SOME BOUNDARY-VALUE PROBLEMS FOR THE EQUATION $\nabla.(|\nabla\phi|^N\nabla\phi)=0$

By C. ATKINSON† *and* C. R. CHAMPION

(*Department of Mathematics, Imperial College, London* SW7 2BZ)

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SUMMARY

Boundary-value problems for the equation $\nabla.(|\nabla\phi|^N \nabla\phi)=0$, $N>-1$, in two space dimensions, are considered which relate to notch and crack problems in the theory of deformation plasticity under conditions of anti-plane strain. The results are also of relevance to problems in nonlinear theories of diffusion and filtration.

1. Introduction

THE equation

$$\nabla.(|\nabla\phi|^N \nabla\phi)=0, \qquad N>-1, \tag{1.1}$$

occurs in the theory of deformation plasticity (two space dimensions) under longitudinal shear (anti-plane strain deformation). Equation (1.1) also occurs in steady-state heat conduction (or diffusion) in a medium with a nonlinear heat conductivity (diffusivity). The version of (1.1) with $\partial\phi/\partial t$ on the right-hand side has been called N-diffusion by Philip (**1**) and models of this kind have been proposed in polymer chemistry. See (**2**), (**3**) and (**4**) for a discussion and qualitative properties of some transient diffusion problems of this type. Equations similar to (1.1) also occur in the nonlinear theory of filtration.

Our interest in this paper is with longitudinal shear problems of deformation plasticity but it is worth noting, for situations involving geometrical singularities such as wedges (notches) (to be considered in section 2), that the angular dependence of the dominant part of the solution near the wedge tip should be the same for both steady-state and transient problems.

We begin, in section 2, by considering the eigenvalue problem of solving (1.1) in two space dimensions with homogeneous boundary conditions given on an infinite wedge of included angle $2\theta_0$. Making the assumption that we can separate variables as a product of a function of θ and a function of r in an (r,θ)-coordinate system we obtain a nonlinear eigenvalue problem involving a nonlinear ordinary differential equation in θ. It can be shown by explicit calculation, excluding certain trivial solutions, that there are in general precisely two such eigenvalues. One of these eigenvalues gives a value of ϕ which is physically reasonable (i.e. finite) at the wedge tip, the

† At present at Schlumberger Cambridge Research, Cambridge, England.

other eigenvalue giving a value of ϕ which is infinite as we approach the wedge vertex. The second of these solutions can be interpreted as the solution at infinity for certain problems involving wedges and is, of course, the exact solution for a blunted wedge with particular conditions on the end region. The first of these solutions is relevant to the dominant field at the edge tip for a variety of loading conditions and is the exact solution in a finite wedge with appropriate boundary conditions. Each of the exact solutions obtained in section 2 can be used as comparison functions when deriving qualitative properties of the solutions of more complicated problems via comparison theorems.

The rest of the paper, section 3, is concerned with deriving exact solutions for cracks in regions in which conditions are specified on boundaries at a finite distance from the crack tip. We begin by giving a complete analysis of a semi-infinite crack in a strip for which fixed displacements are applied to the strip sides (Fig. 1). The leading term in an expansion of the near-crack stress and displacement field can be obtained for this geometry by an alternative method based on a simple energy argument. However, by using the hodograph transformation we show that this problem maps into a region of the hodograph plane such that a complete solution can be obtained merely by inverting an appropriate Mellin transform. It is perhaps worth noting that the problem in the hodograph plane is much simpler than that in the real plane even for the linear case $N=0$. The next problem we solve is that of an edge crack in a three-sided box (see Fig. 3) with displacement boundary conditions on the sides $y=\pm h$ and the side $x=-a$ stress free. With this configuration we have a mapped region in the hodograph plane for which the edge crack problem can be analysed by means of Mellin transform methods and the Wiener–Hopf technique. It should be noted, however, that an unknown boundary point in the mapped plane must also be accounted for. A similar method has been used by Amazigo ((**5**), (**6**) and (**7**)) in a series of papers treating anti-plane crack problems in media with powerlaw stress-strain relationships. The significant feature of the problem treated in section 3.3 that distinguishes it from those discussed by Amazigo in (**5**), (**6**) and (**7**) is that the displacement loading is applied at a finite distance from the crack (note that a notch problem could also be treated in an analogous way) and thus a realistic loading situation is analysed. Results for stress intensity as a function of crack length to strip width are presented in Fig. 5. A natural generalisation of the problem in section (3.3) is to consider a crack in a finite box. However, such a problem leads to a functional equation in the hodograph plane for which explicit solution does not seem possible. Work is in progress to develop efficient numerical methods of treating such problems in the mapped plane.

2. Eigensolutions for wedges and notches in nonlinear power-law elasticity

In the analysis of nonlinear diffusion, or power-law elastic problems where notched regions are involved, the dominant behaviour at the tip of a wedge is important. Even though the details of the boundary conditions on the wedge sides may vary we expect that the angular dependence of the dominant field near the notch (or wedge) tip will have a unique form. In order to determine this expression and the associated power of the radial field variation we attempt to solve the nonlinear eigenvalue problem outlined here. It turns out that an exact solution to this eigenvalue problem can be obtained and that in general there are two distinct eigenvalues. Only one of these eigenvalues corresponds to a finite displacement at the wedge vertex. The other eigenvalue, giving a singular behaviour at the vertex, is, we believe, connected with the behaviour at infinity for problems of wedges (or infinite notches) loaded appropriately on the wedge (or notch) sides.

2.1. *Analysis*

We consider a wedge (or notch)-shaped region occupying ($-\theta_0 \leqslant \theta \leqslant \theta_0$, $r > 0$) where $0 < \theta_0 \leqslant \pi$. If we are close enough to the apex of the wedge to be away from the applied stresses, the edges of the wedge must be stress-free. Also, using symmetry, we can let the displacement be zero on the line $\theta = 0$. Hence the boundary conditions near the wedge apex become

$$w_\theta(r, \theta_0) = 0, \tag{2.1}$$

and

$$w(r, 0) = 0. \tag{2.2}$$

The function $w(r, \theta)$ satisfies equation (1.1) with ϕ replaced by w. Correspondingly, we seek solutions of (1.1) in the form $w(r, \theta) = r^\lambda F(\theta)$, with $F(\theta)$ satisfying the conditions

$$F'(\theta_0) = 0, \tag{2.3}$$

and

$$F(0) = 0. \tag{2.4}$$

Substitution into (1.1) leads to the following nonlinear ordinary differential equation in $F(\theta)$

$$\{\lambda^2 F^2 + (N+1)(F')^2\}F'' + \{(2N+1)\lambda^2 - N\lambda\}F(F')^2 + \lambda^3\{(N+1)\lambda - N\}F^3 = 0. \tag{2.5}$$

(2.5) leads to

$$d(F')/dF = \frac{-[\lambda\{(2N+1)\lambda - N\}(F'/F) + \lambda^3\{(N+1)\lambda - N\}(F/F')]}{[\lambda^2 + (N+1)(F'/F)^2]} \tag{2.6}$$

Writing $P = F'/F$ and integrating we obtain

$$F(\theta) = K\,|P^2(\theta) + \lambda^2|^{-\lambda/2}\,|P^2(\theta) + \lambda^2 - N\lambda/(N+1)|^{(\lambda-1)/2} \tag{2.7}$$

where K is an arbitrary constant. As $d\theta/dP = PF/(dF/dP)$, (2.7) gives

$$d\theta/dP = -\lambda/(P^2 + \lambda^2) + (\lambda - 1)/\{P^2 + \lambda^2 - N\lambda/(N+1)\}. \tag{2.8}$$

If we look for solutions for which $\lambda^2 - N\lambda/(N+1) > 0$, then (2.8) yields

$$\theta(P) = -\tan^{-1}(P/\lambda) - [(1-\lambda)/\mu]\tan^{-1}(P/\mu) + C, \tag{2.9}$$

where C is an arbitrary constant and

$$\mu^2 = \lambda^2 - N\lambda/(N+1) \tag{2.10}$$

with the principal value of $\tan^{-1} x$ being implied.

