Arch. Rational Mech. Anal. 123 (1993) 51–74. © Springer-Verlag 1993

Uniqueness of Lipschitz Extensions: Minimizing the Sup Norm of the Gradient

ROBERT JENSEN

Communicated by D. KINDERLEHRER

Abstract

In this paper we examine the problem of minimizing the sup norm of the gradient of a function with prescribed boundary values. Geometrically, this can be interpreted as finding a minimal Lipschitz extension. Due to the weak convexity of the functional associated to this problem, solutions are generally nonunique. By adopting G. ARONSSON's notion of absolutely minimizing we are able to prove uniqueness by characterizing minimizers as the unique solutions of an associated partial differential equation. In fact, we actually prove a weak maximum principle for this partial differential equation, which in some sense is the Euler equation for the minimization problem. This is significantly difficult because the partial differential equation is both fully nonlinear and has very degenerate ellipticity. To overcome this difficulty we use the weak solutions of M. G. CRANDALL and P.-L. LIONS, also known as viscosity solutions, in conjunction with some arguments using integration by parts.

§ 0. Introduction

Although our story properly begins with G. ARONSSON's papers [1, 2], the motivation for his work and ours is better understood by recalling the context in which his papers appeared. Prior to [1, 2] ARONSSON had been studying variational problems in L^∞, and it was as an outgrowth of these studies that he considered minimal Lipschitz extensions. Explicitly, a minimal Lipschitz extension is a function $u \in W^{1,\infty}(\Omega)$ such that

$$(0.1) \quad \|\mathbf{D}u\|_{L^\infty(\Omega;\mathbb{R}^n)} \leqq \|\mathbf{D}w\|_{L^\infty(\Omega;\mathbb{R}^n)} \quad \text{for all } w \text{ with } (u-w) \in W_0^{1,\infty}(\Omega)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded, connected domain. (If the boundary of Ω is sufficiently well behaved, then $g = u|_{\partial\Omega}$ is well defined, and in this case we say that u is a minimal Lipschitz extension of g into Ω.)

The connection between minimal Lipschitz extensions and variational problems in L^∞ is readily evident. Indeed, it is apparent from (0.1) that the minimal Lipschitz extension *is* a solution to the variational problem: Find a minimizer of the L^∞ norm of the gradient over all functions with prescribed boundary values. In a sense to be made precise later

$$\sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\,\partial x_j} = 0 \quad \text{in } \Omega \tag{0.2}$$

is the Euler equation of the minimal Lipschitz extension problem, and this equation is our point of attack in the present study. By analogy with variational problems in L^2 we view

$$\sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\,\partial x_j}$$

as the fundamental operator for the calculus of variations in L^∞, just as the Laplacian is viewed as the fundamental operator for the calculus of variations in L^2. This perspective suggests using the minimal Lipschitz extension problem as the basis for a systematic investigation of the calculus of variations in L^∞.

The importance of variational problems in L^∞ is due to their frequent appearance in applications. The following examples give just a small sample of these. In the engineering of a load-bearing column it is preferable to minimize the *maximal* stress (i.e., the L^∞ norm of the stress) in the column rather than some average of the stress. When constructing a rocket, the *maximal* acceleration applied to the payload is an important factor in the design. Optimal operation of a heating-cooling system for an office building requires control of the *maximal* and *minimal* temperature within the building rather than the average temperature. Windows on airplanes are made without corners to prevent high *pointwise* stress concentrations.

These considerations motivate the study of the issues of existence, uniqueness, and regularity for minimal Lipschitz extensions. Existence is not too difficult, but (as we later demonstrate) minimal Lipschitz extensions are in general neither unique nor smooth. In 1967 G. ARONSSON [1] introduced a formally "canonical" minimal Lipschitz extension, and he called this object an absolutely minimizing Lipschitz extension or AMLE for short. In his paper ARONSSON proved that the Euler equation for *smooth* AMLE's *is* in fact (0.2). Our primary goal is to rigorously demonstrate the existence and uniqueness of AMLE's under very general assumptions; as we previously indicated, our approach uses (0.2) intensively.

However, it is apparent that (0.2) is fully nonlinear and highly degenerate (in its ellipticity). The particular manifestation of these features precludes the use of standard methods such as integration by parts, a priori L^p estimates (including subelliptic estimates), and Schauder estimates. Therefore, resolving the question of uniqueness is a challenge requiring new techniques and arguments. I feel that these new ideas are intrinsically noteworthy and potentially useful in other nonlinear problems, particularly in those arising from variational problems in L^∞.

In Section 1 we prove that AMLE's are indeed solutions of (0.2). Section 2 provides a maximum principle for solutions of (0.2). Finally, in Section 3 we generalize our previous results to cover as wide a field as possible. The remainder of this section is devoted to notation, definitions, and examples intended to facilitate our exposition.

As noted initially, Ω denotes a bounded, connected domain in $\mathbb{R}^n$. Given Ω we use $d_\Omega : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$ to denote the distance metric relative to Ω. (In Section 1 we define d_Ω precisely.) We define $\mathrm{Lip}_\partial(\Omega)$ by

$$\mathrm{Lip}_\partial(\Omega) = \left\{ g \in C(\partial\Omega) : \sup_{x,y \in \partial\Omega} \left(\frac{|g(x) - g(y)|}{d_\Omega(x, y)} \right) < \infty \right\}.$$

If $\partial\Omega$ is smooth, then $\mathrm{Lip}_\partial(\Omega) = W^{1,\infty}(\partial\Omega)$, but $\mathrm{Lip}_\partial(\Omega)$ is defined for more general domains. The important property held by $\mathrm{Lip}_\partial(\Omega)$ is that if $g \in \mathrm{Lip}_\partial(\Omega)$, then there exists a minimal Lipschitz extension of g into Ω. In fact (although we defer verification until Section 1), given $g \in \mathrm{Lip}_\partial(\Omega)$ for K chosen properly, the function

$$w(x) = \inf_{y \in \bar{\Omega}} \left(g(y) + K\, d_\Omega(x, y) \right)$$

is a minimal Lipschitz extension of g into Ω.

Next we establish our previous remark on the nonuniqueness of minimal Lipschitz extensions. Let $\Omega = B(\bar{0}, 1) \subset \mathbb{R}^2$, and let $g(x, y) = 2xy$ for $x^2 + y^2 = 1$, i.e., $(x, y) = \partial\Omega$. For $\alpha \in [0, \frac{1}{2}]$ set

$$u^\alpha(x, y) = \begin{cases} 0 & \text{if } x^2 + y^2 \leqq \alpha^2, \\ \dfrac{2xy\,(\sqrt{2x^2 + y^2} - \alpha)}{(1 - \alpha)\,(x^2 + y^2)} & \text{if } \alpha^2 \leqq x^2 + y^2 \leqq 1; \end{cases}$$

then for each $\alpha \in [0, \frac{1}{2}]$, $u = u^\alpha$ is a minimal Lipschitz extension of g into Ω.

In order to motivate ARONSSON's AMLE we start by considering the analogous L^p problem, i.e., u is a minimal p-extension if

$$\|\mathbf{D}u\|_{L^p(\Omega;\mathbb{R}^n)} \leqq \|\mathbf{D}w\|_{L^p(\Omega;\mathbb{R}^n)} \text{ for all } w \text{ such that } (u - w) \in W_0^{1,p}(\Omega).$$

Such extensions are unique solutions of the p-Laplacian with Dirichlet data. ARONSSON reasoned that a canonical minimal Lipschitz extension would be the limit of minimal p-extensions. For minimal p-extensions, however, the previous condition is equivalent to

$$\|\mathbf{D}u\|_{L^p(\hat{\Omega};\mathbb{R}^n)} \leqq \|\mathbf{D}u\|_{L^p(\hat{\Omega};\mathbb{R}^n)}$$

$$\text{for all } \hat{\Omega} \subset \Omega \text{ and } w \text{ such that } (u - w) \in W_0^{1,p}(\hat{\Omega}).$$

Since the condition holds in the limit, i.e., for $p = \infty$, ARONSSON declared that an AMLE is a function $u \in W^{1,\infty}(\Omega)$ such that

$$\|\mathbf{D}u\|_{L^\infty(\hat{\Omega};\mathbb{R}^n)} \leqq \|\mathbf{D}u\|_{L^\infty(\hat{\Omega};\mathbb{R}^n)} \tag{0.3}$$

$$\text{for all } \hat{\Omega} \subset \Omega \text{ and } w \text{ such that } (u - w) \in W_0^{1,\infty}(\hat{\Omega}).$$

As we already noted, ARONSSON [1] proved that for *smooth* AMLE's the Euler equation of (0.3) *is* (0.2). However, since AMLE's are *not* in general smooth, this still left the general question open.

Our main results show – for a suitable notion of weak solution – that if $g \in \mathrm{Lip}_\partial(\Omega)$, then

i) There exists an AMLE of g into Ω.

ii) Every AMLE on Ω is a solution of (0.2).

iii) Solutions of (0.2) satisfy a maximum principle.

iv) As a consequence, the AMLE of g into Ω is unique.

In deference to ARONSSON's contributions to the subject we refer to (0.2) as Aronsson's Euler equation or AEE for short.

To the best of my knowledge AEE was first studied by G. ARONSSON in [1, 2]. Recently, however, BHATTACHARYA, DIBENEDETTO & MANFREDI [3] have investigated related problems arising from problems in elasticity. In [1, 2] ARONSSON proved several interesting results on *classical* solutions of AEE, i.e., of (0.2). Here we present an elementary proof of his result that for C^2 functions, (0.2) is the Euler equation of the AMLE problem.

