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BARRIER METHODS FOR OPTIMAL CONTROL PROBLEMS WITH CONVEX NONLINEAR GRADIENT STATE CONSTRAINTS*

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Abstract. In this paper we are concerned with the application of interior point methods in function space to gradient constrained optimal control problems, governed by partial differential equations. We will derive existence of solutions together with first order optimality conditions. Afterwards we show continuity of the central path, together with convergence rates depending on the interior point parameter.

Key words. interior point method, necessary optimality conditions, convergence of the central path, gradient constrained optimization

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1. Introduction.

In a large number of processes that are modeled using partial differential equations bounds on the gradient of the state variable are of vital importance for the underlying model: large temperature gradients during cooling or heating processes may lead to destruction of the object that is being cooled or heated; in elasticity the gradient of the deformation determines the change between elastic and plastic material behavior. In any attempt to optimize such processes the gradient therefore has to be regarded. However, not much attention was given to constraints of gradient type; see [5, 6, 7, 8, 13, 19, 22, 31].

Problems with constraints on the state (pointwise or regarding the gradient) form a class of highly nonlinear and nonsmooth problems. A popular approach for their efficient solution are path-following methods, which solve a sequence of easier to tackle problems. These methods are constructed in a way such that the sequence of the solutions converges to the solution of the original problem. Among these methods one can distinguish three main lines of research. Lavrentiev regularization methods due to Tröltzsch et al. [9, 23, 24, 30], Moreau–Yosida approximation methods due to Hintermüller and Kunisch [2, 3, 20, 21, 22], and interior point methods [28, 29]. While the first two candidates abandon feasibility to improve the regularity of the dual variables, interior point methods yield feasible solutions and aim towards smooth systems of equations.

Application of interior point methods to gradient bounds has been proposed in [31] together with a posteriori error estimates with respect to the interior point parameter and the discretization error. A recent and very comprehensive discussion of the application of Moreau–Yosida approximation to a class of problems containing gradient constraints has been given in [22]. To the authors' knowledge, there are currently no contributions on the application of Lavrentiev techniques to gradient bounds.

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In this paper we perform the analysis of the homotopy path generated by barrier methods to problems with gradient bounds. We approach this problem on the base of the analysis in [29], where pointwise state constraints are considered. Although we can build up on techniques and results established there, it will turn out that a number of interesting, additional issues arise in the case of gradient bounds. For example, the topological framework has to be chosen differently with a C^1-norm, and in contrast to pointwise state constraints the gradient bounds considered here are nonlinear.

Our paper is structured as follows. In section 2 we establish an abstract theoretical framework for our analysis and illustrate the application of the framework to some PDE constrained optimal control problems. In section 3 we consider barrier functionals for gradient bounds and characterize their subdifferentials. Then existence of minimizers and first order optimality conditions are established, together with uniform bounds on the barrier gradients. Finally we consider the convergence of the path of minimizers and derive an order of convergence for a typical case.

2. Gradient constrained optimal control problems. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, let $\emptyset \neq \Omega_C \subseteq \Omega$ be an open subset, and let $\overline{\Omega}_C$ be its closure. Define the space of states U as a closed subspace of $C^1(\overline{\Omega}_C) \oplus L^2(\Omega \setminus \overline{\Omega}_C)$, which is clearly a Banach space, and let $W \subset U$ be a dense subspace of U. Consider $W = W^{2,p}(\Omega) \subset U = C^1(\overline{\Omega}_C) \oplus L^2(\Omega \setminus \overline{\Omega}_C)$ with $p > d$ for an example.

Remark 2.1. The choice $U \subset C^1(\overline{\Omega}_C) \oplus L^2(\Omega \setminus \Omega_C)$ means that we consider the norm $\|\cdot\|_U := \|u\|_{C^1(\overline{\Omega}_C)} + \|u\|^2_{L^2(\Omega\setminus\Omega_C)}$. Another possible choice would have been to use $\tilde{U} \subset C^1(\overline{\Omega}_C) \cap L^2(\Omega)$ equipped with the norm $\|\cdot\|_{\tilde{U}} := \|u\|_{C^1(\overline{\Omega}_C)} + \|u\|^2_{L^2(\Omega)}$. It is clear that both spaces can be continuously embedded into the other by the use of the mapping $U \to \tilde{U}$ given by

$$u_1 \oplus u_2 \mapsto \tilde{u} := \begin{cases} u_1 & \text{on } \overline{\Omega}_C, \\ u_2 & \text{otherwise,} \end{cases}$$

where $u_1 \in C^1(\overline{\Omega}_C)$ and $u_2 \in L^2(\Omega \setminus \overline{\Omega}_C)$.

Further, consider two reflexive Banach spaces Q and Z, which will denote the space of controls and the space for the adjoint state, respectively. We denote the corresponding dual spaces by U^*, Q^*, and Z^*. Consider the following abstract linear partial differential equation on Ω:

$$Au = Bq, \tag{2.1}$$

where we require the following properties:

Assumption 1. *Assume that $A : U \supset \operatorname{dom} A = W \to Z^*$ is a densely defined linear operator and possesses a bounded inverse.* Further let $B : Q \to Z^*$ be a continuous linear operator.

We will see later that continuous invertibility of A is equivalent to closedness and bijectivity. The distinction between the state space U and the domain of definition W of A allows us to consider our optimal control problem in a convenient topological framework (the topology of U), while being able to model differential operators by A, which are only defined on a dense subspace W.

To define an optimal control problem, we specify an objective functional J with some basic regularity assumptions:

Assumption 2. Let $J = J_1 + J_2$. We assume that $J_1 : U \to \mathbb{R}$ and $J_2 : Q \to \mathbb{R}$ are lower semicontinuous, convex, and Gâteaux differentiable. In addition let J_1

be bounded from below and J_2 be strictly convex. Assume that the derivatives are uniformly bounded on bounded sets. This means that there exists a continuous $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|J_1'(u)\|_{U^*} \leq g(\|u\|_U)$ and $\|J_2'(q)\|_{Q^*} \leq g(\|q\|_Q)$.

We now consider the following minimization problem:

$$\min_{Q^{\mathrm{ad}} \times W} J(q,u) = J_1(u) + J_2(q) \tag{2.2a}$$

$$\text{such that (s.t.)} \quad Au = Bq, \tag{2.2b}$$

$$\text{and} \quad |\nabla u(x)|^2 \leq \psi(x) \text{ on } \overline{\Omega}_C, \tag{2.2c}$$

where, $|\cdot|$ is the euclidian norm in $\mathbb{R}^d$, $\psi \in C(\overline{\Omega}_C)$ with $\psi \geq \delta > 0$ ($\delta \in \mathbb{R}$) and $Q^{\mathrm{ad}} \subset Q$ closed and convex.

In order to ensure that there exists a solution we require that the following assumption holds:

Assumption 3. We assume that at least one of the following holds:

(1) Q^{ad} is bounded in Q.

(2) J_2 is coercive on Q.

For the discussion of interior point methods for the gradient constraint we have to require an additional property, which is of Slater type.

Assumption 4. Assume there exists a feasible control $\check{q} \in Q^{\mathrm{ad}}$ such that the corresponding state $\check{u}$ given by $A\check{u} = B\check{q}$ is strictly feasible; that is, $|\nabla\check{u}|^2 < \psi$.

Remark 2.2. Slater conditions are a crucial aspect of state constrained optimal control. The presence of a Slater point is fundamental for additional regularity of dual variables, and ultimately of primal variables, results on well-posedness, and discretization error estimates. Interior point methods require a Slater point by construction, while outer regularization methods admit qualitative results without a Slater point; see, e.g., [20] for a certain class of problems. However, also here Slater conditions are the key to derive stronger results for a larger problem class; see, e.g., [22].

We state the following basic continuity result, whose proof can be found, e.g., in [29, Lemma A.1].