If we consider $F(\theta)$ to be positive in $0 < \theta < \theta_0$, then $F'(\theta)$, and hence $P(\theta)$, will also be positive in this range. Hence, using (2.3) and (2.4), we obtain the following conditions on $P(\theta)$

$$P(\theta) \to +\infty \quad \text{as} \quad \theta \to 0^+, \tag{2.11}$$

$$P(\theta) \to 0^+ \quad \text{as} \quad \theta \to \theta_0. \tag{2.12}$$

Consider now the case $\lambda > 0$. Condition (2.12) implies that $C = \theta_0$, (2.11) leads to the following eigenvalue equation

$$(1-\lambda)/\mu = \delta(\theta_0), \tag{2.13}$$

where

$$\delta(\theta_0) = 2(\theta_0 - \pi/2)/\pi, \qquad -1 < \delta \leq 1. \tag{2.14}$$

From (2.13) we find that λ is given by λ_1 where

$$\lambda_1 = [2(N+1) - N\delta^2 - \delta\{N^2\delta^2 + 4(N+1)\}^{\frac{1}{2}}]/2(N+1)(1-\delta^2) \tag{2.15}$$

and the corresponding value of P is, from (2.9), P_1, where

$$\theta = -\tan^{-1}(P_1/\lambda_1) - \delta\tan^{-1}\{\delta P_1/(1-\lambda_1)\} + \theta_0. \tag{2.16}$$

We find from (2.16) that θ must decrease montonically with P for $P \in [0, \infty)$, and hence (2.16) has a unique solution for $P(\theta) > 0$ on $0 < \theta \leq \theta_0$.

Equations (2.7), (2.15) and (2.16) together give the asymptotic behaviour of $w(r, \theta)$ at the tip of a wedge $0 < \theta_0 < \pi/2$ or notch $\pi/2 < \theta_0 < \pi$ when remote anti-plane strain has been applied. When $\theta_0 = \pi$, the solution will correspond to the behaviour of $w(r, \theta)$ near a crack tip.

We now look for the possibility of a solution with $\lambda < 0$. The equation to be satisfied by λ becomes

$$(1-\lambda)/\mu = l(\theta_0), \tag{2.17}$$

where

$$l(\theta_0) = 2(\theta_0 + \pi/2)/\pi, \qquad 1 < l \leq 3. \tag{2.18}$$

We find that λ is given by λ_2, where

$$\lambda_2 = [Nl^2 - 2(N+1) - l\{l^2N^2 + 4(N+1)\}^{\frac{1}{2}}]/2(l^2-1)(N+1) \tag{2.19}$$

and $P = P_2$, where

$$\theta = -\tan^{-1}(P_2/\lambda_2) - l\tan^{-1}[lP_2/(1-\lambda_2)] + \theta_0. \tag{2.20}$$

As before this equation can be shown to yield a unique solution $P(\theta)>0$ for $\theta\in(0,\theta_0)$. The equations (2.7), (2.19) and (2.20) will give the asymptotic behaviour of $w(r,\theta)$ at infinity for a wedge (or infinite notch) subjected to anti-plane shear applied to a finite part of the body.

2.2. *Special results*

(i) For rational values of θ_0/π equations (2.16) and (2.20) reduce to polynomial equations in $P(\theta)$ with coefficients in $\tan\theta$. We now exhibit these equations in the case $\theta_0=\pi$ i.e. the notch becomes a crack. We find

$$\lambda_1=(N+1)/(N+2), \tag{2.21}$$

$$\tan\theta=(N+2)^2P_1/\{(N+2)^2P_1^2-(N+1)\}, \tag{2.22}$$

$$\lambda_2=[(2N-2)-3\{9N^2+4N+4\}^{\frac{1}{2}}]/16(N+1), \tag{2.23}$$

$$\tan\theta=\frac{P_2^3+\mu_2^2(8\mu_2-3)P_2}{P_2^4-3\mu_2(1-2\mu_2^2)P_2^2+(1-3\mu_2)\mu_2^3}, \tag{2.24}$$

where

$$\mu_2=\tfrac{1}{3}(1-\lambda_2). \tag{2.25}$$

From (2.7), (2.21) and (2.22) we obtain the already known result for $w(r,\theta)$ as $r\to 0$, namely

$$w(r,\theta)\sim Kr^{\lambda_1}\{P_1^2(\theta)+\lambda_1^2\}^{-\frac{1}{2}\lambda_1}\{P_1^2(\theta)+(1-\lambda_1)^2\}^{-\frac{1}{2}(1-\lambda_1)} \tag{2.26}$$

with $\lambda_1=(N+1)/(N+2)$ and

$$P_1(\theta)=\left[\cos\theta+\left\{1-\left(\frac{N}{N+2}\right)^2\sin^2\theta\right\}^{\frac{1}{2}}\right]\Big/2\sin\theta. \tag{2.27}$$

(2.24) has an exact solution for $P_2(\theta)$ but we omit it here due to its complexity.

It should be noted that the θ-dependence of $w(r,\theta)$ exhibited in (2.25) is equivalent to that of (3.29) (see later) with n replaced by $1/(N+1)$. It should also be noted that (1.1) is derived from the power law relating stress and strain given in equation (3.3), again with n replaced by $1/(N+1)$.

(ii) We exhibit here the limiting behaviour of $w(r,\theta)$ on the wedge sides, and also the limiting behaviour of the y-component of strain $\gamma_{yz}[=(1/r)\,\partial w/\partial\theta]$ on the line $\theta=0$:

$$\left.\begin{aligned} w(r,\theta_0)&\sim\frac{K_i}{\mu_i}\left(\frac{\mu_i}{\lambda_i}\right)^{\lambda_i}r^{\lambda_i},\\ w_\theta(r,0)/r&\sim K_ir^{\lambda_i-1},\end{aligned}\right\}\quad\text{with}\quad\begin{cases} i=1, & r\to 0,\\ i=2, & r\to\infty.\end{cases} \tag{2.28}$$

3. Two problems involving cracks in nonlinear elastic strips loaded by longitudinal shear

In this section we solve two problems in anti-plane strain by the use of Mellin transforms in the hodograph plane. Both problems consist of a fully-plastic centred crack in a longitudinal strip subject to equal and opposite constant displacements along its edges. In the first problem (3.2) the crack is semi-infinite and the strip infinite. The second problem (3.3), of which the first is the limiting case as the crack length becomes infinite, has the crack finite and the strip semi-infinite, with the side of the strip stress-free. In the latter case we follow the method used by Amazigo (**5**, **6**, **7**) and proceed to a solution via the Wiener–Hopf technique.

