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Two-scale methods for the normalized infinity Laplacian: rates of convergence

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We propose a monotone and consistent numerical scheme for the approximation of the Dirichlet problem for the normalized infinity Laplacian, which could be related to the family of the so-called two-scale methods. We show that this method is convergent and prove rates of convergence. These rates depend not only on the regularity of the solution, but also on whether or not the right-hand side vanishes. Some extensions to this approach, like obstacle problems and symmetric Finsler norms, are also considered.

Keywords: normalized infinity Laplacian; optimal Lipschitz extensions; tug of war games; monotonicity; obstacle problems; viscosity solutions; degenerate elliptic equations; Finsler norms.

1. Introduction

We begin our discussion by presenting two motivations for the type of problems we wish to study.

1.1 *Motivation: optimal Lipschitz extensions*

The fundamental problem in the calculus of variations (Dacorogna, 2008) is the minimization of a functional $\mathcal{I}$, i.e., we wish to find

$$\min \{\mathbb{I}[w] : w \in \mathcal{A}\},$$

where $\mathcal{A}$ is the *admissible set*, that is a subset of a function space over a domain $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ and encodes, for instance, boundary behavior of the function. A simple example of this scenario is

$$p \in (1,\infty), \qquad \mathbb{I}_p[w] = \frac{1}{|\Omega|}\int_\Omega |\mathrm{D}w(x)|^p \,\mathrm{d}x, \qquad \mathcal{A}_g^p = \left\{ w \in W^{1,p}(\Omega) : w_{|\partial\Omega} = g \right\}, \tag{1.1}$$

whose Euler Lagrange equation gives rise to the boundary value problem (BVP)

$$\Delta_p u = 0, \text{ in } \Omega, \qquad u_{|\partial\Omega} = g. \tag{1.2}$$

Here we introduced the so-called p-Laplacian

$$\Delta_p w(x) = \mathrm{D} \cdot (|\mathrm{D}w(x)|^{p-2}\mathrm{D}w(x)) = |\mathrm{D}w(x)|^{p-2}\Delta w(x) + (p-2)|\mathrm{D}w(x)|^{p-4}\mathrm{D}w(x) \otimes \mathrm{D}w(x) : \mathrm{D}^2 w(x),$$

where $\otimes$ denotes tensor product, and : is the Frobenius inner product, i.e.,

$$(\mathbf{a} \otimes \mathbf{b})_{i,j} = \mathbf{a}_i \mathbf{b}_j, \qquad \mathbf{A} : \mathbf{B} = \mathrm{tr}(\mathbf{A}^\intercal \mathbf{B}) = \sum_{i,j=1}^{d} \mathbf{A}_{i,j}\mathbf{B}_{i,j}.$$

Since $\mathbb{I}_p$ is strictly convex ($p > 1$), to minimize $\mathbb{I}_p$, it is sufficient to solve (1.2). Owing to the divergence structure of the p-Laplacian, (1.2) is a variational problem and it can be tackled with the notion of weak solutions. Thus, the partial differential equation (PDE) theory (Andreianov *et al.*, 2005; Roubíček, 2013; Dahlke *et al.*, 2016; Hartmann & Weimar, 2018) and approximation with either finite differences (Feng *et al.*, 2017; Calder, 2019; Slepčev & Thorpe, 2019; del Teso & Lindgren, 2022) or finite elements (Barrett & Liu, 1993; Ciarlet, 2002; Ebmeyer & Liu, 2005; Diening & Ružička, 2007; Diening & Kreuzer, 2008; Belenki *et al.*, 2012) for this problem is, to a very large extent, well developed.

Let us consider now, instead, the problem of minimizing the L^∞-norm of the gradient,

$$\mathbb{I}_\infty[w] = \|\mathrm{D}w\|_{L^\infty(\Omega;\mathbb{R}^d)} = \mathrm{ess.\,sup}\{|\mathrm{D}w(x)| : x \in \Omega\} \tag{1.3}$$

over the admissible set $\mathcal{A}_g^\infty$. It is not completely evident now how to carry variations for $\mathbb{I}_\infty$, so obtaining the Euler Lagrange equation is not immediate. Using that, as $p \uparrow \infty$, we have $\mathbb{I}_p[w]^{1/p} \to \mathbb{I}_\infty[w]$ one may argue that the Euler Lagrange equation for (1.3) is the limit, as $p \uparrow \infty$, of (1.2). Divide $\Delta_p w$ by $(p-2)|\mathrm{D}w(x)|^{p-4}$ to obtain that, as $p \uparrow \infty$,

$$\begin{aligned}\frac{1}{(p-2)|\mathrm{D}w|^{p-4}}\Delta_p w(x) &= \frac{|\mathrm{D}w(x)|^2}{p-2}\Delta w(x) + \mathrm{D}w(x) \otimes \mathrm{D}w(x) : \mathrm{D}^2 w(x) \\ &\to \Delta_\infty w(x) = \mathrm{D}w(x) \otimes \mathrm{D}w(x) : \mathrm{D}^2 w(x).\end{aligned}$$

The operator Δ_∞ is the ∞-*Laplacian*. Heuristically, to minimize (1.3), we solve the BVP

$$\Delta_\infty u = 0, \text{ in } \Omega, \qquad u_{|\partial\Omega} = g. \tag{1.4}$$

This derivation was originally discussed by G. Aronsson (Aronsson, 1965, 1966, 1969) (see also Aronsson, 1967, 1968), who considered the problem of *minimal Lipschitz extensions*: given $g \in C^{0,1}(\partial\Omega)$, find $u \in C^{0,1}(\bar{\Omega})$ such that $u_{|\partial\Omega} = g$ and its Lipschitz constant is as small as possible. He arrived at (1.4) via the aforementioned approximation procedure. He showed existence of solutions and provided explicit examples that show the optimal expected regularity for solutions of (1.4): $u \in C^{1,1/3}_{\mathrm{loc}}$. Uniqueness, stability, regularity and further properties of solutions were not known to him.

The reason for these shortcomings is that, since the ∞-Laplacian does not have divergence structure, the correct way of interpreting (1.4) is via the theory of viscosity solutions (Crandall *et al.*, 1992; Koike, 2004), which did not exist at the time. This theory made it possible to obtain uniqueness, stability and some properties of solutions (Katzourakis, 2015; Lindqvist, 2016). However, the regularity theory for (1.4) is limited: the best known results being interior $C^{1,\alpha}$ regularity (Juutinen *et al.*, 2001; Evans & Savin, 2008; Siljander *et al.*, 2017) and C^1 regularity in two dimensions (Savin, 2005; Wang & Yu, 2012; Hong, 2013, 2014; Lindgren, 2014; Crasta & Fragalà, 2016). For further historical comments, we refer to Crandall (2008, Section 8).

Although a full theory for problem (1.4) is lacking, we see that this problem appears as a prototype for the *calculus of variations in* L^∞ (Katzourakis, 2015). The minimal Lipschitz extension problem can also serve as a model for inpainting (Casas & Torres, 1996; Caselles *et al.*, 1998).

The ensuing applications justify the need for numerical schemes, which can be proved to converge, and some results exist in this direction. For instance, since solutions to (1.4) can be obtained as limits, when $p \uparrow \infty$, of solutions to (1.2), one idea is to discretize (1.2) with standard techniques and then pass to the limit. This has been explored in Pryer (2018), see also Lakkis & Pryer (2015); Katzourakis & Pryer (2016), where convergence of this method is established. The nature of this passage to the limit, and the limited available regularity of the solution to (1.4), makes obtaining rates of convergence very difficult. In fact, the only reference that tackles this issue appeared as a preprint while this work was under review (Bungert, 2023). This reference shows that the rate of convergence is at least $p^{-1/4}$. The direct approximation of (1.4) is established in Oberman (2005, 2013) via wide stencil finite difference methods. To ascertain convergence, the classical theory of Barles and Souganidis (Barles & Souganidis, 1991) is invoked; see however Jensen & Smears (2018). Rates of convergence for this method are provided in the PhD dissertation (Smart, 2010). It is shown that if the stencil size (coarse scale) $\varepsilon > 0$ behaves like $\varepsilon = \mathcal{O}(h^{1/5})$, where $h > 0$ is the mesh size (fine scale), then the error is of order $\mathcal{O}(h^{1/5})$ in general, and $\mathcal{O}(h^{1/3})$ if the gradient of the solution does not vanish.

The idea of obtaining a minimal Lipschitz extension into a domain from data defined in a subset of this domain finds application in the now popular learning problem, where it is known as *Lipschitz learning*. It is no surprise then that great efforts have been recently focused into the analysis of the Lipschitz learning problem, which in turn can yield numerical methods *and their analyses*, for problem (1.4). In Roith & Bungert (2023), a graph (finite difference) scheme for this problem was proposed, and under certain conditions between the graph connectivity (coarse scale $\varepsilon > 0$) and number of vertices in the graph (mesh size $h > 0$), it is shown that a discrete energy related to this scheme Γ-converges to (1.3) subject to $u \in \mathcal{A}_g^\infty$. Moreover, convergence of minimizers is also established. In addition, while this manuscript was under review reference Bungert *et al.* (2023) and preprint Bungert *et al.* (2022) appeared. Reference Bungert *et al.* (2023) provides rates of convergence for graph schemes for the Lipschitz learning problem. Under the assumption that the scales verify that $\frac{h^2}{\varepsilon^3} \to 0$, a rate of convergence of order $\mathcal{O}(h^{1/9})$ was obtained. Using probabilistic techniques, Bungert *et al.* (2022) improves on the relation between scales to the limiting case and shows that, with high probability, the same rate of convergence can be obtained.

1.2 *Motivation: tug of war games*

A variant of the infinity Laplacian also appears in some *tug of war games* (Peres *et al.*, 2009; Armstrong & Smart, 2012; Lewicka & Manfredi, 2014; Blanc & Rossi, 2019) as we now describe. Fix a step size $\varepsilon > 0$, and place a token at $x_0 \in \Omega$. Two players, at stage $k \geq 0$, are allowed to choose points $x_k^I, x_k^{II} \in \bar{B}_\varepsilon(x_k) \cap \bar{\Omega}$, respectively. A fair coin is tossed: heads means $x_{k+1} = x_k^I$, while tails gives $x_{k+1} = x_k^{II}$. If $x_{k+1} \in \Omega$ the game continues, but if $x_{k+1} \in \partial\Omega$, the game ends and Player II pays Player I the amount

$$P = g(x_{k+1}) + \frac{\varepsilon^2}{2} \sum_{j=0}^{k} f(x_j),$$

where the terminal payoff g, and running cost f are given. Player I attempts to maximize P, while Player II to minimize it. The expected payoff (value) u^ε of this game satisfies the relation

$$2u^\varepsilon(x) - \left(\sup_{y\in\overline{B}_\varepsilon(x)\cap\bar{\Omega}} u^\varepsilon(y) + \inf_{y\in\overline{B}_\varepsilon(x)\cap\bar{\Omega}} u^\varepsilon(y) \right) = \varepsilon^2 f(x), \tag{1.5}$$

which in the (formal) limit $\varepsilon \downarrow 0$ gives

$$-\Delta_\infty^\diamond u = f, \text{ in } \Omega, \qquad u_{|\partial\Omega} = g. \tag{1.6}$$

The operator $\Delta_\infty^\diamond$ is defined as

$$\Delta_\infty^\diamond w(x) = \frac{1}{|\mathrm{D}w(x)|^2}\Delta_\infty w(x) = \frac{\mathrm{D}w(x)}{|\mathrm{D}w(x)|} \otimes \frac{\mathrm{D}w(x)}{|\mathrm{D}w(x)|} : \mathrm{D}^2 w(x),$$

and, for obvious reasons, is known as the *normalized* ∞-Laplacian.

