Existence of Classical Solutions to a Free Boundary Problem for the p-Laplace Operator: (II) The Interior Convex Case

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ABSTRACT. In this paper, we prove the existence of convex classical solutions for a Bernoulli-type free boundary problem, in the interior of a convex domain. The governing operator considered is the p-Laplacian. This work is inspired by the pioneering work of A. Beurling where he proves the existence for the harmonic case in the plane, using the notion of sub- and super-solutions in a geometrical sense.

1. INTRODUCTION. The aim of this paper is to prove the existence of classical solutions to nonlinear Bernoulli-type free boundary problems.

The paper is the "interior" counterpart to [H-Sh2]. The departing point will be a bounded convex domain $K \subset \mathbb{R}^N$ ($N \geq 2$), and we ask for a smaller domain Ω such that the gradient of the p-capacitary potential of $K \setminus \overline{\Omega}$ has a prescribed magnitude on $\partial\Omega$ (the boundary of Ω).

Mathematically the problem is formulated as follows: Given a bounded convex domain $K \subset \mathbb{R}^N$, we look for a function u and a domain Ω ($\subset K$) satisfying, for a given constant $\lambda > 0$,

$$(1.1) \qquad \begin{cases} \Delta_p u = 0, & \text{in } K \setminus \overline{\Omega} \\ u = 1, & \text{on } \partial K \\ u = 0, & \text{on } \partial\Omega \\ |\nabla u| = \lambda, & \text{on } \partial\Omega \end{cases}$$

where Δ_p denotes the p-Laplace operator, i.e., $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. The overdetermined boundary condition $|\nabla u| = \lambda$ is to be understood in the following classical sense

$$\lim_{\substack{y \to x \\ y \in K \setminus \bar{\Omega}}} | \nabla u(y) | = \lambda \text{ for every } x \in \partial\Omega. \tag{1.2}$$

This problem in general (even for the linear model $p = 2$) may fail to have a solution, as the simple case of $K = B(0, R)$ shows by explicit calculations (see e.g. [Fl-R]). More precisely, we will prove here, in section 3, that there exists a critical value λ_K^p, depending on K and p, such that there exists a solution to problem (1.1) if and only if $\lambda \geq \lambda_K^p$. We also give some bounds for this constant λ_K^p.

The problem arises in various nonlinear flow laws, and several physical situations, e.g. electrochemical machining and potential flow in fluid mechanics. For an exhaustive list of references and recent study in the linear case $p = 2$, we refer to [Fl-R] (see also [F]). In the linear case a classical approach for such a problem consists in considering a variational formulation (see e.g. [Ac1], [C-S], [Al-C]), this can also be done in the nonlinear case, without major difficulties. Nevertheless, in this approach it is quite difficult to prove that the overdetermined condition holds in a strong sense, as we will do here. Another approach, inspired by the pioneering work of A. Beurling [Be] consists in using the notion of sub and super-solutions in a geometrical sense. It was generalized by Acker in [Ac2] for this problem in the case $p = 2$. For $1 < p < \infty$, the problem is more or less in a foster stage, and there are, to the authors' knowledge, not many known results, see [Ac-M] and [H-Sh2] for the corresponding exterior problem.

2. EXISTENCE.

Before stating our main result we need to define a certain class of functions.

Definition 2.1 Let $D \subset \mathbb{R}^N$, we define $\mathcal{F} = \mathcal{F}(D, \lambda)$ to be the class of all continuous functions v on $\bar{D}$ such that:

(1) v is Lipschitz continuous on $\bar{D}$;
(2) $v = 1$ on ∂D;
(3) $\Delta_p v \leq 0$ in $\{v > 0\} \cap D$;
(4) $|\nabla v| \leq \lambda$ on $\partial\{v > 0\} \cap D$.

The set $\{v = 0\}$ will be denoted by $\Omega = \Omega_v$. The dependence of $\mathcal{F}$ on D and λ will be suppressed if there is no ambiguity. The requirement (3) in Definition 2.1 is to be understood in a weak sense, i.e.

$$0 \leq \int |\nabla v|^{p-2} \nabla v \cdot \nabla \psi,$$

for all $0 \leq \psi \in C_o^\infty(\{v > 0\} \cap D)$.