Given an arbitrary point x' in some domain Ω we need to show that if $u \in C^2(\Omega)$ is an AMLE, then

$$\frac{\partial u}{\partial x_i}(x')\,\frac{\partial u}{\partial x_j}(x')\,\frac{\partial^2 u}{\partial x_i\,\partial x_j}(x') = 0, \tag{0.2'}$$

where we use the summation convention on repeated indices. By the translation invariance of (0.2) we may assume without loss of generality that $x' = \bar{0}$. Since Ω is a domain, there is a positive constant ε' such that $B(\bar{0}, \varepsilon) \subset \Omega$ if $\varepsilon < \varepsilon'$. Define $w \in C^2(B(\bar{0}, \varepsilon))$ by

$$w(x) = u(x) + \frac{\gamma}{2}\varepsilon^2 - \frac{\gamma}{2}|x|^2 \quad \text{for } x \in B(\bar{0}, \varepsilon).$$

Taylor expansions yield

$$u(x) = a + p_i x_i + \frac{1}{2}\mu_{ij}x_i x_j + o(|x|^2),$$

$$w(x) = a + \frac{\gamma}{2}\varepsilon^2 + p_i x_i + \frac{1}{2}\hat{\mu}_{ij}x_i x_j + o(|x|^2)$$

where $\hat{\mu}_{ij} = (\mu_{ij} - \gamma\delta_{ij})$. In terms of p_i and μ_{ij}, (0.2)′ becomes

$$p_i p_j \mu_{ij} = 0. \tag{0.4}$$

If $\mu_{ij}p_j = 0$ for all $i = 1, \ldots, n$, then clearly (0.4) is true, so we now assume that

$$\mu_{ik}p_k\mu_{ij}p_j > 0.$$

This in turn implies that if γ is sufficiently small, then

$$\hat{\mu}_{ik}p_k\hat{\mu}_{ij}p_j \geqq C_0 > 0. \tag{0.5}$$

If v denotes either u or w, we see that

$$|\mathbf{D}v|^2(x) = p_i p_i + 2\tilde{\mu}_{ij} p_i x_j + o(|x|)$$

where $\tilde{\mu}_{ij}$ is either μ_{ij} or $\hat{\mu}_{ij}$ depending on whether v is respectively u or w. Calculus then establishes

$$\|\mathbf{D}v\|^2_{L^\infty(B(\bar{0},\varepsilon);\mathbb{R}^n)} = p_i p_i + 2\varepsilon\sqrt{\tilde{\mu}_{ij} p_j \tilde{\mu}_{ik} p_k} + o(\varepsilon).$$

Using this estimate and (0.3) we calculate

$$\begin{aligned} 0 &\leqq \|\mathbf{D}w\|^2_{L^\infty(B(\bar{0},\varepsilon);\mathbb{R}^n)} - \|\mathbf{D}u\|^2_{L^\infty(B(\bar{0},\varepsilon);\mathbb{R}^n)} \\ &= 2\varepsilon\left(\sqrt{\hat{\mu}_{ij} p_j \hat{\mu}_{ik} p_k} - \sqrt{\mu_{ij} p_j \mu_{ik} p_k}\right) + o(\varepsilon). \end{aligned}$$

Dividing this inequality by 2ε and letting $\varepsilon \searrow 0$ we obtain

$$0 \leqq \left(\sqrt{\hat{\mu}_{ij} p_j \hat{\mu}_{ik} p_k} - \sqrt{\mu_{ij} p_j \mu_{ik} p_k}\right) = \mathscr{F}(\gamma),$$

and note that $\mathscr{F}(0) = 0$. Consequently, $\dfrac{d\mathscr{F}}{d\gamma}(0) = 0$, which translates into

$$-\frac{\mu_{ij} p_i p_j}{\sqrt{\mu_{ij} p_j \mu_{ik} p_k}} = 0,$$

and establishes (0.4) as desired. $\square$

The problem with the preceding argument and in general with ARONSSON's results is that they all require C^2 solutions or extensions. By Using the L^∞ weak solutions introduced by CRANDALL & LIONS [4] we can define solutions of AEE in the generality desired for our analysis. This is important, in fact, necessary, for both the result on existence of solutions and the argument used to prove uniqueness. The necessity of this approach can best be understood by examining the consequences of a theorem in [2].

Possibly the most important result proved by ARONSSON in [2] is

Theorem. *Let $u \in C^2(\Omega)$ for some $\Omega \subset \mathbb{R}^2$. If u is a nonconstant solution of AEE, then*

$$|\mathbf{D}u| > 0 \quad \textit{in } \Omega.$$

It follows as a corollary that classical solutions of AEE (in two dimensions) are unique. Furthermore, as ARONSSON proved, this also implies that there *cannot* be classical solutions for all Dirichlet problems of AEE.

Indeed, let $\Omega = B(\bar{0}, 1)$ and $g(x, y) = 2xy$ for $x^2 + y^2 = 1$. If u is a C^2 solution of AEE with boundary data g, then by uniqueness, $u(x, y) = u(-x, -y)$. Consequently $\mathbf{D}u(0, 0) = \bar{0}$, but this contradicts ARONSSON's theorem. Hence, there is *no* classical solution in this case.

§ 1. The Euler equation and existence of solutions

In this section we begin by recalling the definition of viscosity solutions. Next we characterize minimal Lipschitz extensions of g into Ω using a seminorm introduced on $\mathrm{Lip}_\partial(\Omega)$. With this characterization we proceed to rigorously establish (0.2) as the Euler equation of the AMLE. We close this section by introducing two auxiliary equations and proving existence of solutions for AEE and the auxiliary equations.

The fully nonlinear nature of AEE necessitates the adoption of a new view of solution. Indeed, we have already indicated some of the failings of classical solutions. Conventional notions of weak solutions based on integration by parts fare no better. In fact, integrating AEE by parts just seems to complicate the problem further. So we have chosen to define solutions as the viscosity solutions of CRANDALL & LIONS.

Given a function $F \in C(\mathbb{R}^n \times S(n))$ where $S(n)$ is the space of $n \times n$ symmetric matrices we recall the definition of viscosity solutions of the partial differential equations

$$F(\mathbf{D}u, \mathbf{D}^2 u) = 0 \quad \text{in } \Omega. \tag{1.1}$$

Definition 1.2. (i) An upper semicontinuous function $u : \bar{\Omega} \to \mathbb{R}$ is a *subsolution* of (1.1) if

$$F(\mathbf{D}\phi(x), \mathbf{D}^2\phi(x)) \geqq 0$$

for every $(x, \phi) \in \Omega \times C^2(\Omega)$ such that $(u - \phi)\ (x) \geqq (u - \phi)\ (y)$ for all $y \in \Omega$.

(ii) A lower semicontinuous function $u : \bar{\Omega} \to \mathbb{R}$ is a *supersolution* of (1.1) if

$$F(\mathbf{D}\phi(x), \mathbf{D}^2\phi(x)) \leqq 0$$

for every $(x, \phi) \in \Omega \times C^2(\Omega)$ such that $(u - \phi)\ (x) \leqq (u - \phi)\ (y)$ for all $y \in \Omega$.

(iii) A continuous function $u : \bar{\Omega} \to \mathbb{R}$ is a *solution* of (1.1) if it is both a subsolution and supersolution.

Information on such solutions may be found in [6–12]. However, one obvious fact is that if (1.1) is elliptic (possibly degenerate), then every classical solution is also a solution in the sense of Definition 1.2, i.e., a viscosity solution.

In order to avoid any confusion we also display the definition of d_Ω. (Remember, we have stipulated that for the rest of this paper Ω is a bounded, connected domain in $\mathbb{R}^n$.)

Definition 1.3.

$$d_\Omega(x, y) = \liminf_{(\xi,\eta)\to(x,y)} \left(\inf \left\{ \int_0^1 \left| \frac{d\zeta}{dt} \right| dt : \zeta \in C^1([0, 1]; \Omega), \xi(0) = \zeta,\ \zeta(1) = \eta \right\} \right).$$

Remark. d_Ω satisfies all of the properties of a metric except the triangle inequality. Instead, we have the following modified form of the triangle inequality

(1.4) $$d_\Omega(x, y) \leqq d_\Omega(x, z) + d_\Omega(z, y) \quad \text{for all } x, y \in \bar{\Omega}, \text{ and } z \in \Omega.$$

We finish our preliminary definitions with the introduction of the seminorm $|||\cdot|||$ on $\mathrm{Lip}_\partial(\Omega)$.

Definition 1.5. For $g \in \mathrm{Lip}_\partial(\Omega)$,

$$|||g||| = |||g|||_\Omega = \sup_{x \neq y \in \partial\Omega} \left(\frac{|g(x) - g(y)|}{d_\Omega(x, y)} \right).$$

With our preparations now complete, we begin the task of characterizing the minimal Lipschitz extensions.

Lemma 1.6. *If* $g \in \mathrm{Lip}_\partial(\Omega)$, $w \in W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$, *and* $w|_{\partial\Omega} = g$, *then*

(1.7) $$\|\mathbf{D}w\|_{L^\infty(\Omega;\mathbb{R}^n)} \geqq |||g|||.$$

Proof. Given an $\varepsilon > 0$, let $x \neq y$ be points in $\partial\Omega$ such that

$$\frac{|g(x) - g(y)|}{d_\Omega(x, y)} \geqq |||g||| - \frac{\varepsilon}{2}.$$

Using the continuity of w on $\bar{\Omega}$ and the definition of $d_\Omega(x, y)$ we conclude that there exists a curve $\zeta \in C^1([0, 1]; \Omega)$ such that

$$\frac{|w(\zeta(0)) - w(\zeta(1))|}{\int_0^1 \left|\frac{d\zeta}{dt}\right| dt} \geqq \frac{|g(x) - g(y)|}{d_\Omega(x, y)} - \frac{\varepsilon}{2} \geqq |||g||| - \varepsilon.$$

This last inequality clearly implies (1.7). □

In the next theorem we present a construction of minimal Lipschitz extensions. This construction is a minor modification of previous work by G. Aronsson [1].

