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Geometric and Transformational Properties of Lipschitz Domains, Semmes-Kenig-Toro Domains, and Other Classes of Finite Perimeter Domains

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ABSTRACT. *In the first part of this article we give intrinsic characterizations of the classes of Lipschitz and C^1 domains. Under some mild, necessary, background hypotheses (of topological and geometric measure theoretic nature), we show that a domain is Lipschitz if and only if it has a continuous transversal vector field. We also show that if the geometric measure theoretic unit normal of the domain is continuous, then the domain in question is of class C^1. In the second part of the article, we study the invariance of various classes of domains of locally finite perimeter under bi-Lipschitz and C^1 diffeomorphisms of the Euclidean space. In particular, we prove that the class of bounded regular SKT domains (previously called chord-arc domains with vanishing constant, in the literature) is stable under C^1 diffeomorphisms. A number of other applications are also presented.*

1. Introduction

Analysis on rough domains has become a prominent area of research over the past few decades. Much of the literature has been devoted to domains in Euclidean space with rough boundary, such as Lipschitz domains and chord-arc domains. However, as treatments of partial differential equations with variable coefficients on such domains has advanced, it has become natural as well as geometrically significant to work on rough domains in Riemannian manifolds. Works on this include [19], [18], and [11], among others. The original definitions of various classes of domains, such as strongly Lipschitz domains, are tied to the linear structure of Euclidean space, and there arises the issue of how to define such classes of domains in the manifold setting.

One viable approach, taken in the articles mentioned above, is to call an open set Ω in a smooth manifold M locally strongly Lipschitz, (for example), if for each $p \in \partial\Omega$ there exists a smooth coordinate chart on a neighborhood U of p such that $\partial\Omega \cap U$ is a Lipschitz graph in this

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coordinate system. One can give similar definitions of chord-arc domains in M, etc. However, this approach leaves aside a number of interesting issues, which we take up in this article. These issues center about whether one can establish the invariance of various classes of rough domains (with their original definitions) under C^1-diffeomorphisms (and, for certain classes of finite perimeter domains, even under bi-Lipschitz maps), and, closely related, whether one can provide intrinsic geometrical characterizations of Lipschitz domains and other classes.

The first issue we treat here is the characterization of locally strongly Lipschitz domains as those domains Ω of locally finite perimeter for which there are continuous (or, equivalently, smooth) vector fields that are transverse to the boundary, and that also satisfy the necessary, mild topological condition $\partial\Omega = \partial\overline{\Omega}$. See Section 2 for the definition of transverse in this setting. The fact that strongly Lipschitz domains possess such transverse vector fields is well known, and has played a significant role in analysis on this class of domains. Results of Section 2 establish the converse.

The analytical significance of the existence of such transversal vector fields is that it leads to Rellich identities. Excellent illustrations of the use of these identities include [12], giving estimates on harmonic measure, and [21], providing boundary Gårding inequalities in strongly Lipschitz domains. Another case where the use of transversal vector fields arises is to establish the invertibility of boundary integral operators of layer potential type on strongly Lipschitz domains, starting with [25]. In this case, the Rellich identities are applied in concert with two other tools:

(a) Green formulas for appropriate classes of functions on strongly Lipschitz domains;
(b) the Calderón-Coifman-McIntosh-Meyer theory for singular integral operators on strongly Lipschitz surfaces.

These tools permit one to reduce various elliptic boundary problems to certain boundary integral equations, and to solve these equations.

In recent years, (a) and (b) above have been extended to a much more general class of domains than that of Lipschitz domains. See, e.g., [9, 17] for (a), and [5, 6] for (b). Quite recently, in [11], we have further refined some of these results and used them to treat boundary value problems in chord-arc domains with vanishing constant (in the terminology of [15, 16]), which we call regular SKT domains. (See Section 5 for a definition.) It follows from the characterization stated above that there is not a corresponding extension of the use of transverse vector fields to a class of domains bigger than the class of strongly Lipschitz domains, so one will need to seek other methods to add to (a) and (b) to tackle elliptic boundary problems on such domains, as has been done in the case of regular SKT domains in [11].

Among other results established in Section 2, we mention the following. Let $\Omega \subset \mathbb{R}^n$ be a bounded, nonempty, open set of finite perimeter for which $\partial\Omega = \partial\overline{\Omega}$, and denote by ν, σ, respectively, the (geometric measure theoretic) outward unit normal and surface measure on $\partial\Omega$. Then the quantity

$$\rho(\Omega) := \inf \big\{ \|\nu - f\|_{L^\infty(\partial\Omega, d\sigma)} : f \in C^0(\partial\Omega, \mathbb{R}^n),\ |f| = 1 \text{ on } \partial\Omega \big\} \tag{1.1}$$

can be used to characterize the membership of Ω both in the class of strongly Lipschitz domains and in the class of C^1 domains. Specifically,

$$\Omega \text{ is a strongly Lipschitz domain} \Longleftrightarrow \rho(\Omega) < \sqrt{2}, \tag{1.2}$$

$$\Omega \text{ is a } C^1\text{-domain} \Longleftrightarrow \rho(\Omega) = 0\,. \tag{1.3}$$

This can be compared with the recent result proved in [11], to the effect that for a bounded NTA domain, with an Ahlfors regular boundary (cf. Section 5 for the relevant terminology),

$$\Omega \text{ is a regular SKT domain} \Longleftrightarrow \inf\left\{\|\nu - f\|_{\mathrm{BMO}(\partial\Omega, d\sigma)} : f \in C^0\big(\partial\Omega, \mathbb{R}^n\big)\right\} = 0\,. \tag{1.4}$$

Section 3 studies the images of locally finite perimeter domains in $\mathbb{R}^n$. We show that this class of domains is invariant under bi-Lipschitz maps. If Ω is such a domain and F is a map that is not only bi-Lipschitz but actually a C^1 diffeomorphism, we relate the (measure-theoretic) outward unit normal ν and surface measure σ on $\partial\Omega$ to the outward unit normal $\widetilde{\nu}$ and surface measure $\widetilde{\sigma}$ on $\partial F(\Omega)$. Specifically, here we prove that

$$\widetilde{\nu} = \frac{\big(DF^{-1}\big)^{\top}\big(\nu \circ F^{-1}\big)}{\big\|\big(DF^{-1}\big)^{\top}\big(\nu \circ F^{-1}\big)\big\|} \quad \text{and} \tag{1.5}$$

$$\widetilde{\sigma} = \big\|\big(DF^{-1}\big)^{\top}\big(\nu \circ F^{-1}\big)\big\| \, \big|\det(DF) \circ F^{-1}\big| \, F_*\sigma\,, \tag{1.6}$$

where DF is the Jacobian matrix of F, and $F_*\sigma$ denotes the push-forward of the measure σ via the mapping F.

In Section 4 we use this result together with the transversality characterization from Section 2, to prove the invariance of the class of locally strongly Lipschitz domains under C^1 diffeomorphisms. Here, as in Sections 2–3, most of our work is done for domains in Euclidean space $\mathbb{R}^n$, but once these invariance results are established, their analogues in the manifold setting are fairly straightforward. For example, both the class of Lipschitz domains, as well as the class of regular SKT domains, have natural definitions in the setting of manifolds, which rely only on the intrinsic C^1 structure of the manifold.

Other topics treated in Section 4 include the recollection of several examples of bi-Lipschitz maps taking bounded strongly Lipschitz domains to domains which fail to be, themselves, strongly Lipschitz. We then proceed to establish the invariance of the class of regular SKT domains under C^1 diffeomorphisms. Furthermore, making use of (1.5)–(1.6), we devise a general approximation scheme for domains of locally finite perimeter. When specialized to the case of bounded strongly Lipschitz domains, this yields an approximation result akin to the work of Calderón in [3].

Section 5 is an Appendix, consisting of three subsections. The purpose of the first, is to collect definitions and basic properties of SKT domains and regular SKT domains, needed for application in Section 4. In the second subsection, we deduce a number of useful formulas in linear algebra, used in Section 2, and in Section 5.3 we collect a number of useful results in elementary topology.

2. Finite perimeter domains with continuous transversal fields

Throughout this article, we shall assume that $n \geq 2$ is a fixed integer. Call an open set $\Omega \subset \mathbb{R}^n$ of *locally finite perimeter* provided

$$\mu := \nabla \mathbf{1}_{\Omega} \tag{2.1}$$

is a locally finite $\mathbb{R}^n$-valued measure. (Hereafter, we denote by $\mathbf{1}_E$ the characteristic function of a set E.) For a domain of locally finite perimeter which has a compact boundary we agree to drop the adverb 'locally.' Given an open set $\Omega \subset \mathbb{R}^n$ of locally finite perimeter we denote by σ the total variation measure of μ; σ is then a locally finite positive measure, supported on $\partial\Omega$, and clearly each component of μ is absolutely continuous with respect to σ. It follows from the Radon-Nikodym theorem that

$$\mu = \nabla \mathbf{1}_{\Omega} = -\nu\sigma\,, \tag{2.2}$$

where

$$\nu \in L^\infty(\partial\Omega, d\sigma) \text{ is an } \mathbb{R}^n\text{-valued function, satisfying } |\nu(x)| = 1, \text{ for } \sigma\text{-a.e. } x \in \partial\Omega\,. \tag{2.3}$$

In the sequel, we shall frequently identify σ with its restriction to $\partial\Omega$, with no special mention. We shall refer to ν and σ, respectively, as the (geometric measure theoretic) *outward unit normal* and the *surface measure* on $\partial\Omega$.

Note that ν defined by (2.2) can only be specified up to a set of σ-measure zero. To eliminate this ambiguity, we redefine $\nu(x)$, for every x, as being

$$\lim_{r\to 0} \fint_{B(x,r)} \nu\, d\sigma \tag{2.4}$$

whenever the limit exists, and zero otherwise. In doing so, the following convention is employed. We set $\fint_{B(x,r)} \nu\, d\sigma := (\sigma(B(x,r)))^{-1} \int_{B(x,r)} \nu\, d\sigma$ if $\sigma(B(x,r)) > 0$, and zero otherwise. The Besicovitch Differentiation Theorem (cf., e.g., [8]) ensures that ν in (2.2) agrees with (2.4) for σ-a.e. x.

The *reduced boundary* of Ω is then defined as

$$\partial^*\Omega := \Big\{x : |\nu(x)| = 1\Big\}\,. \tag{2.5}$$

This is essentially the point of view adopted in [27] (cf. Definition 5.5.1 on p. 233). Let us remark that this definition is slightly different from that given on p. 194 of [8]. The reduced boundary introduced there depends on the choice of the unit normal in the class of functions agreeing with it σ-a.e. and, consequently, can be pointwise specified only up to a certain set of zero surface measure. Nonetheless, any such representative is a subset of our $\partial^*\Omega$ and differs from it by a set of σ-measure zero.

Moving on, it follows from (2.5) and the Besicovitch Differentiation Theorem that σ is supported on $\partial^*\Omega$, in the sense that $\sigma(\mathbb{R}^n \setminus \partial^*\Omega) = 0$. From the work of Federer and of De Giorgi it is also know that, if $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$,

$$\sigma = \mathcal{H}^{n-1}\lfloor\partial^*\Omega\,. \tag{2.6}$$

Recall that, generally speaking, given a Radon measure μ in $\mathbb{R}^n$ and a set $A \subset \mathbb{R}^n$, the restriction of μ to A is defined as $\mu\lfloor A := \mathbf{1}_A\,\mu$. In particular, $\mu\lfloor A \ll \mu$ and $d(\mu\lfloor A)/d\mu = \mathbf{1}_A$. Thus,

$$\sigma \ll \mathcal{H}^{n-1} \quad \text{and} \quad \frac{d\sigma}{d\mathcal{H}^{n-1}} = \mathbf{1}_{\partial^*\Omega}\,. \tag{2.7}$$

Furthermore (cf. Lemma 5.9.5 on p. 252 in [27], and p. 208 in [8]), one has

$$\partial^*\Omega \subseteq \partial_*\Omega \subseteq \partial\Omega, \quad \text{and} \quad \mathcal{H}^{n-1}\big(\partial_*\Omega \setminus \partial^*\Omega\big) = 0\,, \tag{2.8}$$

where $\partial_*\Omega$, the *measure-theoretic boundary* of Ω, is defined by

$$\partial_*\Omega := \Big\{x \in \partial\Omega : \limsup_{r\to 0^+} r^{-n}\mathcal{H}^n\big(B(x,r)\cap\Omega^\pm\big) > 0\Big\}\,. \tag{2.9}$$

Above, $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure (i.e., the Lebesgue measure) in $\mathbb{R}^n$, and we have set $\Omega^+ := \Omega$, $\Omega^- := \mathbb{R}^n \setminus \Omega$ (later on, instead of Ω^- we shall use the notation Ω^c). Let us also record here a useful criterion for deciding whether a Lebesgue measurable subset E of $\mathbb{R}^n$ is of locally finite perimeter in $\mathbb{R}^n$ (cf. [8], p. 222):

$$E \text{ has locally finite perimeter} \iff \mathcal{H}^{n-1}(\partial_*E \cap K) < \infty, \quad \forall\, K \subset \mathbb{R}^n, \text{ compact}\,. \tag{2.10}$$

A moment's reflection shows that this can be rephrased as

$$E \text{ has locally finite perimeter} \iff \forall x \in \partial E\ \exists r > 0 \text{ so that } \mathcal{H}^{n-1}(\partial_* E \cap B(x,r)) < \infty\,. \tag{2.11}$$

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be an open set of locally finite perimeter, with outward unit normal ν and surface measure σ, and a point $x_0 \in \partial\Omega$. Then, it is said that Ω has a *continuous transversal vector field near* x_0 provided there exist $r > 0$, $\kappa > 0$ and a continuous vector field X on $B(x,r) \cap \partial\Omega$ which is (outwardly) transverse to $\partial\Omega$ near x_0, in the sense that

$$\nu \cdot X \geq \kappa \quad \sigma\text{-a.e. on} \quad B(x,r) \cap \partial\Omega\,. \tag{2.12}$$

Next, it is said that Ω has *continuous transversal vector fields* provided Ω has a *continuous transversal vector field near* x for each point $x \in \partial\Omega$.

Finally, Ω is said to have *continuous globally transversal vector fields* if there exist a vector field $X \in C^0(\partial\Omega, \mathbb{R}^n)$ and a number $\kappa > 0$ (called the transversality constant of X) with the property that $\nu \cdot X \geq \kappa$ at σ-a.e. point on $\partial\Omega$.

Lemma 2.2. *Assume that $\Omega \subset \mathbb{R}^n$ is a domain of finite perimeter, whose boundary is compact, and which has continuous locally transversal vector fields. Then Ω has, in fact, global continuous transversal vector fields.*

Proof. The argument is standard. From compactness, there exist $x_j \in \partial\Omega$, r_j, $\kappa_j > 0$, $1 \leq j \leq m$, along with $X_j \in C^0(B(x_j, r_j) \cap \partial\Omega, \mathbb{R}^n)$, $1 \leq j \leq m$, with the property that $\partial\Omega \subseteq \bigcup_{1\leq j\leq m} B(x_j, r_j)$, and $\nu \cdot X_j \geq \kappa_j$ at σ-a.e. point on $B(x_j, r_j) \cap \partial\Omega$, for each $j = 1, \ldots, m$. If we now consider $\{\psi_j\}_{1\leq j\leq m}$, a partition of unity in a neighborhood of $\partial\Omega$ consisting of smooth, nonnegative functions for which $\operatorname{supp}\psi_j \subset B(x_j, r_j)$, $1 \leq j \leq m$, then $X := \sum_{j=1}^m \psi_j X_j \in C^0(\partial\Omega, \mathbb{R}^n)$ satisfies

$$\nu \cdot X = \sum_{j=1}^m \psi_j\, \nu \cdot X_j \geq \sum_{j=1}^m \kappa_j \psi_j \geq \kappa\,, \tag{2.13}$$

where $\kappa := \min\{\kappa_1, \ldots, \kappa_m\} > 0$. Thus, Ω has global continuous transversal vector fields. □

Below we collect several equivalent formulations of the above definition. Here and elsewhere, we shall denote the standard norm in $\mathbb{R}^n$ by either $|\cdot|$, or $\|\cdot\|$. Also, C^0, $C^1, \ldots, C^\infty$ stand, respectively, for the classes of continuous functions, continuously differentiable functions, etc., infinitely many times differentiable functions.

Proposition 2.3. *For an open set of locally finite perimeter, $\Omega \subset \mathbb{R}^n$, the following two conditions are equivalent:*

(i) *Ω has continuous locally transversal vector fields;*
(ii) *for every point $x \in \partial\Omega$ there exist $r > 0$, $\kappa > 0$ and $X \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that* (2.12) *holds.*

Furthermore, the local versions of (i)–(ii) above are also equivalent.

If the domain Ω also satisfies

$$\mathcal{H}^{n-1}(\partial_*\Omega \cap B(x,r)) > 0, \qquad \forall x \in \partial\Omega,\ \forall r > 0\,, \tag{2.14}$$

then (i)–(ii) above are also equivalent to:

(iii) *For every $x \in \partial\Omega$ there exist $\kappa > 0$, $r > 0$ and $X \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $|X| = 1$ on $B(x, r/2) \cap \partial\Omega$ and* (2.12) *holds.*

Given (2.14), *then the local versions of (i)–(ii) are equivalent with the local version of (iii).*

Finally, if additionally to (2.14), *$\partial\Omega$ is compact, then (i)–(iii) above are also equivalent to:*

(iv) *There exists $X \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ which is globally transversal to Ω and such that $|X| = 1$ on $\partial\Omega$;*

(v) *there exists $X \in C^0(\partial\Omega, \mathbb{R}^n)$ satisfying $|X| = 1$ on $\partial\Omega$ and $\|\nu - X\|_{L^\infty(\partial\Omega, d\sigma)} < \sqrt{2}$, where ν, σ stand, respectively, for the outward unit normal and surface measure on $\partial\Omega$.*

Proof. The fact that (i)$\Leftrightarrow$(ii) is an easy consequence of the fact that small L^∞ perturbations of a transversal field are also transversal, plus a standard mollification argument. The same argument also works for the local versions of (i) and (ii). To further show that (ii)$\Leftrightarrow$(iii), note that (2.14) and (2.6)–(2.8) imply

$$\sigma(B(x, r) \cap \partial\Omega) > 0, \qquad \forall x \in \partial\Omega, \;\; \forall r > 0\,. \tag{2.15}$$

Hence, a continuous field satisfying (2.12) cannot vanish on $B(x, r) \cap \partial\Omega$, since this would violate (2.15). Consequently, if X is as in (ii), then its (pointwise) normalized version remains transversal to Ω on $B(x, r/2) \cap \partial\Omega$. If $\partial\Omega$ is compact, Lemma 2.2 shows that there exists $X \in C^\infty(\partial\Omega, \mathbb{R}^n)$ which is globally transversal to $\partial\Omega$. Then the same reasoning as above proves that X can be normalized to unit on $\partial\Omega$. This takes care of the claim made about (iv). As for (v), it suffices to observe that if $X \in C^0(\partial\Omega, \mathbb{R}^n)$ satisfies $|X| = 1$ on $\partial\Omega$, then $\nu \cdot X = \frac{1}{2}(2 - |\nu - X|^2)$ pointwise a.e. on $\partial\Omega$. Thus, the field X is globally transversal to $\partial\Omega$ if and only if $\|\nu - X\|_{L^\infty(\partial\Omega, d\sigma)} < \sqrt{2}$. □

Remark 2.4. It is worth pointing out that, for a set $\Omega \subseteq \mathbb{R}^n$ of locally finite perimeter, condition (2.14) is equivalent to

$$\partial_*\Omega \;\text{ is dense in }\; \partial\Omega\,. \tag{2.16}$$

Indeed, on the one hand, it is clear that (2.14) implies (2.16). On the other hand, it is known that for each $x \in \partial^*\Omega$

$$0 < C_1 \le \liminf_{r\to 0^+} r^{1-n}\sigma(B(x, r) \cap \partial\Omega) \le \limsup_{r\to 0^+} r^{1-n}\sigma(B(x, r) \cap \partial\Omega) \le C_2 < \infty\,, \tag{2.17}$$

for some dimensional constants C_1, C_2 (cf. Lemma 2 on p. 196 in [8]). It is then easy to derive (2.14) based on (2.16) and (2.17) (for this, (2.6)–(2.8) are also useful). Furthermore, a slight variation of the argument above shows that (2.14) is further equivalent to $\partial^*\Omega$ being dense in $\partial\Omega$.

A large class of domains for which continuous locally transversal fields exist is the collection of all strongly Lipschitz domains in $\mathbb{R}^n$, with compact boundary. For the clarity of the exposition we record here a formal definition (recall that the superscript c is the operation of taking the complement of a set, relative to $\mathbb{R}^n$).

Definition 2.5. Let Ω be a nonempty, proper open subset of $\mathbb{R}^n$. Also, fix $x_0 \in \partial\Omega$. Call Ω a *strongly Lipschitz domain near* x_0 if there exist $b, c > 0$ with the following significance. There exist an $(n-1)$-plane $H \subset \mathbb{R}^n$ passing through x_0, a choice N of the unit normal to H, and an open cylinder $\mathcal{C}_{b,c} := \{x' + tN : x' \in H, \; |x' - x_0| < b, \; |t| < c\}$ (called coordinate cylinder near x_0) such that

$$\mathcal{C}_{b,c} \cap \Omega = \mathcal{C}_{b,c} \cap \{x' + tN : x' \in H,\ t > \varphi(x')\}, \tag{2.18}$$

$$\mathcal{C}_{b,c} \cap \partial\Omega = \mathcal{C}_{b,c} \cap \{x' + tN : x' \in H,\ t = \varphi(x')\}, \tag{2.19}$$

$$\mathcal{C}_{b,c} \cap \overline{\Omega}^c = \mathcal{C}_{b,c} \cap \{x' + tN : x' \in H,\ t < \varphi(x')\}, \tag{2.20}$$

for some Lipschitz function $\varphi : H \to \mathbb{R}$ satisfying

$$\varphi(x_0) = 0 \quad \text{and} \quad |\varphi(x')| < d \ \text{ if } \ |x' - x_0| \leq b. \tag{2.21}$$

Finally, call Ω a *locally strongly Lipschitz domain* if it is a locally strongly Lipschitz domain near every point $x \in \partial\Omega$.

Remark 2.6.

(i) It should be noted that the conditions (2.18)–(2.20) are *not* independent since, in fact, (2.18) implies (2.19)–(2.20). In this vein, let us also mention that, (2.19) implies (2.18), (2.20) (up to changing N into $-N$) if, for example, $x_0 \notin (\overline{\Omega})^\circ$ (where, generally speaking, E° stands for the interior of the set $E \subseteq \mathbb{R}^n$). The latter condition is guaranteed if it is known a priori that

$$\partial\Omega = \partial\overline{\Omega}. \tag{2.22}$$

(ii) Whenever conditions (2.18)–(2.21) hold and we find it necessary to emphasize the role of the unit normal N, we shall say that $\partial\Omega$ *is a Lipschitz graph near* x_0 *in the direction of* N.

The classes of bounded $C^{1+\alpha}$ and $C^{1,1}$ domains is defined analogously, requiring that the defining functions φ have first order derivatives of class C^α (the Hölder space of order α), and Lipschitz, respectively.

In the sequel, we shall refer to a locally strongly Lipschitz domain with compact boundary simply as a strongly Lipschitz domain. Given a bounded strongly Lipschitz domain $\Omega \subset \mathbb{R}^n$, the number and size of coordinate cylinders in a finite covering of $\partial\Omega$, along with the quantity $\max \|\nabla\varphi\|_{L^\infty}$ (called the Lipschitz constant of Ω), where the supremum is taken over all Lipschitz functions φ associated with these coordinate cylinders) make up what is called the *Lipschitz character* of Ω.

Definition 2.5 shows that if $\Omega \subset \mathbb{R}^n$ is a strongly Lipschitz domain near a boundary point x_0 then, in a neighborhood of x_0, $\partial\Omega$ agrees with the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$, considered in a suitably chosen system of coordinates (which is isometric with the original one). Then the outward unit normal has an explicit formula in terms of $\nabla'\varphi$ namely, in the new system of coordinates,

$$\nu(x', \varphi(x')) = \frac{(\nabla'\varphi(x'), -1)}{\sqrt{1 + |\nabla'\varphi(x')|^2}}, \qquad \text{if } \ (x', \varphi(x')) \text{ is near } x_0, \tag{2.23}$$

where ∇' denotes the gradient with respect to $x' \in \mathbb{R}^{n-1}$. This readily implies that the constant unit vector which is vertically downward pointing in this new system of coordinates is transversal to $\partial\Omega$ near x_0. As a corollary, locally strongly Lipschitz domains have continuous locally transversal fields. This and Lemma 2.2 then further show that any strongly Lipschitz domain has a global continuous transversal field.

It is also clear that if $\Omega \subset \mathbb{R}^n$ is a strongly Lipschitz domain with compact boundary then Ω satisfies a uniform cone property. This asserts that there exists an open, circular, truncated, one-component cone Γ with vertex at $0 \in \mathbb{R}^n$ such that for every $x_0 \in \partial\Omega$ there exist $r > 0$ and a rotation $\mathcal{R}$ about the origin such that

$$x + \mathcal{R}(\Gamma) \subseteq \Omega, \qquad \forall\, x \in B(x_0, r) \cap \overline{\Omega}\,. \tag{2.24}$$

Let us point out that if Ω satisfies a uniform cone property, as described above, then also

$$x_0 \in \partial\Omega \Longrightarrow x_0 - \mathcal{R}(\Gamma) \subseteq \Omega^c\,, \tag{2.25}$$

at least if the height of Γ is sufficiently small relative to r (appearing in (2.24)). Indeed, the existence of a point $y \in (x_0 - \mathcal{R}(\Gamma)) \cap \Omega$ would entail $x_0 \in y + \mathcal{R}(\Gamma)$. Since $y \in \Omega$ is also close to x_0 (assuming that Γ has small height, relative to r), (2.24) further implies that x_0 belongs to the interior of Ω, in contradiction with $x_0 \in \partial\Omega$.

Granted (2.25), it is not difficult to see that the converse statement regarding strong Lipschitzianity implying a uniform cone condition is also true. That is, a bounded open set $\Omega \subset \mathbb{R}^n$ satisfying a uniform cone property is, necessarily, strongly Lipschitz. See, e.g., Theorem 1.2.2.2 on p. 12 in [10] for a proof. Here we wish to establish yet another useful intrinsic geometrical characterization of the class of locally strongly Lipschitz domains. More specifically, we prove the following theorem.

Theorem 2.7. *Let Ω be a nonempty, proper open subset of $\mathbb{R}^n$ which has locally finite perimeter. Then Ω is a locally strongly Lipschitz domain if and only if it has continuous locally transversal vector fields and* (2.22) *holds.*

Let us note that some hypothesis like (2.22) is necessary for the validity of Theorem 2.7. Indeed, in one direction, it can be verified with the help of Definition 2.5 that

$$\Omega \text{ locally strongly Lipschitz domain} \Longrightarrow \partial\Omega = \partial\overline{\Omega}\,. \tag{2.26}$$

In the opposite direction, let Ω_0 be a strongly Lipschitz domain in $\mathbb{R}^n$, let K be a compact subset of Ω_0 such that $\mathcal{H}^n(K) = 0$, and consider $\Omega = \Omega_0 \setminus K$. Then Ω is a finite perimeter domain, but $\sigma(K) = 0$, $\partial_*\Omega = \partial_*\Omega_0$, and any continuous vector field on $\mathbb{R}^n$ which is locally transversal to $\partial\Omega_0$ is also, according to Definition 2.1, locally transversal to $\partial\Omega$. Nonetheless, Ω is not strongly Lipschitz and, of course, (2.22) also fails. Furthermore, it is clear that the continuity of locally transversal vector fields cannot be weakened to mere boundedness, as ν is globally transversal to any domain of locally finite perimeter. In summary, Theorem 2.7 is sharp.

In fact, a local version of Theorem 2.7 is valid as well. Specifically, we have the following.