We state now the main result of this paper.

Theorem 2.1 *Let K be a regular (C^1-boundary) convex domain, and suppose $\mathcal{F}(K,\lambda)$ is not empty. Then there exists a classical solution u to the free boundary problem (1.1). Moreover the level sets $\{u < t\}$ ($t \leq 1$) of u are convex and $\partial\Omega$ is $C^{2,\alpha}$.*

The proof of Theorem 2.1 involves a number of technical tools that we are going to describe now. We define the operator L_u to be

$$L_u := |\nabla u|^{p-2}\Delta + (p-2)|\nabla u|^{p-4}\sum_{k,l=1}^{N}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_l}\frac{\partial^2}{\partial x_k \partial x_l}. \tag{2.1}$$

This operator is uniformly elliptic where $|\nabla u| \geq c > 0$ (se [Pa-Ph1] and [Pa-Ph2]) and moreover, we have $L_u(u) = 0$ when u is p-harmonic. This fact will be used later in the proof of Lemma 2.1.

For two nested convex sets $D_1 \subset D_2$, and for $x \in \partial D_1$ we denote by $T_{x,a}$ a supporting hyperplane at x with the normal a pointing away from D_1. Obviously, $T_{x,a}$ is not necessarily unique, depending on the geometry of ∂D_1. Now for each $x \in \partial D_1$ there corresponds a point y_x (not necessarily unique) on $\partial D_2 \cap \{z : \ a\cdot(z-x) > 0\}$ and such that $a\cdot(y_x - x) = \max a\cdot(z-x)$, where the maximum has been taken over all $z \in \partial D_2 \cap \{z : a\cdot(z-x) > 0\}$.

The following result is fundamental in this work and it has its own interest for capacitory functions in convex rings.

Lemma 2.1 *Let D_1 and D_2 be two nested open convex domains ($D_1 \subset D_2$) and denote by u the p-capacitary potential of $D_2 \setminus \overline{D_1}$, i.e. the solution of*

$$\begin{cases} \Delta_p u = 0 & \text{in } D_2\setminus\overline{D_1} \\ u = c_1 & \text{on } \partial D_1 \\ u = c_2 & \text{on } \partial D_2 \end{cases}$$

where c_1 and c_2 are two given constants $c_1 \neq c_2 \geq 0$. Then

$$\limsup_{\substack{z\to x\\ z\in D_2\setminus\overline{D_1}}} |\nabla u(z)| \geq \limsup_{\substack{z\to y_x\\ z\in D_2\setminus\overline{D_1}}} |\nabla u(z)| \qquad \forall\ x \in \partial D_1, \tag{2.2}$$

where y_x is the point indicated in the discussion preceding this lemma.

Proof. We may assume $c_1 > c_2$ else we replace u by $c_2 - u$ and obtain new constants $c_1' = c_2 - c_1 > c_2' = 0$. Fix $x \in \partial D_1$. We can also assume x and y_x to be regular points of the boundary: otherwise, due to the convexity of D_1 and D_2, we would have a gradient which tends to infinity at x (at least for a subsequence) or 0 at y_x (see e.g. [K]), and the result is clear. Then using rotation and translation invariance we may assume that $\{x_1 = 0\}$ is a supporting plane to D_1 at x, and $D_1 \subset \{x_1 < 0\}$. Set now $D_2' = D_2 \cap \{x_1 > 0\}$ and introduce a new function $v = u + \alpha x_1$ where

$$\alpha = \lim_{\substack{z \to y_x \\ z \in D_2 \setminus \overline{D_1}}} |\nabla u(z)| - \varepsilon$$

with ε a small positive number.