Theorem 1.8. *Assume* $g \in \mathrm{Lip}_\partial(\Omega)$. *Define* $u : \bar{\Omega} \to \mathbb{R}$ *by*

(1.9) $$u(x) = \inf_{\eta \in \partial\Omega} (g(\eta) + |||g||| \cdot d_\Omega(x, \eta)).$$

Then

(1.10) $$u|_{\partial\Omega} = g, \quad u \in W^{1,\infty}(\Omega) \cap C(\bar{\Omega}), \quad \|\mathbf{D}u\|_{L^\infty(\Omega;\mathbb{R}^n)} \leqq |||g|||$$

and consequently u *is a minimal Lipschitz extension of* g *into* Ω.

Remark. The minimal Lipschitz extension constructed in Theorem 1.8 is *not* in general the AMLE of g into Ω. In fact,

$$v(x) = \sup_{\eta \in \partial\Omega} \left(g(\eta) - |||g||| \cdot d_\Omega(x, \eta) \right)$$

is also a minimal Lipschitz extension of g into Ω; u is the AMLE of g into Ω if and only if $u = v$.

Proof. In view of Lemma 1.6 it is sufficient to prove (1.10). Examination of (1.9) with Definition 1.5 leads to immediate verification of $u|_{\partial\Omega} = g$. In proving that $u \in W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$ we start by proving that u is continuous on $\partial\Omega$. Since $g \in C(\partial\Omega)$, it suffices to prove that if $\{\xi_i\} \subset \Omega$ and $\xi_i \to x \in \partial\Omega$ as $i \to \infty$, then

$$u(\xi_i) \to u(x) = g(x) \quad \text{as } i \to \infty. \tag{1.11}$$

We start our proof of (1.11) by choosing $x_i \in \partial\Omega$ to be a closest point to ξ_i. This implies that

$$d_\Omega(x_i, \xi_i) = |x_i - \xi_i|.$$

We assert that for all i,

$$|u(\xi_i) - g(x_i)| \leqq |||g||| \cdot |x_i - \xi_i|. \tag{1.12}$$

Indeed, one side of inequality (1.12) is easily derived. Due to (1.9) we have

$$u(\xi_i) - g(x_i) \leqq |||g||| \cdot |x_i - \xi_i|.$$

To establish the opposite inequality let $z \in \partial\Omega$ be a point such that

$$u(\xi_i) = g(z) + |||g||| \cdot d_\Omega(z, \xi_i).$$

By the limited triangle inequality (1.4),

$$\begin{aligned} u(\xi_i) - g(x_i) &\geqq g(z) - g(x_i) + |||g||| \cdot d_\Omega(z, x_i) - |||g||| \cdot d_\Omega(x_i, \xi_i) \\ &\geqq -|||g||| \cdot |x_i - \xi_i|. \end{aligned}$$

This completes the proof of our assertion. Finishing the proof of (1.11) is now straightforward. Since $|x - \xi_i| \geqq |x_i - \xi_i|$, it follows that $|x - x_i| \leqq 2|x - \xi_i|$ and

$$\begin{aligned} |u(\xi_i) - g(x)| &\leqq |u(\xi_i) - g(x_i)| + |g(x_i) - g(x)| \\ &\leqq |||g||| \cdot |x - \xi_i| + |g(x_i) - g(x)|. \end{aligned}$$

The preceding inequality clearly implies (1.11).

We finish the proof of (1.10) by proving simultaneously that $u \in C(\Omega)$, $u \in W^{1,\infty}(\Omega)$, and $\|\mathbf{D}u\|_{L^\infty(\Omega;\mathbb{R}^n)} \leqq |||g|||$. To accomplish this let us start with points $x, y \in \Omega$ such that $d_\Omega(x, y) < \frac{1}{4} \min\{\text{distance}(x, \partial\Omega), \text{distance}(y, \partial\Omega)\}$. Choose $z \in \partial\Omega$ such that

$$u(x) = g(z) + |||g||| \cdot d_\Omega(z, x).$$

By our assumption on x and y we have $d_\Omega(x, y) = |x - y|$ and so

$$\begin{aligned} u(y) &\leqq g(z) + |||g||| \cdot d_\Omega(z, y) \\ &\leqq g(z) + |||g||| \cdot d_\Omega(z, x) + |||g||| \cdot d_\Omega(x, y) \\ &\leqq u(x) + |||g||| \cdot |x - y| . \end{aligned}$$

It follows that

$$\frac{u(y) - u(x)}{|x - y|} \leqq |||g||| .$$

Reversing the roles of x and y we conclude that

$$\frac{|u(x) - u(y)|}{|x - y|} \leqq |||g||| \text{ if } d_\Omega(x, y) \leqq \tfrac{1}{4} \min\{\text{distance}(x, \partial\Omega), \text{ distance}(y, \partial\Omega)\}.$$

Our trifold claim follows easily from the preceding inequality. $\square$

The following corollary (which is an immediate consequence of Theorem 1.8 and Lemma 1.6) characterizes the minimal Lipschitz extensions.

Corollary 1.13. *A function* $w \in W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$ *is a minimal Lipschitz extension if and only if*

$$\|\mathbf{D}w\|_{L^\infty(\Omega; \mathbb{R}^n)} = |||w|_{\partial\Omega}||| . \tag{1.14}$$

With the aid of Corollary 1.13 we are now ready to establish (0.2) as the Euler equation of AMLE's (in the sense of viscosity solutions, i.e., of Definition 1.2).

Theorem 1.15. *If* u *is an AMLE, then* u *is a solution of* (0.2).

Proof. We provide a proof by contradiction, and so we start by assuming that u is an AMLE but not a solution of (0.2). This implies that there exists a pair $(x, \phi) \in \Omega \times C^2(\Omega)$ such that either

$$0 = (u - \phi)(x) \geqq (u - \phi)(y) \quad \text{for all } y \in \Omega,$$

$$\mathbf{D}^2\phi(x) \langle \mathbf{D}\phi(x), \mathbf{D}\phi(x)\rangle < 0;$$

or

$$0 = (u - \phi)(x) \leqq (u - \phi)(y) \quad \text{for all } y \in \Omega,$$

$$\mathbf{D}^2\phi(x) \langle \mathbf{D}\phi(x), \mathbf{D}\phi(x)\rangle > 0 .$$

Since the arguments for the cases are symmetric, we may assume without loss of generality that the first case holds. We may further simplify matters (without loss of generality) by assuming that $x = 0$ and

$$\phi(x) = px_1 - \frac{\delta}{2} x_1^2 + \frac{K}{2} \sum_{j=2}^{n} s_j^2 \quad \text{in } B(\bar{0}, \varepsilon),$$

where p and δ are positive constants.

We consider the two domains

$$\Omega = B(0, \varepsilon), \quad \hat{\Omega} = B(0, \varepsilon)\backslash\{0\}.$$

Set $g = \phi|_{\partial\Omega}$ and $\hat{g} = \phi|_{\partial\hat{\Omega}}$; our first goal is to calculate $|||g|||_{\Omega}$ and $|||\hat{g}|||_{\hat{\Omega}}$. Due to the symmetry of g we find

$$\begin{aligned}|||g|||_{\Omega} = \sup\{&(2\varepsilon\sqrt{1-\lambda^2})^{-1}\,|\,(2p\varepsilon\sqrt{1-\lambda^2}\cos(\theta)\\ &+ 2\delta\varepsilon^2\lambda\sqrt{1-\lambda^2}\sin(\theta)\cos(\theta)\\ &+ 2K\varepsilon^2\lambda\sqrt{1-\lambda^2}\sin(\theta)\cos(\theta))\,|\,: -1 \leqq \lambda \leqq 1, 0 \leqq \theta \leqq 2\pi\}\\ = \sup\{&|p\cos\theta + \varepsilon\lambda(\delta + K)\cos(\theta)\sin(\theta)| : -1 \leqq \lambda \leqq 1, 0 \leqq \theta \leqq 2\pi\}\\ = \sup\{&p\cos\theta + \varepsilon(\delta + K)\sin(\theta)\cos(\theta) : -\tfrac{\pi}{2} \leqq \theta \leqq \tfrac{\pi}{2}\}.\end{aligned}$$

Based on the preceding calculations we conclude that

$$|||g|||_{\Omega} = p + \frac{(\delta + K)^2}{p}\varepsilon^2 + O(\varepsilon^3). \tag{1.16}$$

Calculating $|||\hat{g}|||_{\hat{\Omega}}$ leads (by similar considerations) to

$$\begin{aligned}&|||\hat{g}|||_{\hat{\Omega}}\\ &= \max\left\{|||g|||_{\Omega}, \sup\left\{\frac{\left|p\varepsilon\cos(\theta) - \frac{\delta}{2}\varepsilon^2\cos^2(\theta) + \frac{K}{2}\varepsilon^2\sin^2(\theta)\right|}{\varepsilon} : 0 \leqq \theta \leqq 2\pi\right\}\right\}\\ &= \max\left\{|||g|||_{\Omega}, \left(p + \frac{\delta}{2}\varepsilon\right)\right\}.\end{aligned}$$

We conclude from this and (1.16) that if $\varepsilon > 0$ is sufficiently close to zero, then

$$|||\hat{g}|||_{\hat{\Omega}} = \left(p + \frac{\delta}{2}\varepsilon\right). \tag{1.17}$$