Theorem 2.8. *Assume that Ω is a nonempty, proper open subset of $\mathbb{R}^n$ which has locally finite perimeter, and fix $x_0 \in \partial\Omega$. Then Ω is a strongly Lipschitz domain near x_0 if and only if it has a continuous transversal vector field near x_0 and there exists $r > 0$ such that*

$$\partial(\Omega \cap B(x_0, r)) = \partial\big(\overline{\Omega \cap B(x_0, r)}\big)\,. \tag{2.27}$$

As a preamble to the proofs of Theorems 2.7–2.8, we establish an useful auxiliary result, to the effect that (2.22) implies (2.14) for sets of locally finite perimeter. The fact that (2.22) implies the weaker fact that $\mathcal{H}^{n-1}(\partial\Omega \cap B(x, r)) > 0$ for all $x \in \partial\Omega$ and $r > 0$, is actually elementary. To see this, take parallel $(n-1)$-dimensional disks in Ω and in the complement of the closure of Ω, in $B(x, r)$, and note that corresponding lines connecting these disks must all intersect $\partial\Omega$. However, establishing (2.14), in which $\partial_*\Omega$ is used, seems less elementary.

Lemma 2.9. *Let $\Omega \subset \mathbb{R}^n$ be an open set of locally finite perimeter, and for which* (2.22) *holds. Then* (2.14) *also holds.*

Proof. Suppose Ω satisfies the hypotheses stated above, but

$$\mathcal{H}^{n-1}(\partial_*\Omega \cap B(x,r)) = 0\,, \tag{2.28}$$

for some $x \in \partial\Omega$ and some $r > 0$. The hypothesis (2.22) implies that $B(x,s)$ has nonempty intersection with both Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ for each $s \in (0,r)$. A basic result about finite perimeter domains (cf. [8], p. 195) is that

$$\Omega \cap B(x,s) \text{ is a domain of finite perimeter for almost every } s \in (0,r)\,. \tag{2.29}$$

In addition, if we set $\mathcal{O}_{x,s} := \Omega \cap B(x,s)$, then for a.e. $s \in (0,r)$,

$$-\nabla \mathbf{1}_{\mathcal{O}_{x,s}} = N\,\mathcal{H}^{n-1} \lfloor (\Omega \cap \partial B(x,s)) + \nu\,\mathcal{H}^{n-1} \lfloor (\partial_*\Omega \cap B(x,s))\,, \tag{2.30}$$

where N is the outward unit normal to $\partial B(x,s)$.

If (2.28) holds, we have

$$-\nabla \mathbf{1}_{\mathcal{O}_{x,s}} = N\,\mathcal{H}^{n-1} \lfloor (\Omega \cap \partial B(x,s))\,. \tag{2.31}$$

Now denoting by ψ the restriction of $\mathbf{1}_{\mathcal{O}_{x,s}}$ to $B(x,s)$, we deduce from (2.31) that

$$\nabla\psi = 0 \quad \text{in the sense of distributions in } \; B(x,s)\,. \tag{2.32}$$

Hence, ψ is equal a.e. to a constant on $B(x,s)$. However, the construction given above forces $\psi = 1$ on $B(x,s) \cap \Omega$ and $\psi = 0$ on $B(x,s) \setminus \overline{\Omega}$, each a nonempty open set [by (2.22)]. This contradiction implies (2.28) is impossible, and proves the lemma. □

Parenthetically, we wish to point out that, as far as a partial converse to Lemma 2.9 is concerned, it is easy to show that (2.14) plus the hypothesis that $\mathcal{H}^n(\partial\Omega) = 0$ implies (2.22), via the use of (2.9). This is, of course, of lesser significance for our current purposes.

Theorem 2.7 is going to be a consequence of its own local version, Lemma 2.9, and the purely topological result discussed in Lemma 5.12. For now, we choose to record the proof of the fact that the following.

Theorem 2.8 implies Theorem 2.7. Let Ω be a nonempty, proper open subset of $\mathbb{R}^n$ which has locally finite perimeter and satisfies (2.22).

To prove one direction of the equivalence stated in the conclusion of Theorem 2.7, assume that Ω has continuous locally transversal fields. Fixing $x_0 \in \partial\Omega$, this implies that Ω has a continuous transversal field near x_0. Then (2.22) along with Lemma 5.12 used for $\Omega_1 := \Omega$ and $\Omega_2 := B(x_0, r)$, $r > 0$ arbitrary, show that (2.27) holds (for any $r > 0$). Theorem 2.8 then gives that Ω is a strongly Lipschitz domain near x_0 and, since $x_0 \in \partial\Omega$ was arbitrary, we conclude that Ω is a locally strongly Lipschitz domain.

Finally, the opposite implication of the equivalence stated in the conclusion of Theorem 2.7 follows from the discussion centered around (2.23). □

Hence, there remains to give the following.

Proof of Theorem 2.8. In one direction, if Ω is a strongly Lipschitz domain near x_0, it is then clear from our earlier considerations and Definition 2.5 that Ω has a continuous transversal field near x_0 and that (2.27) holds if $r > 0$ is sufficiently small (relative to the size of the coordinate cylinder near x_0).

The main issue is establishing the converse statement. To get started, pick $x_0 \in \partial\Omega$, along with some $r > 0$ for which (2.27) holds. Then, if $s \in (0, r)$, Lemma 5.12 with $\Omega_1 := \Omega \cap B(x_0, r)$ and $\Omega_2 := B(x_0, s)$, gives that (2.27) also holds with r replaced by s. Recalling (2.29), we can then find some $s \in (0, r)$ for which $\Omega \cap B(x_0, s)$ is a domain of finite perimeter with the property that $x_0 \in \partial(\Omega \cap B(x_0, s)) = \partial(\overline{\Omega \cap B(x_0, s)})$.

Re-denoting $\Omega \cap B(x_0, s)$ by Ω, it follows that Ω is a nonempty, proper open subset of $\mathbb{R}^n$ which has locally finite perimeter, (2.22) holds, and which has a continuous vector field X transversal near x_0. Our goal is to prove that Ω is a strongly Lipschitz domain near x_0.

Translating and rotating we can assume $x_0 = 0$ and $X(x_0) = e_n$. Here, Lemma 2.9 and the local version of the characterization in (iii) of Proposition 2.3 is used. Since X is continuous, it follows that e_n is transverse to $\partial\Omega$ near x_0. To express this in a more convenient way, recall that since Ω has locally finite perimeter we have (2.2) with σ the surface measure on $\partial\Omega$, and ν a unit vector field defined σ-a.e. on $\partial\Omega$. Then the transversality hypothesis (2.12) implies that there exists $a \in (1, \infty)$ such that, with $\nu' = \nu - (e_n \cdot \nu)e_n$,

$$e_n \cdot \nu \geq \frac{1}{a}\left|\nu'\right|, \quad \sigma\text{-a.e.}, \tag{2.33}$$

on a neighborhood of $x_0 \equiv 0$, say on an open cylinder

$$\mathcal{C}_{b,c} := B_b \times (-c, c), \quad \text{where} \quad B_b := \left\{x' \in \mathbb{R}^{n-1} : \left|x'\right| < b\right\}, \quad b, c > 0\,. \tag{2.34}$$

Fix $b_1 \in (0, b)$ and $c_1 \in (0, c)$ satisfying

$$ab_1 < c_1\,. \tag{2.35}$$

We will show that for some $b_2 \in (0, b_1)$ and $c_2 \in (0, c_1)$, to be specified later, the set $\partial\Omega \cap \mathcal{C}_{b_2,c_2}$ is the graph of a Lipschitz function from B_{b_2} to $(-c_2, c_2)$, with Lipschitz constant $\leq a$. To proceed, take $\varphi \in C_0^\infty(B(0,1))$ such that $\varphi \geq 0$, $\int \varphi(x)\,dx = 1$, and for each $\delta > 0$ set $\varphi_\delta(x) := \delta^{-n}\varphi(x/\delta)$, $x \in \mathbb{R}^n$. Also, introduce

$$\chi_\delta(x) := \varphi_\delta * \mathbf{1}_\Omega(x), \qquad x \in \mathbb{R}^n\,. \tag{2.36}$$

We have

$$\nabla\chi_\delta(x) = (\varphi_\delta * \mu)(x) = -\int_{\partial\Omega} \varphi_\delta(x-y)\nu(y)\,d\sigma(y), \qquad x \in \mathbb{R}^n\,, \tag{2.37}$$

so as long as $\delta < \min(b/2, c/2)$ and $b_1 < b - \delta$, $c_1 < c - \delta$ (demanded to ensure that C_{b_1,c_1} is a nonempty neighborhood of x_0), estimate (2.33) and the representation (2.37) imply

$$-\frac{\partial}{\partial x_n}\chi_\delta(x) \geq \frac{1}{a}\left|\nabla_{x'}\chi_\delta(x)\right|, \quad \forall\, x \in \mathcal{C}_{b_1,c_1}\,. \tag{2.38}$$

Now take

$$x \in \mathcal{C}_{b_1,c_1} \cap \Omega, \quad y \in \mathcal{C}_{b_1,c_1} \cap \overline{\Omega}^c\,. \tag{2.39}$$

Since $x_0 \in \partial\Omega$ and we assume (2.22), such points exist. We claim that for all such x and y,

$$x_n - y_n < a\left|x' - y'\right|\,. \tag{2.40}$$

To see this, note that since the two sets appearing in (2.39) are open, if δ is sufficiently small we have:

$$\chi_\delta(x) = 1 \quad \text{and} \quad \chi_\delta(y) = 0\,. \tag{2.41}$$

Hence,

$$\begin{aligned} 1 &= \chi_\delta(x) - \chi_\delta(y) \\ &= \int_0^1 (x-y)\cdot\nabla\chi_\delta\big(y + t(x-y)\big)\,dt\,. \end{aligned} \tag{2.42}$$

However, we claim that

$$x_n - y_n \geq a\big|x' - y'\big| \Longrightarrow (x-y)\cdot\nabla\chi_\delta(z) \leq 0, \quad \forall\, z \in \mathcal{C}_{b_1,c_1}\,. \tag{2.43}$$

To prove this claim, if (x', x_n), (y', y_n) and z are as above, then

$$\begin{aligned} (x-y)\cdot\nabla\chi_\delta(z) &= (x_n - y_n)\partial_n\chi_\delta(z) + \big(x' - y'\big)\cdot\nabla_{x'}\chi_\delta(z) \\ &\leq (x_n - y_n)\partial_n\chi_\delta(z) + \big|x' - y'\big|\nabla_{x'}\chi_\delta(z)\big| \\ &\leq (x_n - y_n)\partial_n\chi_\delta(z) + \frac{x_n - y_n}{a}|\nabla_{x'}\chi_\delta(z)| \\ &\leq (x_n - y_n)\partial_n\chi_\delta(z) - (x_n - y_n)\partial_n\chi_\delta(z) = 0\,, \end{aligned} \tag{2.44}$$

where in the last inequality we have used (2.38). This proves (2.43) which, in turn, contradicts (2.42). Hence, (2.40) is proven.

From here, the proof proceeds as follows. First, elementary topology gives that, for an open set $\Omega \subset \mathbb{R}^n$,

$$\partial\Omega = \partial\overline{\Omega} \Longleftrightarrow \overline{\big[\overline{\Omega}^c\big]} = \Omega^c\,. \tag{2.45}$$

Let us now fix

$$0 < b_2 < b_1, \quad 0 < c_2 < c_1, \qquad \text{with} \quad ab_2 < c_2\,. \tag{2.46}$$

Since $A^\circ \cap \bar{B} \subseteq \overline{A \cap B}$ for any two sets $A, B \subset \mathbb{R}^n$, and since $\overline{\mathcal{C}_{b_2,c_2}} \subset (\mathcal{C}_{b_1,c_1})^\circ$, it follows that

$$\overline{\mathcal{C}_{b_2,c_2}} \cap \overline{E} \subseteq \overline{\mathcal{C}_{b_1,c_1} \cap E}, \qquad \forall\, E \subset \mathbb{R}^n\,. \tag{2.47}$$

Utilizing this with $E := \Omega$, and $E := \overline{\Omega}^c$ (in which case (2.45) ensures that $\overline{E} = \Omega^c$), we obtain

$$\overline{\mathcal{C}_{b_2,c_2}} \cap \overline{\Omega} \subseteq \overline{\mathcal{C}_{b_1,c_1} \cap \Omega}, \qquad \overline{\mathcal{C}_{b_2,c_2}} \cap \Omega^c \subseteq \overline{\mathcal{C}_{b_1,c_1} \cap \overline{\Omega}^c}\,. \tag{2.48}$$

In turn, (2.39)–(2.40), the inclusions in (2.48), and a limiting argument give

$$x_n - y_n \leq a\,\big|x' - y'\big|, \qquad \forall\, x \in \overline{\mathcal{C}_{b_2,c_2}} \cap \overline{\Omega}, \quad \forall\, y \in \overline{\mathcal{C}_{b_2,c_2}} \cap \Omega^c\,. \tag{2.49}$$

At this stage we make the claim that

$$\overline{B_{b_2}} \times \{-c_2\} \subset \Omega \quad \text{and} \quad \overline{B_{b_2}} \times \{+c_2\} \subset \overline{\Omega}^c\,. \tag{2.50}$$

To prove the first inclusion we reason by contradiction and assume that there exist $y' \in \overline{B_{b_2}}$ such that $y := (y', -c_2)$ belongs to Ω^c. It follows that $y \in \overline{\mathcal{C}_{b_2,c_2}} \cap \Omega^c$. Since $0 \equiv x_0 \in \mathcal{C}_{b_2,c_2} \cap \overline{\Omega}$, writing (2.49) for this y and $x := 0$ gives $c_2 \leq a|y'| \leq ab_2$, contradicting the last inequality in (2.46). This justifies the first inclusion in (2.50), and the second one can be checked in a similar fashion.

For each $x' \in \overline{B_{b_2}}$ consider the closed segment $I_{x'} := \{(x', t) : -c_2 \leq t \leq c_2\}$, whose endpoints belong to Ω and $\overline{\Omega}^c$, respectively, by (2.50). Since $I_{x'} \subseteq \Omega \cup \overline{\Omega}^c \cup \partial\Omega$, a simple connectivity argument shows that $I_{x'} \cap \partial\Omega \neq \emptyset$. This further implies $J_{x'} \cap \partial\Omega \neq \emptyset$, where $J_{x'} := \{(x', t) : -c_2 < t < c_2\}$. We now claim that the cardinality of $J_{x'} \cap \partial\Omega$ is one. Indeed, if there exist $t_1, t_2 \in (-c_2, c_2)$ with $t_1 \neq t_2$ and such that

$$(x', t_1),\ (x', t_2) \in J_{x'} \cap \partial\Omega \subseteq \left(\overline{\mathcal{C}_{b_2,c_2}} \cap \overline{\Omega}\right) \cap \left(\overline{\mathcal{C}_{b_2,c_2}} \cap \Omega^c\right), \tag{2.51}$$

we obtain from (2.49) (written first for $x := (x', t_1)$, $y := (x', t_2)$, then for $x := (x', t_2)$, $y := (x', t_1)$), that

$$t_1 - t_2 \leq 0 \quad \text{and} \quad t_2 - t_1 \leq 0\,. \tag{2.52}$$

Hence, $t_1 = t_2$. This proves that, given $x' \in \overline{B_{b_2}}$ there exists a unique $\varphi(x') \in (-c_2, c_2)$ such that

$$\begin{aligned} \{(x', t) : -c_2 \leq t < \varphi(x')\} &\subseteq \Omega\,, \\ \{(x', t) : \varphi(x') < t \leq c_2\} &\subseteq \overline{\Omega}^c\,, \\ (x', \varphi(x')) &\in \partial\Omega\,. \end{aligned} \tag{2.53}$$

Furthermore, the same reasoning shows that the application

$$B_{b_2} \ni x' \mapsto (x', \varphi(x')) \in \mathcal{C}_{b_2,c_2} \cap \partial\Omega \tag{2.54}$$

is onto and, since $x_0 \equiv 0$, we also have $\varphi(0) = 0$. Furthermore, from (2.49) we obtain

$$x', y' \in B_{b_2} \Longrightarrow |\varphi(x') - \varphi(y')| \leq a\,|x' - y'|\,, \tag{2.55}$$

so φ is Lipschitz with Lipschitz constant $\leq a$. From (2.53)–(2.54) it is then easy to deduce that

$$\begin{aligned} \mathcal{C}_{b_2,c_2} \cap \Omega &= \{(x', t) \in \mathcal{C}_{b_2,c_2} : t < \varphi(x')\}\,, \\ \mathcal{C}_{b_2,c_2} \cap \overline{\Omega}^c &= \{(x', t) \in \mathcal{C}_{b_2,c_2} : t > \varphi(x')\}\,, \\ \mathcal{C}_{b_2,c_2} \cap \partial\Omega &= \{(x', t) \in \mathcal{C}_{b_2,c_2} : t = \varphi(x')\}\,. \end{aligned} \tag{2.56}$$

To fully match the demands stipulated in Definition 2.5, there remains to extend $\varphi : B_{b_2} \to \mathbb{R}$ to a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$. That this is possible is well-known. Indeed, Kirszbraun's Theorem asserts that any Lipschitz function defined on a subset of a metric space can be extended to a Lipschitz function on the entire space with the same Lipschitz constant (see, e.g., [26]; for a more elementary result which will, nonetheless, do in the current context see Theorem 5.1 on p. 29 in [24]). This shows that Ω is a strongly Lipschitz domain near x_0, hence concluding the proof of the theorem. □

Remark 2.10.

(i) If Ω has compact boundary, then the Lipschitz character of Ω is controlled in terms of the transversality constant of a continuous globally transversal unit vector X [hence, ultimately, on the constant a appearing in (2.33)], along with the modulus of continuity of X.

(ii) An inspection of the above proof reveals that, as a bonus feature, the following result holds:

If Ω is a nonempty, proper open subset of $\mathbb{R}^n$, of locally finite perimeter, for which (2.22) *holds, and if X is a continuous transversal vector field near $x_0 \in \partial\Omega$, then $\partial\Omega$ is a Lipschitz graph near x_0 in the direction of $-X(x_0)$.* As a consequence, (whose significance will become clearer later), for each $t \in (0, t_o)$ and $x \in \partial\Omega$ we have

$$x - t\,X(x) \in \Omega, \qquad x + t\,X(x) \in \mathbb{R}^n \setminus \overline{\Omega}, \tag{2.57}$$

whenever Ω is a bounded strongly Lipschitz domain, X is a continuous globally transversal vector field to $\partial\Omega$, and $t_o > 0$ is sufficiently small (depending on Ω and X).

An immediate consequence of Proposition 2.3 and Theorem 2.7 is the following.

Corollary 2.11. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set of finite perimeter for which* (2.22) *holds. Then Ω is a strongly Lipschitz domain if and only if*

$$\inf\left\{\|\nu - f\|_{L^\infty(\partial\Omega, d\sigma)} : f \in C^0\big(\partial\Omega, \mathbb{R}^n\big),\ |f| = 1 \text{ on } \partial\Omega\right\} < \sqrt{2}\,. \tag{2.58}$$

Another characterization of locally strongly Lipschitz domains can be given in terms of local containment of the unit normal in a fixed cone.

Corollary 2.12. *Let Ω be a proper open subset of $\mathbb{R}^n$, of locally finite perimeter and for which* (2.22) *holds. Denote by ν and σ the outward unit normal and surface measure on $\partial\Omega$. Then Ω is a locally strongly Lipschitz domain if and only if*

$$\begin{aligned}&\forall x \in \partial\Omega,\ \ \exists r > 0 \text{ and } \exists\,\Gamma \text{ circular cone, with vertex at } 0, \text{ of aperture} < \pi \\ &\text{with the property that } \nu(y) \in \Gamma \text{ for } \sigma\text{-almost every } y \in B(x,r) \cap \partial\Omega\,.\end{aligned} \tag{2.59}$$

Proof. In one direction, if Ω is a locally strongly Lipschitz domain, then (2.59) is readily seen from (2.23). In the opposite direction, assume that (2.59) holds. Then, if v is the unit vector along the vertical axis in Γ, it follows that $X \equiv v$ is a continuous vector field which is transversal to $\partial\Omega$ near x. Thus, Theorem 2.7 applies and gives that Ω is a locally strongly Lipschitz domain. □

We say that $\Omega \subset \mathbb{R}^n$ satisfies the *interior corkscrew condition* if there are constants $M > 1$ and $R > 0$ such that for each $x \in \partial\Omega$ and $r \in (0, R)$ there exists $y = y(x, r) \in \Omega$, called corkscrew point relative to x, such that $|x - y| < r$ and $\operatorname{dist}(y, \partial\Omega) > M^{-1}r$. Also, $\Omega \subset \mathbb{R}^n$ satisfies the *exterior corkscrew condition* if $\Omega^c := \mathbb{R}^n \setminus \Omega$ satisfy the interior corkscrew condition. Finally, Ω satisfies the *two sided corkscrew condition* if it satisfies both the interior and exterior corkscrew conditions.

It is clear from (2.9) and the above definition that, for an open set $\Omega \subset \mathbb{R}^n$,

$$\Omega \text{ satisfies the two sided corkscrew condition} \Longrightarrow \partial_*\Omega = \partial\Omega\,. \tag{2.60}$$

We complement this with the following elementary topological result:

$$\Omega \text{ satisfies the exterior corkscrew condition} \Longrightarrow \partial\Omega = \partial\overline{\Omega}\,. \tag{2.61}$$

See Lemma 5.13 for a proof.

One of the virtues of the corollary below is that it makes it clear that a bounded NTA domain (cf. Section 5 for a definition) of finite perimeter is a strongly Lipschitz domain if and only if has a continuous, globally transversal vector field.

Corollary 2.13. *For each nonempty, bounded open subset Ω of $\mathbb{R}^n$, the following are equivalent:*

(i) *Ω is a strongly Lipschitz domain;*
(ii) *Ω is a domain of finite perimeter, satisfying an exterior corkscrew condition, and having a continuous globally transversal vector field.*

Proof. The implication (i)$\Rightarrow$(ii) is well-known. In the opposite direction, it follows from (2.61) that the hypotheses of Theorem 2.7 are satisfied. The desired conclusion follows. $\square$

Our next results establishes a link between the cone property and the direction of the unit normal.

Proposition 2.14. *Let Ω be a proper, nonempty open subset of $\mathbb{R}^n$, of locally finite perimeter. Fix $x_0 \in \partial^*\Omega$ with the property that there exists a (circular, open, truncated, one-component) cone Γ with vertex at 0 and having aperture $\theta \in (0, \pi)$, for which*

$$x_0 + \Gamma \subseteq \Omega\,. \tag{2.62}$$

Denote by Γ_* the (circular, open, infinite, one-component) cone with vertex at 0, of aperture $\pi - \theta$, having the same axis as Γ and pointing in the opposite direction to Γ. Then, if ν denotes the outward unit normal to $\partial\Omega$, there holds

$$\nu(x_0) \in \Gamma_*\,. \tag{2.63}$$

Proof. Consider the half-space

$$H(x_0) := \big\{y \in \mathbb{R}^n : \nu(x_0)\cdot(y - x_0) < 0\big\} \tag{2.64}$$

and, for each $r > 0$ and $E \subseteq \mathbb{R}^n$, set

$$E_r := \big\{y \in \mathbb{R}^n : r(y - x_0) + x_0 \in E\big\}\,. \tag{2.65}$$

Also, denote by $\widetilde{\Gamma}$ the (circular, open, infinite) cone which coincides with Γ near its vertex. The theorem concerning the blow-up of the reduced boundary of a set of locally finite perimeter (cf. e.g., p. 199 in [8]) gives that

$$\mathbf{1}_{\Omega_r} \longrightarrow \mathbf{1}_{H(x_0)} \quad \text{in} \quad L^1_{\text{loc}}\big(\mathbb{R}^n\big), \quad \text{as} \quad r \to 0^+\,. \tag{2.66}$$

On the other hand, it is clear that $(x_0+\Gamma)_r \subset \Omega_r$ and $\mathbf{1}_{(x_0+\Gamma)_r} \longrightarrow \mathbf{1}_{x_0+\widetilde{\Gamma}}$ in $L^1_{\text{loc}}(\mathbb{R}^n)$ as $r \to 0^+$. This and (2.65) then allow us to write

$$\begin{aligned}
\mathbf{1}_{x_0+\widetilde{\Gamma}} &= \lim_{r\to 0^+} \mathbf{1}_{(x_0+\Gamma)_r} = \lim_{r\to 0^+}\Big(\mathbf{1}_{(x_0+\Gamma)_r}\cdot \mathbf{1}_{\Omega_r}\Big) \\
&= \Big(\lim_{r\to 0^+} \mathbf{1}_{(x_0+\Gamma)_r}\Big)\cdot\Big(\lim_{r\to 0^+} \mathbf{1}_{\Omega_r}\Big) = \mathbf{1}_{x_0+\widetilde{\Gamma}}\cdot \mathbf{1}_{H(x_0)} \\
&= \mathbf{1}_{(x_0+\widetilde{\Gamma})\cap H(x_0)}\,,
\end{aligned} \tag{2.67}$$

in a pointwise a.e. sense in $\mathbb{R}^n$. In turn, this implies

$$x_0 + \widetilde{\Gamma} \subseteq \overline{H(x_0)}\,. \tag{2.68}$$

Now, (2.63) readily follows from this, (2.64), the definition of Γ_* and simple geometrical considerations. $\square$

Corollary 2.15. *Assume that Ω is a proper, nonempty open subset of $\mathbb{R}^n$, of locally finite perimeter, and for which (2.22) holds. Denote by σ the surface measure on $\partial\Omega$.*

Then Ω is a locally strongly Lipschitz domain if and only if the following condition is verified. For every $x \in \partial\Omega$ there exist $r > 0$ along with a (circular, open, truncated, one-component) cone Γ with vertex at $0 \in \mathbb{R}^n$ such that

$$y + \Gamma \subseteq \Omega \quad \text{for } \sigma\text{-a.e. } y \in B(x, r) \cap \partial\Omega . \tag{2.69}$$

Proof. In one direction, it is clear that if $\Omega \subset \mathbb{R}^n$ is a locally strongly Lipschitz domain then Ω satisfies (2.69). Consider next the opposite implication, which is the crux of the matter here. Fix an arbitrary point $x \in \partial\Omega$, and let $r > 0$, Γ be such that (2.69) holds. One can, of course, assume that the aperture of Γ is $< \pi$. In concert with the fact that σ is supported on $\partial^*\Omega$, condition (2.69) implies $y + \Gamma \subseteq \Omega$ for σ-a.e. $y \in B(x, r) \cap \partial^*\Omega$. In light of Proposition 2.14, this further entails $\nu(y) \in \Gamma_*$ for σ-a.e. $y \in B(x, r) \cap \partial\Omega$, where ν stands for the outward unit normal to $\partial\Omega$. Then the desired conclusion follows from Corollary 2.12. □

Let us now revisit the uniform cone condition and consider a related, weaker version of (2.24). Specifically, we say that $D \subseteq \mathbb{R}^n$ satisfies a (local, uniform) *weak cone property* if the following holds. For every $x_0 \in \partial D$ there exist $r > 0$ along with an open, circular, truncated, one-component cone Γ with vertex at $0 \in \mathbb{R}^n$ such that

$$x + \Gamma \subseteq D, \qquad \forall x \in B(x_0, r) \cap \partial D . \tag{2.70}$$

Proposition 2.16. *Any proper, nonempty open subset Ω of $\mathbb{R}^n$ whose complement satisfies a (local, uniform) weak cone property is a locally strongly Lipschitz domain.*

Proof. To begin with, based on the two-sided weak cone property and a reasoning very similar to that in the proof of Lemma 5.13, we may conclude that (2.22) holds. Our goal is to show that Ω has locally finite perimeter. To set the stage, recall that generally speaking, $\mathcal{H}^{n-1}(E) \leq C_n \lim_{\delta\to 0^+} \mathcal{H}^{n-1}_\delta(E)$, where $\mathcal{H}^{n-1}_\delta(E)$ denotes the infimum of all sums $\sum_{B\in\mathcal{B}}(\text{radius } B)^{n-1}$, associated with all covers $\mathcal{B}$ of E with balls B of radii $\leq \delta$.