Since the second derivatives of v and u coincide, we have

$$L_u v = L_u u = 0 \qquad \text{in } D_2'.$$

Therefore v has its maximum on the boundary of D_2', and by the definition at either of the points x and y_x. The latter is excluded since else

$$0 \leq \lim \frac{\partial v}{\partial x_1}(y_x) = -\lim |\nabla u(y_x)| + \alpha = -\varepsilon < 0.$$

Hence the maximum of v is achieved at x and we have

$$0 \leq -\lim \frac{\partial v}{\partial x_1}(x) = \lim |\nabla u(x)| - \alpha. \tag{2.3}$$

Since (2.3) holds for all $\varepsilon > 0$ the lemma follows. ☐

Lemma 2.2 *Let $v \in \mathcal{F}$ and assume that $\Omega = \{v = 0\}$ is convex. Then for every boundary point $x \in \partial\Omega$, we have $d(x, \partial K) \geq 1/\lambda$.*

Proof. Applying Lemma 2.1, we obtain that $|\nabla v| \leq \lambda$ in the whole domain $K \setminus \bar{\Omega}$. Now, the result follows immediately thanks to the mean value inequality between x and any point on ∂K. ☐

Remark 2.1 Let Ω be a solution of the free boundary problem (1.1). Let L be any line intersecting the free boundary $\partial\Omega$ at (at least) two points x and x' and let us denote by y and y' the intersection of L with ∂K. The previous Lemma implies that the length of the segments $[xy]$ and $[x'y']$ are larger than $1/\lambda$. Therefore, the minimal width of K (defined as the diameter of the larger ball inscribed in K) is certainly larger than $2/\lambda$.

The following lemma is proved in [Le].

Lemma 2.3 *Hypotheses as in Lemma 2.1. Then there exists a constant* $\lambda_0 > 0$ *such that* $|\nabla u| \geq \lambda_0$ *in* $D_2 \setminus \bar{D}_1$.

The following lemma results immediately from the definition of the class $\mathcal{F}$.

Lemma 2.4 *For* v_1, v_2 *in* $\mathcal{F}$ *there holds* $\min(v_1, v_2) \in \mathcal{F}$.

Next we recall the definition of extremal points of a convex set.

Definition 2.2 (Extremal points) A point $x \in \partial\Omega$ is said to be extremal if there exists a supporting plane to Ω touching $\partial\Omega$ at x only.

Lemma 2.5 *Let* v_1, v_2 *in* $\mathcal{F}$, *with corresponding* $\Omega_j = \{v_j = 0\} \subset K$, *and set* $\Omega = \Omega_1 \cup \Omega_2$. *Denote by* Ω^* *the convex hull of* Ω *and let* u^* *be the* p*-capacitary potential of* $K \setminus \overline{\Omega^*}$. *Then* $|\nabla u^*| \leq \lambda$ *in* $K \setminus \overline{\Omega^*}$, *and in particular* $u^* \in \mathcal{F}$.

Proof. First observe that $\partial\Omega^* \cap \partial\Omega$ contains all the extremal points of Ω^*. Since also by comparison principle $u^* \leq \min(u_1, u_2)$, with $u^* = \min(u_1, u_2) = 0$ on $\partial\Omega^* \cap \partial\Omega$ we conclude that on extremal points of Ω^* there holds $|\nabla u^*| \leq \lambda$. Fix a point y in $K \setminus \Omega^*$, and let $\mathcal{L}_y = \{x : u^*(x) < u^*(y)\}$. Then by results of [Le] the level sets of u^* are convex. In particular $\mathcal{L}_y$ is convex. Now let $x = x_y$ be a point on $\partial\Omega^*$ which corresponds to minimum distance from y to Ω^*. Because of the geometry, it is obvious that we can choose x_y to be an extremal point of Ω^*. Applying Lemma 2.1 with $D_1 = \Omega^*$, $D_2 = \mathcal{L}_y$, $y_x = y$ and $x = x_y$ we have $|\nabla u^*(y)| \leq \limsup_{z \to x_y} |\nabla u^*(z)| \leq \lambda$. This completes the proof. $\square$

