The best approximation of a given function in L^2-norm by Lipschitz functions with gradient constraint

THE BEST APPROXIMATION OF A GIVEN FUNCTION IN L^2-NORM BY LIPSCHITZ FUNCTIONS WITH GRADIENT CONSTRAINT

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ABSTRACT. The starting point of this paper is the study of the asymptotic behavior, as $p \to \infty$, of the following minimization problem

$$(0.1) \qquad \min \left\{ \frac{1}{p} \int_\Omega |\nabla v|^p + \frac{1}{2} \int_\Omega (v-f)^2, \quad v \in W^{1,p}(\Omega) \right\}.$$

We show that the limit problem provides the best approximation, in the L^2-norm, of the datum f among all Lipschitz functions with Lipschitz constant less or equal than one. Moreover such approximation verifies a suitable PDE in the viscosity sense.

After the analysis of the model problem (0.1), we consider the asymptotic behavior of a related family of nonvariational equations and, finally, we also deal with some functionals involving the $(N-1)$-Hausdorff measure of the jump set of the function.

1. INTRODUCTION

Let us assume that $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth, say C^1, boundary. The main goal of this work is to study the optimal approximation in the L^2-norm of a given function $f \in L^2(\Omega)$ by functions in $W^{1,\infty}(\Omega)$ with a constraint on the gradient. Namely, we consider the following minimization problem

$$(1.1) \qquad \inf_{\mathbb{X}(\Omega)} J_\infty(v) \qquad \text{with} \qquad J_\infty(v) = \frac{1}{2} \int_\Omega (v-f)^2,$$

where the set $\mathbb{X}(\Omega)$ is given by

$$\mathbb{X}(\Omega) := \left\{ v \in W^{1,\infty}(\Omega) \, : \, |Dv| \leq 1 \text{ for } a.e. \ x \in \Omega \right\}.$$

In order to obtain a minimizer to (1.1) one can argue by direct methods (taking a minimizing sequence), or rather noticing that such a problem appears naturally as Γ-limit, for $n \to \infty$, of the following classical energy functional

$$(1.2) \qquad J_n(v) = \frac{1}{p_n} \int_\Omega |\nabla v|^{p_n} + \frac{1}{2} \int_\Omega (v-f)^2, \qquad v \in W^{1,p_n}(\Omega),$$

where p_n is a sequence of numbers that diverges to $+\infty$. Setting u_n the minimizer of J_n, the heuristic of the limiting process is that the measure of the set $\{|\nabla u_n| > 1\}$ has to go to zero in order to keep the energy bounded (notice that $J_n(u_n) \leq J_n(0) = \|f\|_{L^2(\Omega)}^2$). Therefore, we expect that u_n converge to the unique minimizer $u \in \mathbb{X}(\Omega)$ of J_∞ (see Proposition 2.2 below).

An interesting feature of the minimizer of (1.1) that we find here, is that it satisfies a kind of representation formula by cones inside the region $\{u \neq f\}$.

Before stating our first result, in order to gain some intuition, let us consider three concrete examples in the special case $\Omega = (-1, 1)$.

- Let $f_1(x) = k\chi_{(-r,r)}$ with $r \in (0, 1]$ and $k \in \mathbb{R}$. Observe that if $r = 1$ then $f_1 \in \mathbb{X}(-1, 1)$ and so $u_1 \equiv f_1$ is the minimizer. Otherwise, it is not hard to see that

$$u_1 = \min\left\{(r + \frac{k}{2} - |x|)_+, k\right\}$$

is the minimizer. In Figure 1 the case $r > \frac{k}{2}$ is represented.
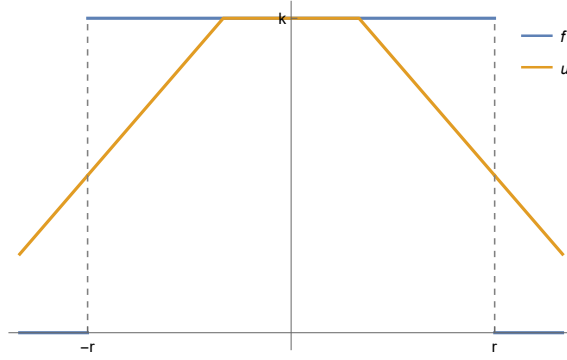

Figure 1.

- Let $f_2(x) = 2|x|$, then the minimizer is given by
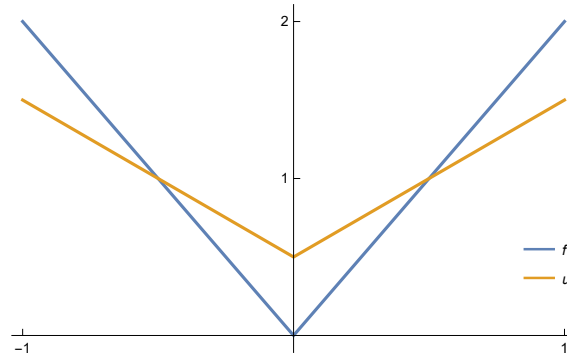
$$u_2(x) = |x| + \frac{1}{2}.$$

See Figure 2.

Figure 2.

- Let $f_3(x) = \sqrt{|x|}$, then is is not hard to see that the minimizer is

$$u_3(x) = \begin{cases} x + \frac{2}{9} & \text{for } |x| \leq \frac{4}{9}, \\ \sqrt{|x|} & \text{for } |x| \geq \frac{4}{9}. \end{cases}$$
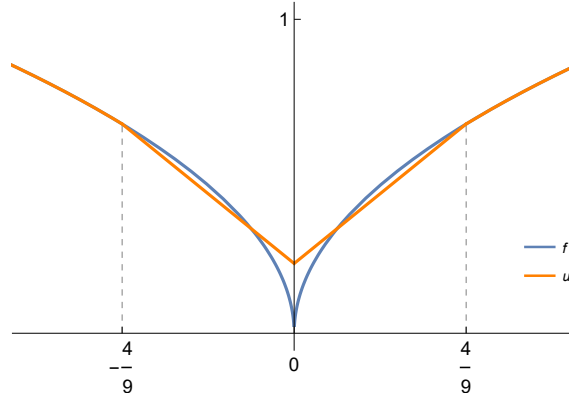
See Figure 3.

Figure 3.

In the three examples above (see Section 6 for the explicit computations) the behavior of the solution u_i, $i = 1, 2, 3$ follows the same idea: any u_i tries to be as close as possible to the datum f_i as long as f_i is smooth and has gradient bounded by one. Otherwise, the best that the solution can do in order to minimize the L^2 norm of the difference, is to growth as much as it is allowed, that is, as a line with slope ± 1.

More in general we have the following result.

Theorem 1.1. *Let $f \in L^2(\Omega)$, let $\{u_n\} \subset W^{1,p_n}(\Omega)$ be the sequence of minimizers of J_n (see (1.2)), and let u be the unique minimizer of (1.1). Then*

$$u_n \longrightarrow u \qquad \text{uniformly in } \overline{\Omega}.$$

In addition, the following representations formulas hold true:

$$u(x) = \max_{y \in \partial A^+} \left[u(y) - d(x,y) \right], \quad \forall x \in A^+$$

(1.3) *and*

$$u(x) = \min_{y \in \partial A^-} \left[u(y) + d(x,y) \right], \quad \forall x \in A^-,$$

where

(1.4) $$A^- = supp\,(u-f)^-, \qquad A^+ = supp\,(u-f)^+,$$

and $d(x,y)$ denotes the geodesic distance inside Ω (see Definition (1.22) below).

After considering the problem from a variational viewpoint, we want to address it from a PDE perspective. Since it is not clear a priori which equetion does the minimizer of (1.1) solve, we go back to the

approximating sequence of minimizers of J_n (see (1.2)). The minimizers of (1.2) are solutions to

$$(1.5) \qquad \begin{cases} u_n(x) - \Delta_{p_n} u_n(x) = f(x) & \text{in } \Omega\,, \\ \dfrac{\partial u_n}{\partial \nu}(x) = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $\Delta_p v = \operatorname{div}(\nabla v |\nabla v|^{p-2})$, with $p > 1$, is the p-Laplacian and we denote by ν the external unit normal to $\partial\Omega$. As usual, we consider such solutions in the weak sense (see Definition 2.1). Our aim is, now, to pass to the limit in (1.5) to find the PDE solved by the limit function, u.

Usually, the limit as $p \to \infty$ of solutions to equations related to the p-Laplacian yields to equations that involve the infinity Laplacian, namely,

$$\Delta_\infty u = \langle D^2 u \nabla u, \nabla u \rangle = \sum_{i,j=1}^N u_{x_i x_j}\, u_{x_i}\, u_{x_j}\,.$$

In particular we refer to the pioneer paper by T. Bhattacharya, E. Di Benedetto and J. Manfredi (see [9]) that first studied such type of problem. We recall that the infinity Laplacian is a second order differential operator (in nondivergence form) that appears in many contexts. For instance, infinity harmonic functions (solutions to $-\Delta_\infty u = 0$) appear naturally as limits of p-harmonic functions (solutions to $-\Delta_p u = -\operatorname{div}(|Du|^{p-2} Du) = 0$) and they have applications to optimal transport problems, image processing, etc. See, among the others, [4], [9], [25] and references therein. Moreover, the infinity Laplacian plays a fundamental role in the calculus of variations of L^∞ functionals, see e.g. [3], [6], [7], [18], [21], [22], [23], [27] and the survey [4]. Notice that such operator is degenerate elliptic (that is non-degenerate only in the direction of the gradient).

Before stating our next result, let us introduce some notation: here and in the rest of the paper for $u, f \in C(\overline{\Omega})$ we denote

$$\Omega_u^+ = \{x \in \Omega \ : \ u(x) - f(x) > 0\} \quad \text{and} \quad \Omega_u^- = \{x \in \Omega \ : \ u(x) - f(x) < 0\}$$

and analogously,

$$(\partial\Omega)_u^+ = \{x \in \partial\Omega \ : \ u(x) - f(x) > 0\} \quad \text{and} \quad (\partial\Omega)_u^- = \{x \in \partial\Omega \ : \ u(x) - f(x) < 0\}.$$

Now we are ready to write the problem solved by u.