Next, fix $x_0 \in \partial\Omega$ and assume that the number $r > 0$ and the cone Γ are so that $x + \Gamma \subseteq \Omega^c$ for every $x \in B(x_0, r) \cap \partial\Omega$. Let $\theta \in (0, \pi)$, L line in $\mathbb{R}^n$, and $h > 0$ be, respectively, the aperture, axis and height of Γ. For some fixed $\lambda \in (0, 1)$, to be specified later, consider $\Gamma_\lambda \subset \Gamma$ to be the open, truncated, circular, one-component cone of aperture $\lambda\,\theta$ with vertex at $0 \in \mathbb{R}^n$ and having the same height h and symmetry axis L as Γ. Elementary geometry gives

$$|x - y| < h,\ x \notin y + \Gamma,\ y \notin x + \Gamma \Longrightarrow |x - y| \leq \frac{\operatorname{dist}(x + L\,,\ y + L)}{\sin(\theta/2)} . \tag{2.71}$$

In subsequent considerations, it can be assumed that r is smaller than a fixed fraction of h; in order to fix ideas, suppose whenceforth that $r \leq h/10$.

In order to continue, select a small number $\delta \in (0, r)$ and cover $\partial\Omega \cap B(x_0, r)$ by a family of balls $\{B(x_j, r_j)\}_{j\in J}$ with $x_j \in \partial\Omega$, $0 < r_j \leq \delta$, such that $\{B(x_j, r_j/5)\}_{j\in J}$ are mutually disjoint. Then $\mathcal{H}^{n-1}_\delta(\partial\Omega \cap B(x_0, r)) \leq C_n \sum_{j\in J} r_j^{n-1}$. Let π be a fixed $(n-1)$-plane perpendicular to the axis of Γ and denote by A_j the projection of $(x_j + \Gamma_\lambda) \cap B(x_j, r_j/5)$ onto π. Clearly, $\mathcal{H}^{n-1}(A_j) \approx r_j^{n-1}$, for every $j \in J$, and there exists a $(n-1)$-dimensional ball of radius $3r$ in π containing all A_j's.

We now claim that $\lambda > 0$ can be chosen sufficiently small as to ensure that the A_j's are mutually disjoint. Indeed, if $A_{j_1} \cap A_{j_2} \neq \emptyset$, for some $j_1, j_2 \in J$, then $\operatorname{dist}(x_{j_1} + L\,,\ x_{j_2} + L) \leq$

$(r_{j_1}+r_{j_2})\sin(\lambda\theta/2)$. Also, $|x_{j_1}-x_{j_2}|\geq(r_{j_1}+r_{j_2})/5$, as $B(x_{j_1},r_{j_1}/5)\cap B(x_{j_2},r_{j_2}/5)=\emptyset$. Note that $|x_{j_1}-x_{j_2}|\leq 4r<h$. Since also $\partial\Omega\ni x_{j_1}\notin x_{j_2}+\Gamma\subseteq(\Omega^c)^\circ$ plus a similar condition with the roles of j_1 and j_2 reversed, it follows from (2.71) that $(r_{j_1}+r_{j_2})/5\leq(r_{j_1}+r_{j_2})\sin(\lambda\theta/2)/\sin(\theta/2)$, or $\sin(\theta/2)<5\sin(\lambda\theta/2)$. Taking $\lambda\in(0,1)$ sufficiently small this leads to a contradiction. This finishes the proof of the claim that the A_j's are mutually disjoint if λ is small enough.

Assuming that this is the case, we thus obtain $\sum_{j\in J}r_j^{n-1}\leq C\sum_{j\in J}\mathcal{H}^{n-1}(A_j)\leq C\mathcal{H}^{n-1}(\cup A_j)\leq Cr^{n-1}$, given the containment condition on the A_j's. As a consequence, $\mathcal{H}_\delta^{n-1}(\partial\Omega\cap B(x_0,r))\leq Cr^{n-1}$, so by taking the supremum over $\delta>0$ we arrive at $\mathcal{H}^{n-1}(\partial\Omega\cap B(x_0,r))\leq Cr^{n-1}$. In particular, $\mathcal{H}^{n-1}(\partial\Omega_*\cap B(x_0,r))\leq\mathcal{H}^{n-1}(\partial\Omega\cap B(x_0,r))<\infty$ so, by (2.11), Ω has locally finite perimeter. With this in hand, Corollary 2.15 applies and gives that Ω is a locally strongly Lipschitz domain. □

Remark 2.17. The same type of argument as above shows that a proper, nonempty open subset Ω of $\mathbb{R}^n$ satisfying (2.22) as well as a (local, uniform) weak cone property is, in fact, a locally strongly Lipschitz domain.

Definition 2.18. A nonempty, bounded open subset Ω of $\mathbb{R}^n$ is called a *bounded C^1-domain* if it is a strongly Lipschitz domain and the Lipschitz functions $\varphi:\mathbb{R}^{n-1}\to\mathbb{R}$ whose graphs locally describe $\partial\Omega$, in the sense of Definition 2.5, can be taken to be of class C^1.

We conclude this section with an intrinsic characterization of the class of bounded C^1 domains in $\mathbb{R}^n$. Specifically, we shall prove the following.

Theorem 2.19. *Assume that Ω is a nonempty, bounded open subset of $\mathbb{R}^n$, of locally finite perimeter, for which* (2.22) *holds, and denote by ν the geometric measure theoretic outward unit normal to $\partial\Omega$, as defined in* (2.2)–(2.3). *Then Ω is a C^1 domain if and only if, after altering ν on a set of σ-measure zero,*

$$\nu\in C^0\big(\partial\Omega,\mathbb{R}^n\big)\,. \tag{2.72}$$

Proof. In one direction, assume that Ω is a bounded C^1 domain, and fix $x_0\in\partial\Omega$. If $\varphi:\mathbb{R}^{n-1}\to\mathbb{R}$ is a function of class C^1 whose graph, in a suitable system of coordinates, (x',t), isometric to the standard one, matches $\partial\Omega$ near x_0, then (2.23) holds. Then (2.72) can be read off this.

The main issue here is the opposite implication. Assuming that (2.72) holds, it follows that ν is a continuous globally transversal vector field to Ω. Theorem 2.7 then gives that Ω is a strongly Lipschitz domain. Then, if the point $x_0\in\partial\Omega$ (identified with $0\in\mathbb{R}^n$) and the Lipschitz function $\varphi:\mathbb{R}^{n-1}\to\mathbb{R}$ are as in Definition 2.5, it follows from (2.23) that $\nu_n(x',\varphi(x'))\neq 0$ and

$$\partial_j\varphi\big(x'\big)=-\frac{\nu_j\big(x',\varphi(x')\big)}{\nu_n\big(x',\varphi(x')\big)},\qquad j=1,\dots,n-1\,, \tag{2.73}$$

granted that x' is near $0\in\mathbb{R}^{n-1}$. Since φ is continuous and (2.72) holds, this further implies that all first-order partial derivatives of φ are continuous functions near $0\in\mathbb{R}^{n-1}$. With this in hand, it is then easy to conclude that Ω is, in fact, a C^1 domain. □

In closing, we wish to point out that, under the same hypotheses as Theorem 2.19, the argument in the proof above shows that, in fact,

$$\Omega\text{ is a }C^{1+\alpha}\text{-domain}\Longleftrightarrow\nu\in C^\alpha\big(\partial\Omega,\mathbb{R}^n\big)\,, \tag{2.74}$$

for every $\alpha \in (0, 1)$, and

$$\Omega \text{ is a } C^{1,1}\text{-domain} \Longleftrightarrow \nu \text{ is Lipschitz}. \tag{2.75}$$

3. Finite perimeter domains under bi-Lipschitz and C^1 diffeomorphisms

As is well-known, the class of topological boundaries is invariant under topological homeomorphisms. Our first result clarifies how the measure theoretic boundaries, as well as the reduced boundaries, of sets of locally finite perimeter in $\mathbb{R}^n$ transform under bi-Lipschitz maps. Before stating it, we take care of a number of prerequisites.

If $\mathcal{O} \subseteq \mathbb{R}^n$ and $F : \mathcal{O} \to \mathbb{R}^n$ is a Lipschitz function, set

$$\mathrm{Lip}\,(F, \mathcal{O}) := \sup\Big\{|F(x) - F(y)|/|x - y| : x, y \in \mathcal{O},\ x \neq y\Big\}. \tag{3.1}$$

Then, with $\mathcal{H}^s$ denoting the s-dimensional Hausdorff measure in $\mathbb{R}^n$, we have (cf. Theorem 1 on p. 75 in [8])

$$\mathcal{H}^s(F(E)) \leq [\mathrm{Lip}\,(F, \mathcal{O})]^s\, \mathcal{H}^s(E), \qquad \forall\, E \subset \mathcal{O}, \qquad s \geq 0. \tag{3.2}$$

As is well-known, if $\mathcal{O} \subseteq \mathbb{R}^n$ is open and $F = (F_1, \dots, F_n) : \mathcal{O} \to \mathbb{R}^n$ is a Lipschitz function then the Jacobian matrix of F, i.e., $DF := (\partial_k F_j)_{1\leq j,k\leq n}$, exists a.e. (cf. [22]) and

$$\|DF\| \leq \mathrm{Lip}\,(F, \mathcal{O}) \quad \text{a.e. in } \mathcal{O}, \tag{3.3}$$

where, given a matrix A, $\|A\|$ denotes the norm of A viewed as a linear operator. Recall that for any $n \times n$ matrix A, $|\det A|$ is the volume of the parallelopiped spanned by the vectors $Ae_1, \dots, Ae_n$, so $|\det A| \leq \|Ae_1\| \cdots \|Ae_n\| \leq \|A\|^n$. Consequently,

$$|\det DF(x)| \leq [\mathrm{Lip}\,(F, \mathcal{O})]^n \quad \text{for a.e.} \quad x \in \mathcal{O}. \tag{3.4}$$

Going further, call a Lipschitz function $F : \mathcal{O} \to \mathbb{R}^n$ *bi-Lipschitz* if F is one-to-one and $\mathrm{Lip}\,(F^{-1}, F(\mathcal{O})) < \infty$. It is known that bi-Lipschitz functions are open; in particular, $F(\mathcal{O})$ is open and $F : \mathcal{O} \to F(\mathcal{O})$ is a topological homeomorphism. Furthermore, while the Chain Rule may, generally speaking, fail for Lipschitz functions, we do have (with $I_{n\times n}$ denoting the $n \times n$ identity matrix),

$$\big[(DF^{-1}) \circ F\big][DF] = I_{n\times n}, \qquad \text{a.e. in } \mathcal{O}, \tag{3.5}$$

if $\mathcal{O} \subseteq \mathbb{R}^n$ is open and $F : \mathcal{O} \to \mathbb{R}^n$ is bi-Lipschitz. Hence, in this case we also have the lower bound

$$\big[\mathrm{Lip}\,\big(F^{-1}, F(\mathcal{O})\big)\big]^{-n} \leq |\det DF(x)| \quad \text{for a.e.} \quad x \in \mathcal{O}. \tag{3.6}$$

In addition, as observed by Rademacher (cf. p. 354 in [22]),

$$\mathcal{O} \text{ connected} \Longrightarrow \text{ either } \det(DF) > 0 \text{ a.e. in } \mathcal{O}, \text{ or } \det(DF) < 0 \text{ a.e. in } \mathcal{O}. \tag{3.7}$$

In the sequel, whenever the context is clear, we shall lighten the notation and simply write $\mathrm{Lip}\,(F)$, $\mathrm{Lip}\,(F^{-1})$ in place of $\mathrm{Lip}\,(F, \mathcal{O})$, $\mathrm{Lip}\,(F^{-1}, F(\mathcal{O}))$. A case in point is the statement that if the function $F : \mathcal{O} \to \mathbb{R}^n$ is bi-Lipschitz then, for every $x \in \mathcal{O}$ and $r > 0$,

$$B(F(x), \big(\mathrm{Lip}\, F^{-1}\big)^{-1} r) \cap F(\mathcal{O}) \subseteq F(B(x, r) \cap \mathcal{O}) \subseteq B(F(x), (\mathrm{Lip}\, F)\, r) \cap F(\mathcal{O}). \tag{3.8}$$

Call $F : \mathcal{O} \to \mathbb{R}^n$ *locally Lipschitz* (respectively, *locally bi-Lipschitz*) if for every $x \in \mathcal{O}$ there exists $r > 0$ with the property that $F : B(x,r) \cap \mathcal{O} \to \mathbb{R}^n$ is Lipschitz (respectively, bi-Lipschitz).

Next, we briefly review the concept of the push-forward of a measure. Let $\mathcal{O}, \widetilde{\mathcal{O}} \subseteq \mathbb{R}^n$ be two open sets and let $F : \mathcal{O} \to \widetilde{\mathcal{O}}$ be a continuous, proper map. If μ is a Borelian measure on $\mathcal{O}$ we define the Borelian measure $F_*\mu$ on $\widetilde{\mathcal{O}}$, the *push-forward of* μ *via* F, as

$$F_*\mu(E) := \mu\big(F^{-1}(E)\big), \qquad \forall\, E \subseteq \mathcal{O} \text{ Borel set}. \tag{3.9}$$

Note that this entails

$$\int f\, dF_*\mu = \int f \circ F\, d\mu, \qquad \forall\, f \in C^0\big(\widetilde{\mathcal{O}}\big), \text{ compactly supported}, \tag{3.10}$$

$$(F_*\mu)\lfloor E = F_*(\mu\lfloor F^{-1}(E)), \qquad \forall\, E \subseteq \mathcal{O} \text{ Borel set}, \tag{3.11}$$

$$F_*(f\mu) = \big(f \circ F^{-1}\big) F_*\mu \quad \text{if } F \text{ is a topological homeomorphism}, \tag{3.12}$$

$$G_*(F_*\mu) = (G \circ F)_*\mu, \qquad \text{if } G : \widetilde{\mathcal{O}} \to \widetilde{\widetilde{\mathcal{O}}} \text{ is a continuous, proper function}. \tag{3.13}$$

Finally, we make the following definition. Given a Radon measure μ in $\mathbb{R}^n$ and two sets $A, B \subseteq \mathbb{R}^n$, we write $A \equiv B$ *modulo* μ, if $\mu(A \triangle B) = 0$, where $A \triangle B := (A \setminus B) \cup (B \setminus A)$ is the symmetric difference of A and B.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set of locally finite perimeter, $\mathcal{O}$ an open neighborhood of $\overline{\Omega}$, and $F : \mathcal{O} \to \mathbb{R}^n$ an injective, locally bi-Lipschitz mapping. Then $\widetilde{\Omega} := F(\Omega)$ is also an open set of locally finite perimeter and, in addition,*

$$\partial_*\widetilde{\Omega} = F(\partial_*\Omega). \tag{3.14}$$

Moreover,

$$\partial^*\widetilde{\Omega} \equiv F\big(\partial^*\Omega\big) \ \text{ modulo } \mathcal{H}^{n-1}, \tag{3.15}$$

so that, in particular,

$$\widetilde{\sigma}\big(\mathbb{R}^n \setminus F\big(\partial^*\Omega\big)\big) = 0, \tag{3.16}$$

where $\widetilde{\sigma}$ denotes the surface measure on $\partial\widetilde{\Omega}$.

Finally, if σ stands for the surface measure on $\partial\Omega$, then

$$\widetilde{\sigma} \text{ and } F_*\sigma \text{ are mutually absolutely continuous}. \tag{3.17}$$

Proof. Formula (3.14) is a consequence of definition (2.9) and the fact that an injective bi-Lipschitz mapping is a topological homeomorphism that changes the Lebesgue measure of the subsets of a given compact set at most by a factor [that is bounded and bounded away from zero — cf. (3.2)]. Then the fact that $\widetilde{\Omega}$ has locally finite perimeter is a consequence of (3.14), (3.2), and (2.10).

Turning our attention to (3.15), using (2.8), (3.14), and (3.2), we compute

$$\begin{aligned}\mathcal{H}^{n-1}\big(\partial^*\widetilde{\Omega} \setminus F\big(\partial^*\Omega\big)\big) &= \mathcal{H}^{n-1}\big(\partial_*\widetilde{\Omega} \setminus F\big(\partial^*\Omega\big)\big)\\ &= \mathcal{H}^{n-1}\big(F(\partial_*\Omega) \setminus F\big(\partial^*\Omega\big)\big) \le \mathcal{H}^{n-1}\big(F\big(\partial_*\Omega \setminus \partial^*\Omega\big)\big) = 0,\end{aligned} \tag{3.18}$$

since $\mathcal{H}^{n-1}(\partial_*\Omega\setminus\partial^*\Omega)=0$ and the class of sets of $\mathcal{H}^{n-1}$-measure zero is invariant under locally bi-Lipschitz mappings. Also,

$$\begin{aligned}\mathcal{H}^{n-1}\big(F(\partial^*\Omega)\setminus\partial^*\widetilde{\Omega}\big)&=\mathcal{H}^{n-1}\big(F(\partial^*\Omega)\setminus\partial_*\widetilde{\Omega}\big)\\&=\mathcal{H}^{n-1}\big(F(\partial^*\Omega)\setminus F(\partial_*\Omega)\big)=\mathcal{H}^{n-1}(\emptyset)=0\,.\end{aligned}\tag{3.19}$$

In concert, (3.18)–(3.19) give that $\partial^*\widetilde{\Omega}\equiv F(\partial^*\Omega)$ modulo $\mathcal{H}^{n-1}$. With this in hand, (3.16) follows from (2.7).

Finally, $E\subseteq\partial\widetilde{\Omega}$ is $\widetilde{\sigma}$-measurable if and only if $F^{-1}(E)$ is σ-measurable and, granted what we have proved up to this point,

$$\begin{aligned}(F_*\sigma)(E)=0\quad&\Leftrightarrow\quad\sigma\big(F^{-1}(E)\big)=0\Leftrightarrow\mathcal{H}^{n-1}\big(\partial^*\Omega\setminus F^{-1}(E)\big)=0\\&\Leftrightarrow\quad\mathcal{H}^{n-1}\big(\partial_*\Omega\setminus F^{-1}(E)\big)=0\Leftrightarrow\mathcal{H}^{n-1}\big(F^{-1}(\partial_*\widetilde{\Omega})\setminus F^{-1}(E)\big)=0\\&\Leftrightarrow\quad\mathcal{H}^{n-1}\big(F^{-1}(\partial_*\widetilde{\Omega}\setminus E)\big)=0\Leftrightarrow\mathcal{H}^{n-1}\big(\partial_*\widetilde{\Omega}\setminus E\big)=0\\&\Leftrightarrow\quad\mathcal{H}^{n-1}\big(\partial^*\widetilde{\Omega}\setminus E\big)=0\Leftrightarrow\widetilde{\sigma}(E)=0\,.\end{aligned}\tag{3.20}$$

This gives (3.17), completing the proof of the proposition. □

In the context of Proposition 3.1, (3.17) raises the issue of computing the Radon-Nikodym derivatives $d\widetilde{\sigma}/dF_*\sigma$ and $d(F^{-1})_*\widetilde{\sigma}/d\sigma$. Our next two theorems are devoted to addressing this issue. To state the first, we need to introduce some more notation. Given a $n\times n$ matrix A, denote by $A^\top$ the transposed of A, and by $\operatorname{adj}A$ the adjunct matrix (sometimes denoted $\operatorname{Cof}(A)$, whose entries are the cofactors of A). In particular,

$$A^\top(\operatorname{adj}A)=(\operatorname{adj}A)\,A^\top=(\det A)\,I_{n\times n}\,.\tag{3.21}$$

We also let $\operatorname{tr}A$ denote the trace of the $n\times n$ matrix A, and equip the space of such matrices with the inner product $\langle A,B\rangle:=\operatorname{tr}(A^\top B)$. Finally, if $A=(a_{jk})_{1\le j,k\le n}$ is a matrix with variable entries, we set

$$\operatorname{Div}A:=(\partial_k a_{jk})_{1\le j\le n}\,,\tag{3.22}$$

i.e., $\operatorname{Div}A$ is the vector whose components are the divergences of the lines of the matrix A.

Theorem 3.2. *Let $\Omega\subset\mathbb{R}^n$ be a domain of locally finite perimeter, $\mathcal{O}$ an open neighborhood of $\overline{\Omega}$, and let $F:\mathcal{O}\to\mathbb{R}^n$ be an orientation preserving C^1-diffeomorphism.*

Then $\widetilde{\Omega}:=F(\Omega)$ is a domain of locally finite perimeter and if ν, $\widetilde{\nu}$ and σ, $\widetilde{\sigma}$ are, respectively, the outward unit normals and surface measures on $\partial\Omega$ and $\partial\widetilde{\Omega}$, then

$$\widetilde{\nu}=\frac{(DF^{-1})^\top(\nu\circ F^{-1})}{\big\|(DF^{-1})^\top(\nu\circ F^{-1})\big\|}\,,\tag{3.23}$$

(with the convention that the right side of (3.23) is zero whenever $\nu\circ F^{-1}=0$), and

$$\widetilde{\sigma}=\big\|(DF^{-1})^\top(\nu\circ F^{-1})\big\|\,(J_F\circ F^{-1})\,F_*\sigma\,,\tag{3.24}$$

where

$$J_F(x):=|\det DF(x)|\,.\tag{3.25}$$

For certain purposes, it is convenient to rephrase (3.23)–(3.24) in a slightly different form. Specifically, since $(DF^{-1})^\top = [(\det(DF))^{-1}\mathrm{adj}\,(DF)] \circ F^{-1}$, we obtain the following.

Corollary 3.3. *In the context of Theorem 3.2,*

$$\widetilde{\nu} \circ F = \frac{\mathrm{adj}\,(DF)\,\nu}{\|\mathrm{adj}\,(DF)\,\nu\|}, \tag{3.26}$$

$$F_*^{-1}\widetilde{\sigma} = \|\mathrm{adj}\,(DF)\,\nu\|\,\sigma\,. \tag{3.27}$$

Proof of Theorem 3.2. We already know, from Proposition 3.1, that $\widetilde{\Omega}$ is a set of locally finite perimeter. To prove (3.23)–(3.24), fix $\vec{\varphi} \in C_0^\infty(F(\mathcal{O}), \mathbb{R}^n)$ and compute

$$\begin{aligned}\big\langle \nabla \mathbf{1}_{F(\Omega)}\,,\,\vec{\varphi}\big\rangle &= -\big\langle \mathbf{1}_{F(\Omega)}\,,\,\mathrm{div}\,\vec{\varphi}\big\rangle = -\big\langle \mathbf{1}_{\Omega} \circ F^{-1}\,,\,\mathrm{div}\,\vec{\varphi}\big\rangle \\ &= -\big\langle \mathbf{1}_{\Omega}\,,\,\big[\big(\mathrm{div}\,\vec{\varphi}\big)\circ F\big]\det(DF)\big\rangle\,. \end{aligned}\tag{3.28}$$

To continue, use the Chain Rule to write

$$D\big(\vec{\varphi}\circ F\big) = [(D\varphi)\circ F](DF) \Longrightarrow (D\varphi)\circ F = \big[D\big(\vec{\varphi}\circ F\big)\big](DF)^{-1} \tag{3.29}$$

from which we further deduce

$$\big(\mathrm{div}\,\vec{\varphi}\big)\circ F = \mathrm{tr}\,[(D\varphi)\circ F] = \mathrm{tr}\big[D\big(\vec{\varphi}\circ F\big)(DF)^{-1}\big] = \big\langle\big[(DF)^{-1}\big]^\top\,,\,D\big(\vec{\varphi}\circ F\big)\big\rangle\,. \tag{3.30}$$

Consequently,

$$[\det(DF)]\big(\mathrm{div}\,\vec{\varphi}\big)\circ F = \big\langle \det(DF)\big[(DF)^{-1}\big]^\top\,,\,D\big(\vec{\varphi}\circ F\big)\big\rangle = \big\langle \mathrm{adj}\,(DF)\,,\,D\big(\vec{\varphi}\circ F\big)\big\rangle\,. \tag{3.31}$$

Returning with this in (3.31) then yields

$$\big\langle \nabla \mathbf{1}_{F(\Omega)}\,,\,\vec{\varphi}\big\rangle = -\Big\langle \mathbf{1}_{F(\Omega)}\,,\,\big\langle \mathrm{adj}\,(DF)\,,\,D\big(\vec{\varphi}\circ F\big)\big\rangle\Big\rangle\,. \tag{3.32}$$

For every matrix $A = (a_{jk})_{1\leq j\leq n}$ with reasonable variable entries and a sufficiently regular vector field $\vec{\varphi} = (\varphi_j)_{1\leq j\leq n}$, we compute (with the summation convention over repeated indices understood):

$$\begin{aligned}\big\langle A\,,\,D\big(\vec{\varphi}\circ F\big)\big\rangle &= a_{jk}\partial_k(\varphi_j\circ F) = \partial_k[a_{jk}(\varphi_j\circ F)] - (\partial_k a_{jk})(\varphi_j\circ F) \\ &= \mathrm{div}\,\big(A^\top\vec{\varphi}\circ F\big) - \big\langle \mathrm{Div}\,A\,,\,\vec{\varphi}\circ F\big\rangle\,. \end{aligned}\tag{3.33}$$

We intend to use the identity (3.33) for the matrix $A := \mathrm{adj}\,(DF)$, a scenario in which it is helpful to bring in the identity

$$\mathrm{Div}\,(\mathrm{adj}\,(DF)) = 0 \quad \text{in the sense of distributions}\,. \tag{3.34}$$

See [20] for a proof of (3.34) by induction, and pp. 440–441 in [7]. Given the importance of this formula for our purposes, we present a short, self-contained argument at the end of the current proof, based on the exterior calculus for differential forms (this proof will also play a role, later in this section as well as in Section 4.3). For now, granted (3.34), we obtain

$$\big\langle \nabla \mathbf{1}_{F(\Omega)}\,,\,\vec{\varphi}\big\rangle = -\big\langle \mathbf{1}_{\Omega}\,,\,\mathrm{div}\,\big(((\mathrm{adj}\,(DF))^\top\vec{\varphi}\circ F\big)\big\rangle\,. \tag{3.35}$$

Consider now a vector field $\vec{\psi} \in C_0^0(\mathcal{O}, \mathbb{R}^n)$ and such that $\mathrm{div}\,\vec{\psi} \in L^1(\mathcal{O})$, and recall the mollifiers φ_δ introduced just above (2.36). If we then set $\vec{\psi}_\delta := \varphi_\delta * \vec{\psi}$, it follows that $\vec{\psi}_\delta \to \vec{\psi}$

uniformly and $\operatorname{div}\vec{\psi}_\delta \to \operatorname{div}\vec{\psi}$ in $L^1(\mathcal{O})$ as $\delta \to 0^+$. Hence, based on the fact that Ω has locally finite perimeter [cf. (2.2)], we may write:

$$-\langle \mathbf{1}_\Omega, \operatorname{div}\vec{\psi}\rangle = -\lim_{\delta\to 0^+}\langle \mathbf{1}_\Omega, \operatorname{div}\vec{\psi}_\delta\rangle = \lim_{\delta\to 0^+}\langle \nabla\mathbf{1}_\Omega, \vec{\psi}_\delta\rangle = -\lim_{\delta\to 0^+}\langle \nu\sigma, \vec{\psi}_\delta\rangle = -\langle \nu\sigma, \vec{\psi}\rangle . \quad (3.36)$$

By using this for $\vec{\psi} := (\operatorname{adj}(DF))^\top\vec{\varphi}\circ F$ we arrive at the identity

$$\langle \nabla\mathbf{1}_{F(\Omega)}, \vec{\varphi}\rangle = -\Big\langle \sigma, \big\langle \nu, (\operatorname{adj}(DF))^\top\vec{\varphi}\circ F\big\rangle\Big\rangle . \quad (3.37)$$

Upon recalling (3.9)–(3.10), as well as (3.21) and the fact that $(DF)^{-1}\circ F^{-1} = DF^{-1}$, the right-hand side of (3.37) can be further transformed into

$$\begin{aligned}
-\Big\langle \sigma, &\big\langle \nu, (\operatorname{adj}(DF))^\top\vec{\varphi}\circ F\big\rangle\Big\rangle \\
&= -\Big\langle \big(J_F\circ F^{-1}\big)F_*\sigma, \big\langle \nu\circ F^{-1}, \big(\det(DF)^{-1}\big)\circ F^{-1}(\operatorname{adj}(DF))^\top\circ F^{-1}\vec{\varphi}\big\rangle\Big\rangle \\
&= -\Big\langle \big(J_F\circ F^{-1}\big)F_*\sigma, \big\langle \nu\circ F^{-1}, \big(DF^{-1}\big)\vec{\varphi}\big\rangle\Big\rangle \\
&= -\Big\langle \big(J_F\circ F^{-1}\big)F_*\sigma, \big\langle \big(DF^{-1}\big)^\top\big(\nu\circ F^{-1}\big), \vec{\varphi}\big\rangle\Big\rangle ,
\end{aligned} \quad (3.38)$$

from which (3.23)–(3.24) now follow (cf. [8]). Thus, we are done, except for the promised justification of (3.34).