Let v and $\hat{v}$ respectively be the minimal Lipschitz extensions of g into Ω and $\hat{g}$ into $\hat{\Omega}$ constructed by Theorem 1.8. By Corollary 1.13,

$$\|\mathbf{D}v\|_{L^\infty(\Omega;\mathbb{R}^n)} = |||g|||_{\Omega}, \quad \|\mathbf{D}\hat{v}\|_{L^\infty(\hat{\Omega};\mathbb{R}^n)} = |||\hat{g}|||_{\hat{\Omega}},$$

which by (1.16) and (1.17) yields

$$\|\mathbf{D}v\|_{L^\infty(\Omega;\mathbb{R}^n)} = p + O(\varepsilon^2), \quad \|\mathbf{D}\hat{v}\|_{L^\infty(\hat{\Omega};\mathbb{R}^n)} = p + \frac{\delta}{2}\varepsilon$$

for ε sufficiently close to zero. Let us note that the preceding implies

$$\begin{aligned}&\hat{v}((x_1\ 0, \ldots, 0)) = px_1 + \frac{\delta}{2}\varepsilon x_1 \quad \text{for } -\varepsilon \leqq x_1 \leqq 0,\\ &v(0) < 0.\end{aligned} \tag{1.18}$$

From the previous facts we first conclude that there is a domain, $\tilde{\Omega}$, containing $\bar{0}$ such that

$$u > v \quad \text{in } \tilde{\Omega}, \qquad u|_{\partial\tilde{\Omega}} = v|_{\partial\tilde{\Omega}}.$$

Applying Corollary 1.13 to u and w on $\tilde{\Omega}$ we conclude that

$$\|\mathbf{D}u\|_{L^\infty(\tilde{\Omega};\mathbb{R}^n)} \leqq p + O(\varepsilon^2);$$

and in particular, for ε sufficiently close to zero,

$$\|\mathbf{D}u\|_{L^\infty(\tilde{\Omega};\mathbb{R}^n)} \leqq p + \frac{\delta}{4}\varepsilon. \tag{1.19}$$

On the other hand, the construction in Theorem 1.8 implies that $v \leqq \hat{v}$ and consequently

$$u|_{\partial\Omega} \leqq \hat{v}|_{\partial\tilde{\Omega}}.$$

This fact in conjunction with the equality in (1.18) implies that

$$\|\mathbf{D}u\|_{L^\infty(\tilde{\Omega};\mathbb{R}^n)} \geqq p + \frac{\delta}{2}\varepsilon.$$

The preceding inequality obviously contradicts (1.19) and so we have proved Theorem 1.15. $\square$

In order to prove uniqueness for solutions of AEE we make use of two auxiliary equations which we now introduce. The upper auxiliary equation

$$\max\{\varepsilon - |\mathbf{D}u|,\ \mathbf{D}^2u\langle\mathbf{D}u, \mathbf{D}u\rangle\} = 0 \quad \text{in } \Omega \tag{1.20}$$

provides us with supersolutions of (0.2). The lower auxiliary equation

$$\min\{|\mathbf{D}u| - \varepsilon,\ \mathbf{D}^2u\langle\mathbf{D}u, \mathbf{D}u\rangle\} = 0 \quad \text{in } \Omega \tag{1.21}$$

provides us with subsolutions of (0.2). The important feature in each of these equations is the fact that formally the gradients of solutions are nonvanishing. The remainder of this section is devoted to proving the existence of solutions to (0.2) (i.e., AEE), (1.20), and (1.21).

Theorem 1.22. *Assume that* $g \in \mathrm{Lip}_\partial(\Omega)$. *Then*
(i) *There exists a solution* u *of* (0.2) *such that* $u|_{\partial\Omega} = g$.
(ii) *There exists a solution* u^+ *of* (1.20) *such that* $u^+|_{\partial\Omega} = g$.
(iii) *There exists a solution* u^- *of* (1.21) *such that* $u^-|_{\partial\Omega} = g$.

Proof. (i) Since $g \in \mathrm{Lip}_\partial(\Omega)$, Theorem 1.8 guarantees the existence of a function $w \in W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$ such that $w|_{\partial\Omega} = g$. This implies that for $p > 1$ there is a minimizer of the functional

$$J_p(v) = \int_\Omega |\mathbf{D}v|^p\,dx \quad \text{for } v \text{ such that } (w - v) \in W_0^{1,p}(\Omega).$$

Denote the unique minimizer of this functional by u_p. This minimizer is a solution of the p-Laplacian equation

$$\frac{\partial}{\partial x_i}\left(p|\mathbf{D}u|^{p-2}\frac{\partial u}{\partial x_i}\right)=0 \quad \text{in } \Omega. \tag{1.23}$$

For $p \geqq n+1$ the family of minimizers $\{u_p\}_{p\geqq n+1}$ is bounded and equicontinuous. Let $\{u_j\}$ be a convergent sequence from the family $\{u_p\}_{p\geqq n+1}$ with the property that

$$u_j = u_{p_j}, \quad p_j \to \infty \text{ as } j \to \infty.$$

We assert that $u = \lim_{j\to\infty} u_j$ is a solution of (0.2) such that $u|_{\partial\Omega} = g$. Indeed, by the uniform convergence of $\{u_j\}$ it follows that $u|_{\partial\Omega} = g$. Due to the symmetric nature of the problem it is sufficient to prove that u is a subsolution of (0.2) (in the sense of Definition 1.2). To this end let $(x, \phi) \in \Omega \times C^2(\Omega)$ be a pair such that

$$(u-\phi)(x) \geqq (u-\phi)(y) \quad \text{for all } y \in \Omega.$$

We need to show that

$$\mathbf{D}^2\phi\langle\mathbf{D}\phi, \mathbf{D}\phi\rangle\,(x) \geqq 0. \tag{1.24}$$

By modifying ϕ if necessary we may assume without loss of generality that

$$(u-\phi)(x) > (u-\phi)(y) \quad \text{if } x \neq y.$$

There is a constant j_0 such that for $j > j_0$ there are points $x_j \in \Omega$ at which $(u_j - \phi)$ is maximized. Due to our previous assumption,

$$x_j \to x \quad \text{as } j \to \infty.$$

From the general theory of viscosity solutions and the p-Laplacian we know that

$$\frac{\partial}{\partial x_i}\left(p_j|\mathbf{D}\phi|^{p_j-2}\frac{\partial \phi}{\partial x_i}\right)(x_j) \geqq 0,$$

and after expansion this yields

$$p_j|\mathbf{D}\phi|^{p_j-2}\Delta\phi(x_j) + p_j(p_j-2)\,|\mathbf{D}\phi|^{p_j-4}\,\mathbf{D}^2\phi\langle\mathbf{D}\phi, \mathbf{D}\phi\rangle\,(x_j) \geqq 0. \tag{1.25}$$

If $\mathbf{D}\phi(x) = \bar{0}$, then (1.24) is clearly true, so we assume that $|\mathbf{D}\phi(x)| > 0$ and consequently we may assume j_0 is large enough that $|\mathbf{D}\phi(x_j)| \geqq C_0 > 0$ if $j > j_0$. So for $j > j_0$ we may divide (1.25) by $p_j(p_j-2)\,|\mathbf{D}\phi(x_j)|^{p_j-4}$. This leaves us with

$$\mathbf{D}^2\phi\langle\mathbf{D}\phi, \mathbf{D}\phi\rangle\,(x_j) \geqq \frac{|\mathbf{D}\phi(x_j)|^2}{p_j-2}\,\Delta\phi(x_j).$$

Letting $j \to \infty$ and using the continuity of $\mathbf{D}\phi$ and $\mathbf{D}^2\phi$ we conclude that (1.24) holds. This completes the proof of case (i).

(ii) The construction of a solution to (1.20) is slightly more complicated than that for the preceding case. To begin with, we use minimizers of the functional

$$J_p^+(v) = \int_\Omega (|\mathbf{D}v|^p - \varepsilon^{p-1}v)\,dx \quad \text{for } v \text{ such that } (u-v) \in W_0^{1,p}(\Omega).$$

These minimizers are also unique; we denote them by u_p^+. They are solutions of the nonhomogeneous p-Laplacian

$$\frac{\partial}{\partial x_i}\left(p\,|\mathbf{D}u|^{p-2}\frac{\partial u}{\partial x_i}\right) = -\varepsilon^{p-1} \quad \text{in } \Omega. \tag{1.26}$$

As in the previous case the family of functions $\{u_p^+\}_{p \geqq n+1}$ is bounded and equicontinuous. Let $\{u_j^+\}$ be a convergent sequence from $\{u_p^+\}_{p \geqq n+1}$ with the property that

$$u_j^+ = u_{p_j}^+, \quad p_j \to \infty \text{ as } j \to \infty.$$

We assert that $u^+ = \lim_{j\to\infty} u_j^+$ is a solution of (1.20) such that $u^+|_{\partial\Omega} = g$. Since it is clear that $u^+|_{\partial\Omega} = g$, we need to prove that u^+ is a solution of (1.20) (in the sense of Definition 1.2). In this case, however, the problem is no longer symmetric. We shall need different arguments to prove that u^+ is a subsolution and a supersolution. We begin by showing that u^+ is a subsolution.

Let $(x, \phi) \in \Omega \times C^2(\Omega)$ be a pair such that

$$(u^+ - \phi)(x) \geqq (u^+ - \phi)(y) \quad \text{for all } y \in \Omega.$$

We need to show that

$$\max\{\varepsilon - |\mathbf{D}\phi(x)|,\ \mathbf{D}^2\phi\langle \mathbf{D}\phi, \mathbf{D}\phi\rangle(x)\} \geqq 0. \tag{1.27}$$

As in the previous case we may assume without loss of generality that

$$(u^+ - \phi)(x) > (u^+ - \phi)(y) \quad \text{if } x \neq y.$$

Furthermore, since $\mathbf{D}\phi(x) = \bar{0}$ clearly implies the validity of (1.27), we may assume $|\mathbf{D}\phi(x)| > 0$.