LEMMA 2.1. *Let U be a Banach space. An operator $A : U \supset W \to Z^*$ has a continuous inverse if and only if A is closed and bijective.*

If Assumption 1 *holds, then there exists a continuous "control-to-state" mapping*

$$S : Q \to U, \quad S := A^{-1}B.$$

Using the Assumptions 1–4 it follows from standard arguments (coercivity, weak seq. compactness, convexity) that (2.2) admits a unique solution $(\overline{q}, \overline{u}) \in Q^{\mathrm{ad}} \times W$.

For the discussion of the adjoint operator A^* of A we exploit density of W in U and reflexivity of Z. A^* possesses a domain of definition $\operatorname{dom} A^*$, given by

$$\operatorname{dom} A^* = \{z \in Z \mid \exists\, c_z : \langle Au, z\rangle_{Z^*,Z} \leq c_z\|u\|_U \quad \forall\, u \in \operatorname{dom} A = W\}.$$

Because W is dense in U for each $z \in \operatorname{dom} A^*$ the linear functional $\langle A\,\cdot\,, z\rangle_{Z^*,Z}$ has a unique continuous extension to a functional on the whole space U. This defines a linear operator $A^* : Z \supset \operatorname{dom} A^* \to U^*$ and it holds

$$\langle u, A^*z\rangle_{U,U^*} = \langle Au, z\rangle_{Z^*,Z} \quad \forall\, u \in \operatorname{dom} A, z \in \operatorname{dom} A^*.$$

LEMMA 2.2. *The operator A^* defined above has a continuous inverse, and it holds*

$$\left(A^{-1}\right)^* = \left(A^*\right)^{-1}. \tag{2.3}$$

Proof. Since Z^* is complete and A is surjective, we can apply [17, Theorem II.3.13], which states that A^* has a bounded inverse under these conditions. Hence, both $(A^{-1})^*$ and $(A^*)^{-1}$ exist, and by [17, Theorem II.3.9] they are equal. ∎

2.1. Examples. Let us apply our abstract framework to optimal control problems with PDEs. First we consider two variants of modeling an elliptic partial differential operator of second order: via the strong form and via the weak form. It will turn out that the strong form yields a more convenient representation of A^* and is thus preferable.

Example 2.1 (Second order elliptic PDE in strong form). Let $\overline{\Omega}_C = \overline{\Omega} \subset \mathbb{R}^d$, $U = C^1(\overline{\Omega}) \cap H_0^1(\Omega)$, $p > d$, and $Z = L^{p'}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Consider $A = -\Delta$ as a mapping from $\operatorname{dom} A = W = W^{2,p}(\Omega) \cap H_0^1(\Omega)$ to $L^p(\Omega)$. This means that A is a differential operator in strong form. We can write this as an integral equation in the following form:

$$\langle Au, z\rangle_{L^p(\Omega),L^{p'}(\Omega)} = \int_\Omega -\Delta u z\, dx \quad \forall\, u \in W, z \in Z.$$

Assume that the boundary of $\Omega \subset \mathbb{R}^d$ is either of class $C^{1,1}$ or that $\Omega \subset \mathbb{R}^2$ is convex and has a polygonal boundary. Then there exists p with $d < p < \infty$ such that A is an isomorphism from W onto Z^*; see, e.g., [16, Theorem 9.15] for the case of a $C^{1,1}$ boundary or [18] for the polygonal case. In particular, A has a continuous inverse from Z^* onto W. By Sobolev embedding W is continuously embedded into U and thus A^{-1} can also be defined as a continuous mapping from Z^* into U. Because W is dense in U the requirements on A of Assumption 1 are fulfilled.

A simple choice for the control space is $Q^{\text{ad}} = Q = L^p(\Omega) = Z^*$. Then $B = \mathrm{Id}$ is a continuous operator. This corresponds to distributed control. As a second setting for the control we may consider $Q = \mathbb{R}^n$ and $f_i \in L^p(\Omega)$, $i = 1 \ldots n$. Then the operator B defined by $Bq = \sum_{i=1}^n f_i q_i$ satisfies Assumption 1 on B.

In the case of distributed control a simple cost functional might be

$$J(q,u) = J_1(u) + J_2(q) = \frac{1}{2}\|u - u^d\|^2_{L^2(\Omega)} + \frac{1}{p}\|q\|^p_{L^p(\Omega)}$$

with given $u^d \in L^2(\Omega)$, $p > d$. It is easily seen that J_2 is coercive on Q. Thus Assumption 3 is satisfied. By simple calculations Assumption 2 on J is verified.

Since the gradient bound ψ is assumed to be strictly positive, taking $\check{q} = 0$ yields the required Slater condition from Assumption 4.

The adjoint operator $A^* : Z \supset \operatorname{dom} A^* \to U^*$ can be interpreted as a very weak form of the Laplace operator; i.e.,

$$\langle u, A^*z\rangle_{U,U^*} = \langle Au, z\rangle_{Z^*,Z} = \int_\Omega -\Delta u z\, dx \quad \forall\, u \in W, z \in \operatorname{dom} A^*.$$

Lemma 2.2 already yields the continuous invertibility of A^*.

Example 2.2 (Second order elliptic PDE in weak form). Let us discuss an alternative approach to Example 2.1: the weak form of the "same" elliptic operator. Usually one defines the differential operator $A = -\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ by

$$\langle Au, z\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} = \int_\Omega \nabla u^T \nabla z\, dx \quad \forall\, z \in H_0^1(\Omega).$$

Our aim is to redefine the spaces for this operator such that Assumption 1 holds. To this end we have to restrict the image space from $H^{-1}(\Omega)$ to $L^p(\Omega)$. Then the space W is given by

$$W = \left\{ u \in H_0^1 \;\Big|\; \int_\Omega \nabla u^T \nabla z \, dx \le c_u \|z\|_{L^{p'}} \quad \forall\, z \in H_0^1(\Omega) \right\}.$$

Observe that the integral in this expression is not defined for all $z \in L^{p'}(\Omega)$, but only for $z \in H_0^1(\Omega)$. However, if $u \in W$, then by definition of W it follows that Au has a unique continuous extension to an element of $L^p(\Omega)$. It is given canonically by

$$\langle Au, z \rangle_{L^p(\Omega), L^{p'}(\Omega)} = \lim_{\substack{z_k \in H_0^1(\Omega),\\ z_k \to z \text{ in } L^{p'}(\Omega)}} \langle Au, z_k \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \tag{2.4}$$

Under the same regularity assumptions as in Example 2.1 we obtain that $W \subset C^1(\overline{\Omega})$ and $\|u\|_{C^1(\overline{\Omega})} \le c\|Au\|_{L^p(\Omega)}$; thus Assumption 1 is fulfilled.

In spite of the complicated representation of A via (2.4), we may represent the equation $Au = f$ conveniently in the form

$$\int_\Omega \nabla u^T \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \quad \forall \varphi \in H_0^1(\Omega) \tag{2.5}$$

via density.

However, since the linear functional Au is defined in $L^p(\Omega)$ by continuous extension (2.4), the representation of the adjoint operator A^* is quite cumbersome. For $z \in \operatorname{dom} A^* \subset L^{p'}(\Omega)$ it is given by

$$\begin{aligned}
\langle u, A^* z \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} &= \lim_{\substack{z_k \in H_0^1(\Omega),\\ z_k \to z \text{ in } L^{p'}(\Omega)}} \langle Au, z_k \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\
&= \lim_{\substack{z_k \in H_0^1(\Omega),\\ z_k \to z \text{ in } L^{p'}(\Omega)}} \int_\Omega \nabla u^T \nabla z_k \, dx
\end{aligned}$$

and has to be used in the adjoint PDE. In contrast to the weak formulation of the primal equation (2.5), where the limit formulation for the *test functions* can be dropped by density, now the limit formulation applies to elements of the *space of solutions*, and thus cannot be neglected. Continuous invertibility of A^*, which follows from our abstract considerations, only applies to its correct representation. A naive formulation of the adjoint PDE would yield wrong results. This is the reason why we prefer the strong formulation for optimal control problems of second order equations with gradient bounds.