To prove (3.34), we note that it suffices to treat the case when the components of $F = (F_1, \dots, F_n)$ are C^∞ mappings. Standard approximation results in Sobolev spaces then show that formula (3.34) holds when the components of F belong to $W^{1,p}_{\mathrm{loc}}$ with $p \geq n-1$. We make use of common notation in the calculus for differential forms. In particular, 'wedge' and 'backwards wedge' stand, respectively, for the exterior product and its adjoint, respectively. If we denote by A_{jk} the (j,k)-entry in the matrix $\operatorname{adj} A$, then

$$A_{jk}\, dx_1\wedge\ldots\wedge dx_n = (-1)^{j+1}\, dx_k \wedge \big[dF_1\wedge\ldots\wedge \widehat{dF_j}\wedge\ldots\wedge dF_n\big] , \quad (3.39)$$

with the convention that the 'hat' above a term means omission. Hence, if $*$ stands for the Hodge star-isomorphism in $\mathbb{R}^n$, the j-th component of the vector $\operatorname{Div}(\operatorname{adj} A)$ is:

$$\begin{aligned}
*\,(-1)^{j+1} d &\left(\sum_{k=1}^n A_{jk}(-1)^{k-1}\, dx_1\wedge\ldots\wedge\widehat{dx_k}\wedge\ldots\wedge dx_n\right) \\
&= *\,(-1)^{j+1} d\left(\sum_{k=1}^n dx_k \vee \big(dx_k\wedge\big[dF_1\wedge\ldots\wedge\widehat{dF_j}\wedge\ldots\wedge dF_n\big]\big)\right) \\
&= *\,(-1)^{j+1} d\big[dF_1\wedge\ldots\wedge\widehat{dF_j}\wedge\ldots\wedge dF_n\big] = 0 .
\end{aligned} \quad (3.40)$$

The second equality above utilizes the fact that

$$\sum_{k=1}^n dx_k \vee (dx_k\wedge u) = u , \quad (3.41)$$

for any $(n-1)$-form u, which is readily checked when u is of the form $dx_1\wedge\ldots\wedge\widehat{dx_i}\wedge\ldots\wedge dx_n$, then extends by linearity to arbitrary $(n-1)$-forms. Also, the last equality in (3.40) is based on repeated applications of Leibniz's product formula for the exterior differentiation operator d, and the fact that $d^2 = 0$. This finishes the justification of (3.34), and concludes the proof of the theorem. □

The approach to (3.23)–(3.24) taken above could actually be done entirely in the framework of differential forms. In a brief outline, given a vector field φ, we set $A\varphi = \varphi \vee (dx_1 \wedge \cdots \wedge dx_n) = \psi$, defining an isomorphism between vector fields and $(n-1)$-forms, satisfying

$$d\psi = (\operatorname{div} \varphi)\, dx_1 \wedge \cdots \wedge dx_n \,. \tag{3.42}$$

Hence,

$$\begin{aligned}\int_{\partial\tilde{\Omega}} \langle \tilde{\nu}, \varphi \rangle \, d\tilde{\sigma} &= \int_{\tilde{\Omega}} \operatorname{div} \varphi \, dx = \int_{\tilde{\Omega}} d\psi \\ &= \int_{\Omega} F^* d\psi = \int_{\Omega} dF^*\psi = \int_{\Omega} \operatorname{div}\big(A^{-1} F^* A\varphi\big)\, dx \,,\end{aligned} \tag{3.43}$$

the last identity by (3.42). If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping, let $\Lambda^k T$ denote the k-fold exterior product of T with itself. Then the last quantity in (3.43) is equal to $\int_{\partial\Omega} \langle \nu \,,\, A^{-1} \Lambda^{n-1} DF(x)^{\top} A\varphi(F(x)) \rangle \, d\sigma$, and we obtain

$$\tilde{\nu}(y)\, \tilde{\sigma} = \big(A^{-1} \Lambda^{n-1} DF\big(F^{-1}(y)\big)^{\top} A\big)^{\top} \, \nu\big(F^{-1}(y)\big) \, F_* \sigma \,. \tag{3.44}$$

Obtaining the equivalence of (3.44) with (3.23)–(3.24) is then a piece of algebra related to Cramer's formula. We omit the details.

In the approach via (3.43), the role of the somewhat mysterious formula (3.34) is taken by the more familiar identity

$$d\big(F^*\psi\big) = F^*(d\psi) \,, \tag{3.45}$$

where ψ is a differential form (in the current context, an $(n-1)$-form).

It is of interest to present an alternative analysis of the behavior of finite perimeter domains under C^1-diffeomorphisms which avoids the use of identities involving the divergence of vector fields. Here we do that and develop a line of proof which, instead, uses mollifiers, the change of variable formula for continuous integrands, and a limiting argument.

Specifically, let $\Omega \subset \mathbb{R}^n$ be a bounded open set of finite perimeter, and let F be a C^1 diffeomorphism of a neighborhood $\mathcal{O}$ of $\bar{\Omega}$ onto $\tilde{\mathcal{O}} \subset \mathbb{R}^n$, mapping Ω to $\tilde{\Omega}$. We will show that $\tilde{\Omega}$ has finite perimeter and give a formula for $\tilde{\nu}\tilde{\sigma} = -\nabla \mathbf{1}_{\tilde{\Omega}}$ in terms of $\nu\sigma = -\nabla \mathbf{1}_{\Omega}$.

To begin, let φ_δ be a mollifier, with (small) compact support, set $\chi_\delta = \varphi_\delta * \mathbf{1}_\Omega$, and set $\tilde{\chi}_\delta = \chi_\delta \circ F^{-1}$, so

$$\tilde{\chi}_\delta \longrightarrow \mathbf{1}_{\tilde{\Omega}}, \quad \chi_\delta \longrightarrow \mathbf{1}_\Omega, \quad \text{in} \quad L^1\text{-norm} \,, \tag{3.46}$$

as $\delta \to 0$. Hence,

$$\nabla \tilde{\chi}_\delta \longrightarrow \nabla \mathbf{1}_{\tilde{\Omega}}, \quad \nabla \chi_\delta \longrightarrow \nabla \mathbf{1}_\Omega, \quad \text{in} \ \mathcal{D}'(\mathbb{R}^n) \,. \tag{3.47}$$

The chain rule gives

$$\nabla \tilde{\chi}_\delta(F(x)) DF(x) = \nabla \chi_\delta(x), \quad \text{and} \quad \nabla \tilde{\chi}_\delta(y) = \nabla \chi_\delta\big(F^{-1}(y)\big) DF^{-1}(y) \,. \tag{3.48}$$

(To put $DF(x)$ on the left, make it $DF(x)^{\top}$.) Since Ω is assumed to have finite perimeter, if $\mathcal{M}$ denotes the collection of Borel measures in $\mathbb{R}^n$, we have

$$\nabla \chi_\delta \longrightarrow \nabla \mathbf{1}_\Omega, \quad \text{weak}^* \text{ in } \mathcal{M}\,, \tag{3.49}$$

with a bound on $\|\nabla \chi_\delta\|_{L^1}$ for $\delta \in (0, 1]$. Hence, by (3.48), we have a bound on $\|\nabla \tilde{\chi}_\delta\|_{L^1}$. It follows from this and (3.47) that $\widetilde{\Omega}$ has finite perimeter and

$$\nabla \tilde{\chi}_\delta \longrightarrow \nabla \mathbf{1}_{\widetilde{\Omega}}, \text{ weak}^* \text{ in } \mathcal{M}\,. \tag{3.50}$$

That is to say, $\nabla \mathbf{1}_{\widetilde{\Omega}} = -\tilde{\nu}\tilde{\sigma}$ with $\tilde{\sigma}$ surface area on $\partial\widetilde{\Omega}$ and

$$-\int \langle \tilde{\nu}, \varphi \rangle \, d\tilde{\sigma} = \lim_{\delta \to 0} \int \langle \nabla \tilde{\chi}_\delta(y), \varphi(y) \rangle \, dy\,, \tag{3.51}$$

for each $\varphi \in C_0^0(\widetilde{\mathcal{O}}, \mathbb{R}^n)$. Now, with $J_F(x) = |\det DF(x)|$, we have

$$\begin{aligned}
\int \langle \nabla \tilde{\chi}_\delta(y), \varphi(y) \rangle \, dy &= \int \langle \nabla \tilde{\chi}_\delta(F(x)), \varphi(F(x)) \rangle J_F(x) \, dx \\
&= \int \langle \nabla \chi_\delta(x), DF(x)^{-1} \varphi(F(x)) \rangle J_F(x) \, dx \\
&\to -\int \langle \nu(x), J_F(x) DF(x)^{-1} \varphi(F(x)) \rangle \, d\sigma(x)\,.
\end{aligned} \tag{3.52}$$

Hence, with $F_*\sigma$ given as in (3.9), we have from (3.51)–(3.52) that for each $\varphi \in C_0^0(\widetilde{\mathcal{O}}, \mathbb{R}^n)$,

$$\begin{aligned}
\int \langle \tilde{\nu}, \varphi \rangle \, d\tilde{\sigma} &= \int \langle J_F(x) \big(DF(x)^{-1}\big)^{\top} \nu(x), \varphi(F(x)) \rangle \, d\sigma(x) \\
&= \int \langle J_F\big(F^{-1}(y)\big) DF^{-1}(y)^{\top} \nu\big(F^{-1}(y)\big), \varphi(y) \rangle \, dF_*\sigma\,,
\end{aligned} \tag{3.53}$$

so

$$\tilde{\nu}(y)\,\tilde{\sigma} = DF^{-1}(y)^{\top} \nu\big(F^{-1}(y)\big)\, J_F\big(F^{-1}(y)\big)\, F_*\sigma\,, \tag{3.54}$$

again giving (3.23) and (3.24).

We next seek to relate $\tilde{\sigma}$ to $F_*\sigma$ in the more general case where F is merely bi-Lipschitz. In such a more general setting, (3.46)–(3.51) continue to hold but the convergence result in (3.52) might fail, since DF and J_F need not be continuous (and, in fact, the right side of (3.54) might not be well defined). In such a scenario, we shall make use of the (generalized) area formula, as presented in Section 12 of [24]. To set the stage for doing so, for the convenience of the reader we first review a number of definitions.

A set $M \subset \mathbb{R}^n$ is called *countably $(n-1)$-rectifiable* if it is a countable disjoint union

$$M = \bigcup_{j=0}^{\infty} M_j\,, \tag{3.55}$$

where $\mathcal{H}^{n-1}(M_0) = 0$ and each M_j, $j \geq 1$, is a compact subset of an $(n-1)$-dimensional C^1 surface N_j in $\mathbb{R}^n$. A countably rectifiable set $M \subset \mathbb{R}^n$ need not have tangent planes in the ordinary sense, but it will have approximate tangent planes. By definition, an $(n-1)$-plane $T_xM \subset \mathbb{R}^n$ passing through $x \in M$ is called the *approximate tangent $(n-1)$-plane to M at x* provided

$$\begin{aligned} &\limsup_{r\searrow 0} r^{-n}\,\mathcal{H}^{n-1}\big(M\cap B(x,r)\big) > 0, \qquad \text{and} \\ &\limsup_{r\searrow 0} r^{-n}\,\mathcal{H}^{n-1}\big(\{y\in M\cap B(x,r):\, \mathrm{dist}\,(y,T_xM) > \lambda\,|x-y|\}\big) = 0, \quad \forall\,\lambda>0\,. \end{aligned} \tag{3.56}$$

Note that if such an $(n-1)$-plane exists, then it is unique (so the notation T_xM is justified). Furthermore, the existence of an approximate tangent $(n-1)$-plane $\mathcal{H}^{n-1}$-almost everywhere is, for $\mathcal{H}^{n-1}$-measurable sets of locally finite Hausdorff measure, equivalent to countably $(n-1)$-rectifiability. See Theorem 1.5 in [9]. In the context of (3.55),

$$T_xM = T_xN_j \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x\in N_j\,, \tag{3.57}$$

where T_xN_j is the differential geometric tangent plane to the C^1 surface N_j at x. See Remark 11.7 on p. 61 in [24].

Assume next that f is a locally bi-Lipschitz, real-valued function defined in an open neighborhood $\mathcal{O} \subseteq \mathbb{R}^n$ of M. Then Rademacher's differentiability theorem ensures that there exists a unique locally bounded function on M, called the *gradient of f relative to M*, such that

$$\nabla^M f: M \longrightarrow \mathbb{R}^n, \qquad \nabla^M f(x) = \nabla^{N_j} f(x) \tag{3.58}$$

for $\mathcal{H}^{n-1}$-a.e. $x \in M_j$ with the property that $f|_{N_j}$ is differentiable at x. Above, ∇^{N_j} represents the differential geometric gradient on the C^1 surface N_j. From (3.57) (cf. also Remark 12.2 on p. 67 in [24]), we then have:

$$\nabla^M f(x) \in T_xM \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. points } x\in M\,. \tag{3.59}$$

Going further, we define the *differential of f on M* by

$$d_Mf_x : T_xM \longrightarrow \mathbb{R}, \quad d_Mf_x(\tau) := \big\langle \tau, \nabla^M f(x)\big\rangle, \quad \tau\in T_xM\,, \tag{3.60}$$

at all points $x \in M$ where T_xM and $\nabla^M f(x)$ exist (hence, $\mathcal{H}^{n-1}$-a.e.). If instead of being real-valued, $F = (F_1,\dots,F_n)$ takes values in $\mathbb{R}^n$, we define

$$d_MF_x : T_xM \longrightarrow \mathbb{R}^n, \quad d_MF_x(\tau) := \sum_{i=1}^n \big\langle \tau, \nabla^M F_i(x)\big\rangle e_i, \quad \tau\in T_xM\,, \tag{3.61}$$

where $e_i = (\delta_{ik})_{1\le k\le n}$, $1 \le i \le n$, are the vectors in the standard orthonormal basis in $\mathbb{R}^n$. Finally, we introduce the *Jacobian determinant of F on M* as

$$J_MF(x) := \sqrt{\det\big[(d_MF_x)^*\circ(d_MF_x)\big]}\,, \tag{3.62}$$

where $(d_MF_x)^* : \mathbb{R}^n \to T_xM$ is the adjoint of (3.61).

In this terminology, and assuming that F is injective and locally bi-Lipschitz from some open neighborhood of the countably $(n-1)$-rectifiable set $M \subset \mathbb{R}^n$ into $\mathbb{R}^n$, the area formula proved in Section 12 of [24] reads (after correcting a typo) as follows:

$$\mathcal{H}^{n-1}(F(E)) = \int_E J_MF\,d\mathcal{H}^{n-1}, \qquad \text{whenever } E\subseteq M \text{ is } \mathcal{H}^{n-1}\text{-measurable}\,. \tag{3.63}$$

See also Theorem 2.91 on p. 100 in [1]. According to a famous theorem of De Giorgi (cf. Theorem 14.3 on p. 72 of [24]), if $\Omega \subseteq \mathbb{R}^n$ is an open set of locally finite perimeter then $\partial^*\Omega$ is countably $(n-1)$-rectifiable, so the above considerations apply to this set.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a domain of locally finite perimeter, $\mathcal{O}$ an open neighborhood of $\overline{\Omega}$, and let $F : \mathcal{O} \to \mathbb{R}^n$ be an injective, locally bi-Lipschitz function. Set $\widetilde{\Omega} := F(\Omega)$ and denote by $\sigma, \widetilde{\sigma}$ the surface measures on $\partial\Omega$ and $\partial\widetilde{\Omega}$, respectively. Then*

$$\widetilde{\sigma} = \left[\left(J_{\partial^*\Omega}F\right) \circ F^{-1}\right] F_*\sigma, \qquad \sigma = \left[\left(J_{\partial^*\widetilde{\Omega}}F^{-1}\right) \circ F\right] \left(F^{-1}\right)_* \widetilde{\sigma}, \tag{3.64}$$

and

$$\left(J_{\partial^*\Omega}F\right)^{-1} = \left(J_{\partial^*\widetilde{\Omega}}F^{-1}\right) \circ F . \tag{3.65}$$

Proof. To begin with, Proposition 3.1 ensures that $\widetilde{\Omega}$ is a set of locally finite perimeter, so $\widetilde{\sigma}$ is well-defined. To proceed, let us recast (3.63) in the form

$$(J_M F)\, \mathcal{H}^{n-1}\lfloor M = \left(F^{-1}\right)_* \left(\mathcal{H}^{n-1}\lfloor F(M)\right) . \tag{3.66}$$

We then write

$$\begin{aligned} (J_{\partial^*\Omega}F)\,\sigma &= (J_{\partial^*\Omega}F)\,\mathcal{H}^{n-1}\lfloor\partial^*\Omega && \text{by (2.6)},\\ &= \left(F^{-1}\right)_*\left(\mathcal{H}^{n-1}\lfloor F\left(\partial^*\Omega\right)\right) && \text{by (3.66) with } M = \partial^*\Omega\,,\\ &= \left(F^{-1}\right)_*\left(\mathcal{H}^{n-1}\lfloor\partial^*\widetilde{\Omega}\right) && \text{by (3.15)},\\ &= \left(F^{-1}\right)_*\widetilde{\sigma} && \text{by (2.6) with } \widetilde{\Omega} \text{ in place of } \Omega\,. \end{aligned} \tag{3.67}$$

This and (3.12)–(3.13) in turn imply $\widetilde{\sigma} = F_*[(J_{\partial^*\Omega}F)\,\sigma] = [(J_{\partial^*\Omega}F) \circ F^{-1}]\, F_*\sigma$, as desired. Then the second formula in (3.64) is a consequence of this, reasoning with the roles of Ω, $\widetilde{\Omega}$ reversed. Finally, (3.65) follows from the second identity in (3.64) and (3.67). □

In the context of Theorem 3.2, a comparison of (3.24) and (3.64) shows that, although not obvious from definitions, formula

$$J_{\partial^*\Omega}F = \left\|\left[\left(DF^{-1}\right)^{\top} \circ F\right]\nu\right\|\, |\det(DF)|, \qquad \text{if } F \text{ is a } C^1\text{-diffeomorphism}\,, \tag{3.68}$$

must, nonetheless, be true. It would be therefore instructive to present a direct proof of (3.68), which does not rely on Theorems 3.2–3.4. To this end, fix a point $x \in \partial^*\Omega$ with the property that $T_x\partial^*\Omega$. Since $F = (F_1, \ldots, F_n)$ is of class C^1 in a neighborhood of $\partial\Omega$, it follows from definitions that for each $i = 1, \ldots, n$,

$$\nabla^{\partial^*\Omega}F_i = \pi_x \nabla F_i, \qquad \pi_x : \mathbb{R}^n \longrightarrow T_x\partial^*\Omega \quad \text{orthogonal projection}\,. \tag{3.69}$$

Consequently, for each $\tau \in T_x\partial^*\Omega$,

$$d_{\partial^*\Omega}F_x(\tau) = \sum_{i=1}^{n}\left\langle\tau, \nabla^{\partial^*\Omega}F_i\right\rangle e_i = \sum_{i=1}^{n}\langle\tau, \nabla F_i\rangle e_i = [DF(x)]\tau\,. \tag{3.70}$$

To continue, abbreviate $W := T_x\partial^*\Omega$ $\mathcal{A} := DF(x)$, $A := d_{\partial^*\Omega}F_x$. Hence, W is a $(n-1)$-plane in $\mathbb{R}^n$ and $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$, $A : W \to \mathbb{R}^n$, are linear mappings with the property that $A = \mathcal{A}|_W$.

Fix an orthonormal basis $\{\tau_i\}_{1\leq i\leq n-1}$ in $T_x\partial^*\Omega$. Using notation and results summarized in the Appendix, we then write

$$\begin{aligned}
J_{\partial^*\Omega}F(x) &= \sqrt{\det\left(A^*A\right)} && \text{by (3.62)},\\
&= |A\tau_1 \times \cdots \times A\tau_{n-1}| && \text{by (5.14)},\\
&= |\mathcal{A}\tau_1 \times \cdots \times \mathcal{A}\tau_{n-1}| && \text{since } A = \mathcal{A}|_W,\\
&= |\det \mathcal{A}|\, \left\|\left(\mathcal{A}^{-1}\right)^{\top}(\tau_1 \times \cdots \times \tau_{n-1})\right\| && \text{by (5.11)},\\
&= |\det \mathcal{A}|\, \left\|\left(\mathcal{A}^{-1}\right)^{\top}\nu(x)\right\| && \text{by (5.7)},\\
&= |\det (DF)(x)|\, \left\|\left[(DF(x))^{-1}\right]^{\top}\nu(x)\right\| && \text{by the definition of } \mathcal{A},\\
&= |\det (DF)(x)|\, \left\|\left[\left(DF^{-1}\right)F(x)\right]^{\top}\nu(x)\right\| && \text{since } (DF)^{-1} = \left(DF^{-1}\right)\circ F,
\end{aligned} \tag{3.71}$$

proving (3.68). However, before concluding the digression pertaining to identity (3.68), we wish to point out that by combining formula (***) on p. 147 in [24] with the definition given at the bottom of p. 138 in [24], we arrive at

$$\widetilde{\nu}(F(x)) = \frac{\left(d_{\partial^*\Omega}F_x\right)\tau_1 \times \cdots \times \left(d_{\partial^*\Omega}F_x\right)\tau_{n-1}}{J_{\partial^*\Omega}F(x)}, \tag{3.72}$$

for $\mathcal{H}^{n-1}$-a.e. $x \in \partial^*\Omega$, if $\tau_1, \ldots, \tau_{n-1}$ form a positively oriented orthonormal basis in $T_x\partial^*\Omega$ (so that, in particular, $\nu(x) = \tau_1 \times \cdots \times \tau_{n-1}$). A similar type of argument as above can then be used to show that this agrees with (3.23) if F is actually a C^1-diffeomorphism.

Formula (3.72) suggests that, in the context of Theorem 3.2, one should be able to relate $\widetilde{\nu}(F(x))$ to $\nu(x)$ using only the "tangential" gradients

$$\nabla_{\tan}F_j = \nabla F_j - (\nu \cdot \nabla F_j)\,\nu, \tag{3.73}$$

of the components of F, instead of the "full" gradients ∇F_j, $1 \leq j \leq n$. In this regard, we shall prove the following.

Proposition 3.5. *Retain the same hypotheses as in Theorem 3.2 and let $N := (N_1, \ldots, N_n)$ be the vector with components*

$$N_j := \det \begin{vmatrix} (\nabla_{\tan}F_1)_1 & (\nabla_{\tan}F_1)_2 & \ldots & (\nabla_{\tan}F_1)_n \\ \vdots & \vdots & \vdots & \vdots \\ \nu_1 & \nu_2 & \ldots & \nu_n \\ \vdots & \vdots & \vdots & \vdots \\ (\nabla_{\tan}F_n)_1 & (\nabla_{\tan}F_n)_2 & \ldots & (\nabla_{\tan}F_n)_n \end{vmatrix}, \qquad j = 1, \ldots, n, \tag{3.74}$$

where the j-th line, $\nu_1, \ldots, \nu_n$, consists of the components of the outward unit normal ν. Then

$$\widetilde{\nu} \circ F = \frac{N}{\|N\|}, \qquad \sigma\text{-a.e. on } \partial\Omega, \tag{3.75}$$

and

$$F_*^{-1}\widetilde{\sigma} = \|N\|\,\sigma. \tag{3.76}$$

Proof. Keeping in mind (3.26), the goal is to express $\operatorname{adj}(DF)\,\nu$ so that only components of $\nabla_{\tan}F_j$ appear instead of components of ∇F_j, $1 \leq j \leq n$. To this end, let us agree to identify vectors $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ with 1-forms $v^\# := v_1\, dx_1 + \cdots + v_n\, dx_n$. In particular, $\nu^\# = \nu_1\, dx_1 + \cdots + \nu_n\, dx_n$. Next, if $\operatorname{adj}(DF) = (A_{jk})_{1\le j,k\le n}$ then (3.39) holds and, analogously to what we have done in (3.40), for every $j \in \{1, \dots, n\}$ we write:

$$\begin{aligned}
\big(\operatorname{adj}(DF)\,\nu\big)_j &= \sum_{k=1}^n A_{jk}\nu_k = *\,\nu^\# \wedge \Big(\sum_{k=1}^n A_{jk}(-1)^{k-1}\, dx_1 \wedge \ldots \wedge \widehat{dx_k} \wedge \ldots \wedge dx_n\Big) \\
&= *\,(-1)^{j+1}\,\nu^\# \wedge \Big(\sum_{k=1}^n dx_k \vee (dx_k \wedge \big[dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n\big]\Big) \\
&= *\,(-1)^{j+1}\,\nu^\# \wedge \big(dF_1 \wedge \ldots \wedge \widehat{dF_j} \wedge \ldots \wedge dF_n\big)\,, \qquad (3.77)
\end{aligned}$$

where the fourth equality is based on (3.41). To continue, for each $k \in \{1, \dots, n\}$, decompose

$$dF_k = \nu^\# \vee \big(\nu^\# \wedge dF_k\big) + \nu^\# \wedge \big(\nu^\# \vee dF_k\big) = (\nabla_{\tan} F_k)^\# + \nu^\# \wedge \big(\nu^\# \vee dF_k\big) \qquad (3.78)$$

and note that, in the context of the last expression in (3.77), the contribution coming from each $\nu^\# \wedge (\nu^\# \vee dF_k)$, $1 \le k \le n$, is zero since $\nu^\# \wedge \nu^\# = 0$. Consequently, the j-th component of $\operatorname{adj}(DF)\,\nu$ has the form:

$$\begin{aligned}
&*\,(-1)^{j+1}\,\nu^\# \wedge \big[(\nabla_{\tan} F_k)^\# \wedge \ldots \wedge (\nabla_{\tan} F_{j-1})^\# \wedge (\nabla_{\tan} F_{j+1})^\# \ldots \wedge (\nabla_{\tan} F_n)^\#\big] \\
&\quad = *\,\big[(\nabla_{\tan} F_k)^\# \wedge \ldots \wedge (\nabla_{\tan} F_{j-1})^\# \wedge \nu^\# \wedge (\nabla_{\tan} F_{j+1})^\# \ldots \wedge (\nabla_{\tan} F_n)^\#\big] \\
&\quad = N_j\,. \qquad (3.79)
\end{aligned}$$

Thus, $\operatorname{adj}(DF)\,\nu = N$ so that (3.75) follows from this and (3.26), whereas (3.76) follows from this and (3.27). □

In the case when $\Omega \subset \mathbb{R}^n$ is a locally strongly Lipschitz domain, then the action of the tangential gradient $\nabla_{\tan}$, originally introduced in (3.73), can be extended in a meaningful way to any Lipschitz function F defined on $\partial\Omega$. In particular, if $F : \partial\Omega \to \mathbb{R}^n$ is of class C^1, then

$$(DF)\Big|_{T_x\partial\Omega} = d_{\partial^*\Omega} F_x = \nabla_{\tan} F(x)\,, \qquad (3.80)$$

for a.e. x on $\partial\Omega$, where $\nabla_{\tan} F$ is taken to be the matrix whose rows are the tangential gradients of the components of F.

