Exact augmented Lagrangians for constrained optimization problems in Hilbert spaces II: Applications

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Abstract

This two-part study is devoted to the analysis of the so-called exact augmented Lagrangians, introduced by Di Pillo and Grippo for finite dimensional optimization problems, in the case of optimization problems in Hilbert spaces. In the second part of our study we present applications of the general theory of exact augmented Lagrangians to several constrained variational problems and optimal control problems, including variational problems with additional constraints at the boundary, isoperimetric problems, problems with nonholonomic equality constraints (PDE constraints), and optimal control problems for linear evolution equations. We provide sufficient conditions for augmented Lagrangians for these problems to be globally/completely exact, that is, conditions under which a constrained variational problem/optimal control problem becomes equivalent to the problem of unconstrained minimization of the corresponding exact augmented Lagrangian in primal and dual variables simultaneously.

1 Preliminaries

Let us introduce notation and recall some auxiliary definitions from the first part of our study [14] and the theory of Sobolev spaces that will be utilised throughout this article.

1.1 Exact augmented Lagrangians

Let X and H be real Hilbert spaces, $\langle \cdot, \cdot \rangle$ be the inner products in X, H or \mathbb{R}^n , depending on the context, and $|\cdot|$ be the Euclidean norm. Let also $f, g_i \colon X \to \mathbb{R}$ with $i \in M := \{1, \ldots, m\}$ and $F \colon X \to H$ be given functions. Denote $g(\cdot) = (g_1(\cdot), \ldots, g_m(\cdot))$. Recall that a function $\varphi \colon X \to \mathbb{R}$ is called *coercive*, if $\varphi(x_n) \to +\infty$ for any sequence $\{x_n\} \subseteq X$ such that $\|x_n\| \to +\infty$ as $n \to \infty$.

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The classical Lagrangian for the constrained optimization problem

min
$$f(x)$$
 subject to $F(x) = 0$, $g_i(x) \le 0$, $i \in M$ (\mathcal{P})

is defined as

$$L(x, \lambda, \mu) = f(x) + \langle \lambda, F(x) \rangle + \langle \mu, g(x) \rangle, \quad \lambda \in H, \quad \mu \in \mathbb{R}^m.$$

To define an exact augmented Lagrangian for the problem (\mathcal{P}) , choose a convex non-decreasing lower semicontinuous (l.s.c.) function $\phi \colon [0, +\infty) \to [0, +\infty]$ such that $\phi(t) = 0$ if and only if t = 0 and $\text{dom } \phi \neq \{0\}$. Choose also a continuously differentiable concave function $\psi \colon [0, +\infty)^m \to \mathbb{R}$ such that $\psi(0) > 0$ and zero is a point of global maximum of ψ (some additional assumptions on these function as well as several particular examples can be found in [14]).

For any vectors $y, z \in \mathbb{R}^m$ denote by $\max\{y, z\} \in \mathbb{R}^m$ the coordinate-wise maximum of vectors y and z. Introduce the functions

$$b(x) = \psi(\max\{g(x), 0\}), \quad p(x, \mu) = \frac{b(x)}{1 + |\mu|^2} \quad \forall x \in X, \ \mu \in \mathbb{R}^m$$

and denote $\Omega = \{ x \in X \mid b(x) > 0, \ \phi(\|F(x)\|^2) < +\infty \}.$

Let the functions f, F, and $g_i, i \in M$, be continuously Fréchet differentiable, $DF(x)[\cdot]: X \to H$ be the Fréchet derivative of F at x, and $\nabla_x L(x, \lambda, \mu)$ be the gradient of the function $x \mapsto L(x, \lambda, \mu)$. The exact augmented Lagrangian for the problem (\mathcal{P}) is defined as follows:

$$\mathcal{L}(x,\lambda,\mu,c) = f(x) + \langle \lambda, F(x) \rangle + \frac{c}{2} \left(1 + \|\lambda\|^2 \right) \phi(\|F(x)\|^2)$$

$$+ \left\langle \mu, \max\left\{ g(x), -\frac{1}{c} p(x,\mu)\mu \right\} \right\rangle + \frac{c}{2p(x,\mu)} \left| \max\left\{ g(x), -\frac{1}{c} p(x,\mu)\mu \right\} \right|^2$$

$$+ \eta(x,\lambda,\mu), \quad (1)$$

if $x \in \Omega$, and $\mathcal{L}(x, \lambda, \mu, c) = +\infty$, otherwise. Here $\lambda \in H$ and $\mu \in \mathbb{R}^m$ are Lagrange multipliers, c > 0 is the penalty parameter, and

$$\eta(x,\lambda,\mu) = \frac{1}{2} \left\| DF(x) \left[\nabla_x L(x,\lambda,\mu) \right] \right\|^2 + \frac{1}{2} \sum_{i=1}^m \left(\langle \nabla g_i(x), \nabla_x L(x,\lambda,\mu) \rangle + g_i(x)^2 \mu_i \right)^2.$$
(2)

For the sake of simplicity, below we will consider only the case $\phi(t) \equiv t$ and $\psi(y) \equiv 1$, although exact augmented Lagrangian methods for problems considered in the article might perform better for a different choice of these functions.

For any $\gamma \in \mathbb{R}$ and c > 0 introduce sublevel sets

$$S_c(\gamma) = \left\{ (x, \lambda, \mu) \in \Omega \times H \times \mathbb{R}^m \mid \mathcal{L}(x, \lambda, \mu, c) \leq \gamma \right\},$$

$$Z_c(\gamma) = \left\{ x \in \Omega \mid f(x) + c \left(\|F(x)\|^2 + |\max\{g(x), 0\}^2| \right) \leq \gamma \right\}.$$
 (3)

Let us finally recall the definition of global exactness of the augmented Lagrangian. Suppose that there exists a globally optimal solution of the problem

 (\mathcal{P}) . Recall that a triplet $(x, \lambda, \mu) \in X \times H \times \mathbb{R}^m$ is called a KKT point of the problem (\mathcal{P}) , if x is feasible for this problem, $\nabla_x L(x, \lambda, \mu) = 0$, and for all $i \in M$ one has $\mu_i g_i(x) = 0$ and $\mu_i \geq 0$. Suppose that for any globally optimal solution x_* of the problem (\mathcal{P}) there exist Lagrange multipliers $\lambda_* \in H$ and $\mu_* \in \mathbb{R}^m$ such that the triplet (x_*, λ_*, μ_*) is a KKT point of the problem (\mathcal{P}) .

Definition 1. The augmented Lagrangian $\mathcal{L}(x,\lambda,\mu,c)$ is called *globally exact*, if there exists $c_* > 0$ such that for all $c \geq c_*$ a triplet (x_*,λ,μ_*) is a globally optimal solution of the problem

$$\min_{(x,\lambda,\mu)\in X\times H\times\mathbb{R}^m} \mathcal{L}(x,\lambda,\mu,c) \tag{4}$$

if and only if x_* is a globally optimal solution of the problem (\mathcal{P}) and (x_*, λ_*, μ_*) is a KKT point of this problem.

Remark 1. It should be noted that under some natural assumptions the augmented Lagrangian $\mathcal{L}(x,\lambda,\mu,c)$ is globally exact if and only if there exists c_* such that for any $c \geq c_*$ the optimal value of problem (4) is equal to the optimal value of the problem (\mathcal{P}) (see [14, Lemma 5.2]).

Sufficient conditions for the global exactness of the augmented Lagrangian $\mathcal{L}(x, \lambda, \mu, c)$ are expressed in terms of the function

$$Q(x)[\lambda, \mu] = \frac{1}{2} \left\| DF(x) \left[DF(x)^* [\lambda] + \sum_{i=1}^m \mu_i \nabla g_i(x) \right] \right\|^2 + \frac{1}{2} \left| \nabla g(x) \left(DF(x)^* [\lambda] + \sum_{i=1}^m \mu_i \nabla g_i(x) \right) + \operatorname{diag}(g_i(x)^2) \mu \right|^2$$
(5)

where $DF(x)^*[\cdot]$ is the adjoint operator of the linear operator $DF(x)[\cdot]$ and $\nabla g(x)y \in \mathbb{R}^m$ is the vector whose *i*-th coordinate is $\langle \nabla g_i(x), y \rangle$ for any $y \in X$. Note that the function $Q(x)[\cdot]$ is quadratic in (λ, μ) .

Definition 2. The function $Q(x)[\cdot]$ is called *positive definite* at a point $x \in X$, if there exists $\sigma > 0$ such that

$$Q(x)[\lambda, \mu] \ge \sigma(\|\lambda\|^2 + |\mu|^2) \quad \forall \lambda \in H, \ \mu \in \mathbb{R}^m$$

The supremum of all those $\sigma \geq 0$ for which the inequality above holds true is denoted by $\sigma_{\max}(Q(x))$.

Remark 2. Throughout this article we implicitly assume that all optimization problems under consideration have finite optimal value and a globally optimal solution.

1.2 L^p and Sobolev spaces

Let \mathscr{U} be a Hilbert space and $\Omega \subset \mathbb{R}^n$ be an open set. For any $1 \leq p \leq \infty$ we denote by $L^p(\Omega; \mathscr{U})$ the space consisting of all those measurable functions $u: \Omega \to \mathscr{U}$, for which

$$\begin{aligned} \|u\|_{L^p(\Omega;\mathscr{U})} &:= \left(\int_{\Omega} \|u(x)\|_{\mathscr{U}}^p \, dx\right)^{\frac{1}{p}} < +\infty, \quad \text{ if } 1 \leq p < \infty, \\ \|u\|_{L^{\infty}(\Omega;\mathscr{U})} &:= \mathrm{esssup}_{x \in \Omega} \, \|u(x)\|_{\mathscr{U}} < +\infty, \end{aligned}$$

endowed with the norm $\|\cdot\|_{L^p(\Omega);\mathscr{U})}$. If \mathcal{U} is the space \mathbb{R}^d equipped with the Euclidean norm $|\cdot|$, then we denote the corresponding norm on $L^p(\Omega;\mathbb{R}^d)$ by $\|\cdot\|_p$.