Example 2.3 (Fourth order elliptic PDE). As a different example we consider once again $\overline{\Omega}_C = \overline{\Omega}$ but choose different spaces. Let $U = \{v \in C^1(\overline{\Omega}) \mid v(x) = |\nabla v(x)| = 0 \; \forall x \in \partial\Omega\}$, $Z = W_0^{2,p'}(\Omega)$. We consider the biharmonic operator $A = \Delta^2$ as a mapping from $\operatorname{dom} A = W = W_0^{2,p}(\Omega)$ to $Z^* = W^{-2,p}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$.

Assume that the domain $\Omega \subset \mathbb{R}^2$ is convex with polygonal boundary; then it is well known [4, Theorem 2] that A has a continuous inverse from Z^* onto W. As it has already been remarked for $d < p < \infty$ the embedding from W into U exists and is dense.

Note that in this case both dual and primal operators can be represented by

$$\langle Au, z\rangle_{Z^*,Z} = \langle u, A^*z\rangle_{U,U^*} = \int_\Omega \Delta u \Delta z \, dx \quad \forall\, u \in W_0^{2,p}(\Omega),\, z \in W_0^{2,p'}(\Omega).$$

By the choice $Q = L^2(\Omega)$ with B the embedding from $L^2(\Omega)$ into $W^{-2,p}(\Omega)$ we see that Assumption 1 is fulfilled.

3. Barrier functional and its subdifferentiability. In this section we are concerned with the analysis of barrier functionals for the problem under consideration. We proceed as in [29]:

Definition 3.1. *For $r \geq 1$ and $\mu > 0$ we define barrier functions l of order r by*

$$l(v;\mu;r) \;:\; \mathbb{R}_+ \to \overline{\mathbb{R}},$$
$$l(v;\mu;r) := \begin{cases} -\mu \ln(v) & r = 1, \\ \frac{\mu^r}{(r-1)v^{r-1}} & r > 1. \end{cases}$$

We extend their domain of definition to $\mathbb{R}$ by setting $l(v;\mu;r) = \infty$ for $x \leq 0$. We denote the pointwise derivative of $l(v;\mu;r)$ by $l'(v;\mu;r)$ if $v > 0$. This yields

$$l'(v;\mu;r) = \frac{-\mu^r}{v^r}.$$

With this we define a barrier functional b for the constraint $v \geq 0$ by

$$b(\,\cdot\,;\mu;r) \;:\; C(\overline{\Omega}_C) \to \overline{\mathbb{R}},$$
$$v \mapsto \int_{\overline{\Omega}_C} l(v(x);\mu;r)\, dx.$$

Its formal derivative $b'(v,\mu;r) \in C(\overline{\Omega}_C)^*$, is defined as

$$\langle b'(v;\mu;r), \delta v\rangle := \int_{\overline{\Omega}_C} l'(v(x);\mu;r)\delta v(x)\, dx$$

if the right hand side exists.

Obviously, if $0 < \varepsilon \leq v \in C(\overline{\Omega}_C)$, then b is differentiable with respect to v, and b' is the Fréchet derivative of b. If $v(x) = 0$ for some $x \in C(\overline{\Omega}_C)$, then the situation is more involved, and techniques of subdifferential calculus have to be applied.

In contrast to the case of state constraints, we may not use $\psi = 0$ to ease notation. This is due to the fact that in this case $u = 0$ would be the only admissible state. Therefore we introduce the following shifted barrier functional.

Definition 3.2. *We define the barrier functional for the constraint $|\nabla u|^2 \leq \psi$ on a compact set $\overline{\Omega}_C \subseteq \overline{\Omega}$ by*

$$b_\psi(\,\cdot\,;\mu;r) \;:\; C^1(\overline{\Omega}_C) \to \overline{\mathbb{R}},$$

$$u \mapsto b_\psi(u;\mu;r) := b(\psi - |\nabla u|^2;\mu;r). \tag{3.1}$$

In several cases we are only interested in a barrier functional of a fixed given order r, and sometimes even for only one fixed value of μ; in those cases we write $b(\cdot;\mu)$ or even $b(\cdot)$ if no confusion can occur.

LEMMA 3.3. *The barrier functional b_ψ defined in* (3.1) *is well defined, convex, and lower semicontinuous.*

Proof. By [29, Proposition 4.3] the outer function $b(\cdot;\mu;r)$ is well defined and lower semicontinuous. Since the inner function $\psi - |\nabla u|^2$ is well defined and continuous on U, the composition of both functions is well defined and lower semicontinuous.

Moreover, we know that $b(\cdot;\mu;r)$ is convex and monotonically decreasing. Further, the mapping $T(u) := \psi - |\nabla u|^2$ is pointwise concave. With these properties we can proof convexity of $b_\psi = b \circ T$ by the following computation which holds for every x in $\overline{\Omega}_C$:

$$l\big(T(\lambda u+(1-\lambda)\tilde{u})\big)(x) \le l\big(\lambda T(u)+(1-\lambda)T(\tilde{u})\big)(x) \le \lambda l\big(T(u)\big)(x)+(1-\lambda)l\big(T(\tilde{u})\big)(x).$$

By monotonicity of the integral we obtain that b_ψ is convex. ☐

We approach subdifferentiability of $b_\psi = b \circ (\psi - |\nabla \cdot|^2)$ via the following chain rule.

LEMMA 3.4. *Let U, V be Banach spaces, $f : V \to \overline{\mathbb{R}}$ be a convex, lower semicontinuous function, and $T : U \to V$ be a continuously differentiable mapping with first derivative T'. Assume that the composite mapping $f \circ T$ is also convex.*

Let u be given and let $T'(u)$ be bounded. Assume that there is $\breve{u} \in U$ such that f is bounded above in a neighborhood of $T(u) + T'(u)\breve{u}$. Then

$$\partial(f \circ T)(u) = (T'(u))^* \partial f(T(u)). \tag{3.2}$$

Proof. This is a slight extension of the well known chain rule of convex analysis (cf. [14, Proposition I.5.7]), which is, however, hard to find in the literature. We thus derive this result from a more general theorem in nonsmooth analysis due to Clarke and Rockafellar (cf. [11, Theorem 2.9.9] or [25, Theorem 3]). Although the construction of the corresponding generalized differential is rather complicated in general, it reduces to the convex subdifferential in the case of convex functions (cf. [26, Theorem 5]).

First of all, we may assume that $f(T(u))$ is finite. Otherwise, $\partial(f \circ T)(u) = \partial(f(T(u)) = \emptyset$ holds trivially, because $\partial g(u) := \emptyset$ in case $g(u) = +\infty$ for every convex function g.

Now, we may argue as in [25, Corollary 1], which shows that the chain rule [25, Theorem 3] can be applied to show our assertion under the additional assumption that T is linear. However, inspection of its (short) proof shows that the same argumentation is still true in the case that T is "strictly differentiable" at u and $f \circ T$ is convex, as long as $\breve{u}$ exists that satisfies our assumptions. Now the corollary subsequent to [11, Proposition 2.2.1] asserts that "strict differentiability" is implied by continuous differentiability, and our assertion follows. ☐

Remark 3.1. Lemma 3.3 and Lemma 3.4 are also useful in the context of pointwise state constraints of the form $g(y(x), x) \le 0$ if g is convex and differentiable in y.

With the help of this lemma we can now characterize the subdifferential for barrier functionals with respect to gradient bounds in terms of the known subdifferential of a barrier functional in $C(\overline{\Omega}_C)$; see [29].