Corollary 3.6. *Assume that $\Omega \subset \mathbb{R}^n$ is a locally strongly Lipschitz domain, $\mathcal{O}$ an open neighborhood of $\overline{\Omega}$, and that $F : \mathcal{O} \to \mathbb{R}^n$ is an injective, locally bi-Lipschitz function. Consider $\widetilde{\Omega} := F(\Omega)$ and denote by σ, $\widetilde{\sigma}$, respectively, the surface measures on $\partial\Omega$ and $\partial\widetilde{\Omega}$. Then formulas* (3.75)–(3.76) *continue to be valid in this setting.*

Proof. In the current context, it is possible to construct a family of C^∞ mappings $\{F^\varepsilon\}_{\varepsilon>0}$ with the property that

$$\nabla_{\tan} F^\varepsilon \longrightarrow \nabla_{\tan} F \quad \text{as } \varepsilon \to 0^+, \quad \text{a.e. on } \partial\Omega\,. \qquad (3.81)$$

Recall N from (3.74) and denote by N^ε the vector defined in an analogous fashion with F replaced by F^ε. Also, pick $x \in \partial\Omega$ such that the tangent plane $T_x\Omega$ exists and fix (a positively oriented) orthonormal basis $\tau_1, \dots, \tau_{n-1}$ in $T_x\Omega$. We may then write

$$\begin{aligned}
&(d_{\partial^*\Omega}F_x\tau_1 \times \cdots \times (d_{\partial^*\Omega}F_x)\tau_{n-1} \\
&= \lim_{\varepsilon\to 0^+} \big(\nabla_{\tan}F^\varepsilon(x)\big)\tau_1 \times \cdots \times \big(\nabla_{\tan}F^\varepsilon(x)\big)\tau_{n-1} && \text{by (3.80) and (3.82)}, \\
&= \lim_{\varepsilon\to 0^+} \big(DF^\varepsilon(x)\big)\tau_1 \times \cdots \times \big(DF^\varepsilon(x)\big)\tau_{n-1} && \text{by (3.80)}, \\
&= \lim_{\varepsilon\to 0^+} \big(\mathrm{adj}\,\big(DF^\varepsilon\big)(x)\big)(\tau_1 \times \cdots \times \tau_{n-1}) && \text{by (5.12)}, \qquad (3.82) \\
&= \lim_{\varepsilon\to 0^+} \big(\mathrm{adj}\,\big(DF^\varepsilon\big)(x)\big)\nu(x) && \text{since } \nu(x) = \tau_1 \times \cdots \times \tau_{n-1}, \\
&= \lim_{\varepsilon\to 0^+} N^\varepsilon(x) && \text{by the proof of Proposition 3.5}, \\
&= N(x) && \text{by (3.74) and (3.82)}.
\end{aligned}$$

In particular,

$$J_{\partial^*\Omega}F(x) = \|(d_{\partial^*\Omega}F_x)\tau_1 \times \cdots \times (d_{\partial^*\Omega}F_x)\tau_{n-1}\| = \|N(x)\|. \tag{3.83}$$

The desired conclusion now follows from (3.64), (3.72), and (3.82)–(3.83). □

Moving on, recall that a locally positive and finite Borelian measure μ in $\mathbb{R}^n$ is said to be *doubling* if the doubling constant of μ,

$$[\mu] := \sup_{r>0,\, x\in\mathbb{R}^n} \frac{\mu\big(B(x,2r)\big)}{\mu(B(x,r))}, \tag{3.84}$$

is finite.

Proposition 3.7. *Assume that $\Omega \subseteq \mathbb{R}^n$ is a set of locally finite perimeter, and that $\mathcal{O} \subseteq \mathbb{R}^n$ is an open neighborhood of $\overline{\Omega}$. For a bi-Lipschitz mapping $F : \mathcal{O} \to \mathbb{R}^n$ set $\widetilde{\Omega} := F(\Omega)$. If the surface measure on $\partial\Omega$ is doubling then the surface measure on $\partial\widetilde{\Omega}$ is also doubling.*

Prior to presenting the proof of this proposition we discuss two auxiliary results of independent interest.

Lemma 3.8. *Let $M \subset \mathbb{R}^n$ be countably $(n-1)$-rectifiable, and assume that f is a real-valued Lipschitz function defined on M. Then*

$$\big\|\nabla^M f\big\|_{L^\infty(M,d\mathcal{H}^{n-1})} \le \mathrm{Lip}\,(f, M). \tag{3.85}$$

Proof. Assume that (3.55) holds, with $M_j \subset N_j$, N_j surface of class C^1 in $\mathbb{R}^n$. If $x \in N_j \subseteq M$ is such that $T_xM = T_xN_j$ and $f|_{N_j}$ is differentiable at x, then for any $\tau \in T_xM$ we can pick a C^1 curve $\gamma : (-1, 1) \to N_j$ with $\gamma(0) = x$ and $\dot\gamma(0) = \tau$. We may then compute

$$\begin{aligned}
\big|\nabla^M f(x)\cdot\tau\big| &= \big|\nabla^{N_j} f(x)\cdot\dot\gamma(0)\big| = \Big|\tfrac{d}{dt}\,f(\gamma(t))\Big| \\
&= \Big|\lim_{t\to 0^+} \tfrac{f(\gamma(t))-f(\gamma(0))}{t}\Big| \le \mathrm{Lip}\,(f, M)\,\lim_{t\to 0^+}\Big|\tfrac{\gamma(t)-\gamma(0)}{t}\Big| \\
&\le \mathrm{Lip}\,(f, M)\,|\dot\gamma(0)| = \mathrm{Lip}\,(f, M)\,|\tau|.
\end{aligned} \tag{3.86}$$

Granted (3.59) and since $\tau \in T_x M$ was arbitrary, this clearly implies (3.85). $\square$

Lemma 3.9. *Let $\Omega \subseteq \mathbb{R}^n$ be a set of locally finite perimeter, $\mathcal{O} \subseteq \mathbb{R}^n$ an open neighborhood of $\overline{\Omega}$, and $F : \mathcal{O} \to \mathbb{R}^n$ a bi-Lipschitz mapping. Set $\widetilde{\Omega} := F(\Omega)$. Then for some dimensional constants $C_n, c_n > 0$,*

$$c_n \left(\mathrm{Lip}\, F^{-1}\right)^{1-n} \leq (J_{\partial^*\Omega} F)(x) \leq C_n \left(\mathrm{Lip}\, F\right)^{n-1}, \qquad \mathcal{H}^{n-1} - \text{a.e. } x \in \partial^*\Omega\,. \tag{3.87}$$

Proof. The upper bound in (3.87) is seen from (3.62), with the help of Lemma 3.8. Then the lower bound follows from this, written with Ω, F replaced by $\widetilde{\Omega}$, F^{-1}, and (3.65). $\square$

Having established Lemma 3.9, we are now ready to tackle the following.

Proof of Proposition 3.7. Denote by $\sigma, \widetilde{\sigma}$ the surface measures on $\partial\Omega$ and $\partial\widetilde{\Omega}$, respectively. From (3.8), (3.64)–(3.65), (3.87), and that the fact that σ and $\widetilde{\sigma}$ are supported on $\partial\Omega$ and $\partial\widetilde{\Omega}$, respectively, we then deduce

$$\begin{aligned}
\widetilde{\sigma}(B(F(x_0), r)) &= \widetilde{\sigma}(B(F(x_0), r) \cap F(\mathcal{O})) \\
&\leq \widetilde{\sigma}(F(B(x_0, cr)) \cap F(\mathcal{O})) = \widetilde{\sigma}(F(B(x_0, cr) \cap \mathcal{O})) \\
&= \left((F^{-1})_* \widetilde{\sigma}\right)(B(x_0, cr) \cap \mathcal{O}) = \int_{B(x_0,cr)\cap\mathcal{O}} \left[\left(J_{\partial^*\widetilde{\Omega}} F^{-1}\right) \circ F\right]^{-1} d\sigma \\
&= \int_{B(x_0,cr)} J_{\partial^*\Omega} F \, d\sigma \leq C\sigma(B(x_0, cr))\,,
\end{aligned} \tag{3.88}$$

for some finite constants $C, c > 0$, depending only on F. A similar type of argument shows that $\sigma(B(x_0, r)) \leq C\widetilde{\sigma}(B(F(x_0), cr))$. In turn, this and (3.88) readily imply that if σ is doubling then so is $\widetilde{\sigma}$. $\square$

4. Further applications

4.1. Bounded Lipschitz domains are invariant under C^1 diffeomorphisms

It is an elementary exercise to show that a bounded, open set $\Omega \subset \mathbb{R}^n$ is a C^1 domain (in the sense of Definition 2.18) if and only if for every $x_0 \in \partial\Omega$ there exist an open neighborhood U of x_0 in $\mathbb{R}^n$ and a mapping $F = (F_1, \ldots, F_n) : U \to \mathbb{R}^n$ with the following properties:

(i) $F(U)$ is open and $F : U \to F(U)$ is a C^1-diffeomorphism;

(ii) $\Omega \cap U = \{x \in U : F_n(x) > 0\}$.

To see this, one direction is clear and in the opposite one it suffices to observe that there exists $j \in \{1, \ldots, n\}$ such that, for x near x_0, one has

$$F_n(x) = 0 \Longleftrightarrow x_j = \varphi(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)\,, \tag{4.1}$$

for some C^1 function φ. That such an index j exists follows from the standard Implicit Function Theorem for C^1 functions. Indeed, if F is a C^1-diffeomorphism then $DF(x_0)$ is an invertible matrix, so necessarily $\partial_j F_n(x_0) \neq 0$ for some j.

When dealing with the case when F is only bi-Lipschitz, what changes is the nature of the Implicit Function Theorem. More specifically, if F is Lipschitz, a sufficient condition validating the equivalence (4.1) for some Lipschitz function φ is

$$C\big|x_j^1-x_j^2\big|\leq\big|F_n\big(x_1,\dots,x_{j-1},x_j^1,x_{j+1},\dots,x_n\big)-F_n\big(x_1,\dots,x_{j-1},x_j^2,x_{j+1},\dots,x_n\big)\big|\,, \tag{4.2}$$

uniformly for $(x_1,\dots,x_{j-1},x_j^1,x_{j+1},\dots,x_n)$, $(x_1,\dots,x_{j-1},x_j^2,x_{j+1},\dots,x_n)$ near x_0. This, however, is not necessarily implied by the fact that $F=(F_1,\dots,F_n)$ is bi-Lipschitz. In fact, in the latter setting, the equivalence (4.1) may fail altogether. To further shed light on this issue, we next discuss some concrete examples, in which the aforementioned failure is implicit, showing that the class of Lipschitz domains is not stable under bi-Lipschitz homeomorphisms.

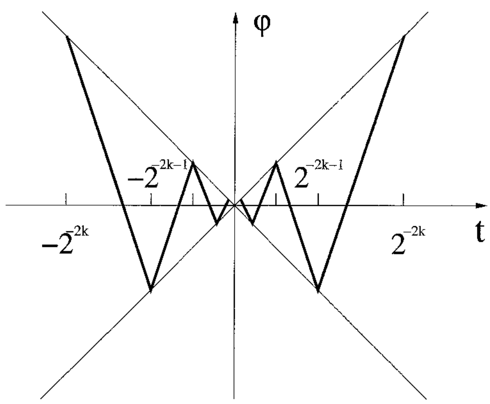
We start with an interesting example from (pp. 7–9 in) [10], where this is attributed to Zerner. Concretely, consider the bi-Lipschitz homeomorphism

$$F:\mathbb{R}^2\longrightarrow\mathbb{R}^2,\qquad F(x_1,x_2):=(x_1,\varphi(x_1)+x_2)\,, \tag{4.3}$$

where $\varphi:\mathbb{R}\to\mathbb{R}$ is the Lipschitz function

$$\varphi(t):=\begin{cases}3|t|-\frac{1}{2^{2k-1}} & \text{for}\quad \frac{1}{2^{2k+1}}\leq|t|\leq\frac{1}{2^{2k}},\\ -3|t|+\frac{1}{2^{2k}} & \text{for}\quad \frac{1}{2^{2k+2}}\leq|t|\leq\frac{1}{2^{2k+1}}\,.\end{cases} \tag{4.4}$$

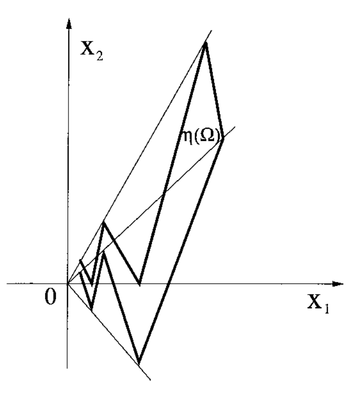
As is also visible from the picture below, the graph of φ is a zigzagged of lines of slopes ± 3:



If one now considers the bounded Lipschitz domain $\Omega\subset\mathbb{R}^2$,

$$\Omega:=\{(x_1,x_2):\,0<x_1<1,\ \ 0<x_2<x_1\}\,, \tag{4.5}$$

then $F(\Omega)$, depicted below



fails to be a strongly Lipschitz domain, since the cone property [cf. (2.24)] is violated at the origin.

In fact, the construction described above can be refined to show that *bi-Lipschitz functions may fail to map even bounded C^∞ planar domains into strongly Lipschitz domains*. Concretely, pick $x_0 \in \Omega$ and let $\varphi : S^1 \to (0, \infty)$ be the Lipschitz function uniquely determined by the requirement that $G : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $G(x) := \varphi((x - x_0)/|x - x_0|)(x - x_0)$ if $x \neq x_0$ and $G(x_0) := 0$, maps $\partial B(x_0, r)$ onto $\partial\Omega$ (for some fixed, sufficiently small $r > 0$). Then $F \circ G$ maps the bounded, C^∞ domain $B(x_0, r)$ onto the domain shown in the picture above.

There are many other interesting examples of strongly Lipschitz domains $\Omega \subset \mathbb{R}^n$ and bi-Lipschitz maps $F : \mathbb{R}^n \to \mathbb{R}^n$ with the property that $F(\Omega)$ fails to be strongly Lipschitz. A large category of such examples can be found within the class of *conical domains*. In order to be more specific, let S^{n-1} stand for the unit sphere in $\mathbb{R}^n$ and denote by S^{n-1}_+ its upper hemisphere. Pick a bi-Lipschitz homeomorphism $\psi : S^{n-1} \to S^{n-1}$ along with an arbitrary Lipschitz function $\varphi : S^{n-1} \to (0, \infty)$, and set

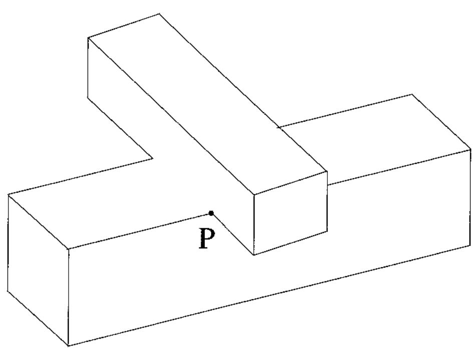
$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \qquad F(r\omega) := r\varphi(\omega)\psi^{-1}(\omega), \quad r \geq 0, \ \ \omega \in S^{n-1}, \tag{4.6}$$

$$\Omega := \left\{r\omega : \ \omega \in S^{n-1}_+, \ \ 0 < r < \varphi(\omega)\right\}. \tag{4.7}$$

Using $|r_1\omega_1 - r_2\omega_2|^2 = |r_1 - r_2|^2 + r_1 r_2 |\omega_1 - \omega_2|^2$ for every $\omega_1, \omega_2 \in S^{n-1}$, $r_1, r_2 \geq 0$, and the fact that the inverse of (4.6) is $F^{-1}(r\omega) = r\varphi(\omega)^{-1}\psi(\omega)$, it can be easily checked that F above is bi-Lipschitz. However, while $\Omega \subset \mathbb{R}^n$ is clearly a strongly Lipschitz domain in $\mathbb{R}^n$,

$$F(\Omega) = \left\{\rho w : \ w \in \psi\big(S^{n-1}_+\big), \ \ 0 < \rho < \varphi(\omega)\right\}, \tag{4.8}$$

may fail to be a strongly Lipschitz domain. In fact, near $0 \in \partial F(\Omega)$, the surface $\partial F(\Omega)$ may fail to be the graph of *any* real-valued function of $n - 1$ variables, in any system of coordinates which is a rigid motion of the standard one (i.e., $\partial F(\Omega)$ is a non-Lipschitz cone). A concrete example, which can be produced using the above recipe, is Maz'ya's so-called *two-brick domain*:



A moment's reflection shows that, indeed, near the point P, the boundary of the above domain is not the graph of any function (as it fails the vertical line test) in any system of coordinates isometric to the original one.

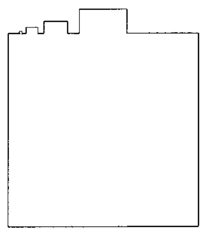
As observed in [2], images of bounded strongly Lipschitz domains via bi-Lipschitz maps can also develop spiral-like singularities, such as:

$$\begin{aligned} F(\Omega) &= \left\{re^{i(\theta - \ln r)} : \ 0 < \theta < \pi/4, \ \ 0 < r < 1\right\} \subset \mathbb{R}^2 \equiv \mathbb{C}, \\ \Omega &:= \left\{re^{i\theta} : \ 0 < r < 1, \ \ 0 < \theta < \pi/4\right\}, \quad F\big(re^{i\theta}\big) := re^{i(\theta - \ln r)}. \end{aligned} \tag{4.9}$$

Another interesting example of the phenomenon described above is as follows. Let

$$\widetilde{\Omega} := \big[(0,1)\times(-1,0)\big] \cup \left[\bigcup_{k=1}^{\infty} \big(3\cdot 2^{-k-2}, 5\cdot 2^{-k-2}\big)\times\big[0, 2^{-k-2}\big)\right] \tag{4.10}$$

be the planar domain in the picture below:



It is not difficult to see that the uniformity of the cone condition is violated in any neighborhood of the origin, so $\widetilde{\Omega}$ is not a strongly Lipschitz domain. Nonetheless, on p. 19 of [17], Maz'ya has constructed a bi-Lipschitz map $F : \mathbb{R}^2 \to \mathbb{R}^2$ with the property that $\widetilde{\Omega} = F((0,1)\times(0,1))$.

The examples presented thus far raise the following interesting issue: *give an intrinsic description of the class of images of bounded strongly Lipschitz domains under bi-Lipschitz mappings.* From the discussion in the next subsection (and Section 5.1) it follows that this is a subclass of the collection of all bounded NTA domains with Ahlfors regular boundaries, and which have Lipschitz reflections in a collar neighborhood of their boundaries. As a related matter, it is natural to conjecture that any bounded strongly Lipschitz domain is the bi-Lipschitz image of a bounded C^∞ domain. This is certainly true for bounded, starlike strongly Lipschitz domains.

Turning to positive results, we shall now prove that bi-Lipschitz homeomorphisms which are also C^1-diffeomorphisms of the space do preserve the class of bounded Lipschitz domains. As the above discussion shows, this result is in the nature of best possible.

Theorem 4.1. *Assume that $\Omega \subset \mathbb{R}^n$ is an open set, $\mathcal{O} \subseteq \mathbb{R}^n$ is an open neighborhood of $\overline{\Omega}$, and $F : \mathcal{O} \to \mathbb{R}^n$ is a C^1-diffeomorphism onto its image.*

Then, if Ω is strongly Lipschitz near $x_0 \in \partial\Omega$, it follows that $F(\Omega)$ is strongly Lipschitz near $F(x_0) \in \partial F(\Omega)$ as well. Consequently, if Ω is locally strongly Lipschitz, then so is $\widetilde{\Omega} := F(\Omega)$.

In particular, if $\Omega \subset \mathbb{R}^n$ is a bounded strongly Lipschitz domain, $\mathcal{O} \subseteq \mathbb{R}^n$ is an open neighborhood of $\overline{\Omega}$, and $F : \mathcal{O} \to \mathbb{R}^n$ is a C^1-diffeomorphism then $\widetilde{\Omega} := F(\Omega)$ is a strongly Lipschitz domain. Furthermore, the Lipschitz character of $\widetilde{\Omega}$ is controlled in terms of the Lipschitz character of Ω, $\mathrm{Lip}\,(F, \overline{\Omega})$ and $\mathrm{Lip}\,(F^{-1}, \overline{F(\Omega)})$.

Proof. Working with Ω replaced by $\Omega \cap \mathcal{C}$, where $\mathcal{C}$ is a suitable coordinate cylinder near x_0, there is no loss of generality in assuming that Ω itself is a bounded strongly Lipschitz domain. Note that $\widetilde{\Omega} := F(\Omega) \subseteq \mathbb{R}^n$ is a bounded, open set which, thanks to (2.26), satisfies $\partial\widetilde{\Omega} = \partial(\overline{\widetilde{\Omega}})$. By Theorem 3.2, this set is also of locally finite perimeter, and we denote by $\widetilde{\nu}$, $\widetilde{\sigma}$ the outward unit normal and surface measure on $\partial\widetilde{\Omega}$.

Consequently, by Theorem 2.8 and Lemma 5.12, it suffices to show that, given $x \in \partial\Omega$, there is a continuous vector field transversal to $\partial\widetilde{\Omega}$ near $F(x)$. However, if X is a continuous vector field which is transversal to $\partial\Omega$ near x, then $\widetilde{X} := [(DF)\circ F^{-1}](X \circ F^{-1})$ will do the job (assuming that F is orientation preserving). Indeed, if ν is the outward unit normal on $\partial\Omega$

then (3.23), (2.12), (3.3), and (3.24) imply that, for a sufficiently small compact neighborhood U_x of x,

$$\begin{aligned}\widetilde{X}\cdot\widetilde{\nu} &= \big[(DF)\circ F^{-1}\big]\big(X\circ F^{-1}\big)\cdot\frac{\big(DF^{-1}\big)^{\top}\big(\nu\circ F^{-1}\big)}{\big\|\big(DF^{-1}\big)^{\top}\big(\nu\circ F^{-1}\big)\big\|}\\ &= \frac{\big(X\circ F^{-1}\big)\cdot\big(\nu\circ F^{-1}\big)}{\big\|\big(DF^{-1}\big)^{\top}\big(\nu\circ F^{-1}\big)\big\|}\geq\frac{\kappa}{\big\|(DF)\circ F^{-1}\big\|}\geq\frac{\kappa}{\mathrm{Lip}\,(F,U_x)}>0\,,\end{aligned}\tag{4.11}$$

$F_*\sigma$-a.e. [hence, $\widetilde{\sigma}$-a.e., by (3.17)] near $F(x)$. □

We conclude this subsection by presenting the following result, which should be compared with the criterion (i)–(ii), characterizing the class of bounded C^1 domains, discussed near the beginning of Section 4.1.

Corollary 4.2. *For a nonempty, proper open subset Ω of $\mathbb{R}^n$, the following are equivalent:*

(a) *Ω is a locally strongly Lipschitz domain;*

(b) *for every point $x_0\in\partial\Omega$ there exist an open neighborhood U of x_0, a C^1-diffeomorphism $F:\mathbb{R}^n\to\mathbb{R}^n$, and numbers $b,c>0$, satisfying the following properties:*

 (i) *$F(x_0)=0$, and $F(U)$ is the open cylinder $C_{b,c}:=\{(x',t):|x'|<b,\ |t|<c\}$;*

 (ii) *there exists a Lipschitz function $\varphi:\mathbb{R}^{n-1}\to\mathbb{R}$ with $\varphi(0)=0$ and $|\varphi(x')|<c$ if $|x'|\leq b$, and for which*

$$F(U\cap\Omega)=C_{b,c}\cap\big\{(x',t):\,x'\in\mathbb{R}^{n-1},\ t>\varphi(x')\big\}\,.\tag{4.12}$$

(c) *For every point $x_0\in\partial\Omega$ there exist an open neighborhood U of x_0 along with a C^1-diffeomorphism $F:U\to F(U)$ such that $F(U\cap\Omega)$ is strongly Lipschitz near $F(x_0)$.*

Proof. If Ω is a locally strongly Lipschitz domain then, by virtue of Definition 2.5, conditions (i)–(ii) in (b) can be satisfied by choosing F to be a suitable isometry of $\mathbb{R}^n$. This proves that (a)$\Rightarrow$(b). Trivially, (b)$\Rightarrow$(c). As for the remaining implication, assume that (c) holds. Since the C^1-diffeomorphism F^{-1} maps $F(U\cap\Omega)$ into $U\cap\Omega$, it follows from Theorem 4.1 that the latter domain is strongly Lipschitz near x_0. Being a locally strongly Lipschitz domain is, however, a local property of the boundary, so we may further conclude that Ω itself is a locally strongly Lipschitz domain. □

Let $\mathcal{M}$ be a topological manifold of (real) dimension n, equipped with a C^1 atlas $\mathcal{A}$. Call an open set $\Omega\subseteq\mathcal{M}$ a *locally strongly Lipschitz domain relative to* $\mathcal{A}$ if for every $x_0\in\partial\Omega$ there exists a local chart $(U,h)\in\mathcal{A}$ with $x_0\in U$ and such that $h(U\cap\Omega)\subseteq\mathbb{R}^n$ is a locally strongly Lipschitz domain near $h(x_0)$. Recall that two C^1 atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ are called equivalent if $\mathcal{A}_1\cup\mathcal{A}_2$ is also a C^1 atlas.

Theorem 4.3. *Assume that $\mathcal{M}$ is a topological manifold of (real) dimension n, and that $\mathcal{A}$ is a C^1 atlas on $\mathcal{M}$. Also, let Ω be an open subset of $\mathcal{M}$ which is a locally strongly Lipschitz domain relative to $\mathcal{A}$. Then Ω is a locally strongly Lipschitz domain relative to any other C^1 atlas on $\mathcal{M}$ which is equivalent to $\mathcal{A}$.*

Proof. Let $\mathcal{A}'$ be a C^1 atlas on $\mathcal{M}$ which is equivalent to $\mathcal{A}$. Then the desired conclusion follows from Theorem 4.1 applied to the transition maps between the charts of $\mathcal{A}'$ and $\mathcal{A}$ (which are C^1 diffeomorphisms). □

4.2. Regular SKT domains are invariant under C^1 diffeomorphisms

We remind the reader that a closed set $\Sigma \subset \mathbb{R}^n$ is said to be *Ahlfors regular* provided there exist $0 < a \leq b < \infty$ (called Ahlfors constants of Σ) such that

$$a\,r^n \leq \mathcal{H}^{n-1}\big(B(x,r) \cap \Sigma\big) \leq b\,r^n\,, \tag{4.13}$$

for each $x \in \Sigma$ and $r \in (0,\infty)$. If Σ is compact, we require (4.13) only for $r \in (0,1]$. Nonetheless, (4.13) continues to hold in this case (albeit with possibly different constants) for each $0 < r < \operatorname{diam}\Sigma$. An open set $\Omega \subset \mathbb{R}^n$ is said to be an *Ahlfors regular domain* provided $\partial\Omega$ is Ahlfors regular. Note that, by (2.10), an Ahlfors regular domain $\Omega \subset \mathbb{R}^n$ satisfying

$$\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0 \tag{4.14}$$

is of locally finite perimeter and $\sigma = \mathcal{H}^{n-1}\lfloor\partial\Omega$.

Recall that every locally strongly Lipschitz domain is locally starlike. For our purposes here, we shall need a curvilinear, scale invariant version of this property. Following [11], we make the following.