If j_0 is sufficiently large, then for $j > j_0$ there are points $x_j \in \Omega$ such that $(u_j^+ - \phi)$ achieves its maximum at x_j. As in the previous case, $x_j \to x$ as $j \to \infty$. We may also assume without loss of generality that j_0 is so large that

$$|\mathbf{D}\phi(x_j)| \geqq C_0 > 0 \quad \text{if } j > j_0.$$

It follows from the general theory of viscosity solutions and the p-Laplacian that

$$\frac{\partial}{\partial x_i}\left(p_j\,|\mathbf{D}\phi|^{p_j-2}\frac{\partial \phi}{\partial x_i}\right)(x_j) \geqq -\varepsilon^{p_j-1}.$$

After expansion this yields

$$p_j\,|\mathbf{D}\phi|^{p_j-2}\Delta\phi(x_j) + p_j(p_j-2)\,|\mathbf{D}\phi|^{p_j-4}\,\mathbf{D}^2\phi\langle \mathbf{D}\phi, \mathbf{D}\phi\rangle(x_j) \geqq -\varepsilon^{p_j-1}.$$

Dividing (1.28) by $p_j(p_j-2)\,|\mathbf{D}\phi|^{p_j-4}(x_j)$ gives

(1.28)
$$\mathbf{D}^2\phi\langle\mathbf{D}\phi,\mathbf{D}\phi\rangle\,(x_j) \geqq -\frac{\varepsilon^3}{p_j(p_j-2)}\left(\frac{\varepsilon}{|\mathbf{D}\phi|\,(x_j)}\right)^{p_j-4} - \frac{|\mathbf{D}\phi(x_j)|^2}{(p_j-2)}\,\Delta\phi(x_j)\,.$$

If $|\mathbf{D}\phi(x)| \leqq \varepsilon$, then (1.27) is clearly true and if $|\mathbf{D}\phi(x)| > \varepsilon$, then (1.28) implies that

$$\mathbf{D}^2\phi\langle\mathbf{D}\phi,\mathbf{D}\phi\rangle\,(x) \geqq 0\,,$$

and once again (1.27) holds. This proves that u^+ is a subsolution of (1.20).

To finish the proof of case (ii) we need to show that u^+ is a supersolution of (1.20). So now we let $(x,\phi)\in\Omega\times C^2(\Omega)$ be a pair such that

$$(u^+-\phi)(x) \leqq (u^+-\phi)(y) \quad \text{for all } y\in\Omega\,.$$

We need to show that

$$\max\{\varepsilon-|\mathbf{D}\phi|\,(x),\ \mathbf{D}^2\phi\langle\mathbf{D}\phi,\mathbf{D}\phi\rangle\,(x)\} \leqq 0\,. \tag{1.29}$$

As we have seen before, we may assume without loss of generality that

$$(u^+-\phi)(x) < (u^+-\phi)(y) \quad \text{if } x\neq y\,.$$

If j_0 is sufficiently large, then for $j>j_0$ there are points $x_j\in\Omega$ at which $(u_j^+-\phi)$ attains its minimum. We know $x_j\to x$ as $j\to\infty$ and

$$\frac{\partial}{\partial x_i}\left(p_j\,|\mathbf{D}\phi|^{p_j-2}\frac{\partial\phi}{\partial x_i}\right)(x_j) \leqq -\varepsilon^{p_j-1}$$

for reasons expressed previously. Expanding this inequality yields

$$p_j\,|\mathbf{D}\phi|^{p_j-2}\,\Delta\phi(x_j) + p_j(p_j-2)\,|\mathbf{D}\phi|^{p_j-4}\,\mathbf{D}^2\phi\langle\mathbf{D}\phi,\mathbf{D}\phi\rangle\,(x_j) \leqq -\varepsilon^{p_j-1}\,.$$

Clearly $|\mathbf{D}\phi|\,(x_j)>0$, and dividing by $p_j(p_j-2)\,|\mathbf{D}\phi|^{p_j-4}\,(x_j)$ gives us

$$\mathbf{D}^2\phi\langle\mathbf{D}\phi,\mathbf{D}\phi\rangle(x_j) \leqq -\frac{\varepsilon^3}{p_j(p_j-2)}\left(\frac{\varepsilon}{|\mathbf{D}\phi|\,(x_j)}\right)^{p_j-4} - \frac{|\mathbf{D}\phi|^2\,(x_j)}{(p_j-2)}\,\Delta\phi(x_j)\,. \tag{1.30}$$

If $|\mathbf{D}\phi(x)| < \varepsilon$, then for some $\delta\in(0,1)$, $|\mathbf{D}\phi(x_j)| \leqq \delta\varepsilon$ for all j sufficiently large. However, this implies that the right-hand side of (1.30) goes to $-\infty$. Since the left-hand side of (1.30) is bounded, we have a contradiction. Consequently $|\mathbf{D}\phi(x)| \geqq \varepsilon$ or $\varepsilon-|\mathbf{D}\phi(x)| \leqq 0$. It is clear that (1.30) implies

$$\mathbf{D}^2\phi\langle\mathbf{D}\phi,\mathbf{D}\phi\rangle\,(x) \leqq 0\,.$$

These last inequalities show that (1.29) is true. This proves that u^+ is a supersolution of (1.20) and thus completes the proof of case (ii).

(iii) It would be redundant to present a proof of this case inasmuch as it is dual to case (ii). □

Remark. An alternative approach in proving the existence of solutions to (0.2), (1.20), and (1.21) may be found in [6, 7]. This approach uses the theory of viscosity solutions more heavily, but has the advantage of proving existence also in the case of $g\in C(\partial\Omega)$.

§ 2. The maximum principle

This section is devoted to proving a maximum principle for solutions of AEE (i.e., equation (0.2)). The approach is unusual in that we derive the maximum principle indirectly from results on solutions of (1.20) and (1.21). This circuitous route appears necessary due to certain technical problems encountered in treating (0.2) directly. We begin with a maximum principle for solutions of (1.20).

Theorem 2.1. *If* v *is a subsolution of* (1.20) *and* w *is a supersolution, then*

$$\sup_{x\in\Omega}\,(v-w)(x)=\sup_{x\in\partial\Omega}\,(v-w)(x)\,. \tag{2.2}$$

Proof. We present a proof by contradiction, and as it is rather lengthy, we break it up into several stages. Since this is a proof by contradiction, we begin with the premise that (2.2) fails, i.e.,

$$\sup_{x\in\Omega}\,(v-w)>\sup_{x\in\partial\Omega}\,(v-w)\,. \tag{*2.2}$$

As our first step in deriving a contradiction we construct a pair of semiconvex functions $\tilde v$ and $-\tilde w$ in $C(\bar\Omega)$ with the following properties

$$\sup_{x\in\Omega}\,(\tilde v-\tilde w)>\sup_{x\in\partial\Omega}\,(\tilde v-\tilde w)\,, \tag{2.3}$$

$$\tilde v \text{ is a subsolution of (1.20)}, \tag{2.4}$$

$$\tilde w \text{ is a supersolution of both}$$
$$\mathbf{D}^2u\langle\mathbf{D}u,\mathbf{D}u\rangle+\mu=0 \text{ in } \Omega \quad\text{and}\quad \varepsilon+\gamma-|\mathbf{D}u|=0 \text{ in } \Omega \tag{2.5}$$

where μ and γ are positive constants.

By using the so-called inf/sup convolutions (e.g., see [11]) we may assume without loss of generality that v and $-w$ are continuous and semiconvex, i.e., for some $K>0$ the functions $v(x)+K|x|^2$ and $-w(x)+K|x|^2$ are convex. (Since $-w$ is semiconvex, this implies that w is semiconcave.) The assumption that w is a supersolution of (1.20) means it is a supersolution of both (0.2) and

$$\varepsilon-|\mathbf{D}u|=0 \quad\text{in } \Omega\,. \tag{2.6}$$

For $\delta>0$ sufficiently small we assert that $w_\delta=w-\dfrac{\delta}{2}w^2$ is a supersolution of

$$\mathbf{D}^2u\langle\mathbf{D}u,\mathbf{D}u\rangle+\tilde K\delta=0 \quad\text{in } \Omega \tag{2.7}$$

where $\tilde K$ is a positive constant. To accomplish this we must show that

$$\mathbf{D}^2\phi\langle\mathbf{D}\phi,\mathbf{D}\phi\rangle\,(x)+\tilde K\delta\leqq 0 \tag{2.8}$$

for any pair $(x,\phi)\in\Omega\times C^2(\Omega)$ such that

$$(w_\delta-\phi)(x)\leqq(w_\delta-\phi)(y)\quad\text{for all } y\in\Omega\,.$$

The boundedness of w implies that

$$\sup_{y \in \Omega} \big(|w_\delta|(y) \big) \leqq C_0 < \infty \quad \text{for } \delta < 1 .$$

Therefore, we may assume without loss of generality that

$$\sup_{y \in \Omega} \big(|\phi|(y) \big) \leqq 2C_0 .$$

By the inverse function theorem if δ is sufficiently close to 0, there is a uniquely defined function $\psi \in C^2(\Omega)$ such that $\psi - \frac{\delta}{2} \psi^2 = \phi$ and

$$\sup_{y \in \Omega} \big(|\psi|(y) \big) \leqq 3C_0 .$$

For such a δ it is also true that

$$(\psi - w)(x) \leqq (\psi - w)(y) \quad \text{for all } y \in \Omega$$

and consequently

$$\mathbf{D}^2\psi \langle \mathbf{D}\psi, \mathbf{D}\psi \rangle (x) \leqq 0 .$$

Expressing $\mathbf{D}\psi$ and $\mathbf{D}^2\psi$ in terms of $\mathbf{D}\phi$ and $\mathbf{D}^2\phi$ gives

$$\mathbf{D}\psi = \frac{1}{1 + \delta\psi} \mathbf{D}\phi ,$$

$$\mathbf{D}^2\psi = \frac{1}{1 + \delta\psi} \mathbf{D}^2\phi + \frac{\delta}{(1 + \delta\psi)^3} \mathbf{D}\phi \otimes \mathbf{D}\phi .$$

Using this in the previous inequality gives

$$\frac{1}{(1 + \delta\psi)^3} \mathbf{D}^2\phi \langle \mathbf{D}\phi, \mathbf{D}\phi \rangle (x) + \frac{\delta}{(1 + \delta\psi)^5} |\mathbf{D}\phi|^4(x) \leqq 0 ,$$

which proves (2.8) if we can show that $|\mathbf{D}\phi|(x)$ is bounded away from 0.