Proof of Theorem 2.1. Let us consider $\inf\{v : v \in \mathcal{F}\}$. Let u_n be a minimizing sequence. Then by Lemmas 2.4–2.5 we may assume $\{u_n < t\}$ is convex for all $0 < t \leq 1$, u_n is decreasing in n, $|\nabla u_n| \leq \lambda$ in $K \setminus \overline{\Omega_n}$ ($\Omega_n = \{u_n = 0\}$) and $\Delta_p u_n = 0$ in $K \setminus \overline{\Omega_n}$. The increasing sequence of convex sets Ω_n converges (for example in the Hausdorff topology) to a convex set Ω. By Ascoli Theorem, u_n converges to a function u uniformly on $K \setminus \bar{\Omega}$. Moreover, Lemma 2.2 ensures that u is not identically 0 in K (i.e. Ω is strictly included in K), because $\partial\Omega_n$ stays away from ∂K. It is easy to verify that u satisfies $\Delta_p u = 0$ in $K \setminus \bar{\Omega}$, $u = 1$ on ∂K and $u = 0$ on $\partial\Omega$. Moreover, by Harnack-type inequalities, $|\nabla u_n|$ converges uniformly to $|\nabla u|$ on every compact subset of $K \setminus \bar{\Omega}$. Now, according to Lemma 2.1 above, we know that $|\nabla u_n| \leq \lambda$ in $K \setminus \overline{\Omega_n}$ and then, passing to the limit $|\nabla u| \leq \lambda$ in $K \setminus \bar{\Omega}$. At last the level sets $\{u < t\}$ are convex.

At this point we need some other technical lemmas.

Lemma 2.6 *Hypotheses as in Lemma 2.1, with* $|\nabla u| \leq \lambda$ *on* ∂D_1. *Then* ∂D_1 *is* C^1.

Proof. Since D_1 is convex, we first prove that there is a unique supporting plane at every boundary point. Suppose there is a point $x^0 \in \partial D_1$ such that we have two supporting planes T_1 and T_2. By rotation and translation we may assume that x^0 is the origin and

$$\begin{aligned} T_1 &= \{x_1 = 0\}, \\ T_2 &= \{x_1 + \varepsilon x_2 = 0\}, \\ D_1 &\subset \{x_1 < 0\} \cap \{x_1 + \varepsilon x_2 < 0\} \end{aligned}$$

for some $\varepsilon > 0$. Now define Λ to be the cone

$$\Lambda := \{x_1 > 0\} \cup \{x_1 + \varepsilon x_2 > 0\}.$$

Then $D_1 \cap \Lambda = \varnothing$ and $0 \in \partial D_1 \cap \partial\Lambda$. Now by results of Krol I.N. [K] (cf. also [D; Theorem 1]) there exists (explicit) positive p-harmonic function v in Λ with zero Dirichlet data on the boundary $\partial\Lambda$. Moreover v is C^α across $\partial\Lambda$ with $\alpha < 1$; the exact value of α depends on the corners opening. Let now M be such that $Mu \geq v$ on $\partial B(0,r)$ (for some fixed $r > 0$ small). Then $Mu \geq v$ inside the ball $B(0,r)$ by the comparison principle. Now u being above a non-Lipschitz function v/M gives a contradiction to the assumption $|\nabla u| \leq \lambda$. Thus, each point on ∂D_1 has a unique tangent plane. To see that these planes change continuously, we observe that if x_n converges to x on the boundary and if the sequences of tangent planes T_n has two subsequences with different limits T and T', then by convexity of D_1, T and T' must be supporting planes to ∂D_1 at x. This proves the lemma. ☐

Lemma 2.7 *Retain the hypotheses of Lemma 2.6 and let* x^0 *be a boundary point of* $\partial\Omega$. *Assume that the origin is at* x^0 *and that the exterior normal* *vector to $\partial\Omega$ is directed by the first coordinate vector. Let r_j be any decreasing sequence converging to 0 and define the blow-up sequence of functions $u_{r_j}(x) = 1/(r_j)\, u(r_j x)$ (where u is the p-capacitary potential of Ω). Then, there exists a positive number α and a subsequence, also denoted by u_{r_j}, which uniformly converges to $u_0(x) := \alpha x_1$*