Definition 4.4. Let $\Omega \subset \mathbb{R}^n$ be an open set. This is said to satisfy a *local John condition* if there exist $\theta \in (0,1)$ and $R > 0$ (required to be ∞ if $\partial\Omega$ is unbounded), called the John constants of Ω, with the following significance. For every $p \in \partial\Omega$ and $r \in (0,R)$ one can find $p_r \in B(p,r) \cap \Omega$, called John center relative to $\Delta(p,r) := B(p,r) \cap \partial\Omega$, such that $B(p_r,\theta r) \subset \Omega$ and with the property that for each $x \in \Delta(Q,r)$ one can find a rectifiable path $\gamma_x : [0,1] \to \overline{\Omega}$, whose length is $\leq \theta^{-1} r$ and such that

$$\gamma_x(0) = x, \quad \gamma_x(1) = p_r, \quad \text{and} \quad \operatorname{dist}(\gamma_x(t), \partial\Omega) > \theta\,|\gamma_x(t) - x| \quad \forall\, t \in (0,1]\,. \tag{4.15}$$

Finally, Ω is said to satisfy a two-sided local John condition if both Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ satisfy a local John condition.

Lemma 4.5. *Bi-Lipschitz mappings preserve the class of Ahlfors regular domains, the class of domains for which* (4.14) *holds, as well as the class of bounded domains satisfying a two-sided local John condition.*

Proof. This is a direct consequence of (3.14) and (3.2). □

Our next result deals with the class of regular SKT (Semmes-Kenig-Toro) domains in $\mathbb{R}^n$. Although intuitively suggestive, the actual definition of this class of domains is somewhat technical. Thus, in order to avoid a lengthy digression we defer such a discussion to the Appendix, Section 5 (we will, however, employ already the terminology introduced there). Our goal here is to show that the class of bounded regular SKT domains is invariant under C^1-diffeomorphisms.

Theorem 4.6. *If $\Omega \subset \mathbb{R}^n$ is a bounded regular SKT domain and F is a C^1-diffeomorphism of $\mathbb{R}^n$, then $\widetilde{\Omega} := F(\Omega)$ is also a (bounded) regular SKT domain.*

In order to facilitate the subsequent presentation, we introduce the following notation. Given $x \in \partial\Omega$, $0 < R < \operatorname{diam}\partial\Omega$ and $f \in L^1_{\mathrm{loc}}(\partial\Omega, d\sigma)$, define

$$\Delta(x,R) := B(x,R) \cap \partial\Omega, \qquad f_{\Delta(x,R)} := \fint_{\Delta(x,R)} f\,d\sigma, \tag{4.16}$$

$$\|f\|_{*,R} := \sup_{x\in\partial\Omega}\ \sup_{\rho\in(0,R)} \Big(\fint_{\Delta(x,\rho)} |f(y) - f_{\Delta(x,\rho)}|^2\,d\sigma(y)\Big)^{1/2}\,. \tag{4.17}$$

When the center x is understood from the context, or irrelevant, we abbreviate $\Delta_R := \Delta(x, R)$. Finally, set

$$\|f\|_{\mathrm{BMO}(\partial\Omega, d\sigma)} := \sup\{\|f\|_{*,R} : R \in (0, \operatorname{diam}\Omega)\} . \tag{4.18}$$

The main estimate used in the proof of Theorem 4.6 is contained in the proposition below, which is itself of independent interest.

Proposition 4.7. *Assume that $\Omega \subset \mathbb{R}^n$ is a domain of locally finite perimeter with the property that its surface measure, σ, is doubling. Denote by $[\sigma]$ the doubling constant of σ and by ν the outward unit normal on $\partial\Omega$. Next, fix an open neighborhood $\mathcal{O} \subseteq \mathbb{R}^n$ of $\overline{\Omega}$ and assume that $F : \mathcal{O} \to \mathbb{R}^n$ is a bi-Lipschitz C^1-diffeomorphism. Set $\widetilde{\Omega} := F(\Omega)$ and denote by $\widetilde{\sigma}$, $\widetilde{\nu}$ the surface measure and outward unit normal on $\partial\widetilde{\Omega}$. Finally, assume that there exists $R_o > 0$ with the property that*

$$\|\nu\|_{*,R_o} \le \delta_1, \qquad \text{for some } \delta_1 \text{ sufficiently small relative to } [\sigma], \tag{4.19}$$

$$\|DF\|_{*,R_o} \le \delta_2 \quad \text{for some } \delta_2 \text{ sufficiently small relative to } [\sigma], \operatorname{Lip} F, \operatorname{Lip} F^{-1} . \tag{4.20}$$

Then there exist $C_0, C > 0$ and $\delta > 0$, depending only on the Lipschitz constants of F, F^{-1} and $[\sigma]$ with the property that

$$\|\widetilde{\nu}\|_{*,R} \le C_0(\|DF\|_{*,CR} + \|\nu\|_{*,CR}) , \qquad \forall R \in (0, \delta R_o) . \tag{4.21}$$

Proof. To get started, recall from Proposition 3.7 that $\widetilde{\sigma}$ is also doubling, and that in fact (3.88) holds. Also, generally speaking,

$$\begin{aligned}
&⨍_{\Delta_R}\Big|(DF)^{-1} - ⨍_{\Delta_R}(DF)^{-1}\,d\sigma\Big|^2 d\sigma \\
&\quad \le ⨍_{\Delta_R}|(DF)^{-1}|\Big|I_{n\times n} - (DF)\Big(⨍_{\Delta_R}(DF)^{-1}\,d\sigma\Big)\Big|^2 d\sigma \\
&\quad \le C⨍_{\Delta_R}\Big|(DF - (DF)_{\Delta_R})\Big(⨍_{\Delta_R}(DF)^{-1}\,d\sigma\Big)\Big|^2 d\sigma \\
&\qquad + C\Big|⨍_{\Delta_R}\big(I_{n\times n} - (DF)_{\Delta_R}(DF)^{-1}\big)\,d\sigma\Big|^2 \\
&\quad \le C\|DF\|_{*,R}^2 + C⨍_{\Delta_R}|DF - (DF)_{\Delta_R}|^2\big|(DF)^{-1}\big|^2\,d\sigma \\
&\quad \le C\|DF\|_{*,R}^2 ,
\end{aligned} \tag{4.22}$$

for every $R > 0$, which shows that whenever (4.20) holds we also have

$$\big\|(DF)^{-1}\big\|_{*,R_o} \le C\delta_2, \qquad \forall R \in (0, R_o) , \tag{4.23}$$

where C depends only on Ω and $\operatorname{Lip} F^{-1}$.

Next, for each $R \in (0, \operatorname{diam}\Omega)$ and $x \in \partial\Omega$, set

$$\nu_{x,R}^*(y) := \sup_{\rho\in(0,R)} ⨍_{\Delta(x,\rho)}|\nu(z) - \nu_{\Delta(x,2R)}|\,d\sigma(z), \qquad y \in \partial\Omega . \tag{4.24}$$

We then obtain

$$\nu^*_{x,R}(y) \leq M\Big(|\nu - \nu_{\Delta(x,2R)}|\mathbf{1}_{\Delta(x,2R)}\Big)(y), \qquad \forall\, y \in \Delta(x, R)\,, \tag{4.25}$$

where M is the Hardy-Littlewood maximal function on $\partial\Omega$. Note that $\partial\Omega$, when equipped with the Euclidean distance and measure σ, becomes a space of homogeneous type. Thus, from the boundedness of M on $L^2(\partial\Omega, d\sigma)$, John-Nirenberg's inequality and (4.25), we may conclude that

$$\Big(⨍_{\Delta(x,R)} \big|\nu^*_{x,R}(y)\big|^2\, d\sigma(y)\Big)^{1/2} \leq C\Big(⨍_{\Delta(x,2R)} |\nu(y) - \nu_{\Delta(x,2R)}|^2\, d\sigma(y)\Big)^{1/2} \leq C\|\nu\|_{*,2R}. \tag{4.26}$$

Fix now $x_0 \in \partial\Omega$ and $R \in (0, \delta\, R_o)$, where $\delta \in (0, 1/2)$ is a sufficiently small constant, depending only on Ω, Lip F and Lip F^{-1}, which will be specified later. Then (4.19) and (4.26) show that there exists $y_0 \in \Delta(x_0, R)$ such that $|\nu(y_0)| = 1$ and $\nu^*_{x_0,R}(y_0) \leq 1/2$, assuming δ_1 small. Since by Lebesgue's Differentiation Theorem $|\nu(y_0) - \nu_{\Delta(x_0,2R)}| \leq \nu^*_{x_0,R}(y_0)$, this forces

$$\tfrac{1}{2} \leq |\nu_{\Delta(x_0,2R)}| \leq 1, \qquad \forall\, R \in (0, \delta\, R_o)\,. \tag{4.27}$$

Going further, set

$$A(x) := \big(DF^{-1}\big)^{\top}(F(x)) = \big[(DF(x))^{-1}\big]^{\top}, \qquad x \in \mathbb{R}^n\,. \tag{4.28}$$

Running an argument similar to the one used in the paragraph above for the matrix-valued function A, with the help of (4.22) we also obtain $|A(y_0) - A_{\Delta(x_0,2R)}| \leq C\delta_2$, where C depends only on the bi-Lipschitz constants of F and $[\sigma]$. Thus, assuming that δ_2 is small, relative to these quantities, we see that

$$A_{\Delta(x_0,R)} \quad \text{is an invertible } n \times n \text{ matrix, for every } R \in (0, \delta\, R_o)\,. \tag{4.29}$$

For $R \in (0, \delta\, R_o)$, $x_0 \in \partial\Omega$ and a vector $\vec{c} \in \mathbb{R}^n$ with $\|\vec{c}\| = 1$, to be specified momentarily, we may use (3.88) and (3.24) in order to estimate

$$\begin{aligned}
&⨍_{B(F(x_0),R)\cap\partial\widetilde{\Omega}} |\widetilde{\nu} - \vec{c}|^2\, d\widetilde{\sigma}\\
&\leq C⨍_{\Delta(x_0,CR)} \Big|\big(DF^{-1}\big)^{\top}(F(x))\nu(x) - \big\|\big(DF^{-1}\big)^{\top}(F(x))\nu(x)\big\|\,\vec{c}\Big|^2\, |\det(DF)(x)|\, d\sigma(x)\\
&\leq C⨍_{\Delta(x_0,CR)} \Big|A(x)\nu(x) - \|A(x)\nu(x)\|\,\vec{c}\Big|^2\, d\sigma(x)\\
&\leq C⨍_{\Delta(x_0,CR)} \Big|A_{\Delta(x_0,CR)}\nu(x) - \|A_{\Delta(x_0,CR)}\nu(x)\|\,\vec{c}\Big|^2\, d\sigma(x) + C\|A\|_{*,CR}\\
&\leq C⨍_{\Delta(x_0,CR)} \Big|A_{\Delta(x_0,CR)}\nu_{\Delta(x_0,CR)} - \|A_{\Delta(x_0,CR)}\nu_{\Delta(x_0,CR)}\|\,\vec{c}\Big|^2\, d\sigma + C\|A\|_{*,CR} + C\|\nu\|_{*,CR}\\
&= C(\|A\|_{*,CR} + \|\nu\|_{*,CR})\,,
\end{aligned} \tag{4.30}$$

for some C which depends exclusively on the Lipschitz constants of F and F^{-1}, if we take

$$\vec{c} := \frac{A_{\Delta(x_0,CR)}\nu_{\Delta(x_0,CR)}}{\|A_{\Delta(x_0,CR)}\nu_{\Delta(x_0,CR)}\|}\,. \tag{4.31}$$

Note that by (4.27) and (4.29), $\vec{c}$ is well-defined, granted that δ is sufficiently small.

To summarize, (4.30), (4.28), and (4.22) give

$$\|\widetilde{\nu}\|_{*,R} \le C_0(\|DF\|_{*,CR} + \|\nu\|_{*,CR}) \tag{4.32}$$

if $R \in (0, \delta R_o)$ with $\delta > 0$ small, for some constants C_0, $C > 0$ depending only on the Lipschitz constants of F and F^{-1}. □

Let us digress for a moment and point out that, as the above proof shows, $\|\widetilde{\nu}\|_{\mathrm{BMO}\,(\partial\widetilde{\Omega}, d\widetilde{\sigma})}$ is small if $\|DF\|_{\mathrm{BMO}\,(\partial\Omega, d\sigma)}$ and $\|\nu\|_{\mathrm{BMO}\,(\partial\Omega, d\sigma)}$ are sufficiently small (relative to the Lipschitz constants of F, F^{-1}, and the doubling constant of σ). Note that $\|\nu\|_{\mathrm{BMO}\,(\partial\Omega, d\sigma)} \le 1$ can only happen when $\partial\Omega$ is not compact (since, otherwise, $\int_{\partial\Omega} \nu\, d\sigma = 0$ forces the opposite inequality).

With Proposition 4.7 in hand, it is easy to finish the following proof.

Proof of Theorem 4.6. Lemma 4.5 ensures that $\widetilde{\Omega}$ is a bounded Ahlfors regular domain, which satisfies a local two-sided John condition and for which (4.14) holds. Keeping in mind the characterization from Theorem 5.8, it remains to show that $\widetilde{\nu} \in \mathrm{VMO}\,(\partial\widetilde{\Omega}, d\widetilde{\sigma})$, i.e., that

$$\limsup_{R\to 0^+} \left(\sup_{x_0\in\partial\widetilde{\Omega}} \inf_{\vec{c}\in\mathbb{R}^n} \left(\fint_{B(F(x_0),R)\cap\partial\widetilde{\Omega}} |\widetilde{\nu} - \vec{c}|^2\, d\widetilde{\sigma} \right)^{1/2} \right) = 0\,. \tag{4.33}$$

This, however, is a consequence of (4.21), the membership of ν in $\mathrm{VMO}\,(\partial\Omega, d\sigma)$, and the fact that, since DF is continuous, its oscillations converge to zero as the radii of the balls go to zero. □

Remark 4.8. The invariance result for the class of bounded regular SKT domains presented in Theorem 4.6 is sharp, as this class is not, generally speaking, preserved under bi-Lipschitz mappings. To see this, consider a Lipschitz map $\varphi : S^{n-1} \longrightarrow (0, 1)$ and define

$$F : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \qquad F(r\omega) := r\varphi(\omega)\omega, \quad r > 0, \;\; \omega \in S^{n-1}\,. \tag{4.34}$$

As with (4.6), F is readily seen to be bi-Lipschitz, but the domain $F(B(0, 1))$ is, generally speaking, no more regular than a generic strongly Lipschitz domain (starlike with respect to the origin). In particular, the image of the regular SKT domain $B(0, 1)$ under such a mapping F may fail to be itself regular SKT.

Remark 4.9. Given $m \in \mathbb{N}$, define $\mathrm{VMO}_1(\mathbb{R}^m)$ as the space of locally integrable functions in $\mathbb{R}^m$ with the property that (the components of) their distributional gradients belong to Sarason's space $\mathrm{VMO}(\mathbb{R}^m)$. Next, define the class of *bounded* VMO_1 *domains* following the same recipe as in Definition 2.5 except that, instead of asking that φ is Lipschitz, this time we stipulate that $\varphi \in \mathrm{VMO}_1(\mathbb{R}^{n-1})$. Clearly, bounded C^1 domains are bounded VMO_1 domains which, in turn, are bounded regular SKT domains (cf. [14, 11] for the latter claim). However, while both the class of bounded C^1 domains, and the class of bounded regular SKT domains are invariant under C^1-diffeomorphisms of the Euclidean space, it is easy to produce examples showing that, generally speaking, this is *not* the case for the class of bounded VMO_1 domains.

Remark 4.10. By relying on results established in [11], the argument in the proof of Theorem 4.6 can be altered to show that the image of a bounded δ-SKT domain $\Omega \subset \mathbb{R}^n$ under a C^1-diffeomorphism of the space is δ_o-SKT for some $\delta_o := C\delta$ with $C = C(\Omega, F) > 0$, granted that the original δ is, to begin with, small relative to the John and Ahlfors constants of Ω. Furthermore, a suitable version for domains which are not necessarily bounded is valid as well.

A local version of Theorem 4.6 holds as well. More specifically, with terminology introduced in Definition 5.9, the same type of argument as above allows us to conclude the following.

Theorem 4.11. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $\mathcal{O} \subseteq \mathbb{R}^n$ an open neighborhood of $\overline{\Omega}$, and $F : \mathcal{O} \to \mathbb{R}^n$ a C^1-diffeomorphism onto its image. If Ω is a regular SKT domain near $x_0 \in \partial\Omega$, it follows that $F(\Omega)$ is a regular SKT domain near $F(x_0)$.*

In particular, much as we have done in the case of Lipschitz domains, this allows to define regular SKT domains on manifolds and shows that the definition depends only on the intrinsic C^1 structure of the manifold.

4.3. Approximating domains of locally finite perimeter

Let $\Omega \subset \mathbb{R}^n$ be an open set of locally finite perimeter, and fix an open neighborhood $\mathcal{O} \subseteq \mathbb{R}^n$ of $\overline{\Omega}$. As before, we let ν, σ denote, respectively, the outward unit normal and surface measure on $\partial\Omega$. Going further, assume that $F_j : \mathcal{O} \to \mathbb{R}^n$, $j \in \mathbb{N}$, is a family of global bi-Lipschitz, orientation preserving, C^1-diffeomorphisms, satisfying

$$\sup_{j\in\mathbb{N}} \mathrm{Lip}\,(F_j, \mathcal{O}) < \infty, \qquad \sup_{j\in\mathbb{N}} \mathrm{Lip}\,\big(F_j^{-1}, F_j(\mathcal{O})\big) < \infty\,. \tag{4.35}$$

Set

$$\Omega_j := F_j(\Omega), \qquad j \in \mathbb{N}\,. \tag{4.36}$$

Then, Proposition 3.1 ensures that each Ω_j is a set of locally finite perimeter, and we denote by ν_j, σ_j, respectively, the outward unit normal and surface measure on $\partial\Omega_j$, $j \in \mathbb{N}$.

Proposition 4.12. *Retain the above hypotheses and, in addition, assume that*

$$DF_j(x) \longrightarrow I_{n\times n} \quad \text{as } j \to \infty, \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega, \tag{4.37}$$

$$F_j(x) \longrightarrow x \quad \text{as } j \to \infty, \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega\,. \tag{4.38}$$

Then

$$\nu_j(F_j(x)) \longrightarrow \nu(x) \quad \text{as } j \to \infty, \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega\,, \tag{4.39}$$

and

$$\big(F_j^{-1}\big)_*\sigma_j \longrightarrow \sigma \quad \text{as } j \to \infty, \quad \text{weakly in } \mathcal{M}\,. \tag{4.40}$$

Furthermore, there exist constants $0 < c_1 < c_2 < \infty$, depending only on the quantities in (4.35), *along with functions $\omega_j \in L^\infty(\partial\Omega, d\sigma)$, $j \in \mathbb{N}$, for which*

$$c_1 \le \omega_j \le c_2 \;\; \sigma\text{-a.e. on } \partial\Omega, \;\; \forall\, j \in \mathbb{N}\,, \tag{4.41}$$

$$\omega_j \longrightarrow 1 \;\; \text{as } j \to \infty, \quad \sigma\text{-a.e. on } \partial\Omega\,, \tag{4.42}$$

and which satisfy the following additional property for every $j \in \mathbb{N}$:

$$\sigma_j(F_j(E)) = \int_E \omega_j\, d\sigma \qquad \forall\, E \subset \partial\Omega, \;\; \text{Borel set}\,. \tag{4.43}$$

Proof. From (3.23), for each $j \in \mathbb{N}$ and σ-a.e. $x \in \partial\Omega$, we have

$$\nu_j(F_j(x)) = \frac{\left(DF_j^{-1}\right)^{\top}(F_j(x))\nu(x)}{\left\|\left(DF_j^{-1}\right)^{\top}(F_j(x))\nu(x)\right\|} = \frac{\left[(DF_j)^{\top}(x)\right]^{-1}\nu(x)}{\left\|\left[(DF_j)^{\top}(x)\right]^{-1}\nu(x)\right\|} . \tag{4.44}$$

It is then elementary to deduce (4.39) from (4.37)–(4.38) and (4.35).

Next, formula (3.24) gives that for every $f \in C^0(\mathbb{R}^n)$ with compact support contained in $\mathcal{O}$, and every $j \in \mathbb{N}$,

$$\begin{aligned}\int f\, d\left(F_j^{-1}\right)_* \sigma_j &= \int f \circ F_j^{-1}\, d\sigma_j \\ &= \int \left(f \circ F_j^{-1}\right)\left\|\left(DF_j^{-1}\right)^{\top}\left(\nu \circ F_j^{-1}\right)\right\| \left|\det(DF_j) \circ F_j^{-1}\right| d(F_j)_*\sigma \\ &= \int f \left\|\left(DF_j^{-1}\right)^{\top} \circ F_j\, \nu\right\| \left|\det(DF_j)\right| d\sigma . \end{aligned} \tag{4.45}$$

Hence, the desired conclusion follows with the help of (4.37)–(4.38), (4.35), and Lebesgue's Dominated Convergence Theorem.

In fact, (4.45) also shows that for every Borel set $E \subset \partial\Omega$ and $j \in \mathbb{N}$,

$$\sigma_j(F_j(E)) = \int_E \left\|\left(DF_j^{-1}\right)^{\top} \circ F_j\, \nu\right\| \left|\det(DF_j)\right| d\sigma \tag{4.46}$$

so that (4.43) is valid if we set

$$\omega_j := \left\|\left[\left(DF_j^{-1}\right)^{\top} \circ F_j\right]\nu\right\| \left|\det(DF_j)\right| \quad \text{on} \quad \partial\Omega . \tag{4.47}$$

For this choice it is then easy to verify that (4.41)–(4.42) hold, again, by relying on (4.37)–(4.38) and (4.35). □

Remark 4.13. Of course, formula (4.43) further entails that composition with F_j^{-1} is an isomorphism from $L^1(\partial\Omega_j, d\sigma_j)$ onto $L^1(\partial\Omega, d\sigma)$ and

$$\int_{\partial\Omega_j} f \circ F_j^{-1}\, d\sigma_j = \int_{\partial\Omega} f\, \omega_j\, d\sigma, \qquad \forall f \in L^1(\partial\Omega, d\sigma) . \tag{4.48}$$

Remark 4.14. Let $\Omega \subset \mathbb{R}^n$ be arbitrary, and assume that $\mathcal{O} \subseteq \mathbb{R}^n$ is an open neighborhood of $\overline{\Omega}$. If $F_j : \mathcal{O} \to \mathbb{R}^n$, $j \in \mathbb{N}$, are bi-Lipschitz maps satisfying (4.35) and for which $|F_j(x)-x| \to 0$ as $j \to \infty$, *uniformly* for $x \in \overline{\Omega}$, it is then straightforward to check that

$$\partial\Omega_j \longrightarrow \partial\Omega \quad \text{and} \quad \Omega_j \longrightarrow \Omega \quad \text{as } j \to \infty , \tag{4.49}$$

in the Hausdorff distance sense (cf. (4.53) below for the definition of the Hausdorff distance between two subsets of $\mathbb{R}^n$). If, on the other hand, $DF_j(x) \to I_{n\times n}$ as $j \to \infty$, *uniformly* for $x \in \partial\Omega$, it follows from (4.44) that

$$\|\nu_j(F_j(\cdot)) - \nu\|_{L^\infty(\partial\Omega, d\sigma)} \longrightarrow 0 \quad \text{as } j \to \infty . \tag{4.50}$$

In the Appendix of [16], Kenig and Toro developed an approximation scheme of a δ-Reifenberg flat domain Ω by $\Omega_j := \phi_j(\Omega)$, $j \in \mathbb{N}$, where the ϕ_j's are certain smooth diffeomorphisms of the space, of bounded bi-Lipschitz character. In the process, results such as (4.39)–(4.43) are established via somewhat ad hoc methods. An inspection of their proofs reveals, however, that the ϕ_j's satisfy similar conditions to (4.37)–(4.38), so Proposition 4.12 could be used to substantially simplify their arguments. In the class of strongly Lipschitz domains, approximation schemes similar to the one discussed in Proposition 4.12 have been developed in [3, 4, 25]. Below we consider in more detailed the case when Ω is a bounded strongly Lipschitz domain, and give a concrete recipe for constructing a sequence of C^1-diffeomorphisms satisfying (4.37)–(4.38).

To set the stage, assume that $\Omega \subset \mathbb{R}^n$ is a bounded strongly Lipschitz domain, and fix a vector field

$$h \in C^1\big(\mathbb{R}^n, S^{n-1}\big), \qquad h \text{ globally transversal to } \partial\Omega\,. \tag{4.51}$$

Next, for each $t > 0$, define the mapping:

$$F_t(x) := x - t\,h(x), \qquad F_t : \mathbb{R}^n \longrightarrow \mathbb{R}^n\,. \tag{4.52}$$

Finally, we remind the reader that the Hausdorff distance between two sets $A, B \subset \mathbb{R}^n$ is defined as

$$D[A, B] := \max\big\{\sup\{\text{dist}\,(a, B) : a \in A\}\,,\ \sup\{\text{dist}\,(b, A) : b \in B\}\big\}\,. \tag{4.53}$$

Proposition 4.15. *In the context given above, there exist $t_o > 0$ along with an open neighborhood $\mathcal{O}$ of $\overline{\Omega}$, both depending only on Ω and h, with the property that if $0 < t < t_o$ then $F_t(\Omega)$ is open, $F_t : \mathcal{O} \to F_t(\Omega)$ is a C^1-diffeomorphism. In addition,*

$$\overline{F_t(\Omega)} \subset \Omega, \qquad \forall\, t \in (0, t_o)\,, \tag{4.54}$$

and

$$D[\partial\Omega\,,\, \partial F_t(\Omega)] \le t, \qquad \forall\, t \in (0, t_o)\,. \tag{4.55}$$

In order to facilitate the proof of this result, we isolate in the following lemma an important step in this direction.

Lemma 4.16. *Retain the same context as before and, for each $t > 0$, define*

$$\mathcal{O}_t := \{x - s\,h(x) : x \in \partial\Omega,\ \ -t < s < t\}\,. \tag{4.56}$$

Then there exists $t_o > 0$, depending only on Ω and h, with the property that

$$0 < t < t_o \Longrightarrow \mathcal{O}_t \text{ is an open set, and } \partial\mathcal{O}_t = \partial\overline{\mathcal{O}_t} = \{x \pm t\,h(x) : x \in \partial\Omega\}\,. \tag{4.57}$$

Proof. Fix an arbitrary point $x_0 \in \partial\Omega$. From Definition 2.5 we know that there exist an isometric system of coordinates (x', s) of $\mathbb{R}^n$, with x_0 as the origin, such that $\partial\Omega$ coincides near 0 with the graph of a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\varphi(0) = 0$. Let us denote by $h_1, \dots, h_n$ the components of h in this new system of coordinates. Thanks to (2.23), the transversality condition satisfied by h ensures that

$$-h_n\big(x'\varphi(x')\big) \ge \kappa > 0, \qquad \text{uniformly for } x' \text{ near } 0 \in \mathbb{R}^{n-1}\,. \tag{4.58}$$

In this new system of coordinates, let us also define

$$G : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \qquad G\big(x', s\big) := \big(x', \varphi(x')\big) - s\,h\big(x', \varphi(x')\big)\,. \tag{4.59}$$

The claim that we make at this stage is that

$$G \text{ is bi-Lipschitz near } 0 \in \mathbb{R}^n . \tag{4.60}$$

To prove this claim, let us first observe that, schematically,

$$DG(x', s) = \left|\begin{array}{ccc} & & \vdots & -h_1 \\ & I_{(n-1)\times(n-1)} & \vdots & -h_2 \\ & & \vdots & \vdots \\ \cdots & \cdots & \vdots & \vdots \\ & \nabla'\varphi(x') & \vdots & -h_n \end{array}\right| + O(s) , \tag{4.61}$$

where the components $h_1, \dots, h_n$ of h are evaluated at $(x', \varphi(x'))$, and the constant implicit in the "big O" symbol depends only on h and $\|\nabla'\varphi\|_{L^\infty}$. As a consequence,

$$\det(DG)(x', s) = -h_n(x', \varphi(x')) + O(s) . \tag{4.62}$$

Hence, had φ been of class C^1, the mapping (4.59) would be invertible near $0 \in \mathbb{R}^n$, by the Inverse Function Theorem. This would clearly entail (4.60).