3.1. *Fundamental equations*

Let the only non-vanishing component of displacement be the z-component $w(x, y)$. The only non-zero strain components will then be $\gamma_{xz} = \partial w/\partial x$ and $\gamma_{yz} = \partial w/\partial y$. For small deformations and an isotropic material the corresponding stresses τ_{xz} and τ_{yz} are the only nonzero stress components. For simplicity we introduce the notation $\gamma_{xz} = \gamma_x$, $\tau_{xz} = \tau_x$ etc. The compatibility and equilibrium equations become respectively

$$\partial\gamma_x/\partial y = \partial\gamma_y/\partial x \tag{3.1}$$

and

$$\partial\tau_x/\partial x + \partial\tau_y/\partial y = 0. \tag{3.2}$$

We assume a pure power-hardening stress-strain law given by

$$\gamma/\gamma_0 = \alpha(\tau/\tau_0)^n, \qquad \gamma_x/\gamma = \tau_x/\tau \quad \text{and} \quad \gamma_y/\gamma = \tau_y/\tau. \tag{3.3}$$

where α is a non-dimensional constant, γ_0 and τ_0 are reference values of the principal strains and principal stresses respectively, and n is the power-hardening parameter. The principal stresses and strains are

$$\tau = (\tau_x^2 + \tau_y^2)^{\frac{1}{2}}; \qquad \gamma = (\gamma_x^2 + \gamma_y^2)^{\frac{1}{2}}. \tag{3.4}$$

From (3.1), (3.2) and (3.3) we obtain the following equation for $w(x, y)$

$$|\nabla w|\,\nabla^2 w = (1 - 1/n)\,\nabla(|\nabla w|).\nabla w \tag{3.5}$$

Equation (3.5) is nonlinear in w and is not readily solvable by analytic methods. However, as shown in (**5**), the use of a hodograph transformation reduces the problem to a linear one. This transformation changes the roles of dependent variables (γ_x, γ_y) and independent variables (x, y) by use of implicit function theory. The compatibility and equilibrium equations become, respectively

$$\partial x/\partial\gamma_y = \partial y/\partial\gamma_x, \tag{3.6}$$

and

$$\partial x/\partial\tau_x + \partial y/\partial\tau_y = 0. \tag{3.7}$$

Equation (3.6) leads to the existence of a potential function ψ such that

$$\mathbf{x} = \nabla_\gamma \psi, \tag{3.8}$$

where $\mathbf{x}$ is the position vector (x, y) and ∇_γ is the gradient operator with respect to the strain vector $\boldsymbol{\gamma} = (\gamma_x, \gamma_y)$.

Finally, equations (3.3), (3.7) and (3.8) lead to the following linear partial differential equation in ψ

$$\psi_{\gamma_x\gamma_x} + \psi_{\gamma_y\gamma_y} + (n-1)\{\gamma_x^2\psi_{\gamma_x\gamma_x} + 2\gamma_x\gamma_y\psi_{\gamma_x\gamma_y} + \gamma_y^2\psi_{\gamma_y\gamma_y}\}/\gamma = 0. \tag{3.9}$$

3.2. *A semi-infinite centred crack in an infinite longitudinal strip*

We consider the problem of a fully-plastic semi-infinite crack situated centrally in an infinite longitudinal strip, the strip being subjected to equal and opposite constant displacements along its edges. The strip occupies the region $(-\infty < x < \infty, -h \leqslant y \leqslant h, z < \infty)$ (see Fig. 1) and the crack is given by $(-\infty < x < 0, y = 0, z < \infty)$. The edge displacements are $w = \pm w_0$ on $y = \pm h$ respectively, with w_0 and h both positive.

3.2.1. *The hodograph plane*

We proceed by introducing the non-dimensional quantities

$$\rho = h\gamma/w_0; \qquad \Psi = \psi/w_0, \tag{3.10}$$

and a (ρ, ϕ) coordinate system such that

$$h\gamma_x/w_0 = -\rho \sin \phi; \qquad h\gamma_y/w_0 = \rho \cos \phi. \tag{3.11}$$

We obtain the hodograph image of the strip as shown in Fig. 2. The points H, G and F correspond to $(h\gamma_x/w_0, h\gamma_y/w_0) = (0^-, 1^-), (0, 1^+), (0^+, 1^-)$ respectively.

Using (3.11), equation (3.9) becomes

$$n\Psi_{\rho\rho} + \Psi_\rho/\rho + \Psi_{\phi\phi}/\rho^2 = 0. \tag{3.12}$$

The boundary conditions in the real plane are

$$\gamma_x = 0^\mp, \qquad 0 < \gamma_y < w_0 \quad \text{on} \quad y = \pm h \quad (-\infty < x < \infty) \tag{3.13}$$

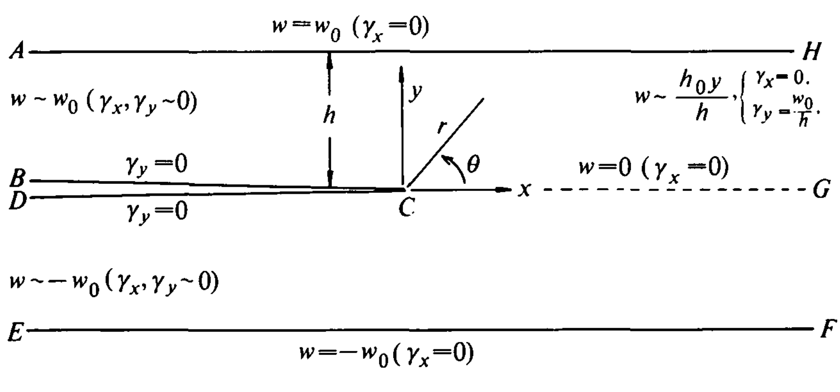

FIG. 1

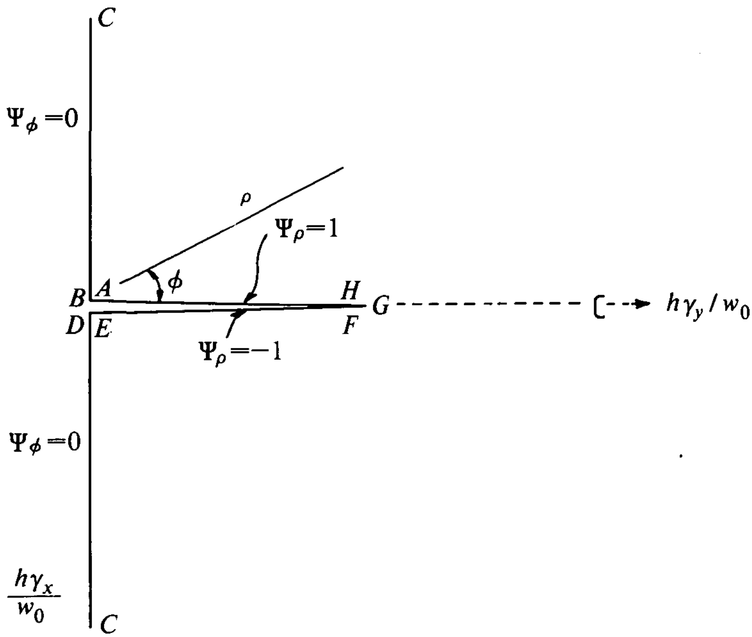

FIG. 2

and

$$\left.\begin{array}{llll}\gamma_y=0^+, & -\infty<\gamma_x<0 & \text{when} & y=0^+ \quad (-\infty<x<0),\\ \gamma_y=0^+, & 0<\gamma_x<\infty & \text{when} & y=0^- \quad (-\infty<x<0).\end{array}\right\} \tag{3.14}$$