PROPOSITION 3.5. *Assume that $\psi \ge \delta > 0$. Define*

$$\begin{aligned} b_\psi \,:\, C^1(\overline{\Omega}_C) &\to \overline{\mathbb{R}}, \\ u &\mapsto b(\psi - |\nabla u|^2) \end{aligned}$$

as in Definition 3.2. *Then the subdifferential $\partial b_\psi(u)$ has the following representation:*

$$\partial b_\psi(u) = (-2\nabla u^T \nabla)^* \partial b(\psi - |\nabla u|^2). \tag{3.3}$$

This means $\tilde{m} \in \partial b_\psi(u)$ *if and only if there is* $m \in \partial b(\psi - |\nabla u|^2)$ *such that*

$$\langle \delta u, \tilde{m}\rangle_{C^1(\overline{\Omega}_C),C^1(\overline{\Omega}_C)^*} = -2\langle \nabla u^T \nabla \delta u, m\rangle_{C(\overline{\Omega}_C),C(\overline{\Omega}_C)^*} \quad \forall \delta u \in C^1(\overline{\Omega}_C).$$

If u *is strictly feasible, then* $m = b'(\psi - |\nabla u|^2)$.

Proof. Let $T : C^1(\overline{\Omega}_C) \to C(\overline{\Omega}_C)$ be defined by $T(u) := \psi - |\nabla u|^2$. Obviously, the mapping $\psi - |\nabla u|^2 : C^1(\overline{\Omega}_C) \to C(\overline{\Omega}_C)$ is continuously differentiable with bounded derivative $(T'(u)\delta u)(x) = -2(\nabla u(x))^T \nabla \delta u(x)$.

We are going to apply Lemma 3.4 to the function $b_\psi : U \to \overline{\mathbb{R}}$, $b_\psi(u) = b \circ T$. By [29, Proposition 4.3], b is convex and lower semicontinuous and by Lemma 3.3 b_ψ is convex, too. Setting $\breve{u} := -0.5u$, we have $T'(u)\breve{u} = |\nabla u|^2$, and $\tilde{v} := T(u)+T'(u)\breve{u} = \psi$. Since $\psi \geq \delta > 0$, b is bounded from above in a $C(\overline{\Omega}_C)$-neighborhood of $\tilde{v}$. Hence, Lemma 3.4 can be applied and yields our representation formula (3.3). Finally [29, Proposition 4.6] shows that $\partial b(v) = \{b'(v)\}$ if v is strictly feasible. ∎

The barrier functional b_ψ can also be analyzed on closed subspaces $\tilde{U}$ of $C^1(\overline{\Omega}_C)$. To this end let $E : \tilde{U} \to C^1(\overline{\Omega}_C)$ be the continuous embedding operator. Then its adjoint $E^* : C^1(\overline{\Omega}_C)^* \to \tilde{U}^*$ is the restriction operator for linear functionals. If $\breve{u}$ in Assumption 4 can be chosen from $\tilde{U}$, then the chain rule of convex analysis applied to $b_\psi \circ E$ yields a characterization of the subdifferential of the restriction of b_ψ to $\tilde{U}$ as restriction of the subdifferential:

$$\partial(b_\psi \circ E)(u) = E^* \partial b_\psi(Eu).$$

Closed subspaces of $C^1(\overline{\Omega}_C)$ may, for example, be spaces that incorporate Dirichlet boundary conditions on $\overline{\Omega}_C \cap \Omega$ or finite dimensional subspaces.

4. Minimizers of barrier problems. With the preparations made in the previous sections we will now show that there exists a unique solution for the barrier problem, and later on some first order necessary conditions that are fulfilled by these.

THEOREM 4.1 (Existence of solutions to barrier problems). *Let Assumptions* 1–4 *be fulfilled. Then the problem*

$$\begin{aligned} \min J_\mu(q,u) \quad &:= J(q,u) + b_\psi(u;\mu) \\ s.t. \qquad & \quad Au = Bq \end{aligned} \tag{4.1}$$

admits a unique minimizer (q_μ, u_μ). *Moreover* u_μ *is strictly feasible almost everywhere in* $\overline{\Omega}_C$, *and for every* $\mu_0 > 0$ *the solutions* $(q_\mu, u_\mu) \in Q \times W$ *of* (4.1) *are uniformly bounded on* $(0, \mu_0]$.

Proof. By continuity of $S = A^{-1}B$ (Lemma 2.1) we may eliminate u and write $\tilde{J}_\mu(q) := J_\mu(q, Sq)$. Using Lemma 3.3 and Assumption 1 it is easy to see that $J_\mu(q)$ is a strictly convex, lower semicontinuous functional on the reflexive space Q. By Assumption 4 $J_\mu(\breve{q},\breve{u}) < \infty$. Further, J_μ is coercive by Assumption 3, by the required lower bound for J_1, and because b_ψ is bounded from below, since ψ is bounded above. This allows to apply standard results on the existence and uniqueness of minimizers (cf., e.g., [14]). Since $J_\mu(q_\mu, u_\mu) < \infty$ it follows that u is strictly feasible almost everywhere in Ω_C. To prove our final assertion, we note that due to Lemma 2.1 and Assumption 1 it is sufficient to show that q_μ is uniformly bounded. To see this we note that (cf. [29])

$$J_\mu(q_\mu, u_\mu) \leq J_\mu(q_{\mu_0}, u_{\mu_0}) \leq J_{\mu_0}(q_{\mu_0}, u_{\mu_0}).$$

From $J(q_\mu, u_\mu) \leq J_\mu(q_\mu, u_\mu)$ together with Assumption 3 we obtain that q_μ is bounded, which concludes the proof. ∎

Usually, if $W \subset C^1(\overline{\Omega}_C)$, the state enjoys the additional regularity $W \subset C^{1,\beta}(\overline{\Omega}_C) \subset C^1(\overline{\Omega}_C)$. This means the gradients are even Hölder continuous of order β. Then we obtain for a sufficiently high order r of the barrier method that the state is in fact strictly feasible everywhere in Ω_C, as the following theorem shows.

THEOREM 4.2. *Let $\Omega_C \subset \mathbb{R}^d$ be compact satisfying a cone property (cf.* [1, *Definition* IV.4.3]*) and for some $\beta \in (0,1)$ let $\psi \in C^{0,\beta}(\overline{\Omega}_C)$ be given. Let Assumptions* 1–4 *be satisfied. If the state has the additional regularity $u_\mu \in C^{1,\beta}(\overline{\Omega}_C)$, then for $r-1 > \frac{d}{\beta}$ the state u_μ is strictly feasible in Ω_C.*

Proof. By Theorem 4.1 we obtain $0 \leq \psi - |\nabla u_\mu|^2 \in C^{0,\beta}(\Omega_C)$. From [29, Lemma 7.1] we obtain that therefore $(\psi - |\nabla u_\mu|^2)^{-1} \in C(\Omega_C)$ which concludes the proof. ∎

Remark 4.1. The proof of [29, Lemma 7.1] uses a radial integral technique, which exploits smoothness of y to pass from L_1-estimates to L_∞-estimates. Dependence on the spatial dimension enters due to the integral transformation formula for polar coordinates.

Remark 4.2. Concerning our examples 2.1–2.3 the existence of such a β follows directly as $W^{2,p}(\Omega)$ can be continuously embedded into $C^{1,\beta}(\overline{\Omega})$ provided $0 \leq \beta < 1 - \frac{d}{p}$.

We are now prepared to derive first order necessary conditions for the minimizer of the barrier problem (4.1).