Let $W^{s,p}(\Omega)$ be the Sobolev space, while $W^{s,p}(\Omega; \mathbb{R}^d)$ be the space consisting of all those vector-valued functions $u: \Omega \to \mathbb{R}^d$, $u = (u_1, \ldots, u_d)$, for which $u_i \in W^{s,p}(\Omega)$ for all $i \in \{1, \ldots, d\}$. The space $W^{1,2}(\Omega; \mathbb{R}^d)$ is equipped with the inner product

$$\langle u, v \rangle = \int_{\Omega} (\langle u(x), v(x) \rangle + \langle \nabla u(x), \nabla v(x) \rangle) dx \quad \forall u, v \in W^{1,2}(\Omega; \mathbb{R}^d).$$

and the corresponding norm. The closure of the space $C_0^\infty(\Omega; \mathbb{R}^d)$ of all infinitely continuously differentiable functions $u \colon \Omega \to \mathbb{R}^d$ with compact support in the space $W^{1,2}(\Omega; \mathbb{R}^d)$ is denoted by $W_0^{1,2}(\Omega; \mathbb{R}^d)$.

As is well known (see, e.g. [16]), in the case when $\Omega = (a, b) \subset \mathbb{R}$ the space $W^{1,2}((a, b); \mathbb{R}^d)$ is isomorphic to the space of all absolutely continuous functions $u: [a, b] \to \mathbb{R}^d$ such that $u' \in L^2((a, b); \mathbb{R}^d)$, which we denote by $W^{1,2}([a, b]; \mathbb{R}^d)$.

We will also use a Sobolev-like function space introduced by the author in [12]. Let us recall the definition of this space. Any n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ of nonnegative integers α_i is called a multi-index. Its absolute value is defined as $|\alpha| = \alpha_1 + \ldots + \alpha_n$. For any multi-index α denote by $D^{\alpha} = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$ a differential operator of order $|\alpha|$, where $D_i = \partial/\partial x_i$. If $\alpha = 0$, then $D^{\alpha}u = u$ for any function u. For any $k \in \{0, \ldots, n\}$ define

$$I_k = \left\{ \alpha \in \mathbb{Z}_+^n \mid |\alpha| = k, \ \alpha_i = 0 \text{ or } \alpha_i = 1 \ \forall i \in \{1, \dots, n\} \right\}.$$

Finally, denote by $MW^{n,2}(\Omega)$ the set of all function $u \in L^2(\Omega)$ such that for any $k \in \{1, \ldots, n\}$ and $\alpha \in I_k$ there exists the weak derivative $D^{\alpha}u$ of u that belongs to $L^2(\Omega)$. Thus, $MW^{n,2}$ consists of all those functions $u \in L^2(\Omega)$ for which there exist all weak mixed derivatives of the order $k \in \{1, \ldots, n\}$ that belong to $L^2(\Omega)$. Note that $W^{n,2}(\Omega) \subseteq MW^{n,2}(\Omega)$ and this inclusion turns into an equality in the case n = 1.

The linear space $MW^{n,2}(\Omega)$ equipped with the following inner product

$$\langle u, v \rangle = \sum_{k=0}^{n} \sum_{\alpha \in I_k} \int_{\Omega} D^{\alpha} u(x) D^{\alpha} v(x) dx \quad \forall u, v \in MW^{n,2}(\Omega)$$

and the corresponding norm $\|\cdot; MW^{n,2}\|$ is a separable Hilbert space. The closure of $C_0^\infty(\Omega)$ in $MW^{n,2}(\Omega)$ is denoted by $MW_0^{n,2}(\Omega)$. By [12, Thm. 2] the seminorm on $MW_0^{n,2}(\Omega)$ corresponding to the inner product

$$\langle u, v \rangle = \int_{\Omega} D^{(1,\dots,1)} u(x) D^{(1,\dots,1)} v(x) dx \quad \forall u, v \in MW_0^{n,2}(\Omega)$$

is a norm that is equivalent to the norm $\|\cdot; MW^{n,2}\|$. Therefore, below we suppose that the space $MW_0^{n,2}(\Omega)$ is endowed with this inner product and the corresponding norm.

Functions from the space $MW_0^{n,2}(\Omega)$ admit a very simple and convenient characterisation in the case when $\Omega = \prod_{i=1}^n (a_i, b_i)$ is a bounded open box in \mathbb{R}^n . In this case for any $v \in L^2(\Omega)$ denote

$$(\mathcal{A}v)(x) = \int_{a_1}^{x_1} \dots \int_{a_n}^{x_n} v(s) \, ds \quad \text{for a.e. } x \in \Omega.$$
 (6)

By the Fubini theorem the function $\mathcal{A}v$ is correctly defined and \mathcal{A} is a continuous linear operator mapping $L^2(\Omega)$ to $L^2(\Omega)$. The following result holds true (see [12, Thm. 3]).

Theorem 1. Let $\Omega = \prod_{i=1}^n (a_i, b_i)$. Then a function $u: \Omega \to \mathbb{R}$ belongs to $MW_0^{n,2}(\Omega)$ if and only if there exist a function $v \in L^2(\Omega)$ such that:

1.
$$u(x) = (Av)(x)$$
 for a.e. $x \in \Omega$;

2.
$$\int_{a_i}^{b_i} v(x_1, \dots, x_{i-1}, s_i, x_{i+1}, \dots, x_n) ds_i = 0$$
 for a.e. $x \in \Omega$ and for all $i \in \{1, \dots, n\}$.

Moreover, such function v is uniquely defined and $v = D^{(1,...,1)}u$ in the weak sense.

2 Applications to the calculus of variations

In this section, we consider applications of the theory of exact augmented Lagrangians to constrained problems of the calculus of variations. Namely, we present new exact augmented Lagrangians for variational problems with additional constraints at the boundary, problems with isoperimetric constraints, and variational problems with nonholonomic equality constraints. We provide sufficient conditions on the problem data ensuring global/complete exactness of these augmented Lagrangians. For the sake of simplicity we consider each type of constraints separately, although one can define an exact augmented Lagrangian for variational problems involving all types of the aforementioned constraints and with the use of the results presented below derive sufficient conditions for the global/complete exactness of such augmented Lagrangian.

2.1 Problems with additional constraints at the boundary

Consider the following variational problems with additional nonlinear equality and inequality constraints at the boundary of the domain $(a, b) \subset \mathbb{R}$:

$$\min_{u \in W^{1,2}([a,b];\mathbb{R}^d)} \mathcal{I}(u) = \int_a^b f(u(x), u'(x), x) \, dx + f_0(u(a), u(b))$$
subject to $f_i(u(a), u(b)) = 0, \ j \in J, \ g_i(u(a), u(b)) \le 0, \ i \in M.$

Here $f: \mathbb{R}^d \times \mathbb{R}^d \times [a,b] \to \mathbb{R}$, $f = f(u,\xi,x)$, $f_j: \mathbb{R}^{2d} \to \mathbb{R}$, and $g_i: \mathbb{R}^{2d} \to \mathbb{R}$ are given function, and $J = \{1,\ldots,\ell\}$ and $M = \{1,\ldots,m\}$ are finite index sets any one of which can be empty. We suppose that the functions f_j and g_i are continuously differentiable, while f is a Carathéodoary function (i.e. the function $f(u,\xi,\cdot)$ is measurable for all $u,\xi\in\mathbb{R}^d$ and the function $f(\cdot,x)$ is continuous for a.e. $x\in[a,b]$ that is differentiable in u and ξ and the gradients $\nabla_u f$ and $\nabla_\xi f$ are Carathéodory functions as well.

We also impose the following growth conditions on f and its partial derivatives. Nanely, for any R > 0 there exist $C_R > 0$ and a.e. nonnegative functions $\eta_R \in L^1(a,b)$ and $\theta_R \in L^2(a,b)$ such that

$$|f(u,\xi,x)| \le C_R |\xi|^2 + \eta_R(x), \quad |\nabla_u f(u,\xi,x)| \le C_R |\xi|^2 + \eta_R(x), |\nabla_\xi f(u,\xi,x)| \le C_R |\xi| + \theta_R(x).$$
 (8)

for a.e. $x \in [a, b]$ and any $u, \xi \in \mathbb{R}^d$ such that $|u| \leq R$. These growth conditions ensure that for any $u \in W^{1,2}([a, b]; \mathbb{R}^d)$ the value $\mathcal{I}(u)$ is correctly defined and finite, the functional \mathcal{I} is Gâteaux differentiable and its Gâteaux derivative $\mathcal{I}'(u)[\cdot]$ has the form

$$\mathcal{I}'(u)[w] = \int_{a}^{b} \left(\langle \nabla_{u} f(u(x), u'(x), x), w(x) \rangle + \langle \nabla_{\xi} f(u(x), u'(x), x), w'(x) \rangle \right) dx + \langle \nabla f_{0}(u(a), u(b)), (w(a), w(b)) \rangle$$
(9)

for all $w \in W^{1,2}([a,b]; \mathbb{R}^d)$ (see, e.g. [5, Section 4.3]). Furthermore, it is easily seen that

$$\begin{aligned} \left| \mathcal{I}'(u)[w] - \mathcal{I}'(v)[w] \right| &\leq \left\| \nabla_{u} f(u(\cdot), u'(\cdot), \cdot) - \nabla_{u} f(v(\cdot), v'(\cdot), \cdot) \right\|_{1} \|w\|_{\infty} \\ &+ \left\| \nabla_{\xi} f(u(\cdot), u'(\cdot), \cdot) - \nabla_{\xi} f(v(\cdot), v'(\cdot), \cdot) \right\|_{2} \|w'\|_{2} \\ &+ \left| \nabla f_{0}(u(a), u(b)) - \nabla f_{0}(v(a), v(b)) \right| \|w\|_{\infty} \end{aligned}$$

for any $u, v, w \in W^{1,2}([a, b]; \mathbb{R}^d)$. Hence taking into account the fact that $\|\cdot\|_{\infty} \le C\|\cdot\|_{1,2}$ for some C > 0 (this result follows from the Sobolev imbedding theorem [1, Thm. 5.4]) one gets that

$$\|\mathcal{I}'(u) - \mathcal{I}'(v)\| \le C \|\nabla_u f(u(\cdot), u'(\cdot), \cdot) - \nabla_u f(v(\cdot), v'(\cdot), \cdot)\|_1 + \|\nabla_{\xi} f(u(\cdot), u'(\cdot), \cdot) - \nabla_{\xi} f(v(\cdot), v'(\cdot), \cdot)\|_2 + C |\nabla f_0(u(a), u(b)) - \nabla f_0(v(a), v(b))|.$$

Therefore, applying standard results on the continuity of Nemytskii operators (see, e.g. [2]) one can conclude that the Gâteaux derivative of \mathcal{I} is continuous, which implies that this functional is continuously Fréchet differentiable.