If we now use the symmetry of the problem, we can reduce it to the solution of (3.12) in the region $0<\phi<\pi/2$, with $\Psi(\rho, \phi)$ satisfying the following boundary conditions

$$\left.\begin{array}{ll}\Psi_\rho(\rho, 0)=1, & 0<\rho<1,\\ \Psi(\rho, 0)=0, & 1<\rho<\infty,\end{array}\right\} \tag{3.15}$$

and

$$\Psi_\phi(\rho, \pi/2)=0, \qquad 0<\rho<\infty. \tag{3.16}$$

3.2.2. *The Mellin transform*

We take a Mellin transform of Ψ, namely

$$\bar{\Psi}(s, \phi)=\int_0^\infty \Psi(\rho, \phi)\rho^{s-1}\,d\rho. \tag{3.17}$$

To obtain the strip of regularity of $\bar{\Psi}(s, \phi)$, we look at the behaviour of

$\Psi(\rho, \phi)$ as $\rho \to \infty$ and $\rho \to 0$. From the already known behaviour of $w(r, \theta)$ near the crack tip in the real plane we obtain

$$\Psi \to \rho^{-1/n} \quad \text{as} \quad \rho \to \infty. \tag{3.18}$$

For the validity of the transform we assume that $\Psi = O(\rho^{-1/n})$ as $\rho \to 0$ i.e. we assume there exists an α such that $\bar{\Psi}$ exists in $\alpha < \text{Re}\, s < 1/n$ with $\alpha < 1/n$.

Transforming the differential equation results in the following equation for $\bar{\Psi}$

$$\bar{\Psi}_{\phi\phi} + \omega^2(s)\bar{\Psi} = 0, \qquad \alpha < \text{Re}\, s < 1/n \tag{3.19}$$

where

$$\omega^2(s) = s\{n(s+1) - 1\}. \tag{3.20}$$

Transforming (3.16) we obtain

$$\bar{\Psi}_{\phi}(s, \pi/2) = 0. \tag{3.21}$$

Before we transform (3.15) it must be noted that $\Psi_{\phi}(\rho, 0) = 1$, $0 < \rho < 1$ yields

$$\Psi(\rho, 0) = \rho + c, \qquad 0 < \rho < 1, \tag{3.22}$$

where c is an arbitrary constant. However, an expansion of Ψ about $(\rho = 1, \phi = 0)$ shows that Ψ must be continuous across $\rho = 1$ along the line $\phi = 0$, otherwise the condition that ψ_{γ_y} be bounded as $\rho \to 1$ would be violated. We thus conclude that $c = -1$. Hence, transforming (3.15), we obtain

$$\bar{\Psi}(s, 0) = -1/s(s+1). \tag{3.23}$$

Equations (3.19), (3.21) and (3.23) yield

$$\bar{\Psi}(s, \phi) = -\cos\{\omega(s)(\pi/2 - \phi)\}/s(s+1)\cos\{\pi\omega(s)/2\}, \qquad \alpha < \text{Re}\, s < 1/n. \tag{3.24}$$

The singularity below $s = 1/n$ is $s = 0$ and we conclude that $\alpha = 0$. Hence, from the inversion formula for the Mellin transform we have

$$\Psi(\rho, \phi) = -\frac{1}{2\pi i}\int_{q-i\infty}^{q+i\infty} \frac{\cos\{\omega(s)(\pi/2 - \phi)\}}{s(s+1)\cos\left\{\omega(s)\dfrac{\pi}{2}\right\}}\rho^{-s}\, ds, \qquad 0 < q < 1/n. \tag{3.25}$$

To evaluate the integral we apply Cauchy's residue theorem. The integrand has simple poles at $s = s_k^{\pm}$, $1/n$, 0 where

$$s_k^{\pm} = \tfrac{1}{2}[1/n - 1 \pm \{(1/n - 1)^2 + 4(2k+1)^2/n\}^{\frac{1}{2}}], \qquad k \geq 1, \tag{3.26}$$

and a double pole at $s = -1$.

We find, for $\phi > 0$

$$\left.\begin{aligned}\Psi(\rho, \phi) &= -1 - \frac{4}{\pi(n+1)}\rho \log \rho \sin \phi + \\ &\quad + \frac{\rho}{\pi(n+1)^2}\left[(3+6n-n^2)\sin\phi + 2(n+1)^2\left(\frac{\pi}{2}-\phi\right)\cos\phi\right] + \\ &\quad + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(2k+1)\sin(2k+1)\phi}{s_k^-(s_k^-+1)(2ns_k^-+n-1)}\rho^{-s_k^-}, \qquad 0<\rho<1, \\ \Psi(\rho, \phi) &= -\frac{4n^2}{\pi(n+1)^2}\rho^{-1/n}\sin\phi \\ &\quad - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(2k+1)\sin(2k+1)\phi}{s_k^+(s_k^++1)(2ns_k^++n-1)}\rho^{-s_k^+}, \qquad \rho>1. \end{aligned}\right\} \tag{3.27}$$

3.2.3. *Asymptotic results in the real plane*

(i) Behaviour near the crack tip—the stress intensity factor.

The behaviour of $w(r, \theta)$ near the crack tip is linked to the behaviour of $\Psi(\rho, \phi)$ as $\rho \to \infty$. We find, as $r \to 0$

$$\begin{pmatrix}\tau_x \\ \tau_y\end{pmatrix} \sim K_{\mathrm{III}}^{(1)}\left(\frac{l(\theta)}{r}\right)^{1/(n+1)}\begin{pmatrix}-\sin\phi \\ \cos\phi\end{pmatrix} \tag{3.28}$$

and

$$w(r, \theta) \sim \frac{n+1}{n}\left(\frac{K_{\mathrm{III}}^{(1)}}{\tau_1}\right)^n\left(\frac{l(\theta)}{r}\right)^{-1/(n+1)}\sin\phi, \tag{3.29}$$

where

$$l(\theta) = \frac{n+1}{n}\frac{\sin 2\phi}{2\sin\theta}; \qquad 2\phi = \theta + \sin^{-1}\left[\left(\frac{n-1}{n+1}\right)\sin\theta\right], \tag{3.30}$$

and $K_{\mathrm{III}}^{(1)}$ is the stress-intensity factor, given by

$$K_{\mathrm{III}}^{(1)} = \tau_1\left(\frac{w_0}{h}\right)^{1/n}\left(\frac{4n^2h}{\pi(n+1)^2}\right)^{1/(n+1)}, \qquad \tau_1 = \tau_0(\alpha\gamma_0)^{-1/n}. \tag{3.31}$$

(Note that $l(0) = 1$.)