The authors of Armstrong & Smart (2012) used (1.5) as starting point to develop a semidiscrete scheme for (1.6). Namely, they defined the operator

$$\Delta_{\infty,\varepsilon}^\diamond w(x) = \frac{1}{\varepsilon^2}\left(\sup_{y\in\overline{B}_\varepsilon(x)\cap\bar{\Omega}} w(y) - 2w(x) + \inf_{y\in\overline{B}_\varepsilon(x)\cap\bar{\Omega}} w(y) \right), \tag{1.7}$$

and studied the problem

$$-\Delta_{\infty,\varepsilon}^\diamond u = f, \text{ in } \Omega, \qquad u_{|\partial\Omega} = g. \tag{1.8}$$

Existence and uniqueness of solutions, as well as convergence with rates, as $\varepsilon \to 0$, are established for $f > 0$.

1.3 *Goals and organization*

Despite the fact that the approximation theory for viscosity solutions is rather primitive (Feng *et al.*, 2013; Neilan *et al.*, 2017), the purpose of this manuscript is to develop convergent, with rates, numerical schemes for (1.4), (1.6) and related problems. Let us describe how we organize our discussion in order to achieve these goals.

We begin by establishing the notation and main assumptions we shall operate under in Section 2. In Section 2.1, we also review the existing theory around (1.6). The spatial discretization and our numerical scheme *per se* are presented in Section 2.2 and Section 2.3, respectively. The method belongs to the class of two-scale schemes, which have become rather prevalent in the numerical discretization of nondivergence form and fully nonlinear elliptic equations; see Neilan *et al.* (2017) for an extensive review. The analysis of the method begins in Section 3, where we establish monotonicity and comparison principles. The comparison principles need to distinguish whether f is of constant sign or $f \equiv 0$. These properties expediently give us, as shown in Section 3.2, existence, uniqueness and stability of discrete solutions. The interior consistency of the discrete operator is studied next, in Section 3.3. This, together with a barrier argument, and an approach *à la* Barles and Souganidis, Barles & Souganidis (1991) give us convergence; see Section 3.4.

Having established convergence the next step is to provide, under realistic regularity assumptions, rates of convergence. This is the main goal of Section 4. Once again, we need to distinguish between the cases when f is of constant sign or $f \equiv 0$. In the first scenario, we obtain a rate that is, at best, $\mathcal{O}(h^{1/3})$; whereas in the second $\mathcal{O}(h^{1/4})$. To the best of our knowledge, the error estimates for the case $f \neq 0$ are the first of their kind for this type of problems.

In Section 5, we then turn our attention to some variants of our main theme. First, in Section 5.1, an alternative way of dealing with the boundary conditions is presented. Throughout our main discussion, it is assumed that an extension of the boundary condition to a layer inside the domain is available to us. Our first extension explores the case where this is not possible. Next, a variant of the tug of war game presented in Section 1.2, which leads to an obstacle problem for the normalized infinity Laplacian, is discussed in Section 5.2. A convergent numerical scheme is presented and analyzed. The rates, in this case, are the same as for the Dirichlet problem. These are the first of their kind. The last variation we consider is detailed in Section 5.3, where we discuss the so-called Finsler infinity Laplacian. This is an operator that arises from a variant of our tug of war game, and can be thought of as an anisotropic version of our original operator. For its Dirichlet problem, we sketch the construction of a convergent scheme.

Section 6 details some practical aspects. We explain why the usually adopted approaches for the solution of the ensuing discrete problems may not work, and discuss an alternative. A convergent fixed point iteration is presented and analyzed. Finally, Section 7 presents some numerical experiments that illustrate our theory.

2. Notation and preliminaries

Throughout our work, $\Omega \subset \mathbb{R}^d$, with $d \geq 1$, will be a bounded domain whose boundary is at least continuous; see Grisvard (1985, Definition 1.2.1.2). For a point $z \in \mathbb{R}^d$ and $r > 0$ by $B_r(z)$ and $\overline{B}_r(z)$, we denote, respectively, the open and closed balls centered at z and of radius r. For a vector $\mathbf{v} \in \mathbb{R}^d$, by $|\mathbf{v}|$ we denote its Euclidean norm. If $\mathbf{M} \in \mathbb{R}^{d\times d}$, then $|\mathbf{M}|$ is the induced operator norm.

We will follow standard notation and terminology regarding function spaces. In particular, if $w : \Omega \to \mathbb{R}$ is sufficiently smooth and $x \in \Omega$, we denote by $\mathrm{D}w(x) \in \mathbb{R}^d$ and $\mathrm{D}^2w(x) \in \mathbb{R}^{d\times d}$ the gradient and Hessian, respectively, of w at x.

The relation $A \lesssim B$ shall mean that $A \leq cB$ for a constant c, whose value may change at each occurrence, but does not depend on A, B nor the discretization parameters. By $A \gtrsim B$, we mean $B \lesssim A$, and $A \approx B$ means $A \lesssim B \lesssim A$. The Landau symbol is $\mathcal{O}$.

We assume that our data satisfies:

RHS.1. The right-hand side $f \in C(\Omega) \cap L^\infty(\Omega)$.

RHS.2. Either:

- **RHS**.2a. $\sup\{f(x) : x \in \Omega\} < 0$ or $\inf\{f(x) : x \in \Omega\} > 0$.
- **RHS**.2b. $f \equiv 0$.

In addition,

BC.1. The boundary datum $g \in C(\partial\Omega)$.

BC.2. There is a neighborhood $\mathcal{B}$ of $\partial\Omega$ and $\widetilde{g} \in C(\overline{\mathcal{B}})$ such that $\widetilde{g}_{|\partial\Omega} = g$. Moreover, if g is such that, for some $\alpha \in [0, 1]$, we have $g \in C^{0,\alpha}(\partial\Omega)$, then $\widetilde{g} \in C^{0,\alpha}(\overline{\mathcal{B}})$ with

$$|\widetilde{g}|_{C^{0,\alpha}(\overline{\mathcal{B}})} \leq C_b |g|_{C^{0,\alpha}(\partial\Omega)}.$$

Further assumptions will be introduced as needed.

REMARK 2.1 (extension). We comment that the extension to $\mathcal{B}$ of the boundary data can be realized, for instance, via a closest point projection; see Bungert *et al*. (2023, Assumption 2.4).

2.1 *The normalized infinity Laplacian*

We note that for $x \in \Omega$, even if $w \in C^\infty(\Omega)$ the expression $\Delta^\diamond_\infty w(x)$ does not make sense whenever the gradient of w vanishes at x. For this reason, we define the upper and lower semicontinuous envelopes, respectively, of the mapping $x \mapsto \Delta^\diamond_\infty w(x)$ to be

$$\Delta^+_\infty w(x) = \begin{cases} \Delta^\diamond_\infty w(x), & \mathrm{D}w(x) \neq \mathbf{0}, \\ \lambda_{\max}(\mathrm{D}^2 w(x)), & \mathrm{D}w(x) = \mathbf{0}, \end{cases} \tag{2.1}$$

and

$$\Delta^-_\infty w(x) = \begin{cases} \Delta^\diamond_\infty w(x), & \mathrm{D}w(x) \neq \mathbf{0}, \\ \lambda_{\min}(\mathrm{D}^2 w(x)), & \mathrm{D}w(x) = \mathbf{0}. \end{cases} \tag{2.2}$$

Here, for a symmetric matrix $\mathbf{M} \in \mathbb{R}^{d\times d}$, by $\lambda_{\max}(\mathbf{M})$ and $\lambda_{\min}(\mathbf{M})$, we denote the largest and smallest eigenvalues of $\mathbf{M}$. We recall that these can be characterized as

$$\lambda_{\max}(\mathbf{M}) = \max_{|\mathbf{v}|=1} \mathbf{v}^\mathsf{T}\mathbf{M}\,\mathbf{v}, \qquad \lambda_{\min}(\mathbf{M}) = \min_{|\mathbf{v}|=1} \mathbf{v}^\mathsf{T}\mathbf{M}\,\mathbf{v}.$$

The operators Δ_∞ and $\Delta^\diamond_\infty$ are degenerate elliptic in the sense of Crandall *et al.* (1992). The theory of viscosity solutions yields existence and uniqueness of solutions to (1.4); see Katzourakis (2015). Armed with the envelopes (2.1) and (2.2), existence of solutions can also be asserted for (1.6); see Armstrong & Smart (2012); Lindqvist (2016). Uniqueness can only be guaranteed when Assumption **RHS**.2 holds (Crandall *et al.*, 2007). Finally, if $f \equiv 0$, we have that the viscosity solutions to (1.4) and (1.6) coincide; see Armstrong & Smart (2012, Remark 2.2).

2.2 *Space discretization*

We now begin to describe our method for the approximation to the solution of (1.6). We begin by introducing some notation. For $r > 0$, we set

$$\Omega^{(r)} = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r\}.$$

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of unstructured shape regular meshes that, for each $h > 0$, consist of closed simplices T. This family is quasiuniform in the usual, finite element, sense (Ciarlet, 2002). For each

$h > 0$, we set

$$h_T = \mathrm{diam}(T), \quad h = \max_{T \in \mathcal{T}_h} h_T, \quad \Omega_h = \mathrm{int} \bigcup \{T : T \in \mathcal{T}_h\},$$

where by $\mathrm{int}\, A$ we denote the interior of the set $A \subset \mathbb{R}^d$. The set of nodes of $\mathcal{T}_h$ is denoted by $\mathcal{N}_h$.

We need to quantify how Ω_h approximates Ω. To do so we assume that, for all $h > 0$,

$$\Omega^{(h)} \subset \Omega_h \subset \Omega.$$

For $r > 0$, the set $\mathcal{N}_{h,r}^I = \mathcal{N}_h \cap \Omega^{(2r)}$ is the set of interior nodes, and $\mathcal{N}_{h,r}^b = \mathcal{N}_h \setminus \mathcal{N}_{h,r}^I$ is the set of boundary nodes. Note that the condition above guarantees that, if h is sufficiently small and $r \geq h$, then neither of these sets is empty.