Recall that w is also a supersolution of (2.6) so

$$\varepsilon - |\mathbf{D}\psi|(x) \leqq 0, \quad |\mathbf{D}\psi|(x) \geqq \varepsilon .$$

Expressing $\mathbf{D}\psi$ in terms of $\mathbf{D}\phi$ yields

$$|\mathbf{D}\phi|(x) \geqq \big(1 + \delta\psi(x)\big) \varepsilon \geqq (1 - 3C_0\delta) \varepsilon .$$

Thus we have shown not only that $|\mathbf{D}\phi|(x)$ is bounded away from 0, but also that w_δ is a supersolution of

$$(1 - 3C_0\delta) \varepsilon - |\mathbf{D}u| = 0 \quad \text{in } \Omega . \tag{2.9}$$

Set $\tilde{v} = v$ and $\tilde{w} = (1 + 5C_0\delta) w_\delta$. We assert that if δ is chosen sufficiently close to zero, then $\tilde{v}$ and $\tilde{w}$ satisfy (2.3)–(2.5). Indeed, (2.3) follows from the continuous dependence on δ and (*2.2), and (2.4) is in fact trivially satisfied. To prove the first part of (2.5) note that (from the fact that w_δ is a supersolution of (2.7) it follows that) $\tilde{w}$ is a supersolution of

$$\mathbf{D}^2 u\langle \mathbf{D}u, \mathbf{D}u\rangle + (1 + 5C_0\delta)^3 \tilde{K}\delta = 0 \quad \text{in } \Omega.$$

The second half of (2.5) follows from the fact that w_δ is a solution of (2.9), which implies that $\tilde{w}$ is a supersolution of

$$(1 + 5C_0\delta)\,(1 - 3C_0\delta)\,\varepsilon - |\mathbf{D}u| = 0 \quad \text{in } \Omega$$

and the fact that for δ sufficiently close to 0,

$$(1 + 5C_0\delta)\,(1 - 3C_0\delta) \geqq 1 + C_0\delta.$$

This proves the existence of $\tilde{v}$ and $\tilde{w}$ satisfying the previously stipulated conditions.

We begin our next step in the proof of Theorem 2.1 by choosing $\tilde{x}_0$ to be a point at which $\tilde{v} - \tilde{w}$ achieves its maximum. By (2.3) $\tilde{x}_0 \in \Omega$, and we assert that

$$\mathbf{D}\tilde{v}(\tilde{x}_0),\ \mathbf{D}\tilde{w}(\tilde{x}_0) \text{ both exist},$$

$$\tilde{v}(x) - \tilde{v}(\tilde{x}_0) - \langle \mathbf{D}\tilde{v}(\tilde{x}_0), x - \tilde{x}_0\rangle = O(|x - \tilde{x}_0|^2), \tag{2.10}$$

$$\tilde{w}(x) - \tilde{w}(\tilde{x}_0) - \langle \mathbf{D}\tilde{w}(\tilde{x}_0), x - \tilde{x}_0\rangle = O(|x - \tilde{x}_0|^2).$$

Indeed, since $\tilde{v}$ is semiconvex, there is a vector p_0 such that

$$\tilde{v}(x) - \tilde{v}(\tilde{x}_0) - \langle p_0, x - \tilde{x}_0\rangle \geqq O(|x - \tilde{x}_0|^2)$$

and the choice of $\tilde{x}_0$ means that

$$\begin{aligned} \tilde{w}(x) - \tilde{w}(\tilde{x}_0) &\geqq \tilde{v}(x) - \tilde{v}(\tilde{x}_0) \\ &\geqq \langle p_0, x - \tilde{x}_0\rangle + O(|x - \tilde{x}_0|^2). \end{aligned}$$

Noting that $\tilde{w}$ is semiconcave we conclude that $\mathbf{D}\tilde{w}(\tilde{x}_0)$ exists and equals p_0, and

$$\tilde{w}(x) - \tilde{w}(\tilde{x}_0) - \langle \mathbf{D}\tilde{w}(\tilde{x}_0), x - \tilde{x}_0\rangle = O(|x - \tilde{x}_0|^2).$$

A similar argument establishes these results for $\tilde{v}$ and completes the proof of our assertion (2.10).

In this next stage we construct "blow-ups" of $\tilde{v}$ and $\tilde{w}$ centered at $\tilde{x}_0$. Using properties of these limits we finally derive the contradiction necessary to complete the proof of Theorem 2.1. To this end we define

$$w^\rho(x) = \rho^{-2}\big(\tilde{w}(\tilde{x}_0 + \rho x) - \tilde{w}(\tilde{x}_0) - \rho\langle \mathbf{D}\tilde{w}(\tilde{x}_0), x\rangle\big),$$

$$v^\rho(x) = \rho^{-2}\big(\tilde{v}(\tilde{x}_0 + \rho x) - \tilde{v}(\tilde{x}_0) - \rho\langle \mathbf{D}\tilde{v}(\tilde{x}_0), x\rangle\big).$$

From the choice of $\tilde{x}_0$ we have

$$0 = (v^\rho - w^\rho)(0) \geqq (v^\rho - w^\rho)(y) \quad \text{for all } y \in \Omega_\rho \tag{2.11}$$

where Ω_ρ is the common domain of v^ρ and w^ρ.

For any $R > 0$ there is a value $\rho(R) > 0$ such that $B(0, R) \subset \Omega_\rho$ if $0 < \rho < \rho(R)$. Due to (2.10), $\{v^\rho\}_{0<\rho<\rho(R)}$ and $\{w^\rho\}_{0<\rho<\rho(R)}$ are both uniformly bounded on $B(\bar{0}, R)$. It also follows from its definition that $\{v^\rho\}_{0<\rho}$ is uniformly semiconvex and $\{w^\rho\}_{0<\rho}$ is uniformly semiconcave. As a consequence, $\{v^\rho\}_{0<\rho<\rho(R)}$ and $\{w^\rho\}_{0<\rho<\rho(R)}$ are uniformly Lipschitz continuous in $B(\bar{0}, R/2)$. Precompactness of these families of functions implies that there exists a sequence $\rho_i \to 0$ as $i \to \infty$ such that $v^{\rho_i} \to v^*$ and $w^{\rho_i} \to w^*$ locally uniformly in $\mathbb{R}^n$ as $i \to \infty$.

Using (2.3)–(2.5) and standard limit techniques for viscosity solutions we conclude that v^* is a subsolution of

$$\max\{\varepsilon - |\mathbf{D}\tilde{v}(\tilde{x}_0)|, \mathbf{D}^2u\langle\tilde{v}(\tilde{x}_0), \mathbf{D}\tilde{v}(\tilde{x}_0)\rangle\} = 0 \quad \text{in } \mathbb{R}^n,$$

and that w^* is a supersolution of both

$$\mathbf{D}^2u\langle\mathbf{D}\tilde{w}(\tilde{x}_0), \mathbf{D}\tilde{w}(\tilde{x}_0)\rangle + \mu = 0 \quad \text{in } \mathbb{R}^n \tag{2.12}$$

and

$$\varepsilon + \gamma - |\mathbf{D}\tilde{w}(\tilde{x}_0)| = 0 \quad \text{in } \mathbb{R}^n.$$

From the last equation we conclude that

$$|\mathbf{D}\tilde{w}(\tilde{x}_0)| \geqq \varepsilon + \gamma$$

and from the choice of $\tilde{x}_0$, that $p_0 = \mathbf{D}\tilde{v}(\tilde{x}_0) = \mathbf{D}\tilde{w}(\tilde{x}_0)$ so that

$$|\mathbf{D}\tilde{v}(\tilde{x}_0)| = |\mathbf{D}\tilde{w}(\tilde{x}_0)| \geqq \varepsilon + \gamma.$$

We conclude that v^* is in fact a subsolution of

$$\mathbf{D}^2u\langle\mathbf{D}\tilde{v}(\tilde{x}_0), \mathbf{D}\tilde{v}(\tilde{x}_0)\rangle = 0 \quad \text{in } \mathbb{R}^n. \tag{2.13}$$

Consider $\mathscr{F}(x) = (v^* - w^*)\,(x) - \dfrac{\mu}{4|p_0|^2}|x|^2$, which by (2.11) satisfies

$$\mathscr{F}(x) \leqq -\frac{\mu}{4|p_0|^2}|x|^2 \quad \text{in } \mathbb{R}^n.$$

By the results in [10] we can find a point z_0 such that

$$\mathbf{D}^2\mathscr{F}(z_0) \leqq 0, \qquad \mathbf{D}^2v^*(z_0) \text{ exists}, \qquad \mathbf{D}^2w^*(z_0) \text{ exists}. \tag{2.14}$$

It follows that

$$\mathbf{D}^2v^*(z_0) - \mathbf{D}^2w^*(z_0) \leqq \frac{\mu}{2|p_0|}I,$$

and thus

$$\mathbf{D}^2v^*(z_0)\langle p_0, p_0\rangle - \mathbf{D}^2w^*(z_0)\langle p_0, p_0\rangle \leqq \frac{\mu}{2}.$$

On the other hand, (2.13) and (2.12) imply that

$$\mathbf{D}^2v^*(z_0)\langle p_0, p_0\rangle - \mathbf{D}^2w^*(z_0)\langle p_0, p_0\rangle \geqq \mu.$$

This last inequality obviously contradicts the previous one, and so brings to a close our proof of Theorem 2.1. □

Since the argument is similar, we state without proof the analogous theorem on solutions of (1.21).