(ii) Behaviour of $w(x, y)$ as $x \to -\infty$, $0<y<h$.

The asymptotic behaviour of $w(x, y)$ as $x \to -\infty$, $0<y<h$, is governed by the form of $\Psi(\rho, \phi)$ as $\rho \to 0$, $0<\phi<\pi/2$. We find

$$w(x, y) \sim w_0 - \frac{4w_0}{\pi(n+1)}\exp\left[\frac{\pi(n+1)}{4h}\{x+f(\phi)\}\right]\sin\phi \tag{3.32}$$

where $\phi = \phi(y)$ is given by

$$2\phi - \left(\frac{n-1}{n+1}\right)\sin 2\phi = \pi\left(1 - \frac{y}{n}\right), \qquad 0 < y < h, \tag{3.33}$$

and $f(\phi)$ satisfies

$$f(\phi) = \frac{2n(1-n)}{\pi(1+n)} + \left(\frac{1-n}{1+n}\right)\cos 2\phi. \tag{3.34}$$

(iii) Behaviour as $x \to \infty$.

The form of the solution as $x \to \infty$ can be obtained by considering the behaviour of $\Psi(\rho, \phi)$ near the point $(\rho, \phi) = (1, 0)$. By converting the integral in (3.25) to a Fourier transform and using results from (**8**) to obtain an asymptotic form of $\Psi(\rho, \phi)$ near $(\rho, \phi) = (1, 0)$ we can show that as $x \to \infty$

$$\left.\begin{aligned} \gamma_y &\to \frac{w_0}{h}\left[1 - e^{-\pi x/n^{\frac{1}{2}}h}\right] \quad \text{on} \quad y = h, \\ \gamma_y &\to \frac{w_0}{h}\left[1 + e^{-\pi x/n^{\frac{1}{2}}h}\right] \quad \text{on} \quad y = 0, \\ w &\to \frac{w_0 y}{h} + O\left(e^{-\pi x/n^{\frac{1}{2}}h}\right) \quad \text{on} \quad 0 < y < h. \end{aligned}\right\} \tag{3.34b}$$

3.3. *A finite centred crack in a semi-infinite longitudinal strip*

In this section we consider the strip having a stress-free boundary at $x = -a$ i.e. the crack is finite and of length a (see Fig. 3). Using symmetry this problem could be interpreted as a crack of length $2a$ in the centre of an infinite strip.

3.3.1. *The hodograph plane*

Defining $\gamma_y(-a, h) = \gamma_A > 0$ and $\epsilon = w_0/h\gamma_A > 1$ we obtain the image of the strip in the hodograph plane as shown in Fig. 4, where A and H

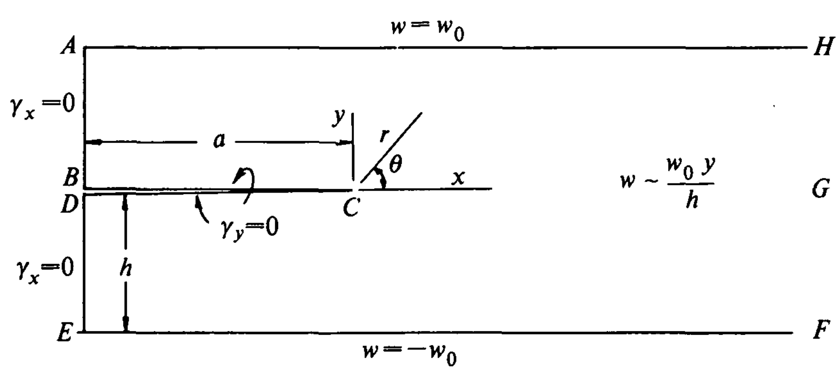

FIG. 3

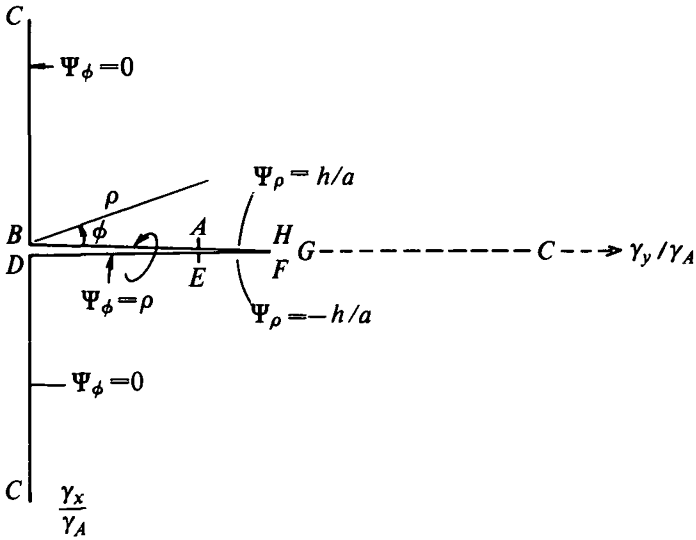

FIG. 4

correspond to the points $(0^-, 1)$ and $(0^-, \epsilon)$ respectively, and

$$\Psi=\psi/ah, \qquad \gamma_x/\gamma_A=-\rho\sin\phi, \qquad \gamma_y/\gamma_A=\rho\cos\phi. \tag{3.35}$$

The problem reduces to solving

$$n\Psi_{\rho\rho}+\Psi_\rho/\rho+\Psi_{\phi\phi}/\rho^2=0 \quad \text{in} \quad 0<\rho<\infty, \qquad 0<\phi<\pi/2, \tag{3.36}$$

with boundary conditions

$$\left.\begin{aligned} \Psi_\phi(\rho,0)&=\rho, & 0&<\rho<1,\\ \Psi(\rho,0)&=h(\rho-\epsilon)/a, & 1&<\rho<\epsilon,\\ \Psi(\rho,0)&=0, & \epsilon&<\rho<\infty, \end{aligned}\right\} \tag{3.37}$$

and

$$\Psi_\phi(\rho,\pi/2)=0, \qquad \rho>0. \tag{3.38}$$

In (3.37) we have used the fact that $\Psi(\rho,0)$ must be continuous at $\rho=\epsilon$ for the solution to be valid.

3.3.2. *The Mellin transform*

As in (3.2), $\Psi\to\rho^{-1/n}$ as $\rho\to\infty$. Also, a consideration of Ψ in the nieghbourhood of the origin (see (5)) shows that Ψ is bounded as $\rho\to 0$. Hence the Mellin transform will exist in the strip $0<\operatorname{Re}s<1/n$.