Let $\mathbb{V}_h$ be the space of continuous piecewise linear functions subject to $\mathcal{T}_h$, and $\mathcal{I}_h : C(\overline{\Omega}) \to \mathbb{V}_h$ be the Lagrange interpolant, i.e.,

$$\mathcal{I}_h w(x) = \sum_{\mathrm{z}_h \in \mathcal{N}_h} w(\mathrm{z}_h) \widehat{\varphi}_{\mathrm{z}_h}(x), \qquad \forall w \in C(\overline{\Omega}).$$

Here $\{\widehat{\varphi}_{\mathrm{z}_h}\}_{\mathrm{z}_h \in \mathcal{N}_h} \subset \mathbb{V}_h$ is the canonical (hat) basis of $\mathbb{V}_h$, i.e., $\widehat{\varphi}_{\mathrm{z}_h}(\mathrm{z}_h') = \delta_{\mathrm{z}_h, \mathrm{z}_h'}$ for all $\mathrm{z}_h, \mathrm{z}_h' \in \mathcal{N}_h$. We recall that, as a consequence, these functions are non-negative and, moreover, form a partition of unity on $\overline{\Omega}$.

2.3 *The numerical method*

Here and in what follows, we will call the meshsize h the fine scale. The coarse scale shall be given by $\varepsilon \in [h, \mathrm{diam}(\Omega)]$. Inspired by (1.7) we could consider a fully discrete operator of the form

$$(w, \mathrm{z}_h) \mapsto -\Delta_{\infty,\varepsilon}^{\diamond} \mathcal{I}_h w(\mathrm{z}_h), \qquad \forall w \in C(\overline{\Omega}), \quad \forall \mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^I.$$

We notice, however, that even for a function $w_h \in \mathbb{V}_h$ the computation of $\max_{x \in \overline{B}_\varepsilon(\mathrm{z}_h)} w_h(x)$ and $\min_{x \in \overline{B}_\varepsilon(\mathrm{z}_h)} w_h(x)$ can be complicated in practice. To simplify this calculation, we observe that, for ε sufficiently small,

$$\max_{x \in \overline{B}_\varepsilon(\mathrm{z}_h)} w(x) \approx \max \left\{ w(\mathrm{z}_h), \max_{x \in \partial B_\varepsilon(\mathrm{z}_h)} w(x) \right\}, \quad \forall w \in C^2(\overline{B}_\varepsilon(\mathrm{z}_h)),$$

and similar for the minimum. Indeed, if the maximum occurs at $x \in \mathrm{int}\, B_\varepsilon(\mathrm{z}_h)$, then we must have $\mathrm{D}w(x) = \mathbf{0}$, so that, as $\varepsilon \downarrow 0$,

$$w(\mathrm{z}_h) = w(x) + \mathrm{D}w(x)^{\intercal}(\mathrm{z}_h - x) + \mathcal{O}(\varepsilon^2) = w(x) + \mathcal{O}(\varepsilon^2).$$

As a final simplification, we discretize $\partial \overline{B}_\varepsilon(\mathrm{z}_h)$. Let $\mathbb{S}$ be the unit sphere in $\mathbb{R}^d$. For $\theta \leq 1$, we introduce a discretization $\mathbb{S}_\theta \subset \mathbb{S}$: for any $\mathbf{v} \in \mathbb{S}$, there is $\mathbf{v}_\theta \in \mathbb{S}_\theta$ such that

$$|\mathbf{v} - \mathbf{v}_\theta| \leq \theta.$$

We further assume that $\mathbb{S}_\theta$ is symmetric, i.e., if $\mathbf{v}_\theta \in \mathbb{S}_\theta$, then $-\mathbf{v}_\theta \in \mathbb{S}_\theta$ as well. We will then, instead of $\partial B_\varepsilon(\mathrm{z}_h)$, only consider points of the form $\mathrm{z}_h + \varepsilon \mathbf{v}_\theta$ for $\mathbf{v}_\theta \in \mathbb{S}_\theta$.

With the previous simplifications at hand set $\mathfrak{h} = (h, \varepsilon, \theta)$. For $\mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^I$, we define

$$\mathcal{N}_\mathfrak{h}(\mathrm{z}_h) = \{\mathrm{z}_h\} \cup \{\mathrm{z}_h + \varepsilon \mathbf{v}_\theta : \mathbf{v}_\theta \in \mathbb{S}_\theta\},$$

and, for $w \in C(\overline{\Omega})$,

$$S_\mathfrak{h}^+ w(\mathrm{z}_h) = \frac{1}{\varepsilon}\left(\max_{x \in \mathcal{N}_\mathfrak{h}(\mathrm{z}_h)} w(x) - w(\mathrm{z}_h)\right), \quad S_\mathfrak{h}^- w(\mathrm{z}_h) = \frac{1}{\varepsilon}\left(w(\mathrm{z}_h) - \min_{x \in \mathcal{N}_\mathfrak{h}(\mathrm{z}_h)} w(x)\right).$$

Finally, our fully discrete operator is defined, for $w \in C(\overline{\Omega})$, by

$$-\Delta_{\infty,\mathfrak{h}}^\diamond w(\mathrm{z}_h) = -\frac{1}{\varepsilon(\mathrm{z}_h)^2}\left(S_\mathfrak{h}^+ \mathcal{I}_h w(\mathrm{z}_h) - S_\mathfrak{h}^- \mathcal{I}_h w(\mathrm{z}_h)\right), \qquad \forall \mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^I. \tag{2.3}$$

For future reference, we define, for $\mathrm{z}_h \in \mathcal{N}_h$,

$$\widetilde{\mathcal{N}}_\mathfrak{h}(\mathrm{z}_h) = \left\{\mathrm{z}_h\right\} \cup \left\{\mathrm{z}_h' \in \mathcal{N}_h : \exists \mathbf{v}_\theta \in \mathbb{S}_\theta, \widehat{\varphi}_{\mathrm{z}_h'}(\mathrm{z}_h + \varepsilon \mathbf{v}_\theta) > 0\right\} \subset \mathcal{N}_h.$$

This set is such that, for $w_h \in \mathbb{V}_h$, $S_\mathfrak{h}^\pm w_h(\mathrm{z}_h)$ is uniquely defined by the restriction of w_h to $\widetilde{\mathcal{N}}_\mathfrak{h}(\mathrm{z}_h)$.

Under Assumption **BC**.2, we will now define our numerical scheme. Our numerical method reads: find $u_\mathfrak{h} \in \mathbb{V}_h$ such that

$$-\Delta_{\infty,\mathfrak{h}}^\diamond u_\mathfrak{h}(\mathrm{z}_h) = f(\mathrm{z}_h), \quad \forall \mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^I, \qquad u_\mathfrak{h}(\mathrm{z}_h) = \widetilde{g}(\mathrm{z}_h), \quad \forall \mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^b. \tag{2.4}$$

Of importance in our analysis will be the concepts of discrete sub- and supersolution.

Definition 2.2 (subsolution). The function $w_h \in \mathbb{V}_h$ is a discrete subsolution (supersolution) of (2.4) if

$$-\Delta_{\infty,\mathfrak{h}}^\diamond w_h(\mathrm{z}_h) \le (\ge) f(\mathrm{z}_h), \quad \forall \mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^I, \qquad w_h(\mathrm{z}_h) \le (\ge) \widetilde{g}(\mathrm{z}_h), \quad \mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^b.$$

Therefore, a discrete solution of (2.4) is both a discrete sub- and supersolution.

3. The discrete problem

We now begin to study the properties of our scheme. The definition of (2.3) immediately implies the monotonicity of the operator $-\Delta_{\infty,\mathfrak{h}}^\diamond$ in the following sense.

Lemma 3.1 (monotonicity). Let $\mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^I$ be an interior node and $w_h, v_h \in \mathbb{V}_h$. If

$$(w_h - v_h)(\mathrm{z}_h) \ge (w_h - v_h)(\mathrm{z}_h'), \quad \forall \mathrm{z}_h' \in \widetilde{\mathcal{N}}_\mathfrak{h}(\mathrm{z}_h), \tag{3.1}$$

then

$$S_{\mathfrak{h}}^{+} w_h(z_h) \leq S_{\mathfrak{h}}^{+} v_h(z_h), \quad S_{\mathfrak{h}}^{-} w_h(z_h) \geq S_{\mathfrak{h}}^{-} v_h(z_h),$$

and

$$-\Delta_{\infty,\mathfrak{h}}^{\diamond} w_h(z_h) \geq -\Delta_{\infty,\mathfrak{h}}^{\diamond} v_h(z_h). \tag{3.2}$$

Proof. It simply follows by definition of the operators.

We observe, first of all, that (3.1) implies the same condition over $x \in \mathcal{N}_{\mathfrak{h}}(z_h)$. Indeed, let $\psi_h = w_h - v_h \in \mathbb{V}_h$ and $x \in \mathcal{N}_{\mathfrak{h}}(z_h)$, so that

$$\psi_h(x) = \sum_{z_h' \in \widetilde{\mathcal{N}}_{\mathfrak{h}}(z_h)} \psi_h(z_h') \widehat{\varphi}_{z_h'}(x) \leq \sum_{z_h' \in \widetilde{\mathcal{N}}_{\mathfrak{h}}(z_h)} \psi_h(z_h) \widehat{\varphi}_{z_h'}(x) = \psi_h(z_h) \sum_{z_h' \in \widetilde{\mathcal{N}}_{\mathfrak{h}}(z_h)} \widehat{\varphi}_{z_h'}(x) = \psi_h(z_h),$$

where we used the partition of unity property of the hat basis functions.

Let now $x_0 \in \mathcal{N}_{\mathfrak{h}}(z_h)$ satisfy

$$w_h(x_0) = \max_{x \in \mathcal{N}_{\mathfrak{h}}(z_h)} w_h(x),$$

so that

$$\varepsilon S_{\mathfrak{h}}^{+} v_h(z_h) = \max_{x \in \mathcal{N}_{\mathfrak{h}}(z_h)} v_h(x) - v_h(z_h) \geq v_h(x_0) - v_h(z_h) \geq w_h(x_0) - w_h(z_h) = \varepsilon S_{\mathfrak{h}}^{+} w_h(z_h).$$

Similarly, $S_{\mathfrak{h}}^{-} v_h(z_h) \leq S_{\mathfrak{h}}^{-} w_h(z_h)$. These two relations immediately imply (3.2). □

3.1 *Comparison*

The monotonicity presented in Lemma 3.1 leads to the following discrete comparison principle.