THEOREM 4.3. *Let the Assumptions* 1–4 *be fulfilled. Then $(q_\mu, u_\mu) \in Q^{ad} \times U$ is a solution to* (4.1) *if and only if there exist $m_\mu \in \partial b(\psi - |\nabla u_\mu|^2) \subset C(\overline{\Omega}_C)^*$ and $z_\mu \in Z$, $q_\mu^* \in Q^*$ such that the following holds:*

$$Au_\mu = Bq_\mu \qquad \text{in } Z^*, \tag{4.2a}$$

$$A^* z_\mu = J_1'(u_\mu) + (-2(\nabla u_\mu)^T \nabla)^* m_\mu \qquad \text{in } U^*, \tag{4.2b}$$

$$J_2'(q_\mu) = -B^* z_\mu - q_\mu^* \qquad \text{in } Q^*, \tag{4.2c}$$

$$\langle q - q_\mu, q_\mu^* \rangle_{Q,Q^*} \leq 0 \qquad \forall\, q \in Q^{ad}. \tag{4.2d}$$

Proof. We consider the following minimization problem where we omit the dependence on the parameter μ:

$$\min_{q \in Q} F(q) = \chi_{Q^{\mathrm{ad}}}(q) + j_\mu(q) := \chi_{Q^{\mathrm{ad}}}(q) + J_\mu(q, Sq), \tag{4.3}$$

where $\chi_{Q^{\mathrm{ad}}}$ is the indicator function for the admissible set of the controls, and S is the control-to-state mapping defined by (2.2b). Clearly $(q_\mu, u_\mu) = (q_\mu, Sq_\mu)$ is a solution to (4.1) if and only if q_μ is a solution to (4.3), which is in turn equivalent to $0 \in \partial F(q_\mu)$. In order to utilize this we will split the subdifferential by the sum-rule of convex analysis:

$$\partial F(q_\mu) = \partial(\chi_{Q^{\mathrm{ad}}})(q_\mu) + \partial j_\mu(q_\mu). \tag{4.4}$$

For its application note that Assumption 4 asserts the existence of a point

$$\check{q} \in \operatorname{dom} \chi_{Q^{\mathrm{ad}}} \cap \operatorname{dom} j_\mu$$

such that j_μ is continuous in $\check{q}$. In addition the function $\chi_{Q^{\mathrm{ad}}}$ is convex and lower semicontinuous, thus it coincides with its "Γ-regularization" [14, Chapter I, Proposition 3.1]. We can therefore apply the sum-rule of convex analysis (cf. [14, Chapter I, Proposition 5.6]) to obtain (4.4).

Since j defined by $j(q) := J(q, Sq)$ is continuous in q_μ we obtain by the same argument that

$$\partial j_\mu(q_\mu) = \partial j(q_\mu) + \partial(b_\psi \circ S)(q_\mu),$$

where we recall the definition $b_\psi(u) = b(\psi - |\nabla u|^2)$. Now we note that

$$j(q) = J \circ (1, S)(q)$$

with the linear mapping

$$(1, S) : Q \to Q \times U, \qquad q \mapsto (q, Sq).$$

Together with Assumption 4 we are able to apply the linear chain rule and obtain

$$\begin{aligned} \partial j(q_\mu) &= (1, S^*)\partial J(q_\mu, u_\mu), \\ \partial(b_\psi \circ S)(q_\mu) &= S^* \partial b_\psi(Sq_\mu). \end{aligned}$$

Inserting the representation for the subdifferential of the barrier function b_ψ in Proposition 3.5 our computations have shown so far that

$$0 \in \partial(\chi_{Q^{\mathrm{ad}}})(q_\mu) + (1, S^*)\partial J(q_\mu, u_\mu) + S^*(-2(\nabla u_\mu)^T \nabla)^* \partial b(\psi - |\nabla u_\mu|^2) \tag{4.5}$$

is equivalent to (q_μ, u_μ) being a solution to (4.1). Since the cost functional is differentiable we obtain (cf. [14, Chapter I, Proposition 5.3])

$$\partial J(q_\mu, u_\mu) = \{J_1'(u_\mu) + J_2'(q_\mu)\}.$$

Equation (4.5) means there exist $q_\mu^* \in \partial\chi_{Q^{\mathrm{ad}}}(q_\mu)$ and $m_\mu \in \partial b(\psi - |\nabla u_\mu|^2)$ such that

$$0 = q_\mu^* + J_2'(q_\mu) + S^*\big(J_1'(u_\mu) + (-2(\nabla u_\mu)^T \nabla)^* m_\mu\big) \in Q^*. \tag{4.6}$$

Note that $S^* = (A^{-1}B)^* = B^*(A^{-1})^* = B^*(A^*)^{-1}$, where $A^* : Z \supset \operatorname{dom} A^* \to U^*$ is well defined with continuous inverse due to Lemma 2.2. Define

$$z_\mu = (A^*)^{-1}\big(J_1'(u_\mu) + (-2(\nabla u_\mu)^T \nabla)^* m_\mu\big). \tag{4.7}$$

Then $z_\mu \in \operatorname{dom} A^* \subset Z$ and satisfies (4.2b) by definition. Equation (4.2c) now follows from (4.6). Further note that q_μ^* fulfills (see, e.g., [14, Chapter I, Proposition 5.1])

$$\sup_{q \in Q^{\mathrm{ad}}} \langle q, q_\mu^* \rangle_{Q,Q^*} = \langle q_\mu, q_\mu^* \rangle_{Q,Q^*} \tag{4.8}$$

which is equivalent to (4.2d). ∎

Example 4.1. Let us apply our abstract results to Example 2.1 in the case of distributed control ($B = \mathrm{Id}$). Using the notation from there the first order optimality conditions have the following form. Let (q_μ, u_μ) be a solution to (4.1); then there exists $z_\mu \in Z$, $m_\mu \in \partial b(\psi - |\nabla u_\mu|^2; \mu)$ such that

$$\int_\Omega -\Delta u_\mu \, \varphi \, dx = \int_\Omega q_\mu \varphi \, dx \qquad \forall \varphi \in Z, \tag{4.9a}$$

$$\int_\Omega -\Delta \varphi \, z_\mu \, dx = \int_\Omega (u_\mu - u^d)\varphi \, dx - 2\int_{\overline{\Omega}} (\nabla u_\mu)^T \nabla \varphi \, dm_\mu \qquad \forall \varphi \in W, \tag{4.9b}$$

$$|q_\mu|^{p-2} q_\mu = -z_\mu \qquad \text{a.e. in } \Omega. \tag{4.9c}$$

For a discussion of the first two equations and in particular the representation of A and A^* we refer to Example 2.1. The barrier gradient m_μ is an element of $\partial b(u_\mu;\mu;r)$, and a measure in general. If u_μ is strictly feasible, which can usually be guaranteed a priori by a proper choice of the order r, then $m_\mu = b'(y;\mu;r)$ and thus is a function; cf. [29, Proposition 4.6].

Equation (4.9c) holds pointwise almost everywhere since it holds in $L^p(\Omega)$. The multiplier q_μ^* does not appear due to the fact that $Q^{\mathrm{ad}} = Q$.

After having studied the necessary optimality conditions we will now discuss the behavior of the dual variables. The hard part is showing the boundedness of the measure obtained from the subdifferential of the barrier functional.

Theorem 4.4. *Let the assumptions of Theorem* 4.3 *be fulfilled. Then for each* $\mu_0 > 0$

$$\sup_{\mu\in(0,\mu_0]} \|m_\mu\|_{C(\overline{\Omega}_C)^*} \le C.$$

Proof. Let (q_μ, u_μ) be the solution to (4.1) and $(\check{q}, \check{u})$ be a Slater point; e.g., let $\psi - |\nabla \check{u}|^2 \ge \tau > 0$. Then, following [29], we multiply (4.2b) with $\delta u = u_\mu - \check{u}$ and (4.2c) with $\delta q = q_\mu - \check{q}$ and obtain

$$\begin{aligned} 0 &= \langle \delta u, -A^* z_\mu + J_1'(u_\mu) + (-2(\nabla u_\mu)^T\nabla)^* m_\mu\rangle_{U,U^*} + \langle \delta q, J_2'(q_\mu) + B^* z_\mu + q_\mu^*\rangle_{Q,Q^*} \\ &= \langle \delta u, J_1'(u_\mu) + (-2(\nabla u_\mu)^T\nabla)^* m_\mu\rangle_{U,U^*} + \langle \delta q, J_2'(q_\mu) + q_\mu^*\rangle_{Q,Q^*} \\ &\quad + \langle A\delta u - B\delta q, -z_\mu\rangle_{Z^*,Z}. \end{aligned}$$