From (3.36), and (3.38) we find,

$$\bar{\Psi}(s,\phi)=C(s)\frac{\cos[\omega(s)(\pi/2-\phi)]}{\cos\pi\omega(s)/2}, \qquad 0<\operatorname{Re}s<1/n, \tag{3.39}$$

where $C(s)$ is an arbitrary constant. Writing

$$\Psi(\rho, 0) = g(\rho), \qquad 0<\rho<1, \tag{3.40}$$

and

$$\Psi_\phi(\rho) = u(\rho), \qquad 1<\rho<\infty, \tag{3.41}$$

we obtain

$$\bar{\Psi}(s, 0) = \frac{h}{a}\left(\frac{\epsilon^{s+1}-1}{s+1}+\frac{\epsilon-\epsilon^{s+1}}{s}\right)+\bar{g}_+(s), \tag{3.42}$$

and

$$\bar{\Psi}_\phi(s, 0) = \frac{1}{s+1}+\bar{u}_-(s), \tag{3.43}$$

where

$$\bar{u}_-(s) = \int_1^\infty u(\rho)\rho^{s-1}\,d\rho, \qquad \operatorname{Re} s<1/n, \tag{3.44}$$

and

$$\bar{g}_+(s) = \int_0^1 g(\rho)\rho^{s-1}\,d\rho, \qquad \operatorname{Re} s>0. \tag{3.45}$$

Equations (3.39), (3.42) and (3.43) lead to

$$\frac{h}{a}\left(\frac{\epsilon^{s+1}-1}{s+1}+\frac{\epsilon-\epsilon^{s+1}}{s}\right)_+ +\bar{g}_+(s) = \left[\left(\frac{1}{s+1}\right)_+ +\bar{u}_-(s)\right]P(s), \qquad 0<\operatorname{Re} s<1/n, \tag{3.46}$$

where

$$P(s) = \omega^{-1}(s)\cot \pi\omega(s)/2 \tag{3.47}$$

and a plus sign denotes a function analytic in $\operatorname{Re} s>0$, a minus sign a function analytic in $\operatorname{Re} s<1/n$.

We now apply the Wiener–Hopf technique (see e.g. (**9**)) to equation (3.46). Note that the first term on the left-hand side is an entire function and can in principle be retained as an entity during our analysis. However the application of the Wiener–Hopf technique will yield an entire function with complicated behaviour at $s=\infty$ if this term is not split up.

3.3.3. *Solution of the Wiener–Hopf problem*

We proceed by first decomposing $P(s)$ into the quotient

$$P(s) = N_-(n, s)/R_+(n, s), \tag{3.48}$$

where $N_-(n, s)$ has no poles or zeros for $\operatorname{Re} s<1/n$, and $R_+(n, s)$ has no poles or zeros for $\operatorname{Re} s<0$. As shown in (**5**), a convenient decomposition has

$$N_-(n, s) = 2^{-s\sqrt{n}}\prod_{k=1}^{\infty}(\gamma_{2k-1}^+ - a_{2k-1}s)\exp(a_{2k-1}\bar{s})\Big/\prod_{k=1}^{\infty}(\gamma_{2k}^+ - a_{2k}s)\exp(a_{2k}\bar{s}) \tag{3.49}$$

and

$$R_+(n,s)=\frac{\pi}{2}ns(s+1-1/n)2^{-s\sqrt{n}}\prod_{k=1}^{\infty}(a_{2k}s-\gamma_{2k}^-)\exp(-a_{2k}\bar{s})\Big/\prod_{k=1}^{\infty}(a_{2k-1}s-\gamma_{2k-1}^-)\exp(-a_{2k-1}\bar{s}) \quad (3.50)$$

where

$$\bar{s}=s+(n-1)/2n, \qquad a_m=n^{\frac{1}{2}}/m, \quad (3.51)$$

and

$$\gamma_m^{\pm}=n^{\frac{1}{2}}[1/n-1\pm\{(1-1/n)^2+4m^2/n\}^{\frac{1}{2}}]/2m. \quad (3.52)$$

Asymptotically we have

$$N_-(n,s)\sim(-\pi s/2)^{\frac{1}{2}}n^{\frac{1}{4}}2^{(n-1)/2\sqrt{n}}, \qquad |s|\to\infty, \qquad \operatorname{Re}s<1/n \quad (3.53)$$

and

$$R_+(n,s)\sim(\pi/2)^{\frac{1}{2}}s^{\frac{3}{2}}n^{\frac{3}{4}}2^{(n-1)/2\sqrt{n}}, \qquad |s|\to 0, \qquad \operatorname{Re}s>0 \quad (3.54)$$

Substituting (3.48) into (3.46) gives

$$\bar{g}_+(s)R_+(n,s)+\frac{h}{a}\left(\frac{\epsilon}{s}-\frac{1}{s+1}\right)_+R_+(n,s)-\frac{h}{a}\frac{\epsilon^{s+1}}{s(s+1)}R_+(n,s)$$
$$=\frac{N_-(n,s)}{s+1}+\bar{u}_-(s)N_-(n,s), \qquad 0<\operatorname{Re}s<1/n. \quad (3.55)$$

The first term on the right-hand side of (3.55) can be written as

$$\frac{N_-(n,s)}{s+1}=\left(\frac{N_-(n,-1)}{s+1}\right)_++\left(\frac{N_-(n,s)-N_-(n,-1)}{s+1}\right)_-. \quad (3.56)$$

Consider the third term on the left-hand side of (3.55). We decompose this term by subtracting away the singular parts of its Laurent expansions at each pole of the function. We note that the function has simple poles at $s=b_{2k+1}^-$, $(k=1,2,\ldots)$, and a double pole at $s=b_1^-=-1$, where

$$b_m^{\pm}=\gamma_m^{\pm}/a_m. \quad (3.57)$$

By writing $R_+(n,s)=N_-(n,s)\omega(s)\sin\{\omega(s)\pi/2\}/\cos\{\omega(s)\pi/2\}$, we obtain the singular parts of $\epsilon^{s+1}R_+(n,s)/s(s+1)$ in the form

$$\sum_{k=1}^{\infty}\frac{D_k}{s-b_{2k-1}^-}+\frac{D_0}{(s+1)^2}, \quad (3.58)$$

where

$$D_k=-\frac{4}{\pi}\frac{\epsilon^{1+b_{2k-1}^-}N_-(n,b_{2k-1}^-)(2k-1)^2}{b_{2k-1}^-(b_{2k-1}^-+1)(2nb_{2k-1}^-+n-1)}, \qquad k=2,3,\ldots \quad (3.59)$$

$$D_1=-\frac{4}{\pi(n+1)}\left[N_-'(n,-1)+\left\{\log\epsilon+\frac{1+2n-3n^2}{4(n+1)}\right\}N_-(n,-1)\right] \quad (3.60)$$

and

$$D_0 = -\frac{4}{\pi}\frac{N_-(n,-1)}{n+1}. \tag{3.61}$$

Note that $N'_-(n, s)$ denotes $(\partial/\partial s)N_-(n, s)$.

Hence we can write (3.55) as

$$\bar{g}_+(s)R_+(n,s)+\frac{h}{a}\left(\frac{\epsilon}{s}-\frac{1}{s+1}\right)_+ R_+(n,s)-\frac{h}{a}\left(\sum_{k=1}^{\infty}\frac{D_k}{s-b_{2k-1}^-}+\frac{D_0}{(s+1)^2}\right)_+$$
$$-\left(\frac{N_-(n,-1)}{s+1}\right)_+ = \bar{u}_-(s)N_-(n,s)+\left(\frac{N_-(n,s)-N_-(n,-1)}{s+1}\right)_- +$$
$$+\frac{h}{a}\left(\frac{\epsilon^{s+1}R_+(n,s)}{s(s+1)}-\left[\sum_{k=1}^{\infty}\frac{D_k}{s-b_{2k-1}^-}+\frac{D_0}{(s+1)^2}\right]\right)_-, \qquad 0<\operatorname{Re} s<1/n. \tag{3.62}$$

Since the left-hand side of (3.62) is analytic for $\operatorname{Re} s>0$ and the right-hand side is analytic for $\operatorname{Re} s<1/n$, each side is the analytic continuation of the other. Hence each side represents the same entire function $E(s)$, say. $E(s)$ is found by obtaining the asymptotic behaviour as $|s|\to\infty$ of each side of (3.62) in the respective half-planes, and applying Liouville's theorem.

