}{p} = 0\,,$$

that implies the uniform convergence of u_p to $d(x, \partial\Omega)$.

Concerning the gradients, direct computations show that

$$u_p'(x) = -x|x|^{\frac{2-p}{p-1}} \to -\text{sign}\, x, \qquad \text{a.e. and strongly in } L^q(-1,1) \text{ for every } 1 \le q < \infty,$$

but since $u_p'(0) = 0$ we conclude that the gradients do not converge uniformly.

Notice that u_p attains its maximum at $x = 0$, and consequently we have $u_p'(0) = 0$, and hence (since u_p is $C^{1,\alpha}$) the derivative of u_p is small in a neighborhood to such a point. On the other hand, the limit u_∞ is the distance to the boundary that is not differentiable at $x = 0$ and it has a derivative with modulus equal to one in the whole $(-1, 1) \setminus \{0\}$. This is what excludes the uniform convergence.

3. Proofs of the main results

Let us begin with a lemma that shows that the gradients are uniformly bounded in L^q. This result follows from [5] but we include it here for completeness.

Lemma 3.1. *Let u_p be the solution to (1.1) with $f \geq 0$ and bounded. There exists a constant C independent of p such that*

$$\|\nabla u_p\|_{L^q(\Omega)} \leq C,$$

for every p large enough.

Proof. Along this proof we denote by K a generic constant independent of p. Just use u_p as test function in the weak form of (1.1) to obtain, using Hölder's inequality,

$$\int_\Omega |\nabla u_p|^p = \int_\Omega f u_p \leq \|f\|_{L^{p'}(\Omega)} \|u\|_{L^p(\Omega)}.$$

Now we use the Sobolev embedding (and the arguments in [11] that imply that the Sobolev constant can be bounded as pK with K independent of p) to obtain

$$\int_\Omega |\nabla u_p|^p \leq pK \|f\|_{L^{p'}(\Omega)} \|\nabla u\|_{L^p(\Omega)}.$$

Hence, we arrive to

$$\|\nabla u\|_{L^p(\Omega)} \leq \Big(pK \|f\|_{L^{p'}(\Omega)}\Big)^{\frac{1}{p-1}} := C(p)$$

Notice that

$$\lim_{p\to\infty} C(p) = \lim_{p\to\infty} \Big(pK \|f\|_{L^{p'}(\Omega)}\Big)^{\frac{1}{p-1}} = 1.$$

Hence we have that

$$\|\nabla u\|_{L^p(\Omega)} \leq 2$$

for p large.

Now, for $1 \leq q < \infty$ we observe that for $p > q$, from Hölder's inequality, we obtain

$$\left(\int_\Omega |\nabla u_p|^q\right)^{\frac{1}{q}} \leq |\Omega|^{\frac{1}{q}-\frac{1}{p}} \left(\int_\Omega |\nabla u_p|^p\right)^{\frac{1}{p}} \leq |\Omega|^{\frac{1}{q}-\frac{1}{p}} C(p), \tag{3.1}$$

with $|\Omega|^{\frac{1}{q}-\frac{1}{p}} \to |\Omega|^{\frac{1}{q}}$ and $C(p) \to 1$ as $p \to \infty$. □

As a consequence of this bound we obtain uniform convergence of u_p to a limit u_∞ (possibly extracting a subsequence) and *weak* convergence of the gradients in $L^q(\Omega)$ (and moreover, we obtain that the limit u_∞ is Lipschitz).

Theorem 3.2. *Let u_p be the solution to (1.1) with $f \geq 0$ and bounded. Then, along a subsequence u_p converges uniformly in $\overline{\Omega}$ to a limit $u_\infty \in W^{1,\infty}(\Omega)$ with*

$$\|\nabla u_\infty\|_{L^\infty(\Omega)} \leq 1.$$

Moreover, for any $q \geq 1$

$$\nabla u_p \rightharpoonup \nabla u_\infty \qquad \text{weakly in } L^q(\Omega).$$

Proof. From our previous estimates we know that

$$\left(\int_\Omega |\nabla u_p|^p dx \right)^{\frac{1}{p}} \leq C(p),$$

with $C(p) \to 1$ as $p \to \infty$. Therefore we conclude that $\{u_p\}$ is uniformly bounded and has a uniform modulus of continuity for p large. Hence u_p converges uniformly in $\overline{\Omega}$ to a limit u_∞. In addition, from (3.1) we obtain weak convergence of u_p to u_∞ in $W^{1,q}(\Omega)$. Also from our previous uniform bound (3.1) this limit u_∞ verifies for any $q \geq 1$

$$\left(\int_\Omega |\nabla u_\infty|^q \right)^{\frac{1}{q}} \leq |\Omega|^{\frac{1}{q}}.$$