Proof. Thanks to the assumptions, both u_{r_j} and $|\nabla u_{r_j}|$ are bounded uniformly by the Lipschitz norm of u. Hence there is a subsequence converging to a limit function u_0, which is p-harmonic in the upper half space (after suitable rotation and translation) and has concave level sets seen from the plane $\{x_1 = 0\}$. This implies that the level sets of u_0 are hyperplanes and $u_0(x)$ depends on x_1 only. It is also elementary that u_0 is not identically zero since, by Lemma 2.3,

$$\lim_{x \to 0} |\nabla u| \geq \lambda_0 > 0,$$

implies the nondegeneracy of u_{r_j}. Now, the problem being one dimensional, we can simply calculate the solution to find that $u_0(x) = \alpha x_1$ for some $\alpha > 0$. ☐

Now to finish the proof of the theorem, it remains to prove that the gradient of u is exactly λ at every boundary point of $\partial\Omega$. Assume, for a contradiction, that it is not the case, that is there exists a point $x^0 \in \partial\Omega$, such that

$$\liminf_{z \to x^0} |\nabla u(z)| = \alpha < \lambda. \tag{2.4}$$

In particular it implies, according to the previous lemma, that there exists a sequence r_j such that the blow-up sequence of functions u_{r_j} converges to αx_1.

Lemma 2.8 *Let $D_R = \{x_1 < 1\} \setminus B(x^R, R)$, where $x^R = (-R, 0, \ldots, 0)$, and take $\varepsilon_1 > 0$ small enough such that $\alpha + \varepsilon_1 < \lambda$ (α is the constant in (2.4)). Let u_R be the solution of*

$$\begin{cases} \Delta_p u_R = 0 & \text{in } D_R \\ u_R = \alpha + \varepsilon_1 & \text{on } \{x_1 = 1\} \\ u_R = 0, & \text{on } \partial B(x^R, R). \end{cases}$$

Then for any $\varepsilon_2 > \varepsilon_1$ there is a R such that $|\nabla u_R| \leq \alpha + \varepsilon_2$ on $\partial B(x^R, R)$.

Proof. Let $C_R = B(x^R, R+1) \setminus B(x^R, R)$ and set

$$\begin{aligned} v_R &= (\alpha+\varepsilon_1)\frac{|x-x^R|^{(p-N)/(p-1)} - R^{(p-N)/(p-1)}}{(R+1)^{(p-N)/(p-1)} - R^{(p-N)/(p-1)}} \quad (p \neq N), \\ v_R &= (\alpha+\varepsilon_1)\frac{\log|x-x^R| - \log R}{\log(R+1) - \log R} \quad (p = N), \end{aligned}$$

Then v_R is p-harmonic in C_R and $|\nabla v_R| \leq \alpha + \varepsilon_2$ on $\partial B(x^R, R)$, if R is large enough. Now by comparison principle $u_R \leq v_R$ and hence $|\nabla u_R| \leq |\nabla v_R| \leq \alpha + \varepsilon_2$ on $\partial B(x^R, R)$, if R is large enough. □

Proof of Theorem 2.1 (continued). Let us remark that for ε_3 small enough, we will have by continuity

$$|\nabla u_R| \leq \alpha + 2\varepsilon_2 \quad \text{on } \partial\{u_R \leq \varepsilon_3\}. \tag{2.5}$$

Let now

$$w_R = (\alpha+\varepsilon_1)\left(\frac{u_R - \varepsilon_3}{\alpha + \varepsilon_1 - \varepsilon_3}\right)_+ .$$

Then $w_R = \alpha + \varepsilon_1$ on $\{x_1 = 1\}$, $w_R = 0$ on $\{u_R = \varepsilon_3\}$, and by (2.5)

$$|\nabla w_R| \leq (\alpha+\varepsilon_1)\frac{\alpha + 2\varepsilon_2}{\alpha + \varepsilon_1 - \varepsilon_3} = \alpha + \varepsilon_4 < \lambda, \qquad \text{on } \partial\{u_R \leq \varepsilon_3\} = \partial\{w_R = 0\},$$

where

$$\varepsilon_4 = \frac{\alpha(\varepsilon_3 + 2\varepsilon_2) + 2\varepsilon_1\varepsilon_2}{\alpha + \varepsilon_1 - \varepsilon_3}$$

which is small if we choose ε_1, ε_2, ε_3 small enough.