THEOREM 3.2 (comparison I). Assume that $w_h, v_h \in \mathbb{V}_h$ satisfy

$$-\Delta_{\infty,\mathfrak{h}}^{\diamond} w_h(z_h) \leq -\Delta_{\infty,\mathfrak{h}}^{\diamond} v_h(z_h) \quad \forall z_h \in \mathcal{N}_{h,\varepsilon}^{I}, \tag{3.3}$$

and, moreover, one of the following holds:

(1) For every $z_h \in \mathcal{N}_{h,\varepsilon}^{I}$, (3.3) holds with strict inequality or $-\Delta_{\infty,\mathfrak{h}}^{\diamond} w_h(z_h) < 0$.

(2) For every $z_h \in \mathcal{N}_{h,\varepsilon}^{I}$, (3.3) holds with strict inequality or $-\Delta_{\infty,\mathfrak{h}}^{\diamond} v_h(z_h) > 0$.

Then,

$$\max_{z_h \in \mathcal{N}_h} \left[w_h(z_h) - v_h(z_h) \right] = \max_{z_h \in \mathcal{N}_{h,\varepsilon}^{b}} \left[w_h(z_h) - v_h(z_h) \right].$$

Proof. We first prove that the theorem is true if (3.3) holds with strict inequality for every $z_h \in \mathcal{N}_{h,\varepsilon}^I$. We assume by contradiction that

$$m = \max_{z_h \in \mathcal{N}_h} \left[w_h(z_h) - v_h(z_h) \right] > \max_{z_h \in \mathcal{N}_{h,\varepsilon}^b} \left[w_h(z_h) - v_h(z_h) \right],$$

and consider the set of points

$$E = \{ z_h \in \mathcal{N}_h : (w_h - v_h)(z_h) = m \} \subset \mathcal{N}_{h,\varepsilon}^I.$$

Clearly, the set E is nonempty because of the definition of m. For any $z_h \in E$, we have

$$w_h(z_h) - v_h(z_h) \geq w_h(z_h') - v_h(z_h') \quad \forall z_h' \in \mathcal{N}_h. \tag{3.4}$$

By Lemma 3.1, this implies that

$$-\Delta_{\infty,\mathfrak{h}}^{\diamond} w_h(z_h) \geq -\Delta_{\infty,\mathfrak{h}}^{\diamond} v_h(z_h),$$

which is a contradiction to our assumption.

Next, consider the case that (3.3) does not hold with strict inequality. By symmetry, we only need to prove the result under the assumption that $-\Delta_{\infty,\mathfrak{h}}^{\diamond} w_h(z_h) < 0$ for all $z_h \in \mathcal{N}_{h,\varepsilon}^I$. Then, for any constant $\beta > 0$ that is small enough, we have

$$-\Delta_{\infty,\mathfrak{h}}^{\diamond} \left((1+\beta) w_h \right)(z_h) < -\Delta_{\infty,\mathfrak{h}}^{\diamond} v_h(z_h),$$

for any $z_h \in \mathcal{N}_{h,\varepsilon}^I$. Applying the comparison principle we just proved to the functions $(1+\beta)w_h$ and v_h, we obtain that

$$\max_{z_h \in \mathcal{N}_h} \left[(1+\beta) w_h(z_h) - v_h(z_h) \right] = \max_{z_h \in \mathcal{N}_{h,\varepsilon}^b} \left[(1+\beta) w_h(z_h) - v_h(z_h) \right].$$

Letting $\beta \downarrow 0$ finishes our proof. □

The comparison of Theorem 3.2 will be enough to study the case when the right-hand side f satisfies Assumption **RHS**.2a. However, when Assumption **RHS**.2b holds, we need a refinement. If this is the case, we require the set of vertices, $\mathcal{N}_h$, to satisfy an additional condition, regarding its connectivity.

M.1 For any $\emptyset \neq S \subset \mathcal{N}_{h,\varepsilon}^I$, there are $z_h \in S$ and $z_h' \in \mathcal{N}_h \setminus S$ so that

$$z_h' \in \widetilde{\mathcal{N}}_{\mathfrak{h}}(z_h).$$

We comment that Assumption **M**.1 is essentially a connectivity assumption, which is guaranteed when h, ε are sufficiently small for a fixed connected domain Ω.

With this condition at hand, we can now prove another comparison principle. The following result is a discrete counterpart of Armstrong & Smart (2012, Corollary 2.8).

THEOREM 3.3 (comparison II). Let $w_h, v_h \in \mathbb{V}_h$ satisfy

$$-\Delta^{\diamond}_{\infty,\mathfrak{h}} w_h(z_h) \le -\Delta^{\diamond}_{\infty,\mathfrak{h}} v_h(z_h), \quad \forall z_h \in \mathcal{N}^I_{h,\varepsilon}. \tag{3.5}$$

Further, assume that $-\Delta^{\diamond}_{\infty,\mathfrak{h}} w_h(z_h) \le 0$ or $-\Delta^{\diamond}_{\infty,\mathfrak{h}} v_h(z_h) \ge 0$ for every $z_h \in \mathcal{N}^I_{h,\varepsilon}$. If Assumption **M**.1 holds, then

$$\max_{z_h \in \mathcal{N}_h} \big[w_h(z_h) - v_h(z_h)\big] = \max_{z_h \in \mathcal{N}^b_{h,\varepsilon}} \big[w_h(z_h) - v_h(z_h)\big].$$

Proof. The proof proceeds in the same way as Armstrong & Smart (2012, Proof of Theorem 2.7). By symmetry, we only need to prove the theorem under the assumption $-\Delta^{\diamond}_{\infty,\mathfrak{h}} w_h(z_h) \le 0$. We assume by contradiction that

$$m = \max_{z_h \in \mathcal{N}_h} \big[w_h(z_h) - v_h(z_h)\big] > \max_{z_h \in \mathcal{N}^b_{h,\varepsilon}} \big[w_h(z_h) - v_h(z_h)\big].$$

Consider the set of points

$$E = \{z_h \in \mathcal{N}_h : (w_h - v_h)(z_h) = m\} \subset \mathcal{N}^I_{h,\varepsilon}.$$

Clearly, the set E is nonempty because of the definition of m. For any $z_h \in E$, we have

$$w_h(z_h) - v_h(z_h) \ge w_h(z'_h) - v_h(z'_h), \quad \forall z'_h \in \mathcal{N}_h. \tag{3.6}$$

By Lemma 3.1, this implies that

$$S^+_{\mathfrak{h}} w_h(z_h) \le S^+_{\mathfrak{h}} v_h(z_h), \qquad S^-_{\mathfrak{h}} w_h(z_h) \ge S^-_{\mathfrak{h}} v_h(z_h).$$

However, from (3.5) and the definition of $-\Delta^{\diamond}_{\infty,\mathfrak{h}}$, we have

$$S^-_{\mathfrak{h}} w_h(z_h) - S^+_{\mathfrak{h}} w_h(z_h) \le S^-_{\mathfrak{h}} v_h(z_h) - S^+_{\mathfrak{h}} v_h(z_h).$$

Therefore, we must have

$$S^+_{\mathfrak{h}} w_h(z_h) = S^+_{\mathfrak{h}} v_h(z_h), \qquad S^-_{\mathfrak{h}} w_h(z_h) = S^-_{\mathfrak{h}} v_h(z_h), \qquad \forall z_h \in E. \tag{3.7}$$

Furthermore, let $l = \max_{z_h \in E} w_h(z_h)$ and consider the set

$$F = \{z_h \in E : w_h(z_h) = l\}.$$

We claim that $S^+_{\mathfrak{h}} w_h(\mathrm{z}_h) = 0$ for any $\mathrm{z}_h \in F$. We argue by contradiction and suppose that there exists such a point $\mathrm{z}_h \in F$ with $S^+_{\mathfrak{h}} w_h(\mathrm{z}_h) > 0$. Then, either there exists $\mathrm{z}_h' \in \widetilde{\mathcal{N}}_{\mathfrak{h}}(\mathrm{z}_h)$ with

$$w_h(\mathrm{z}_h') - w_h(\mathrm{z}_h) = \varepsilon S^+_{\mathfrak{h}} w_h(\mathrm{z}_h) > 0, \tag{3.8}$$

or there is $\mathbf{v}_\theta \in \mathbb{S}_\theta$ such that

$$w_h(\mathrm{z}_h + \varepsilon \mathbf{v}_\theta) - w_h(\mathrm{z}_h) = \sum_{\mathrm{z}_h' \in \widetilde{\mathcal{N}}_{\mathfrak{h}}(\mathrm{z}_h) \cap T} \widehat{\varphi}_{\mathrm{z}_h'}(\mathrm{z}_h + \varepsilon \mathbf{v}_\theta) \left(w_h(\mathrm{z}_h') - w_h(\mathrm{z}_h) \right) = \varepsilon S^+_{\mathfrak{h}} w_h(\mathrm{z}_h) > 0, \tag{3.9}$$

where T is a simplex containing $\mathrm{z}_h + \varepsilon \mathbf{v}_\theta$. If (3.8) holds, then $w_h(\mathrm{z}_h') > w_h(\mathrm{z}_h) = l$ and thus $\mathrm{z}_h' \notin E$. This implies that

$$w_h(\mathrm{z}_h) - v_h(\mathrm{z}_h) > w_h(\mathrm{z}_h') - v_h(\mathrm{z}_h'),$$

which leads to

$$\varepsilon S^+_{\mathfrak{h}} w_h(\mathrm{z}_h) = w_h(\mathrm{z}_h') - w_h(\mathrm{z}_h) < v_h(\mathrm{z}_h') - v_h(\mathrm{z}_h) \le \varepsilon S^+_{\mathfrak{h}} v_h(\mathrm{z}_h), \tag{3.10}$$

but this contradicts (3.7). If (3.9) holds, then choose $\mathrm{z}_h' \in T \cap \widetilde{\mathcal{N}}_{\mathfrak{h}}(\mathrm{z}_h)$ with

$$\widehat{\varphi}_{\mathrm{z}_h'}(\mathrm{z}_h + \varepsilon \mathbf{v}_\theta) \left(w_h(\mathrm{z}_h') - w_h(\mathrm{z}_h) \right) > 0,$$

i.e.,

$$\widehat{\varphi}_{\mathrm{z}_h'}(\mathrm{z}_h + \varepsilon \mathbf{v}_\theta) > 0, \quad w_h(\mathrm{z}_h') > w_h(\mathrm{z}_h) = l.$$

So we, once again, deduce that $\mathrm{z}_h' \notin E$ and

$$w_h(\mathrm{z}_h) - v_h(\mathrm{z}_h) > w_h(\mathrm{z}_h') - v_h(\mathrm{z}_h').$$

The previous inequality, combined with (3.6), gives

$$\begin{aligned} \varepsilon S^+_{\mathfrak{h}} w_h(\mathrm{z}_h) &= \sum_{\mathrm{z}_h' \in T \cap \widetilde{\mathcal{N}}_{\mathfrak{h}}(\mathrm{z}_h)} \widehat{\varphi}_{\mathrm{z}_h'}(\mathrm{z}_h + \varepsilon \mathbf{v}_\theta) \left(w_h(\mathrm{z}_h') - w_h(\mathrm{z}_h) \right) \\ &< \sum_{\mathrm{z}_h' \in T \cap \widetilde{\mathcal{N}}_{\mathfrak{h}}(\mathrm{z}_h)} \widehat{\varphi}_{\mathrm{z}_h'}(\mathrm{z}_h + \varepsilon \mathbf{v}_\theta) \left(v_h(\mathrm{z}_h') - v_h(\mathrm{z}_h) \right) \le \varepsilon S^+_{\mathfrak{h}} v_h(\mathrm{z}_h), \end{aligned} \tag{3.11}$$

which contradicts (3.7). This proves the claim that $S^+_{\mathfrak{h}} w_h(\mathrm{z}_h) = 0$ for any $\mathrm{z}_h \in F$.