As the above inequality holds for every q, we get that $u_\infty \in W^{1,\infty}(\Omega)$ and moreover, taking the limit as $q \to \infty$, we obtain

$$\|\nabla u_\infty\|_{L^\infty(\Omega)} \leq 1.$$

This ends the proof. □

3.1. **Convergence of the gradients. The case $f > 0$.** Now we are ready to proceed with the first proof of our main result, that first improves the convergence of the gradients showing convergence almost-everywhere.

The first proof of Theorem 1.1 follows ideas in [5].

First Proof of Theorem 1.1. We modify slightly the argument of Proposition 2.1 of [5]. Using that $|\nabla u_\infty| = 1$ a.e., we have that

(3.2)
$$\frac{1}{4} \int_\Omega |\nabla(u_p + u_\infty)|^2 + \frac{1}{4} \int_\Omega |\nabla(u_p - u_\infty)|^2 = \frac{1}{2} \left[\int_\Omega |\nabla u_p|^2 + \int_\Omega |\nabla u_\infty|^2 \right]$$
$$\leq \frac{1}{2} \left[\left(\int_\Omega |\nabla u_p|^p \right)^{\frac{2}{p}} |\Omega|^{1-\frac{2}{p}} + |\Omega| \right] \leq \frac{1}{2} \left[\left(pK \|f\|_{L^{p'}(\Omega)} \right)^{\frac{2}{p-1}} |\Omega|^{1-\frac{2}{p}} + |\Omega| \right],$$

where the last inequality comes from Lemma 3.1. Notice that the right hand side above converges to $|\Omega|$ as p diverges, hence we get

$$\limsup_{p\to\infty} \frac{1}{4} \int_\Omega |\nabla(u_p + u_\infty)|^2 \leq |\Omega|.$$

Moreover, since $\nabla u_p + \nabla u_\infty$ converges weakly in $L^2(\Omega)$ to $2\nabla u_\infty$ we have that

$$\liminf_{p\to\infty} \frac{1}{4} \int_\Omega |\nabla(u_p + u_\infty)|^2 \geq \int_\Omega |\nabla u_\infty|^2 = |\Omega|.$$

Thus we deduce from (3.2) strong convergence in $L^2(\Omega)$,

$$\lim_{p\to\infty} \int_\Omega |\nabla(u_p - u_\infty)|^2 = 0,$$

that implies, up to a subsequence, the required almost-everywhere convergence. Due to the uniqueness of the limit, we obtain the convergence of the whole sequence as $p \to \infty$. $\square$

Now we present an alternative proof of almost-everywhere convergence of the gradients, that takes some ideas from the classical paper [18] (see also [6]).

Second proof of Theorem 1.1. First of all, we choose $u_p - u_\infty$ as test function in the weak formulation of (1.1) in order to get

$$\int_\Omega \langle |\nabla u_p|^{p-2}\nabla u_p, \nabla(u_p - u_\infty)\rangle = \int_\Omega f(u_p - u_\infty).$$

Since u_p converges uniformly to u_∞ and $\nabla(u_p - u_\infty)$ weakly converges to 0 in $L^q(\Omega)$ for any $q \geq 1$, and $|\nabla u_\infty| \leq 1$ a.e., we have that

$$\begin{aligned}
&\int_\Omega \langle (|\nabla u_p|^{p-2}\nabla u_p - |\nabla u_\infty|^{p-2}\nabla u_\infty), \nabla(u_p - u_\infty)\rangle \\
&\qquad = \int_\Omega f(u_p - u_\infty) - \int_\Omega \langle |\nabla u_\infty|^{p-2}\nabla u_\infty, \nabla(u_p - u_\infty)\rangle = \varepsilon_p,
\end{aligned}$$

with $\varepsilon_p \to 0$ as $p \to \infty$.