Now, we claim

$$w_R > \alpha x_1 + \varepsilon_5 \quad \text{on } \partial B(0,1) \cap \{x_1 > -\delta\}, \tag{2.6}$$

where δ is small enough such that $\{u_R = \varepsilon_3\} \cap \partial B(0,1) \subset \{x_1 < -2\delta\}$, for some small ε_5, provided ε_3 is small. Indeed, on the upper hemisphere of the unit sphere we have by comparison principle $u_R > \alpha x_1 + \varepsilon$ for some small ε (more precisely $\varepsilon = \inf(u_R(x) - \alpha x_1)$ where the infimum is taken over all x in the unit sphere). Then $w_R \geq u_R - \varepsilon_3 > \alpha x_1 + \varepsilon - \varepsilon_3 > \alpha x_1 + \varepsilon_5$ if ε_3 is small enough.

Now, we put the origin at x^0 and we assume that the outward normal vector to Ω at x^0 is the x_1-axis. Then by Lemma 2.7 the sequence $u_{r_j} = u(r_j x)/r_j$ converges to αx_1. Hence, for r_j small enough,

$$u < \alpha x_1 + r_j \varepsilon_5, \tag{2.7}$$

inside a ball of center 0 and radius r_j. Let us now introduce $\tilde{w} = r_j w_R(x/r_j)$. According to (2.6) and (2.7), we have $\tilde{w} > \alpha x_1 + r_j \varepsilon_5 > u$ on $\partial B(0, r_j)$.

Let now

$$w = \begin{cases} \min(u, \tilde{w}) & \text{inside } B(0, r_j) \\ u & \text{outside } B(0, r_j) \end{cases}$$

Then $w \in \mathcal{F}$ and since w is identically zero in a small neighbourhood of the origin, we will have a contradiction to the minimal property of u.

Now the $C^{2,\alpha}$ regularity of $\partial\Omega$ is a consequence of a result of A.L. Vogel (see [V; Theorem 1]). The proof is now completed. □

3. THE BERNOULLI CONSTANT. The key point to be able to apply Theorem 2.1 is to know whether the class $\mathcal{F}(K, \lambda)$ is empty or not. Since we have the obvious inclusion

$$\lambda_1 \le \lambda_2 \Rightarrow \mathcal{F}(K, \lambda_1) \subset \mathcal{F}(K, \lambda_2)$$

we can introduce a constant λ_K^p defined as

$$\lambda_K^p := \inf\{\lambda > 0 : \mathcal{F}(K, \lambda) \text{ is not empty}\}.$$

We claim:

Theorem 3.1 *For every* $\lambda \ge \lambda_K^p$, *the problem* (1.1) *has a solution. For every* $\lambda < \lambda_K^p$, *the problem* (1.1) *has no solution.*

Proof. The case $\lambda > \lambda_K^p$ comes immediately from the definition of λ_K^p and Theorem 2.1. In the case $\lambda < \lambda_K^p$ any solution of problem (1.1) would provide an element in the class $\mathcal{F}(K, \lambda)$ contradicting the definition of λ_K^p. Let us prove now that for $\lambda = \lambda_K^p$, there also exists a solution. Let λ_n a sequence converging to λ_K^p from above. According to Theorem 2.1, for each n, there exists a convex solution, let Ω_n to problem (1.1) with the constant λ_n. Now, we can substract a subsequence Ω_{n_k} which converge, for the Hausdorff topology, to a convex domain Ω. According to classical results in nonlinear potential theory (cf e.g. [Bu-Tr] or [Da-De]), the p-capacitary potential u_{n_k} of Ω_{n_k} converges in $W_0^{1,p}(K)$ to u the p-capacitary potential of Ω. Now, according to Lemma 2.1 above, we know that $|\nabla u_{n_k}| \le \lambda_{n_k}$ in $K \setminus \bar{\Omega}_{n_k}$ and then, passing to the limit $|\nabla u| \le \lambda_K^p$ in $K \setminus \bar{\Omega}$. It follows that the function u belongs to the class $\mathcal{F}(K, \lambda_K^p)$ and the result follows from Theorem 2.1. $\square$

