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APPROXIMATION OF SHAPE OPTIMIZATION PROBLEMS WITH NONSMOOTH PDE CONSTRAINTS: AN OPTIMALITY CONDITIONS POINT OF VIEW*

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Abstract. This paper is concerned with a shape optimization problem governed by a nonsmooth PDE; i.e., the nonlinearity in the state equation is not necessarily differentiable. We follow a functional variational approach where the set of admissible shapes is parametrized by a large class of continuous mappings. This methodology allows for both boundary and topological variations. It has the advantage that one can rewrite the shape optimization problem as a control problem in a function space. To overcome the lack of convexity of the set of admissible controls, we provide an essential density property. This permits us to show that each parametrization associated to the optimal shape is the limit of global optima of nonsmooth distributed optimal control problems. The admissible set of the approximating minimization problems is a convex subset of a Hilbert space of functions. Moreover, its structure is such that one can derive strong stationary optimality conditions. This opens the door to future research concerning sharp first-order necessary optimality conditions in the form of a qualified optimality system.

Key words. optimal control of nonsmooth PDEs, shape optimization, topical variations, functional variational approach, approximation scheme of fixed domain type

MSC codes. 49Q10, 35Q93, 49N99

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1. Introduction. Shape optimization problems arise in many engineering applications [6] and are often formulated as minimization problems governed by one or more PDEs or VIs [32]. Their main particularity is the fact that these equations are solved on unknown domains [28, 32]. The goal is to find the optimal shape, i.e., that domain for which the distance to a certain desired state is minimized. Thus, shape optimization problems exhibit similarities with optimal control problems [39], with the essential difference and difficulty being that the admissible control set consists of variable geometries. Such problems are highly nonconvex, which makes their investigation challenging from both theoretical and computational points of view.

There is an enormous amount of literature devoted to the study of optimal design problems; see, e.g., the classical contributions [28, 32] and the references therein. To keep the depiction concise, we just focus on the works that deal with these problems at a theoretical level, with an emphasis on optimality systems in qualified form. These resemble the classical Karush–Kuhn–Tucker conditions, and in the case of optimal control problems, they involve an adjoint equation. In [31], topology and boundary variations are combined for the first time in order to derive necessary optimality conditions. In [2], the considered shape and topology optimization problem is governed by a cone constraint, and the existence of Lagrange multipliers is shown, with the respective optimality conditions being expressed in the form of a complementarity system. Methods for computing the shape derivative of the cost functional without

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resorting to the differentiability of the control-to-state map have been developed in [12, 14, 33]. The techniques therein are applied for shape optimization problems governed by smooth PDEs.

Shape optimization problems with nonsmooth constraints have been investigated at a theoretical level mostly w.r.t. existence of optimal shapes [7, 8] and sensitivity analysis [10, 22, 30, 32]. While there is an increasing number of contributions concerning optimal shape design problems governed by VIs (see [7, 8, 10, 16, 17, 22] and the references therein), to the best of the author's knowledge there are no papers that address the case where the governing equation is a *nonsmooth PDE.* In [16, 17], the authors resort to smoothing techniques, and optimality systems in qualified form are obtained just for the smoothed problem [16], or for the original problem [17], but only under certain assumptions. If smoothing is not involved, optimality conditions for the nonsmooth shape optimization problem do not involve an adjoint equation, unless linearity w.r.t. direction is assumed [22]. Otherwise, these are stated just in a primal form [10]; i.e., the respective optimality condition only asserts the nonnegativity of the shape derivative of the reduced objective functional in feasible directions. We point out that the approaches in all the aforementioned papers (see also the references therein) are based on variations of the geometry. Two of the most common notions in this context are the shape derivative [32] and the topological derivative [26].

A more novel technique for deriving optimality conditions in qualified form, where general functional variations instead of geometrical ones are involved, can be found in [38]. Therein, an optimal design problem governed by a linear PDE with Neumann boundary conditions is investigated. By means of the implicit parametrization theorem [25, 35] combined with Hamiltonian systems [36], the equivalence of the shape and topology optimization problem with an optimal control problem in function space is established, provided that the set of admissible geometries is generated by a certain class of continuous functions. The respective control problem in function space is amenable to the derivation of optimality conditions by a classical Lagrange multipliers approach. The same functional variational method will be adopted (to some extent) in the present work.

The aim of this paper is to provide a first essential step towards the derivation of optimality systems for the optimal shape associated to the following *nonsmooth* shape optimization problem:

$$(P_\Omega) \qquad \left.\begin{aligned} \min_{\Omega\in\mathcal{O}, E\subset\Omega} \quad & \int_E (y_\Omega(x)-y_d(x))^2\,dx + \alpha\int_\Omega dx \\ \text{s.t.} \quad & -\triangle y_\Omega + \beta(y_\Omega) = f \quad \text{in } \Omega, \\ & \qquad\qquad\quad y_\Omega = 0 \quad \text{on } \partial\Omega. \end{aligned}\right\}$$

The main particularities of (P_Ω) are as follows:

- the nonsmooth character of β, which is supposed to be locally Lipschitz continuous and directionally differentiable but *not* necessarily *differentiable,* e.g., $\beta = \max\{\cdot, 0\}$ (we underline that we do not intend to replace β by a differentiable function);
- the fact that the governing PDE is solved on the unknown (variable) domain Ω (which plays the role of the control).

The admissible control set $\mathcal{O}$ consists of (unknown) subdomains of a given, fixed domain D. These are the so-called *admissible shapes.* In this article, they are generated by a class of continuous functions; see Definition 2.5. As it turns out, $\mathcal{O}$ covers all subdomains of D of class C^2 that contain the observation set E and whose boundaries do not touch ∂D; cf. Proposition 2.8. The holdall domain $D \subset \mathbb{R}^2$ is bounded andof class $C^{1,1}$, and $f \in L^2(D)$. In the current work, the nonsmoothness β can be thought of as a piecewise linear mapping. In [5], which specifically deals with the lack of differentiability (see below), we will consider much more general situations. The symbol $-\triangle : H_0^1(D) \to H^{-1}(D)$ denotes the Laplace operator in the distributional sense; note that $H_0^1(D)$ is the closure of the set $C_c^\infty(D)$ w.r.t. the $H^1(D)$-norm. The desired state y_d is an L^2-function, which is defined on the observation set E (Assumption 2.3). The parameter α appearing in the objective is supposed to satisfy $\alpha \geq 0$.

The main purpose of this paper is to develop an approximation scheme for (P_Ω) by means of more approachable optimal control problems where the underlying control space consists of functions; see (P_ε) below. We show that each parametrization associated to the optimal shape of (P_Ω) is the limit of global minimizers of (P_ε) (Corollary 5.4). The approximating control problem is governed by a nonsmooth PDE in the holdall domain D. Its structure is such that, under certain requirements on the given data, it permits the derivation of strong stationary optimality systems [5], i.e., complete first-order necessary conditions. The goal of an upcoming work will be to carry out the passage to the limit $\varepsilon \to 0$ in these optimality systems.

The starting point of our analysis is the functional variational method introduced by [23], where the admissible set $\mathcal{O}$ is generated by a large class of functions [37]. We point out that this is different from the level set method of [27]. It allows one to switch from the shape optimization problem to a control problem in a function space, with the latter being more amenable to a rigorous mathematical investigation. This technique involves the description of the boundary of the unknown domain as the solution to a Hamiltonian system, and it is based on the implicit function theorem [25, 35] and Poincaré–Bendixson theory [11, Chap. 10]. The idea that the admissible shapes are parametrized by so-called shape functions turned out to be successful in various papers [18, 19, 20, 21, 23, 24, 37, 38]. The variations used in all these works have no prescribed geometric form (as is usual in the literature) and the methodology therein provides a unified analytic framework allowing for both boundary and topological variations. Penalization methods were developed in [37] and extended in [18, 19]. We mention in particular [19] where the shape and topology optimization problem governed by a PDE with Neumann boundary conditions is equivalently rewritten as a control problem in a function space where the state equation is solved on the fixed domain, while the boundary condition appears penalized in the objective. The functional variational approach was also the fundament for the derivation of optimality conditions for optimal shape design problems governed by linear PDEs with Neumann boundary conditions in [38]. We also refer the reader to the more recent contributions [21] (clamped plates) and [20] (Navier–Stokes equations).

Given a continuous function g on the holdall domain D, we define

$$\Omega_g := \operatorname{int}\{x \in D : g(x) \leq 0\}. \tag{1.1}$$

To ensure that $E \subset \Omega_g$, one requires

$$g<0 \quad \text{in } E.$$

Two other key properties which need to be imposed on the parametrization (of the unknown domain) are

$$|\nabla g| + |g| > 0 \quad \text{in } D, \qquad g > 0 \quad \text{on } \partial D.$$

If $g \in C^2(\bar{D})$, these imply that $\partial\Omega_g$ is a finite union of closed disjoint C^2 curves, without self intersections, that do not intersect ∂D [37, Prop. 2]. Let us stress that this result is valid in two dimensions only, and this is why we stick to the two-dimensional framework in this paper. Together, the above conditions give rise to the definition of the set of admissible shape functions, called $\mathcal{F}_{\mathfrak{s}}$; see (2.3) and Remark 2.6 below. We note that Ω_g may have many connected components. The admissible shape (domain) that we use in the definition of $\mathcal{O}$ below (see (2.4)) is the component that contains the subdomain E. Since this may not be simply connected, the approach we discuss in this paper is related to topological optimization too.

By proceeding like this, we arrive at a reformulation of (P_Ω) in terms of a control problem in a function space. This reads as

$$(P) \qquad \left.\begin{aligned} \min_{g\in\mathcal{F}_{\mathfrak{s}}} \quad & \int_E (y_g(x)-y_d(x))^2\,dx + \alpha\int_D 1-H(g)\,dx \\ \text{s.t.} \quad & -\triangle y_g + \beta(y_g) = f \quad \text{in } \Omega_g, \\ & \qquad\qquad y_g = 0 \quad \text{on } \partial\Omega_g, \end{aligned}\right\}$$

where $H:\mathbb{R}\to\{0,1\}$ is the Heaviside function; see (2.7) below.

We underline the difficulty of the problem (P). The application of traditional optimization methods is excluded, as we deal with a control problem governed by a nonsmooth PDE with the additional challenge that the admissible set $\mathcal{F}_{\mathfrak{s}}$ is nonconvex, while the control does not appear on the right-hand side of the nonsmooth PDE but in the definition of the variable domain on which this is solved. Even in the case of classical control problems (where the domain is fixed), the nonsmooth character is particularly challenging, as the standard KKT theory cannot be directly employed if the differentiability of the control-to-state map is not available. When we exclude the prominent smoothening techniques from [3], the derivation of (strong stationary) optimality conditions for nonsmooth control problems in the presence of control constraints [4, 40] is restricted to certain situations (so-called constraint qualifications). As far as we know, there is no literature on the topic of optimality conditions for nonsmooth optimal control problems with nonconvex admissible set that does not involve smoothing. We emphasize that we do not intend to resort to such techniques. In two subsequent papers that continue the findings from the present paper, all the problems examined ((P_Ω), (P), and (P_ε); see below) feature the same nondifferentiable mapping β.