In order to define an exact augmented Lagrangian for problem (7), we need to compute the gradient of the objective function (the functional \mathcal{I}) and all constraints of this problem in the space $W^{1,2}([a,b],\mathbb{R}^d)$. To simplify this problem, we will use a trick (a change of variables) called *transition into the space of derivatives*, that was widely utilised by Demyanov in his works on the calculus of variations [6–11].

For any $y \in \mathbb{R}^d$ and $v \in L^2((a,b);\mathbb{R}^d)$ denote $\mathcal{A}(y,v)(x) = y + \int_a^x v(s) \, ds$ for all $x \in [a,b]$. The linear operator \mathcal{A} continuously maps $\mathbb{R}^d \times L^2((a,b);\mathbb{R}^d)$ to $W^{1,2}([a,b];\mathbb{R}^d)$. Furthermore, the Lebesgue differentiation theorem implies that \mathcal{A} is a one-to-one correspondence between these spaces with the inverse operator $\mathcal{A}^{-1}(u) = (u(a), u'(\cdot))$ (see, e.g. [16]). Therefore, by applying the change of variables $u = \mathcal{A}(y,v)$ we can convert problem (7) into an equivalent variational problem of the form:

$$\min_{(y,v)\in\mathbb{R}^d\times L^2((a,b);\mathbb{R}^d)} \widehat{\mathcal{I}}(y,v) := \mathcal{I}(\mathcal{A}(y,v))$$
subject to $\widehat{f}_j(y,v) = 0, \ j\in J, \ \widehat{g}_i(y,v) \leq 0, \ i\in M.$

where $\widehat{f}_j(y,v) := f_j(y,y + \int_a^b v(s) \, ds)$ and $\widehat{g}_i(y,v) := g_i(y,y + \int_a^b v(s) \, ds)$. We will define an exact augmented Lagrangian for problem (10) formulated in the "space of derivatives".

Denote $X=\mathbb{R}^d\times L^2((a,b);\mathbb{R}^d)$ and endow the space X with the inner product

$$\langle (y,v),(z,w)\rangle = \langle y,z\rangle + \int_a^b \langle v(x),w(x)\rangle dx \quad \forall (y,v),(z,w) \in X$$

and the corresponding norm. Fix some $(y,v) \in X$ and for the sake of convenience denote $u = \mathcal{A}(y,v)$. From equality (9) it follows that the functional $\widehat{\mathcal{I}}$ is continuously Fréchet differentiable and its Fréchet derivative $D\widehat{\mathcal{I}}(y,v)[\cdot]$ has the form

$$D\widehat{\mathcal{I}}(y,v)[z,w] = \int_{a}^{b} \left(\left\langle \nabla_{u} f(u(x), v(x), x), z + \int_{a}^{x} w(s) \, ds \right\rangle + \left\langle \nabla_{\xi} f(u(x), v(x), x), w(x) \right\rangle \right) dx + \left\langle \nabla_{u_{a}} f_{0}(y, u(b)) + \nabla_{u_{b}} f_{0}(y, u(b)), z \right\rangle + \left\langle \nabla_{u_{b}} f_{0}(y, u(b)), \int_{a}^{b} v(x) \, dx \right\rangle$$

for any $(z, w) \in X$, where $f_0 = f_0(u_a, u_b)$. Integrating by parts and rearranging the terms one gets

$$\begin{split} D\widehat{\mathcal{I}}(y,v)[z,w] \\ &= \Big\langle \int_a^b \nabla_u f(u(x),v(x),x) \, dx + \nabla_{u_a} f_0(y,u(b)) + \nabla_{u_b} f_0(y,u(b)),z \Big\rangle \\ &+ \int_a^b \left\langle P[y,v](x),w(x)\right\rangle dx \quad \forall (z,w) \in X, \end{split}$$

where

$$P[y, v](x) = \int_{x}^{b} \nabla_{u} f(u(s), v(s), s) \, ds + \nabla_{\xi} f(u(x), v(x), x) + \nabla_{u_{b}} f_{0}(y, u(b))$$

for a.e. $x \in (a,b)$. With the use of the growth conditions (8) one can readily check that $P[y,v] \in L^2((a,b);\mathbb{R}^d)$. Therefore, one can conclude that the gradient $\nabla \widehat{\mathcal{I}}(y,v)$ of the functional $\widehat{\mathcal{I}}$ has the form

$$\nabla \widehat{\mathcal{I}}(y,v) = \left(\int_{a}^{b} \nabla_{u} f(u(x), v(x), x) dx + \nabla_{u_{a}} f_{0}(y, u(b)) + \nabla_{u_{b}} f_{0}(y, u(b)), P[y, v] \right) \in X$$

$$\tag{11}$$

for all $(y,v) \in \mathbb{R}^d \times L^2((a,b); \mathbb{R}^d)$. Arguing in a similar way one can readily verify that

$$\nabla \widehat{f}_{j}(y,v) = \left(\nabla_{u_{a}} f_{j}(y,u(b)) + \nabla_{u_{b}} f_{j}(y,u(b)), \nabla_{u_{b}} f_{j}(y,u(b))\right) \in X,$$

$$\nabla \widehat{g}_{i}(y,v) = \left(\nabla_{u_{a}} g_{i}(y,u(b)) + \nabla_{u_{b}} g_{i}(y,u(b)), \nabla_{u_{b}} g_{i}(y,u(b))\right) \in X$$
(12)

for all $j \in J$ and $i \in M$, where $f_j = f_j(u_a, u_b)$ and $g_i = g_i(u_a, u_b)$.

Remark 3. It should be pointed out that with the use of the "transition into the space of derivatives" technique we were able to obtain simple and explicit analytical expressions for the gradients of the objective functional and constraints in a straightforward manner. Without this technique, computation of the gradients of these functions is a challenging task.

Denote $\widehat{F} = (\widehat{f}_1, \dots, \widehat{f}_\ell)$ and $\widehat{G} = (\widehat{g}_1, \dots, \widehat{g}_m)$. The classical Lagrangian for problem (10) has the form

$$L(y, v, \lambda, \mu) = \widehat{\mathcal{I}}(y, v) + \langle \lambda, \widehat{F}(y, v) \rangle + \langle \mu, \widehat{G}(y, v) \rangle \quad \forall \lambda \in \mathbb{R}^{\ell}, \ \mu \in \mathbb{R}^{m}.$$

Its gradient $\nabla_{(u,v)}L(y,v,\lambda,\mu)$ in (y,v) can be easily computed with the use of (11) and (12). An exact augmented Lagrangian $\mathcal{L}(y,v,\lambda,\mu,c)$ with $\lambda \in \mathbb{R}^{\ell}$ and $\mu \in \mathbb{R}^m$ for problem (10) is defined according to equalities (1) and (2). For the sake of brevity and convenience, we will write in explicitly in terms of the original problem (7):

$$\begin{split} \mathscr{L}(u,\lambda,\mu,c) &= \int_{a}^{b} f(u(x),u'(x),x) \, dx + f_{0}(u(a),u(b)) \\ &+ \langle \lambda, F(u(a),u(b)) \rangle + \frac{c}{2} (1+|\lambda|^{2}) |F(u(a),u(b))|^{2} \\ &+ \left\langle \mu, \max \left\{ G(u(a),u(b)), -\frac{1}{c(1+|\mu|^{2})} \mu \right\} \right\rangle \\ &+ \frac{c}{2} (1+|\mu|^{2}) \Big| \max \left\{ G(u(a),u(b)), -\frac{1}{c(1+|\mu|^{2})} \mu \right\} \Big|^{2} + \eta(u,\lambda,\mu), \end{split}$$

where $F = (f_1, ..., f_{\ell}), G = (g_1, ..., g_m),$ and

$$\begin{split} \eta(u,\lambda,\mu) &= \frac{1}{2} \sum_{j=1}^{c} \left(\left\langle \nabla_{u_a} f_j(u(a),u(b)) + \nabla_{u_b} f_j(u(a),u(b)), \nabla_y L(u,\lambda,\mu) \right\rangle \right. \\ &+ \int_{a}^{b} \left\langle \nabla_{u_b} f_j(u(a),u(b)), \nabla_v L(u,\lambda,\mu)(x) \right\rangle dx \bigg)^2 \\ &+ \frac{1}{2} \sum_{i=1}^{m} \left(\left\langle \nabla_{u_a} g_i(u(a),u(b)) + \nabla_{u_b} g_i(u(a),u(b)), \nabla_y L(u,\lambda,\mu) \right\rangle \right. \\ &+ \int_{a}^{b} \left\langle \nabla_{u_b} g_i(u(a),u(b)), \nabla_v L(u,\lambda,\mu)(x) \right\rangle dx + g_i(u(a),u(b)^2 \mu_i \bigg)^2 \end{split}$$

and

$$\nabla_{y}L(u,\lambda,\mu) = \int_{a}^{b} \nabla_{u}f(u(x),u'(x),x) dx + \nabla_{u_{a}}f_{0}(u(a),u(b)) + \nabla_{u_{b}}f_{0}(u(a),u(b))$$

$$+ \sum_{j=1}^{\ell} \lambda_{j} \left(\nabla_{u_{a}}f_{j}(u(a),u(b)) + \nabla_{u_{b}}f_{j}(u(a),u(b)) \right)$$

$$+ \sum_{i=1}^{m} \mu_{i} \left(\nabla_{u_{a}}g_{i}(u(a),u(b)) + \nabla_{u_{b}}g_{i}(u(a),u(b)) \right)$$

and

$$\nabla_v L(u, \lambda, \mu)(x) = P[u](x) + \sum_{j=1}^{\ell} \lambda_j \nabla_{u_b} f_j(u(a), u(b)) + \sum_{i=1}^{m} \mu_i \nabla_{u_b} g_i(u(a), u(b))$$

and

$$P[u](x) = \int_{x}^{b} \nabla_{u} f_{0}(u(s), u'(s), s) ds + \nabla_{\xi} f_{0}(u(x), u'(x), x) + \nabla_{u_{b}} f_{0}(u(a), u(b))$$

for a.e. $x \in (a, b)$.