Thanks to the assumption that $-\Delta^\diamond_{\infty,\mathfrak{h}} w_h(\mathrm{z}_h) \le 0$, we have

$$\varepsilon S^-_{\mathfrak{h}} w_h(\mathrm{z}_h) \le \varepsilon S^+_{\mathfrak{h}} w_h(\mathrm{z}_h).$$

This leads to

$$\varepsilon S_{\mathfrak{h}}^{-} w_h(\mathrm{z}_h) \le \varepsilon S_{\mathfrak{h}}^{+} w_h(\mathrm{z}_h) = 0, \quad \forall \mathrm{z}_h \in F.$$

So for any $\mathrm{z}_h \in F$, we have $\varepsilon S_{\mathfrak{h}}^{-} w_h(\mathrm{z}_h) = \varepsilon S_{\mathfrak{h}}^{+} w_h(\mathrm{z}_h) = 0$. This implies that

$$w_h(\mathrm{z}_h + \varepsilon \mathbf{v}_\theta) = w_h(\mathrm{z}_h), \quad \forall \mathbf{v}_\theta \in \mathbb{S}_\theta. \tag{3.12}$$

We further claim that $\widetilde{\mathcal{N}}_{\mathfrak{h}}(\mathrm{z}_h) \subset E$. In fact, if there exists some $\mathrm{z}_h' \in \widetilde{\mathcal{N}}_{\mathfrak{h}}(\mathrm{z}_h) \setminus E$, we argue in the same way as we did in (3.10) or (3.11) to get a contradiction. From $\widetilde{\mathcal{N}}_{\mathfrak{h}}(\mathrm{z}_h) \subset E$, we see that $w_h(\mathrm{z}_h') \le w_h(\mathrm{z}_h) = l$ for any $\mathrm{z}_h' \in \widetilde{\mathcal{N}}_{\mathfrak{h}}(\mathrm{z}_h)$. Combine this with (3.12) to obtain that

$$w_h(\mathrm{z}_h') = w_h(\mathrm{z}_h) = l, \quad \forall \mathrm{z}_h' \in \widetilde{\mathcal{N}}_{\mathfrak{h}}(\mathrm{z}_h)$$

and thus $\widetilde{\mathcal{N}}_{\mathfrak{h}}(\mathrm{z}_h) \subset F \subset \mathcal{N}_{h,\varepsilon}^{I}$. To summarize, we have shown that if $\mathrm{z}_h \in F$, then we necessarily must have that $\widetilde{\mathcal{N}}_{\mathfrak{h}}(\mathrm{z}_h) \subset F$.

To conclude, we observe that, by Assumption **M**.1, we can find $\mathrm{z}_h \in F$ and $\mathrm{z}_h' \in \mathcal{N}_h \setminus F$, for which

$$\mathrm{z}_h' \in \widetilde{\mathcal{N}}_{\mathfrak{h}}(\mathrm{z}_h),$$

which is a contradiction. □

REMARK 3.4 (discrete comparison). An easy consequence of the comparison principle of Theorem 3.2 or Theorem 3.3 is that

$$\max_{\mathrm{z}_h \in \mathcal{N}_h} \Big(u_{\mathfrak{h},1}(\mathrm{z}_h) - u_{\mathfrak{h},2}(\mathrm{z}_h) \Big) \le \max_{\mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^{b}} \big(\widetilde{g}_1(\mathrm{z}_h) - \widetilde{g}_2(\mathrm{z}_h) \big) = C, \tag{3.13}$$

where $u_{\mathfrak{h},i}$ is the solution to (2.4) with boundary data given by $\widetilde{g}_i$ and the same right-hand side f. To prove this, notice that $u_{\mathfrak{h},2} + C$ is a supersolution to (2.4) with boundary data given by $\widetilde{g}_1$. Then (3.13) follows from the comparison principle.

3.2 *Existence, uniqueness and stability*

We are now in position to study the existence, uniqueness and stability of $u_{\mathfrak{h}} \in \mathbb{V}_h$, the solution of (2.4). We begin with an *a priori*, that is stability, estimate.

LEMMA 3.5 (stability). Assume **RHS**.1–2, and **BC**.1–2. Moreover, if **RHS**.2b suppose, in addition, that for all $h > 0$, the mesh $\mathcal{T}_h$ is such that **M**.1 holds. If $u_{\mathfrak{h}} \in \mathbb{V}_h$ solves (2.4), then we have

$$\|u_{\mathfrak{h}}\|_{L^\infty(\Omega_h)} \le C\|f\|_{L^\infty(\Omega_h)} + \max_{\mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^{b}} |\widetilde{g}(\mathrm{z}_h)|,$$

where the constant C depends only on Ω, but is independent of the parameters $\mathfrak{h} = (h, \varepsilon, \theta)$ that define our scheme.

Proof. Without loss of generality, we may assume that $0 \in \Omega$. Let us define the barrier function $\varphi(x) = \mathbf{p}^\intercal x - \frac{1}{2}|x|^2 + A$, where $\mathbf{p} \in \mathbb{R}^d$ and $A \in \mathbb{R}$ are to be specified. We choose the vector $\mathbf{p}$ to guarantee that, for any $\mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^I$,

$$\max_{x \in \mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)} \varphi(x) = \max\left\{ \max_{\mathbf{v}_\theta \in \mathbb{S}_\theta} \varphi(\mathrm{z}_h + \varepsilon \mathbf{v}_\theta), \varphi(\mathrm{z}_h) \right\} = \max_{\mathbf{v}_\theta \in \mathbb{S}_\theta} \varphi(\mathrm{z}_h + \varepsilon \mathbf{v}_\theta). \tag{3.14}$$

To achieve this, it suffices to have

$$\max_{\mathbf{v}_\theta \in \mathbb{S}_\theta} \varphi(\mathrm{z}_h + \varepsilon \mathbf{v}_\theta) > \varphi(\mathrm{z}_h).$$

Since, for $x \in \mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)$,

$$\begin{aligned} \varphi(x) &= \mathbf{p}^\intercal (x - \mathrm{z}_h) + \mathbf{p}^\intercal \mathrm{z}_h - \frac{1}{2}\left(|x - \mathrm{z}_h|^2 + 2(x - \mathrm{z}_h)^\intercal \mathrm{z}_h + |\mathrm{z}_h|^2\right) + A \\ &= (\mathbf{p} - \mathrm{z}_h)^\intercal (x - \mathrm{z}_h) - \frac{1}{2}|x - \mathrm{z}_h|^2 + \left(\mathbf{p}^\intercal \mathrm{z}_h - \frac{1}{2}|\mathrm{z}_h|^2 + A\right), \end{aligned}$$

we then have

$$\varphi(\mathrm{z}_h + \varepsilon \mathbf{v}_\theta) - \varphi(\mathrm{z}_h) = \varepsilon (\mathbf{p} - \mathrm{z}_h)^\intercal \mathbf{v}_\theta - \frac{1}{2}\varepsilon^2.$$

By the definition of $\mathbb{S}_\theta$, there is $\bar{\mathbf{v}}_\theta \in \mathbb{S}_\theta$ such that

$$\left| \bar{\mathbf{v}}_\theta - \frac{\mathbf{p} - \mathrm{z}_h}{|\mathbf{p} - \mathrm{z}_h|} \right| \leq \theta \leq 1.$$

With this choice, we then have

$$\varepsilon (\mathbf{p} - \mathrm{z}_h)^\intercal \bar{\mathbf{v}}_\theta - \frac{1}{2}\varepsilon^2 \geq \frac{\varepsilon}{2}|\mathbf{p} - \mathrm{z}_h| - \frac{1}{2}\varepsilon^2.$$

Consequently, if we are able to choose $\mathbf{p}$ such that $|\mathbf{p} - \mathrm{z}_h| > \varepsilon$, for all $\mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^I$, we will have shown that

$$\max_{\mathbf{v}_\theta \in \mathbb{S}_\theta} \varphi(\mathrm{z}_h + \varepsilon \mathbf{v}_\theta) > \varphi(\mathrm{z}_h).$$

To this aim, one could simply choose $\mathbf{p} \in \mathbb{R}^d$ such that for any $x \in \Omega$

$$|\mathbf{p}| \geq 2\,\mathrm{diam}(\Omega) \geq |x| + \varepsilon,$$

where we, trivially, assumed that $\varepsilon \leq \mathrm{diam}(\Omega)$. Once $\mathbf{p}$ is chosen, we let A be sufficiently big so that $\varphi \geq 0$ on $\overline{\Omega}$.

