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First Order Limiting Optimality Conditions in the Coefficient Control of an Obstacle Problem

Nicolai Simon[1,*] and **Winnifried Wollner**[1,**]

[1] Universität Hamburg, Fachbereich Mathematik, Bundesstr. 55, 20146 Hamburg, Germany

We consider an optimal control problem governed by an obstacle problem. The novelty and focus of this work will be the introduction of a control variable into the coefficients of this optimal control problem.

In particular, we discuss the effects of the multiplicative coupling between control and, derivatives of, the state. This prevents the use of standard weak convergence arguments in limit arguments. As it is known, H-convergence needs to be utilized, e.g., to prove existence of a solution to the problem. To obtain optimality conditions, we present a regularization approach to circumvent the non-differentiability of the solution operator of the obstacle problems. We then discuss the limiting optimality conditions obtained by the proposed regularization.

1 Introduction

One of the fundamental examples in the study of variational inequalities are obstacle problems. As free boundary problems they naturally occur in a variety real world applications and thus have been a subject in the research of variational inequalities by researchers, e.g., [1, 2] for an overview. From a theoretical standpoint obstacle problems themselves are also interesting because the obstacle introduces a complementarity constraint and the solution operator of the problem is not Gateaux-differentiable, see, e.g., [3].

Problems containing complementarity constraints cannot, in general, fulfill standard constraint qualifications, see, e.g., [4]. As a result several alternative approaches for stationarity arguments have been proposed, see, e.g., [5, 6]. In our case we focus on Clarke stationarity, see [7], which requires weaker constraint qualifications. The second challenge when considering obstacle problems is the aforementioned lack of Gateaux-differentiability of the solution operator. Again there are several approaches to dealing with these kinds of problems, see, e.g., [8, 9]. In this work we utilize a regularization approach as seen in, e.g., [5, 8, 10, 11]. This approach allows us to compute optimality conditions by considering the limits of optimality conditions for a regularized problem.

Our own contribution to this topic is focused on the effects of introducing a matrix valued coefficient control into the obstacle problem. This consideration introduces additional challenges to the approach described above. Notably the multiplicative coupling of control and state in the main part of the operator forces us to consider products of weakly converging sequences when formulating limit arguments. As products of weakly converging sequences do not need to converge against the product of the weak limits of control and state respectively, we introduce new arguments based on H-convergence. H-convergence as presented by Murat and Tartar, see, e.g., [12,13], allows us to obtain limits that satisfy the, limiting, variational inequality. This is done by utilizing H-sequential compactness and H-semicontinuity properties of the considered problems, see, e.g., [14, 15]. The same approach allows to prove existence for the considered problem as it will be discussed in the forthcoming work [16].

As a result, computing optimality conditions can be quite challenging. The reason is that the H-limit is not given as an, a.e., pointwise limit, and thus extraction of the H-limit from a numerical computation is challenging. However, utilizing a bootstrapping argument, we can show that, given some natural properties, we can achieve strong convergence of the coupled term in the appropriate L^p-space thus allowing us to numerically obtain close approximations to the limit objects.

We begin by introducing the basic problem formulation and the H-convergence property in Section 2. Based on that we also introduce the regularization of the problem and its properties. In Section 3, we provide optimality conditions subject to, semidefinite, box constraints on the control and provide convergence results of the regularized solutions and optimality conditions. The detailed proofs will be provided in a forthcoming publication.