Setting

$$g_p = \langle (|\nabla u_p|^{p-2}\nabla u_p - |\nabla u_\infty|^{p-2}\nabla u_\infty), \nabla(u_p - u_\infty)\rangle$$

we have that $g_p \geq 0$ (this follows as a simple consequence of the elementary fact that $\langle (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta), (\xi - \eta)\rangle \geq 0$ for any two vectors ξ and η in $\mathbb{R}^N$) and therefore g_p converges to 0 strongly in $L^1(\Omega)$. Then, up to a (not relabeled) subsequence, g_p also converges to 0 almost-everywhere.

Since $u_\infty(x) = d(x, \partial\Omega)$, we have that u_∞ is almost-everywhere differentiable with $|\nabla u_\infty| = 1$ almost-everywhere in Ω. Let us set now

$$Z = \Big\{ x \in \Omega \;:\; |\nabla u_\infty| \neq 1 \Big\} \cup \Big\{ x \in \Omega \;:\; \lim_{p\to\infty} g_p \neq 0 \Big\}$$

and notice that by construction $|Z| = 0$. Take now any $x_0 \in \Omega \setminus Z$ so that we have

$$g_p(x_0) = \langle (|\nabla u_p(x_0)|^{p-2}\nabla u_p(x_0) - \nabla u_\infty(x_0)), \nabla(u_p(x_0) - u_\infty(x_0))\rangle \to 0$$

and

$$|\nabla u_\infty(x_0)| = 1.$$

Let us denote $\nabla u_p(x_0) = \xi_p$, that, up to a subsequence (still not relabeled) converges to $\xi \in \mathbb{R}^N \setminus \{0\}$ (otherwise $x_0 \in Z$). Suppose on the one hand that $\lim\limits_{p\to\infty} \xi_p = \xi$ with $|\xi| > 1$; thus

$$\begin{aligned}
&\lim_{p\to\infty} \langle (|\xi_p|^{p-2}\xi_p - \nabla u_\infty(x_0)), (\xi_p - \nabla u_\infty(x_0))\rangle \\
&= \lim_{p\to\infty} |\xi_p|^p + \langle |\xi_p|^{p-2}\xi_p, \nabla u_\infty(x_0)\rangle - \langle \nabla u_\infty(x_0), (\xi_p - \nabla u_\infty(x_0))\rangle = +\infty,
\end{aligned}$$

that yields to a contradiction.

On the other hand, suppose that $\lim\limits_{p\to\infty} \xi_p = \xi$ with $|\xi| < 1$, so that

$$\begin{aligned}
&\lim_{p\to\infty} \langle (|\xi_p|^{p-2}\xi_p - \nabla u_\infty(x_0)), (\xi_p - \nabla u_\infty(x_0))\rangle \\
&\qquad = \lim_{p\to\infty} |\xi_p|^p + \langle |\xi_p|^{p-2}\xi_p, \nabla u_\infty(x_0)\rangle - \langle \nabla u_\infty(x_0)), (\xi_p - \nabla u_\infty(x_0))\rangle \\
&\qquad = -\langle \nabla u_\infty(x_0), \xi\rangle + |\nabla u_\infty(x_0)|^2 \neq 0,
\end{aligned}$$

since $|\xi| < 1$ while $|\nabla u_\infty(x_0)| = 1$, that is also a contradiction.

This proves that at any point $x_0 \in \Omega \setminus Z$ we have that

$$|\nabla u_p(x_0)| \to |\nabla u_\infty(x_0)| = 1.$$

In order to obtain that $\xi = \nabla u_\infty(x_0)$, let us suppose that, along a subsequnece

$$\lim_{p\to\infty} |\xi_p|^{p-2} = \alpha > 1.$$

Thus, up to subsequences,

$$\lim_{p\to\infty} \langle (|\xi_p|^{p-2}\xi_p - \nabla u_\infty(x_0)), (\xi_p - \nabla u(x_0))\rangle$$
$$= \langle (\alpha\xi - \nabla u_\infty(x_0)), (\xi - \nabla u_\infty(x_0))\rangle = \alpha - (\alpha+1)\langle \xi, \nabla u_\infty(x_0)\rangle + 1$$

and the right hand side cannot vanish since $\alpha > 1$ and $\langle \xi, \nabla u_\infty(x_0)\rangle \in (-1, 1)$.