Let us provide sufficient conditions for the exactness of the augmented Lagrangian $\mathcal{L}(y, v, \lambda, \mu, c)$, which ensure that the problem

$$\min_{(y,v,\lambda,\mu)\in\mathbb{R}^d\times L^2((a,b);\mathbb{R}^d)\times\mathbb{R}^{\ell+m}} \mathcal{L}(y,\mu,\lambda,\mu,c) \tag{13}$$

is (in some sense) equivalent to problem (7) for any sufficiently large c. We will formulate such conditions in terms of the functions $f_0, f_j, j \in J, g_i, i \in M$, and their derivatives. To avoid cumbersome assumptions on the function f_0 and its first and second order derivatives, we will assume that f_0 is quadratic in ξ and bounded below in u, although the following theorem can be proved under less restrictive assumptions on the function f_0 .

Theorem 2. Let the following assumptions be valid:

1. the function f has the form

$$f(u,\xi,x) = \langle \xi, h_2(x)\xi \rangle + \langle h_1(u,x), \xi \rangle + h_0(u,x) \quad \forall u, \xi \in \mathbb{R}^d, \ x \in [a,b]$$

for some function $h_2 \in L^{\infty}((a,b); \mathbb{R}^{d \times d})$ and some continuous functions $h_1 \colon \mathbb{R}^d \times [a,b] \to \mathbb{R}^d$, and $h_0 \colon \mathbb{R}^d \times [a,b] \to \mathbb{R}$ that are twice continuously differentiable in u and such that

$$\langle \xi, h_2(x)\xi \rangle \ge \varkappa |\xi|^2 \quad \forall \xi \in \mathbb{R}^d, \ x \in [a, b]$$

for some $\varkappa > 0$, and $|h_1(u,x)| \le \eta(x)$ and $h_0(u,x) \ge -\eta(x)$ for some a.e. nonnegative function $\eta \in L^2(a,b)$ and all $u \in \mathbb{R}^d$ and $x \in [a,b]$;

- 2. the functions f_j , $j \in J \cup \{0\}$, and g_i , $i \in M$, are twice continuously differentiable and the function $f_0(\cdot) + c(|F(\cdot)|^2 + |\max\{G(\cdot), 0\}|^2)$ is coercive for some c > 0;
- 3. the gradients $\nabla f_j(z)$, $j \in J$, and $\nabla g_i(z)$, $i \in \{i \in M \mid g_i(z) = 0\}$ are linearly independent for any $z \in \mathbb{R}^{2d}$.

Then the augmented Lagrangian $\mathcal{L}(y, v, \lambda, \mu, c)$ is globally exact and for any $\gamma \in \mathbb{R}$ there exists $c(\gamma) > 0$ such that for all $c \geq c(\gamma)$ the following statements hold true:

- 1. the sublevel set $S_c(\gamma)$ is bounded;
- 2. if $(y_*, v_*, \lambda_*, \mu_*) \in S_c(\gamma)$ is a point of local minimum of problem (13), then $u_* = \mathcal{A}(y_*, v_*)$ is a locally optimal solution of problem (7);
- 3. a quadruplet $(y_*, v_*, \lambda_*, \mu_*) \in S_c(\gamma)$ is a stationary point of problem (13) if and only if (u_*, λ_*, μ_*) with $u_* = \mathcal{A}(y_*, v_*)$ is a KKT point of problem (7) with $\mathcal{I}(u_*) \leq \gamma$.

Proof. Note that the first assumption of the theorem implies that the growth conditions (8) hold true. Furthermore, one can readily check that the first assumptions also ensures that the functional $\widehat{\mathcal{I}}$ is twice continuously Fréchet differentiable and all its first and second order derivatives are bounded on bounded subsets of the space X.

In turn, by [5, Thm. 3.23] the conditions on the functions h_2 , h_1 , and h_0 guarantee that the functional \mathcal{I} (and hence the functional $\widehat{\mathcal{I}}$ as well) is weakly sequentially lower semicontinuous (l.s.c.), while the mappings

$$(y,v) \mapsto \nabla_u f(\mathcal{A}(y,v)(\cdot),v(\cdot),\cdot), \quad (y,v) \mapsto \nabla_{\varepsilon} f(\mathcal{A}(y,v)(\cdot),v(\cdot),\cdot)$$

are weakly sequentially continuous, since they are affine in v (see [4]). Note also that if a sequence $\{(y_n, v_n)\}$ weakly converges in X, then the corresponding sequence $\{(y_n, u_n(b))\}$ converges in \mathbb{R}^{2d} . With the use of these facts one can readily check that the augmented Lagrangian $\mathcal{L}(y, v, \lambda, \mu, c)$ is weakly sequentially lower semicontinuous in (y, v).

In addition, the first assumption of the theorem implies that

$$f(u,\xi,x) \ge \varkappa |\xi|^2 - \eta(x)|\xi| - \eta(x) \quad \forall \xi, u \in \mathbb{R}^d, \text{ a.e. } x \in (a,b)$$

Therefore, if a sequence $\{(y_n, v_n)\}\subset X$ is such that $\|v_n\|_2\to +\infty$ as $n\to\infty$, then $\int_a^b f(u_n(x), v_n(x), x)\,dx\to +\infty$ as $n\to\infty$. In turn, if the sequence $\{v_n\}$ is bounded in $L^2((a,b);\mathbb{R}^d)$, but $|y_n|\to +\infty$ as $n\to\infty$, then the sequence $\{\int_a^b f(u_n(x), v_n(x), x)\,dx\}$ is bounded below, while

$$f_0(y_n, u_n(b)) + c\Big(|F(y_n, u_n(b))|^2 + |\max\{G(y_n, u_n(b)), 0\}|^2\Big) \to +\infty$$

as $n \to \infty$ by the second assumption of the theorem. Therefore, the set $Z_c(\gamma)$ (see (3)) is bounded for any c > 0 and $\gamma \in \mathbb{R}$.

Let us finally check that the third assumption of the theorem guarantees that for any bounded set $V \subset X$ there exists $\sigma > 0$ such that $\sigma_{\max}(Q(y,v)) \geq \sigma$ for all $(y,v) \in V$. Then applying [14, Cor. 3.6 and Thms. 5.3 and 5.7] one obtains the required result.

First, we show that $Q(y,v)[\cdot]$ is positive definite for any $(y,v) \in X$. Indeed, fix any $(y,v) \in X$. By [14, Lemma 3.3] the function $Q(y,v)[\cdot]$ is positive definite if and only if the linear operator $\mathcal{T} \colon X \to \mathbb{R}^{\ell+m(y,v)}$ defined as

$$\mathcal{T}(z,w) = \prod_{j=1}^{\ell} \left\{ \langle \nabla \widehat{f}_j(y,v), (z,w) \rangle \right\} \times \prod_{i \in M(y,v)} \left\{ \langle \nabla \widehat{g}_i(y,v), (z,w) \rangle \right\}$$

is surjective, where $M(y,v) = \{i \in M \mid \widehat{g}_i(y,v) = 0\}$ and m(y,v) is the cardinality of the set M(y,v).

As is easily seen, the linear operator \mathcal{T} is surjective if and only if the vectors

$$\nabla \widehat{f}_j(y,v), \quad j \in \{1,\dots,\ell\}, \quad \nabla \widehat{g}_i(y,v), \quad i \in M(y,v),$$
 (14)

are linearly independent. Therefore, the function $Q(y,v)[\cdot]$ is positive definite, provided these vectors are linearly independent. Let us check that this is indeed the case.

Suppose by contradiction that the vectors (14) are linearly dependent. Then bearing in mind (12) one gets that there exist real numbers α_j , $j \in \{1, ..., \ell\}$, and β_i , $i \in M(y, v)$, not all zero, such that

$$\sum_{j=1}^{\ell} \alpha_j (\nabla_{u_a} + \nabla_{u_b}) f_j(y, u(b)) + \sum_{i \in M(y, v)} \beta_j (\nabla_{u_a} + \nabla_{u_b}) g_i(y, u(b)) = 0,$$

$$\sum_{j=1}^{\ell} \alpha_j \nabla_{u_b} f_j(y, u(b)) + \sum_{i \in M(y, v)} \beta_j \nabla_{u_b} g_i(y, u(b)) = 0,$$

where $u = \mathcal{A}(y, v)$. Consequently, one has

$$\sum_{j=1}^{\ell} \alpha_j \nabla f_j(y, u(b)) + \sum_{i \in M(y, v)} \beta_j \nabla g_i(y, u(b)) = 0,$$

which contradicts the third assumption of the theorem. Thus, the function $Q(y,v)[\cdot]$ is positive definite for any $(y,v) \in X$.

Observe that from the definition of the function $Q(\cdot)$ (see (5)) it follows that in the case of the problem under consideration it has the form

$$Q(y,v)[\lambda,\mu] = \langle \begin{pmatrix} \lambda \\ \mu \end{pmatrix}, E_Q(y,u(b)) \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \rangle \quad \forall \lambda \in \mathbb{R}^{\ell}, \ \mu \in \mathbb{R}^m$$

for some matrix $E_Q(y, u(b))$ defined via the function G and the partial derivatives of F and G and, therefore, continuously depending on (y, u(b)).