The constant λ_K^p could be referred as the Bernoulli constant of the domain K for the p-Laplacian. In the case of a ball of radius R, explicit computations give:

$$\lambda_K^p = \begin{cases} \dfrac{\left|\dfrac{p-N}{p-1}\right|}{\left|\left(\dfrac{p-1}{N-1}\right)^{(N-1)/(N-p)} - \left(\dfrac{p-1}{N-1}\right)^{(p-1)/(N-p)}\right|} \dfrac{1}{R} & \text{if } p \neq N, \\ \dfrac{e}{R} & \text{if } p = N. \end{cases}$$

Here is an open problem stated by Flucher and Rumpf in [Fl-Ru]: Is $\lambda_K^p \ge \lambda_{K^*}^p$, where K^* is any ball with the same volume as that of K?

More generally, it seems difficult, even in the linear case, to have some explicit bounds for λ_K^p. Nevertheless, we have seen in Lemma 2.2 and its subsequent remark that the distance between Ω and the boundary of K is always larger than $1/\lambda$. So, using this necessary condition, we have indeed the following estimation from below for λ_K^p:

Theorem 3.2 *Let l_K be the width of K defined as the diameter of the larger ball inscribed in K. Then*

$$\lambda_K^p \ge \frac{2}{l_K}.$$

For an estimation from above, it is more difficult. At least, in the linear case $p = 2$ and in dimension 2 ($N = 2$), we can proceed as follow. Let x_0 be the conformal center of K defined e.g. in [Ba-Fl] or [Po-Sz]. Since K is convex, x_0 is unique according to Haegi ([Ha]). Let Φ be the conformal map from the unit disk to K such that $\Phi(0) = x_0$. Applying the Koebe distorsion theorem to the univalent function $(\Phi(z) - \Phi(0))/\Phi'(0)$ (see [Co]) yields

$$|\Phi'(z)| \ge \frac{1 - |z|}{(1 + |z|)^3} |\Phi'(0)|. \tag{3.1}$$

By definition, $|\Phi'(0)|$ is the maximal conformal radius of K, we will denote it by r_K.

Now, let us introduce $\omega = \Phi(D_\rho)$ where D_ρ is the disk centered at the origin with radius ρ. The capacitary potential of ω is given by

$$u(z) = \frac{\log|\varphi^{-1}(z)|}{\log \rho}$$

so, for $z \in \partial\omega,\ z = \Phi(\rho e^{i\theta})$, we have

$$|\nabla u(z)| = -\frac{1}{|\Phi'(\rho e^{i\theta})|\rho \log \rho}. \tag{3.2}$$

Now (3.1) and (3.2) yield

$$\forall z \in \partial\omega \qquad |\nabla u(z)| \leq -\frac{(1+\rho)^3}{\rho \log \rho (1-\rho) r_K}.$$

Now, the quantity $-(1+\rho)^3/\rho \log \rho (1-\rho)$ is minimal for $\rho = 0.13277$ and its value is 6.252. So, if $\lambda \geq 6.252/r_K$, we have found a function u in the class $\mathcal{F}(K, \lambda)$, therefore we can conclude:

Theorem 3.3 *For $N = 2$ and $p = 2$, the minimal value λ_K^2, for which problem* (1.1) *has a solution, is estimated from above by*

$$\lambda_K^2 \leq \frac{6.252}{r_K} \tag{3.3}$$

where r_K is the maximal conformal radius of K.

Of course, the inequality (3.3) is not optimal, since when K is a disk of radius R (its maximal conformal radius is also equal to R) we obtain $\lambda_K^2 = e/R = 2.71828/R$. It comes from the fact that the Koebe distorsion inequality is not an equality in the case of the disk.

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Antoine Henrot thanks the Göran Gustafsson Foundation for several visiting appointments, to RIT in Stockholm. Henrik Shahgholian was partially supported by the Swedish Natural Science Research Council.

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SUBJECT CLASSIFICATION: (1991 Mathematics Subject Classification) Primary: 35R35; secondary: 35N05, 35J65.
Submitted: November 18th, 1998; revised: March 8th, 1999.