* Corresponding author: e-mail nicolai.simon@uni-hamburg.de

** e-mail winnifried.wollner@uni-hamburg.de

2 Obstacle Problem with Control in the Coefficients

We consider an optimal control problem governed by an obstacle problem for a bounded domain $\Omega \subset \mathbb{R}^2$ similar to the problem considered in [16]. It is given by

$$\begin{aligned} \min_{q,u}\ & J(q,u) = \frac{1}{2}\|u-u_d\|^2 + \frac{\alpha}{2}\|q\|^2 \\ \text{s.t. } & (q\nabla u, \nabla(v-u)) \geq (f, v-u) \qquad \forall v \in K \\ & u \in K, \qquad q \in Q^{\mathrm{ad}} \end{aligned} \tag{$\mathrm{P_{VI}}$}$$

with $u_d \in L^2(\Omega)$. The set of admissible states K and admissible controls Q^{ad} is given by

$$K = \left\{u \in H_0^1(\Omega) \,\middle|\, u \geq \psi\right\},$$
$$\text{and} \quad Q^{\mathrm{ad}} = \left\{q \in L^2\left(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}}\right) \,\middle|\, 0 \prec q_{\min}\,\mathrm{I} \preccurlyeq q(x) \preccurlyeq q_{\max}\,\mathrm{I} \text{ a.e.}\right\}.$$

Here, $\psi \in \mathbb{R}$ with $\psi < 0$ is a given obstacle and $0 < q_{\min} < q_{\max} \in \mathbb{R}$ are given control bounds. For symmetric 2×2 matrices A, B the relation $A \prec B$, and $A \preccurlyeq B$, state that $B - A$ is, required to be, positive definite, or semi-definite, respectively.

Remark 2.1 It is important to note that any $q \in Q^{\mathrm{ad}}$ is also a uniformly positive definite matrix function and $q \in L^\infty(\Omega; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$. Hence, for any $q \in Q^{\mathrm{ad}}$, unique solvability of the variational inequality

$$(q\nabla u, \nabla(v-u)) \geq (f, v-u) \quad \forall v \in K, \tag{1}$$

to obtain the corresponding state u is ensured.

Since $K - \psi$ is a convex cone, we can equivalently write problem (1) as a complementarity system using, e.g., [1, Theorem I.5.5]. This gives the equivalent reformulation of $(\mathrm{P_{VI}})$ as an MPCC in function space:

$$\begin{aligned} \min_{q \in Q^{\mathrm{ad}}, u \in H_0^1(\Omega)} \ & J(q,u) \\ \text{s.t. } & -\nabla\cdot(q\nabla u) = f + \lambda, \\ & u \geq \psi \qquad\qquad \text{a.e. in } \Omega, \\ & \lambda \geq 0 \qquad\qquad \text{in } H^{-1}(\Omega), \\ & (\lambda, \psi - u) = 0. \end{aligned} \tag{$\mathrm{P_{CC}}$}$$

Due to the multiplicative coupling of control and state, direct methods using weak convergence to prove the existence of an optimal solution to this problem no longer apply as we can no longer verify (1) when considering limits of a minimizing sequence (q_k, u_k). Instead we will make use of H-convergence, see [12, Definition 1].

Definition 2.2 A sequence $q_k \in Q^{\mathrm{ad}}$ H-converges to $q^H \in Q^{\mathrm{ad}}$ (in symbols $q_k \xrightarrow{H} q^H$) if

$$q_k\nabla u_k \rightharpoonup q^H \nabla u \qquad \text{in } L^2(\Omega)$$

for any sequence $u_k \in H_0^1(\Omega)$ satisfying

$$u_k \rightharpoonup u \qquad \text{in } H_0^1(\Omega)$$
$$\text{and } \nabla\cdot(q_k\nabla u_k) \to g \qquad \text{in } H^{-1}(\Omega)$$

for some $u \in H_0^1(\Omega)$ and some $g \in H^{-1}(\Omega)$.

Using H-convergence it can be shown, see [16], that solutions to problem $(\mathrm{P_{CC}})$ exist, providing the following existence result.

Theorem 2.3 *There exists at least one global solution of* $(\mathrm{P_{CC}})$.

2.1 Regularization

The original problem is not Gateaux differentiable, so we want to regularize the obstacle constraint in $(\mathrm{P_{CC}})$. In this subsection we want to present such a regularization and some crucial properties relating to bounds on the different variables.