Theorem 1.2. *Assume $f \in C(\overline{\Omega})$, then the unique minimizer $u \in \mathbb{X}(\Omega)$ of (1.1), is a viscosity solution to*

$$(1.6) \qquad \begin{cases} 1 - |\nabla u| = 0 & \text{in } \Omega_u^+\,, \\ |\nabla u| - 1 = 0 & \text{in } \Omega_u^-\,. \end{cases}$$

Moreover it satisfies the following boundary conditions:

$$(1.7) \qquad \begin{cases} 1 - |\nabla u| \le 0 & \text{on } (\partial\Omega)_u^+\,, \\ \max\left\{1 - |\nabla u|, \dfrac{\partial u}{\partial \nu}\right\} \ge 0 & \text{on } (\partial\Omega)_u^+\,, \end{cases} \quad \text{and} \quad \begin{cases} |\nabla u| - 1 \ge 0 & \text{on } (\partial\Omega)_u^-\,, \\ \min\left\{|\nabla u| - 1, \dfrac{\partial u}{\partial \nu}\right\} \le 0 & \text{on } (\partial\Omega)_u^-\,. \end{cases}$$

Quite surprisingly, in the limit the second order operator is somehow lost: indeed there is no trace of the infinity Laplacian and in fact u satisfies a first order equation. Actually, equation (1.6) is the proper one in order to reflect the information carried out by (1.3): in the region $\{u - f < 0\}$ the function u looks like a cone pointing upwards and therefore it solves the eikonal equation (and analogously in $\{u - f > 0\}$, with

the eikonal equation with the reverse sign). As far as at the boundary conditions are concerned, it is natural to wonder if it would be possible to extend the equation in (1.6) up to $\overline{\Omega}$. However the answer is negative, and, to get convinced that (1.7) are the natural conditions to impose on $\partial\Omega$, it is enough to look at Example 2 above.

For the sake of clarity, notice that in Theorem 1.2 f is continuous and the open set Ω_u^+ coincides with the interior of A^+. We will provide an explanation for the absence of second order operators in (1.6) in Proposition 1.6 below.

It is also worth to mention explicitly that the regions Ω_u^+ and Ω_u^- (and their boundary counterparts) depend on the solution u itself. A consequence of this fact is that, even if the minimizer of (1.1) is unique, problem (1.6)-(1.7) does not posses a unique solution. In fact, we can construct minimal and maximal solutions to problem (1.6)-(1.7) that, in general, do not coincide with the minimizer of J_∞.

Theorem 1.3. *Assume that $f \in C(\overline{\Omega})$, then there exists a unique maximal (viscosity) solution $\overline{u} \in \mathbb{X}(\Omega)$ to the following obstacle problem*

$$
(1.8) \qquad \begin{cases} \overline{u} \geq f & in\ \Omega, \\ 1 - |\nabla\overline{u}| \geq 0 & in\ \Omega, \\ 1 - |\nabla\overline{u}| = 0 & in\ \Omega_{\overline{u}}^+, \end{cases}
$$

with boundary conditions

$$
(1.9) \qquad \begin{cases} 1 - |\nabla\overline{u}| \leq 0 & on\ (\partial\Omega)_{\overline{u}}^+, \\ \max\left\{ 1 - |\nabla\overline{u}|, \dfrac{\partial\overline{u}}{\partial\nu} \right\} \geq 0 & on\ (\partial\Omega)_{\overline{u}}^+. \end{cases}
$$

Clearly the solution $\overline{u}$ provided by Theorem 1.3 solves (1.6)-(1.7), but it is not a minimizer of (1.6) (unless $f \in \mathbb{X}(\Omega)$). Indeed, as the intuition suggests, since the minimizer u has to be close to f in L^2, then $u - f$ has to change sign in Ω (see Figure 4 and Proposition 2.4 for more details).
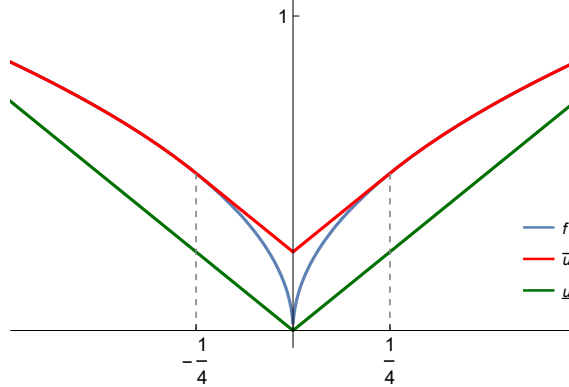

Figure 4.

The limit procedure showed above can be, in fact, generalized to a larger class of differential equations that involve lower order terms that have a polynomial growth with respect to the gradient. More precisely,

we consider the following problem

$$\text{(1.10)} \qquad \begin{cases} v_n(x) - \Delta_{p_n} v_n(x) = H_n(x, \nabla v_n(x)) + f(x) & \text{in } \Omega, \\ \dfrac{\partial v_n}{\partial \nu}(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where the Hamiltonian term $H_n \in C(\overline{\Omega} \times \mathbb{R}^N)$ satisfies the following growth conditions

$$\text{(1.11)} \qquad \exists K_1, K_2 > 0 : \qquad |H_n(x, \xi)| \leq p_n K_1 |\xi|^{p_n} + K_2, \qquad \forall x \in \Omega, \ \forall \xi \in \mathbb{R}^N$$

and

$$\text{(1.12)} \quad \exists H_\infty \in C(\overline{\Omega} \times \mathbb{R}^N) : \qquad \lim_{n\to\infty} \frac{H_n(x, \xi)}{p_n |\xi|^{p_n - 4}} = H_\infty(x, \xi), \qquad \text{uniformly with respect to } x \in \Omega.$$

Remark 1.4. *Notice that the second assumption in* (1.12) *implies that*

$$\text{(1.13)} \qquad \lim_{n\to\infty} H_n(x, \xi) = 0 \quad \text{for any } |\xi| < 1, \quad \text{uniformly with respect to } x \in \Omega.$$

Among all the possible choice of H_n that satisfy (1.11), let us mention two important examples:

$$H_n(x, \xi) = p_n |\xi|^{p_n - 2} \xi \cdot B(x) \quad \text{with} \quad B \in (C(\overline{\Omega}))^N \quad \text{and} \quad H_n(x, \xi) = p_n |\xi|^{p_n}.$$

These two types of lower order terms, accounting for the "linear $(p-1)$ growth" and "natural growth", are thoroughly studied in literature (for fixed p). Typically, the presence of the Hamiltonian affects the coercivity of the operator and some specific techniques are required to obtain apriori estimates: the first case can be dealt with symmetrization procedures [8], log-type estimates [10], slicing techniques [16] or arguing by contradiction [11]. For the case of natural growth we mention the exponential test function approach developed in [12]–[14], again symmetrization in [1], and an argument based on Schauder fixed point Theorem in [31].

Let us notice that, under assumption (1.11), for a smooth f and a fixed n, the existence of a solution $v_n \in W^{1,p_n}(\Omega) \cap L^\infty(\Omega)$ to problem (1.10) is a straightforward consequence of the results contained in [14] (see Theorem 3.1 below for more details). We also explicitly point out that the presence of the zeroth lower order term allows us to solve (1.12) without any smallness assumption on the datum f.

The mail difficulty when one wants to pass to the limit as p diverges in (1.10) is to obtain apriori estimates for v_n that are stable with respect to n. Up to our knowledge, the only related result for quasilinear equations with gradient lower order terms is contained in [17], where a large solution problem is considered.

Our main result concerning problem (1.10) is the following.

Theorem 1.5. *Assume* (1.11)–(1.12) *and let* $f \in C(\overline{\Omega})$. *Then there exists* $v \in W^{1,\infty}(\Omega) \cap L^\infty(\Omega)$ *such that*

$$v_n \to v \quad \text{uniformly in } \overline{\Omega}$$

which solves (in viscosity sense) the following equation

$$\text{(1.14)} \qquad \begin{cases} \max\left\{1 - |\nabla v|, -\Delta_\infty v - H_\infty(x, \nabla v)\right\} = 0 & \text{in } \Omega_v^+, \\ \min\left\{|\nabla v| - 1, -\Delta_\infty v - H_\infty(x, \nabla v)\right\} = 0 & \text{in } \Omega_v^-. \end{cases}$$

together with the following boundary conditions:

$$\text{(1.15)} \quad \min\left\{\max\left\{1 - |\nabla v|, -\Delta_\infty v - H_\infty(x, \nabla v)\right\}, \frac{\partial v}{\partial \nu}\right\} \leq 0 \quad on \ (\partial\Omega)_v^+ \, ,$$

$$\max\left\{\max\left\{1 - |\nabla v|, -\Delta_\infty v - H_\infty(x, \nabla v)\right\}, \frac{\partial v}{\partial \nu}\right\} \geq 0 \quad on \ (\partial\Omega)_v^+ \, ,$$

and

$$\text{(1.16)} \quad \min\left\{\min\left\{|\nabla v| - 1, -\Delta_\infty v - H_\infty(x, \nabla v)\right\}, \frac{\partial v}{\partial \nu}\right\} \leq 0 \quad on \ (\partial\Omega)_v^- \, ,$$

$$\max\left\{\min\left\{|\nabla v| - 1, -\Delta_\infty v - H_\infty(x, \nabla v)\right\}, \frac{\partial v}{\partial \nu}\right\} \geq 0 \quad on \ (\partial\Omega)_v^- \, .$$

At the first glance we notice that the structure of (1.14) is more involved then the one in (1.6). First of all v does not belong in general to $\mathbb{X}(\Omega)$ since the bound on ∇v depends also on the Hamiltonian (see Section 3). Moreover, as expected, the presence of the Hamiltonian term changes both the limit equation and the boundary conditions.

Let us now go back to the minimizer $u \in \mathbb{X}(\Omega)$ of (1.1) and notice that it also solves the equation (1.14) with $H_\infty \equiv 0$. Apparently, this provides some more information about $-\Delta_\infty u$ that is missing in (1.6). However, we have the following result.

Proposition 1.6. *Assume that $f \in C(\overline{\Omega})$, and that $w \in \mathbb{X}(\Omega)$. Then w solves (1.6) if and only if it solves*

$$\begin{cases} \max\left\{1 - |\nabla w|, -\Delta_\infty w\right\} = 0 & in \ \Omega_w^+ \, , \\ \min\left\{|\nabla w| - 1, -\Delta_\infty w\right\} = 0 & in \ \Omega_w^- \, . \end{cases}$$

Roughly speaking this is due to the bound $|\nabla w| \leq 1$ and the fact that cones are solutions to the eikonal equation and are also infinity harmonic away from their singular points.

Finally, we include some considerations about a very well known family of functionals that allow *free discontinuities*. Unfortunately, we are able to provide only partial results in this case and we left a more complete analysis of the limiting behavior of such functionals for $p \to \infty$ for future research. Let us consider the functional $\mathcal{I}_p : SBV(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined as

$$\text{(1.17)} \quad \mathcal{I}_p(v) = \frac{1}{p}\int_\Omega |\nabla v|^p + \frac{1}{r}\int_\Omega |v - f|^r + \mathcal{H}^{N-1}(S_v) \, ,$$

with $r \geq 1$. Here $SBV(\Omega)$ is the space of Special Functions of Bounded Variation, ∇v is the absolutely continuous part of Dv with respect to the Lebesgue measure, S_v is the singular set of v and $\mathcal{H}^{N-1}$ is the $N - 1$ dimensional Hausdorff measure (more details and formal definitions can be found in Section 5). We set $\mathcal{I}_p(v) = +\infty$ when $\nabla v \notin L^p(\Omega)$.