Analogously, if along a subsequence we have

$$\lim_{p\to\infty} |\xi_p|^{p-2} = \beta < 1$$

we would get that (up to subsequences)

$$\lim_{p\to\infty} \langle (|\xi_p|^{p-2}\xi_p - \nabla u_\infty(x_0)), (\xi_p - \nabla u_\infty(x_0))\rangle$$
$$= \langle (\beta\xi - \nabla u_\infty(x_0)), (\xi - \nabla u_\infty(x_0))\rangle = \beta - (\beta+1)\langle \xi, \nabla u_\infty(x_0)\rangle + 1$$

that cannot vanish. Thus, we have proved that necessarily

$$\lim_{p\to\infty} |\xi_p|^p = 1.$$

Finally, passing to the limit in g_p we obtain

$$0 = \lim_{p\to\infty} g_p(x_0) = |\xi - \nabla u_\infty(x_0)|^2$$

that vanishes if and only if $\xi = \nabla u_\infty(x_0)$.

We have proved that at every point $x_0 \in \Omega \setminus Z$ we have that

$$\nabla u_p(x_0) \to \nabla u_\infty(x_0).$$

As before, the uniqueness of the limit implies the convergence of the whole sequence as $p \to \infty$. The proof is complete. □

As a corollary we obtain strong convergence of the gradients in $L^q(\Omega)$.

Proof of Corollary 1.2. We use Vitali's Theorem.

We have almost-everywhere convergence thanks to our previous result.

Now we need to check that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_A |\nabla u_p|^q < \varepsilon$$

for every $A \subset \Omega$ measurable with $|A| < \delta$. This property holds since, taking $\delta < 1$, from our previous bounds we have

$$\int_A |\nabla u_p|^q \le |A|^{1-\frac{q}{p}} \left(\int_\Omega |\nabla u_p|^p\right)^{\frac{q}{p}} \le 2^q |A|^{\frac{1}{2}} \le \varepsilon.$$

Then, from Vitali's theorem we conclude that $\nabla u_p \to \nabla u$ strongly in $L^q(\Omega)$. □

3.2. **The case $f \geq 0$.** With the same ideas we can prove the result where the norm of the gradient of the limit function attains its maximum.

Proof of Theorem 1.4. The previous results are also valid for the case $f \geq 0$ (f may vanish in some region of Ω) but only inside the subset of Ω where the modulus of the gradient attains its maximum, $\{x \in \Omega \,:\, |\nabla u_\infty(x)| = 1\}$. This set includes the set $\overline{\{x : f(x) > 0\}}$, see [5], and therefore has positive measure.

The first proof of Theorem 1.4 can be adapted to cover this case just integrating in the set $\{x \in \Omega \,:\, |\nabla u_\infty(x)| = 1\}$.

Looking carefully at the second proof of Theorem 1.1, one realizes that the main ingredient of the argument is that $|\nabla u_\infty| \leq 1$ (this still holds for $f \geq 0$), to deduce that $|\nabla u_\infty|^{p-2} \leq 1$. Hence, one can use the same arguments but arguing only in the set $\{x \in \Omega \,:\, |\nabla u_\infty(x)| = 1\}$ to prove that $\nabla u_p \to \nabla u_\infty$ almost-everywhere in $\{x \in \Omega \,:\, |\nabla u_\infty(x)| = 1\}$.

After we have a.e. convergence we can use as before Vitali's theorem to conclude that $\nabla u_p \to \nabla u_\infty$ strongly in $L^q(\{|\nabla u_\infty| = 1\})$ for $1 \leq q < \infty$.

Since the limit is unique, we obtain the convergence of the whole sequence as $p \to \infty$. □

Remark 3.3. When $f \equiv 0$ and we take the limit as $p \to \infty$ in

$$\begin{cases} -\Delta_p u_p = 0 & \text{in} \quad \Omega \\ u_p = g & \text{on} \quad \partial\Omega \end{cases}$$

it turns out (see [4, 8, 15] and the survey [3]) that there is a uniform limit, u_∞, that is the unique solution to

$$\begin{cases} -\Delta_\infty u_\infty = 0 & \text{in} \quad \Omega \\ u_\infty = g & \text{on} \quad \partial\Omega \end{cases}$$

where $\Delta_\infty u_\infty = \langle D^2 u_\infty \nabla u_\infty, u_\infty \rangle$ is the well-known $\infty-$Laplacian (this operator also appears related to applications, for example, random turn games [19] and mass transportation problems [13], image reconstruction and enhancement [7], and the study of shape metamorphism [10].).