The asymptotic behaviours of $\bar{u}_-(s)$ and $\bar{g}_+(s)$ are dominated by the natures of $u(\rho)$ and $g(\rho)$ as $\rho\to 1^{\pm}$ respectively on the line $\phi=0$. Following Amazigo (**5**) we introduce new variables (r_1, β) such that

$$\gamma_x/\gamma_A = -(r_1/n^{\frac{1}{2}})\sin\beta; \qquad \gamma_y/\gamma_A = 1+r_1\cos\beta, \tag{3.63}$$

and look for an expansion of Ψ in the form

$$\Psi \sim P_0(\beta)+P_1(\beta)r_1^{\frac{1}{2}}+P_2(\beta)r_1+O(r_1^{\frac{3}{2}}), \tag{3.64}$$

we find that

$$P_0=h(1-\epsilon)/a, \qquad P_1=0, \qquad P_2=(1/n^{\frac{1}{2}})\sin\beta+(n/a)\cos\beta \tag{3.65}$$

and conclude from (3.65) and (3.64) that

$$g(\rho)\sim h(1-\epsilon)/a, \qquad \rho\to 1^-, \tag{3.66}$$

and

$$u(\rho)\sim 1, \qquad \rho\to 1^+. \tag{3.67}$$

Equations (3.66) and (3.67) imply, respectively, that

$$\bar{g}_+(s)\sim h(1-\epsilon)/as, \qquad |s|\to\infty, \qquad \operatorname{Re} s>0, \tag{3.68}$$

and

$$\bar{u}_-(s)\sim -1/s, \qquad |s|\to\infty, \qquad \operatorname{Re} s<1/n. \tag{3.69}$$

Equations (3.68) and (3.69), together with (3.53) and (3.54) imply that the entire function $E(s)\to 0$ as $|s|\to\infty$, and hence by Liouville's theorem it must be zero. Thus, from (3.62)

$$\bar{g}_+(s) = \frac{h}{a}\left(\frac{1}{s+1} - \frac{\epsilon}{s}\right) + \frac{1}{R_+(n, s)} \times$$
$$\times\left(\frac{h}{a}\left\{\sum_{k=1}^{\infty} \frac{D_k}{s - b_{2k-1}^-} + \frac{D_0}{(s+1)^2}\right\} + \frac{N_-(n, -1)}{s+1}\right). \tag{3.70}$$

Using (3.70), (3.42) and (3.39) and taking the inverse Mellin transform we obtain

$$\Psi(\rho, \phi) = \frac{1}{2\pi i}\int_{q-i\infty}^{q+i\infty} \rho^{-s}\left(\frac{1}{R_+(n, s)}\left\{\frac{N_-(n, -1)}{s+1} + \frac{h}{a}\sum_{k=1}^{\infty}\frac{D_k}{s - b_{2k-1}^-} + \frac{h}{a}\frac{D_0}{(s+1)^2}\right\}\right.$$
$$\left. - \frac{h}{a}\frac{\epsilon^{s+1}}{s(s+1)}\right)\frac{\cos[\omega(s)(\pi/2 - \phi)]}{\cos \pi\omega(s)/2}\,ds, \qquad 0 < q < 1/n. \tag{3.71}$$

The integrand has simple poles at $s = 0$, $1/n - 1$, -1, b_{2m-1}^+ and b_{2m}^- $(m = 1, 2, \ldots)$.

Near $\rho = 0$, the pole at $s = 1/n - 1$ will give a contribution to Ψ which does not satisfy the condition that Ψ_ρ is bounded as $\rho \to 0$ (this condition comes from the finite boundary in the real plane). Hence we must make the residue at this point zero, which gives the following equation for ϵ.

$$\sum_{k=1}^{\infty} \frac{D_k(\epsilon)}{1/n - 1 - b_{2k-1}^-} + nN_-(n, -1)\left(\frac{a}{h} - \frac{4n}{\pi(n+1)}\right) = 0. \tag{3.72}$$

From the theory of residues we find, for $\phi > 0$,

$$\Psi(\rho, \phi) = \frac{2}{\pi(n-1)N_-(n, 0)}\left(N_-(n, -1) + \frac{h}{a}\left\{D_0 - \sum_{k=1}^{\infty}\frac{D_k}{b_{2k-1}^-}\right\}\right) - \frac{h\epsilon}{a} + \rho\sin\phi +$$
$$+ \frac{4}{\pi}\sum_{m=1}^{\infty}\left(\frac{N_-(n, -1)}{1 + b_{2m}^-} + \frac{h}{a}\left\{\sum_{k=1}^{\infty}\frac{D_k}{b_{2m}^- - b_{2k-1}^-} + \frac{D_0}{(b_{2m}^- + 1)^2}\right\}\right) \times$$
$$\times \frac{\cos 2m\phi\, \rho^{-b_{2m}^-}}{(2nb_{2m}^- + n - 1)N_-(n, b_{2m}^-)}, \qquad 0 < \rho < 1, \tag{3.73$_1$}$$

$$\Psi(\rho, \phi) = -\frac{h}{a} - \frac{4h\rho \log \rho \sin\phi}{\pi a(n+1)} + \frac{h\rho}{a\pi(n+1)^2} \times$$
$$\times\left[\{3 + 6n - n^2 + 4(n+1)\log\epsilon\}\sin\phi + 2(n+1)^2\left(\frac{\pi}{2} - \phi\right)\cos\phi\right] +$$
$$+ \frac{4h\epsilon}{\pi a}\sum_{m=2}^{\infty}\frac{(2m-1)\sin(2m-1)\phi}{b_{2m-1}^-(b_{2m-1}^- + 1)(2nb_{2m-1}^- + n - 1)}\left(\frac{\rho}{\epsilon}\right)^{b_{2m-1}^-} +$$
$$+ \frac{4}{\pi}\sum_{m=1}^{\infty}\left(\frac{N_-(n, -1)}{1 + b_{2m-1}^+} + \frac{h}{a}\left\{\sum_{k=1}^{\infty}\frac{D_k}{b_{2m-1}^+ - b_{2k-1}^-} + \frac{D_0}{(b_{2m-1}^+ + 1)^2}\right\}\right) \times$$
$$\times \frac{(2m-1)\sin(2m-1)\phi\, \rho^{-b_{2m-1}^+}}{(2nb_{2m-1}^+ + n - 1)R_+(n, b_{2m-1}^+)}, \qquad 1 < \rho < \epsilon, \tag{3.73$_2$}$$

$$\Psi(\rho,\phi)=\frac{4}{\pi}\sum_{m=1}^{\infty}\left(\frac{N_-(n,-1)}{1+b_{2m-1}^+}+\frac{h}{a}\left\{\sum_{k=1}^{\infty}\frac{D_k}{b_{2m-1}^+-b_{2k-1}^-}+\frac{D_0}{(b_{2m-1}^++1)^2}\right.\right.$$
$$\left.\left.-\frac{\epsilon^{1+b_{2m-1}^+}R_+(n,b_{2m-1}^+)}{b_{2m-1}^+(b_{2m-1}^++1)}\right\}\right)\times$$
$$\times\frac{(2m-1)\sin(2m-1)\phi\,\rho^{-b_{2m-1}^+}}{(2nb_{2m-1}^++n-1)R_+(n,b_{2m-1}^+)},\qquad \epsilon<\rho<\infty. \tag{3.73$_3$}$$

We note that $n\geqslant 1$ has been assumed throughout this analysis (i.e. we have taken the point $s=1/n-1$ to lie outside the strip of regularity $0<\operatorname{Re}s<1/n$). However, it can be seen that the results are also valid for $0<n<1$.