Now, owing to (3.14), for any $z_h \in \mathcal{N}_{h,\varepsilon}^I$, there exists $\mathbf{v}_\theta \in \mathbb{S}_\theta$ such that

$$\varphi(z_h + \varepsilon \mathbf{v}_\theta) = \max_{x \in \mathcal{N}_\mathfrak{h}(z_h)} \varphi(x).$$

Since we assumed that $\mathbb{S}_\theta$ is symmetric, then $-\mathbf{v}_\theta \in \mathbb{S}_\theta$, and we have

$$\varphi(z_h - \varepsilon \mathbf{v}_\theta) \geq \min_{x \in \mathcal{N}_\mathfrak{h}(z_h)} \varphi(x),$$

and thus

$$\begin{aligned} 2\varphi(z_h) - \max_{x \in \mathcal{N}_\mathfrak{h}(z_h)} \varphi(x) - \min_{x \in \mathcal{N}_\mathfrak{h}(z_h)} \varphi(x) &= 2\varphi(z_h) - \varphi(z_h + \varepsilon \mathbf{v}_\theta) - \min_{x \in \mathcal{N}_\mathfrak{h}(z_h)} \varphi(x) \\ &\geq 2\varphi(z_h) - \varphi(z_h + \varepsilon \mathbf{v}_\theta) - \varphi(z_h - \varepsilon \mathbf{v}_\theta) = \varepsilon^2. \end{aligned}$$

Combine this with the concavity of φ to arrive at

$$\begin{aligned} -\Delta_{\infty,\mathfrak{h}}^\diamond \varphi(z_h) &= \frac{1}{\varepsilon^2}\left(2\varphi(z_h) - \max_{x \in \mathcal{N}_\mathfrak{h}(z_h)} \mathcal{I}_h \varphi(x) - \min_{x \in \mathcal{N}_\mathfrak{h}(z_h)} \mathcal{I}_h \varphi(x) \right) \\ &\geq \frac{1}{\varepsilon^2}\left(2\varphi(z_h) - \max_{x \in \mathcal{N}_\mathfrak{h}(z_h)} \varphi(x) - \min_{x \in \mathcal{N}_\mathfrak{h}(z_h)} \varphi(x) \right) \geq \frac{1}{\varepsilon^2}\varepsilon^2 = 1. \end{aligned}$$

This implies that the function

$$w_h = \|f\|_{L^\infty(\Omega_h)} \mathcal{I}_h \varphi + \max_{z_h \in \mathcal{N}_{h,\varepsilon}^b} \widetilde{g}(z_h) \in \mathbb{V}_h,$$

satisfies

$$-\Delta_{\infty,\mathfrak{h}}^\diamond w_h(z_h) \geq f(z_h), \quad \forall z_h \in \mathcal{N}_{h,\varepsilon}^I, \qquad w_h(z_h) \geq \widetilde{g}(z_h), \quad \forall z_h \in \mathcal{N}_{h,\varepsilon}^b,$$

i.e., it is a discrete supersolution. If **RHS**.2a holds, then we invoke Theorem 3.2; otherwise, i.e., when **RHS**.2b holds, by Theorem 3.3, we have

$$\max_{z_h \in \mathcal{N}_h} \left(u_\mathfrak{h}(z_h) - w_h(z_h) \right) = \max_{z_h \in \mathcal{N}_{h,\varepsilon}^b} \left(u_\mathfrak{h}(z_h) - w_h(z_h) \right) = \max_{z_h \in \mathcal{N}_{h,\varepsilon}^b} \left(\widetilde{g}(z_h) - w_h(z_h) \right) \leq 0.$$

Therefore,

$$\begin{aligned} \max_{z_h \in \mathcal{N}_h} u_\mathfrak{h}(z_h) &\leq \|f\|_{L^\infty(\Omega_h)} \max_{z_h \in \mathcal{N}_h} \varphi(z_h) + \max_{z_h \in \mathcal{N}_{h,\varepsilon}^b} \widetilde{g}(z_h) \\ &\leq C\|f\|_{L^\infty(\Omega_h)} + \max_{z_h \in \mathcal{N}_{h,\varepsilon}^b} \widetilde{g}(z_h), \end{aligned}$$

where the constant C depends only on the domain Ω. This gives an upper bound for $u_{\mathfrak{h}}$. Similarly, we can obtain a lower bound by using $-\|f\|_{L^\infty(\Omega_h)}\mathcal{I}_h\varphi+\min_{z_h\in\mathcal{N}^b_{h,\varepsilon}}\widetilde{g}(z_h)$ as a discrete subsolution. Combining the lower and upper bounds together finishes the proof. □

Let us now show that (2.4) has a unique solution.

LEMMA 3.6 (existence, uniqueness). Assume **RHS**.1–2, and **BC**.1–2. Moreover, if **RHS**.2b holds, suppose that for all $h > 0$ the mesh $\mathcal{T}_h$ satisfies **M**.1. For any choice of parameters $\mathfrak{h} = (h, \varepsilon, \theta)$, there exists a unique $u_{\mathfrak{h}} \in \mathbb{V}_h$ that solves (2.4).

Proof. Uniqueness follows, when **RHS**.2a holds, immediately from Theorem 3.2. Similarly, when **RHS**.2b is valid, we invoke Theorem 3.3 to get uniqueness. To show existence, we employ a, somewhat standard, discrete version of Perron's method (Han & Lin, 2011, Section 6.1). Consider the set of discrete subsolutions

$$W_h = \left\{w_h \in \mathbb{V}_h :\ -\Delta^\diamond_{\infty,\mathfrak{h}} w_h(z_h) \le f(z_h)\ \forall z_h \in \mathcal{N}^I_{h,\varepsilon};\ w_h(z_h) \le \widetilde{g}(z_h)\ \forall z_h \in \mathcal{N}^b_{h,\varepsilon}\right\}.$$

From the proof of Lemma 3.5, we see that the set W_h is nonempty and bounded from above. Define $v_h \in \mathbb{V}_h$ as

$$v_h(z_h) = \sup_{w_h\in W_h} w_h(z_h), \qquad \forall z_h \in \mathcal{N}_h.$$

A standard argument based on discrete monotonicity implies that v_h is also a discrete subsolution, i.e., $v_h \in W_h$. In addition, it satisfies

$$-\Delta^\diamond_{\infty,\mathfrak{h}} v_h(z_h) = f(z_h), \quad \forall z_h \in \mathcal{N}^I_{h,\varepsilon}, \qquad v_h(z_h) = \widetilde{g}(z_h), \quad \forall z_h \in \mathcal{N}^b_{h,\varepsilon}, \tag{3.15}$$

i.e., v_h is a discrete solution. To see this assume, for instance, that the equation is not satisfied at some $z_h \in \mathcal{N}^I_{h,\varepsilon}$. Then, we can consider the function

$$\widetilde{v}_h = v_h + \beta\widehat{\varphi}_{z_h},$$

with some positive $\beta > 0$. For $\beta > 0$ small enough, we have that $\widetilde{v}_h \in W_h$ since, by monotonicity,

$$-\Delta^\diamond_{\infty,\mathfrak{h}}\widetilde{v}_h(z'_h) \le -\Delta^\diamond_{\infty,\mathfrak{h}} v_h(z'_h) \le f(z'_h),$$

for any $z'_h \neq z_h$. This contradicts with our definition of v_h since $\widetilde{v}_h(z_h) > v_h(z_h)$ and $\widetilde{v}_h \in W_h$. Similarly, if there is $z_h \in \mathcal{N}^b_{h,\varepsilon}$ where $v_h(z_h) < \widetilde{g}(z_h)$, we can again define

$$\widetilde{v}_h = v_h + \beta\widehat{\varphi}_{z_h},$$

which for $\beta > 0$, but small enough, is a subsolution. Once again we reach a contradiction, since $\widetilde{v}_h \in W_h$ and $\widetilde{v}_h(z_h) > v_h(z_h)$. Consequently, (3.15) holds and v_h is a discrete solution. □

3.3 *Interior consistency*

Let us now study the interior consistency of scheme (2.4). We must consider two cases, depending on whether the gradient of our function vanishes or not.

LEMMA 3.7 (consistency I). Let $\mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^I$ and $\varphi \in C^3(\overline{B}_\varepsilon(\mathrm{z}_h))$ be such that $\mathrm{D}\varphi(\mathrm{z}_h) \neq \mathbf{0}$. There exists a constant $C > 0$ depending only on φ such that

$$\left| -\Delta_\infty^\diamond \varphi(\mathrm{z}_h) + \Delta_{\infty,\mathfrak{h}}^\diamond \varphi(\mathrm{z}_h)\right| \leq C\big(\varepsilon + \theta + (h/\varepsilon)^2\big).$$

Proof. The proof follows that of Armstrong & Smart (2012, Lemma 4.2). It suffices to show

$$-\Delta_\infty^\diamond \varphi(\mathrm{z}_h) \leq -\Delta_{\infty,\mathfrak{h}}^\diamond \varphi(\mathrm{z}_h) + C\big(\varepsilon + \theta + (h/\varepsilon)^2\big). \tag{3.16}$$

The other direction can be proved in a similar way. Now, to prove (3.16), let

$$\mathbf{v} = \frac{\mathrm{D}\varphi(\mathrm{z}_h)}{|\mathrm{D}\varphi(\mathrm{z}_h)|}, \quad \mathbf{v}_\theta \in \operatorname*{arg\,min}_{\mathbf{v}_\theta \in \mathbb{S}_\theta} |\mathbf{v} - \mathbf{v}_\theta|, \quad \mathbf{w} \in \operatorname*{arg\,max}_{\mathbf{v} \in \mathbb{S}_\theta \cup \{\mathbf{0}\} : \mathrm{z}_h + \varepsilon\mathbf{v} \in \mathcal{N}_\mathfrak{h}(\mathrm{z}_h)} \varphi(\mathrm{z}_h + \varepsilon\mathbf{v}),$$

then $|\mathbf{v}| = |\mathbf{v}_\theta| = 1$ and $|\mathbf{w}| \leq 1$. By definition of $\mathbb{S}_\theta$, we have $|\mathbf{v}_\theta - \mathbf{v}| \leq \theta$ and, in addition,

$$0 \leq (\mathbf{v} - \mathbf{v}_\theta)^\intercal \mathbf{v} = \frac{1}{2}|\mathbf{v} - \mathbf{v}_\theta|^2 \leq \frac{1}{2}\theta^2.$$

From the choice of $\mathbf{w}, \mathbf{v}_\theta$, we apply Taylor expansion and obtain

$$\begin{aligned}
0 &\leq \varphi(\mathrm{z}_h + \varepsilon\mathbf{w}) - \varphi(\mathrm{z}_h + \varepsilon\mathbf{v}_\theta) \\
&\leq \varepsilon \mathrm{D}\varphi(\mathrm{z}_h)^\intercal(\mathbf{w} - \mathbf{v}_\theta) + C\varepsilon^2 \|\mathrm{D}^2\varphi\|_{L^\infty(B_\varepsilon(\mathrm{z}_h);\mathbb{R}^{d\times d})} |\mathbf{w} - \mathbf{v}_\theta| \\
&= \varepsilon |\mathrm{D}\varphi(\mathrm{z}_h)| (\mathbf{w} - \mathbf{v}_\theta)^\intercal \mathbf{v} + C\varepsilon^2 \|\mathrm{D}^2\varphi\|_{L^\infty(B_\varepsilon(\mathrm{z}_h);\mathbb{R}^{d\times d})} |\mathbf{w} - \mathbf{v}_\theta| \\
&\leq \varepsilon |\mathrm{D}\varphi(\mathrm{z}_h)| (\mathbf{w} - \mathbf{v})^\intercal \mathbf{v} + C\varepsilon^2 \|\mathrm{D}^2\varphi\|_{L^\infty(B_\varepsilon(\mathrm{z}_h);\mathbb{R}^{d\times d})} |\mathbf{w} - \mathbf{v}| \\
&\quad + \varepsilon |D\varphi(\mathrm{z}_h)| (\mathbf{v} - \mathbf{v}_\theta)^\intercal \mathbf{v} + C\varepsilon^2 \|\mathrm{D}^2\varphi\|_{L^\infty(B_\varepsilon(\mathrm{z}_h);\mathbb{R}^{d\times d})} |\mathbf{v}_\theta - \mathbf{v}| \\
&\leq \varepsilon |\mathrm{D}\varphi(\mathrm{z}_h)| (\mathbf{w} - \mathbf{v})^\intercal \mathbf{v} + C\varepsilon^2 \|\mathrm{D}^2\varphi\|_{L^\infty(B_\varepsilon(\mathrm{z}_h);\mathbb{R}^{d\times d})} |\mathbf{w} - \mathbf{v}| + \varepsilon\theta^2 |\mathrm{D}\varphi(\mathrm{z}_h)| \\
&\quad + C\varepsilon^2\theta \|\mathrm{D}^2\varphi\|_{L^\infty(B_\varepsilon(\mathrm{z}_h);\mathbb{R}^{d\times d})}.
\end{aligned}$$