In order to deal with the control problem (P) and thus the shape optimization problem (P_Ω), we introduce an approximation scheme of fixed domain type [23, 24]. That is, we extend the state equation on the whole reference domain D. We only smooth the Heaviside function, since this has jumps, unlike the nonsmoothness β which is supposed to be continuous. By proceeding like this, we preserve the nonsmooth character and arrive, for $\varepsilon > 0$ small, fixed, at the following approximating optimal control problem:

$$(P_\varepsilon) \qquad \left.\begin{aligned} \min_{g\in\mathcal{F}} \quad & \int_E (y(x)-y_d(x))^2\,dx + \alpha\int_D (1-H_\varepsilon(g))(x)\,dx + \frac{1}{2}\|g-\bar{g}_{\mathfrak{s}}\|_{\mathcal{W}}^2 \\ \text{s.t.} \quad & -\triangle y + \beta(y) + \frac{1}{\varepsilon}H_\varepsilon(g)y = f + \varepsilon g \quad \text{in } D, \\ & \qquad\qquad y = 0 \quad \text{on } \partial D. \end{aligned}\right\}$$

In the objective of (P_ε), $\bar{g}_{\mathfrak{s}}$ is a local optimum of (P) which we intend to approximate by local optima of (P_ε) (adapted penalization [3]), in the sense that $\bar{g}_{\mathfrak{s}}$ will arise as the limit of a sequence of local minima of (P_ε) (Theorem 5.2). The mapping H_ε is the regularization of the Heaviside function; cf. (4.2) below. We underline the fact that we replaced the nonconvex set $\mathcal{F}_\mathfrak{s}$ with a convex subset of the Hilbert space $\mathcal{W} := L^2(D) \cap H^s(D \setminus \bar{E})$, $s > 1$, namely

$$\mathcal{F} := \{g \in \mathcal{W} : g \leq 0 \text{ a.e. in } E\}. \tag{1.2}$$

The formulation of (P_ε) is slightly reminiscent of the minimization problem recently studied in [4], with the additional feature that the control enters the state equation in a nonlinear fashion (via the mapping H_ε). However, the structure of (P_ε) allows us to state conditions on the given data under which the derivation of a complete (so-called strong stationary) optimality system is possible [5]. For more comments regarding the particularities of (P_ε), see section 4.

Our main goal is to show that local optima of (P) can be approximated by local optima of (P_ε) (Theorem 5.2). One of the fundamental challenges will arise from the fact that in (P_ε) we replaced the nonconvex set $\mathcal{F}_\mathfrak{s}$ from the original problem (P) by $\mathcal{F}$. To bridge this gap, we prove that $\mathcal{F}_\mathfrak{s}$ is dense in $\mathcal{F}$ w.r.t. the $L^2(D)$-norm (Proposition 3.1). Since the control-to-state operator of (P) is defined on $\mathcal{F}_\mathfrak{s}$ only and not on $\mathcal{F}$, meanwhile the standard arguments of the adapted penalization method [3] must be accordingly changed. This requires establishing certain convergences of the control-to-state operator $g \mapsto y_g$ of (P) (subsection 4.2), which are more difficult to obtain compared to the classical case where the domain is fixed and the control is distributed. The involved techniques, especially the density $\mathcal{F}_\mathfrak{s} \overset{d}{\hookrightarrow} \mathcal{F}$, may be employed in the study of other similar problems (Remark 3.2).

The paper is organized as follows. After introducing the precise assumptions at the beginning of section 2, subsection 2.1 deals with the definition of the admissible set $\mathcal{O}$. Here we establish that this consists of a certain class of C^2 domains that contain the fixed subdomain E (Proposition 2.8). Then, in subsection 2.2, we reformulate the nonsmooth shape optimization problem (P_Ω) as a nonsmooth control problem in a function space, i.e., as (P). Here, we also introduce the notion of local optimum for the latter. Section 3 is dedicated to the density property $\mathcal{F}_\mathfrak{s} \overset{d}{\hookrightarrow} \mathcal{F}$, which is essential for making the connection between (P) and the approximating nonsmooth control problem with fixed domain (P_ε). This is introduced in section 4. In subsection 4.1 we show that the unique solution of the state equation in (P) is the limit of solutions to state equations in (P_ε). Subsection 4.2 is dedicated to the convergence properties of the control-to-state maps of (P) and (P_ε). In subsection 4.3 we establish that (P_ε) admits optimal solutions. Finally, section 5 contains the main results of this paper, namely Theorem 5.2 and Corollary 5.4. Theorem 5.2 states that the approximation of (P) by (P_ε) is meaningful in the sense that local minima of (P) arise as limits of local optima of (P_ε). As a consequence, each parametrization of the optimal shape of (P_Ω) is approximable by global minimizers of (P_ε) (Corollary 5.4). The findings in this last section provide the basis for future investigations with respect to the derivation of qualified optimality systems for (P_Ω).

2. The shape optimization problem. We begin this section by stating the precise assumptions on the nonsmoothness β appearing in the state equation in (P_Ω). We also mention the requirements needed for the observation set E. Then, in the upcoming subsections we will introduce the set of admissible shapes $\mathcal{O}$ for (P_Ω). This will allow us to "rewrite" (P_Ω) as an optimal control problem in a function space (Proposition 2.10).

Assumption 2.1 (The nonsmoothness). The function $\beta:\mathbb{R}\to\mathbb{R}$ is monotone increasing and locally Lipschitz continuous in the following sense: For all $M>0$, there exists a constant $L_M>0$ such that

$$|\beta(z_1)-\beta(z_2)|\le L_M|z_1-z_2| \quad \forall\, z_1,z_2\in[-M,M].$$

Remark 2.2. At this stage, we do not make any assumptions concerning the limited differentiability properties of β that were advertised in the introduction. These do not play any role in the present paper, but in [5], which is a continuation of the present work. We just mention that, therein, β is supposed to be semi-differentiable only, i.e., the right and left side derivative of β may not be equal at certain points, e.g. $\beta=\max\{\cdot,0\}$ or $\beta=|\cdot|$. This is precisely one of the main challenges in [5].

By Assumption 2.1, it is straight forward to see that the Nemytskii operator $\beta: L^\infty(D)\to L^\infty(D)$ is well-defined. Moreover, this is Lipschitz continuous on bounded sets in the following sense: for every $M>0$, there exists $L_M>0$ so that

$$\|\beta(y_1)-\beta(y_2)\|_{L^q(D)}\le L_M\,\|y_1-y_2\|_{L^q(D)} \quad \forall\, y_1,y_2\in\overline{B_{L^\infty(D)}(0,M)},\ \forall\, 1\le q\le\infty. \tag{2.1}$$

Assumption 2.3 (The observation set). The set $E\subset D$ is a domain with boundary of measure zero. Moreover, $\operatorname{dist}(\overline{E},\partial D)>0$, where

$$\operatorname{dist}(\overline{E},\partial D):=\inf_{(x_1,x_2)\in\overline{E}\times\partial D}\operatorname{dist}(x_1,x_2).$$

We point out that the last condition in Assumption 2.3 ensures that the set of admissible shape functions $\mathcal{F}_{\mathfrak{s}}$ is not empty, cf. (2.3) below.

In the rest of the paper, one tacitly supposes that Assumptions 2.1 and 2.3 are always fulfilled without mentioning them every time.

2.1. The admissible set. As pointed out in the introduction, we will work in the framework of [18, 19, 23, 24, 37, 38] (see also the references therein), where the admissible set $\mathcal{O}$ consists of a family of subdomains of D that are generated by a certain class of continuous functions, called $\mathcal{F}_{\mathfrak{s}}$, see (2.3) below. For a general mapping $g\in C(D)$, we will use the notation Ω_g to describe the following open subset of the holdall domain D:

$$\Omega_g:=\operatorname{int}\{x\in D: g(x)\le 0\}. \tag{2.2}$$

Note that Ω_g is not necessarily connected and the choice of g is not unique, in fact there are infinitely many functions generating the same Ω_g, see for instance the end of the proof of Lemma 2.9. While Ω_g is an open set and may have many connected components, the admissible shape (domain) that we use in the definition of $\mathcal{O}$ below, see (2.4), is the component that contains the subdomain E. Its existence is guaranteed by imposing that the parametrization g satisfies $g<0$ in E.

Definition 2.4 (the set of admissible shape functions). *We define*

$$\begin{aligned}\mathcal{F}_{\mathfrak{s}}:=\{g\in C^2(\bar D): g(x)<0\ \forall\, x\in E,\ |\nabla g(x)|+|g(x)|>0\ \forall\, x\in D,\\ g(x)>0\ \forall\, x\in\partial D\}.\end{aligned} \tag{2.3}$$

$\mathcal{F}_{\mathfrak{s}}$ will be later the control set for the optimization problem (P) below, as we will switch from the topology of subsets in the Euclidean space to the one in a function space; note that (P_Ω) and (P) are 'equivalent' in the sense of Proposition 2.10. Instead of C^2 functions, we may take $C^{k,l}$ mappings in (2.3), $k \geq 2, l \in [0,1]$ (this is possible due to Proposition 3.1). Switching to $C^{k,l}$ shape functions impacts only the regularity of the boundary of Ω_g, which is as smooth as g is (see Lemma 2.7 or [6, Thm. 4.2]). Since we want to keep the admissible set $\mathcal{F}_\mathfrak{s}$ as large as possible and at the same time ensure the H^2 regularity of the state y_g in (P), we choose to stick to the C^2-setting.

Definition 2.5 (the set of admissible domains). *The admissible set for the shape optimization problem* (P_Ω) *is given by*

$$\mathcal{O} := \{\textit{the component of } \Omega_g \textit{ that contains the set } E : g \in \mathcal{F}_\mathfrak{s}\}. \tag{2.4}$$

At the end of this subsection we will show that $\mathcal{O}$ covers all subdomains of D of class C^2 that contain E and whose boundaries do not intersect ∂D. These domains may not be simply connected, that is, topological and boundary variations are both included in our investigations.

Remark 2.6 (comments regarding the definition of $\mathcal{F}_\mathfrak{s}$). Let us underline that all the conditions appearing in the definition of $\mathcal{F}_\mathfrak{s}$ are meaningful, as they guarantee that the admissible shapes have the properties stated in Lemma 2.7 below. These are mostly used for (2.8), in the proof of Proposition 4.6 and in subsection 4.2. Lemma 2.7 will also play a central role when establishing the adjoint equation (which will be addressed in an upcoming work).

The inequality $g<0$ in E implies that all admissible shapes contain the set E, as already mentioned.

Since $g \in C^2(\bar{D})$, the requirements $|\nabla g| + |g| > 0$ in D and $g > 0$ on ∂D ensure that the set $\{x \in D : g(x) = 0\}$ is a finite union of disjoint closed C^2 curves, without self intersections, not intersecting ∂D [37, Prop. 2]. The fact that we work in two dimensions only is due to the viability of this assertion. If the three-dimensional case is considered, we need to include more conditions in $\mathcal{F}_\mathfrak{s}$ [25, 35].

Lemma 2.7 (Properties of admissible shapes and Ω_g). *Let* $g \in \mathcal{F}_\mathfrak{s}$ *and denote by* $\Omega \in \mathcal{O}$ *the relevant component of* Ω_g*, that is, the component that contains* E*. Then,*

1. Ω *is a domain of class* C^2*;*
2. $\partial\Omega_g = \{x \in D : g(x) = 0\}$ *and* $\Omega_g = \{x \in D : g(x) < 0\}$*;*
3. $\mu\{x \in D : g(x) = 0\} = 0$.

Proof. We observe that $\partial\Omega_g \subset \{x \in D : g(x) = 0\}$. As $g \in C^2(\bar{D})$ and since it satisfies the gradient condition and the condition on the boundary of D in (2.3), one deduces from [37, Prop. 2] that $\{x \in D : g(x) = 0\}$ is a finite union of disjoint closed C^2 curves, without self intersections, disjoint from ∂D. This implies the first claim.