The functional $\mathcal{I}_p(v)$ was introduced in the seminal paper by De Giorgi, Carriero and Leaci ([19]) to provide a weak formulation of the Mumford–Shah image segmentation problem, namely, minimizing

$$\text{(1.18)} \quad F_p(K, v) = \frac{1}{p}\int_\Omega |\nabla v|^p + \frac{1}{r}\int_\Omega |v - f|^r + \mathcal{H}^{N-1}(K) \, ,$$

among all $K \subset \overline{\Omega}$ closed sets, with $v \in W^{1,p}(\Omega \setminus K)$. An in-depth overview about the history of these functionals, their relation with applications, and the huge impact that their study has had in the field of

Calculus Variation is definitively out of reach for our contribution. We refer the interested reader to the original paper [19], the classical monograph [2], and to the nice review [26]. For our aims, we simply recall that the main difficulty in dealing directly with (1.18) is that the Hausdorff measure is not lower semicontinuous with respect to any reasonable metric the collection of closed set of Ω can be equipped with. Therefore, the strategy of [19] was to find a minimizer $u \in SBV(\Omega)$ of $\mathcal{I}_p$ and then show that $(\overline{S_u}, u)$ was also a minimizer to (1.18). The most delicate step in their argument was to prove that $\mathcal{H}^{N-1}(\overline{S_u} \cap \Omega) = \mathcal{H}^{N-1}(S_u)$ and this crucial step was achieved by a lower bound on the density of S_u, namely

$$(1.19) \qquad \exists \theta > 0 \quad \mathcal{H}^{N-1}(S_u \cap B_\rho(x)) > \theta \rho^{N-1}, \qquad \forall x \in \overline{S_u}, \forall \rho > 0$$

(see Theorems 7.15, 7.21 and 7.22 in [2]).

At this point it is natural to wonder what happens with the limiting minimization problem relative to (1.18) as $n \to \infty$. The answer is given by the following Proposition.

Proposition 1.7. *Assume that $f \in L^\infty(\Omega)$ and that $p_n \to +\infty$, then the sequence of functionals $\mathcal{I}_{p_n} : SBV(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined in (1.18) Γ-converges to $\mathcal{I}_\infty : SBV(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined as*

$$(1.20) \qquad \mathcal{I}_\infty(v) = \begin{cases} \dfrac{1}{r} \displaystyle\int_\Omega |v - f|^r + \mathcal{H}^{N-1}(S_v) & \text{if } |\nabla v| \leq 1, \\ +\infty & \text{otherwise.} \end{cases}$$

The next step would be to prove that a minimizer of (1.20) is also a minimizer to

$$(1.21) \qquad F_\infty(K, u) = \frac{1}{r} \int_\Omega |u - f|^r + \mathcal{H}^{N-1}(K),$$

with $K \subset \overline{\Omega}$ a closed set, $u \in W^{1,\infty}(\Omega \setminus K)$ and $|\nabla u| \leq 1$. Unfortunately, we are not able to show such a property since we do not know how to obtain a lower density bound, as in (1.19), for the limit function u. We leave this as an interesting open problem.

NOTATIONS AND BASIC DEFINITIONS

Let us introduce the notion of geodesic distance (see [32], for instance): for $x, y \in \Omega$ we define

$$\Gamma = \{\gamma_{x,y} \in C^1([0,1]; \Omega) \text{ with } \gamma(0) = x, \ \gamma(1) = y\}$$

$$(1.22) \qquad \text{and}$$

$$d(x, y) = \inf_{\gamma \in \Gamma} \{\mathcal{L}(\gamma_{x,y})\}, \quad \text{where } \mathcal{L}(\gamma_{x,y}) = \text{lenght}(\gamma_{x,y}).$$

Of course, if Ω is convex $d(x, y) \equiv |x - y|$, but for general domains we just have that $d(x, y) \geq |x - y|$. The geodesic distance is the natural quantity to consider in our setting since the condition

$$v \in \mathbb{X}(\Omega) := \{v \in W^{1,\infty}(\Omega) : |Dv| \leq 1 \ \ a.e. \ x \in \Omega\}$$

can be rephrased as

$$v \in \{v \in W^{1,\infty}(\Omega) : |v(x) - v(y)| \leq d(x, y), \quad \forall \, x, y \in \Omega\}.$$

Now, let us introduce some extra notations that will be used in the rest of the paper. Let us recall that (see Proposition 4.17 in [15]) the support of a general function $f : \Omega \to \mathbb{R}$ is defined as

$$\text{supp}(f) = \left(\bigcup_{i \in I} \omega_i \right)^c \qquad \text{where} \qquad I = \Big\{ \omega \text{ open subset of } \Omega \text{ such that } f = 0 \ a.e. \ \text{in } \omega \Big\}.$$

By definition $\text{supp}(f)$ is a closed set with respect to the subset topology relative to Ω. For any $f \in L^2(\Omega)$, and for any continuous function u we define the following sets

$$(1.23) \qquad\qquad A^- = \text{supp}\,(u - f)^-, \qquad A^+ = \text{supp}\,(u - f)^+ .$$

For future use we need to give the precise definition of a solution to a partial differential equation in the viscosity sense.

Definition 1.8. *Let D be a locally compact subset of $\mathbb{R}^{\mathbb{N}}$ and let $F : D \times \mathbb{R} \times \mathbb{R}^{\mathbb{N}} \times \mathcal{S}(N)$ be a continuous function that satisfies*

$$F(x, r, \xi, X) \leq F(x, s, \xi, Y) \ \ \text{whenever} \ \ r \leq s \ \ \text{and} \ \ Y \leq X.$$

We say that $u \in C(D)$ is a viscosity sub (super) solution to

$$(1.24) \qquad\qquad F(x, u, \nabla u, D^2 u) = 0 \qquad \text{at } x \in D$$

if, for any $C^2(D)$ function φ such that $u - \varphi$ has a strict local maximum (minimum) at x_0, it holds true

$$F(x_0, \varphi(x_0), \nabla \varphi(x_0), D^2 \varphi(x_0)) \leq (\geq) \, 0.$$

Finally, $u \in C(D)$ is a viscosity solution of (1.24) if it is both sub and super solution.

Let us denote by $\delta(x) = \text{dist}(x, \partial\Omega)$ and let us recall that $\delta(x) \in W_0^{1,\infty}(\Omega)$ and that it satisfies (in the viscosity sense) the eikonal equation

$$|\nabla \delta| - 1 = 0 \quad \text{in } \Omega.$$

In the sequel we will use the following result.

Proposition 1.9. *Any function in $\mathbb{X}(\Omega)$ satisfies both*

$$(1.25) \qquad\qquad |\nabla u| - 1 \leq 0 \qquad \text{and} \quad -|\nabla u| + 1 \geq 0 \qquad \text{in } \Omega$$

in the viscosity sense.

Proof of Proposition 1.9. Let us observe that (1.25) is equivalent to say that the following inequalities hold true at any $x_0 \in \Omega$:

- for any $\varphi \in C^2(\Omega)$ such that there exists $r > 0$ with $u(x) - \varphi(x) \leq u(x_0) - \varphi(x_0)$ in $B_r(x_0) \subset \Omega$, we have that $|\nabla \varphi(x_0)| - 1 \leq 0$;
- for any $\varphi \in C^2(\Omega)$ such that there exists $r > 0$ with $u(x) - \varphi(x) \geq u(x_0) - \varphi(x_0)$ in $B_r(x_0) \subset \Omega$, we have that $-|\nabla \varphi(x_0)| + 1 \geq 0$.

In order to prove the first inequality in (1.25) let $x_0 \in \Omega$ and assume that there exists φ, a C^2 function of a neighborhood of x_0, such that $u - \varphi$ has a strict local maximum at x_0, with $u(x_0) = \varphi(x_0)$. Suppose by contradiction that $|\nabla \varphi(x_0)| > 1$. Taking the first order Taylor expansion of $\varphi(x)$ at x_0 it follows that

$$\varphi(x) - \varphi(x_0) = \nabla \varphi(x_0)(x - x_0) + o(|x - x_0|) \ \ \text{as} \ \ x \to x_0.$$

Let us consider

$$x_t = -\frac{\nabla \varphi(x_0)}{|\nabla \varphi(x_0)|} t + x_0, \quad \text{for } t > 0 \text{ small.}$$

We have $|x_t - x_0| = t$ and thus

$$u(x_t) - u(x_0) \le \varphi(x_t) - \varphi(x_0) = \nabla \varphi(x_0) \cdot (x_t - x_0) + o(|x - x_0|) = -|\nabla \varphi(x_0)| t \big(1 + o(1)\big), \quad \text{as } t \to 0.$$

This is contradicts $u \in \mathbb{X}(\Omega)$.

The second inequality in (1.25) follows in the same way. $\qquad\square$

2. The best Lipschitz approximation with gradient constraint

Let p_n be an increasing sequence such that $p_n \to \infty$ as $n \to \infty$. Let us set, for any $f \in L^2(\Omega)$, the following functional

$$(2.1) \qquad J_n(v) = \frac{1}{p_n} \int_\Omega |\nabla v|^{p_n} + \frac{1}{2} \int_\Omega (v - f)^2, \qquad v \in W^{1,p_n}(\Omega).$$

For any fixed n, it is hot hard to see that there exists a unique (exploiting the convexity of (2.1)) minimizer $u_n \in W^{1,p_n}(\Omega)$ to (2.1). Thus, from direct variational arguments, the minimizer, u_n, turns out to be a weak solution to

$$(2.2) \qquad \begin{cases} u_n(x) - \Delta_{p_n} u_n(x) = f(x) & \text{in } \Omega, \\ \dfrac{\partial u_n}{\partial \nu}(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us recall here such a formulation.

Definition 2.1. *A weak solution u_n to (2.2) is a $W^{1,p_n}(\Omega)$ function that satisfies*

$$(2.3) \qquad \int_\Omega u_n(x)\varphi + \int_\Omega |\nabla u_n(x)|^{p_n - 2} \nabla u_n \cdot \nabla \varphi = \int_\Omega f\varphi, \qquad \forall \varphi \in W^{1,p_n}(\Omega).$$

The aim of this section is to deal with the limiting behaviour of these three objects - the minimizer, the functional, and the equation - as p_n diverges.

We start with the following proposition that provides a limit for the sequence of minimizers together with a limiting minimization problem associated to the limit of (2.1).