As $(\delta q, \delta u)$ fulfills the state equation (2.2b) this simplifies to

$$\begin{aligned} 0 = {} & \langle \delta u, J_1'(u_\mu)\rangle_{U,U^*} + \langle \delta q, J_2'(q_\mu)\rangle_{Q,Q^*} \\ & - 2\langle (\nabla u_\mu)^T \nabla \delta u, m_\mu\rangle_{C(\overline{\Omega}_C),C(\overline{\Omega}_C)^*} + \langle \delta q, q_\mu^*\rangle_{Q,Q^*}. \end{aligned} \tag{4.10}$$

From the uniform boundedness of the primal variable (see Theorem 4.1 together with Assumption 2) we obtain that

$$|\langle \delta u, J_1'(u_\mu)\rangle_{U,U^*} + \langle \delta q, J_2'(q_\mu)\rangle_{Q,Q^*}| \le C$$

with a constant C independent of μ. Inserting this estimate into (4.10) yields

$$| -2\langle (\nabla u_\mu)^T \nabla \delta u, m_\mu\rangle_{C(\overline{\Omega}_C),C(\overline{\Omega}_C)^*} + \langle \delta q, q_\mu^*\rangle_{Q,Q^*}| \le C. \tag{4.11}$$

We will split this into the sum of the absolute values. To do so we will show that both terms have essentially the same sign. First, we define the "almost" active set

$$\mathcal{A} = \{x \in \overline{\Omega}_C \,|\, \psi - |\nabla u_\mu|^2 \le 0.5\tau\},$$

and denote by $m|_S$ for a measurable subset of $S \subset \overline{\Omega}_C$ the restriction of the measure m to this set. This is motivated by [29, Corollary 4.7] which implies that $m_\mu|_{\overline{\Omega}_C\setminus\mathcal{A}}$ has a representation as a function on $\overline{\Omega}_C \setminus \mathcal{A}$ and

$$\sup_{\mu\in(0,\mu_0]} \|m_\mu|_{\overline{\Omega}_C\setminus\mathcal{A}}\|_{L_1(\overline{\Omega}_C\setminus\mathcal{A})} \le C. \tag{4.12}$$

Then

$$\begin{aligned} &|\langle (\nabla u_\mu)^T \nabla \delta u, m_\mu|_{\overline{\Omega}_C\setminus\mathcal{A}}\rangle_{C(\overline{\Omega}_C),C(\overline{\Omega}_C)^*}| \\ &\qquad \le \|m_\mu|_{\overline{\Omega}_C\setminus\mathcal{A}}\|_{L_1(\overline{\Omega}_C\setminus\mathcal{A})^*} \|(\nabla u_\mu)^T\nabla\delta u\|_{C(\overline{\Omega}_C)} \le C. \end{aligned} \tag{4.13}$$

Thus it remains to take a look at the behavior of $\langle m_\mu|_{\mathcal{A}}, (\nabla u_\mu)^T\nabla\delta u\rangle_{C(\overline{\Omega}_C)^*,C(\overline{\Omega}_C)}$. We will now show that $0 < c \le (\nabla u_\mu)^T\nabla\delta u$ holds on $\mathcal{A}$. For this we apply Young's inequality and obtain

$$2|(\nabla u_\mu)^T\nabla\breve{u}| \le |\nabla u_\mu|^2 + |\nabla\breve{u}|^2 \le |\nabla u_\mu|^2 + \psi - \tau$$

or

$$\tau - \psi - |\nabla u_\mu|^2 \le -2|(\nabla u_\mu)^T\nabla\breve{u}|.$$

This leads to the following pointwise estimate on $\mathcal{A}$:

$$\begin{aligned}0.25\tau &\le 0.5(|\nabla u_\mu|^2 - \psi) + 0.5\tau\\ &\le 0.5(\tau - \psi - |\nabla u_\mu|^2) + |\nabla u_\mu|^2\\ &\le |\nabla u_\mu|^2 - (\nabla u_\mu)^T\nabla\breve{u} = (\nabla u_\mu)^T\nabla\delta u.\end{aligned}$$

From [29, Proposition 4.6] we obtain that $m_\mu \le 0$ as a measure thus leading to

$$-2\langle(\nabla u_\mu)^T\nabla\delta u, m_\mu|_{\mathcal{A}}\rangle_{C(\overline{\Omega}_C),C(\overline{\Omega}_C)^*} \ge 0.$$

Now we take a look on (4.2d) to see that $\langle q_\mu - \breve{q}, q_\mu^*\rangle_{Q,Q^*} \ge 0$. Together with (4.13) we obtain from (4.11) that

$$|\langle(\nabla u_\mu)^T\nabla\delta u, m_\mu|_{\mathcal{A}}\rangle_{C(\overline{\Omega}_C),C(\overline{\Omega}_C)^*}| \le C.$$

Finally we note that on the compact set $\mathcal{A}$ for any nonpositive measure σ and any positive continuous function f the estimate

$$\int_{\mathcal{A}} f\,d\sigma \le \min_{x\in\mathcal{A}} f(x)\int_{\mathcal{A}} 1\,d\sigma \le 0$$

holds. Since $m_\mu \le 0$ is nonpositive and $(\nabla u_\mu)^T\nabla\delta u$ is positive we conclude from this estimate

$$\begin{aligned}\langle(\nabla u_\mu)^T\nabla\delta u, m_\mu|_{\mathcal{A}}\rangle_{C(\overline{\Omega}_C),C(\overline{\Omega}_C)^*} &\le \langle(\nabla u_\mu)^T\nabla\delta u, m_\mu|_{\mathcal{A}}\rangle_{C(\mathcal{A}),C(\mathcal{A})^*}\\ &\le \min_{x\in\mathcal{A}}\big((\nabla u_\mu(x))^T\nabla\delta u(x)\big)\langle 1, m_\mu|_{\mathcal{A}}\rangle_{C(\mathcal{A}),C(\mathcal{A})^*}\\ &\le -\frac{\tau}{4}\|m_\mu\|_{C(\mathcal{A})^*}.\end{aligned}$$

This implies

$$\|m_\mu\|_{C(\mathcal{A})^*} \le \frac{4}{\tau}|\langle(\nabla u_\mu)^T\nabla\delta u, m_\mu|_{\mathcal{A}}\rangle_{C(\overline{\Omega}_C),C(\overline{\Omega}_C)^*}| \le C$$

and together with (4.12) completes the proof. ∎

COROLLARY 4.5. *Under the Assumptions 1–4 the following holds for every given* $\mu_0 > 0$*:*

$$\sup_{\mu\in(0,\mu_0]}\|z_\mu\|_Z \le C,$$
$$\sup_{\mu\in(0,\mu_0]}\|q_\mu^*\|_{Q^*} \le C.$$

Proof. First we note that the right hand side of (4.2b) is bounded due to Assumption 2, boundedness of u_μ, m_μ, and continuity of $((\nabla u_\mu)^T\nabla)^* : C(\overline{\Omega}_C)^* \to U^*$. The bound for z_μ follows from the boundedness of the right hand side of (4.2b) and continuity of $(A^*)^{-1}$. The bound for q_μ^* then follows from the bound on z_μ and q_μ using (4.2c) and Assumption 2 and continuity of B^*. ∎

5. Properties of the central path. We will now show convergence of the cost functional with rate μ.

THEOREM 5.1. *Let Assumptions* 1–4 *be fulfilled, and* (q_μ, u_μ) *be a solution of the barrier problem* (4.1) *for* $\mu > 0$. *Then the following holds for the minimizer* $(\overline{q}, \overline{u})$ *of* (2.2)*:*

$$J(q_\mu, u_\mu) \le J(\overline{q}, \overline{u}) + C\mu. \tag{5.1}$$

Proof. The proof follows the lines of [29, Lemma 6.1]; however, since we consider nonlinear constraints on the gradient of the states we have to modify the argumentation concerning the multiplier coming from the subdifferential of the barrier functional.