Further, from [6, Thm. 4.2] it follows, by a contradiction argument, that $\operatorname{int}\{x \in D : g(x) \leq 0\} = \{x \in D : g(x) < 0\}$ and $\partial\{x \in D : g(x) < 0\} = \{x \in D : g(x) = 0\}$. This gives us the second statement.

The last assertion is a direct consequence of the first two. It can also be deduced directly from [15, Lemma A.4] and the gradient condition in the definition of $\mathcal{F}_\mathfrak{s}$. ☐

We end this subsection with an essential observation.

Proposition 2.8. *It holds*

$$\mathcal{O} = \{\Omega \subset D : \Omega \textit{ is a domain of class } C^2 \textit{ that contains } E,\ \partial\Omega \cap \partial D \neq \emptyset\}.$$

Proof. While the inclusion $\subset$ is due to Lemma 2.7, the opposite one can be shown by the exact same arguments as in the proof of [6, Thm. 4.1] with one difference: one replaces the distance functions defined therein by appropriate C^∞ functions. Let $\Omega \subset D$ so that Ω is a domain of class C^2that contains E, $\partial\Omega \cap \partial D \neq \emptyset$. To show that there exists $g \in \mathcal{F}_{\mathfrak{s}}$ so that $\Omega_g = \Omega$, one defines $\rho, \rho_c \in C^\infty(\bar{D})$ as follows

$$\rho(y) := \begin{cases} \in (0,1] & \text{for } y \in \Omega^c \setminus \bar{W}, \\ 0 & \text{for } y \in \bar{W} \cup \Omega = \bar{W} \cup \bar{\Omega}, \end{cases} \tag{2.5}$$

$$\rho_c(y) := \begin{cases} \in (0,1] & \text{for } y \in \Omega \setminus \bar{W}, \\ 0 & \text{for } y \in \bar{W} \cup \Omega^c = \bar{W} \cup \bar{\Omega}^c, \end{cases} \tag{2.6}$$

where Ω^c stands for $\bar{D} \setminus \Omega$ and W is an appropriate neighbourhood of $\partial\Omega$, see the proof of [6, Thm. 4.1] for details. Then, by arguing exactly in the same way as in the proof of [6, Thm. 4.1], the desired result follows. ∎

2.2. Reformulation of (P_Ω) as an optimal control problem in a function space. In this subsection we rewrite (P_Ω) as an optimal control problem where the admissible set no longer consists of subsets of D, but of functions. In other words, we change the topology, so that (P_Ω) becomes amenable to optimal control methods. This is possible in light of the definition of the set of admissible shapes, see (2.4). The reformulation of (P_Ω) reads

$$\left.\begin{aligned} \min_{g \in \mathcal{F}_{\mathfrak{s}}} \quad & \int_E (y_g(x) - y_d(x))^2\,dx + \alpha \int_D 1 - H(g)\,dx, \\ \text{s.t.} \quad & -\triangle y_g + \beta(y_g) = f \quad \text{in } \Omega_g, \\ & \qquad\qquad y_g = 0 \quad \text{on } \partial\Omega_g, \end{aligned}\right\} \tag{P}$$

where $H : \mathbb{R} \to \{0,1\}$ is the Heaviside function

$$H(v) := \begin{cases} 0, & \text{if } v \le 0, \\ 1 & \text{if } v > 0. \end{cases} \tag{2.7}$$

Regarding the second term in the objective, we note that

$$\int_D 1 - H(g)\,dx = \int_{\Omega_g} dx \quad \forall\, g \in \mathcal{F}_{\mathfrak{s}}, \tag{2.8}$$

in view of Lemma 2.7 and since $g > 0$ outside $\overline{\Omega_g}$, cf. (2.3).

Lemma 2.9. *For each $\Omega \in \mathcal{O}$ there exists an infinite number of mappings $g \in \mathcal{F}_{\mathfrak{s}}$ so that $\Omega_g = \Omega$.*

Proof. Since $\Omega \in \mathcal{O}$, there exists $\widehat{g} \in \mathcal{F}_{\mathfrak{s}}$ so that Ω is the component of $\Omega_{\widehat{g}}$ that contains E, cf. (2.4). From [37, Prop. 2], see also the proof of Lemma 2.7, we know that $\Omega_{\widehat{g}}$ has a finite number of components, and we define

$$\mathcal{M} := \cup_{\omega \neq \Omega, \omega \text{ is a component of } \Omega_{\widehat{g}}} \overline{\omega}.$$

Note that $\operatorname{dist}(\overline{\Omega}, \overline{\omega}) > 0$ for each component ω of $\Omega_{\widehat{g}}$ which is different from Ω (the boundary of $\Omega_{\widehat{g}}$ consists of disjoint closed curves [37, Prop. 2], see also the proof of Lemma 2.7). Now, define the C^∞ function

$$h := \begin{cases} 0 & \text{for } x \in \overline{\Omega}, \\ \in [0,2] & \text{for } x \in \overline{D} \setminus (\overline{\Omega} \cup \mathcal{M}), \\ 2 & \text{for } x \in \mathcal{M}. \end{cases} \tag{2.9}$$

and

$$g := \widehat{g} - (\min_{x\in\overline{D}} \widehat{g})h.$$

Note that$\min_{x\in\overline{D}} \widehat{g} < 0$,; otherwise, $\widehat{g} \geq 0$ in $\overline{D}$, which contradicts $\widehat{g} \in \mathcal{F}_{\mathfrak{s}}$. We observe that

$$g = \widehat{g} \quad \text{in } \overline{\Omega}, \qquad g > 0 \quad \text{in } \bar{D} \setminus \overline{\Omega},$$

since $\widehat{g}$>0 outside $\overline{\Omega}_{\widehat{g}}$ (Lemma 2.7). As $\widehat{g} \in \mathcal{F}_{\mathfrak{s}}$, we can conclude from the above that $g \in \mathcal{F}_{\mathfrak{s}}$ with $\Omega_g = \Omega$. Finally, we see that one can define an infinite number of mappings with this property, say $g_n := n\, g, n \in \mathbb{N}$. ∎

The shape optimization problem (P_Ω) with the admissible set $\mathcal{O}$ from Definition 2.5 is equivalent to (P) in the following sense.

PROPOSITION 2.10. *Let $\Omega^\star \in \mathcal{O}$ be an optimal shape of (P_Ω). Then, each of the functions $g^\star \in \mathcal{F}_{\mathfrak{s}}$ that satisfy $\Omega_{g^\star} = \Omega^\star$ is a global minimizer of (P). Conversely, if $g^\star \in \mathcal{F}_{\mathfrak{s}}$ minimizes (P), then the component of $\Omega_{g^\star}$ that contains E is an optimal shape for (P_Ω).*

Proof. Let $\Omega^\star \in \mathcal{O}$ be an optimal shape for (P_Ω) with $\mathcal{O}$ as in (2.4), i.e.,

$$\int_E (y_{\Omega^\star}(x) - y_d(x))^2\, dx + \alpha \int_{\Omega^*} dx \leq \int_E (y_\Omega(x) - y_d(x))^2\, dx + \alpha \int_\Omega dx \quad \forall\, \Omega \in \mathcal{O}.$$

Now, let $g^\star \in \mathcal{F}_{\mathfrak{s}}$ with $\Omega_{g^\star} = \Omega^\star$ be fixed (note that, according to Lemma 2.9, there are infinitely many mappings with this property). Then, $y_{\Omega^\star} = y_{g^\star}$ and

$$\int_E (y_{g^\star}(x) - y_d(x))^2\, dx + \alpha \int_{\Omega_{g^\star}} dx \leq \int_E (y_\Omega(x) - y_d(x))^2\, dx + \alpha \int_\Omega dx \quad \forall\, \Omega \in \mathcal{O}.$$

Let $g \in \mathcal{F}_{\mathfrak{s}}$ be arbitrary and fixed, and denote by Ω the component of Ω_g that contains E. Testing with this particular Ω in the above inequality yields

$$\begin{aligned}\int_E (y_{g^\star}(x) - y_d(x))^2\, dx + \alpha \int_{\Omega_{g^\star}} dx &\leq \int_E (y_g(x) - y_d(x))^2\, dx + \alpha \int_\Omega dx \\ &\leq \int_E (y_g(x) - y_d(x))^2\, dx + \alpha \int_{\Omega_g} dx,\end{aligned}$$

where we used the fact that $y_g = y_\Omega$ in E. Since $g^\star \in \mathcal{F}_{\mathfrak{s}}$ and $g \in \mathcal{F}_{\mathfrak{s}}$ was arbitrary, this proves the first statement; see (2.8).

To show the converse assertion, assume that $g^\star \in \mathcal{F}_{\mathfrak{s}}$ satisfies

(2.10)
$$\int_E (y_{g^\star}(x) - y_d(x))^2\, dx + \alpha \int_{\Omega_{g^\star}} dx \leq \int_E (y_g(x) - y_d(x))^2\, dx + \alpha \int_{\Omega_g} dx \quad \forall\, g \in \mathcal{F}_{\mathfrak{s}}.$$

We denote by $\omega_{g^\star}$ the component of $\Omega_{g^\star}$ that contains E. This implies

$$y_{\omega_{g^\star}} = y_{g^\star} \quad \text{in } E.$$

Let $\Omega \in \mathcal{O}$ be arbitrary but fixed. Again, by Lemma 2.9, we can define $\widetilde{g} \in \mathcal{F}_{\mathfrak{s}}$ so that $\Omega_{\widetilde{g}} = \Omega$. Then, $y_\Omega = y_{\widetilde{g}}$. In view of (2.10), where we test with $\widetilde{g}$, we have

$$\begin{aligned}\int_E (y_{\omega_{g^\star}}(x) - y_d(x))^2\, dx + \alpha \int_{\omega_{g^\star}} dx &\leq \int_E (y_{g^\star}(x) - y_d(x))^2\, dx + \alpha \int_{\Omega_{g^\star}} dx \\ &\leq \int_E (y_\Omega(x) - y_d(x))^2\, dx + \alpha \int_\Omega dx.\end{aligned}$$

Since $\omega_{g^\star} \in \mathcal{O}$ and $\Omega \in \mathcal{O}$ was arbitrary, this proves the second assertion. ∎

From now on, all our findings concern the optimal control problem formulated as (P). We focus not only on global minimizers but also on the much larger class of local optima in the $L^2(D)$ sense, which are defined as follows.

DEFINITION 2.11. *We say that* $\bar{g}_{\mathfrak{s}} \in \mathcal{F}_{\mathfrak{s}}$ *is locally optimal for* (P) *in the* $L^2(D)$ *sense if there exists* $r > 0$ *such that*

$$\mathcal{J}(\bar{g}_{\mathfrak{s}}) \le \mathcal{J}(g) \quad \forall\, g \in \mathcal{F}_{\mathfrak{s}} \text{ with } \|g - \bar{g}_{\mathfrak{s}}\|_{L^2(D)} \le r, \tag{2.11}$$

where

$$\mathcal{J}(g) := \int_E (\mathcal{S}(g)(x) - y_d(x))^2\, dx + \alpha \int_D (1 - H(g))(x)\, dx$$

is the reduced cost functional associated to the control problem (P), *and*

$$\mathcal{S} : g \in \mathcal{F}_{\mathfrak{s}} \mapsto y_g \in H_0^1(\Omega_g) \cap H^2(\Omega_g)$$

denotes the control-to-state map. This will be introduced below; see Definition 4.9.