Proposition 2.2. *Assume $f \in L^2(\Omega)$ and let $\{u_n\} \subset W^{1,p_n}(\Omega)$ be the sequence of minimizers of (2.1), then there exists $u \in \mathbb{X}(\Omega)$ such that*

$$u_n \longrightarrow u \qquad \text{uniformly in } \overline{\Omega}.$$

Moreover, u is the unique minimizer of the functional

$$(2.4) \qquad J_\infty(v) = \frac{1}{2} \int_\Omega (v - f)^2, \quad v \in \mathbb{X}(\Omega).$$

Proof. The minimality of u_n gives us that

$$\frac{1}{2} \int_\Omega (u_n - f)^2 + \frac{1}{p_n} \int_\Omega |\nabla u_n|^{p_n} \le J_n(v) \le J_n(0) = \frac{1}{2} \int_\Omega |f|^2.$$

This implies that

$$(2.5) \qquad \|u_n\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)} \qquad \text{and} \qquad \|\nabla u_n\|_{L^{p_n}(\Omega)} \le \left(p_n\|f\|^2_{L^2(\Omega)}\right)^{\frac{1}{p_n}}.$$

For $N < p_n$, let us recall that Morrey inequality (see [15, Corollary 9.14]) implies that, for any $x, y \in \Omega$,

$$(2.6) \qquad\qquad\qquad |u_n(x) - u_n(y)| \le C_M\|\nabla u_n\|_{L^p(\Omega)},$$

where $C_M = C_M(N, \Omega)$ can be chosen independent on p (see in particular formula (28) in the proof of [15, Theorem 9.12]). Therefore, we deduce that

$$(2.7) \qquad\qquad |u_n(x)| \le |u_n(y)| + C_M\|\nabla u_n\|_{L^{p_n}(\Omega)}, \quad \forall\, x, y \in \Omega,$$

and consequently

$$(2.8) \qquad\qquad \|u_n\|_{L^\infty(\Omega)} \le |\Omega|^{\frac{1}{2}}\|u_n\|_{L^2(\Omega)} + \tilde{C}_M\|\nabla u_n\|_{L^{p_n}(\Omega)}.$$

Combining (2.5) with (2.8) we conclude that

$$(2.9) \qquad\qquad \|u_n\|_{L^\infty(\Omega)} \le |\Omega|^{\frac{1}{2}}\|f\|_{L^2(\Omega)} + c_\Omega\|f\|^{\frac{2}{p_n}}_{L^2(\Omega)}.$$

On the other hand, using Hölder's inequality, it follows that, for any q such that $N < q < p_n$,

$$\|\nabla u_n\|_{L^q(\Omega)} \le \|\nabla u_n\|_{L^{p_n}(\Omega)}|\Omega|^{\frac{1}{q}-\frac{1}{p_n}} \le p_n^{\frac{1}{p_n}}\|f\|^{\frac{2}{p_n}}_{L^2(\Omega)}|\Omega|^{\frac{1}{q}-\frac{1}{p_n}}.$$

Thanks to the two estimates above, we deduce that the sequence $\{u_n\}$ is bounded in $W^{1,q}(\Omega)$ and then up to a (not relabeled) subsequence, $u_n \to u$ weakly in $W^{1,q}(\Omega)$ and uniformly in Ω. The lower semi continuity of the norm also implies that for any $q \ge 1$ we have

$$\|\nabla u\|_{L^q(\Omega)} \le |\Omega|^{\frac{1}{q}}.$$

Passing to the limit as $q \to \infty$ we conclude that $u \in \mathbb{X}(\Omega)$.

In order to show that u is, in fact, a minimizer to (2.4), let us notice that, by definition, u_n satisfies

$$\frac{1}{2}\int_\Omega (u_n - f)^2 \le \frac{1}{2}\int_\Omega (u_n - f)^2 + \frac{1}{p_n}\int_\Omega |\nabla u_n|^{p_n} \le \frac{1}{2}\int_\Omega (z - f)^2 + \frac{1}{p_n}\int_\Omega |\nabla z|^{p_n},$$

$\forall z \in \mathbb{X}(\Omega) \subset W^{1,p_n}(\Omega)$. Since $|\nabla z| \le 1$ a.e. in Ω we get

$$\frac{1}{2}\int_\Omega (u_n - f)^2 \le \frac{1}{2}\int_\Omega (z - f)^2 + \frac{1}{p_n}|\Omega|, \quad \forall z \in \mathbb{X}(\Omega) \subset W^{1,p_n}(\Omega).$$

We can now pass to the limit in the previous inequality (recall that $u_n \to u$ uniformly in Ω), and we deduce that u is a minimizer. The uniqueness follows by a the convexity of J_∞. $\qquad\square$

Remark 2.3. *When $f \in L^\infty(\Omega)$ we have a explicit uniform bound for u_n,*

$$|u_n| \le \|f\|_{L^\infty(\Omega)} =: \overline{k}.$$

Indeed, $T_{\overline{k}}(u_n)$ is a competitor for u_n for the energy (2.1) and the uniqueness of the minimizer of J_n implies $u_n \equiv T_{\overline{k}}(u_n)$.

In other words, the minimizer of (2.4) is the best approximation of f among all the functions in $\mathbb{X}(\Omega)$. The gradient constraint imposes to the minimizer a specific geometric structure. The goal of the following results is to clarify such a structure.

Proposition 2.4. *Assume that $f \in L^2(\Omega)$ and let $u \in \mathbb{X}(\Omega)$ be the minimizer to* (2.4)*. Then*

$$|A^-||A^+| = 0 \qquad \text{if and only if} \qquad f \in \mathbb{X}(\Omega).$$

Proof. We observe that if $f \in \mathbb{X}(\Omega)$, then $u \equiv f$ and hence $|A^-| = 0$ and $|A^+| = 0$. Suppose now, that $f \in L^2(\Omega) \setminus \mathbb{X}(\Omega)$ and by contradiction suppose that $|A^-||A^+| \neq 0$. If both $|A^-|$ and $|A^+|$ are zero then $u = f$ and we reach a contradiction.

Assume now that $|A^-| > 0$ and $|A^+| = 0$ (the case $|A^+| > 0$ and $|A^-| = 0$ is analogous). Then, for any $\varepsilon > 0$, $u + \varepsilon \in \mathbb{X}(\Omega)$ satisfies

$$J_\infty(u + \varepsilon) = J_\infty(u) + \frac{1}{2}\varepsilon^2|\Omega| + \varepsilon \int_{A^-} (u - f) = J_\infty(u) + \frac{1}{2}\varepsilon \left(\varepsilon|\Omega| - 2 \int_{A^-} |u - f| \right),$$

and the last term in the right hand side is negative for ε small enough. This contradicts the minimality of u. $\qquad \square$

Remark 2.5. In fact, the proof shows that

$$|A^-| = 0 \text{ and } |A^+| = 0 \qquad \text{if and only if} \qquad f \in \mathbb{X}(\Omega).$$

However, note that the statement of the proposition highlights that it is enough to have $|A^-| = 0$ or $|A^+| = 0$ to obtain that $u = f \in \mathbb{X}(\Omega)$.

Next we deduce a representation formula for the unique minimizer to (2.4).

Theorem 2.6. *Assume $f \in L^2(\Omega)$ and let u be the unique minimizer to* (2.4)*, then the following representations formulas hold true:*

$$(2.10) \qquad u(x) = \max_{y \in \partial A^+} \left[u(y) - d(x, y) \right], \quad \forall\, x \in A^+$$

and

$$(2.11) \qquad u(x) = \min_{y \in \partial A^-} \left[u(y) + d(x, y) \right], \quad \forall\, x \in A^-.$$

Proof. We assume that $f \notin \mathbb{X}(\Omega)$, otherwise $u \equiv f$ and there is nothing to prove. Proposition 2.4 assures us that both A^+ and A^- are nontrivial, then the max and the min in (2.10) and (2.11) are well defined. We start by proving the second formula above. Let us define

$$v(x) = \begin{cases} \min_{y \in \partial A^-} \left[u(y) + d(x, y) \right] & \text{in} \quad A^-, \\ u(x) & \text{in} \quad \Omega \setminus A^-. \end{cases}$$

The strategy of the proof is to show that such a v belongs to $\mathbb{X}(\Omega)$ and that it is a competitor for u in the minimization of (2.4). Then, uniqueness of the minimizer implies that

$$v \equiv u.$$

The first step is to show that

$$(2.12) \qquad |v(x) - v(z)| \leq d(x, z), \qquad \forall\, x, z \in A^-.$$

By definition we have that

$$v(x) = u(\bar{x}) + d(x, \bar{x}) = \min_{y \in \partial A^-} \Big[u(y) + d(x, y) \Big]$$

(2.13)   and

$$v(z) = u(\bar{z}) + d(z, \bar{z}) = \min_{y \in \partial A^-} \Big[u(y) + d(x, y) \Big],$$

for some $\bar{x}, \bar{z} \in \partial A^-$. If $\bar{x} = \bar{z}$ we have that

$$|v(x) - v(z)| = d(x, \bar{x}) - d(z, \bar{x}) \le d(x, z).$$

On the other hand, if $\bar{x} \ne \bar{z}$, we get

$$v(x) = u(\bar{x}) + d(x, \bar{x}) \le u(\bar{z}) + d(x, \bar{z})$$

(2.14)   and

$$v(z) = u(\bar{z}) + d(z, \bar{z}) \le u(\bar{x}) + d(z, \bar{x}).$$

Thus by (2.13)–(2.14), we deduce that

$$-d(x, z) \le d(x, \bar{x}) - d(z, \bar{x}) \le v(x) - v(z) \le d(x, \bar{z}) - d(z, \bar{z}) \le d(x, z),$$

and (2.12) follows.

Next we prove that

$$v \equiv u \quad \text{on } \partial A^-.$$

By contradiction, assume that there exists $x \in \partial A^-$ such that, say, $u(x) > v(x)$. Thus, $\exists \bar{x} \in \partial A^-$, $x \ne \bar{x}$ such that $v(x) = u(\bar{x}) + d(x, \bar{x}) < u(x)$ that contradicts $u \in \mathbb{X}(\Omega)$.

We need to prove now that $v \in \mathbb{X}(\Omega)$. The only relevant case we have to consider is when $y \in \Omega \setminus A^-$ and $x \in A^-$. Consider $\gamma \in \Gamma$ (see (1.22) for the definition), let $t^* := \inf\{t \in (0, 1) \ : \ \gamma(t) \notin \Omega\}$, and $z = \gamma(t^*)$. Thus, since $z \in \partial A^-$, then $u(z) = v(z)$ and

$$|v(x) - v(y)| \le |u(x) - u(z)| + |v(z) - v(y)| \le d(x, z) + d(z, y) \le \mathcal{L}(\gamma_1) + \mathcal{L}(\gamma_2) = \mathcal{L}(\gamma),$$

where $\gamma_1 = \gamma \, \llcorner \, \big(t \in [0, t^*]\big)$ and $\gamma_2 = \gamma \, \llcorner \, \big(t \in [t^*, 1]\big)$. Minimizing the above expression with respect to any $\gamma \in \Gamma$ we deduce that $v \in \mathbb{X}(\Omega)$.