In the more general situation when φ is merely Lipschitz, mollify φ to produce a sequence of C^∞ functions φ_j, $j \in \mathbb{N}$, such that:

$$\varphi_j \to \varphi \text{ in } L^\infty, \quad \nabla'\varphi_j \to \nabla'\varphi \text{ pointwise a.e., and } \sup_{j\in\mathbb{N}} \|\nabla'\varphi_j\|_{L^\infty} \le \|\nabla'\varphi\|_{L^\infty} . \tag{4.63}$$

If for each $j \in \mathbb{N}$ we now consider G_j defined analogously to (4.59) but with φ replaced by φ_j, then there exists a neighborhood U of $0 \in \mathbb{R}^n$ with the property that $G_j \to G$ uniformly in U. Furthermore, as the previous discussion shows, U can be chosen such that

$$G_j \text{ is bi-Lipschitz in } U, \text{ with constants independent of } j . \tag{4.64}$$

Now (4.60) follows easily from this and a limiting argument. In turn, (4.60) further implies that

$$G \text{ maps sufficiently small neighborhoods of } 0 \in \mathbb{R}^n \text{ into neighborhoods of } x_0 . \tag{4.65}$$

Next, for each $t > 0$, let us now consider the mapping

$$H : \partial\Omega \times (-t, t) \longrightarrow \mathbb{R}^n, \qquad H(x, s) := x - s\,h(x) . \tag{4.66}$$

We claim that there exists $t_o > 0$ such that

$$t \in (0, t_o) \Longrightarrow H \text{ is a bi-Lipschitz mapping} . \tag{4.67}$$

To justify this, observe first that if an arbitrary point $x_0 \in \partial\Omega$ has been fixed, along with a new system of coordinates (x', s), isometric with the standard one, as at the beginning of the current proof, then in this new system of coordinates

$$H(x, s) = G(x', s), \quad \text{if } x = (x', x_n) \in \partial\Omega \text{ is near } x_0 . \tag{4.68}$$

From this and (4.60) we then deduce that

$$\begin{gathered} |H(x, s_1) - H(y, s_2)| \ge C(|x' - y'| + |s_1 - s_2|), \quad \text{whenever} \\ x = (x', x_n),\ y = (y', y_n) \in \partial\Omega \text{ are near } x_0, \text{ and } s_1, s_2 \text{ are near } 0 . \end{gathered} \tag{4.69}$$

Since, in the above context, $|x_n - y_n| = |\varphi(x') - \varphi(y')| \leq C|x' - y'|$, (4.69) further improves to

$$
\begin{aligned}
&|H(x, s_1) - H(y, s_2)| \geq C|(x, s_1) - (y, s_2)|, \\
&\text{if} \quad x,\ y \in \partial\Omega \ \text{ are near } x_0, \text{ and } s_1, s_2 \text{ are near } 0\,.
\end{aligned}
\tag{4.70}
$$

Since the reverse inequality in (4.70) is a simple consequence of (4.66), this shows that H in (4.66) is locally bi-Lipschitz. It is then easy to show that the claim in (4.67) holds, as soon as we prove that H in (4.66) is one-to-one.

To this end, assume that $x_1, x_2 \in \partial\Omega$, $s_1, s_2 \in (-t, t)$ are such that $H(x_1, s_1) = H(x_2, s_2)$. It follows that $|x_1 - x_2| = |s_1\, h(x_1) - s_2\, h(x_2)| \leq |s_1| + |s_2| \leq 2t$. Hence, if $t \in (0, t_o)$ for a sufficiently small $t_o > 0$, this forces x_1, x_2 to belong to the same coordinate cylinder near a boundary point x_0 (in the terminology of Definition 2.5). Moreover, this coordinate cylinder can be made small by taking t_o small. Once this has been established, the local bi-Lipschitzianity of H gives that $(x_1, s_1) = (x_2, s_2)$, proving that H is one-to-one. This concludes the proof of (4.67).

Going further, note that:

$$
\mathcal{O}_t = H(\partial\Omega \times (-t, t))\,. \tag{4.71}
$$

As a consequence of this, (4.65) and (4.68), we may conclude that $\mathcal{O}_t$ is open if $t \in (0, t_o)$, if $t_o > 0$ is sufficiently small. Furthermore, regarding $\partial\Omega \times (-t, t)$ as an open set in the topological space $\partial\Omega \times (-t_1, t_1)$, with $0 < t < t_1 < t_o$, $t_o > 0$ small, then (4.67) gives

$$
\begin{aligned}
\partial\overline{\mathcal{O}_t} &= \partial\big[\overline{H(\partial\Omega \times (-t, t))}\big] = H\big(\partial\overline{(\partial\Omega \times (-t, t))}\big) \\
&= H(\partial(\partial\Omega \times [-t, t])) = H(\partial\Omega \times \{\pm t\}) = \{x \pm t\, h(x) : x \in \partial\Omega\}\,.
\end{aligned}
\tag{4.72}
$$

A similar argument shows that $\partial\mathcal{O}_t = \{x \pm t\, h(x) : x \in \partial\Omega\}$, finishing the proof of (4.57). □

Remark 4.17. The proof of Lemma 4.16 is more resourceful than its actual statement indicates. For example, it gives that for $t_o > 0$ small and $t \in (0, t_o)$,

$$
\mathcal{O}_t^{\pm} := \{x \mp s\, h(x) : x \in \partial\Omega,\ \ 0 < s < t\} \tag{4.73}
$$

are open sets with $\partial\mathcal{O}_t^{\pm} := \{x \mp t\, h(x) : x \in \partial\Omega\}$. Also, the application

$$
R : \mathcal{O}_t \longrightarrow \mathcal{O}_t, \quad R(x - s\, h(x)) := x + s\, h(x),\ \ x \in \partial\Omega,\ \ s \in (-t, t)\,, \tag{4.74}
$$

is well-defined, Lipschitz, involutive, fixes $\partial\Omega$, and maps $\mathcal{O}_t^{\pm}$ onto $\mathcal{O}_t^{\mp}$ (that is, R is a *Lipschitz reflection across the boundary*, $\partial\Omega$, in a collar neighborhood of it).

We are now prepared to present the following proof.

Proof of Proposition 4.15. To begin with, we note that there exists $t_o > 0$ such that F_t is a C^1-diffeomorphism mapping an open neighborhood $\mathcal{O}$ of $\overline{\Omega}$ onto an open subset of $\mathbb{R}^n$. Indeed, F_t is of class C^1 in $R\mathbb{R}^n$ for each t and, given a bounded, open neighborhood $\mathcal{O}$ of $\overline{\Omega}$, we have $|DF_t(x) - I_{n\times n}| \leq t\,\|Dh\|_{L^\infty(\mathcal{O})}$, for any $x \in \mathcal{O}$. Thus, DF_t is invertible at every point in $\mathcal{O}$ if $t \in (0, t_o)$ with $t_o > 0$ sufficiently small. As a consequence, the Inverse Function Theorem can be invoked to conclude that F_t is locally a C^1-diffeomorphism, if $t \in (0, t_o)$. In particular, $F_t(\mathcal{O})$ is an open subset of $\mathbb{R}^n$ for each $t \in (0, t_o)$. To conclude that actually $F_t : \mathcal{O} \to F_t(\mathcal{O})$ is a C^1-diffeomorphism, it suffices to show that this map is one-to-one. Given that, we know this at the local level, this can be easily arranged [using the explicit formula in (4.52)] by ensuring that t_o is small enough.

Next, from (4.52) and the fact that F is a C^1-diffeomorphism (hence, in particular, a topological homeomorphism), we obtain:

$$\partial F_t(\Omega) = F_t(\partial\Omega) = \{x - t\,h(x) : x \in \partial\Omega\}, \qquad t \in (0, t_o)\,. \tag{4.75}$$

From this and (4.53), the estimate (4.55) readily follows.

Next, retaining notation used in Lemma 4.16, let us write

$$\partial\big(\Omega \setminus \overline{\mathcal{O}_t}\big) = \overline{\Omega} \cap \partial\overline{\mathcal{O}_t} = \{x - t\,h(x) : x \in \partial\Omega\} = \partial F_t(\Omega)\,. \tag{4.76}$$

Above, the first equality is a consequence of Lemma 5.14 (specifically, (5.23) applied with $A := \Omega$ and $B := \mathcal{O}_t$). The second equality then follows from (4.57) and (2.57), while the third is contained in (4.75).

As far as (4.54) is concerned, working in each connected component of Ω, there is no loss of generality in assuming that Ω itself is connected. Assuming that this is the case, we now bring in Lemma 5.15, considered for the open sets $\mathcal{O}_1 := F_t(\Omega)$ and $\mathcal{O}_2 := \Omega \setminus \overline{\mathcal{O}_t}$. Note that $\mathcal{O}_1$ is connected, since Ω is connected and F_t is continuous. The conclusion then is that: Either (i) $F_t(\Omega)$ and $\Omega \setminus \overline{\mathcal{O}_t}$ are disjoint, or (ii) $F_t(\Omega) \subseteq \Omega \setminus \overline{\mathcal{O}_t}$. To rule out the first eventuality, it suffices to observe that if $x \in \Omega$ is such that $r := \mathrm{dist}\,(x, \partial\Omega) > 2t_o$, then for each $t \in (0, t_o)$ we have $F_t(x) \in B(x, r/2) \subset \Omega$, and $\mathrm{dist}\,(F_t(x), \partial\Omega) > t_o$. The latter condition ensures that $F_t(x)$ does not belong to

$$\overline{\mathcal{O}_t} = \mathcal{O}_t \cup \partial\mathcal{O}_t = \{x - s\,h(x) : x \in \partial\Omega,\ \ -t \le s \le t\}\,, \tag{4.77}$$

since any point in the last set above is at distance $\le t_o$ from $\partial\Omega$, granted that $0 < t < t_o$. Thus altogether, $F_t(x) \in \Omega \setminus \overline{\mathcal{O}_t}$, proving that (i) above cannot happen if $0 < t < t_o$, with t_o small enough. Consequently,

$$F_t(\Omega) \subseteq \Omega \setminus \overline{\mathcal{O}_t}, \qquad \forall\, t \in (0, t_o)\,, \tag{4.78}$$

if t_o small. Thus, $\overline{F_t(\Omega)} \subseteq \overline{\Omega \setminus \overline{\mathcal{O}_t}}$ if $t \in (0, t_o)$, and (4.54) follows from this and Lemma 5.16 used with $\mathcal{O} := \mathcal{O}_t$. This concludes the proof of Proposition 4.15. □

Remark 4.18. By further refining some of the arguments above, it is possible to show that, in fact,

$$F_t(\Omega) = \Omega \setminus \overline{\mathcal{O}_t}, \qquad \forall\, t \in (0, t_o)\,, \tag{4.79}$$

if t_o small. To see this, we invoke (4.66)–(4.67), (4.71) and the fact that $\partial\Omega$ is locally (up to an isometry) a Lipschitz graph, in order to conclude that for each $x \in \partial(\Omega \setminus \overline{\mathcal{O}_t})$ there exists $r > 0$ with the property that $B(x, r) \setminus \partial(\Omega \setminus \overline{\mathcal{O}_t})$ consists of two open connected components, one contained in $\Omega \cap \mathcal{O}_t$ and the other in $\Omega \setminus \overline{\mathcal{O}_t}$. Given that, as we have already shown, $F_t(\Omega)$ is a Lipschitz domain for which $\partial F_t(\Omega) = \partial(\Omega \setminus \overline{\mathcal{O}_t})$ [cf. (4.76)], we may conclude that the sets $F_t(\Omega)$ and $\Omega \setminus \overline{\mathcal{O}}_t$ agree in the neighborhood of any point on their common boundary. Then (4.79) follows from this and Lemma 5.17.

We are now in a position to state an approximation result in the spirit of Calderón's work in [3].

Proposition 4.19. *Consider a bounded strongly Lipschitz domain Ω in $\mathbb{R}^n$ with surface measure σ and outward unit normal ν, and let h be a C^1 vector field in $\mathbb{R}^n$ satisfying*

$$|h(x)| = 1 \quad \text{and} \quad \langle h(x), \nu(x)\rangle \ge \kappa \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega\,, \tag{4.80}$$

where $\kappa \in (0,1)$ is a fixed constant. Also, for each $t > 0$, let Ω_t be the subset of Ω defined by:

$$\Omega_t := \{x - t\,h(x) : x \in \Omega\}\,. \tag{4.81}$$

Then, there exists $t_o > 0$, depending only on the Lipschitz character of Ω, the Lipschitz constant of h in a compact neighborhood of $\overline{\Omega}$, n and κ, such that the following properties hold.

(i) *Whenever $0 < t < t_o$, Ω_t is a bounded strongly Lipschitz domain with*

$$\overline{\Omega_t} \subseteq \Omega \quad \text{and} \quad \partial\Omega_t = \{x - t\,h(x) : x \in \partial\Omega\}\,. \tag{4.82}$$

(ii) *There exists a covering of $\partial\Omega$ with finitely many coordinate cylinders which also form a family of coordinate cylinders for $\partial\Omega_t$, for each $t \in (0, t_o)$. Moreover, for each such cylinder $\mathcal{C}$, if φ and φ_t are the corresponding Lipschitz functions whose graphs describe the boundaries of Ω and Ω_t, respectively, in $\mathcal{C}$, then $\|\nabla\varphi_t\|_{L^\infty} \le C\|\nabla\varphi\|_{L^\infty}$ and $\nabla\varphi_t \to \nabla\varphi$ pointwise σ-a.e. as $t \to 0^+$.*

(iii) *Consider the mapping defined by*

$$\Lambda_t : \partial\Omega \longrightarrow \partial\Omega_t, \qquad \Lambda_t(x) := x - t\,h(x), \quad x \in \partial\Omega\,. \tag{4.83}$$

Then Λ_t is a bi-Lipschitz map for each $t \in (0, t_o)$ and the Lipschitz constants of Λ_t and ${\Lambda_t}^{-1}$ are uniformly bounded in t.

(iv) *For every $t \in (0, t_o)$ and every $x \in \partial\Omega$, $\Lambda_t(x)$ approaches x as $t \to 0^+$ along a transversal direction (hence, nontangentially),*

$$\sup_{x\in\partial\Omega} |x - \Lambda_t(x)| \le Ct\,, \tag{4.84}$$

for some finite, positive constant $C = C(\Omega, h)$.

(v) *For each $t \in (0, t_o)$, there exist positive functions $\omega_t : \partial\Omega \to \mathbb{R}_+$, bounded away from zero and infinity uniformly in t, such that, for any measurable set $E \subset \partial\Omega$,*

$$\int_E \omega_t\,d\sigma = \int_{\Lambda_t(E)} d\sigma_t\,, \tag{4.85}$$

where $d\sigma_t$ denotes the surface measure on $\partial\Omega_t$. In addition,

$$\sup_{x\in\partial\Omega} |1 - \omega_t(x)| \le Ct, \qquad \forall\, t \in (0, t_o)\,, \tag{4.86}$$

for some finite, positive constant $C = C(\Omega, h)$.

(vi) *If ν_t is the outward unit normal vector to $\partial\Omega_t$ then, with C as above,*

$$\sup_{x\in\partial\Omega} |\nu(x) - \nu_t(\Lambda_t(x))| \le Ct, \qquad \forall\, t \in (0, t_o)\,. \tag{4.87}$$

Finally, a similar approximation result from the outside, i.e., with domains $\Omega_t \supset \Omega$, holds as well.

Proof. This is largely a direct consequence of Theorem 4.1, Proposition 4.12, and Proposition 4.15. The pointwise convergence in (iii) can be seen from (2.73) and (3.23). The fact that $\Lambda_t(x) \to x$ nontangentially as $t \to 0^+$ is seen from (2.57) and the remark made on that occasion. $\square$

We conclude by discussing a refined version of Proposition 4.12, using results from the second part of Section 3. Recall definition (3.61).

Proposition 4.20. *Retain the same context as in Proposition 4.12 and, instead of* (4.37)–(4.38), *assume that there is a bi-Lipschitz mapping $F : \mathcal{O} \to \mathbb{R}^n$ for which*

$$d_{\partial^*\Omega} F_j(x) \longrightarrow d_{\partial^*\Omega} F(x) \quad \text{as } j \to \infty, \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega, \tag{4.88}$$

$$F_j(x) \longrightarrow F(x) \quad \text{as } j \to \infty, \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega\,. \tag{4.89}$$

Let $\widetilde{\nu}, \widetilde{\sigma}$ denote the outward unit normal and surface measure on the boundary of $\widetilde{\Omega} := F(\Omega)$. Then

$$\nu_j(F_j(x)) \longrightarrow \widetilde{\nu}(F(x)) \quad \text{as } j \to \infty, \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega\,, \tag{4.90}$$

and

$$\left(F_j^{-1}\right)_* \sigma_j \longrightarrow \left(F^{-1}\right)_* \widetilde{\sigma} \quad \text{as } j \to \infty, \quad \text{weakly, as Radon measures}\,. \tag{4.91}$$

If Ω is a bounded Lipschitz domain, then (4.90)–(4.91) *hold when* (4.88) *is replaced by*

$$\nabla_{\tan} F_j(x) \longrightarrow \nabla_{\tan} F(x) \quad \text{as } j \to \infty, \quad \text{for } \sigma\text{-a.e. } x \in \partial\Omega\,, \tag{4.92}$$

where $\nabla_{\tan}$ is the tangential gradient on the C^1 surface $\partial\Omega$. In this latter scenario, formulas (3.75)–(3.76) *hold.*

Proof. The first part follows much as Proposition 4.12, with the help of Theorem 3.4 and (3.72). Then the second part is a consequence of (4.90)–(4.91) and (3.75)–(3.76). □

Below is a corollary of this result which is quite useful in applications.

Corollary 4.21. *Assume that $\varphi : S^{n-1} \to (0, \infty)$ is a Lipschitz function and consider the open set $\Omega := \{r\omega : 0 \le r < \varphi(\omega),\ \omega \in S^{n-1}\} \subseteq \mathbb{R}^n$. Then Ω is a strongly Lipschitz domain, with outward unit normal given by*

$$\nu(\varphi(\omega)\omega) = \frac{\varphi(\omega)\omega - (\nabla_{\tan}\varphi)(\omega)}{\sqrt{|(\nabla_{\tan}\varphi)(\omega)|^2 + |\varphi(\omega)|^2}}, \quad \text{for a.e. } \omega \in S^{n-1}\,, \tag{4.93}$$

(where $\nabla_{\tan}\varphi$ denotes the tangential gradient of φ on S^{n-1}), and surface measure σ satisfying

$$\int_{\partial\Omega} f\, d\sigma = \int_{S^{n-1}} f(\omega\varphi(\omega))\varphi(\omega)\sqrt{|(\nabla_{\tan}\varphi)(\omega)|^2 + |\varphi(\omega)|^2}\, d\omega\,, \tag{4.94}$$

for every nonnegative, measurable function f on $\partial\Omega$.

Proof. Define $F_\varphi : \mathbb{R}^n \to \mathbb{R}^n$ by setting $F_\varphi(r\omega) := r\varphi(\omega)\omega$, if $r \ge 0$, $\omega \in S^{n-1}$. It can then be easily checked that F_φ is a bi-Lipschitz mapping with the property that $(F_\varphi)^{-1} = F_{1/\varphi}$. Also, $\Omega = F_\varphi(B(0,1))$. Thus, Ω is a bounded domain of finite perimeter by Proposition 3.1. For $t \in (0,1)$, consider the kernel

$$k_t(x,y) := \frac{1}{a_n}\frac{1-t^2}{|tx-y|^n}, \qquad x, y \in \mathbb{R}^n\,, \tag{4.95}$$

where a_n is the area of S^{n-1}. Since $k_t(x, y) = K(tx, y)$, where $K(x, y) = \frac{1}{a_n}\frac{1-|x|^2}{|x-y|^n}$ is the (harmonic) Poisson kernel for the unit ball, it follows that $k_t(\omega, \omega')$ acts as an approximate identity on S^{n-1} as $t \to 1^-$. Set

$$\varphi_t(\omega) := \int_{S^{n-1}} k_t(\omega, \omega')\varphi(\omega')\,d\omega', \qquad \omega \in S^{n-1}, \tag{4.96}$$

so that $\varphi_t \in C^\infty(S^{n-1})$. As $(\frac{x_j}{|x|}\partial_{x_k} - \frac{x_k}{|x|}\partial_{x_j})k_t(x, y) = -(\frac{y_j}{|y|}\partial_{y_k} - \frac{y_k}{|y|}\partial_{y_j})k_t(x, y)$ for each pair of indices $j, k \in \{1, \dots, n\}$, we also obtain

$$\varphi_t(\omega) \longrightarrow \varphi(\omega) \text{ and } \nabla_{\tan}\varphi_t(\omega) \longrightarrow \nabla_{\tan}\varphi(\omega) \text{ as } t \to 1^- \text{ for a.e. } \omega \in S^{n-1}. \tag{4.97}$$

Thus, $F_{\varphi_t} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is a C^∞ diffeomorphism with the property that $F_{\varphi_t} \to F_\varphi$ pointwise on S^{n-1}. Also, a direct calculation shows that for each $j \in \{1, \dots, n\}$,

$$\nabla_{\tan}(F_{\varphi_t})_j(\omega) = \varphi_t(\omega)(e_j - \omega_j\omega) + \omega_j(\nabla_{\tan}\varphi_t)(\omega), \qquad \omega \in S^{n-1}, \tag{4.98}$$

which, in turn, can be used to conclude that $\nabla_{\tan}F_{\varphi_t}(\omega) \to \nabla_{\tan}F_\varphi(\omega)$ as $t \to 1^-$ for a.e. $\omega \in S^{n-1}$. Using the readily checked formula

$$(DF_{\varphi_t})(\omega) = \varphi_t(\omega)I_{n\times n} + \omega \otimes (\nabla_{\tan}\varphi_t)(\omega), \qquad \omega \in S^{n-1}, \tag{4.99}$$

one can then employ (3.23) in order to conclude that the outward unit normal to $\Omega_t := F_{\varphi_t}(B(0, 1))$ is given by

$$\nu_t(F_{\varphi_t}(\omega)) = \frac{\varphi_t(\omega)\omega - (\nabla_{\tan}\varphi_t)(\omega)}{\sqrt{|(\nabla_{\tan}\varphi_t)(\omega)|^2 + |\varphi_t(\omega)|^2}}, \quad \text{for } \omega \in S^{n-1}. \tag{4.100}$$

With this in hand, (4.93) follows from (4.90), by letting $t \to 1^-$ and invoking (4.97). The argument for (4.94) is similar, with (3.24) and (4.91) involved this time.

Finally, to prove that Ω is a strongly Lipschitz domain, it suffices to observe $\partial\Omega = \partial\overline{\Omega}$ (itself a consequence of the fact that Ω is the image of $B(0, 1)$ under the bi-Lipschitz map F_φ) and that $X(x) := x/|x|$, $x \in \mathbb{R}^n \setminus \{0\}$, is a continuous vector field near $\partial\Omega$ satisfying

$$\begin{aligned} X(\varphi(\omega)\omega) \cdot \nu(\varphi(\omega)\omega) &= \frac{\varphi(\omega)}{\sqrt{|(\nabla_{\tan}\varphi)(\omega)|^2 + |\varphi(\omega)|^2}} \\ &\geq \frac{\inf_{S^{n-1}}\varphi}{\sqrt{\|\nabla_{\tan}\varphi\|_\infty^2 + \|\varphi\|_\infty^2}} > 0, \end{aligned} \tag{4.101}$$

for a.e. $\omega \in S^{n-1}$. Then the desired conclusion is provided by Theorem 2.7. □

In the lemma on p. 17 of [17], it is established that if $\Omega \subset \mathbb{R}^n$ is an open set which is starlike with respect to the ball $B(0, \rho)$, for some $\rho > 0$, then there exists a Lipschitz function $\varphi : S^{n-1} \to (0, \infty)$ such that $\Omega = \{r\omega : 0 \leq r < \varphi(\omega),\ \omega \in S^{n-1}\}$. As a consequence of this and Corollary 4.21 we then obtain:

Proposition 4.22. *If Ω is an open, proper subset of $\mathbb{R}^n$, which is starlike with respect to some ball, then Ω is a strongly Lipschitz domain.*

5. Appendix

5.1. Reifenberg flat, nontangentially accessible, and Semmes-Kenig-Toro domains

For the convenience of the reader, here we summarize the definitions of Reifenberg flat, NTA and SKT domains. To keep the technicalities to a minimum, we shall only consider here the case of bounded domains. Our presentation follows closely that of [15, 16], with also some influence from [11]. We momentarily digress for the purpose of explaining the terminology used in this subsection. What we here call SKT domains have been previously called in the literature *chord arc domains*. The latter notion originated in the two-dimensional setting, where the defining condition is that the length of a boundary arc between two points does not exceed a fixed multiple of the length of a chord between these points. In higher dimensions, where this phenomenon becomes somewhat more sophisticated, this notion originated in Semmes [23] and was further developed in [14]–[16]. In the higher-dimensional setting, this "chord arc" designation no longer adequately captures the essential features of such domains, and in [11] we have proposed to call them SKT (Semmes-Kenig-Toro) domains. Likewise, we have relabeled what was previously called in these articles *chord arc domains with vanishing constant*, calling them *regular SKT domains*.

Recall the definition of the Hausdorff distance from (4.53).

Definition 5.1. Let $\Sigma \subset \mathbb{R}^n$ be a compact set and let $\delta \in (0, \frac{1}{4\sqrt{2}})$. We say that Σ is δ-*Reifenberg flat* if there exists $R > 0$ such that for every $x \in \Sigma$ and every $r \in (0, R]$ there exists a $(n-1)$-dimensional plane $L(x, r)$ which contains x and such that

$$\frac{1}{r}\, D[\Sigma \cap B(x,r)\,,\ L(x,r) \cap B(x,r)] \leq \delta\,. \tag{5.1}$$

Definition 5.2. We say that a bounded open set $\Omega \subset \mathbb{R}^n$ has the *separation property* if there exists $R > 0$ such that for every $x \in \partial\Omega$ and $r \in (0, R]$ there exists an $(n-1)$-dimensional plane $\mathcal{L}(x, r)$ containing x and a choice of unit normal vector to $\mathcal{L}(x, r)$, $\vec{n}_{x,r}$, satisfying

$$\begin{aligned} &\{y + t\,\vec{n}_{x,r} \in B(x,r) : \ y \in \mathcal{L}(x,r),\ t < -\tfrac{r}{4}\} \subset \Omega\,, \\ &\{y + t\,\vec{n}_{x,r} \in B(x,r) : \ y \in \mathcal{L}(x,r),\ t > \tfrac{r}{4}\} \subset \mathbb{R}^n \setminus \Omega\,. \end{aligned} \tag{5.2}$$

Moreover, if Ω is unbounded, we also require that $\partial\Omega$ divides $\mathbb{R}^n$ into two distinct connected components and that $\mathbb{R}^n \setminus \Omega$ has a nonempty interior.

Definition 5.3. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $\delta \in (0, \delta_n)$. Call Ω a δ-*Reifenberg flat domain* if Ω has the separation property and $\partial\Omega$ is δ-Reifenberg flat.

For example, given $\delta > 0$, a strongly Lipschitz domain with a sufficiently small Lipschitz constant is a δ-Reifenberg flat domain.

Definition 5.4. A bounded open set $\Omega \subset \mathbb{R}^n$ is called an *NTA domain* provided Ω satisfies a two-sided corkscrew condition (defined in Section 2), along with a Harnack chain condition.

The Harnack chain condition is defined as follows (with reference to M and R as above). First, given $x_1, x_2 \in \Omega$, a Harnack chain from x_1 to x_2 in Ω is a sequence of balls $B_1, \dots, B_K \subset \Omega$ such that $x_1 \in B_1$, $x_2 \in B_K$ and $B_j \cap B_{j+1} \neq \emptyset$ for $1 \leq j \leq K-1$, and such that each B_j has a radius r_j satisfying $M^{-1} r_j < \operatorname{dist}(B_j, \partial\Omega) < M r_j$. The length of the chain is K. Then the Harnack chain condition on Ω is that if $\varepsilon > 0$ and $x_1, x_2 \in \Omega \cap B_{r/4}(z)$ for some $z \in \partial\Omega$,

$r \in (0, R)$, and if $\mathrm{dist}(x_j, \partial\Omega) > \varepsilon$ and $|x_1 - x_2| < 2^k\varepsilon$, then there exists a Harnack chain $B_1, \dots, B_K$ from x_1 to x_2, of length $K \le Mk$, having the further property that the diameter of each ball B_j is $\ge M^{-1}\min\big(\mathrm{dist}(x_1, \partial\Omega), \mathrm{dist}(x_2, \partial\Omega)\big)$.