3.3.4. *The stress intensity factor*

The behaviour of $w(r,\theta)$ near the crack tip is governed by the asymptotic form of $\Psi(\rho,\phi)$ as $\rho\to\infty$. We find

$$\Psi(\rho,\phi)\sim -A(n,\epsilon)\rho^{-1/n}\sin\phi\quad\text{as}\quad\rho\to\infty \tag{3.74}$$

where

$$A(n,\epsilon)=\frac{4}{(n+1)R_+(n,1/n)}\left\{\frac{h}{a}\left(\frac{n^2\epsilon^{1+1/n}R_+(n,1/n)}{n+1}-\frac{n^2D_0}{(n+1)^2}\right.\right.$$
$$\left.\left.-\sum_{k=1}^{\infty}\frac{D_k}{1/n-b_{2k-1}^-}\right)-\frac{nN_-(n,-1)}{n+1}\right\}. \tag{3.75}$$

Equations (3.74), (3.35), (3.8) and (3.3) imply that, as $r\to 0$

$$\begin{pmatrix}\tau_x\\ \tau_y\end{pmatrix}\sim K_{\rm III}^{(2)}\left\{\frac{l(\theta)}{r}\right\}^{1/(n+1)}\begin{pmatrix}-\sin\phi\\ \cos\phi\end{pmatrix}, \tag{3.76}$$

$$w(r,\theta)\sim\frac{n+1}{n}\left\{\frac{K_{\rm III}^{(2)}}{\tau_1}\right\}^n\left\{\frac{l(\theta)}{r}\right\}^{-1/(n+1)}\sin\phi, \tag{3.77}$$

where the stress intensity factor $K_{\rm III}^{(2)}$ satisfies

$$K_{\rm III}^{(2)}=\tau_1\left(\frac{w_0}{h\epsilon}\right)^{1/n}[aA(n,\epsilon)]^{1/(n+1)}, \tag{3.78}$$

and τ_1 and $l(\theta)$ are as defined in section 3.2.3.

From equation (3.72) we obtain the asymptotic result

$$\epsilon\sim\exp\left(\frac{(n+1)}{4}\frac{a}{h}-\frac{N_-'(n,-1)}{N_-(n,-1)}-\frac{(n^2+6n+1)}{4(n+1)}\right)\quad\text{as}\quad\frac{a}{h}\to\infty. \tag{3.79}$$

Consequently (3.75) implies that

$$A(n,\epsilon)\sim\frac{4n^2}{\pi(n+1)^2}\frac{h}{a}\epsilon^{1+1/n}\quad\text{as}\quad\frac{a}{h}\to\infty. \tag{3.80}$$

Use of (3.80) in (3.78) leads to the expected result

$$K_{\text{III}}^{(2)} \sim K_{\text{III}}^{(1)} \quad \text{as} \quad \frac{a}{h} \to \infty \tag{3.81}$$

where $K_{\text{III}}^{(1)}$ is defined in equation (3.31). A convenient normalization of $K_{\text{III}}^{(2)}$ is $\bar{K}$, where

$$\bar{K} = K_{\text{III}}^{(2)}/K_{\text{III}}^{(1)} = \left(\frac{\pi(n+1)^2 A(n, \epsilon)}{4n^2 \epsilon^{1+1/n}} \frac{a}{h}\right)^{1/(n+1)}. \tag{3.82}$$

By scaling the problem on a, it can be shown that

$$(\bar{K})^{(n+1)} \sim \frac{4n^2}{(n+1)^2} \frac{N_-(n, -1)}{R_+(n, 1/n)} \frac{a}{h} \quad \text{as} \quad \frac{a}{h} \to 0.$$

In Fig. 5 we display the variation of $(\bar{K})^{(n+1)}$ with (a/h) for various values of n.

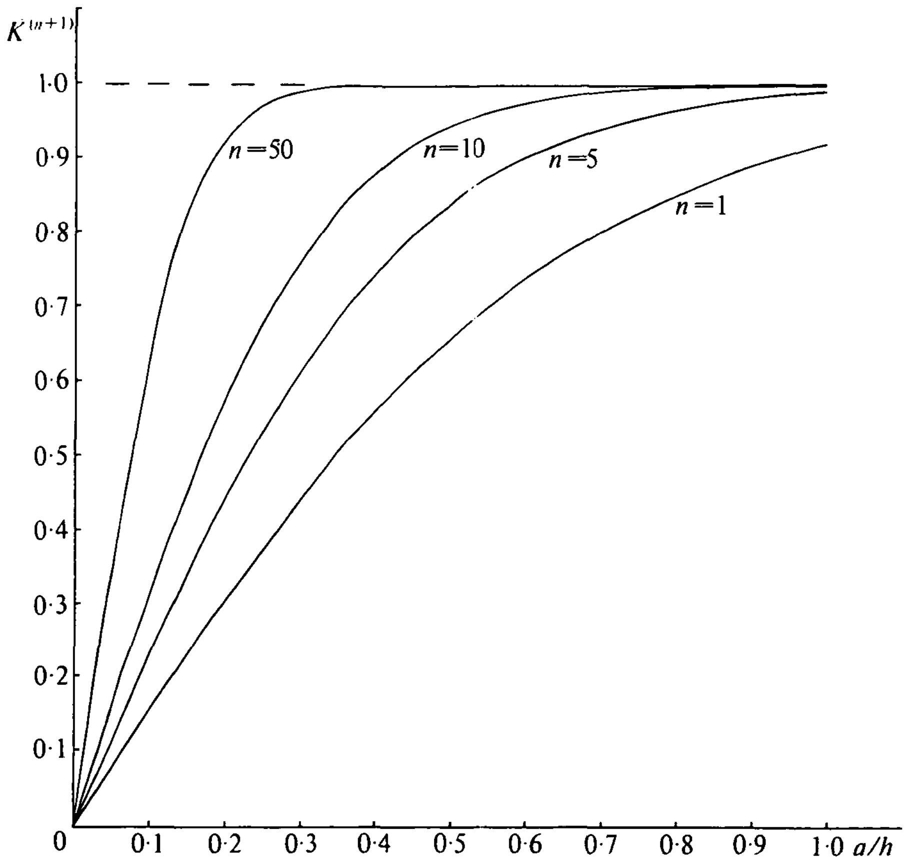

FIG. 5

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