We consider the following regularized optimal control problem for a bounded domain $\Omega \subset \mathbb{R}^2$ given by

$$\begin{aligned} \min_{q_\gamma, u_\gamma}\ & J(q_\gamma, u_\gamma) \\ \text{s.t. } & -\nabla\cdot(q_\gamma\nabla u_\gamma) + r(\gamma; u_\gamma) = f \\ & u_\gamma \in H_0^1(\Omega), \quad q_\gamma \in Q^{\mathrm{ad}} \end{aligned} \tag{$\mathrm{P_\gamma}$}$$

where $r : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is based on a quadratic penalization of the energy functional for (1), i.e.,

$$r(\gamma; u_\gamma) := -\left[\max\left(\gamma\left(\psi - u_\gamma\right), 0\right)\right].$$

Remark 2.4 Note that, similar to the penalization in [8], the penalty

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \ni u_\gamma \mapsto r(\gamma; u_\gamma) \in L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$$

is a locally Lipschitz continuous and monotone Nemyzkii operator. Further, the control is a positive definite and symmetric operator, therefore the left-hand side of the PDE

$$-\nabla \cdot (q_\gamma \nabla u_\gamma) + r(\gamma; u_\gamma) = f \tag{2}$$

is, for each $q_\gamma \in Q^{\mathrm{ad}}$, Lipschitz-continuous and monotone as a mapping $H_0^1(\Omega) \to H^{-1}(\Omega)$. Applying the Browder-Minty theorem, see [17, Theorem 10.49] allows us to conclude that for each $q_\gamma \in Q^{\mathrm{ad}}$ the PDE has a unique solution $u_\gamma \in H_0^1(\Omega)$.

As usual, since the data of the problem are more regular than necessary, it is possible to obtain improved smoothness of the solutions. The following bounds are crucial in our limit considerations of the problem, for details, see, [16].

Lemma 2.5 *The solutions $u_\gamma \in H_0^1(\Omega)$ of (2) are bounded independent of the choice $q_\gamma \in Q^{\mathrm{ad}}$. In particular, it holds*

$$\|\nabla u_\gamma\| \le \frac{c}{q_{\min}} \|f\|$$

with a constant c independent of γ and $q_\gamma \in Q^{\mathrm{ad}}$.

Lemma 2.6 *For every $q_\gamma \in Q^{\mathrm{ad}}$ and corresponding solution $u_\gamma \in H_0^1(\Omega)$ of (2) it holds $\nabla \cdot (q_\gamma \nabla u_\gamma) \in L^2(\Omega)$ and the bound*

$$\|\nabla \cdot (q_\gamma \nabla u_\gamma)\| + \|r(\gamma, u_\gamma)\| \le c\|f\|$$

holds true with a constant c independent of γ and $q_\gamma \in Q^{\mathrm{ad}}$.

For the regularized problem we can also employ the notion of H-convergence to prove the existence of an optimal solution to the optimal control problem, similar to Theorem 2.3.

Theorem 2.7 *There exists at least one solution for* (P_γ).

Further, given the assumption that our problem is regular in the sense of Gröger, see [19], we can utilize this property to conclude that the regularized state u_γ satisfies the additional regularity $u_\gamma \in W^{1,p}(\Omega)$ for some $2 \le p < \infty$, see [20].

3 Stationarity Conditions

Now we provide stationarity conditions for the regularized problem (P_γ) in Proposition 3.1. Starting from this, we will discuss the limit in the regularization in Theorems 3.4 and 3.5. Finally, we provide limiting necessary optimality conditions in Theorem 3.5. The approach is similar to, e.g., [5, 8, 11].

First we provide the optimality conditions for the regularized problem.