Remark 2.12. Clearly, in the study of local optima in the $L^2(D)$ sense of (P) we cover global optima and thus the associated (global) optimal shapes of (P_Ω), cf. Proposition 2.10.

Note that the set of local optima in the $H^1(D)$ sense is larger than the set of local optima in the $L^2(D)$ sense. However, we choose to work with the concept of local optima in the $L^2(D)$ sense, in view of the essential density property from Proposition 3.1 below. This does not hold w.r.t. the $H^1(D)$-norm because of the boundary condition on ∂D in (2.3). Proposition 3.1 is, however, essential in the proof of Theorem 5.2, as a closer inspection shows; it ensures that $\{\widehat{g}_\varepsilon\} \subset \mathcal{F}_{\mathfrak{s}}$ is contained in the ball of local optimality of $\bar{g}_{\mathfrak{s}}$. If this ball of local optimality is taken w.r.t. the $H^1(D)$-norm, then it is necessary that Proposition 3.1 also be true w.r.t. the $H^1(D)$-norm, which is not the case, as explained above.

3. Density of the set of admissible shape functions in $L^p(D)$, $p \in [1, \infty)$.

As already mentioned in the introduction, we cannot tackle (P) in a direct manner, so that we will consider an approximation scheme; cf. sections 4 and 5 below. One of the main challenges that arise in the investigation of (P) is the structure of the set $\mathcal{F}_{\mathfrak{s}}$. It is a nonconvex cone, while the governing PDE is nonsmooth and the control appears as the parametrization of the unknown domain. The admissible set $\mathcal{F}_{\mathfrak{s}}$ has, however, a density property, which allows us to examine the approximating optimal control problem (P_ε) below on a convex subset of a Hilbert space, see (4.1).

The focus of this subsection is to prove the aforementioned density property, which is stated in the following (in a slightly more general form).

PROPOSITION 3.1. *Let*

$$\widetilde{\mathcal{F}} := \{g \in L^2(D) : g \le 0 \text{ a.e. in } E\}.$$

For each $g \in \widetilde{\mathcal{F}}$, *there exists a sequence* $\{g_m\} \subset \mathcal{F}_{\mathfrak{s}} \cap C^\infty(\bar{D})$ *so that*

$$g_m \to g \quad \text{in } L^2(D) \quad \text{as } m \to \infty.$$

Remark 3.2. The assertion in Proposition 3.1 remains true if $L^2(D)$ is replaced by $L^p(D)$, $p \in [1, \infty)$; see (3.2) and (3.6) below.

It has different applications for all sorts of problems where the solution operator is defined on $\mathcal{F}_{\mathfrak{s}}$ only, i.e., where conditions as those in Lemma 2.7 need to be fulfilled by the admissible shapes; see, e.g., [19] and the references therein.

The proof of Proposition 3.1 will be conducted in three steps as follows. We introduce the sets

$$\mathcal{F}_c := \{g \in C^\infty(\bar{D}) : g(x) \le 0 \ \forall x \in E\},$$

$$\mathcal{F}_c^+ := \{g \in C^\infty(\bar{D}) : g(x) \le 0 \ \forall x \in E, \ g(x) > 0 \ \forall x \in \partial D\}$$

and prove Proposition 3.1 by showing the set of densities

$$\begin{aligned} \mathcal{F}_c &\overset{d}{\hookrightarrow} \widetilde{\mathcal{F}} \quad \text{(Lemma 3.3 below)}, \\ \mathcal{F}_c^+ &\overset{d}{\hookrightarrow} \mathcal{F}_c \quad \text{(Lemma 3.4 below)}, \\ \mathcal{F}_{\mathfrak{s}} \cap C^\infty(\bar{D}) &\overset{d}{\hookrightarrow} \mathcal{F}_c^+ \quad \text{(Lemma 3.5 below)} \end{aligned}$$

w.r.t. the $L^2(D)$-norm.

LEMMA 3.3. *For each $g \in \widetilde{\mathcal{F}}$, there exists a sequence $\{g_m\} \subset \mathcal{F}_c$ so that*

$$g_m \to g \quad \text{in } L^2(D) \quad \text{as } m \to \infty.$$

Proof. Let $g \in L^2(D)$ with $g \le 0$ a.e. in E be arbitrary but fixed, and denote by $\chi_D g : \mathbb{R}^2 \to \mathbb{R}$ the extension of g by zero outside D. For every $m \in \mathbb{N}$ we define the $L^2(D)$-function $\widehat{g}_m : \mathbb{R}^2 \to \mathbb{R}$ as

$$\widehat{g}_m := \begin{cases} \min\{g, 0\} & \text{on } E_m, \\ \chi_D g & \text{otherwise}, \end{cases} \tag{3.1}$$

where

$$E_m := \{v \in \mathbb{R}^2 : d(v, E) \le 1/m\},$$

and d is the distance induced by the l_∞-norm; note that $E_m \subset D$ for m large enough, by Assumption 2.3. Since $g \le 0$ a.e. in E, we have

$$\|\widehat{g}_m - g\|^2_{L^2(D)} = \|\min\{g, 0\} - g\|^2_{L^2(E_m \setminus E)} = \int_{E_m \setminus E} (-\max\{g, 0\})^2 \to 0 \quad \text{as } m \to \infty, \tag{3.2}$$

since $(\max\{g, 0\})^2 \in L^1(D)$ and$\mu(E_m \setminus E) \to 0$ as $m \to \infty$,; cf., e.g., [1, Lemma A.1.17]. Now we define the C^∞-function $g_m : \mathbb{R}^2 \to \mathbb{R}$ as

$$g_m(v) := \int_{\mathbb{R}^2} \widehat{g}_m \left(v - \frac{1}{m} s \right) \psi(s) \, ds,$$

where $\psi \in C_c^\infty(\mathbb{R}^2)$, $\psi \ge 0$, $\operatorname{supp} \psi \subset [-1, 1]^2$, and $\int_{\mathbb{R}^2} \psi(s) \, ds = 1$. Then, by the definition of E_m and $\widehat{g}_m$, it holds that

$$g_m(v) = \int_{[-1,1]^2} \widehat{g}_m \left(v - \frac{1}{m} s \right) \psi(s) \, ds \le 0 \quad \forall v \in E,$$

i.e., $g_m \in \mathcal{F}_c$. In view of (3.2), the proof is now complete. ∎

LEMMA 3.4. *For each $g \in \mathcal{F}_c$, there exists a sequence $\{g_m\} \subset \mathcal{F}_c^+$, so that*

$$g_m \to g \quad \text{in } L^2(D) \qquad \text{as } m \to \infty.$$

Proof. We recall that

$$\mathcal{F}_c := \{g \in C^\infty(\bar{D}) : g(x) \le 0 \ \forall x \in E\}$$

and

$$\mathcal{F}_c^+ := \{g \in C^\infty(\bar{D}) : g(x) \le 0 \ \forall x \in E,\ g(x) > 0 \ \forall x \in \partial D\}. \tag{3.3}$$

Let $g \in \mathcal{F}_c$ be arbitrary but fixed, and let $m \in \mathbb{N}$. We define $h_m \in C^\infty(\bar{D})$ as

$$h_m(x) := \begin{cases} -2\min\left\{\min\limits_{x\in\partial D} g, 0\right\} + 1/m & \text{for } x \in \partial D, \\ \in [0, -2\min\left\{\min\limits_{x\in\partial D} g, 0\right\} + 1/m] & \text{for } x \in D \text{ where } d(\partial D, x) \in (0, 1/m), \\ 0 & \text{for } x \in D \text{ where } d(\partial D, x) \ge 1/m. \end{cases} \tag{3.4}$$

Then, the mapping

$$g_m := g + h_m \in C^\infty(\bar{D}) \tag{3.5}$$

satisfies

$$\|g_m - g\|_{L^2(D)} = \|h_m\|_{L^2(D_m)} \to 0 \quad \text{as } m \to \infty, \tag{3.6}$$

where we use the shorthand notation

$$D_m := \{x \in D : d(\partial D, x) \in (0, 1/m)\}.$$

Let us now check if g_m fulfills the inequalities appearing in the definition of $\mathcal{F}_c^+$.

(i) $g_m \le 0$ *in* E: By Assumption 2.3, we can choose m large enough so that $1/m < d(\bar{E}, \partial D)$. This implies that $h_m = 0$ in $\bar{E}$, which in turn gives $g_m = g \le 0$ in $\bar{E}$.

(ii) $g_m > 0$ *on* ∂D: By (3.5), we have, for $x \in \partial D$,

$$g_m(x) \ge \min\left\{\min_{x\in\partial D} g, 0\right\} - 2\min\left\{\min_{x\in\partial D} g, 0\right\} + 1/m = -\min\left\{\min_{x\in\partial D} g, 0\right\} + 1/m \ge 1/m,$$

where for the first estimate we used the definition of h_m, i.e., (3.4). This allows us to conclude the desired assertion. ∎

LEMMA 3.5. *For each $g \in \mathcal{F}_c^+$, there exists a sequence $\{g_m\} \subset \mathcal{F}_\mathfrak{s} \cap C^\infty(\bar{D})$, so that*

$$g_m \to g \quad \text{in } C(\bar{D}) \qquad \text{as } m \to \infty.$$

Proof. We recall the definitions of $\mathcal{F}_c^+$ and $\mathcal{F}_\mathfrak{s}$:

$$\mathcal{F}_c^+ := \{g \in C^\infty(\bar{D}) : g(x) \le 0 \ \forall x \in E,\ g(x) > 0 \ \forall x \in \partial D\},$$

$$\begin{aligned}\mathcal{F}_\mathfrak{s} := \{g \in C^2(\bar{D}) : g(x) < 0 \ \forall x \in E,\ |\nabla g(x)| + |g(x)| > 0 \ \forall x \in D, \\ g(x) > 0 \ \forall x \in \partial D\}.\end{aligned}$$

Let $g \in \mathcal{F}_c^+$ be arbitrary but fixed. According to Sard's theorem [29, Thm. 7.2], the set

$$\{g(x) : \nabla g(x) = 0\} \subset \mathbb{R} \quad \text{has measure zero.} \tag{3.7}$$

Since g is continuous and $g > 0$ on ∂D, it holds that $\min_{x\in\partial D} g(x) > 0$. In light of (3.7), there exists $\delta_1 \in (0, \min_{x\in\partial D} g(x))$ so that

$$\nabla g(x) \neq 0 \quad \forall\, x \in g^{-1}(\delta_1).$$

Note that $g^{-1}(\delta_1)$ may be empty, in which case the above assertion is still valid. Sard's theorem (see (3.7)) further ensures the existence of a sequence $\{\delta_m\}$ satisfying

$$\delta_m \in \left(0, \min\left\{\delta_{m-1}, \frac{1}{m}\right\}\right), \quad m \in \mathbb{N},$$

and

$$\nabla g(x) \neq 0 \quad \forall\, x \in g^{-1}(\delta_m) \quad \forall\, m \in \mathbb{N}. \tag{3.8}$$

We note that the above constructed sequence satisfies

$$\min_{x\in\partial D} g(x) > \delta_1 > \delta_2 > \cdots > \delta_m > 0, \quad \delta_m \to 0 \quad \text{as } m \to \infty. \tag{3.9}$$

Further, we define $f_m \in C^\infty(\bar{D})$ as

$$f_m := \begin{cases} \in [0,1] & \text{for } x \in \bar{D} \setminus \overline{\Omega_{g-\delta_m}}, \\ 0 & \text{for } x \in \overline{\Omega_{g-\delta_m}}, \end{cases} \tag{3.10}$$

where we recall that, according to (2.2), the set $\Omega_{g-\delta_m}$ is defined as

$$\Omega_{g-\delta_m} = \operatorname{int}\{x \in D : g(x) - \delta_m \leq 0\}. \tag{3.11}$$

Note that functions f_m as in (3.10) exist, since we can choose a compact subset of $\mathbb{R}^2$, disjoint with $\bar{D}$, so that $f_m = 1$ there. Then, the mapping

$$g_m := g - \delta_m + \frac{1}{m} f_m \in C^\infty(\bar{D}) \tag{3.12}$$

satisfies

$$\|g_m - g\|_{C(\bar{D})} \to 0 \quad \text{as } m \to \infty; \tag{3.13}$$

see (3.9). Let us now check if g_m fulfills the inequalities appearing in the definition of $\mathcal{F}_{\mathfrak{s}}$.