Theorem 2.15. *If v is a solution and w is a supersolution of* (1.21), *then*

$$\sup_{x\in\Omega} (v-w)(x) = \sup_{x\in\partial\Omega} (v-w)(x). \tag{2.16}$$

Given $g \in \mathrm{Lip}_\partial(\Omega)$ and solutions u, u^+, and u^- of (0.2), (1.20), and (1.21) respectively such that $u|_{\partial\Omega} = u^+|_{\partial\Omega} = u^-|_{\partial\Omega} = g$, Theorems 2.1 and 2.15 imply that $u^- \leqq u \leqq u^+$. In the following lemma we construct particular solutions u^- and u^+ such that $\sup_{x\in\Omega}(u^+ - u^-) \to 0$ as $\varepsilon \to 0$ (where ε is the parameter appearing in (1.20) and (1.21)). From this we shall be able to prove a maximum principle for (0.2).

Lemma 2.17. *Given $g \in \mathrm{Lip}_\partial(\Omega)$, there exists a continuous, nondecreasing function $\gamma: [0,\infty) \to [0,\infty)$ such that $\gamma(0) = 0$ and there are solutions u^+ and u^- of* (1.20) *and* (1.21) *respectively for which*

$$u^-(x) \leqq u^+(x) \leqq u^-(x) + \gamma(\varepsilon) \quad \textit{for all } x \in \Omega \tag{2.18}$$

where ε is the parameter appearing in (1.20) *and* (1.21).

Proof. Recall the construction given in the proof of Theorem 1.22 showing that u^+ is obtained as a limit of solutions w_{p_j} of

$$\frac{\partial}{\partial x_i}\left(p_j |\mathbf{D}u|^{p_j-2} \frac{\partial u}{\partial x_i}\right) = -\varepsilon^{p_j-1} \quad \text{in } \Omega \tag{2.19}$$

and u^- as a limit of solutions $v_{\tilde{p}_j}$ of

$$\frac{\partial}{\partial x_i}\left(\tilde{p}_j |\mathbf{D}u|^{\tilde{p}_j-2} \frac{\partial u}{\partial x_i}\right) = \varepsilon^{\tilde{p}_j-1} \quad \text{in } \Omega. \tag{2.20}$$

We may assume without loss of generality that $\tilde{p}_j = p_j$ for all $j \in \mathbb{Z}^+$.

Since $(v_{p_j} - w_{p_j}) \in W_0^{1,p_j}(\Omega)$, we may multiply (2.19) and (2.20) by $(v_{p_j} - w_{p_j})$, integrate by parts and substract the results to obtain

$$\begin{aligned}\int_\Omega \Bigg[\frac{p_j}{2}\left(|\mathbf{D}v_{p_j}|^{p_j-2} - |\mathbf{D}w_{p_j}|^{p_j-2}\right)\left(|\mathbf{D}v_{p_j}|^2 - |\mathbf{D}w_{p_j}|^2\right) \\ + \frac{p_j}{2}\left(|\mathbf{D}v_{p_j}|^{p_j-2} + |\mathbf{D}w_{p_j}|^{p_j-2}\right)|\mathbf{D}v_{p_j} - \mathbf{D}w_{p_j}|^2\Bigg]\,dx \\ = 2\varepsilon^{p_j-1}\int_\Omega (v_{p_j} - w_{p_j})\,dx.\end{aligned}$$

From this we can see that

$$\int_\Omega p_j\left(\frac{|\mathbf{D}v_{p_j}|^{p_j-2} + |\mathbf{D}w_{p_j}|^{p_j-2}}{2}\right)|\mathbf{D}v_{p_j} - \mathbf{D}w_{p_j}|^2\,dx \leqq 2\varepsilon^{p_j-1}\,\|v_{p_j} - w_{p_j}\|_{L^1(\Omega)}.$$

Since v_{p_j} and w_{p_j} are bounded independently of j, we have

$$\int_{\Omega_\varepsilon} |\mathbf{D}v_{p_j} - \mathbf{D}w_{p_j}|^2 \, dx \leqq \frac{2}{p_j} \varepsilon C_1$$

where $\Omega_\varepsilon = \{x \in \Omega : \max\{|\mathbf{D}v_{p_j}(x)|, |\mathbf{D}w_{p_j}(x)|\} \geqq \varepsilon\}$. Consequently

$$\int_{\Omega} |\mathbf{D}v_{p_j} - \mathbf{D}w_{p_j}|^2 \, dx \leqq \frac{2}{p_j} \varepsilon C_1 + \varepsilon^2 C_2 .$$

Taking $j \to \infty$ we have

$$\|\mathbf{D}u^- - \mathbf{D}u^+\|_{L^2(\Omega; \mathbb{R}^n)} \leqq C_2 \varepsilon^2 \tag{2.21}$$

where u^- and u^+ are solutions of (1.20) and (1.21) respectively such that $u^-|_{\partial\Omega} = u^+|_{\partial\Omega} = g$. In view of the fact that u^- and u^+ are in $W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$ it follows from (2.21) that the function $\gamma : [0, \infty) \to [0, \infty)$ must exist as asserted. This completes the proof of our lemma. □

We are now in a position to prove a maximum principle for solutions of AEE. The following theorem gives such a result for boundary values in $\mathrm{Lip}_\partial(\Omega)$. Later, after we have developed some properties of solutions of AEE we shall be able to extend the maximum principle to include solutions with continuous boundary values.

Theorem 2.22. *Assume v is a subsolution and w is a supersolution of AEE. If $v|_{\partial\Omega}$, $w|_{\partial\Omega} \in \mathrm{Lip}_\partial(\Omega)$, then*

$$\sup_{x\in\Omega} (v - w) = \sup_{x\in\partial\Omega} (v - w) . \tag{2.23}$$

Proof. Let $\tilde{v}^+$ and $\tilde{v}^-$ be solutions of (1.20) and (1.21) respectively such that $\tilde{v}^+|_{\partial\Omega} = \tilde{v}^-|_{\partial\Omega} = v|_{\partial\Omega}$. Let $\tilde{w}^-$ be a solution of (1.21) such that $\tilde{w}^-|_{\partial\Omega} = w|_{\partial\Omega}$. By Theorems 2.1 and 2.15 we see

$$v - w \leqq \tilde{v}^+ - \tilde{w}^- \leqq (\tilde{v}^+ - \tilde{v}^-) + (\tilde{v}^- - \tilde{w}^-) .$$

Using Theorem 2.15 and Lemma 2.17 we find

$$\begin{aligned} \sup_{x\in\Omega} (v - w)(x) &\leqq \sup_{x\in\Omega} (\tilde{v}^+ - \tilde{v}^-)(x) + \sup_{x\in\Omega} (\tilde{v}^- - \tilde{w}^-) \\ &\leqq \gamma(\varepsilon) + \sup_{x\in\partial\Omega} (\tilde{v}^- - \tilde{w}^-)(x) \\ &= \gamma(\varepsilon) + \sup_{x\in\partial\Omega} (v - w)(x) . \end{aligned}$$

Since ε is arbitrary, we conclude that (2.23) is indeed valid. □

Corollary 2.24. *Given $g \in \mathrm{Lip}_\partial(\Omega)$, u is the AMLE of g into Ω if and only if u is the solution of AEE with $u|_{\partial\Omega} = g$.*

Proof. One direction has already been established, i.e., Theorem 1.15. Furthermore, it is clear that the solution of (0.2) constructed in Theorem 1.22 is also an AMLE of g into Ω. By Theorem 2.22 solutions of (0.2) with given boundary values are unique. Consequently the only solutions of (0.2) are AMLE's. □

§ 3. Properties of solutions

In this, the final section of our paper, we provide a direct proof of the interior Lipschitz regularity of solutions of (0.2). We then use this to prove a maximum principle for solutions with continuous boundary data. Next we give a proof that such solutions of (0.2) satisfy the absolutely, minimizing Lipschitz-extension property (0.3). We close with a simple example of n-dimensional solutions which are not in C^2. (For $n = 2$ such examples follow from results in [2].)