To conclude the proof we need to show that $v \equiv u$ in A^-. If we assume that there exists $\bar{x} \in A^-$ such that $u(\bar{x}) > v(\bar{x})$, then, by definition of v, we have that $v(\bar{x}) = u(\bar{y}) + d(\bar{x}, \bar{y})$ for some $\bar{y} \in \partial A^-$. It follows that

$$u(\bar{x}) - u(\bar{y}) > v(\bar{x}) - v(\bar{y}) = d(\bar{x}, \bar{y}),$$

that contradicts the fact that $u \in \mathbb{X}(\Omega)$. Therefore, let us assume, arguing again by contradiction, that there exists $\bar{x} \in A^-$ such that

$$\sup_{y \in A^-} (v - u)(y) = (v - u)(\bar{x}) = \mu > 0.$$

We set

$$D = \left\{ x \in A^- \ : \ v(x) - u(x) > \frac{\mu}{2} \right\}, \quad \eta = \operatorname*{ess\,inf}_{y \in D} (f - u) \ge 0$$

and we define

$$\tilde{u} = \begin{cases} \max\{u, v - \varepsilon\} & \text{in } D\,, \\ u & \text{in } \Omega \setminus D\,, \end{cases}$$

with $\varepsilon = \max\left\{\mu - \frac{\eta}{2}, \frac{\mu}{2}\right\}$. Since both u and v are continuous it follows that $\tilde{u}$ is continuous in Ω, and moreover $\tilde{u} \in \mathbb{X}(\Omega)$ (since both u and v are). Observe now that that $\tilde{u}$ satisfies $\tilde{u} < f$ in A^-: indeed in $A^- \setminus D$ this follows by the definition of A^-, while

$$\tilde{u}(x) - f(x) = v(x) - \varepsilon - f(x) = \underbrace{v(x) - u(x)}_{\leq \mu} - \varepsilon + \underbrace{u(x) - f(x)}_{\leq -\eta} \leq \mu - \varepsilon - \eta < 0 \qquad \text{in } D\,.$$

Consequently

(2.15) $$\big(u(x) - f(x)\big)^2 \geq \big(\tilde{u}(x) - f(x)\big)^2 \quad \text{in } \Omega,$$

since the two quantities above coincide in $\Omega \setminus D$ while

$$u(x) - f(x) < \max\{u(x), v(x) - \varepsilon\} - f(x) < 0 \qquad \text{in } D.$$

Then, we have that

$$J_\infty(\tilde{u}) \; < \; J_\infty(u)$$

that is in contradiction with the fact that u minimizes J_∞.

Thus, we have that $v(x) = u(x)$ in Ω and in particular $v(x) = u(x)$ in A^- and consequently in A^- the representation formula given by (2.11) holds true.

Analogously one can prove that also (2.10) is in force. $\square$

Proof of Theorem 1.1. The proof follows immediately putting together the results of Proposition 2.2 and Theorem 2.6. $\square$

Remark 2.7. *Let us observe that a similar version of the previous results can be obtained working in* $W_0^{1,p_n}(\Omega)$*, i.e. considering homogeneous Dirichlet boundary conditions. Actually, this latter case can be recovered as a sort of corollary of the* $W^{1,p_n}(\Omega)$*. To get convinced of this fact notice that the limit space for the Dirichlet problem is*

$$\mathbb{X}_0(\Omega) := \left\{v \in W^{1,\infty}(\Omega) \; : \; |Dv| \leq 1 \; \text{a.e. in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega\right\}.$$

We consider the two minimization problems

$$\inf_{\mathbb{X}_0(\Omega)} \frac{1}{2}\int_\Omega (v - f)^2 \qquad \inf_{\mathbb{X}(\Omega)} \frac{1}{2}\int_\Omega (v - \tilde{f})^2$$

where $\tilde{f}(x) = \max\{-\delta(x), \max\{f(x), \delta(x)\}\}$*. We claim that the minimizer* $u \in \mathbb{X}_0(\Omega)$ *of the first functional coincides with the minimizer* $\tilde{u} \in \mathbb{X}_0(\Omega)$ *of the second one. Indeed, thanks to the definition of* $\tilde{f}(x)$ *one easily deduce that* $\tilde{u} \in \mathbb{X}_0(\Omega)$*. Moreover if* u *and* $\tilde{u}$ *would differ in some region one could build a competitor for one of the two, contradicting uniqueness.*

3. THE PROBLEM WITH GRADIENT LOWER ORDER TERM

In this section we address the limit, as $n \to \infty$, of the family of problems (1.10).

Our first result provides existence and uniqueness of a solution for problem (1.10), for any fixed $p_n \in (1, \infty)$, together with some estimates useful in order to study the asymptotic behaviour as $n \to \infty$.

Theorem 3.1. *Let us assume* (1.11) *and* $f \in L^\infty(\Omega)$. *Then, for any fixed* $n \in \mathbb{N}$, *there exists a unique solution* $v_n \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *to problem* (1.10) *in the following weak sense*

$$(3.1) \qquad \int_\Omega v_n \phi + \int_\Omega |\nabla v_n|^{p_n-2} \nabla v_n \nabla \phi = \int_\Omega \left(H_n(x, \nabla v_n) + f \right) \phi, \quad \forall \phi \in W^{1,p_n}(\Omega) \cap L^\infty(\Omega).$$

Moreover the following estimates hold true

$$(3.2) \quad \|v_n\|_{L^\infty(\Omega)} \leq A := K_2 + \|f\|_{L^\infty(\Omega)} \quad and \quad \|\nabla v_n\|_{L^{p_n}(\Omega)} \leq B_n := A^{\frac{1}{p_n}} e^{(K_1 + 1/p_n)A} |\Omega|^{\frac{1}{p_n}}.$$

Remark 3.2. Observe that

$$\lim_{n \to +\infty} B_n = e^{K_1(K_2 + \|f\|_\infty)} =: B_\infty > 1.$$

Proof. The existence of a bounded weak solution to (3.1) follows by [12] – [14], while the uniqueness is a consequence of [28, Theorem 1.2].

In order to prove that $\|v_n\|_{L^\infty(\Omega)} \leq A$, assume by contradiction that

$$\mathrm{meas}(\{x \in \Omega : \ |v_n| > A\}) > 0;$$

therefore, there exists $k > A$ such that

$$\mathrm{meas}(\{x \in \Omega : \ |v_n| > k\}) > 0.$$

Let us consider $\mu_n = p_n K_1 + 1$ an let us choose for $k \geq 0$,

$$\phi = (e^{\mu_n |G_k(v_n)|} - 1)\mathrm{sign}(v_n)$$

as a test function in (3.1). Using assumption (1.11), we get that

$$\int_\Omega |v_n|(e^{\mu_n |G_k(v_n)|} - 1) + \mu_n \int_\Omega |\nabla G_k(v_n)|^{p_n} e^{\mu_n |G_k(v_n)|} \leq \int_\Omega \left(|H(x, \nabla v_n)| + |f| \right)(e^{\mu_n |G_k(v_n)|} - 1)$$

$$\leq p_n K_1 \int_\Omega |\nabla G_k(v_n)|^{p_n} e^{\mu_n |G_k(v_n)|} + \left(K_2 + \|f\|_{L^\infty(\Omega)} \right) \int_\Omega (e^{\mu_n |G_k(v_n)|} - 1),$$

that implies

$$(3.3) \qquad \int_\Omega |\nabla G_k(v_n)|^{p_n} + (k - A) \int_{\{|v_n| > k\}} (e^{\mu_n |G_k(v_n)|} - 1) \leq 0.$$

If we choose any $k > A$ we get a contradiction, since the left hand side above turns out to be strictly positive. Consider now $k = 0$ in (3.3), then we get

$$\int_\Omega |\nabla v_n|^{p_n} \leq (K_2 + \|f\|_{L^\infty(\Omega)}) \int_\Omega (e^{\mu_n |v_n|} - 1) \leq B_n^{p_n}.$$

Gathering together the above estimates, we obtain (3.2). $\qquad \square$

Now we are ready to prove the main result of the section.

Proof of Theorem 1.5. Thanks to (3.2), we have that

$$\|v_n\|_{L^\infty(\Omega)} \le A \quad \text{and} \quad \|\nabla v_n\|_{L^{p_n}(\Omega)} \le B_n.$$

Using Hölder's inequality, it follows that for any $N < q < p_n$ we have

$$\|\nabla v_n\|_{L^q(\Omega)} \le \|\nabla v_n\|_{L^{p_n}(\Omega)} |\Omega|^{\frac{1}{q} - \frac{1}{p_n}}.$$

Since $B_n \to B_\infty$ (see Remark 3.2) and thanks to the L^∞ bound, we deduce that the sequence $\{v_n\}$ is bounded in $W^{1,q}(\Omega)$ and, up to a (not relabeled) sequence, we get that $v_n \to v$ weakly in $W^{1,q}(\Omega)$ and uniformly in Ω. The lower semicontinuity of the norm also implies that

$$\|\nabla v\|_{L^q(\Omega)} \le B_\infty, \qquad \forall q > 1.$$

Therefore we conclude that $v(x)$ satisfies

$$\|v\|_{W^{1,\infty}(\Omega)} \le A + B_\infty.$$

Let us now focus on the equation solved by v. We start by proving that

(3.4) $$\max\left\{1 - |\nabla v|, -\Delta_\infty v - H_\infty(x, \nabla v)\right\} \le 0 \qquad \text{in } \Omega_v^+.$$

Indeed, let us pick any $x_0 \in \Omega_v^+$ and any $\varphi \in C^2(\Omega)$ such that $v(x) - \varphi(x)$ has a strict local maximum at x_0 with $v(x_0) - \varphi(x_0) = 0$. We need to check that

$$\max\left\{1 - |\nabla \varphi(x_0)|, -\Delta_\infty \varphi(x_0) - H_\infty(x_0, \nabla \varphi(x_0))\right\} \le 0.$$

Let us recall that v is the (uniform) limit of solutions to (3.1), so that there exist a sequence of real numbers $\varepsilon_n \to 0$ and a sequence of points $x_n \to x_0$ such that $v_n(x) - \varphi(x) - \varepsilon_n$ has a local maximum at x_n and $v_n(x_n) - \varphi(x_n) = \varepsilon_n$.