Proposition 3.1 *Let $(\bar{q}_\gamma, \bar{u}_\gamma) \in Q^{\mathrm{ad}} \times H_0^1(\Omega)$ be a local optimum of Problem (P_γ). Then there exists $\bar{\lambda}_\gamma, \bar{\mu}_\gamma \in L^2(\Omega)$ and $\bar{p}_\gamma, \bar{\theta}_\gamma \in H_0^1(\Omega)$ such that*

$$-\nabla \cdot (\bar{q}_\gamma \nabla \bar{u}_\gamma) = f + \bar{\lambda}_\gamma, \tag{3a}$$

$$\bar{\lambda}_\gamma + r(\gamma, \bar{u}_\gamma) = 0, \tag{3b}$$

$$-\nabla \cdot (\bar{q}_\gamma \nabla \bar{p}_\gamma) = \bar{u}_\gamma - u_d - \bar{\mu}_\gamma, \tag{3c}$$

$$\bar{\mu}_\gamma - \partial_u r(\gamma, \bar{u}_\gamma)\bar{\theta}_\gamma = 0, \tag{3d}$$

$$\bar{p}_\gamma - \bar{\theta}_\gamma = 0, \tag{3e}$$

$$(\alpha\bar{q}_\gamma - \nabla\bar{u}_\gamma \otimes \nabla\bar{p}_\gamma, q - \bar{q}_\gamma) \ge 0 \qquad \forall q \in Q^{\mathrm{ad}}. \tag{3f}$$

Remark 3.2 The variables λ_γ, θ_γ and μ_γ can be eliminated from the conditions above. They are introduced here to allow us to more easily map the optimality conditions to our original problem, and compare the later introduced first order limiting optimality conditions with similar conditions found in literature, see, e.g., [4, 8].

Remark 3.3 Finally, we note that we can reformulate the control condition (3f) as a local projection onto Q^{ad} using the techniques introduced in [18]. As usual, we note that (3f) is equivalent to the inequality

$$(\alpha\bar{q}_\gamma(x) - \nabla\bar{u}_\gamma(x) \otimes \nabla\bar{p}_\gamma(x), q - \bar{q}_\gamma(x)) \geq 0 \qquad \forall q_{\min}\,\mathrm{I} \preccurlyeq q \preccurlyeq q_{\max}\,\mathrm{I},\ \text{a.e. } x \in \Omega$$

by the usual localization argument. Hence, for almost all $x \in \Omega$,

$$\bar{q}_\gamma(x) = P_{\mathrm{ad}}\left(\frac{1}{\alpha}\nabla\bar{u}_\gamma(x) \otimes \nabla\bar{p}_\gamma(x)\right)$$

where $P_{\mathrm{ad}} \colon \mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ is the orthogonal projection onto the set

$$\mathcal{Q}_{\mathrm{ad}} = \{q \in \mathbb{R}^{2\times 2}_{\mathrm{sym}} \mid q_{\min}\,\mathrm{I} \preccurlyeq q \preccurlyeq q_{\max}\,\mathrm{I}\}.$$

To provide an explicit form for this projection, we consider the eigenvalue decomposition $q = S^T D S$ for a symmetric matrix q where $D = \mathrm{diag}(\lambda_1(q), \lambda_2(q))$ is a diagonal matrix consisting of the eigenvalues of q and S is an orthogonal matrix. Then we map this matrix onto $\mathcal{Q}_{\mathrm{ad}}$ using the component wise projection $P_{[q_{\min}, q_{\max}]}(x) = \max\{q_{\min}, \min\{x, q_{\max}\}\}$ of the diagonal elements, i.e., we define

$$\widetilde{D} = \mathrm{diag}\left(P_{[q_{\min}, q_{\max}]}(\lambda_1(q)), P_{[q_{\min}, q_{\max}]}(\lambda_2(q))\right) \in \mathcal{Q}_{\mathrm{ad}}.$$