From the proof of Theorem 4.3 together with the relation

$$\partial b(\psi - |\nabla u_\mu|^2; \mu) = \mu^r \partial b(\psi - |\nabla u_\mu|^2; 1)$$

(cf. [14, Chapter I, equation (5.21)]) we obtain that there exists $m \in \partial b(\psi - |\nabla u_\mu|^2; 1)$ and $\varphi \in \partial \chi_{Q^{\mathrm{ad}}}(q_\mu) + \partial j(q_\mu) = \partial(\chi_{Q^{\mathrm{ad}}} + j)(q_\mu)$ such that

$$\varphi - 2\mu^r S^*((\nabla u_\mu)^T \nabla)^* m = 0.$$

This shows that

$$2\mu^r S^*((\nabla u_\mu)^T \nabla)^* m \in \partial(\chi_{Q^{\mathrm{ad}}} + j)(q_\mu).$$

From convexity of $\chi_{Q^{\mathrm{ad}}} + j$ we obtain that for every $l \in \partial(\chi_{Q^{\mathrm{ad}}} + j)(q_\mu)$ the following holds:

$$j(q_\mu) \le j(\overline{q}) + \langle l, q_\mu - \overline{q} \rangle_{Q^*,Q}.$$

Applied to $2\mu^r S^*((\nabla u_\mu)^T \nabla)^* m$ we obtain

$$J(q_\mu, u_\mu) \le J(\overline{q}, \overline{u}) + 2\mu^r \langle m, (\nabla u_\mu)^T \nabla(u_\mu - \overline{u}) \rangle_{C(\overline{\Omega}_C)^*, C(\overline{\Omega}_C)}.$$

Since b is monotonically decreasing, the measure m is negative; cf. [29, Proposition 4.6]. Thus we can estimate further

$$2\mu^r \langle m, (\nabla u_\mu)^T \nabla(u_\mu - \overline{u}) \rangle_{C(\overline{\Omega}_C)^*, C(\overline{\Omega}_C)} \le 2\mu^r \langle m|_{\Omega_S}, (\nabla u_\mu)^T \nabla(u_\mu - \overline{u}) \rangle_{C(\overline{\Omega}_C)^*, C(\overline{\Omega}_C)},$$

where we define $\Omega_S := \{x \in \overline{\Omega}_C \,|\, (\nabla u_\mu)^T \nabla(u_\mu - \overline{u}) < 0\}$. From the Cauchy–Schwarz inequality it follows that $|\nabla u_\mu(x)| < |\nabla \overline{u}(x)| \le \psi(x)$ on Ω_S and thus $\Omega_S \subset \{x \in \overline{\Omega}_C \,|\, |\nabla u_\mu|^2 < \psi\}$. Hence we obtain from [29, Proposition 4.6.]

$$2\mu^r \langle m|_{\Omega_S}, \nabla u_\mu \nabla(u_\mu - \overline{u}) \rangle_{C(\overline{\Omega}_C)^*, C(\overline{\Omega}_C)} = -2 \int_{\Omega_S} \frac{\mu^r}{(\psi - |\nabla u_\mu|^2)^r} (\nabla u_\mu)^T \nabla(u_\mu - \overline{u})\, dx.$$

From $(\nabla u_\mu)^T \nabla \overline{u} \le |\nabla u_\mu|\, |\nabla \overline{u}| \le \psi$ we see that

$$\frac{-(\nabla u_\mu)^T \nabla(u_\mu - \overline{u})}{\psi - |\nabla u_\mu|^2} = \frac{(\nabla u_\mu)^T \nabla \overline{u} - |\nabla u_\mu|^2}{\psi - |\nabla u_\mu|^2} \le 1$$

and thus

$$2\mu^r \langle m|_{\Omega_S}, (\nabla u_\mu)^T \nabla(u_\mu - \overline{u}) \rangle_{C(\overline{\Omega}_C)^*, C(\overline{\Omega}_C)} \le 2\mu \int_{\Omega_S} \frac{\mu^{r-1}}{(\psi - |\nabla u_\mu|^2)^{r-1}}\, dx. \tag{5.2}$$

From Theorem 4.4 and boundedness of the domain $\overline{\Omega}_C$ we obtain for the function $f := \mu/(\psi - |\nabla u_\mu|^2)$ that

$$\|f^{r-1}\|_{L^1(\overline{\Omega}_C)}^{1/(r-1)} = \|f\|_{L^{r-1}(\overline{\Omega}_C)} \le C\,\|f\|_{L^r(\overline{\Omega}_C)} = C\,\|f^r\|_{L^1(\overline{\Omega}_C)}^{1/r} \le C.$$

Thus the integral on the right hand side of (5.2) is bounded independent of μ. Hence the assertion follows. ∎

COROLLARY 5.2. *Let $\mu > 0$, (q_μ, u_μ) be a solution to the barrier problem* (4.1) *and $(\overline{q}, \overline{u})$ be the solution to the minimization problem* (2.2). *Further assume that there exist $c > 0$, $p \ge 2$, and a norm $\|\cdot\|$ such that*

$$c\|q_1 - q_2\|^p \le J_2(q_1) + J_2(q_2) - 2J_2\left(\frac{q_1+q_2}{2}\right).$$

Then the following estimate holds:

$$\|q_\mu - \overline{q}\| = O(\mu^{1/p}). \tag{5.3}$$

Proof. By assumption and convexity of J_1 the following proves the assertion

$$\begin{aligned} c\|q_\mu - \overline{q}\|^p &\le J_2(q_\mu) + J_2(\overline{q}) - 2J_2\left((q_\mu + \overline{q})/2\right) \\ &\le J(q_\mu, u_\mu) + J(\overline{q}, \overline{u}) - 2J((q_\mu + \overline{q})/2, (u_\mu + \overline{u})/2) \\ &\le J(q_\mu, u_\mu) + J(\overline{q}, \overline{u}) - 2J(\overline{q}, \overline{u}) = O(\mu). \end{aligned}$$ ∎

Remark 5.1. By an analogous assumption on J_1 a similar result for the state u_μ can be obtained. In addition, if $\|\cdot\|$ is stronger than $\|\cdot\|_Q$, the convergence of u_μ in U (with the same rate $O(\mu^{1/p})$) follows by continuity of S.

Example 5.1. We finally return to Example 2.1. We apply the Clarkson inequality [12, Theorem 2, equation (3)] for L^p-spaces with $p > 2$, which yields

$$\left\|\frac{f-g}{2}\right\|_{L^p(\Omega)}^p \le \frac{1}{2}\,\|f\|_{L^p(\Omega)}^p + \frac{1}{2}\,\|g\|_{L^p(\Omega)}^p - \left\|\frac{f+g}{2}\right\|_{L^p(\Omega)}^p.$$

From this we see that $\|q\|_{L^p(\Omega)}^p$ matches the assumption of Theorem 5.2.

With the same techniques as in Theorem 5.1 it is possible to show for $\mu_0 > \mu > 0$ that $J_\mu(q_{\mu_0}, u_{\mu_0}) \le J_\mu(q_\mu, u_\mu) + C(\mu_0 - \mu)$. Then continuity of the central path follows via Theorem 5.2.

6. Numerical results.

Here we will demonstrate our findings on two numerical examples corresponding to Example 2.1 and Example 2.3. First we will discuss an example already considered in the literature. Then we will consider a generic optimal control problem with a fourth order PDE. The results are computed using the Finite Element Toolkit Gascoigne [15] and the Optimization Toolbox RoDoBo [27]. In both examples we choose the order of the barrier method $r = 6$.