Dividing by $\varepsilon|\mathrm{D}\varphi(\mathrm{z}_h)|$ implies

$$(\mathbf{v} - \mathbf{w})^\intercal \mathbf{v} \leq |\mathrm{D}\varphi(\mathrm{z}_h)|^{-1} \left(C\varepsilon|\mathbf{w} - \mathbf{v}| + \varepsilon\theta\right) \|\mathrm{D}^2\varphi\|_{L^\infty(B_\varepsilon(\mathrm{z}_h);\mathbb{R}^{d\times d})} + \theta^2.$$

Combining with the fact that, since $|\mathbf{w}| \leq 1 = |\mathbf{v}|$,

$$(\mathbf{v} - \mathbf{w})^\intercal \mathbf{v} = \frac{1}{2}|\mathbf{v} - \mathbf{w}|^2 + \frac{1}{2}\big(|\mathbf{v}|^2 - |\mathbf{w}|^2\big) \geq \frac{1}{2}|\mathbf{v} - \mathbf{w}|^2$$

we obtain

$$\frac{1}{2}|\mathbf{v}-\mathbf{w}|^2 \le |\mathrm{D}\varphi(\mathrm{z}_h)|^{-1}\left(C\varepsilon|\mathbf{w}-\mathbf{v}|+\varepsilon\theta\right)\|\mathrm{D}^2\varphi\|_{L^\infty(B_\varepsilon(\mathrm{z}_h);\mathbb{R}^{d\times d})}+\theta^2,$$

which leads to

$$|\mathbf{v}-\mathbf{w}| \lesssim \varepsilon|\mathrm{D}\varphi(\mathrm{z}_h)|^{-1}+\theta\big(\|\mathrm{D}^2\varphi\|_{L^\infty(B_\varepsilon(\mathrm{z}_h);\mathbb{R}^{d\times d})}+1\big).$$

Therefore, using Taylor expansion again, we see that

$$\begin{aligned}\varphi(\mathrm{z}_h+\varepsilon\mathbf{w})+\varphi(\mathrm{z}_h-\varepsilon\mathbf{w})-\big(\varphi(\mathrm{z}_h+\varepsilon\mathbf{v})+\varphi(\mathrm{z}_h-\varepsilon\mathbf{v})\big) &\lesssim \varepsilon^2|\mathbf{w}-\mathbf{v}|\|\mathrm{D}^2\varphi\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}\\ &\lesssim \varepsilon^3|\mathrm{D}\varphi(\mathrm{z}_h)|^{-1}\|D^2\varphi\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}+\varepsilon^2\theta\big(\|\mathrm{D}^2\varphi\|^2_{L^\infty(\Omega;\mathbb{R}^{d\times d})}+1\big).\end{aligned}$$

Combine this with the facts that

$$\left|-\Delta^\diamond_\infty\varphi(\mathrm{z}_h)-\frac{1}{\varepsilon^2}\left(2\varphi(\mathrm{z}_h)-\varphi(\mathrm{z}_h+\varepsilon\mathbf{v})-\varphi(\mathrm{z}_h-\varepsilon\mathbf{v})\right)\right| \lesssim \varepsilon\|\mathrm{D}^3\varphi\|_{L^\infty(B_\varepsilon(\mathrm{z}_h);\mathbb{R}^{d\times d\times d})},$$

and that, since $\mathcal{N}_\mathfrak{h}(\mathrm{z}_h)$ is symmetric,

$$\begin{aligned}2\varphi(\mathrm{z}_h)-\max_{x\in\mathcal{N}_\mathfrak{h}(\mathrm{z}_h)}\varphi(x)-\min_{x\in\mathcal{N}_\mathfrak{h}(\mathrm{z}_h)}\varphi(x) &= 2\varphi(\mathrm{z}_h)-\varphi(\mathrm{z}_h+\varepsilon\mathbf{w})-\min_{x\in\mathcal{N}_\mathfrak{h}(\mathrm{z}_h)}\varphi(x)\\ &\ge 2\varphi(\mathrm{z}_h)-\varphi(\mathrm{z}_h+\varepsilon\mathbf{w})-\varphi(\mathrm{z}_h-\varepsilon\mathbf{w}),\end{aligned}$$

we arrive at

$$\begin{aligned}&-\Delta^\diamond_\infty\varphi(\mathrm{z}_h)-\frac{1}{\varepsilon^2}\left(2\varphi(\mathrm{z}_h)-\max_{x\in\mathcal{N}_\mathfrak{h}(\mathrm{z}_h)}\varphi(x)-\min_{x\in\mathcal{N}_\mathfrak{h}(\mathrm{z}_h)}\varphi(x)\right)\\ &\quad\lesssim \varepsilon\|\mathrm{D}^3\varphi\|_{L^\infty(B_\varepsilon(\mathrm{z}_h);\mathbb{R}^{d\times d\times d})}+\varepsilon|\mathrm{D}\varphi(\mathrm{z}_h)|^{-1}\|\mathrm{D}^2\varphi\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}+\theta\big(\|\mathrm{D}^2\varphi\|^2_{L^\infty(\Omega;\mathbb{R}^{d\times d})}+1\big).\end{aligned} \tag{3.17}$$

Finally, recalling the definition of $-\Delta^\diamond_{\infty,\mathfrak{h}}$, we have

$$\begin{aligned}\left|\left(2\varphi(\mathrm{z}_h)-\max_{x\in\mathcal{N}_\mathfrak{h}(\mathrm{z}_h)}\varphi(x)-\min_{x\in\mathcal{N}_\mathfrak{h}(\mathrm{z}_h)}\varphi(x)\right)+\Delta^\diamond_{\infty,\mathfrak{h}}\varphi(\mathrm{z}_h)\right| &\le 2\|\varphi-\mathcal{I}_h\varphi\|_{L^\infty(B_\varepsilon(\mathrm{z}_h))}\\ &\lesssim h^2\|\mathrm{D}^2\varphi\|_{L^\infty(B_\varepsilon(\mathrm{z}_h);\mathbb{R}^{d\times d})},\end{aligned}$$

which combined with (3.17) implies the desired result, i.e.,

$$-\Delta_\infty^\diamond \varphi(z_h) + \Delta_{\infty,\mathfrak{h}}^\diamond \varphi(z_h) \le C\left(\varepsilon + \theta + \frac{h^2}{\varepsilon^2}\right).$$

The constant C depends on $|D\varphi(z_h)|$, $\|D^2\varphi\|^2_{L^\infty(\Omega;\mathbb{R}^{d\times d})}$, $\|D^3\varphi\|^2_{L^\infty(B_\varepsilon(z_h);\mathbb{R}^{d\times d\times d})}$. □

Lemma 3.7 controls the consistency error at an interior vertex z_h in terms of the discretization parameters provided that $D\varphi(z_h) \neq \mathbf{0}$. When $D\varphi(z_h) = \mathbf{0}$, the derivation of a consistency error is not standard. For simplicity, we assume that φ is a quadratic function and obtain the following estimate between the discrete operator $-\Delta_{\infty,\mathfrak{h}}^\diamond$ and the lower and upper semicontinuous envelope operators $-\Delta_\infty^+, -\Delta_\infty^+$.

Lemma 3.8 (consistency II). Let $z_h \in \mathcal{N}_{h,\varepsilon}^I$, and φ be a quadratic polynomial. If φ has no strict local maximum in $B_\varepsilon(z_h)$, then

$$-\Delta_{\infty,\mathfrak{h}}^\diamond \varphi(z_h) \ge -\lambda_{\max}(D^2\varphi) - C\left(\frac{h^2}{\varepsilon^2} + \theta^2\right),$$

where the constant C depends on the dimension d, $D^2\varphi$ and the shape regularity constant of the mesh $\mathcal{T}_h$.

Proof. Without loss of generality, we can assume that $\varphi(x) = \mathbf{p}^\intercal x + \frac{1}{2}x^\intercal \mathbf{M} x$. Since φ has no strict local maximum in $B_\varepsilon(z_h)$, we have

$$\max_{x\in\overline{B}_\varepsilon(z_h)} \varphi(x) = \varphi(z_h + \varepsilon\mathbf{v}),$$

for some $|\mathbf{v}| = 1$. Let $\mathbf{v}_\theta \in \mathbb{S}_\theta$ be such that $|\mathbf{v} - \mathbf{v}_\theta| \le \theta$, and we claim that

$$\varphi(z_h + \varepsilon\mathbf{v}_\theta) \ge \varphi(z_h) - \frac{3}{4}\varepsilon^2\theta^2|\mathbf{M}|. \tag{3.18}$$

Without loss of generality, we could further assume that $z_h = 0$. This implies that $\varphi(z_h) = 0$ and we only need to show that

$$\varphi(\varepsilon\mathbf{v}_\theta) \ge -\frac{3\varepsilon^2\theta^2}{4}|\mathbf{M}|.$$

From the choice of $\mathbf{v}$, we have

$$D\varphi(\varepsilon\mathbf{v}) = \mathbf{p} + \varepsilon\mathbf{M}\mathbf{v} = \lambda\mathbf{v}, \tag{3.19}$$

for some $\lambda \ge 0$. Moreover,

$$0 = \varphi(0) \le \varphi(\varepsilon\mathbf{v}) = \mathbf{p}^\intercal\varepsilon\mathbf{v} + \frac{1}{2}(\varepsilon\mathbf{v})^\intercal\mathbf{M}(\varepsilon\mathbf{v}) = (\lambda\mathbf{v} - \varepsilon\mathbf{M}\mathbf{v})^\intercal\varepsilon\mathbf{v} + \frac{\varepsilon^2}{2}\mathbf{v}^\intercal\mathbf{M}\mathbf{v} = \lambda\varepsilon - \frac{\varepsilon^2}{2}\mathbf{v}^\intercal\mathbf{M}\mathbf{v}. \tag{3.20}$$