Now, we can define the projection

$$P_{\mathrm{ad}}(q) = S^T \widetilde{D} S \in \mathcal{Q}_{\mathrm{ad}}.$$

To see that this is indeed the case, let $w \in \mathcal{Q}_{\mathrm{ad}}$ be an arbitrary matrix. Then

$$\begin{aligned}(P_{\mathrm{ad}}(q) - q, w - P_{\mathrm{ad}}(q)) &= (S^T\widetilde{D}S - S^T D S, w - S^T\widetilde{D}S)\\ &= (\widetilde{D}S - DS, Sw - \widetilde{D}S)\\ &= (\widetilde{D} - D, SwS^T - \widetilde{D})\\ &= \sum_{i=1}^{2}\left(P_{[q_{\min}, q_{\max}]}\left(\lambda_i(q)\right) - \lambda_i(q)\right)\left(\tilde{w}_{ii} - P_{[q_{\min}, q_{\max}]}\left(\lambda_i(q)\right)\right)\end{aligned} \tag{4}$$

where $\tilde{w} = SwS^T$. To proceed, we need to see that the diagonal elements of $\tilde{w}$ remain within the bounds $[q_{\min}, q_{\max}]$. To see this, notice, that

$$\begin{aligned} q_{\min} &\leq \lambda_{\min}(\tilde{w})\\ &= \inf_{\|v\|=1}(v, \tilde{w}v)\\ &\leq (e_i, \tilde{w}e_i)\\ &= \tilde{w}_{ii}\\ &\leq \sup_{\|v\|=1}(v, \tilde{w}v)\\ &= \lambda_{\max}(w) \leq q_{\max}.\end{aligned}$$

With this, we see that the summands in (4) are zero whenever $\lambda_i \in [q_{\min}, q_{\max}]$ since the first difference is zero. In the other cases, either $P_{[q_{\min}, q_{\max}]}\left(\lambda_i(q)\right) = q_{\min}$ and both factors are non-negative or $P_{[q_{\min}, q_{\max}]}\left(\lambda_i(q)\right) = q_{\max}$ and both factors are non-positive. From this we conclude that our definition of P_{ad} in (4) yields

$$(P_{\mathrm{ad}}(q) - q, w - P_{\mathrm{ad}}(q)) \geq 0 \quad \forall w \in \mathcal{Q}_{\mathrm{ad}}$$

and thus the formula for the projection is shown.

Based on the stationarity conditions for the regularized problem, we now want to pass to the limit to find optimality conditions for the original problem. First we note that a series of optimal controls of the regularized problems H-converges towards a global minimizer of the original problem.

Theorem 3.4 *For every $\gamma > 0$ there is a globally optimal solution to the penalized problem denoted by $(\bar{q}_\gamma, \bar{u}_\gamma)$. If $\gamma \to \infty$ then every sequence $\{\bar{q}_\gamma\}$ of global minimizers of (P_γ) admits a H-accumulation point $q^H \in Q^{\mathrm{ad}}$. The corresponding solution $(q^H, \bar{u})$ is a global minimizer of Problem* $(\mathrm{P_{CC}})$.

Note that without further arguments H-convergence only provides the weak convergence

$$\bar{q}_\gamma \nabla \bar{u}_\gamma \rightharpoonup q^H \nabla \bar{u} \text{ in } L^2(\Omega)$$

where the control $\bar{q}_\gamma$ H-converges towards the H-limit q^H. While the H-limit of the control does not necessarily coincide with its weak limit, we can utilize the properties of our problem to bootstrap H-convergence and prove stronger convergence properties of the control and the state resulting in strong convergence of the coupled term. In turn, similar arguments can be applied to the adjoint equation as well.

On basis of these improved convergence results, we consider the limits of the optimality conditions given in Proposition 3.1 to formulate a set of first order limiting optimality conditions for the obstacle problem with control in the coefficients.