Finally, call a bounded open set $\Omega \subset \mathbb{R}^n$ a *two-sided NTA domain* provided both Ω and $\mathbb{R}^n \setminus \overline{\Omega}$ are NTA domains.

The following result is proved in Section 3 of [14].

Proposition 5.5. *There exists a dimensional constant $\delta_n \in (0, \frac{1}{4\sqrt{2}})$ with the property that any bounded domain $\Omega \subset \mathbb{R}^n$ that has the separation property and whose boundary is a δ-Reifenberg flat set, $\delta \in (0, \delta_n)$, is an* NTA*-domain.*

Definition 5.6. Let $\delta \in (0, \delta_n)$, where δ_n is as in Theorem 5.5. A bounded set $\Omega \subset \mathbb{R}^n$ of finite perimeter is said to be a *δ-SKT domain* if Ω is a δ-Reifenberg flat domain, $\partial\Omega$ is Ahlfors regular and there exists $r > 0$ such that

$$\sup_{x\in\partial\Omega}\left(\sup_{\Delta\subset\Delta(x,r)}\left(\fint_{\Delta}|\nu - \nu_\Delta|^2\, d\sigma\right)^{1/2}\right) < \delta\,, \tag{5.3}$$

with the supremum taken over all surface balls Δ contained in $\Delta(x, r) := \partial\Omega \cap B(x, r)$. Here, as before, ν is the measure-theoretic outward unit normal to $\partial\Omega$ and $\nu_\Delta := \fint_\Delta \nu\, d\sigma$.

Definition 5.7. Call a bounded open set $\Omega \subset \mathbb{R}^n$ a *regular SKT domain* if Ω is a δ-SKT domain for some $\delta \in (0, \delta_n)$ and, in addition, $\nu \in \mathrm{VMO}(\partial\Omega, d\sigma)$. The last condition means that

$$\limsup_{r\to 0^+}\left(\sup_{x\in\partial\Omega}\left(\fint_{\Delta(x,r)}|\nu - \nu_{\Delta(x,r)}|^2\, d\sigma\right)^{1/2}\right) = 0\,. \tag{5.4}$$

We conclude by recalling a useful, natural characterization of the class of bounded regular SKT domains, recently established in [11].

Theorem 5.8. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Then the following are equivalent:*

(i) *Ω is a regular SKT domain;*
(ii) *Ω is a two-sided* NTA *domain, $\partial\Omega$ is Ahlfors regular, and $\nu \in$* VMO *$(\partial\Omega, d\sigma)$;*
(iii) *Ω is a domain with an Ahlfors regular boundary, satisfying a two-sided local John condition (cf. Definition 4.4), and for which $\nu \in$* VMO *$(\partial\Omega, d\sigma)$.*

Note that any domain with a compact, Ahlfors regular boundary has finite perimeter. In particular, it makes sense to talk about its outward unit normal ν and surface measure σ. In fact, there holds $\sigma = \mathcal{H}^n\lfloor\partial\Omega$. Let us also remark that if the condition $\nu \in \mathrm{VMO}\,(\partial\Omega, d\sigma)$ in Theorem 5.8 is strengthened to $\nu \in C^0(\partial\Omega, \mathbb{R}^n)$, then this result becomes a characterization of C^1 domains (compare with Theorem 2.19).

Recall that the conditions demanded in Definition 5.7 are that the domain is δ-Reifenberg flat, has an Ahlfors regular boundary, and its unit normal is in VMO. Compared with the original definition of a regular SKT domain, the last two characterizations given in Theorem 5.8 have the advantage of being more economical, in that they avoid stipulating *a priori* two sources of "regularity" for the boundary, namely that the unit normal has vanishing mean oscillations, plus a certain degree of Reifenberg flatness, e.g., in the setting of (iii) in Theorem 5.8, Reifenberg flatness has been replaced by a two-sided local John condition which is not a flatness/regularity condition per se. In the context of the current article, this is useful inasmuch as it is not clear that the image of a (bounded) δ-Reifenberg flat domain Ω under a C^1-diffeomorphism F is a δ'-Reifenberg flat domain, where $\delta' := C\delta$, with C depending on F and the geometry of Ω.

We conclude this subsection with a local version of SKT regularity.

Definition 5.9. Let Ω be an open, nonempty, proper subspace of $\mathbb{R}^n$, and assume that $x_0 \in \partial\Omega$. Call Ω a regular SKT domain near x_0 if the following conditions are fulfilled:

(i) There exist $r_0 > 0$ and $\lambda > 0$ such that $B(x_0, r_0) \cap \Omega$ is a set of finite perimeter (with surface measure σ and outward unit normal ν), such that $\mathcal{H}^{n-1}(\Delta_r) \approx r^{n-1}$ uniformly for any surface ball $\Delta_r \subset B(x_0, \lambda r_0) \cap \partial\Omega$;

(ii) there holds

$$\limsup_{r\to 0^+}\left(\sup_{\Delta_r\subseteq B(x_0,\lambda r_0)\cap\partial\Omega}\left(⨍_{\Delta_r}|\nu(y)-\nu_{\Delta_r}|^2\,d\sigma(y)\right)^{1/2}\right)=0\,; \tag{5.5}$$

(iii) all points $p \in B(x_0, \lambda r_0) \cap \partial\Omega$ satisfy a condition analogous to the two-sided local John condition from Definition 4.4.

This is natural in the sense that a nonempty, bounded open set $\Omega \subset \mathbb{R}^n$ is a regular SKT domain if and only if Ω is a regular SKT domain near each $x_0 \in \partial\Omega$.

5.2. Cross products and determinants

Let us define the vector product $v_1 \times v_2 \times \cdots \times v_{n-1}$ of a collection of $n-1$ vectors in $\mathbb{R}^n$, $v_1 = (v_{11}, \dots, v_{1n}), \dots, v_{n-1} = (v_{n-1\,1}, \dots, v_{n-1\,n})$, as

$$v_1\times v_2\times\cdots\times v_{n-1}=\det\begin{vmatrix} v_{11} & v_{12} & \dots & v_{1n}\\ v_{21} & v_{22} & \dots & v_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ v_{n-1\,1} & v_{n-1\,2} & \dots & v_{n-1\,n}\\ e_1 & e_2 & \dots & e_n\end{vmatrix}, \tag{5.6}$$

where the determinant is understood as being computed by formally expanding it with respect to the last row, the result being a vector in $\mathbb{R}^n$. As before, $e_1, \dots, e_n$ are the vectors of the standard orthonormal basis of $\mathbb{R}^n$. From this, it easily follows that if $v_1, \dots, v_n$ is a positively oriented orthonormal basis in $\mathbb{R}^n$ then:

$$v_1,\dots,v_n \text{ is a positively oriented orthonormal basis in } \mathbb{R}^n \Longrightarrow v_1\times\cdots\times v_{n-1}=v_n\,. \tag{5.7}$$

Let us also point out that, if vectors $v \in \mathbb{R}^n$ are identified with 1-forms $v^{\#} := \sum_{i=1}^n \langle v, e_i\rangle\, dx_i$, then:

$$v_1\times v_2\times\cdots\times v_{n-1}=*\big((v_1)^{\#}\wedge\cdots\wedge(v_{n-1})^{\#}\big)\,. \tag{5.8}$$

It is also useful to observe that if $v_1 = (v_{11}, \dots, v_{1n}), \dots, v_n = (v_{n1}, \dots, v_{nn})$ are n vectors in $\mathbb{R}^n$, then:

$$\langle v_1 \times v_2 \times \cdots \times v_{n-1}, v_n \rangle = \det \begin{vmatrix} v_{11} & v_{12} & \dots & v_{1n} \\ v_{21} & v_{22} & \dots & v_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ v_{n-1\,1} & v_{n-1\,2} & \dots & v_{n-1\,n} \\ v_{n1} & v_{n2} & \dots & v_{nn} \end{vmatrix}, \tag{5.9}$$

i.e., $\langle v_1 \times \cdots \times v_{n-1}, v_n \rangle$ is the (oriented) volume of the parallelopiped spanned by $v_1, \dots, v_n$ in $\mathbb{R}^n$.

Lemma 5.10. *For any $n \times n$ matrix A, and any collection of $n-1$ vectors $v_1, v_2, \dots, v_{n-1} \in \mathbb{R}^n$,*

$$A^\top (Av_1 \times \cdots \times Av_{n-1}) = (\det A)\, v_1 \times \cdots \times v_{n-1} . \tag{5.10}$$

Thus, in the particular case when A is an invertible $n \times n$ matrix,

$$Av_1 \times \cdots \times Av_{n-1} = (\det A) \left(A^{-1}\right)^\top \left(v_1 \times \cdots \times v_{n-1}\right), \tag{5.11}$$

for every collection of vectors $v_1, v_2, \dots, v_{n-1} \in \mathbb{R}^n$. Furthermore, for any $n \times n$ matrix A and any collection of vectors $v_1, v_2, \dots, v_{n-1} \in \mathbb{R}^n$,

$$Av_1 \times \cdots \times Av_{n-1} = (\operatorname{adj} A)(v_1 \times \cdots \times v_{n-1}) . \tag{5.12}$$

Proof. Fix an arbitrary collection of $n-1$ vectors $v_1, v_2, \dots, v_{n-1}$ in $\mathbb{R}^n$. Then, making use of the remark just preceding the statement of this lemma, for each vector $v_n \in \mathbb{R}^n$ we may write:

$$\begin{aligned} \left\langle A^\top (Av_1 \times \cdots \times Av_{n-1}),\, v_n \right\rangle &= \langle Av_1 \times \cdots \times Av_{n-1}\,,\, Av_n \rangle \\ &= \text{oriented volume}\,(\text{span}\,\{Av_1, \dots, Av_n\}) \\ &= (\det A)\, \text{oriented volume}\,(\text{span}\,\{v_1, \dots, v_n\}) \\ &= (\det A)\, \langle v_1 \times \cdots \times v_{n-1}\,,\, v_n \rangle . \end{aligned} \tag{5.13}$$

In the third equality above, we have interpreted the volume in question as an integral and have used a well-known "change of variable" formula (cf. e.g., Lemma 1 on p. 92 in [8]). Since $v_n \in \mathbb{R}^n$ was arbitrary, (5.10) follows from (5.13). Next, (5.11) is a direct consequence of (5.10), and (5.12) follows directly from (5.11) when A is invertible since, in this case, $\operatorname{adj} A = (\det A)^{-1}(A^{-1})^\top$. Finally, when A is arbitrary, we use the fact that there exists a sequence of invertible $n \times n$ matrices A_j, $j \in \mathbb{N}$, with the property that $A_j \to A$ as $j \to \infty$, and recover (5.12) by passing to the limit in $A_j v_1 \times \cdots \times A_j v_{n-1} = (\operatorname{adj} A_j)(v_1 \times \cdots \times v_{n-1})$. □

Lemma 5.11. *If A is an $n \times (n-1)$ matrix, then*

$$\sqrt{\det\left(A^\top A\right)} = \left|Ae_1' \times \cdots \times Ae_{n-1}'\right|, \tag{5.14}$$

where $e_1', \dots, e_{n-1}'$ are the vectors of the canonical orthonormal basis in $\mathbb{R}^{n-1}$.

Proof. Let $A = O \circ S$ be the polar decomposition of A, where S is a $(n-1) \times (n-1)$ symmetric (i.e., $S = S^\top$) matrix, and O is a $n \times (n-1)$ orthogonal (i.e., inner product preserving) matrix.

Set $V := O\mathbb{R}^{n-1}$, which is a $(n-1)$-plane in $\mathbb{R}^n$, and pick a unit vector $v \in \mathbb{R}^n$ with the property that $V \oplus \langle v \rangle = \mathbb{R}^n$ (orthogonal, direct sum).

For each $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, write $x' := (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and define $\widetilde{x'} := (x_1, \dots, x_{n-1}, 0)$. We can then extend O, originally viewed as a linear operator $O : \mathbb{R}^{n-1} \to \mathbb{R}^n$, to an operator $\mathcal{O} : \mathbb{R}^n \to \mathbb{R}^n$ (subsequently identified with a $n \times n$ matrix) defined by $\mathcal{O}x := Ox' + x_n v$, if $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. With this convention, $\mathcal{O}$ becomes a unitary transformation in $\mathbb{R}^n$. Using the result in Lemma 5.10, we then compute

$$\begin{aligned}
|Ae_1' \times \cdots \times Ae_{n-1}'| &= \big|O(Se_1') \times \cdots \times O(Se_{n-1}')\big| = \big|\mathcal{O}(\widetilde{Se_1'}) \times \cdots \times \mathcal{O}(\widetilde{Se_{n-1}'})\big| \\
&= \big|(\widetilde{Se_1'}) \times \cdots \times (\widetilde{Se_{n-1}'})\big| = |(\det S)\, e_n| = |\det S| \\
&= \sqrt{\det(S^2)} = \sqrt{\det\big((O \circ S)^\top (O \circ S)\big)} \\
&= \sqrt{\det(A^\top A)}\,, \qquad (5.15)
\end{aligned}$$

where the fourth equality above is a direct consequence of (5.6). □

5.3. Some topological lemmas

Lemma 5.12. *Let Ω_1, Ω_2 be two open subsets of $\mathbb{R}^n$, with the property that $\partial\Omega_j = \partial(\overline{\Omega_j})$, $j = 1, 2$. Then*

$$\partial(\Omega_1 \cap \Omega_2) = \partial\big(\overline{\Omega_1 \cap \Omega_2}\big)\,. \qquad (5.16)$$

Proof. Since $\partial(\overline{E}) = \partial E$ for any set $E \subset \mathbb{R}^n$, the right-to-left inclusion in (5.16) always holds, so there remains to shows that

$$\partial(\Omega_1 \cap \Omega_2) \subseteq \partial\big(\overline{\Omega_1 \cap \Omega_2}\big)\,. \qquad (5.17)$$

To this end, recall that

$$\partial(A \cap B) \subseteq \big(\overline{A} \cap \partial B\big) \cup \big(\partial A \cap \overline{B}\big), \qquad \forall\, A, B \subseteq \mathbb{R}^n\,, \qquad (5.18)$$

which further implies

$$\partial(A \cap B) = \Big(\partial(A \cap B) \cap \overline{A} \cap \partial B\Big) \cup \Big(\partial(A \cap B) \cap \overline{B} \cap \partial A\Big)\,. \qquad (5.19)$$

From this, and simple symmetry considerations, we see that (5.17) will follow as soon as we check the validity of the inclusion

$$\partial(\Omega_1 \cap \Omega_2) \cap \big(\overline{\Omega_1} \cap \partial\Omega_2\big) \subseteq \partial\big(\overline{\Omega_1 \cap \Omega_2}\big)\,. \qquad (5.20)$$

To this end, we reason by contradiction and assume that there exist a point x and a number $r > 0$ satisfying

$$\begin{aligned}
&x \in \partial(\Omega_1 \cap \Omega_2), \qquad x \in \partial\Omega_2, \qquad \text{and} \\
&\text{either } B(x, r) \cap (\overline{\Omega_1 \cap \Omega_2}) = \emptyset, \quad \text{or } B(x, r) \subseteq \overline{\Omega_1 \cap \Omega_2}\,.
\end{aligned} \qquad (5.21)$$

Note that if $B(x,r)\cap(\overline{\Omega_1\cap\Omega_2})=\emptyset$ then also $B(x,r)\cap(\Omega_1\cap\Omega_2)=\emptyset$, contradicting the fact that $x\in\partial(\Omega_1\cap\Omega_2)$. Thus, necessarily, $B(x,r)\subseteq\overline{\Omega_1\cap\Omega_2}$. However, this entails $B(x,r)\subset\overline{\Omega_2}$, contradicting the fact that $x\in\partial\Omega_2=\partial(\overline{\Omega_2})$. This shows that the conditions listed in (5.21) are contradictory and, hence, proves (5.20). □

Lemma 5.13. *For every subset Ω of $\mathbb{R}^n$, the implication* (2.61) *holds.*

Proof. From the fact that $\Omega\subset\mathbb{R}^n$ satisfies the exterior corkscrew condition it follows that $\partial\Omega\subseteq\overline{[(\Omega^c)^\circ]}$. This and the readily verified formula $(\Omega^c)^\circ=(\overline{\Omega})^c$, then yield $\partial\Omega\subseteq\overline{[(\bar{\Omega})^c]}$. Hence, $\partial\Omega\subseteq\overline{\Omega}\cap\overline{[(\bar{\Omega})^c]}=\partial\overline{\Omega}$, i.e., $\partial\Omega\subseteq\partial\overline{\Omega}$. Since the opposite inclusion is always true, (2.61) follows. □

Lemma 5.14.
(i) *If $A,B\subset\mathbb{R}^n$ are such that $\partial A\subseteq\overline{B}$ then*

$$\partial(A\setminus B)\subseteq\overline{A}\cap\partial B\,. \tag{5.22}$$

(ii) *If $A,B\subset\mathbb{R}^n$ are two sets with the property that $\partial A\subseteq B$ and $B\cap\partial B=\emptyset$ then*

$$\partial(A\setminus B)=A\cap\partial B\,. \tag{5.23}$$

Proof. Note that $\partial(A\setminus B)\subseteq\overline{A\setminus B}\subseteq\overline{A}$. Also, (5.18) gives $\partial(A\setminus B)\subseteq(\overline{A}\cap\partial(B^c))\cup(\partial A\cap\overline{B^c})$. Since $\partial(B^c)=\partial B$ and, using the hypotheses, $\partial A\cap\overline{B^c}\subseteq\overline{B}\cap\overline{B^c}=\partial B$, we conclude that $\partial(A\setminus B)\subseteq\partial B$. Thus, (5.22) follows.

As for (5.23), note that $\overline{A}\cap\partial B=(A\cap\partial B)\cup(\partial A\cap\partial B)=A\cap\partial B$, by hypotheses. Thus, we only have to show that $A\cap\partial B\subseteq\partial(A\setminus B)=\overline{A\setminus B}\cap\overline{A^c\cup B}$. However, $A\cap\partial B\subseteq\partial B\subseteq\overline{B}\subseteq\overline{A^c\cup B}$ and, if $B\cap\partial B=\emptyset$, then also $A\cap\partial B\subseteq A\setminus B\subseteq\overline{A\setminus B}$. This finishes the proof of the lemma. □

Lemma 5.15. *Let $\mathcal{O}_1,\mathcal{O}_2\subset\mathbb{R}^n$ be two open sets such that $\mathcal{O}_1$ is connected and $\partial\mathcal{O}_1=\partial\mathcal{O}_2$. Then*

$$\textit{either}\;\; \mathcal{O}_1\cap\mathcal{O}_2=\emptyset,\;\; \textit{or}\;\; \mathcal{O}_1\subseteq\mathcal{O}_2\,. \tag{5.24}$$

Proof. Observe that $\mathbb{R}^n=\mathcal{O}_2\cup(\overline{\mathcal{O}_2})^c\cup\partial\mathcal{O}_2$, mutually disjoint unions, and since $\mathcal{O}_1\cap\partial\mathcal{O}_2=\mathcal{O}_1\cap\partial\mathcal{O}_1=\emptyset$, we obtain

$$\mathcal{O}_1=\Big(\mathcal{O}_1\cap\mathcal{O}_2\Big)\cup\Big(\mathcal{O}_1\cap(\overline{\mathcal{O}_2})^c\Big)\,, \tag{5.25}$$

disjoint union. Thus, since $\mathcal{O}_1$ is connected, it follows from (5.25) that $\mathcal{O}_1\cap\mathcal{O}_2$ is either $\mathcal{O}_1$, or the empty set. With this in hand, (5.24) readily follows. □

Lemma 5.16. *Let $\Omega,\mathcal{O}\subset\mathbb{R}^n$ be two open sets with the property that Ω is bounded and $\partial\Omega\subseteq\mathcal{O}$. Then*

$$\overline{\Omega\setminus\overline{\mathcal{O}}}\subseteq\Omega\,. \tag{5.26}$$

Proof. Since $\overline{\Omega\setminus\overline{\mathcal{O}}}\subseteq\overline{\Omega}=\Omega\cup\partial\Omega$, disjoint union, it suffices to show that $\partial\Omega\cap\overline{\Omega\setminus\overline{\mathcal{O}}}=\emptyset$. To this end, let $x\in\partial\Omega$ be arbitrary. Then x belongs to the open set $\mathcal{O}$ and, hence, there exists $r>0$

such that $B(x,r) \subseteq \mathcal{O}$. Consequently, $\overline{\Omega \setminus \overline{\mathcal{O}}} \subseteq \overline{\Omega \setminus B(x,r)} \subseteq \overline{\mathbb{R}^n \setminus B(x,r)} = \mathbb{R}^n \setminus \overline{B(x,r)}$. This makes it clear that $x \notin \overline{\Omega \setminus \overline{\mathcal{O}}}$, proving the lemma. □

Lemma 5.17. *Let $\mathcal{O}_1, \mathcal{O}_2$ be two open subsets of $\mathbb{R}^n$ with the property that $\partial\mathcal{O}_1 = \partial\mathcal{O}_2 \neq \emptyset$. In addition, assume that*

$$\forall\, x \in \partial\mathcal{O}_1\ \exists\, r > 0 \text{ such that } B(x,r) \cap \mathcal{O}_1 = B(x,r) \cap \mathcal{O}_2\,. \tag{5.27}$$

Then $\mathcal{O}_1 = \mathcal{O}_2$.

Proof. Fix $x_o \in \partial\mathcal{O}_1 = \partial\mathcal{O}_2$ and let

$$r_o := \sup\{r > 0 :\ B(x_o,r) \cap \mathcal{O}_1 = B(x_o,r) \cap \mathcal{O}_2\}\,. \tag{5.28}$$

Then (5.27) ensures that r_o is well-defined and satisfies $0 < r_o \leq \infty$. Our goal is to show that $r_o = \infty$, from which the desired conclusion clearly follows. To this end, we reason by contradiction and assume that $r_o < \infty$. In order to facilitate the subsequent exposition, we make the following definition. Call a point $y \in \partial B(x_o,r_o)$ *good* if there exists $r > 0$ such that $B(y,r) \cap \mathcal{O}_1 = B(y,r) \cap \mathcal{O}_2$, and call $y \in \partial B(x_o,r_o)$ *bad* if it is not good.

At this stage, we make the claim that all bad points are on $\partial\mathcal{O}_1 = \partial\mathcal{O}_2$. With the aim of arriving at a contradiction, let us assume that $y_o \in B(x_o,r_o)$ is a bad point with the property that $y_o \notin \partial\mathcal{O}_1$ (and, hence, $y_o \notin \partial\mathcal{O}_2$). Note that if $y_o \notin \mathcal{O}_1 \cup \mathcal{O}_2$, then $y_o \notin (\mathcal{O}_1 \cup \mathcal{O}_2) \cup \partial\mathcal{O}_1 = \overline{\mathcal{O}_1} \cup \overline{\mathcal{O}_2}$. However, in this scenario, it is possible to select $r > 0$ small enough so that $B(y_o,r) \cap \mathcal{O}_1 = \emptyset = B(y_o,r) \cap \mathcal{O}_2$, hence contradicting the fact that y_o is bad. Thus, necessarily, $y_o \in \mathcal{O}_1 \cup \mathcal{O}_2$. To fix ideas, assume that $y_o \in \mathcal{O}_1$ (the other case being analogous). If it happens that $y_o \in \mathcal{O}_2$, then we can choose $r > 0$ small enough so that $B(y_o,r) \cap \mathcal{O}_1 = B(y_o,r) = B(y_o,r) \cap \mathcal{O}_2$, in contradiction with the fact that y_o is bad. Consequently, we must have $y_o \notin \mathcal{O}_2$. Since, by assumption, $y_o \notin \partial\mathcal{O}_1 = \partial\mathcal{O}_2$, it follows that $y_o \notin \mathcal{O}_2 \cup \partial\mathcal{O}_2 = \overline{\mathcal{O}_2}$. In particular, we can select $\rho > 0$ [which can be assumed to be less than r_o, defined in (5.28)] for which

$$B(y_o,\rho) \subseteq \mathcal{O}_1 \quad \text{and} \quad B(y_o,\rho) \cap \mathcal{O}_2 = \emptyset\,. \tag{5.29}$$

Based on this, for some fixed number $r \in (r_o - \rho, r_o)$, we may then write

$$\begin{aligned} \emptyset &\neq B(y_o,\rho) \cap B(x_o,r) = \Big(\mathcal{O}_1 \cap B(y_o,\rho)\Big) \cap B(x_o,r) = B(y_o,\rho) \cap \Big(B(x_o,r) \cap \mathcal{O}_1\Big) \\ &= B(y_o,\rho) \cap \Big(B(x_o,r) \cap \mathcal{O}_2\Big) = B(x_o,r) \cap \Big(B(y_o,\rho) \cap \mathcal{O}_2\Big) = \emptyset\,. \end{aligned} \tag{5.30}$$

This contradiction proves the claim made at the beginning of this paragraph, namely that all bad points are on $\partial\mathcal{O}_1 = \partial\mathcal{O}_2$.

However, from hypothesis and terminology, it is clear that there are no bad points on $\partial\mathcal{O}_1 = \partial\mathcal{O}_2$, to begin with. Hence, all points on $\partial B(x_o,r_o)$ are good. In turn, this implies that

$$\forall\, y \in \partial B(x_o,r_o)\ \ \exists\, \rho_y > 0 \text{ such that } B(y,\rho_y) \cap \mathcal{O}_1 = B(y,\rho_y) \cap \mathcal{O}_2\,. \tag{5.31}$$

A standard compactness argument then shows that it is possible to select a finite family of points and numbers, $y_j \in \partial B(x_o,r_o)$, $\rho_j > 0$, $1 \leq j \leq N$, with the property that

$$\partial B(x_o,r_o) \subseteq \bigcup_{j=1}^{N} B(y_j,\rho_j/2) \text{ and } B(y_j,\rho_j) \cap \mathcal{O}_1 = B(y_j,\rho_j) \cap \mathcal{O}_2,\ \forall\, j \in \{1,\dots,N\}. \tag{5.32}$$

To proceed from here, set $\rho_* := \min\{\rho_j : 1 \le j \le N\}$, so that $\rho_* > 0$. Then simple geometrical considerations show that:

$$B(x_o, r_o + \rho_*/2) \subseteq \left[\bigcup_{j=1}^{N} B(y_j, \rho_j)\right] \cup B(x_o, r_o - \rho_*/2)\,. \tag{5.33}$$

On the other hand, from the second part of (5.32) we have:

$$\Big(\bigcup_{j=1}^{N} B(y_j, \rho_j)\Big) \cap \mathcal{O}_1 = \Big(\bigcup_{j=1}^{N} B(y_j, \rho_j)\Big) \cap \mathcal{O}_2 \tag{5.34}$$

which, given the definition of r_o, further implies:

$$\left[\Big(\bigcup_{j=1}^{N} B(y_j, \rho_j)\Big) \cup B(x_o, r_o - \rho_*/2)\right] \cap \mathcal{O}_1 = \left[\Big(\bigcup_{j=1}^{N} B(y_j, \rho_j)\Big) \cup B(x_o, r_o - \rho_*/2)\right] \cap \mathcal{O}_2\,. \tag{5.35}$$

Thanks to (5.33), we may ultimately conclude from this that:

$$B(x_o, r_o + \rho_*/2) \cap \mathcal{O}_1 = B(x_o, r_o + \rho_*/2) \cap \mathcal{O}_2\,, \tag{5.36}$$

which, in turn, entails $r_o + \rho_*/2 \le r_o$. Given that $r_o < \infty$ and $\rho_* > 0$ this, however, is an impossibility. This contradiction then finishes the proof of the lemma. $\square$

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