Since both f and H_∞ belong to $C^0(\overline{\Omega})$, then (see Theorem 1 in [29]) any v_n belongs to $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ (see also [20]) and thus v_n turns out to be also a viscosity solution to (1.10). Hence we have that

(3.5)
$$\varphi(x_n) + \varepsilon_n - f(x_n) - \Delta\varphi(x_n)|\nabla\varphi(x_n)|^{p_n-2} - (p_n-2)\Delta_\infty\varphi(x_n)|\nabla\varphi(x_n)|^{p_n-4} - H_n(x_n, \varphi(x_n)) \le 0.$$

Assume at first that $1 - |\nabla\varphi(x_0)| \le 0$. It follows that $|\nabla\varphi(x_n)| > 0$ for n large enough and since x_0 belongs to the (open) set Ω_v^+, then $\varphi(x_n) + \varepsilon_n - f(x_n)$ is strictly positive and we can drop it from the previous inequality (3.5). Hence, we divide (3.5) by $(p_n - 2)|\nabla\varphi(x_n)|^{p_n-4}$ and we take the limit with respect to n: exploiting (1.12) we deduce that

$$-\Delta_\infty\varphi(x_0) - H_\infty(x_0, \nabla\varphi(x_0)) \le 0.$$

Conversely, if $|\nabla\varphi(x_0)| < 1$, we can directly pass to the limit in (3.5), and we get $\varphi(x_0) - f(x_0) = u(x_0) - f(x_0) \le 0$, that is in contradiction with the case under consideration.

Next we prove that

(3.6) $$\max\{1 - |\nabla v|, -\Delta_\infty v - H_\infty(x, \nabla v)\} \ge 0 \qquad \text{in } \{v(x) - f(x) > 0\}.$$

Consider any $x_0 \in \Omega_v^+$ and take any $\varphi \in C^2(\Omega)$ such that $v(x) - \varphi(x)$ has a strict local minimum at x_0 with $v(x_0) - \varphi(x_0) = 0$. We want to prove that for such a φ we have that

$$\max\left\{1 - |\nabla\varphi(x_0)|, -\Delta_\infty\varphi(x_0) - H_\infty(x_0, \nabla\varphi(x_0))\right\} \ge 0.$$

If $1 - |\nabla\varphi(x_0)| \geq 0$ the inequality is trivially satisfied, so we deal with the case $|\nabla\varphi(x_0)| > 1$. As before, we recall that there exist a sequence of real numbers $\varepsilon_n \to 0$ and a sequence of points $x_n \to x_0$ such that $v_n(x) - \varphi(x) - \varepsilon_n$ as a local minimum at x_n and $v_n(x_n) - \varphi(x_n) = \varepsilon_n$ such that

$$\varphi(x_n) + \varepsilon_n - f(x_n) - \Delta\varphi(x_n)|\nabla\varphi(x_n)|^{p_n-2} - (p_n - 2)\Delta_\infty\varphi(x_n)|\nabla\varphi(x_n)|^{p_n-4} - H_n(x_n, \varphi(x_n)) \geq 0.$$

Dividing the above inequality by $(p_n - 2)|\nabla\varphi(x_n)|^{p_n-4}$ and taking the limit with respect to n, we conclude that

$$-\Delta_\infty\varphi(x_0) - H_\infty(x_0, \nabla\varphi(x_0)) \leq 0.$$

Gathering (3.4) and (3.6), we conclude that v is a viscosity solution to

$$\max\left\{1 - |\nabla v|, -\Delta_\infty v - H_\infty(x, \nabla v)\right\} = 0 \qquad \text{in } \Omega_v^+ .$$

The proof that v solves

$$\min\left\{|\nabla v| - 1, -\Delta_\infty v - H_\infty(x, \nabla v)\right\} = 0 \qquad \text{in } \Omega_v^- ,$$

follows similarly to the previous case.

Let us focus now on the boundary conditions. For the sake of brevity we only prove the first one of the four inequalities (1.15)–(1.16), since the proofs of the other ones follow in the same way.

Assume that there exists $\varphi \in C^2(\overline{\Omega})$ such that $v - \varphi$ has a strict local maximum at x_0, with $v(x_0) = \varphi(x_0)$. Since $\{v_n\}$ converges uniformly in $\overline{\Omega}$ to v, there exist a sequence of real numbers $\varepsilon_n \to 0$ and a sequence of points $\overline{\Omega} \ni x_n \to x_0$ such that $v_n - \varphi - \varepsilon_n$ attains a local maximum at x_n and $v_n(x_n) = \varphi(x_n) + \varepsilon_n$.

Let us assume that (up to a subsequence) $\{x_n\} \subset (\partial\Omega)_v^+$; we recall that since any v_n belongs to $C^{1,\alpha}(\overline{\Omega})$ then

$$\frac{\partial v_n(x)}{\partial\nu} = 0, \qquad \forall x \in \partial\Omega.$$

Using that $\varphi(x) + \varepsilon_n$ touches v_n from above at x_n, we have that

$$\frac{\partial\varphi(x_n)}{\partial\nu} \leq \frac{\partial v(x_n)}{\partial\nu} = 0.$$

Passing to the limit as $n \to \infty$, it follows

$$\frac{\partial\varphi(x_0)}{\partial\nu} \leq 0.$$

On the contrary, if we assume that $\{x_n\} \subset \Omega_v^+$, we take advantage to the fact that v_n is a viscosity subsolution to (2.2), i.e. inequality (3.5) holds true. Going back to the proof of Theorem 1.5 we observe that $|\nabla\varphi(x_0)| < 1$ yields to a a contradiction and that $|\nabla\varphi(x_0)| - 1 \geq 0$ implies that $-\Delta_\infty\varphi(x_0) - H_\infty\big(x, \nabla\varphi(x_0)\big) \leq 0$, as desired. $\qquad\square$

4. The limit equation for the model problem

Now, let us go back to the limit as $p \to \infty$ to (1.10) with $H \equiv 0$.

Proof of Theorem 1.2. Since $u \in \mathbb{X}(\Omega)$, by Proposition 1.9 we have that both

$$1 - |\nabla u| \geq 0 \qquad \text{and} \quad |\nabla u| - 1 \leq 0 \qquad \text{in } \Omega.$$

The reverse inequalities in Ω_u^+ and Ω_u^- follow directly from Theorem 1.5.

As far as the boundary conditions are concerned, notice that inequality

$$\max\left\{ 1 - |\nabla u|, \frac{\partial u}{\partial \nu} \right\} \geq 0 \quad \text{on } (\partial \Omega)_u^+ ,$$

follows directly from Theorem 1.5. In order to prove that

(4.1) $$1 - |\nabla u| \leq 0 \text{ on } (\partial \Omega)_u^+ ,$$

assume by contradiction that there exists $\varphi \in C^2$ that touches u from above at x_0 and that verifies

$$1 - |\nabla \varphi(x_0)| > 0.$$

Take a small r such that $B_r(x_0) \cap \partial \Omega \subset (\partial \Omega)^+$ and $|\nabla \varphi(y)| < 1$ for all $|y - x_0| \leq r$ and $B_r(x_0)$. Therefore, set

$$\mu := \inf_{x \in \partial B_r(x_0) \cap \Omega} \left(\varphi(x) - u(x) \right) > 0 \quad \text{and} \quad \eta := \inf_{x \in B_r(x_0) \cap \Omega} \left(\varphi(x) - f(x) \right) > 0$$

and consider, for $\varepsilon := \min\{\mu, \eta\}$, the following function

$$\tilde{u} = \begin{cases} \min\{u, \varphi - \varepsilon\} & \text{in } B_r(x_0) \cap \overline{\Omega} , \\ u & \text{in } \overline{\Omega} \setminus B_r(x_0) . \end{cases}$$

Thanks to the definition of ε, $\tilde{u}$ is continuous, $\tilde{u} > f$ in A^+ and $\tilde{u}(x_0) < u(x_0)$. Moreover the choice of r also implies that $\tilde{u} \in \mathbb{X}(\Omega)$. This would imply $J_\infty(\tilde{u}) < J_\infty(u)$, that contradicts the minimality of u. Therefore (4.1) holds true.

To deal with the boundary condition on $(\partial \Omega)^-$, we can follow exactly the same strategy and hence we omit the details. $\qquad \square$

Even if the minimizer of (2.4) is unique, in general problem (1.6)–(1.7) possesses more than one solution. In the following we provide a solution to (1.6), that satisfies the same boundary condition, and that differs from the minimizer of J_∞ (unless $f \in \mathbb{X}(\Omega)$).

Proof of Theorem 1.3. Let us consider the set

$$M = \{v \in \mathbb{X}(\Omega) \ : \ v \geq f\},$$

and notice that it is not empty since the constant $\|f\|_{L^\infty(\Omega)} \in M$, and let us set

$$\alpha = \inf\left\{ \int_\Omega v \ : \ v \in M \right\}.$$

By definition of M, we have that $\alpha \geq \int_\Omega f$ and we consider a sequence $\{v_n\} \subset M$ whose integral on Ω converges to α. Observe that for any $n \in \mathbb{N}$, we have that $v_n \geq f$ and moreover one can assume, without loss of generality, that $v_n \leq \|f\|_{L^\infty(\Omega)}$ (otherwise it is sufficient to truncate v_n at high $\|f\|_{L^\infty(\Omega)}$). Hence the sequence $\{v_n\}$ is equibounded.

Moreover, since $\{v_n\} \subset \mathbb{X}(\Omega)$, then it is also equicontinuous. Thus by Ascoli Arzelà Theorem, we have that v_n converges uniformly in $\overline{\Omega}$ to some $\overline{u} \in \mathbb{X}$, that satisfies $\alpha = \int_\Omega \overline{u}$.

Next, we prove that $\overline{u}$ turns out to be a viscosity solution to (1.8). Thanks to Proposition 1.9, we immediately deduce that $\overline{u}$ satisfies

$$1 - |\nabla \overline{u}| \geq 0 \quad \text{in } \Omega.$$

Hence, to conclude the proof, we take any $x_0 \in \overline{\Omega_u^+} \cap (\partial\Omega)_u^+$ and assume by contradiction that there exists $\varphi \in C^2$ that touches $\overline{u}$ from above at x_0 and that satisfies

$$1 - |\nabla\varphi(x_0)| > 0.$$

Since φ is smooth, there exists a small $r > 0$ such that $|\nabla\varphi(y)| < 1$ for all $y \in B_r(x_0) \cap \overline{\Omega}$. Therefore, we set

$$\mu := \inf_{x \in \partial B_r(x_0) \cap \Omega} \left(\varphi(x) - \overline{u}(x) \right) > 0 \qquad \text{and} \qquad \eta := \inf_{x \in B_r(x_0) \cap \Omega} \left(\varphi(x) - f(x) \right) > 0$$

and we consider

$$\tilde{u} = \begin{cases} \min\{\overline{u}, \varphi - \varepsilon\} & \text{in } B_r(x_0), \\ \overline{u} & \text{in } \Omega \setminus B_r(x_0), \end{cases} \qquad \text{with} \qquad \varepsilon := \min\{\mu, \eta\}.$$

Thus, thanks to the definition of ε, $\tilde{u}$ is continuous, $\tilde{u} > f$ in $\Omega_{\overline{u}}^+$ and $\tilde{u}(x_0) < \overline{u}(x_0)$. Moreover the choice of r also implies that $\tilde{u} \in \mathbb{X}(\Omega)$. However, by construction this would imply that $\int_\Omega \tilde{u} < \alpha$, that yields to a contradiction. Hence, we obtain that $1 - |\nabla\varphi(x_0)| \leq 0$.