In this case our results also hold in the region where the modulus of the gradient attains its maximum. We first consider the sequence (assuming that we are not in the trivial case $u_\infty = cte$)

$$v_p(x) = \frac{u_p(x)}{\|\nabla u_\infty\|_{L^\infty(\Omega)}}$$

that converges uniformly to $v_\infty(x) = \dfrac{u_\infty(x)}{\|\nabla u_\infty\|_{L^\infty(\Omega)}}$ that has $\|v_\infty\|_{L^\infty(\Omega)} = 1$. Then, we can apply our arguments to obtain $\nabla v_p \to \nabla v_\infty$ almost-everywhere inside the set $\{x \in \Omega \,:\, |\nabla v_\infty(x)| = 1\}$ and therefore we also get that $\nabla u_p \to \nabla u_\infty$ almost-everywhere in $\{x \in \Omega \,:\, |\nabla u_\infty(x)| = \|\nabla u_\infty\|_{L^\infty}(\Omega)\}$. After that we conclude that $\nabla u_p \to \nabla u_\infty$ strongly in $L^q(\{|\nabla u_\infty(x)| = \|\nabla u_\infty\|_{L^\infty}(\Omega)\})$ for $1 \leq q < \infty$. Of course, this is only relevant when $\{x \in \Omega \,:\, |\nabla u_\infty(x)| = \|\nabla u_\infty\|_{L^\infty}(\Omega)\}$ has positive measure.

3.3. **The eigenvalue problem.** Finally, we deal with the eigenvalue problem (1.3). Now we have a uniform bound for the gradients of the eigenfunctions (that comes from the normalization $\|\nabla u_p\|_{L^\infty(\Omega)} = 1$). Then we can deduce uniform convergence of u_p to a limit u_∞ (possibly extracting a subsequence) and *weak* convergence of the gradients in $L^q(\Omega)$. This was already proved in [16].

Theorem 3.4. *Let u_p be the sequence of eigenfunction of* (1.3) *corresponding to the first eigenvalue $\lambda_{1,p}$ normalized according to $\|\nabla u_p\|_{L^\infty(\Omega)} = 1$. Then, along a subsequence, u_p converges uniformly to a limit $u_\infty \in W^{1,\infty}(\Omega)$ that verifies $\|\nabla u_\infty\|_{L^\infty} = 1$. Moreover, for any $q \geq 1$*

$$\nabla u_p \rightharpoonup \nabla u_\infty \qquad \textit{weakly in } L^q(\Omega).$$

Proof. The proof is similar to the one of Theorem 3.2 (see also [16]). □

Now we can improve the convergence of the gradients.

Proof of Theorem 1.5. We proceed as in the proof of Theorem 1.1 and choose $u_p - u_\infty$ as test function in the weak formulation of the eigenvalue problem (1.3) in order to obtain

$$\int_\Omega \langle |\nabla u_p|^{p-2}\nabla u_p, \nabla(u_p - u_\infty)\rangle = \lambda_{1,p} \int_\Omega |u_p|^{p-2} u_p (u_p - u_\infty).$$

Since u_p converges uniformly to u_∞, $(\lambda_{1,p})^{\frac{1}{p}} \to \lambda_{1,\infty} = 1/R$ and $\nabla(u_p - u_\infty)$ weakly converges to 0 in any $L^q(\Omega)$ and $|\nabla u_\infty| \leq 1$ a.e., we have that, up to a subsequence,

$$\begin{aligned}&\int_\Omega \langle (|\nabla u_p|^{p-2}\nabla u_p - |\nabla u_\infty|^{p-2}\nabla u_\infty), \nabla(u_p - u_\infty)\rangle \\ &\qquad = \int_\Omega |(\lambda_{1,p})^{\frac{1}{p-2}} u_p|^{p-2} u_p (u_p - u_\infty) - \int_\Omega \langle |\nabla u_\infty|^{p-2}\nabla u_\infty, \nabla(u_p - u_\infty)\rangle = \varepsilon_p,\end{aligned}$$

with $\varepsilon_p \to 0$ as $p \to \infty$.

Letting

$$g_p = \langle (|\nabla u_p|^{p-2}\nabla u_p - |\nabla u_\infty|^{p-2}\nabla u_\infty), \nabla(u_p - u_\infty)\rangle$$

we have that $g_p \geq 0$ and therefore g_p converges to 0 strongly in $L^1(\Omega)$. Then, up to a (not relabeled) subsequence, it also converges to 0 almost-everywhere.

From this point the rest of the proof follows as in the proof of Theorem 1.1. □

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