(i) $g_m < 0$ *in* E: Since $g \in \mathcal{F}_c^+$, it holds that $g \leq 0$ in $\bar{E}$, which implies that

$$\bar{E} \subset \{x \in D : g(x) - \delta_m < 0\},$$

from which we infer

$$\bar{E} \subset \overline{\Omega_{g-\delta_m}}.$$

This yields $f_m = 0$ in $\bar{E}$; see (3.10). Thus, by (3.12), $g_m = g - \delta_m \leq -\delta_m < 0$ in $\bar{E}$.

(ii) $g_m > 0$ *on* ∂D*:* From (3.9) one deduces that for $x \in \partial D$ it holds that

$$g(x) - \delta_m > g(x) - \min_{x \in \partial D} g \geq 0.$$

From (3.12) and (3.10) we now conclude that $g_m > 0$ on ∂D.

(iii) $|\nabla g_m(x)| + |g_m(x)| > 0 \; \forall x \in D$: By (3.12) and (3.10) one knows that

$$g_m(x) > 0 \; \forall x \in \bar{D} \setminus \overline{\Omega_{g-\delta_m}}. \tag{3.14}$$

Thus, we only need to show that

$$|\nabla g_m(x)| + |g_m(x)| > 0 \; \forall x \in \overline{\Omega_{g-\delta_m}}.$$

Note that, due to the regularity and the definition of f_m (see (3.10)), its gradient vanishes on $\overline{\Omega_{g-\delta_m}}$. This means that, by (3.12), it holds that

$$g_m = g - \delta_m \text{ in } \overline{\Omega_{g-\delta_m}}, \quad \nabla g_m = \nabla g \quad \text{ in } \overline{\Omega_{g-\delta_m}}.$$

Now, thanks to (3.8), $g_m(x) = 0$ implies $\nabla g_m(x) \neq 0$ if $x \in \overline{\Omega_{g-\delta_m}}$. This completes the proof. $\square$

4. The approximating optimal control problem. In this section, $\bar{g}_\mathfrak{s} \in \mathcal{F}_\mathfrak{s}$ is a local optimum of (P) in the $L^2(D)$ sense. Inspired by a classical adapted penalization scheme [3] in combination with the fixed domain methodology from [23], our approximating minimization problem (P_ε) preserves the nonsmoothness. We point out that the nonconvex set $\mathcal{F}_\mathfrak{s}$ from (P) is now replaced by a convex subset of a Hilbert space. As shown by one of the main results in the next section (Theorem 5.2), this replacement is reasonable thanks to the findings from the previous section (Proposition 3.1).

Let $\varepsilon > 0$ be fixed. We consider the following approximating control problem:

$$(P_\varepsilon) \qquad \left.\begin{aligned} \min_{g \in \mathcal{F}} \quad & \int_E (y(x) - y_d(x))^2 \, dx + \alpha \int_D (1 - H_\varepsilon(g))(x) \, dx + \frac{1}{2} \|g - \bar{g}_\mathfrak{s}\|^2_{\mathcal{W}} \\ \text{s.t.} \quad & -\triangle y + \beta(y) + \frac{1}{\varepsilon} H_\varepsilon(g) y = f + \varepsilon g \quad \text{in } D, \\ & y = 0 \quad \text{on } \partial D, \end{aligned}\right\}$$

where $\mathcal{W}$ is the Hilbert space $L^2(D) \cap H^s(D \setminus \bar{E})$, $s > 1$, endowed with the norm

$$\|\cdot\|^2_{\mathcal{W}} := \|\cdot\|^2_{L^2(D)} + \|\cdot\|^2_{H^s(D \setminus \bar{E})},$$

and

$$\mathcal{F} := \{g \in \mathcal{W} : g \leq 0 \text{ a.e. in } E\}. \tag{4.1}$$

Definition 4.1. *The nonlinearity* $H_\varepsilon : \mathbb{R} \to [0,1]$ *is defined as follows:*

$$H_\varepsilon(v) := \begin{cases} 0 & \text{if } v \leq 0, \\ \dfrac{v^2(3\varepsilon - 2v)}{\varepsilon^3} & \text{if } v \in (0, \varepsilon), \\ 1 & \text{if } v \geq \varepsilon. \end{cases} \tag{4.2}$$

Remark 4.2. The mapping H_ε introduced in (4.2) is Lipschitz continuous and continuously differentiable. It is obtained as a regularization of the Heaviside function from (2.7), and it is one possible choice that has the aforementioned properties. Note that Heaviside functions and their regularizations play an essential role in the context of shape optimization via fixed domain approaches; see, for instance, [23, 24] and the references therein.

Remark 4.3 (additional term in the state equation). We notice that the state equation in (P_ε) is an approximating extension of the state equation in (P) from Ω_g to D [23]. We point out the presence of the additional term $+\varepsilon g$ on the right-hand side, which is essential for the proof of our main strong stationarity result in [5]; see [5, Rem. 3.16] for details. In all the other upcoming investigations, this can be dropped.

Remark 4.4 (objective in (P_ε)). The presence of the H^s term in the objective of (P_ε) ensures the existence of optimal solutions. We underline that it is necessary to consider the $H^s(D \setminus \bar{E})$-norm (not the full $H^s(D)$-norm) in order to conclude complete (i.e., strong stationary) optimality conditions for local optima of (P_ε), see [5]. If $\mathcal{W} = H^s(D)$, certain sign conditions for the adjoint state are not available, and the optimality system for (P_ε) reduces to a weaker one, namely the limit optimality system obtained via smoothing methods.

Moreover, it is essential that $s > 1$ in order to obtain the convergence of the states in Theorem 5.2; see also Lemma 4.11. This condition ensures that the embedding $H^s(D \setminus \bar{E}) \hookrightarrow L^\infty(D \setminus \bar{E})$ is true [13, p. 88]. It will also be needed when deriving the final optimality system for the optimal shape of (P_Ω) in an upcoming work.

4.1. Solvability of the state equations in (P_ε) and (P). We start this subsection with a result on the unique solvability of the state equation in the approximating control problem. For convenience, we recall here the governing PDE in (P_ε):

$$
\begin{aligned}
-\triangle y + \beta(y) + \frac{1}{\varepsilon} H_\varepsilon(g) y &= f + \varepsilon g \quad \text{in } D, \\
y &= 0 \quad \text{on } \partial D.
\end{aligned}
\tag{4.3}
$$

LEMMA 4.5. *For any right-hand side $g \in L^2(D)$, equation* (4.3) *admits a unique solution $y \in H_0^1(D) \cap H^2(D)$. This satisfies*

$$
\|y\|_{H_0^1(D)\cap C(\bar{D})} \le c_1 + c_2\, \varepsilon \|g\|_{L^2(D)},
$$

where $c_1, c_2 > 0$ are independent of ε and g. The control-to-state mapping $S_\varepsilon : L^2(D) \to H_0^1(D) \cap H^2(D)$ is Lipschitz continuous on bounded sets; i.e., for every $M > 0$, there exists $L_{M,\varepsilon} > 0$ so that

$$
\|S_\varepsilon(g_1) - S_\varepsilon(g_2)\|_{H_0^1(D)\cap H^2(D)} \le L_{M,\varepsilon}\, \|g_1 - g_2\|_{L^2(D)} \quad \forall\, g_1, g_2 \in \overline{B_{L^2(D)}(0, M)}.
\tag{4.4}
$$

Proof. According to [39, Thm. 4.8], equation (4.3) admits a unique solution $y \in H_0^1(D) \cap C(\bar{D})$ with

$$
\|y\|_{H^1(D)\cap L^\infty(D)} \le c\, \|f + \varepsilon g - \beta(0)\|_{L^2(D)},
$$

where $c > 0$ is independent of ε, β, and of g. Then, by [9, Lem. 9.17], the $H^2(D)$ regularity follows. To show the desired Lipschitz continuity of the control-to-state map, one first shows that $S_\varepsilon : L^2(D) \to H_0^1(D)$ is Lipschitz continuous on bounded sets. This is a consequence of the monotonicity of β and the Lipschitz continuity and nonnegativity ofH_ε,; cf. (4.2). In view of the $C^{1,1}$ regularity of D, the same can be concluded for $S_\varepsilon : L^2(D) \to H_0^1(D) \cap H^2(D)$. ∎

Next, we aim at highlighting the connection between (4.3) and the state equation in (P).

PROPOSITION 4.6 (solvability of the state equation in (P)). *Let $g \in C(\bar{D})$ be fixed so that Ω_g is a domain of class C. Moreover, assume that $g > 0$ a.e. in $D \setminus \Omega_g$. Then,*

$$S_\varepsilon(g)|_{\Omega_g} \rightharpoonup y_g \quad \text{in } H^1(\Omega_g) \quad \text{as } \varepsilon \searrow 0,$$

where S_ε is the control-to-state map associated to (4.3), *and $y_g \in H_0^1(\Omega_g)$ is the unique solution to*

$$\begin{aligned} -\triangle y + \beta(y) &= f \quad \text{in } \Omega_g, \\ y &= 0 \quad \text{on } \partial\Omega_g. \end{aligned} \tag{4.5}$$

Remark 4.7. All the admissible domains $\Omega_g \in \mathcal{O}$ (Definition 2.4) satisfy (together with their associated parametrization) the hypotheses of Proposition 4.6 as a result of Lemma 2.7.

Remark 4.8. Given a function $g \in C(\bar{D})$ that does not satisfy the strict positivity hypothesis outside Ω_g in Proposition 4.6, one can construct (as in the proof of Lemma 2.9) an infinite number of functions that coincide with g on $\bar{\Omega}_g$ and are strict positive outside $\bar{\Omega}_g$ (in each point).