It is easily seen that $\sigma_{\max}(Q(y,v))$ coincides with the smallest eigenvalue of the matrix $E_Q(y,u(b))$ that continuously depends on (y,u(b)), since the matrix $E_Q(y,u(b))$ continuously depends on (y,u(b)). Hence taking into account the fact that the matrix $E_Q(\cdot)$ is positive definite (since for all $(y,v) \in X$ one has $\sigma_{\max}(Q(y,v)) > 0$) one can conclude that for any set $V \subset X$ such that the set $V(a,b) = \{(y,u(b)) \in \mathbb{R}^{2d} \mid (y,v) \in V\}$ is bounded, there exists $\sigma > 0$ such that $\sigma_{\max}(Q(y,v)) \geq \sigma$. It remains to note that if a set $V \subset X$ is bounded, then the set V(a,b) is bounded as well.

2.2 Isoperimetric problems

Let us now consider an exact augmented Lagrangian for multidimensional variational problems with isoperimetric constraints. For the sake of simplicity, we restrain our consideration to the case of a single equality isoperimetric constraint, although all results of this subsection can be extended to the case of variational problems with multiple equality and inequality isoperimetric constraints.

Let $\Omega = \prod_{i=1}^n (a_i, b_i) \subset \mathbb{R}^n$ be a bounded open box and $\overline{u} \in W^{1,2}(\Omega; \mathbb{R}^d)$ be a given function. Consider the following variational problem with an isoperimetic equality constraint:

$$\min_{u \in \overline{u} + W_0^{1,2}(\Omega; \mathbb{R}^d)} \mathcal{I}_0(u) = \int_{\Omega} f_0(u(x), \nabla u(x), x) dx$$
subject to $\mathcal{I}_1(u) = \int_{\Omega} f_1(u(x), \nabla u(x), x) dx - \zeta = 0.$ (15)

Here $f_i: \mathbb{R}^d \times \mathbb{R}^{d \times n} \times \Omega \to \mathbb{R}$, $f_i = f_i(u, \xi, x)$ are Carathéodory functions that are differentiable in u and ξ and their partial derivatives $\nabla_u f_i$ and $\nabla_{\xi} f_i$ are

Carathéodory functions as well, while $\zeta \in \mathbb{R}$ is a given constant. Although without loss of generality one can suppose that $\zeta = 0$, in many applications it is more convenient to consider the case $\zeta \neq 0$, instead of redefining the function f_1 . Finally, the function \overline{u} defines boundary conditions, since

$$\{\overline{u}\} + W_0^{1,2}(\Omega; \mathbb{R}^d) = \{u \in W^{1,2}(\Omega; \mathbb{R}^d) \mid \operatorname{Tr}(u) = \operatorname{Tr}(\overline{u})\},$$

where $Tr(\cdot)$ is the trace operator (see [1, Sects. 5.20–5.22]).

We impose the following growth conditions on the functions f_i and their partial derivatives. Namely, there exist C > 0 and a.e. nonnegative functions $\eta \in L^1(\Omega)$ and $\theta \in L^2(\Omega)$ such that

$$|f_{i}(u,\xi,x)| \le C(|u|^{2} + |\xi|^{2}) + \eta(x),$$

$$\max\{|\nabla_{u}f_{i}(u,\xi,x)|, |\nabla_{\xi}f_{i}(u,\xi,x)|\} \le C(|u| + |\xi|) + \theta(x).$$
(16)

for a.e. $x \in \Omega$ and any $u \in \mathbb{R}^d$, $\xi \in \mathbb{R}^{d \times n}$ and $i \in \{0,1\}$. Let us note that these growth conditions can be relaxed with the use of the Sobolev imbedding theorem [1, Thm. 5.4]. In particular, in the case s=1 it is sufficient to assume that the growth conditions of the form (8) hold true. The first inequality in (16) can be replaced with $|f_i(u,\xi,x)| \leq C(|u|^q + |\xi|^2) + \eta(x)$, where q = 2n/(n-2) in the case n > 2, and q is any number in $[2, +\infty)$ in the case n = 2, etc.

The growth conditions (16) ensure that the functionals \mathcal{I}_i , $i \in \{0,1\}$, are Gâteaux differentiable and their Gâteaux derivatives have the form:

$$\mathcal{I}'_{i}(u)[w] = \int_{\Omega} \left(\langle \nabla_{u} f_{i}(u(x), u'(x), x), w(x) \rangle + \langle \nabla_{\xi} f_{i}(u(x), u'(x), x), \nabla w(x) \rangle \right) dx \tag{17}$$

for all $w \in W^{1,2}(\Omega; \mathbb{R}^d)$ (see, e.g. [5, Sect. 3.4.2]). Arguing in the same way as in the previous subsection and utilising the standard results on the continuity of Nemytskii operators [2] one can verify that the growth conditions (16) guarantees that the Gâteaux derivatives $\mathcal{I}'_i(\cdot)$ are continuous and, therefore, the functionals \mathcal{I}_i , $i \in \{0, 1\}$, are continuously Fréchet differentiable.

In order to compute the gradients of the functionals \mathcal{I}_i , we will utilise an extension of the "transition into the space of derivatives" technique to the multidimensional case, developed by the author in [12]. To this end, suppose that all KKT points (u_*, λ_*) of problem (15) one has $u_* \in \overline{u} + MW_0^{n,2}(\Omega; \mathbb{R}^d)$ (see Subsection 1.2). In other words, suppose that solutions of the corresponding Euler-Lagrange equation are more regular than is assumed in the formulation of the problem, in the sense that there exist all weak mixed derivatives of these solutions of all orders $k \in \{2, \ldots, n\}$ and these derivatives belong to $L^2(\Omega)$. Then problem (15) is equivalent to the following variational problem:

$$\min_{u \in MW_0^{n,2}(\Omega; \mathbb{R}^d)} \mathcal{I}_0(\overline{u} + u) = \int_{\Omega} f_0(\overline{u}(x) + u(x), \nabla \overline{u}(x) + \nabla u(x), x) dx$$
s.t.
$$\mathcal{I}_1(\overline{u} + u) = \int_{\Omega} f_1(\overline{u}(x) + u(x), \nabla \overline{u}(x) + \nabla u(x), x) dx - \zeta = 0.$$
(18)

By Theorem 1 this problem is equivalent to the following one:

$$\min_{v \in X} \ \widehat{\mathcal{I}}_0(v) := \mathcal{I}_0(\overline{u} + \mathcal{A}v) \quad \text{subject to } \widehat{\mathcal{I}}_1(v) := \mathcal{I}_1(\overline{u} + \mathcal{A}v), \tag{19}$$

where the linear operator A is defined in (6) and

$$X = \left\{ v \in L^2(\Omega; \mathbb{R}^d) \mid \int_{a_i}^{b_i} v \, ds_i = 0 \text{ for a.e. } x \in \Omega \ \forall i \in \{1, \dots, n\} \right\}.$$
 (20)

Applying (17) and integrating by parts (see [12] for more details) one obtains that the functionals $\widehat{\mathcal{I}}_i$, $i \in \{0, 1\}$, are continuously Fréchet differentiable and their Fréchet derivatives have the form

$$D\widehat{\mathcal{I}}_i[v](h) = \int_{\Omega} \langle P_i(v)(x), h(x) \rangle dx \quad \forall x \in X,$$

where

$$P_{i}(v)(x) = (-1)^{n} \int_{x_{1}}^{b_{1}} \dots \int_{x_{n}}^{b_{n}} \frac{\partial f}{\partial u}(u(s), \nabla u(s), s) ds$$

$$+ (-1)^{n-1} \sum_{i=1}^{n} \int_{x_{n}}^{b_{n}} \dots \int_{x_{i+1}}^{b_{i+1}} \int_{x_{i-1}}^{b_{i-1}} \dots \int_{x_{1}}^{b_{1}} \frac{\partial f}{\partial \xi_{i}}(u(s^{i}), \nabla u(s^{i}), s^{i}) ds_{1} \dots ds_{i-1} ds_{i+1} \dots ds_{n}$$
(21)

for a.e. $x \in \Omega$, $u = \overline{u} + Av$, $\partial f/\partial \xi_i = (\partial f/\partial \xi_{1i}, \dots, \partial f/\partial \xi_{di})$, and $s^i = (s_1, \dots, s_{i-1}, x_i, s_{i+1}, \dots, s_n)$ for any $i \in \{1, \dots, n\}$. With the use of the growth conditions (16) one can readily check that the nonlinear operator $P_i(\cdot)$ continuously maps $L^2(\Omega)$ to $L^2(\Omega)$. Therefore, the gradient of the functional $\widehat{\mathcal{I}}_i$ at a point $v \in X$ in the Hilbert space X is an orthogonal projection $\Pr_X(P_i(v))$ of the function $P_i(v)$ in $L^2(\Omega)$ onto the subspace X. This projection can be computed analytically with the use of the following proposition (see [12, Prop. 3]).

Proposition 1. For any $v \in L^2(\Omega)$ one has

$$\operatorname{Pr}_X v = v + \sum_{k=1}^n \sum_{\alpha \in L} (-1)^{|\alpha|} c_{\alpha} S_{\alpha} v,$$

where for any $\alpha \in I_k$ one has $c_{\alpha} = \prod_{i=1}^n (b_i - a_i)^{-\alpha_i}$ and

$$\left(S_{\alpha}v\right)(x) = \int_{a_{\alpha_{i_1}}}^{b_{\alpha_{i_1}}} \dots \int_{a_{\alpha_{i_k}}}^{b_{\alpha_{i_k}}} v \, ds_{i_k} \dots ds_{i_1},$$

and the indices $1 \le i_1 < \ldots < i_k \le n$ are such that $\alpha_r = 0$ if and only if $r \ne i_k$ (that is, i_j are indices of nonzero components of the multi-index α).

Thus, $\nabla \widehat{\mathcal{I}}_i(v) = \Pr_X P_i(v), i \in \{0,1\}$. The exact augmented Lagrangian $\mathscr{L}(v,\lambda,c)$ with $\lambda \in \mathbb{R}$ for problem (19) is defined according to equalities (1) and (2) as follows:

$$\mathcal{L}(v,\lambda,c) = \int_{\Omega} f_0(u(x), \nabla u(x), x) \, dx + \lambda \left(\int_{\Omega} f_1(u(x), \nabla u(x), x) \, dx - \zeta \right)$$

$$+ \frac{c}{2} (1 + \lambda^2) \left(\int_{\Omega} f_1(u(x), \nabla u(x), x) \, dx - \zeta \right)^2$$

$$+ \frac{1}{2} \left(\int_{\Omega} \langle \operatorname{Pr}_X P_1(v)(x), \operatorname{Pr}_X P_0(v)(x) + \lambda \operatorname{Pr}_X P_1(v)(x) \rangle \, dx \right)^2$$

where $u = \overline{u} + Av$ or, equivalently, $v = D^{(1,\dots,1)}(u - \overline{u})$ in the weak sense.