As a tool in the proof of interior regularity we introduce upper and lower cone functions, $c^+_{x_0,\lambda}(x)$ and $c^+_{x_0,\lambda}(x)$, (for $\lambda > 0$) given by

$$c^+_{x_0,\lambda}(x) = \lambda|x - x_0|, \quad c^-_{x_0,\lambda}(x) = -\lambda|x - x_0|.$$

Lemma 3.1. *Assume v is a subsolution of* (0.2). *If* $B(x_0, r) \subset \Omega$ *and*

$$c^+_{x_0,\lambda}(x) \geqq v(x) - v(x_0) \quad \textit{for } x \in \partial B(x_0, r), \tag{3.2}$$

then

$$c^+_{x_0,\lambda}(x) \geqq v(x) - v(x_0) \quad \textit{for } x \in B(x_0, r). \tag{3.3}$$

Proof. We assume (3.3) is false and derive a proof by contradiction. Indeed, if (3.3) fails, then

$$\sup_{x \in B(x_0, r)} ((v - \tilde{c})(x)) > v(x_0) \tag{3.4}$$

where $\tilde{c}(x) = c^+_{x_0,r}(x) - \varepsilon(|x - x_0|^2 - r|x - x_0|)$ and $\varepsilon > 0$ is sufficiently small. Let $\tilde{x} \in \bar{B}(x_0, r)$ be a point at which the maximum of $v - \tilde{c}^+_{x_0,r}$ occurs. By the definition of $\tilde{c}$ and (3.2) we see that $\tilde{x} \in B(x_0, r)\backslash\{x_0\}$. Since v is a subsolution of (0.2), we conclude that

$$\mathbf{D}^2\tilde{c}\langle \mathbf{D}\tilde{c}, \mathbf{D}\tilde{c}\rangle (\tilde{x}) \geqq 0.$$

This becomes

$$-2\varepsilon \left(\frac{\lambda + r}{|\tilde{x} - x_0|} - 2\varepsilon\right)^2 |\tilde{x} - x_0|^2 \geqq 0,$$

or equivalently,

$$-2\varepsilon(\lambda + r - 2\varepsilon|\tilde{x} - x_0|)^2 \geqq 0,$$

which is clearly a contradiction if $\varepsilon > 0$ is sufficiently small. This proves the lemma. □

We state without proof the analogous result for $c^-_{x_0,\lambda}$.

Lemma 3.5. *Assume* w *is a supersolution of* (0.2). *If* $B(x_0, r) \subset \Omega$ *and*

$$c^-_{x_0,\lambda}(x) \leqq w(x) - w(x_0) \quad \text{for } x \in \partial B(x_0, r). \tag{3.6}$$

then

$$c^-_{x_0,\lambda}(x) \leqq w(x) - w(x_0) \quad \text{for } x \in B(x_0, r). \tag{3.7}$$

As a consequence we obtain the following corollaries, which follow immediately from the preceding lemmas.

Corollary 3.8. *If* v *is a bounded subsolution of* (0.2), *then* v *is locally Lipschitz continuous.*

Proof. Let $M = \sup_{x\in\Omega} |v(x)|$ and let $\Omega_\varepsilon = \{x \in \Omega : \text{distance } (x, \partial\Omega) > \varepsilon\}$. We assert that for $\varepsilon < 1$,

$$\sup_{\xi,\eta\in\Omega_\varepsilon} \frac{|v(\xi) - v(\eta)|}{|\xi - \eta|} \leqq \frac{2M}{\varepsilon}. \tag{3.9}$$

Indeed, for $\lambda = 2M/\varepsilon$ we have

$$c^+_{\xi,\lambda}(x) \geqq v(x) - v(\xi) \quad \text{for } x \in \partial B(\xi, \varepsilon),$$

$$c^+_{\eta,\lambda}(y) \geqq v(y) - v(\eta) \quad \text{for } y \in \partial B(\eta, \varepsilon)$$

due to Lemma 3.1. Consequently

$$c^+_{\xi,\lambda}(\eta) \geqq v(\eta) - v(\xi) \geqq -c^+_{\eta,\lambda}(\xi)$$

or

$$\lambda|\eta - \xi| \geqq v(\eta) - v(\xi) \geqq -\lambda|\xi - \eta|,$$

which proves the corollary. □

We omit the proof of the next corollary since it is analogous to the preceding one.

Corollary 3.10. *If* w *is a bounded supersolution of* (0.2), *then* w *is locally Lipschitz continuous.*

We are now in a position to prove a more general version of Theorem 2.22.

Theorem 3.11. *Assume* v *is a bounded subsolution of AEE and* w *is a bounded supersolution. Then*

$$\sup_{x\in\Omega} (v - w)(x) = \sup_{x\in\partial\Omega} (v - w)(x). \tag{3.12}$$

Proof. Define Ω_ε (as previously done) by

$$\Omega_\varepsilon = \{x \in \Omega : \text{distance } (x, \partial\Omega) > \varepsilon\}.$$

Clearly 3.12 holds if for every $\varepsilon > 0$,

$$\sup_{x \in \Omega_\varepsilon} (v - w)(x) = \sup_{x \in \partial\Omega_\varepsilon} (v - w)(x). \tag{3.13}$$

By Corollaries 3.8 and 3.10 we may apply Theorem 2.22 to $v|_{\Omega_\varepsilon}$ and $w|_{\Omega_\varepsilon}$, which proves that (3.13) holds for every $\varepsilon > 0$. This proves the theorem. □

Corollary 3.14. *Given $g \in C(\partial\Omega)$, there is a unique solution u of AEE with $u|_{\partial\Omega} = g$. Furthermore, u is also the unique AMLE of g into Ω.*

Proof. The uniqueness is clear by Theorems 3.11 and 1.15. We therefore focus our attention on the proof of existence of a solution. To obtain this we start by extending g to a continuous function $G: \mathbb{R}^n \to \mathbb{R}$. Then there exists $\{G_i\} \subset W^{1,\infty}_{\text{loc}}(\mathbb{R}^n)$ such that $G_i \to G$ as $i \to \infty$. In particular $g_i = G_i|_{\partial\Omega} \to g$ as $i \to \infty$ and $\{g_i\} \subset \text{Lip}_\partial(\Omega)$. By Corollary 2.24 there are solutions u_i of (0.2) with $u_i|_{\partial\Omega} = g_i$, and by Theorem 3.11 $\{u_i\}$ is a Cauchy sequence in the sup norm on $C(\bar{\Omega})$. Setting $u = \lim_{i\to\infty} u_i$ we see that u is a solution of AEE with $u|_{\partial\Omega} = g$.

To complete our proof we must show that u is an AMLE of g into Ω. However, for any $\hat{\Omega}$ with closure contained by Ω, Corollary 3.8 shows that $u \in W^{1,\infty}(\hat{\Omega})$. By Theorem 2.22 u is the AMLE of $u|_{\partial\hat{\Omega}}$ into $\hat{\Omega}$, and this completes the proof of the corollary. □

We now devote our attention to constructing an n-dimensional extension of a two-dimensional result found at the end of [2]. We begin by choosing a particular domain

$$\Omega = B(0, 1) \times (-1, 1)^{n-2} \subset \mathbb{R}^n$$

where $B(0, 1) \subset \mathbb{R}^2$. Define $w \in C(\overline{B(0, 1)})$ to be the AMLE of $g_1(x_1, x_2) = 2x_1x_2$ into $B(0, 1)$. Define $g: \partial\Omega \to \mathbb{R}$ by

$$g(x_1, \ldots, x_n) = \begin{cases} w(x_1, x_2) & \text{if } (x_1, \ldots, x_n) \in B(0, 1) \times \partial((-1, 1)^{n-2}), \\ g(x_1, x_2) & \text{if } (x_1, \ldots, x_n) \in \partial B(0, 1) \times (-1, 1)^{n-2}. \end{cases}$$

By our existence and uniqueness results the AMLE of g into Ω is $u(x_1, \ldots, x_n) = w(x_1, x_2)$. By Aronsson's results [2], w and therefore u are not in C^2.

Acknowledgement. This work was supported in part by a grant from the National Science Foundation.

References

1. G. Aronsson, Extension of functions satisfying Lipschitz conditions, *Ark. Math* **6** (1967) 551–561.
2. G. Aronsson, On the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$, *Ark. Math.* **7** (1968) 395–425.

3. T. BHATTACHARYA, E. DIBENEDETTO & J. MANFREDI, Limits as $p \to \infty$ of $\triangle_p(u) = f$ and related extremal problems, *Rend. Sem. Mat. Univ. Pol. Torino, Fascicolo Speciale 1989 Nonlinear PDE's*, 15–68.
4. M. G. CRANDALL & P.-L. LIONS, Viscosity solutions and Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.* **277** (1983) 1–42.
5. D. GILBARG & N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Second edition, Springer-Verlag, New York, 1983.
6. H. ISHII, Perron's method for Hamilton-Jacobi equations, *Duke Math. J.* **55** (1987) 369–384.
7. H. Ishii, On existence and uniqueness of viscosity solutions of fully nonlinear second-order elliptic PDE's, *Comm. Pure Appl. Math.* **42** (1989) 14–45.
8. H. ISHII & P.-L. LIONS, Viscosity solutions of fully nonlinear second-order elliptic partial differential equations, *J. Diff. Equations* **83** (1990) 26–78.
9. R. JENSEN, Uniqueness criteria for viscosity solutions of fully nonlinear elliptic partial differential equations, *Indiana Univ. Math. J.* **38** (1989) 629–667.
10. R. JENSEN, The maximum principle for viscosity solutions of fully nonlinear second order differential equations, *Arch. Rational. Mech. Anal.* **101** (1988) 1–27.
11. R. JENSEN, P.-L. LIONS & P. E. SOUGANIDIS, A uniqueness result for viscosity solutions of second order fully nonlinear partial differential equations, *Proc. Amer. Math. Soc.* **102** (1988) 975–978.
12. P.-L. LIONS, Optimal control of diffusion processes and Hamilton-Jacobi equations. Part 2. Viscosity solutions and uniqueness, *Comm. P.D.E.* **8** (1983) 1229–1276.

Department of Mathematical Sciences
Loyola University of Chicago
6325 North Sheridan Road
Chicago, Illinois 60626

(Accepted November 16, 1992)