It remains to verify that, for any $x_0 \in \partial\Omega_{\overline{u}}^+$ and any $\varphi \in C^2$ that touches $\overline{u}$ from below at x_0, we have

$$\max\left\{ 1 - |\nabla\varphi(x_0)|, \frac{\partial\varphi(x_0)}{\partial\nu} \right\} \geq 0.$$

Assume by contradiction that $1 - |\nabla\varphi(x_0)| < 0$ and $\frac{\partial\varphi(x_0)}{\partial\nu} = -\nabla\varphi(x_0) \cdot \nabla\delta(x_0) < 0$.

This latter condition implies that $\nabla\varphi(x_0)$ directs inside Ω; moreover since $|\nabla\varphi(x_0)| > 1$, we deduce that there exists $y_0 \in \Omega$ (close enough to x_0) such that $\varphi(y_0) - \varphi(x_0) > |y_0 - x_0|$. Using that φ touches $\overline{u}$ from below at x_0 we have that

$$\overline{u}(y_0) - \overline{u}(y_0) \geq \varphi(y_0) - \varphi(y_0) > |y_0 - x_0|,$$

that yields to a contradiction.

Finally, the uniqueness of the solution of the obstacle problem follows from the minimality of $\overline{u}$. $\qquad \square$

Remark 4.1. *Analogously to* (1.8) *one can consider the solution to*

$$(4.2) \qquad \begin{cases} \underline{u} \leq f & \text{in } \Omega, \\ |\nabla\underline{u}(x)| - 1 \leq 0 & \text{in } \Omega, \\ |\nabla\underline{u}(x)| - 1 = 0 & \text{if } \Omega_{\underline{u}}^-, \\ \min\left\{ |\nabla\underline{u}| - 1, \frac{\partial\underline{u}}{\partial\nu} \right\} \leq 0 & \text{on } (\partial\Omega)_{\underline{u}}^-. \end{cases}$$

Arguing as before, we have that also $\underline{u}$ solves (1.6).

Remark 4.2. As already observed, we have that in general

$$\underline{u}(x) \leq f(x) \leq \overline{u}(x) \qquad \text{in } \Omega,$$

while the variational solution u is such that $u - f$ cannot have constant sign (unless $f \in \mathbb{X}(\Omega)$).

Proof of Proposition 1.6. Let us define

(4.3)
$$v^+(x) := \max_{y \in \partial\Omega_w^+} \left[f(y) - d(x, y) \right], \quad \forall x \in \Omega_w^+ ,$$
$$v^-(x) := \min_{y \in \partial\Omega_w^-} \left[f(y) + d(x, y) \right], \quad \forall x \in \Omega_w^- .$$

We prove, at first, that v^+ solves $1 - |\nabla v^+| = 0$ and that $-\Delta_\infty v^+ \le 0$ in Ω_w^+. Thanks to Lemma 2.6 (we recall that $v^+ \in \mathbb{X}(\Omega^+)$) we already now that $1 - |\nabla v^+| \ge 0$. Notice now that for any $y \in \partial\Omega_w^+$, the function

$$x \to f(y) - d(x, y)$$

is locally a cone (recall that the geodesic distance coincides locally with the euclidean one). Therefore, it solves the eikonal equation and it is infinity harmonic outside its vertex. Since v^+ is defined as maximum of such functions, the stability properties of viscosity solutions (see Proposition 4.3 of [24]) imply that $1 - |\nabla v^+| \le 0$ and $-\Delta_\infty v^+ \le 0$ in Ω_w^+.

A similar argument proves that $|\nabla v^-| - 1 = 0$ and $-\Delta_\infty v^- \ge 0$ in Ω_w^-.

Consequently, we found that

$$\max\left\{ 1 - |\nabla v^+|, -\Delta_\infty v^+ \right\} = 0, \quad \text{and} \quad 1 - |\nabla v^+| = 0 \quad \text{in } \Omega_w^+ ,$$

$$\min\left\{ |\nabla v^-| - 1, -\Delta_\infty v^- \right\} = 0 \quad \text{and} \quad |\nabla v^-| - 1 = 0 \quad \text{in } \Omega_w^- .$$

Let us assume now that w solves

(4.4)
$$\begin{cases} \max\left\{ 1 - |\nabla w|, -\Delta_\infty w \right\} = 0 & \text{in} \quad \Omega_w^+ , \\ \min\left\{ |\nabla w| - 1, -\Delta_\infty w \right\} = 0 & \text{in} \quad \Omega_w^- . \end{cases}$$

Thanks to the comparison principle (see Theorem 2.1 of [27]) we have that

$$w \equiv v^\pm \quad \text{in} \quad \Omega_w^\pm ,$$

respectively. Thanks to the properties of $v^\pm$, we deduce that w is also a solution to

(4.5)
$$\begin{cases} 1 - |\nabla w| = 0 & \text{in } \Omega_w^+ , \\ |\nabla w| - 1 = 0 & \text{in } \Omega_w^- . \end{cases}$$

On the other hand if w is a solution to (4.5) then again for the comparison principle (see Theorem 5.9 of [5]) $w \equiv v^\pm$ in $\Omega_w^\pm$ and then it also solves (4.4). $\qquad\square$

5. A FUNCTIONAL ALLOWING JUMPS

We start this section recalling some basic facts abound the *space of special functions of bounded variation $SBV(\Omega)$*. For more details we address the interested reader to the overview [26] or the classical monograph [2].

Let us recall that a function $v \in L^1(\Omega)$ belongs to $BV(\Omega)$ (bounded variation) if and only if there exists a vector valued Radon measure Du such that

$$\int_\Omega v \, \text{div}(\varphi) = -\int_\Omega \varphi \, d(Dv) \quad \forall \, \varphi \in C_c^\infty(\Omega, \mathbb{R}^N).$$

Thanks to the Radon-Nikodym Theorem, we can uniquely decompose Dv as $Dv = Dv^a + Dv^s$ with

$$Dv^a \ll \mathcal{H}^N \quad \text{and} \quad Dv^s \perp \mathcal{H}^N.$$

To give a better description of the absolutely continuous part and the singular one, we need to introduce some more concepts. For any $x \in \Omega$, we say that $v \in BV(\Omega)$ admits approximate limit $\bar{v}(x) \in \mathbb{R}$ if

$$(5.1) \qquad \lim_{\rho \to 0} \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} |v(y) - \bar{v}(x)| dy = 0$$

and approximate gradient $\nabla v(x) \in \mathbb{R}^N$ if

$$\lim_{\rho \to 0} \frac{1}{\rho |B_\rho(x)|} \int_{B_\rho(x)} |v(y) - \bar{v}(x) - \nabla v(x) \cdot (y - x)| dy = 0.$$

It can be proved (see Theorem 3.83 of [2]) that any $v \in BV(\Omega)$ is approximately differentiable $a.e.$ and that $\nabla v \in (L^1(\Omega))^N$ is the density of the absolutely continuous part of Dv with respect to the Lebesgue measure, namely

$$Dv^a = Dv \llcorner \mathcal{H}^N = \nabla v \mathcal{H}^N.$$

In order to describe also the singular part of Dv, let us consider the set $C_v \subset \Omega$ of all points where (5.1) holds true and denote $S_u = \Omega \setminus C_v$. The set S_v is called the set of approximated discontinuity of v and it is $(N-1)$-countably rectifiable. Now, we introduce

$$Dv^j = Dv \llcorner S_u \quad \text{and} \quad Dv^c = Dv \llcorner (\Omega \setminus S_u).$$

The measure Dv^j takes into account the jumps of the function v, while D^c is the Cantor part of Dv. We notice that in [2] Dv^j is defined through the slightly smaller set J_v, the set of approximated jumps point of v; since $\mathcal{H}^{N-1}(S_v \setminus J_v) = 0$ we keep using S_v for the sake of simplicity (see also [26]). At this point it is easy to define the space of special functions of bounded variation as

$$SBV(\Omega) = \{u \in BV(\Omega) \ : \ D^c u \equiv 0\}.$$

Now we are ready to define our functional $\mathcal{I} : SBV(\Omega) \to \mathbb{R}$ as

$$(5.2) \qquad \mathcal{I}_p(v) = \frac{1}{p} \int_\Omega |\nabla v|^p + \frac{1}{2} \int_\Omega |v - f|^2 + \mathcal{H}^{N-1}(S_v).$$

Comparing (5.2) with (1.18), we note that the new funcional framework allows to pass from a two variable functions to a single variable one, where the discontinuity set K is replaced by the singular (jump) set S_u.

We also stress that $BV(\Omega)$ would not be a good ambient space for (5.2) since the functional is not coercive with respect to the cantor part of gradient measure. To be convinced that $SBV(\Omega)$ is the right ambient space to settle the minimization of $\mathcal{I}_p$, it is enough to look at the following result.

Theorem 5.1. *[Theorems 4.7 and 4.8 of [2]] Let $\{u_n\} \subset SBV(\Omega)$ such that*

$$\int_\Omega |\nabla v_n|^p + \mathcal{H}^{N-1}(S_{v_n}) + \|v_n\|_{L^\infty(\Omega)} \leq C.$$

Then there exists $v \in SBV(\Omega)$ such that, up to a subsequence $v_n \to v$ strongly in $L^1(\Omega)$, $\nabla v_n \rightharpoonup \nabla v$ weakly in $L^p(\Omega)$, $Dv_n \rightharpoonup Dv$ weakly star in the sense of measure and

$$\int_\Omega |\nabla v|^p \leq \liminf_{n \to \infty} \int_\Omega |\nabla v_n|^{p_n} \quad \text{and} \quad \mathcal{H}^{N-1}(S_v) \leq \liminf_{n \to \infty} \mathcal{H}^{N-1}(S_{v_n}).$$

Theorem 5.1 easily implies the existence of a minimizer $u_p \in SBV(\Omega)$ for (5.2). Indeed it is easy to see that any minimizing sequence v_n can be chosen so that $\|v_n\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)}$, basically because truncations with a large constant always decrease the energy of the functional.

What is not clear is that if the couple (S_{u_p}, u_p) is a minimizer of (1.18). The main difficulty here is that in general the singular set of a SBV function need not to be closed and can be even dense in Ω. What is possible to prove is that any minimizer $u_p \in SBV(\Omega)$ of (5.2) satisfies $\mathcal{H}^{N-1}(\overline{S}_{u_p} \setminus S_{u_p}) = 0$ and that indeed $(\overline{S}_{u_p}, u_p)$ is a minimizer of (1.18). We gather all this information in the following result.