6.1. Example 1.

First we will consider an example corresponding to Example 2.1. For this purpose we consider an example from [13] with known solution. The problem reads as follows:

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2}\|\,u - u^d\,\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|q\|_{L^2(\Omega)}^2 \\ \text{s.t.} \quad & -\Delta u = q + f \text{ in}\,\Omega, \\ & \quad u = 0 \qquad \text{on}\,\partial\Omega, \\ \text{s.t.} \quad & |\nabla u|^2 \le 0.25 \quad \text{in}\,\overline{\Omega}, \\ & -2 \le q \le 2 \text{ a.e. in}\,\Omega. \end{aligned}$$

Where $\alpha = 1$, the domain $\Omega = \{x \in \mathbb{R}^2 \,|\, |x| < 2\}$ and the data of the problem is

$$f = \begin{cases} 2 & |x| \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$u^d = \begin{cases} 0.25 + 0.5\ln(2) - 0.25|x|^2 & |x| \le 1, \\ 0.5\ln(2) - 0.5\ln(|x|) & \text{otherwise.} \end{cases}$$

The exact solution satisfies $\overline{u} = u^d$ and $\overline{q} = \begin{cases} -1 & |x| \le 1, \\ 0 & \text{otherwise,} \end{cases}$ and the functional value is given as $J(\overline{q}, \overline{u}) = \frac{\pi}{2}$.

For the computation we have chosen an initial $\mu = 1.0$ and then successively reduced μ by $\sqrt{2}$ until $\mu < 10^{-4}$. The barrier subproblems were solved by a Newton's method in the control space which has been globalized using a line search technique, as provided by RoDoBo. In our test problems, strictly feasible starting values were easy to obtain by taking $\breve{q} = -f$.

In Figure 6.1 we have depicted the convergence of the functional value. Here we can see that after an initial phase the functional value is converging with an approximate order $O(\mu^{3/2})$ until it reaches the discretization accuracy. The intermediate kink in the transition between regularization and discretization error is due to cancellation between the two error components.

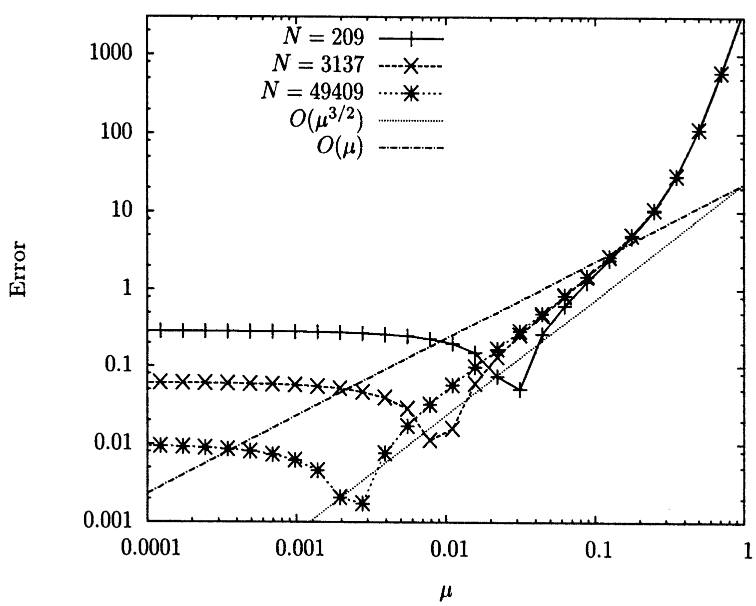
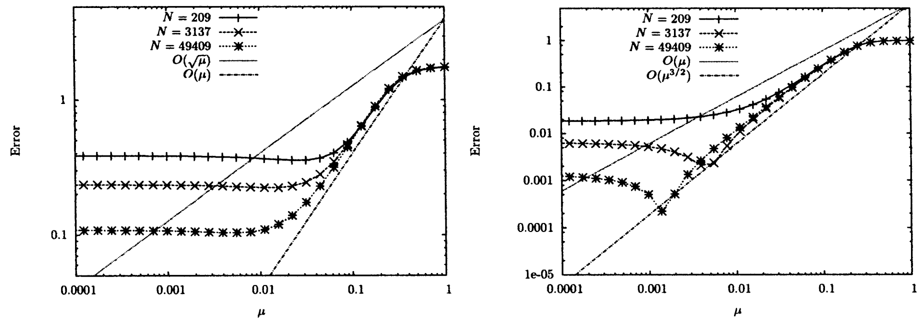


FIG. 6.1. *Error in the cost functional vs. barrier parameter μ on different meshes.*

In Figure 6.2 we can see the convergence behavior of the primal variables. We see that the control variable is in fact converging with order μ instead of the predicted $\sqrt{\mu}$; see Figure 6.2(a). The state variable is converging with approximately the same



(a) L^2-Error of the control variable vs. barrier parameter μ (b) L^2-Error of the state variable vs. barrier parameter μ

FIG. 6.2. *Convergence behavior of the primal variables on different meshes.*

speed as the functional value, namely of order $O(\mu^{3/2})$, where we can see once again the cancellation in the transition between regularization and discretization error.

This rate of convergence exceeds our theoretical findings. It is an interesting question, whether this is an exceptional case, caused by the specific construction of the example, or if our theory can be refined. However, in the next example we will see the predicted behavior.

6.2. Example 2. We will now consider an example corresponding to Example 2.3. Hence we consider the following optimization problem:

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2}\|u-u^d\|_{L^2}^2 + \frac{\alpha}{2}\|q\|_{L^2}^2 \\ \text{s.t.} \quad & \Delta^2 u = q \quad \text{in}\,\Omega, \\ & u = \partial_n u = 0 \quad \text{on}\,\partial\Omega, \\ \text{s.t.} \quad & |\nabla u|^2 \le 0.04 \ \ \text{in}\,\overline{\Omega}. \end{aligned}$$

We choose $\alpha = 10^{-3}$ the domain $\Omega = (-1,1)^2 \subset \mathbb{R}^2$ and

$$u^d = (x^2-1)^2(y^2-1)^2.$$

For the discretization of the state equation we consider a mixed finite element method; e.g., we consider $\sigma = \nabla u$ as an independent variable. Its continuous formulation is for given $q \in L^2(\Omega)$ find $(\sigma, u) \in H^1(\Omega) \times H_0^1(\Omega)$ such that

$$\begin{aligned} (\sigma,\varphi) + (\nabla u, \nabla\varphi) &= 0 && \forall\, \varphi \in H^1(\Omega) \\ (\nabla\sigma, \nabla\varphi) &= (q,\varphi) && \forall\, \varphi \in H_0^1(\Omega) \end{aligned}$$

which is discretized using conforming Q^1 finite elements. For details on this discretization, see [10]. In Figure 6.3 we made a series of computations on different globally refined meshes, where N denotes the number of nodes in the mesh. For these computations the barrier parameter μ was initialized as 0.03 on each mesh and then successively decreased by a factor of $\sqrt{2}$ until it reached a value lower than 10^{-4}. The choice of the initial μ was motivated by the previous example, where an initial phase with low convergence was observed.

We can clearly see the predicted order of convergence of the cost functional. As in this example the exact solution is unknown; we used a reference value obtained on

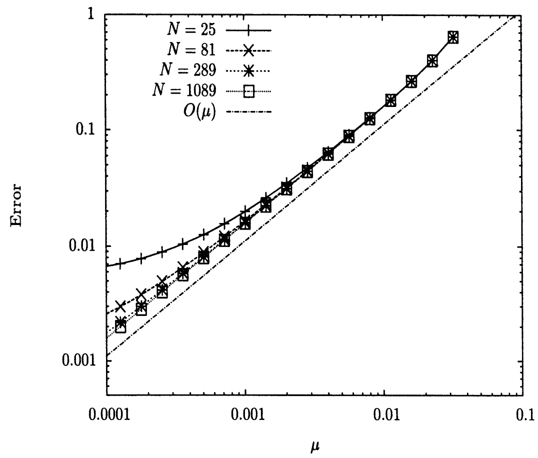


FIG. 6.3. *Error in the cost functional vs. barrier parameter μ on different meshes.*

a mesh with 10^6 nodes and a value $\mu = 10^{-6}$. The approximate functional value is 0.286619. Here we can clearly see the predicted order of convergence, namely $O(\mu)$.

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