Let us provide sufficient conditions for the exactness of the augmented Lagrangian $\mathcal{L}(v, \lambda, c)$. We will formulate these conditions in "abstract terms" and then show how these conditions can be reformulated in terms of assumptions on the functions f_0 and f_1 in some particular cases.

Theorem 3. Let the following assumptions be valid:

- 1. for all KKT points (u_*, λ_*) of problem (15) one has $u_* \in \overline{u} + MW_0^{n,2}(\Omega; \mathbb{R}^d)$;
- 2. the growth conditions (16) hold true;
- 3. the functions f_0 and f_1 are twice differentiable in u and ξ and all corresponding second order derivatives are Carathéodory functions that are essentially bounded on $\mathbb{R}^d \times \mathbb{R}^{d \times n} \times \mathbb{R}^n$;
- 4. the augmented Lagrangian $\mathcal{L}(v,\lambda,c)$ is weakly sequentially l.s.c. in (v,λ) ;
- 5. for any $\gamma \in \mathbb{R}$ there exists c > 0 such that the set

$$Z_c(\gamma) = \left\{ u \in \overline{u} + MW_0^{n,2}(\Omega; \mathbb{R}^d) \mid \mathcal{I}_0(u) + c(\mathcal{I}_1(u))^2 \le \gamma \right\}$$

is bounded in $MW^{n,2}(\Omega; \mathbb{R}^d)$;

6. for any bounded subset $V \subset X$ there exists $\sigma > 0$ such that the inequality $\|\Pr_X P_1(v)\|_2 \ge \sigma$ holds true for all $v \in V$.

Then the augmented Lagrangian $\mathcal{L}(v,\lambda,c)$ is globally exact and for any $\gamma \in \mathbb{R}$ there exists $c(\gamma) > 0$ such that for all $c \geq c(\gamma)$ the following statements hold true:

- 1. the sublevel set $S_c(\gamma)$ is bounded in X;
- 2. if $(v_*, \lambda_*) \in S_c(\gamma)$ is a point of local minimum of $\mathcal{L}(v, \lambda, c)$, then $u_* = \overline{u} + \mathcal{A}v_*$ is locally optimal solution of problem (15);
- 3. a pair $(v_*, \lambda_*) \in S_c(\gamma)$ is a stationary point of $\mathcal{L}(v, \lambda, c)$ if and only if (u_*, λ_*) with $u_* = \overline{u} + \mathcal{A}v_*$ is a KKT point of problem (15) such that $\mathcal{I}_0(u_*) \leq \gamma$.

Proof. The first assumption of the theorem guarantees that the isoperimetric problem (15) in the Sobolev space $W^{1,2}(\Omega;\mathbb{R}^d)$ is equivalent to this problem formulated in the space $MW^{n,2}(\Omega;\mathbb{R}^d)$ (see (18)), which is, in turn, equivalent to problem (19) by Theorem 1.

The second and third assumptions of the theorem ensure that the functionals $\widehat{\mathcal{I}}_0$ and $\widehat{\mathcal{I}}_1$ are twice continuously Fréchet differentiable and their first and second order Fréchet derivatives are bounded on bounded subsets of the space X.

Note that in the case of problem (19) one has $Q(v)[\lambda] = (\|\operatorname{Pr}_X P_1(v)\|_2)^4 \lambda^2$ (see (5)). Consequently, the last assumption of the theorem implies that the function $Q(\cdot)$ is uniformly positive definite on any bounded subset of the space X. Therefore, applying the fifth assumption of the theorem and [14, Cor. 3.6 and Thms. 5.3 and 5.7] one obtains that all three statements of the theorem hold true.

Remark 4. Let us comment on the assumptions of the previous theorem:

(i) With the use of the Sobolev imbedding theorem one can relax the assumption on the essential boundedness of second order derivatives of the functions f_0 and f_1 in some particular cases. For example, in the case n=1 it is sufficient to suppose that for any R>0 there exist $C_R>0$ and a.e. nonnegative functions $\eta_R \in L^1(\Omega)$ and $\theta_R \in L^2(\Omega)$ such that

$$\left| \frac{\partial^2 f}{\partial u^2}(u,\xi,x) \right| \le C_R |\xi|^2 + \eta_R(x), \quad \left| \frac{\partial^2 f}{\partial u \partial \xi}(u,\xi,x) \right| \le C_R |\xi| + \theta_R(x),$$
$$\left| \frac{\partial^2 f}{\partial \xi^2}(u,\xi,x) \right| \le C_R$$

for a.e. $x \in \Omega$ and all $(u, \xi) \in \mathbb{R}^{2d}$ such that $|u| \leq R$.

(ii) The augmented Lagrangian $\mathcal{L}(v,\lambda,c)$ is weakly sequentially l.s.c. in (v,λ) , if f_0 is convex in ξ and quadratic in (u,ξ) , f_1 is affine in (u,ξ) , and the functions f_0 and f_1 satisfy some standard growth conditions. Under these assumptions, by [5, Cor. 3.24 and Thm. 3.26] the functional $\mathcal{I}_0(\cdot)$ is weakly sequentially l.s.c., the functional $\mathcal{I}_1(\cdot)$ is weakly sequentially continuous, while the function

$$(v,\lambda) \mapsto \langle \Pr_X P_1(v)(\cdot), \Pr_X P_0(v)(\cdot) + \lambda \Pr_X P_1(v)(\cdot) \rangle$$

is affine in v and also weakly sequentially continuous (see [4]). With the use of these facts one can readily verify that $\mathcal{L}(v,\lambda,c)$ is weakly sequentially l.s.c. in (v,λ) . It should be mentioned that in various particular cases (e.g. the case n=1) one can prove the weak sequential l.s.c. of the augmented Lagrangian under less restrictive assumptions.

(iii) It should be mentioned that in the case n=1, the fifth assumption of the theorem is satisfied, provided

$$f_0(u,\xi,x) \ge C_1 |\xi|^2 + C_2 |u|^q + \eta(x) \quad \forall (u,\xi,x) \in \mathbb{R}^{2d} \times \Omega$$
 (22)

for some $C_1 > 0$, $C_2 \in \mathbb{R}$, $1 \leq q < 2$, and $\eta \in L^1(\Omega)$, since this inequality implies that the functional $u \mapsto \mathcal{I}_0(\overline{u}+u)$ is coercive on $W_0^{1,2}(\Omega;\mathbb{R}^d) = MW_0^{1,2}(\Omega;\mathbb{R}^d)$. In the case n > 1, the question of when the sublevel set $Z_c(\gamma)$ of the penalty function $\mathcal{I}_0(\cdot) + c(\mathcal{I}_1(\cdot) - \zeta)^2$ is actually bounded in $MW_0^{n,2}(\Omega;\mathbb{R}^d)$ is a challenging open problem for future research. Let us however note that by carefully writing out the proofs of the main results from the first part of our study [14] in the case of the isoperimetric problem (15), one can show that the assumption on the boundedness of the set $Z_c(\gamma)$ in $MW_0^{n,2}(\Omega;\mathbb{R}^d)$ can be replaced by the less restrictive assumption on the boundedness of this set in the space $W^{1,2}(\Omega;\mathbb{R}^d)$. In particular, note that by the growth condition (16) the functions $P_i(\cdot)$ (see (21)) are bounded, provided the set $\{u = \overline{u} + Tv \mid v \in Z_c(\gamma)\}$ is bounded only in $W^{1,2}(\Omega;\mathbb{R}^d)$. In turn, the boundedness of the set $Z_c(\gamma)$ in this Sobolev space can be ensured by imposing the growth condition (22) on the function f_0 . (iv) Let us point out some particular cases in which the last assumption of the theorem holds true. If the function f_1 is affine in (u, ξ) , that is, $f_1(u, \xi, x) =$

(iv) Let us point out some particular cases in which the last assumption of the theorem holds true. If the function f_1 is affine in (u,ξ) , that is, $f_1(u,\xi,x) = \langle h_1(x), \xi \rangle + \langle h_2(x), u \rangle + h_3(x)$, then the operator $P_1(\cdot)$ is constant and the fourth assumption of the theorem holds true, provided the projection of $P_1(v)$ onto X is nonzero. Note that by [12, Prop. 5] the assumption on the projection of P_1 is satisfied in the case when h_1 is differentiable, provided the function $h_2(\cdot) - \sum_{i=1}^n D_i h_1(\cdot)$ is not identically zero. In the case when the function

 f_1 is quadratic in (u, ξ) , the fourth assumption of the theorem is satisfied, if $\Pr_X P_1(v) \neq 0$ for any $v \in X$, since in this case the function $P_1(\cdot)$ is affine, which implies that the norm $\|\Pr_X P_1(\cdot)\|_2$ is weakly lower semicontinuous. With the use of this fact one can prove the existence of the required $\sigma > 0$. Note that by [12, Prop. 5] under some smoothness assumptions the inequality $\Pr_X P_1(v) \neq 0$ means that the Euler-Lagrange equation for the functional \mathcal{I}_1 has no solutions. In the general case, the last assumption of the theorem means that the gradient of the functional \mathcal{I}_1 is bounded away from zero on bounded sets, which under some smoothness assumptions implies that the Euler-Lagrange equation for $\mathcal{I}_1(\cdot)$ has no solutions.