Theorem 5.2. *[Theorems 7.15, 7.21 and 7.22 of [2]] Set $p \in (1, \infty)$, $r \in [1, \infty)$ and $f \in L^\infty(\Omega)$. Then there exists a minimizer $u_p \in SBV(\Omega)$ to (5.2). Moreover there exists θ depending on N and p such that*

$$\mathcal{H}^{N-1}(S_{u_p} \cap B_\rho(x)) > \theta \rho^{N-1}$$

for any $x \in \overline{S_{u_p}}$ and $\rho > 0$.

This lower bounds implies that $\mathcal{H}^{N-1}(\overline{S_u} \setminus S_u) = 0$. Thus the pair given by $K = \overline{S}_{u_p}$ and $u \in W^{1,p}(\Omega \setminus \overline{S}_{u_p})$ and is a minimizer of the classical *Mumford–Shah problem (1.18).*

Our next step is to prove that any sequence $\{u_n\} \subset SBV(\Omega)$ of minimizers of (5.2), associated to a sequence $p_n \to \infty$, converges (up to a subsequence) to a minimizer of

(5.3) $$M_\infty(v) = \frac{1}{r} \int_\Omega |v - f|^r + \mathcal{H}^{N-1}(S_v) \quad \text{with } v \in SBV(\Omega), \ |\nabla v| \leq 1.$$

Theorem 5.3. *Let $\{u_n\} \subset SBV(\Omega)$ be the sequence of minimizer of the functional (5.2) relative to a diverging sequence of real number p_n. Then any $u_\infty \in SBV(\Omega)$, accumulation point of $u_n \to u$ (in $L^1(\Omega)$), is a minimizer of (5.3).*

Proof. Recalling that $M_{p_n}(u_n) \leq M_{p_n}(0)$ and that $\|u_n\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)}$, it results

$$\frac{1}{p_n} \int_\Omega |\nabla u_n|^{p_n} + \|u_n\|_{L^\infty(\Omega)} + \mathcal{H}^{N-1}(S_{u_n}) \leq C\|f\|_{L^\infty(\Omega)}.$$

Using Hölder inequality we get

$$\left(\int_\Omega |\nabla u_n|^q \right)^{\frac{1}{q}} \leq \left(pC\|f\|_{L^\infty(\Omega)} \right)^{\frac{1}{p_n}} |\Omega|^{\frac{1}{q} + \frac{1}{p_n}}$$

for any $q < p_n$. Theorem 5.1 and a diagonal argument then implies that there exists $u_\infty \in SBV(\Omega)$, with $|\nabla u_\infty| \leq 1$ such that $u_n \to u_\infty$ strongly in any $L^q(\Omega)$, $\nabla u_n \rightharpoonup \nabla u_\infty$ weakly in any $(L^q(\Omega))^N$, with $q \in [1, \infty)$, $Du_n \rightharpoonup Du_\infty$ weakly star in the sense of measure and

(5.4) $$M_\infty(u_\infty) \leq \liminf_{n \to \infty} M_{p_n}(u_n).$$

Now we claim that

$$M_\infty(u_\infty) \leq M_\infty(z) \quad \forall z \in SBV(\Omega) \cap L^\infty(\Omega), \ |\nabla z| \leq 1.$$

This concludes the proof since, by a simple truncation argument, if reached, the infimum of J_∞ in $SBV(\Omega) \cap L^\infty(\Omega)$ has to be bounded. By definition of u_n we have that

(5.5) $$M_{p_n}(u_n) \leq M_{p_n}(z) \leq \frac{1}{p_n}|\Omega| + \frac{1}{2} \int_\Omega |z - f|^2 + \mathcal{H}^{N-1}(S_z).$$

Taking the liminf with respect to n we obtain the desired result. □

Notice that in the proof of Theorem 5.3 we have proved that the functional M_{p_n} Γ-converges to the functional M_∞. For completeness let us recall the definition of Γ-convergence.

Definition 5.4. *Let $F_n : \mathbb{X} \to \mathbb{R}$ be a sequence of functionals. We say that F_n Γ–converges to $F : \mathbb{X} \to \mathbb{R}$ if*

i) *for any $x \in \mathbb{X}$ and any $x_n \to x$, we have that*
$$\liminf_{n \to +\infty} F_n(x_n) \geq F(x);$$

ii) *for any $x \in \mathbb{X}$, there exists $x_n \subset \mathbb{X}$ such that $x_n \to x$, and*
$$\limsup_{n \to +\infty} F_n(x_n) \leq F(x).$$

In our case $i)$ has been proved in (5.4) while $ii)$ is trivially obtained taking the limit in the right hand side of (5.5).

6. EXAMPLES

In this section we collect some explicit examples that clarify the behavior of the minimizers of some of the functional presendet in this manuscript.

Examples 6.1. *Suppose that $\Omega = (-1, 1)$, we are interested in considering the following minimization problem:*
$$\inf_{\mathbb{X}(-1,1)} J_\infty(v) \qquad with \qquad J_\infty(v) = \frac{1}{2} \int_{-1}^{1} (v - f)^2,$$

Case 1. *Choose $f_1(x) = k\chi_{(-r,r)}(x)$ with $r \in (0, 1]$ and $k \in \mathbb{R}$ (see Figure 1).*
We first observe that if $r = 1$ then $f_1 \in \mathbb{X}(-1, 1)$ and thus the choice $u_1(x) \equiv f_1(x)$ is allowed.
Conversely, if $r \in (0, 1)$, then $f_1 \notin \mathbb{X}(-1, 1)$, and we look for a solution $u_1(x)$ of the form $u_h(x) = \min\{(h - |x|)_+, k\}$, with h to be determined. Thus $J_\infty(u_h)$ can be explicitly computed and it turns out that
$$J_\infty(u_h) = \int_{h-k}^{r} \left(h - k - x\right)^2 dx + \int_{r}^{h} \left(h - x\right)^2 dx.$$

Minimizing its value with respect to h, we have that the minimum is achieved at $h^* = r + \frac{k}{2}$.
The heuristic that motivates this example is that in a neighbor where f is not smooth the solution is a line with maximum slope allowed (i.e. ± 1) and far away it tries to be as close as possible to f.

Case 2. *Consider $f_2(x) = 2|x|$ (see Figure 2).*
In this case the picture is much more clear. Since $f_2 \notin \mathbb{X}(-1, 1)$, the closest function to f_2 in $\mathbb{X}(-1, 1)$ is a line with slope $+1$ in $(0, 1]$ and with slope -1 in $[-1, 0]$. Since f_2 is even, u_2 inherits the same property, so that
$$J_\infty(u_h) = \int_{0}^{1} \left(2x - (x + h)\right)^2 dx,$$

attains its minimum value for $h = \frac{1}{2}$.

Case 3. *Consider* $f_3(x) = \sqrt{|x|}$ *(see Figure 3).*

By symmetry it follows that the solution is even and that it has the form

$$u_3(x) = \begin{cases} |x| - s + \sqrt{s} & \text{for } |x| \leq s \\ \sqrt{|x|} & \text{for } |x| > s, \end{cases}$$

with $s > \frac{1}{4}$. *To determine the value of* s *we have to compute*

$$\frac{\partial}{\partial s} J_\infty(u) = \frac{\partial}{\partial s} \int_0^s (\sqrt{|x|} - |x| + s - \sqrt{s})^2 dx$$

$$= 2\left(1 - \frac{1}{2\sqrt{s}}\right) \int_0^s (\sqrt{|x|} - |x| + s - \sqrt{s}) dx = 2\left(1 - \frac{1}{2\sqrt{s}}\right)\left(\frac{s^2}{2} - \frac{s^{\frac{3}{2}}}{3}\right).$$

Therefore we deduce that the minimum is achieved for $s = \frac{4}{9}$.

Example 6.2. *Let us underline the importance of minimizing* J_∞ *in* $SBV(\Omega)$ *by showing that for certain* f, *when it is allowed, the minimizer does jump.*

In order to avoid technicalities, we present first full details in dimension 1, being the extension of these ideas to any dimension quite similar.

As already observed in the previous example the minimizer of

$$J_\infty(u) = \frac{1}{2} \int_{-R}^{R} \left(u(x) - k\chi_{(-r,r)}\right)^2 dx \qquad \text{with } u \in \mathbb{X}(-R, R)$$

is

$$u_h(x) = \min\left\{(h - |x|)_+, k\right\},$$

with $h^* = r + \frac{k}{2}$. *In addition, we have*

$$J_\infty(u_{h^*}) = \frac{1}{12}k^3.$$

Observe that, since such an $f \in SBV(-R, R)$, *we can consider*

$$\mathcal{I}(v) = \frac{1}{2} \int_{-R}^{R} \left(u(x) - k\chi_{(-r,r)}\right)^2 dx + \mathcal{H}(S_u) \qquad \text{with } u \in SBV(-R, R)$$

with $\mathcal{H}(S_u)$ *the measure of the singular set of* u. *Observe that* $\mathcal{H}(S_f) = 2$ *and since* $f \in SBV(-R, R)$ *thus* $\mathcal{I}(f) = 2$. *Consequently if*

$$2 < \frac{1}{12}k^3,$$

we have

$$\mathcal{I}(f) < J_\infty(u_h^*),$$

and we conclude that the minimizer indeed has jumps.

As far as the case in dimension $N \geq 2$ *is concerned, we repeat the same argument: in the domian* $\Omega = B_R(0)$ *we set*

$$f(x) = k\chi_{B_r(0)}(x)$$

with $r > 0$ *to be chosen large enough.*

Thus the minimizer of

$$J_\infty(u) = \frac{1}{2} \int_{B_R(0)} (u(x) - k\chi_{B_r(0)})^2 dx \qquad with\ u \in \mathbb{X}(B_R(0))$$

has the form

$$u_h(x) = \min\left\{k, (h - |x|)_+\right\}.$$

The minimum value of $J_\infty(u_h)$ is achieved for a suitable value $h^* > 0$ that is not so easy to be computed explicitly and which satisfies

$$\lim_{r \to \infty} \frac{r}{h^*} = 1.$$

Thus, tedious but not difficult computations yield to

$$J_\infty(u_{h^*}) \sim \frac{k^3}{24}\omega_N r^3 \qquad as\ r \to +\infty\,.$$

Analogously, it is not hard to see that for the functional

$$\mathcal{I}(u) = \frac{1}{2} \int_{B_R(0)} (u(x) - k\chi_{B_r(0)})^2 dx + \mathcal{H}^{N-1}(S_u) \qquad with\ u \in SBV(\Omega)$$

since $f \in SBV(\Omega)$ we have that

$$\mathcal{I}(f) = N\omega_N r^{N-1}\,.$$

Consequently, if r is large enough and $k^3 > 24N$, then it holds that

$$\mathcal{I}(f) = N\omega_N r^{N-1} < J_\infty(u_0),$$

and the minimizer jumps.

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