Proof. We follow the arguments from the proof of [23, Thm. 1]. For the beginning, let $\varepsilon > 0$ be arbitrary but fixed. We abbreviate $y_\varepsilon := S_\varepsilon(g)$, and by multiplying (4.3) with y_ε, we infer

$$\|y_\varepsilon\|^2_{H_0^1(D)} + \int_D \beta(y_\varepsilon) y_\varepsilon \, dx + \frac{1}{\varepsilon} \int_D H_\varepsilon(g) y_\varepsilon^2 \, dx \leq \int_D (f + \varepsilon g) y_\varepsilon \, dx. \tag{4.6}$$

Since β is monotonically increasing and $H_\varepsilon(g) \geq 0$, by assumption, we deduce from (4.6) the estimate

$$\|y_\varepsilon\|^2_{H_0^1(D)} + \underbrace{\int_D [\beta(y_\varepsilon) - \beta(0)] y_\varepsilon \, dx}_{\geq 0} \leq \int_D (f + \varepsilon g) y_\varepsilon \, dx - \int_D \beta(0) y_\varepsilon \, dx.$$

By applying Young's inequality on the right-hand side, we derive uniform bounds w.r.t. ε so that we can extract a weakly convergent subsequence

$$y_\varepsilon \rightharpoonup \widehat{y} \quad \text{in } H_0^1(D). \tag{4.7}$$

In view of (2.1) and the compact embedding $H^1(D) \hookrightarrow\hookrightarrow L^2(D)$, we then have

$$\beta(y_\varepsilon) \to \beta(\widehat{y}) \quad \text{in } L^2(D); \tag{4.8}$$

note that $\|y_\varepsilon\|_{L^\infty(D)} \leq c$, where $c > 0$ is independent of ε, by Lemma 4.5. Further, multiplying (4.6) with ε implies

$$\int_D H_\varepsilon(g) y_\varepsilon^2 \, dx \to 0 \text{ as } \varepsilon \searrow 0.$$

In light of (4.2), it holds that $H_\varepsilon(g) \to H(g)$ in $L^q(D)$, $q \in [1, \infty)$, where $H : \mathbb{R} \to \{0, 1\}$ stands for the Heaviside function (2.7). Thus, by (4.7) and the compact embedding $H^1(D) \hookrightarrow\hookrightarrow L^q(D)$, one obtains

$$\int_{D \setminus \Omega_g} \widehat{y}^2 \, dx = \int_D H(g) \widehat{y}^2 \, dx = \lim_{\varepsilon \searrow 0} \int_D H_\varepsilon(g) y_\varepsilon^2 \, dx = 0,$$

where the first identity is due to (2.2), (2.7), and $g>0$ a.e. in $D\setminus\Omega_g$ by assumption. Hence, $\widehat{y}=0$ a.e. in $D\setminus\Omega_g$. As $\widehat{y}\in H_0^1(D)$ (cf. (4.7)), it can be extended by zero on $\mathbb{R}^2$ (while preserving the H^1 regularity). Since Ω_g is of class C, by assumption, the result in [34, Thm. 2.3] implies that the weak limit from (4.7) satisfies

$$\widehat{y}\in H_0^1(\Omega_g). \tag{4.9}$$

Testing (4.3) with $\phi\in C_c^\infty(\Omega_g)$ further implies

$$\int_{\Omega_g}\nabla y_\varepsilon\nabla\phi\,dx+\int_{\Omega_g}\beta(y_\varepsilon)\phi\,dx=\int_{\Omega_g}(f+\varepsilon g)\phi\,dx,$$

since $H_\varepsilon(g)=0$ on Ω_g (cf. definitions (2.2) and (4.2)). Passing to the limit $\varepsilon\searrow 0$, where one relies on (4.7), (4.8), then results in

$$\int_{\Omega_g}\nabla\widehat{y}\nabla\phi\,dx+\int_{\Omega_g}\beta(\widehat{y})\phi\,dx=\int_{\Omega_g}f\phi\,dx.$$

Since$\widehat{y}\in H_0^1(\Omega_g)$ (cf. (4.9)), we have $\widehat{y}_{|\Omega_g}=y_g$, and the proof is now complete. ∎

The result in Proposition 4.6 allows us to introduce

Definition 4.9 (the control-to-state map associated to the state equation in (P)). *We define*

$$\mathcal{S}:g\in\mathcal{F}_{\mathfrak{s}}\mapsto y_g\in H_0^1(\Omega_g)\cap H^2(\Omega_g), \tag{4.10}$$

where y_g solves equation (4.5) *on the component of Ω_g containing E, i.e., $\Omega_g\in\mathcal{O}$ (with a little abuse of notation, we do not distinguish between the notation for Ω_g and the notation for its relevant component; we just write $\Omega_g\in\mathcal{O}$ when we mean the latter).*

Note that the additional $H^2(\Omega_g)$ regularity is due to the fact that each admissible shape Ω_g is of class C^2 (Lemma 2.7.1) and [9, Lem. 9.17].

Remark 4.10. According to Definition 4.9, $\mathcal{S}(g)$ exists only as an element of $H_0^1(\Omega_g)$. Whenever we write $\mathcal{S}(g)$ as an element of $H^1(D)$ in what follows, we think of its extension by zero outside $\Omega_g\in\mathcal{O}$.

4.2. Convergence properties. This subsection is dedicated to the study of other limit behaviors of solutions to (4.3) when $\varepsilon\to 0$. In addition to the convergence from Proposition 4.6, we will need in the proof of our main result (Theorem 5.2) two convergence results that are contained in Lemmas 4.11 and 4.13 below. Note that, from now on, we simply write $\Omega_g\in\mathcal{O}$ when we mean the relevant component of Ω_g (i.e., the one that contains the set E).

Lemma 4.11. *Let $\{g_\varepsilon\}\subset\mathcal{F}$ and $g\in\mathcal{F}_{\mathfrak{s}}$ so that*

$$g_\varepsilon\to g\quad\text{in } L^2(D)\cap L^\infty(D\setminus\bar{E})\quad\text{as }\varepsilon\searrow 0.$$

Then, for each compact subset K of Ω_g, there exists $\varepsilon_0>0$, independent of x, so that

$$g_\varepsilon\le 0\quad\text{a.e. in }K,\ \forall\,\varepsilon\in(0,\varepsilon_0]. \tag{4.11}$$

Moreover,

$$S_\varepsilon(g_\varepsilon)\rightharpoonup\mathcal{S}(g)\quad\text{in } H_0^1(D)\quad\text{as }\varepsilon\searrow 0.$$

Proof. Since g is continuous, we have

$$g(x) \leq -\delta \quad \forall x \in K \subset\subset \Omega_g$$

for some $\delta > 0$. As $g_\varepsilon \to g$ in $L^\infty(D \setminus \bar{E})$, by assumption, there exists $\varepsilon > 0$, small, independent of x, so that

$$g_\varepsilon(x) \leq g(x) + \delta/2 \leq -\delta/2 < 0 \quad \text{a.e. in } K \cap (D \setminus \bar{E}).$$

Since $g_\varepsilon \in \mathcal{F}$, we can thus conclude with (4.11).

Let us now show the desired convergence. For simplicity we define $y_\varepsilon := S_\varepsilon(g_\varepsilon)$ and, by arguing exactly as in the proof of Proposition 4.6, we show that there exists a weakly convergent subsequence (denoted by the same symbol) with

$$y_\varepsilon \rightharpoonup \widehat{y} \quad \text{in } H_0^1(D) \quad \text{as } \varepsilon \searrow 0, \tag{4.12}$$

$$\beta(y_\varepsilon) \to \beta(\widehat{y}) \quad \text{in } L^2(D) \quad \text{as } \varepsilon \searrow 0. \tag{4.13}$$

Moreover,

$$\int_D H_\varepsilon(g_\varepsilon) y_\varepsilon^2 \, dx \to 0 \text{ as } \varepsilon \searrow 0. \tag{4.14}$$

We also note that, by Lemma 4.5, there exists a constant $C > 0$, independent of ε, so that

$$\|y_\varepsilon\|_{L^\infty(D)} \leq C \quad \forall \varepsilon > 0. \tag{4.15}$$

As a result of $g_\varepsilon \to g$ in $L^2(D)$, we have

$$\begin{aligned} &\mu\{x \in D : g < 0 \text{ and } g_\varepsilon \geq 0\} \to 0 \quad \text{as } \varepsilon \searrow 0, \\ &\mu\{x \in D : g > 0 \text{ and } g_\varepsilon - \varepsilon \leq 0\} \to 0 \quad \text{as } \varepsilon \searrow 0, \end{aligned}$$

and with (4.15), it follows that

$$\lim_{\varepsilon \to 0} \int_{\{g<0, g_\varepsilon \geq 0\}} H_\varepsilon(g_\varepsilon) y_\varepsilon^2 \, dx + \lim_{\varepsilon \to 0} \int_{\{g>0, g_\varepsilon \leq \varepsilon\}} H_\varepsilon(g_\varepsilon) y_\varepsilon^2 \, dx = 0.$$

Thus, by (4.14), Lemma 2.7.3, (4.2), and (4.12) combined with the compact embedding $H^1(D) \hookrightarrow\hookrightarrow L^2(D)$, one deduces

$$\begin{aligned} 0 = \lim_{\varepsilon \to 0} \int_D H_\varepsilon(g_\varepsilon) y_\varepsilon^2 \, dx &= \lim_{\varepsilon \to 0} \int_{\{g<0, g_\varepsilon<0\}} H_\varepsilon(g_\varepsilon) y_\varepsilon^2 \, dx + \lim_{\varepsilon \to 0} \int_{\{g>0, g_\varepsilon>\varepsilon\}} H_\varepsilon(g_\varepsilon) y_\varepsilon^2 \, dx \\ &= \lim_{\varepsilon \to 0} \int_{\{g>0, g_\varepsilon>\varepsilon\}} y_\varepsilon^2 \, dx = \lim_{\varepsilon \to 0} \int_{\{g>0\}} y_\varepsilon^2 \, dx = \lim_{\varepsilon \to 0} \int_{D \setminus \Omega_g} y_\varepsilon^2 \, dx \\ &= \int_{D \setminus \Omega_g} \widehat{y}^2 \, dx. \end{aligned} \tag{4.16}$$

Hence, $\widehat{y} = 0$ a.e. in $D \setminus \Omega_g$, and in view of [34, Thm. 2.3] (applied for the component of Ω_g that contains the set E), this implies that

$$\widehat{y} \in H_0^1(\Omega_g).$$

Note that here we use the fact that $\Omega_g \in \mathcal{O}$ is a domain of class C (even C^2; see Lemma 2.7.1). Testing (4.3) with right-hand side g_ε with $\phi \in C_c^\infty(\Omega_g), \Omega_g \in \mathcal{O}$, further yields

$$\int_{\Omega_g} \nabla y_\varepsilon \nabla \phi \, dx + \int_{\Omega_g} \beta(y_\varepsilon)\phi \, dx + \int_{\Omega_g} \frac{1}{\varepsilon} H_\varepsilon(g_\varepsilon) y_\varepsilon \phi \, dx = \int_{\Omega_g} (f + \varepsilon g_\varepsilon)\phi \, dx.$$

For a fixed $\phi \in C_c^\infty(\Omega_g)$ there exists a compact subset $\widetilde{K}$ of Ω_g so that $\phi \in C_c^\infty(\widetilde{K})$. Hence, by (4.11) and (4.2), the third term in the above variational identity vanishes for $\varepsilon > 0$ small enough, independent of x (dependent on $\widetilde{K}$ and thus on ϕ). Passing to the limit $\varepsilon \searrow 0$, where one relies on (4.12), (4.13), and the uniform boundedness of $\{g_\varepsilon\}$ in $L^2(D)$, then results in

$$\int_{\Omega_g} \nabla \widehat{y} \nabla \phi \, dx + \int_{\Omega_g} \beta(\widehat{y})\phi \, dx = \int_{\Omega_g} f \phi \, dx.$$

As $\widehat{y} \in H_0^1(\Omega_g)$, and since (4.5) is uniquely solvable, we conclude that $\widehat{y} = \mathcal{S}(g)$, and, thanks to (4.12), the proof is now complete. ∎

Assumption 4.12. For the given datum f and the desired state y_d, we require

$$f \geq \beta(0) \quad \text{a.e. in } D, \quad y_d \leq 0 \quad \text{a.e. in } E \tag{4.17}$$

or, alternatively,

$$f \leq \beta(0) \quad \text{a.e. in } D, \quad y_d \geq 0 \quad \text{a.e. in } E. \tag{4.18}$$

Lemma 4.13. *If Assumption* 4.12 *is satisfied, then*

$$\liminf_{\varepsilon \to 0} \int_E (S_\varepsilon(g_\varepsilon)(x) - y_d(x))^2 \, dx \geq \liminf_{\varepsilon \to 0} \int_E (\mathcal{S}(g_\varepsilon)(x) - y_d(x))^2 \, dx$$

for each sequence $\{g_\varepsilon\} \subset \mathcal{F}_\mathfrak{s}$ that is uniformly bounded in $L^2(D)$.