Theorem 3.5 *If $\gamma \to \infty$ then every sequence of global optimizers $\bar{q}_\gamma$ of (P_γ) admits a H-accumulation point $\bar{q}$. Every H-accumulation point $\bar{q}$ is also a strong accumulation point in $L^p(\Omega)$ for all $2 \le p < \infty$, i.e., for a suitable subsequence $\gamma \to \infty$ it holds*

$$\bar{q}_\gamma \to \bar{q} \qquad \text{in } L^p(\Omega) \text{ for all } 2 \le p < \infty, \tag{5a}$$

and the associated sequence of solutions to problem (P_γ) fulfills

$$\begin{aligned}
\bar{u}_\gamma &\to \bar{u} && \text{in } W^{1,p}(\Omega) \text{ for an } 2 < p < \infty, && (5b)\\
\bar{q}_\gamma \nabla \bar{u}_\gamma &\to \bar{q} \nabla \bar{u} && \text{in } L^2(\Omega). && (5c)\\
\bar{p}_\gamma &\to \bar{p} && \text{in } W^{1,p}(\Omega) \text{ for an } 2 < p < \infty, && (5d)\\
\bar{q}_\gamma \nabla \bar{p}_\gamma &\to \bar{q} \nabla \bar{p} && \text{in } L^2(\Omega), && (5e)\\
\nabla \bar{u}_\gamma \otimes \nabla \bar{p}_\gamma &\to \nabla \bar{u} \otimes \nabla \bar{p} && \text{in } L^s(\Omega) \text{ for an } 1 < s < \infty, && (5f)\\
\bar{\theta}_\gamma &\to \bar{\theta} && \text{in } W^{1,p}(\Omega) \text{ for an } 2 < p < \infty, && (5g)\\
\bar{\lambda}_\gamma &\to \bar{\lambda} && \text{in } H^{-1}(\Omega), && (5h)\\
\bar{\mu}_\gamma &\to \bar{\mu} && \text{in } H^{-1}(\Omega) && (5i)
\end{aligned}$$

with suitably defined limit quantities $(\bar{u}, \bar{\lambda}, \bar{p}, \bar{\mu}, \bar{\theta})$. Further, the limit $(\bar{q}, \bar{u}, \bar{\lambda}, \bar{p}, \bar{\mu}, \bar{\theta})$ satisfies the optimality system

$$\begin{aligned}
-\nabla \cdot (\bar{q} \nabla \bar{u}) &= f + \bar{\lambda}, && (6a)\\
\bar{u} \ge \psi \text{ a.e. in } \Omega, \bar{\lambda} &\ge 0 \text{ in } H^{-1}(\Omega), \left(\bar{\lambda}, \bar{u} - \psi\right) = 0, && (6b)\\
-\nabla \cdot (\bar{q} \nabla \bar{p}) &= \bar{u} - u_d - \bar{\mu}, && (6c)\\
\left(\bar{\theta}, \bar{\lambda}\right) = 0, (\bar{\mu}, \psi - \bar{u}) &= 0, \left(\bar{\theta}, \bar{\mu}\right) \ge 0, && (6d)\\
\bar{p} - \bar{\theta} &= 0, && (6e)\\
(\alpha \bar{q} - \nabla \bar{u} \otimes \nabla \bar{p}, q - \bar{q}) &\ge 0 \qquad \forall q \in Q^{\mathrm{ad}}. && (6f)
\end{aligned}$$

4 Conclusion

In this work, we studied the inclusion of an $L^2\left(\Omega\,; \mathbb{R}^{2\times 2}_{\mathrm{sym}}\right)$ control in the coefficients of an obstacle problem. As the solution operator of the obstacle problem is not Gateaux differentiable, we introduced a regularization scheme to compute optimality conditions. To this effect, we computed bounds independent of the regularization parameter for both, the gradient of the state, as well as the main part of the PDE. Based on these bounds, we made use of H-convergence properties to utilize a bootstrapping argument and prove strong $L^2(\Omega)$ convergence of the coupled terms. We then used these convergence properties to present first order limiting optimality conditions for the obstacle problem with coefficient control by considering the limits of the regularized optimality system. The detailed proofs for these results will be made available in a forthcoming publication. Further, we demonstrated how to reformulate the control conditions of the optimality systems, i.e. (3f) and (6f), in terms of a local projection onto Q^{ad}.

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