2.3 Problems with nonholonomic equality constraints

In this subsection we study an exact augmented Lagrangian for multidimensional variational problems with nonholonomic equality constraints (which can be viewed as PDE constraints). Let, as in the previous section, $\Omega = \prod_{i=1}^{n} (a_i, b_i) \subset \mathbb{R}^n$ be a bounded open box and $\overline{u} \in W^{1,2}(\Omega; \mathbb{R}^d)$ be a given function. Consider the following variational problem with nonholonomic equality (PDE) constraints:

$$\min_{u \in \overline{u} + W_0^{1,2}(\Omega; \mathbb{R}^d)} \mathcal{I}(u) = \int_{\Omega} f(u(x), \nabla u(x), x) \, dx$$
subject to $F(u(x), \nabla u(x), x) = 0$ for a.e. $x \in \Omega$.

Here $f: \mathbb{R}^d \times \mathbb{R}^{d \times n} \times \Omega \to \mathbb{R}$, $f = f(u, \xi, x)$, and $F: \mathbb{R}^d \times \mathbb{R}^{d \times n} \times \Omega \to \mathbb{R}^\ell$, $F = F(u, \xi, x)$, are Carathéodory functions that are differentiable in u and ξ and their partial derivatives in u and ξ are Carathéodory functions as well.

Suppose that the function f satisfies the growth conditions of the form (16), while the function F satisfies the following growth conditions. Namely, there exist C > 0 and an a.e. nonnegative function $\eta \in L^2(\Omega)$ such that

$$|F(u,\xi,x)| \le C(|u|+|\xi|) + \eta(x), \quad \max\{|\nabla_u F(u,\xi,x)|, |\nabla_\xi F(u,\xi,x)|\} \le C. \tag{24}$$

These assumptions guarantee that the functional $\mathcal{I}(\cdot)$ is continuously Fréchet differentiable, while the Nemytskii operator $\mathcal{F}(u) = F(u(\cdot), \nabla u(\cdot), \cdot)$ maps the Sobolev space $W^{1,2}(\Omega; \mathbb{R}^d)$ to $L^2(\Omega; \mathbb{R}^\ell)$ and is continuously Fréchet differentiable as well (see, e.g. [2]). The Fréchet derivative of \mathcal{I} has the form (17), while the Fréchet derivative of the Nemytskii operator \mathcal{F} is defined as

$$D\mathcal{F}(u)[w] = \nabla_u F(u(\cdot), \nabla u(\cdot), \cdot) w(\cdot) + \sum_{i=1}^n \frac{\partial F}{\partial \xi^i}(u(\cdot), \nabla u(\cdot), \cdot) \nabla w_i(\cdot)$$
 (25)

for any $u, w \in W^{1,2}(\Omega; \mathbb{R}^d)$, where $\partial/\partial \xi^i = (\partial/\partial \xi_{i1}, \dots, \partial/\partial \xi_{in})$ and $w = (w_1, \dots, w_d)$.

In order to define an exact augmented Lagrangian for problem (23) we will use the same technique as in the case of isoperimetric constraints. Namely, suppose that for all KKT points (u_*, λ_*) of problem (23) one has $u_* \in \overline{u} + MW_0^{n,2}(\Omega; \mathbb{R}^d)$. Then taking into account Theorem 1 one can conclude that problem (23) is equivalent to the following one:

$$\min_{u \in X} \widehat{\mathcal{I}}(v) := \mathcal{I}(\overline{u} + \mathcal{A}v) \text{ subject to } \widehat{\mathcal{F}}(u) := \mathcal{F}(\overline{u} + \mathcal{A}v) = 0,$$
 (26)

where the Hilbert space X is defined in (20). Arguing in the same way as in previous section, one obtains that the gradient of the functional $\widehat{\mathcal{I}}$ at a point $v \in X$ has the form $\Pr_X P(v)$, where P(v) is defined according to the equality (21) with f_i replaced by f. In turn, the Nemytskii operator $\widehat{\mathcal{F}}(\cdot)$ maps X to $L^2(\Omega; \mathbb{R}^\ell)$ and is continuously Fréchet differentiable. Applying (25) and integrating by parts one obtains that the Fréchet derivative of $\widehat{\mathcal{F}}(\cdot)$ has the form $D\widehat{\mathcal{F}}(v)[h] = P_F(v)h$ for any $v, h \in X$, where

$$P_{F}(v)(x) = (-1)^{n} \int_{x_{1}}^{b_{1}} \dots \int_{x_{n}}^{b_{n}} \nabla_{u} F(u(s), \nabla u(s), s) ds$$

$$+ (-1)^{n-1} \sum_{i=1}^{n} \int_{x_{n}}^{b_{n}} \dots \int_{x_{i+1}}^{b_{i+1}} \int_{x_{i-1}}^{b_{i-1}} \dots \int_{x_{1}}^{b_{1}} \frac{\partial F}{\partial \mathcal{E}_{i}}(u(s^{i}), \nabla u(s^{i}), s^{i}) ds_{1} \dots ds_{i-1} ds_{i+1} \dots ds_{n}$$

for a.e. $x \in \Omega$ and $u = \overline{u} + Av$. Note that the growth conditions (24) imply that $P_F(v) \in L^{\infty}(\Omega; \mathbb{R}^{\ell \times d})$ for any $v \in X$.

The exact augmented Lagrangian $\mathcal{L}(v, \lambda, c)$ with $\lambda \in L^2(\Omega; \mathbb{R}^{\ell})$ for problem (26) is defined according to equalities (1) and (2) as follows:

$$\mathcal{L}(v,\lambda,c) = \int_{\Omega} f(u(x), \nabla u(x), x) dx + \int_{\Omega} \langle \lambda(x), F(u(x), \nabla u(x), x) \rangle dx$$
$$+ \frac{c}{2} (1 + ||\lambda||_2^2) \int_{\Omega} |F(u(x), \nabla u(x), x)|^2 dx$$
$$+ \frac{1}{2} \int_{\Omega} |P_F(v)(x) (\operatorname{Pr}_X P(v)(x) + P_F(v)(x)^* \lambda(x))|^2 dx$$

where $u = \overline{u} + Av$ and $P_F(v)(x)^*$ is the transpose of the matrix $P_F(v)(x)$.

Let us provide sufficient conditions for the exactness of the augmented Lagrangian $\mathcal{L}(v, \lambda, c)$. For the sake of simplicity, we will formulate these conditions under the assumption that the mapping F is affine in (u, ξ) (that is, problem (23) is a variational problem with constraints defined by a system of linear PDE), since this is the simplest assumption under which the corresponding Nemytskii operator is twice continuously Fréchet differentiable.

Theorem 4. Let the following assumptions be valid:

- 1. for all KKT points (u_*, λ_*) of problem (23) one has $u_* \in \overline{u} + MW_0^{n,2}(\Omega; \mathbb{R}^d)$;
- 2. the function f is twice differentiable in (u, ξ) , satisfies the growth conditions (16), and its second order derivatives are Carathéodory functions and essentially bounded on $\mathbb{R}^d \times \mathbb{R}^{d \times n} \times \mathbb{R}^n$;
- 3. the augmented Lagrangian $\mathcal{L}(v,\lambda,c)$ is weakly sequentially l.s.c. in (v,λ) ;
- 4. the function F has the form $F(u, \xi, x) = A(x)u + \sum_{i=1}^{n} B_i(x)\xi_i + D(x)$ for some essentially bounded measurable functions A, B_i , and D;
- 5. for any $\gamma \in \mathbb{R}$ there exists c > 0 such that the set

$$Z_c(\gamma) = \left\{ u \in \overline{u} + MW_0^{n,2}(\Omega; \mathbb{R}^d) \mid \mathcal{I}_0(u) + c \| F(u(\cdot), \nabla u(\cdot), \cdot) \|_2^2 \le \gamma \right\}$$

is bounded in $MW^{n,2}(\Omega; \mathbb{R}^d)$:

6. $P_F(v) \neq 0$ for any $v \in X$ (if the functions B_i are continuously differentiable it is sufficient to suppose that $A(x) - \sum_{i=1}^n D_i B_i(x) \neq 0$).

Then the augmented Lagrangian $\mathcal{L}(v,\lambda,c)$ is globally exact and for any $\gamma \in \mathbb{R}$ there exists $c(\gamma) > 0$ such that for all $c \geq c(\gamma)$ the following statements hold true:

- 1. the sublevel set $S_c(\gamma)$ is bounded in X;
- 2. if $(v_*, \lambda_*) \in S_c(\gamma)$ is a point of local minimum of $\mathcal{L}(v, \lambda, c)$, then $u_* = \overline{u} + \mathcal{A}v_*$ is locally optimal solution of problem (23);
- 3. a pair $(v_*, \lambda_*) \in S_c(\gamma)$ is a stationary point of $\mathcal{L}(v, \lambda, c)$ if and only if (u_*, λ_*) with $u_* = \overline{u} + \mathcal{A}v_*$ is a KKT point of problem (23) such that $\mathcal{I}_0(u_*) \leq \gamma$.

The proof of this theorem largely repeats the proof of Theorem 3 and consists in verifying that the assumptions of the theorem ensure that the assumptions of [14, Cor. 3.6 and Thms. 5.3 and 5.7] are satisfied. We omit it for the sake of shortness.

Remark 5. (i) Just like in the case of isoperimetric problems, by carefully writing out the proof of the main results of the first part of our study [14] one can check that instead of the boundedness of the set $Z_c(\gamma)$ in $MW^{n,2}(\Omega; \mathbb{R}^d)$ it is sufficient to suppose that this set is bounded only in $W^{1,2}(\Omega; \mathbb{R}^d)$ to guarantee that the claims of Theorem 4 remain to hold true. Let us also mention that the boundedness of the set $Z_c(\gamma)$ follows, e.g. from the growth condition (22).