Proof. Let $\varepsilon > 0$ be arbitrary but fixed. Arguing as in Lemma 4.5, we see that the equation

$$\begin{aligned} -\triangle y_\varepsilon + \beta(y_\varepsilon) + \frac{1}{\varepsilon} H_\varepsilon(g_\varepsilon) y_\varepsilon &= f \quad \text{in } D, \\ y_\varepsilon &= 0 \quad \text{on } \partial D \end{aligned} \tag{4.19}$$

admits a unique solution $y_\varepsilon := \widehat{S}_\varepsilon(g_\varepsilon) \in H_0^1(D) \cap H^2(D)$. Since $\{g_\varepsilon\}$ is uniformly bounded in $L^2(D)$, by assumption we have

$$\|y_\varepsilon - S_\varepsilon(g_\varepsilon)\|_{H_0^1(D)} \leq \varepsilon \|g_\varepsilon\|_{L^2(D)} \to 0 \quad \text{as } \varepsilon \searrow 0.$$

Thus, it suffices to show

$$\liminf_{\varepsilon \to 0} \int_E (y_\varepsilon(x) - y_d(x))^2 \, dx \geq \liminf_{\varepsilon \to 0} \int_E (\mathcal{S}(g_\varepsilon)(x) - y_d(x))^2 \, dx. \tag{4.20}$$

We use shorthand notation $z_\varepsilon := \mathcal{S}(g_\varepsilon)$, see Definition 4.9 and (4.5). In view of Lemma 2.9, we can define $\widetilde{g_\varepsilon} \in \mathcal{F}_\mathfrak{s}$ so that the entire set $\Omega_{\widetilde{g_\varepsilon}}$ is just the relevant component of Ω_{g_ε}. Hence, z_ε is the unique solution of

$$\begin{aligned} -\triangle z_\varepsilon + \beta(z_\varepsilon) &= f \quad \text{in } \Omega_{\widetilde{g_\varepsilon}}, \\ z_\varepsilon &= 0 \quad \text{on } \partial\Omega_{\widetilde{g_\varepsilon}}. \end{aligned} \tag{4.21}$$

Let $k > 0$ be arbitrary but fixed, and define $z_\varepsilon^k := \widehat{S}_k(\widetilde{g}_\varepsilon)$; i.e., z_ε^k solves

$$\begin{aligned} -\triangle z_\varepsilon^k + \beta(z_\varepsilon^k) + \frac{1}{k} H_k(\widetilde{g}_\varepsilon) z_\varepsilon^k = f \quad \text{in } D, \\ z_\varepsilon^k = 0 \quad \text{on } \partial D. \end{aligned} \tag{4.22}$$

In view of Lemma 2.7, the hypotheses of Proposition 4.6 are satisfied. By arguing in the exact same way as in the proof thereof, we deduce that there exists a (not relabeled) subsequence $\{z_\varepsilon^k\}_k$ so that

$$z_\varepsilon^k \rightharpoonup z_\varepsilon \quad \text{in } H_0^1(D) \quad \text{as } k \to 0. \tag{4.23}$$

With a little abuse of notation, z_ε denotes here its extension by zero on the whole domain D.

(I) In the case when (4.17) is true, it holds that

$$y_\varepsilon \geq 0,\ z_\varepsilon^k \geq 0,\ z_\varepsilon \geq 0 \quad \text{a.e. in } D. \tag{4.24}$$

Our next aim is to compare z_ε with y_ε. To this end, we rewrite (4.22) as

$$\begin{aligned} -\triangle z_\varepsilon^k + \beta(z_\varepsilon^k) + \frac{1}{\varepsilon} H_\varepsilon(g_\varepsilon) z_\varepsilon^k = f + \underbrace{\left[\frac{1}{\varepsilon} H_\varepsilon(g_\varepsilon) - \frac{1}{k} H_k(\widetilde{g}_\varepsilon)\right]}_{\leq 0} \underbrace{z_\varepsilon^k}_{\geq 0} \quad \text{in } D, \\ z_\varepsilon^k = 0 \quad \text{on } \partial D. \end{aligned} \tag{4.25}$$

We observe that by (4.2), it holds that $H_k \geq H_\varepsilon$ for $k \leq \varepsilon$, which implies

$$\frac{1}{\varepsilon} H_\varepsilon(g_\varepsilon) \leq \frac{1}{k} H_k(g_\varepsilon) \leq \frac{1}{k} H_k(\widetilde{g}_\varepsilon) \quad \text{a.e. in } D, \quad \forall\, 0 < k \leq \varepsilon.$$

Note that in the last inequality we employed the monotonicity of H_k (cf. (4.2)) and the fact that $g_\varepsilon \leq \widetilde{g}_\varepsilon$, by construction; see Lemma 2.9. Hence, comparing (4.19) with (4.25) yields

$$z_\varepsilon^k \leq y_\varepsilon \quad \text{a.e. in } D, \quad \forall\, 0 < k \leq \varepsilon.$$

By passing to the limit $k \to 0$, where we use (4.23), we arrive at

$$z_\varepsilon \leq y_\varepsilon \quad \text{a.e. in } D.$$

Then, (4.24) and $y_d \leq 0$ a.e. in E (see (4.17)) lead to

$$0 \leq z_\varepsilon - y_d \leq y_\varepsilon - y_d \quad \text{a.e. in } E,$$

whence (4.20) follows.

(II) The case when (4.18) in Assumption 4.12 is true can be treated completely analogously. One obtains

$$y_\varepsilon \leq 0,\ z_\varepsilon^k \leq 0,\ z_\varepsilon \leq 0 \quad \text{a.e. in } D,$$

$$z_\varepsilon \geq y_\varepsilon \quad \text{a.e. in } D.$$

Then, $y_d \geq 0$ a.e. in E (cf. (4.18)) gives in turn

$$0 \geq z_\varepsilon - y_d \geq y_\varepsilon - y_d \quad \text{a.e. in } E.$$

The proof is now complete. ∎

4.3. Existence of optimal solutions for (P_ε). In the proof of our main result (Theorem 5.2), we will need the fact that the approximating control problem (P_ε) admits optimal solutions (when the admissible set $\mathcal{F}$ is restricted to a certain ball). For this, we gather some useful convergences in the next lemma. Throughout this subsection, $\varepsilon > 0$ is arbitrary but fixed, and the solution operator to (4.3) is denoted by S_ε; cf. Lemma 4.5.

LEMMA 4.14. *Let $\{g_n\} \subset \mathcal{F}$ with*

$$g_n \rightharpoonup g \quad \text{in } \mathcal{W}, \text{ as } n \to \infty.$$

Then,

$$H_\varepsilon(g_n) \to H_\varepsilon(g) \quad \text{in } L^2(D), \quad \text{as } n \to \infty, \tag{4.26}$$

$$S_\varepsilon(g_n) \rightharpoonup S_\varepsilon(g) \quad \text{in } H^2(D) \cap H_0^1(D), \text{ as } n \to \infty. \tag{4.27}$$

Proof. We begin by observing that due to $\{g_n\} \subset \mathcal{F}$, we have $g \in \mathcal{F}$, and thus it holds that

$$H_\varepsilon(g_n) = 0 = H_\varepsilon(g) \quad \text{a.e. in } E, \ \forall n. \tag{4.28}$$

In light of the compact embedding$H^s(D \setminus \bar{E}) \hookrightarrow\hookrightarrow L^2(D \setminus E)$ and the continuity of H_ε, we also have

$$H_\varepsilon(g_n) \to H_\varepsilon(g) \quad \text{in } L^2(D \setminus E), \text{ as } n \to \infty.$$

This proves (4.26). To show the convergence concerning S_ε, for simplicity we define $y_n := S_\varepsilon(g_n)$. As a result of Lemma 4.5, the sequence $\{y_n\}$ is uniformly bounded in $H^2(D) \cap H_0^1(D)$, and we can thus extract a weakly convergent subsequence denoted by the same symbol so that

$$y_n \rightharpoonup \widetilde{y} \quad \text{in } H^2(D) \cap H_0^1(D) \text{ as } n \to \infty.$$

Passing to the limit $n \to \infty$ in (4.3), where one uses (4.26), then yields

$$\begin{aligned} -\triangle \widetilde{y} + \beta(\widetilde{y}) + \frac{1}{\varepsilon} H_\varepsilon(g)\widetilde{y} &= f + \varepsilon g \quad \text{a.e. in } D, \\ \widetilde{y} &= 0 \quad \text{on } \partial D. \end{aligned} \tag{4.29}$$

From Lemma 4.5 we finally deduce $\widetilde{y} = S_\varepsilon(g) \in H^2(D) \cap H_0^1(D)$, and the proof of (4.27) is complete. ∎

By means of the direct method of calculus of variations, we can now show the following result.

PROPOSITION 4.15. *The approximating optimal control problem (P_ε) admits at least one global minimizer in $\mathcal{F}$.*

5. Correlation between (P_ε) and shape optimization. This section contains the main results of the present paper. One of them states that local optima in the $L^2(D)$ sense of (P) (Definition 2.11) can be approximated by local minimizers of the approximating control problems (P_ε) introduced in the last section. As a consequence, we deduce that each parametrization of the optimal shape of (P_Ω) is the limit of global optima of (P_ε) (Corollary 5.4). These two findings open the door to future research on the topic of optimality systems in qualified form for (P_Ω).

Similarly to Definition 2.11, we introduce the notion of local optimality for the approximating control problem.

DEFINITION 5.1. *Let $\varepsilon > 0$ be fixed and $\bar g_{\mathfrak{s}} \in \mathcal{F}_{\mathfrak{s}}$. We say that $\bar g_\varepsilon \in \mathcal{F}$ is locally optimal for (P_ε) in the $L^2(D)$ sense if there exists $r > 0$ such that*

$$j_\varepsilon(\bar g_\varepsilon) \le j_\varepsilon(g) \quad \forall\, g \in \mathcal{F} \text{ with } \|g - \bar g_\varepsilon\|_{L^2(D)} \le r, \tag{5.1}$$

where

$$j_\varepsilon(g) := \int_E (S_\varepsilon(g)(x) - y_d(x))^2\, dx + \alpha \int_D (1 - H_\varepsilon(g))(x)\, dx + \frac{1}{2}\, \|g - \bar g_{\mathfrak{s}}\|^2_{\mathcal{W}}$$

is the reduced cost functional associated to the control problem (P_ε).