(ii) Let us note that the augmented Lagrangian $\mathcal{L}(v,\lambda,c)$ is weakly sequentially l.s.c. in (v,λ) , in particular, if the function $F=F(u,\xi,x)$ does not depends on ξ (i.e. the constraints are holonomic), and the function f has the form

$$f(u,\xi,x) = \langle \xi, h_2(x)\xi \rangle + \langle h_1(x), \xi \rangle + h_0(u,x)$$

for some functions h_2 , h_1 , and h_0 such that the matrix $h_2(x)$ is positive semidefinite for a.e. $x \in \Omega$. Under these assumptions the functional $\mathcal{I}(\cdot)$ is weakly sequentially lower semicontinuous (provided h_2 satisfies appropriate growth conditions) by [5, Cor. 3.24], the map $\Pr_X P(\cdot)$ is weakly sequentially continuous, the map $P_F(\cdot)$ does not depends on v, while the Nemytskii operator $u \mapsto F(u(\cdot), \nabla u(\cdot), \cdot)$ maps weakly continuous sequences to strongly continuous ones by the Rellich-Kondrachov theorem (see, e.g. [1, Thm. 6.2]). Combining these facts one can check that the augmented Lagrangian is indeed weakly sequentially l.s.c. in (v, λ) . The question of when the augmented Lagrangian is weakly sequentially l.s.c. in the case of nonholonomic constraints is an interesting open problem, which is connected with the analysis of compact imbedding of the space $MW_0^{n,2}(\Omega)$.

3 An application to optimal control

Below we use standard definitions and results on control problems for linear evolution equations that can be found in [17]. Let \mathscr{H} and \mathscr{U} be complex Hilbert spaces, \mathbb{T} be a strongly continuous semigroup on \mathscr{H} with generator $\mathcal{A} \colon \mathcal{D}(A) \to \mathscr{H}$. Let also \mathcal{B} be an admissible control operator for \mathbb{T} (see [17, Def. 4.2.1]).

Consider the following fixed-endpoint optimal control problem:

$$\min_{(x,u)} \frac{1}{2} \int_0^T \|u(t)\|_{\mathscr{U}}^2 dt$$
subject to $\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \quad t \in [0,T], \quad x(0) = x_0, \quad x(T) = x_T.$

Here T > 0 and $x_0, x_T \in \mathcal{H}$ are fixed, and controls $u(\cdot)$ belong to the space $L^2((0,T);\mathcal{U})$.

Our aim is to convert the fixed-endpoint problem (27) into an equivalent free-endpoint problem for an exact augmented Lagrangian. To define this augmented Lagrangian, for any $t \geq 0$ introduce the so-called input map $F_t u = \int_0^t \mathbb{T}_{t-\tau} \mathcal{B}u(\tau) d\tau$ corresponding to the pair $(\mathcal{A}, \mathcal{B})$. The assumption that \mathcal{B} is an admissible control operator for \mathbb{T} guarantees that for any $t \in T$ the input map F_t is a bounded linear operator from $L^2((0,T); \mathcal{U})$ to \mathcal{H} by [17, Prop. 4.2.2]. Moreover, by [17, Prop. 4.2.5] for any $x_0 \in \mathcal{H}$ the initial value problem

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \quad x(0) = x_0 \tag{28}$$

had a unique solution $x \in C([0,T]; \mathcal{H})$ given by

$$x(t) = \mathbb{T}_t x_0 + F_t u \quad \forall t \in [0, T]. \tag{29}$$

Therefore, problem (27) can be rewritten as follows:

$$\min_{u \in L^2((0,T);\mathscr{U})} \frac{1}{2} \int_0^T \|u(t)\|_{\mathscr{U}}^2 dt \quad \text{subject to} \quad F_T u = \widehat{x}_T, \tag{30}$$

where $\hat{x}_T = x_T - \mathbb{T}_T x_0$. An exact augmented Lagrangian for this problem has the form

$$\mathcal{L}(u,\lambda,c) = \frac{1}{2} \int_0^T \|u(t)\|_{\mathscr{U}}^2 dt + \langle \lambda, F_T u - \widehat{x}_T \rangle + \frac{c}{2} (1 + \|\lambda\|_{\mathscr{H}}^2) \|F_T u - \widehat{x}_T\|^2 + \frac{1}{2} \|F_T (u + F_T^* \lambda)\|^2,$$

where $\lambda \in \mathcal{H}$ is a Lagrange multiplier corresponding to the end-point constraint $x(T) = x_T$ and F_T is the adjoint operator of F_T .

Remark 6. Note that to compute F_Tv for some $v \in L^2((0,T); \mathcal{U})$ one needs to find a solution of the initial value problem

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}v(t), \quad x(0) = 0$$

and set $F_T v = x(T)$. Let us also point out that $(F_T^*\lambda)(t) = \mathcal{B}^*\mathbb{T}_{T-t}^*\lambda$ for any $t \in [0,T]$ and $\lambda \in \mathscr{H}$. If the operator \mathcal{A} is self-adjoint, then $\mathbb{T}_t = \mathbb{T}_t^*$ for all $t \geq 0$ and $\mathbb{T}_{T-t}^*\lambda = y(T-t)$ for any $t \in [0,T]$, where y is a solution of the initial value problem $\dot{y}(t) = \mathcal{A}y(t), y(0) = \lambda$ (see (29)).

Let us provide sufficient conditions for the exactness of the augmented Lagrangian $\mathcal{L}(u,\lambda,c)$, which ensure that the problem

$$\min_{(u,\lambda)\in L^2((0,T);\mathcal{U})\times\mathcal{H}}\mathcal{L}(u,\lambda,c)$$

is equivalent to the problem (27) for any c greater than some $c_* > 0$. Note that the problem above can be rewritten as the following free-endpoint optimal control problem:

$$\min_{(x,u,\lambda)} \frac{1}{2} \int_{0}^{T} \|u(t)\|_{\mathscr{U}}^{2} dt + \langle \lambda, x(T) - x_{T} \rangle + \frac{c}{2} (1 + \|\lambda\|_{\mathscr{H}}^{2}) \|x(T) - x_{T}\|^{2}
+ \frac{1}{2} \|F_{T}(u + F_{T}^{*}\lambda)\|^{2}
\text{subject to } \dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \quad t \in [0,T], \quad x(0) = x_{0}.$$

To conveniently formulate sufficient conditions for the equivalence of this problem and problem (27), recall that system (28) is called *exactly controllable* using L^2 -controls (see, e.g. [15]), if for any initial state $x_0 \in \mathcal{H}$ and any final state $x_T \in \mathcal{H}$ one can find a control $u \in L^2((0,T);\mathcal{U})$ such that the corresponding solution $x(\cdot)$ of the initial value problem (28) satisfies the equality $x(T) = x_T$. Particular examples of exactly controllable systems can be found in [3,15,17].

Theorem 5. Let system (28) be exactly controllable using L^2 -controls. Then the augmented Lagrangian $\mathcal{L}(u,\lambda,c)$ is globally exact and for any $\gamma \in \mathbb{R}$ there exists $c(\gamma) > 0$ such that for all $c \geq c(\gamma)$ the following statements hold true:

- 1. the sublevel set $S_c(\gamma)$ is bounded;
- 2. if a point $(u_*, \lambda_*) \in S_c(\gamma)$ is a stationary point of $\mathcal{L}(u, \lambda, c)$, then u_* is a globally optimal solution of the problem (30) and (u_*, λ_*) is a KKT-point of this problem.

Proof. In the case of problem (30) for any $u \in L^2((0,T); \mathcal{U})$ the function Q(u) has the form $Q(u)[\lambda] = 0.5 ||F_T(F_T^*\lambda)||^2$ and does not depend on u. Note that from the expression (29) for the solution of the initial value problem (28) it follows that the exact controllability of system (28) is equivalent to the surjectivity of the input map F_T . Therefore, by the assumption of the theorem and [14, Lemma 3.3] the function $Q(\cdot)$ is uniformly positive definite on the entire space $L^2((0,T); \mathcal{U})$.

Since the objective function of the problem (30) is obviously coercive, the set $Z_c(\gamma)$ is bounded for any $c \geq 0$ and $\gamma \in \mathbb{R}$. Hence applying [14, Cor. 3.6] one obtains that the sublevel set $S_c(\gamma)$ is bounded for any sufficiently large $c = c(\gamma)$.

Observe that the augmented Lagrangian $\mathcal{L}(u,\lambda,c)$ can be represented as the sum of the convex in (u,λ) function

$$\frac{1}{2} \int_{0}^{T} \|u(t)\|_{\mathscr{U}}^{2} dt + \langle \lambda, F_{T}u - \widehat{x}_{T} \rangle + \frac{c}{2} \|F_{T}u - \widehat{x}_{T}\|^{2} + \frac{1}{2} \|F_{T}(u + F_{T}^{*}\lambda)\|^{2}$$

and the function $\omega(u,\lambda) = \|\lambda\|_{\mathscr{H}}^2 \|F_T u - \widehat{x}_T\|^2$, which, as one can readily check, is weakly sequentially l.s.c. as the product of two *nonnegative* convex (and therefore sequentially l.s.c.) functions. Consequently, for any $c \geq 0$ the augmented Lagrangian $\mathscr{L}(\cdot,c)$ is weakly sequentially lower semicontinuous. Hence applying [14, Thm. 5.3] one can conclude that $\mathscr{L}(u,\lambda,c)$ is globally exact.

Finally, applying [14, Thm. 5.7] one obtains that for any sufficiently large c stationary points $(u_*, \lambda_*) \in S_c(\gamma)$ of $\mathcal{L}(u, \lambda, c)$ are precisely KKT-point of the problem (30). The convexity of this problem implies that any such u_* is globally optimal, that is, the last statement of the theorem holds true.

Remark 7. Let us note that the assumption on exact controllability of system (28) was used in closely related result on the exactness of penalty function for problem (27). Namely, in [15] it was proved that for any sufficiently large $c \ge 0$ problem (27) is equivalent to the free-endpoint penalized problem

$$\min_{(x,u)} \frac{1}{2} \int_0^T \|u(t)\|_{\mathscr{U}}^2 dt + c \|x(T) - x_T\|_{\mathscr{H}}$$

subject to $\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \quad t \in [0,T], \quad x(0) = x_0$

under the assumption that system (28) is exactly controllable and there exists C>0 such that the control from the definition of exact controllability satisfies the inequality $\|u\|_{L^2((0,T);\mathscr{U})} \leq C(\|x_0\| + \|x_T\|)$. In turn, in [13] it was shown the the assumption on the existence of C can be dropped and, furthermore, the exact controllability assumption can be replaced with the weaker assumption on the closedness of the image of the input map F_T . Whether the closedness of the image of this map is sufficient for the global exactness of the augmented Lagrangian $\mathscr{L}(u,\lambda,c)$ is an open problem.

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