THEOREM 5.2. *Let $\bar g_{\mathfrak{s}} \in \mathcal{F}_{\mathfrak{s}}$ be a local minimizer of (P) in the sense of Definition 2.11. Then, under Assumption 4.12, there exists a sequence of local minimizers $\{\bar g_\varepsilon\}$ of (P_ε) such that*

$$\bar g_\varepsilon \to \bar g_{\mathfrak{s}} \quad \text{in } \mathcal{W}, \text{ as } \varepsilon \searrow 0.$$

Moreover,

$$S_\varepsilon(\bar g_\varepsilon) \rightharpoonup \mathcal{S}(\bar g_{\mathfrak{s}}) \quad \text{in } H_0^1(D), \text{ as } \varepsilon \searrow 0.$$

Proof. The proof is a variation of well-known ideas [3]. However, let us point out that we need to take into account that the control-to-state map $\mathcal{S}$ is defined on $\mathcal{F}_{\mathfrak{s}}$, not on $\mathcal{F}$. To bridge the gap between these two sets, we will make use of the essential Proposition 3.1 and the convergence results from subsection 4.2. Denote by $\overline{B_{L^2(D)}(\bar g_{\mathfrak{s}}, r)}$, $r > 0$, the (closed) ball of local optimality of $\bar g_{\mathfrak{s}} \in \mathcal{F}_{\mathfrak{s}}$. Let $\varepsilon > 0$ be fixed. Thanks to Lemma 4.14, we can show by the direct method of calculus of variations that the optimization problem

$$\min_{g \in \mathcal{F} \cap \overline{B_{L^2(D)}(\bar g_{\mathfrak{s}}, r/2)}} j_\varepsilon(g) \tag{5.2}$$

admits a global minimizer

$$\bar g_\varepsilon \in \mathcal{F} \cap \overline{B_{L^2(D)}(\bar g_{\mathfrak{s}}, r/2)}.$$

Next, we recall that the reduced cost functional associated to the control problem (P) is given by

$$\mathcal{J}(g) = \int_E (\mathcal{S}(g)(x) - y_d(x))^2\, dx + \alpha \int_D (1 - H(g))(x)\, dx, \quad g \in \mathcal{F}_{\mathfrak{s}}.$$

Due to Proposition 4.6 (see also Remark 4.7), it holds that

$$\begin{aligned} \mathcal{J}(\bar g_{\mathfrak{s}}) &= \lim_{\varepsilon \to 0} \int_E (S_\varepsilon(\bar g_{\mathfrak{s}})(x) - y_d(x))^2\, dx + \alpha \int_D (1 - H_\varepsilon(\bar g_{\mathfrak{s}}))(x)\, dx \\ &\ge \limsup_{\varepsilon \to 0} \int_E (S_\varepsilon(\bar g_\varepsilon)(x) - y_d(x))^2\, dx + \alpha \int_D (1 - H_\varepsilon(\bar g_\varepsilon))(x)\, dx \\ &\quad + \frac{1}{2}\, \|\bar g_\varepsilon - \bar g_{\mathfrak{s}}\|^2_{\mathcal{W}}, \end{aligned} \tag{5.3}$$

where for the last inequality we relied on the facts that $\bar g_\varepsilon$ is a global minimizer of (5.2) and that $\bar g_{\mathfrak{s}}$ is admissible for (5.2). Thanks to Proposition 3.1, for each $\varepsilon > 0$, there exists $\widehat g_\varepsilon \in \mathcal{F}_{\mathfrak{s}}$, so that

$$\|\widehat g_\varepsilon - \bar g_\varepsilon\|_{L^2(D)} \le \frac{r\varepsilon}{2} \min\left\{ \frac{1}{L_{H_\varepsilon}}, \frac{1}{L_{\|\bar g_{\mathfrak{s}}\|_{L^2(D)} + r, \varepsilon}} \right\} \le r/2, \tag{5.4}$$

where $L_{H_\varepsilon} > 0$ and $L_{\|\bar g_{\mathfrak s}\|_{L^2(D)}+r,\varepsilon} > 0$ are the Lipschitz constants of H_ε and S_ε (see (4.4)), respectively; note that the second estimate in (5.4) is true for ε small enough. Then,

$$\|H_\varepsilon(\widehat g_\varepsilon) - H_\varepsilon(\bar g_\varepsilon)\|_{L^2(D)} \le r\varepsilon/2, \tag{5.5}$$

and according to (4.4), we also have

$$\|S_\varepsilon(\widehat g_\varepsilon) - S_\varepsilon(\bar g_\varepsilon)\|_{L^2(D)} \le L_{\|\bar g_{\mathfrak s}\|_{L^2(D)}+r,\varepsilon}\|\widehat g_\varepsilon - \bar g_\varepsilon\|_{L^2(D)} \le r\varepsilon/2. \tag{5.6}$$

Here we note that

$$\|\widehat g_\varepsilon\|_{L^2(D)}, \|\bar g_\varepsilon\|_{L^2(D)} \le \|\bar g_{\mathfrak s}\|_{L^2(D)} + r, \tag{5.7}$$

since $\bar g_\varepsilon$ is admissible for (5.2) and in light of (5.4). Going back to (5.3), we can thus write

$$\begin{aligned}
\mathcal J(\bar g_{\mathfrak s}) &= \lim_{\varepsilon\to 0}\int_E (S_\varepsilon(\bar g_{\mathfrak s})(x) - y_d(x))^2\,dx + \alpha\int_D (1 - H_\varepsilon(\bar g_{\mathfrak s}))(x)\,dx\\
&\ge \limsup_{\varepsilon\to 0}\int_E (S_\varepsilon(\bar g_\varepsilon)(x) - y_d(x))^2\,dx + \alpha\int_D (1 - H_\varepsilon(\bar g_\varepsilon))(x)\,dx\\
&\quad + \frac12\|\bar g_\varepsilon - \bar g_{\mathfrak s}\|^2_{\mathcal W}\\
&\ge \liminf_{\varepsilon\to 0}\int_E (S_\varepsilon(\bar g_\varepsilon)(x) - y_d(x))^2\,dx + \alpha\int_D (1 - H_\varepsilon(\bar g_\varepsilon))(x)\,dx\\
&\quad + \frac12\|\bar g_\varepsilon - \bar g_{\mathfrak s}\|^2_{\mathcal W}\\
&= \liminf_{\varepsilon\to 0}\int_E (S_\varepsilon(\widehat g_\varepsilon)(x) - y_d(x))^2\,dx + \alpha\int_D (1 - H_\varepsilon(\widehat g_\varepsilon))(x)\,dx\\
&\quad + \frac12\|\bar g_\varepsilon - \bar g_{\mathfrak s}\|^2_{\mathcal W},
\end{aligned} \tag{5.8}$$

where we relied on (5.5) and (5.6). Thanks to Lemma 4.13 combined with (5.7) and $H_\varepsilon \le H$ (see (4.2) and (2.7)), (5.8) can be continued as

$$\begin{aligned}
\mathcal J(\bar g_{\mathfrak s}) &\ge \liminf_{\varepsilon\to 0}\int_E (\mathcal S(\widehat g_\varepsilon)(x) - y_d(x))^2\,dx + \alpha\int_D (1 - H(\widehat g_\varepsilon))(x)\,dx\\
&\quad + \frac12\|\bar g_\varepsilon - \bar g_{\mathfrak s}\|^2_{\mathcal W}\\
&\ge \int_E (\mathcal S(\bar g_{\mathfrak s})(x) - y_d(x))^2\,dx + \alpha\int_D (1 - H(\bar g_{\mathfrak s}))(x)\,dx\\
&= \mathcal J(\bar g_{\mathfrak s}).
\end{aligned} \tag{5.9}$$

The last estimate in (5.9) is due to the fact that $\widehat g_\varepsilon \in \mathcal F_{\mathfrak s}$ is in the ball of local optimality of $\bar g_{\mathfrak s}$ (in light of (5.4) and since $\bar g_\varepsilon$ is admissible for (5.2)). From (5.8) and (5.9) we can now conclude that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_E (S_\varepsilon(\bar g_\varepsilon)(x) - y_d(x))^2\,dx + \alpha\int_D (1 - H_\varepsilon(\bar g_\varepsilon))(x)\,dx + \frac12\|\bar g_\varepsilon - \bar g_{\mathfrak s}\|^2_{\mathcal W}\\
&\quad = \mathcal J(\bar g_{\mathfrak s}) = \int_E (\mathcal S(\bar g_{\mathfrak s})(x) - y_d(x))^2\,dx + \alpha\int_D (1 - H(\bar g_{\mathfrak s}))(x)\,dx\\
&\quad = \lim_{\varepsilon\to 0}\int_E (S_\varepsilon(\bar g_\varepsilon)(x) - y_d(x))^2\,dx + \alpha\int_D (1 - H_\varepsilon(\bar g_\varepsilon))(x)\,dx,
\end{aligned} \tag{5.10}$$

where the last identity follows by estimating as in (5.8)–(5.9), without taking the term $\frac{1}{2}\|\bar{g}_\varepsilon - \bar{g}_{\mathfrak{s}}\|^2_{\mathcal{W}}$ into account. By (5.10), we finally deduce

$$\bar{g}_\varepsilon \to \bar{g}_{\mathfrak{s}} \quad \text{in } \mathcal{W}. \tag{5.11}$$

To show that $\bar{g}_\varepsilon$ is a local minimizer of (P_ε), let $v \in \mathcal{F}$ with $\|v - \bar{g}_\varepsilon\|_{L^2(D)} \leq r/4$ be arbitrary but fixed. Then, by (5.11), we have, for $\varepsilon > 0$ small enough,

$$\|v - \bar{g}_{\mathfrak{s}}\|_{L^2(D)} \leq \|v - \bar{g}_\varepsilon\|_{L^2(D)} + \|\bar{g}_\varepsilon - \bar{g}_{\mathfrak{s}}\|_{L^2(D)} \leq r/2.$$

As $\bar{g}_\varepsilon$ is global optimal for (5.2), this means that $j_\varepsilon(\bar{g}_\varepsilon) \leq j_\varepsilon(v)$, whence the desired local optimality of $\bar{g}_\varepsilon$ follows (Definition 5.1).

Since $s > 1$, we have $\mathcal{W} \hookrightarrow L^2(D) \cap L^\infty(D \setminus \bar{E})$, and Lemma 4.11 with (5.11) implies

$$S_\varepsilon(\bar{g}_\varepsilon) \rightharpoonup \mathcal{S}(\bar{g}_{\mathfrak{s}}) \quad \text{in } H_0^1(D), \text{ as } \varepsilon \searrow 0.$$

The proof is now complete. ∎

Remark 5.3 (approximation of global optima of (P)). A short inspection of the proof of Theorem 5.2 shows that, if $\bar{g}_{\mathfrak{s}}$ is a global minimizer, then the associated sequence $\{\bar{g}_\varepsilon\}$ inherits this property, i.e., it consists of global optima of (P_ε). Here we pay attention to the fact that (5.7) is replaced by

$$\|\widehat{g}_\varepsilon\|_{L^2(D)}, \|\bar{g}_\varepsilon\|_{L^2(D)} \leq c, \tag{5.12}$$

where $c > 0$ is a constant independent of ε (this can be proven by arguing as in the proof of (5.3)). Thus, we can still make use of Lemma 4.13 to deduce (5.9).

As a consequence of Theorem 5.2 and Proposition 2.10, we then arrive at the following.

COROLLARY 5.4 (approximation of the optimal shape). *Suppose that Assumption* 4.12 *is satisfied. Let $\Omega^\star \in \mathcal{O}$ be an optimal shape for (P_Ω). Then, for each $g^\star \in \mathcal{F}_{\mathfrak{s}}$ with $\Omega_{g^\star} = \Omega^\star$, there exists a sequence of global minimizers $\{\bar{g}_\varepsilon\}$ of (P_ε) such that*

$$\bar{g}_\varepsilon \to g^\star \quad \text{in } \mathcal{W}, \text{ as } \varepsilon \searrow 0.$$

Moreover,

$$S_\varepsilon(\bar{g}_\varepsilon) \rightharpoonup \mathcal{S}(g^\star) \quad \text{in } H_0^1(D), \text{ as } \varepsilon \searrow 0.$$

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