Now write $\varphi(\varepsilon \mathbf{v}_\theta)$ as

$$
\begin{aligned}
\varphi(\varepsilon \mathbf{v}_\theta) &= \varphi(\varepsilon \mathbf{v}) + \varepsilon \mathrm{D}\varphi(\varepsilon \mathbf{v})^\intercal (\mathbf{v}_\theta - \mathbf{v}) + \frac{\varepsilon^2}{2}(\mathbf{v}_\theta - \mathbf{v})^\intercal \mathbf{M}(\mathbf{v}_\theta - \mathbf{v}) \\
&= \left(\lambda\varepsilon - \frac{\varepsilon^2}{2}\mathbf{v}^\intercal \mathbf{M}\mathbf{v}\right) + \varepsilon(\mathbf{p} + \varepsilon \mathbf{M}\mathbf{v})^\intercal (\mathbf{v}_\theta - \mathbf{v}) + \frac{\varepsilon^2}{2}(\mathbf{v}_\theta - \mathbf{v})^\intercal \mathbf{M}(\mathbf{v}_\theta - \mathbf{v}) \\
&= \left(\lambda\varepsilon - \frac{\varepsilon^2}{2}\mathbf{v}^\intercal \mathbf{M}\mathbf{v}\right) + \lambda\varepsilon \mathbf{v}^\intercal (\mathbf{v}_\theta - \mathbf{v}) + \frac{\varepsilon^2}{2}(\mathbf{v}_\theta - \mathbf{v})^\intercal \mathbf{M}(\mathbf{v}_\theta - \mathbf{v}) \\
&= \left(\lambda\varepsilon - \frac{\varepsilon^2}{2}\mathbf{v}^\intercal \mathbf{M}\mathbf{v}\right) - \frac{\lambda\varepsilon}{2}|\mathbf{v}_\theta - \mathbf{v}|^2 + \frac{\varepsilon^2}{2}(\mathbf{v}_\theta - \mathbf{v})^\intercal \mathbf{M}(\mathbf{v}_\theta - \mathbf{v}) \\
&\geq \left(\lambda\varepsilon - \frac{\varepsilon^2}{2}\mathbf{v}^\intercal \mathbf{M}\mathbf{v}\right) - \frac{\lambda\varepsilon\theta^2}{2} - \frac{\varepsilon^2\theta^2}{2}|\mathbf{M}| \\
&= \left(1 - \frac{1}{2}\theta^2\right)\left(\lambda\varepsilon - \frac{\varepsilon^2}{2}\mathbf{v}^\intercal \mathbf{M}\mathbf{v}\right) - \frac{\varepsilon^2\theta^2}{4}\mathbf{v}^\intercal \mathbf{M}\mathbf{v} - \frac{\varepsilon^2\theta^2}{2}|\mathbf{M}| \\
&\geq -\frac{\varepsilon^2\theta^2}{4}\mathbf{v}^\intercal \mathbf{M}\mathbf{v} - \frac{\varepsilon^2\theta^2}{2}|\mathbf{M}| \geq -\frac{3\varepsilon^2\theta^2}{4}|\mathbf{M}|,
\end{aligned}
$$

where we used (3.19), (3.20) and the fact that, since $\theta \leq 1$, we have $1 - \theta^2/2 \geq 0$.

Now, choose $\mathbf{w}_\theta$ that satisfies

$$
\max_{\mathbf{w} \in \mathbb{S}_\theta} \varphi(\mathrm{z}_h + \varepsilon \mathbf{w}) = \varphi(\mathrm{z}_h + \varepsilon \mathbf{w}_\theta).
$$

Thanks to (3.18), we have

$$
\varphi(\mathrm{z}_h + \varepsilon \mathbf{w}_\theta) \geq \max_{x \in \mathcal{N}_\mathfrak{h}(\mathrm{z}_h)} \varphi(x) - \frac{3\varepsilon^2\theta^2}{4}|\mathbf{M}|.
$$

Therefore, since $\mathcal{N}_\mathfrak{h}(\mathrm{z}_h)$ is symmetric,

$$
\begin{aligned}
-\Delta_{\infty,\mathfrak{h}}^\diamond \varphi(\mathrm{z}_h) &= \frac{1}{\varepsilon^2}\left(2\varphi(\mathrm{z}_h) - \min_{x \in \mathcal{N}_\mathfrak{h}(\mathrm{z}_h)} \mathcal{I}_h\varphi(x) - \max_{x \in \mathcal{N}_\mathfrak{h}(\mathrm{z}_h)} \mathcal{I}_h\varphi(x)\right) \\
&\geq \frac{1}{\varepsilon^2}\left(2\varphi(\mathrm{z}_h) - \min_{x \in \mathcal{N}_\mathfrak{h}(\mathrm{z}_h)} \varphi(x) - \max_{x \in \mathcal{N}_\mathfrak{h}(\mathrm{z}_h)} \varphi(x) - Ch^2\right) \\
&\geq \frac{1}{\varepsilon^2}\left(2\varphi(\mathrm{z}_h) - \varphi(\mathrm{z}_h - \varepsilon \mathbf{w}_\theta) - \varphi(\mathrm{z}_h + \varepsilon \mathbf{w}_\theta) - Ch^2 - C\varepsilon^2\theta^2\right) \\
&= -\mathbf{w}_\theta^\intercal \mathbf{M}\, \mathbf{w}_\theta - Ch^2\varepsilon^{-2} - C\theta^2 \geq -\lambda_{\max}(\mathbf{M}) - Ch^2\varepsilon^{-2} - C\theta^2.
\end{aligned}
$$

This finishes the proof of the lemma. $\square$

Let $\{\mathfrak{h}_j = (h_j, \varepsilon_j, \theta_j)\}_{j=1}^{\infty}$ be such that $\mathfrak{h}_j \to (0,0,0)$ as $j \uparrow \infty$. Let $w_j \in \mathbb{V}_{h_j}$. We define the upper and lower semicontinuous envelopes to be

$$\begin{aligned} \overline{w}(x) &= \limsup_{j\uparrow\infty, \mathcal{N}_{h_j} \ni z_{h_j} \to x} w_j(z_{h_j}), \\ \underline{w}(x) &= \liminf_{j\uparrow\infty, \mathcal{N}_{h_j} \ni z_{h_j} \to x} w_j(z_{h_j}). \end{aligned} \tag{3.21}$$

By construction, $\overline{w} \in \mathrm{USC}(\overline{\Omega})$ and $\underline{w} \in \mathrm{LSC}(\overline{\Omega})$.

With the aid of interior consistency, we can show that the lower and upper semicontinuous envelopes of a sequence of discrete solutions to (2.4) are a sub- and supersolution, respectively.

LEMMA 3.9 (envelopes). Assume the right-hand side f satisfies Assumptions **RHS**.1–2. Assume, in addition, that if **RHS**.2b holds, then for all $h > 0$ the mesh $\mathcal{T}_h$ satisfies Assumption **M**.1. Let $\{\mathfrak{h}_j = (h_j, \varepsilon_j, \theta_j)\}_{j=1}^{\infty}$ be a sequence of discretization parameters such that $\mathfrak{h}_j \to (0,0,0)$ as $j \uparrow \infty$ and, in addition,

$$\frac{h_j}{\varepsilon_j} \to 0. \tag{3.22}$$

Let $u_{\mathfrak{h}_j} \in \mathbb{V}_{h_j}$ be the sequence of discrete solutions to (2.4). Then, the function $\overline{u}$, defined in (3.21), is a subsolution, i.e., in the viscosity sense, we have

$$-\Delta_\infty^+ \overline{u} \le f \quad \text{in } \Omega. \tag{3.23}$$

Similarly, $\underline{u}$, defined in (3.21), is a supersolution, i.e., in the viscosity sense, we have

$$-\Delta_\infty^- \underline{u} \ge f \quad \text{in } \Omega.$$

Proof. By symmetry, it suffices to show (3.23), as the proof of the fact that $\underline{u}$ is a supersolution follows in the same way.

We need to prove that for a quadratic polynomial φ such that $\overline{u} - \varphi$ attains a strict local maximum at $x_0 \in \Omega$, then

$$-\Delta_\infty^+ \varphi(x_0) \le f(x_0).$$

Without loss of generality, we may assume that $x_0 = 0$ and $\overline{u}(0) = \varphi(0) = 0$. Therefore, $\varphi(x) = \mathbf{p}^\intercal x + \frac{1}{2} x^\intercal \mathbf{M} x$ where $\mathbf{p} = \mathrm{D}\varphi(0)$ and $\mathbf{M} = \mathrm{D}^2\varphi(0)$. Since the maximum is strict, there is $\delta_0 > 0$ such that $\overline{u}(0) - \varphi(0) > \overline{u}(x) - \varphi(x)$ for any $x \in \overline{B}_{\delta_0}(0) \setminus \{0\}$. By the definition of $\overline{u}$, for any $0 < \delta \le \delta_0$ and large enough j, we can select $z_{h_j} \in \mathcal{N}_{h_j,\varepsilon_j}^I \cap \overline{B}_\delta(0)$ such that

$$u_{\mathfrak{h}_j}(z_{h_j}) - \mathcal{I}_{h_j}\varphi(z_{h_j}) = \max_{\mathcal{N}_{h_j} \cap \overline{B}_\delta(0)} (u_{\mathfrak{h}_j} - \mathcal{I}_{h_j}\varphi),$$

and $\mathrm{z}_{h_j} \to 0$ as $j \uparrow \infty$. Using the monotonicity of $-\Delta^{\diamond}_{\infty,\mathfrak{h}_j}$, presented in Lemma 3.1, we have

$$-\Delta^{\diamond}_{\infty,\mathfrak{h}_j}\varphi\left(\mathrm{z}_{h_j}\right) \le -\Delta^{\diamond}_{\infty,\mathfrak{h}_j}u_{\mathfrak{h}_j}\left(\mathrm{z}_{h_j}\right) \le f\left(\mathrm{z}_{h_j}\right). \tag{3.24}$$

To conclude the proof, we will discuss three cases separately.

Case 1: $\mathbf{p} \neq \mathbf{0}$. By Lemma 3.7 and the upper semicontinuity of Δ^+_∞, we obtain that

$$\begin{aligned}-\Delta^+_\infty\varphi(0) &\le \lim_{j\uparrow\infty} -\Delta^+_\infty\varphi\left(\mathrm{z}_{h_j}\right) \le \limsup_{j\uparrow\infty}\left(-\Delta^{\diamond}_{\infty,\mathfrak{h}_j}\varphi\left(\mathrm{z}_{h_j}\right) + C\left(\varepsilon_j + \theta_j + (h_j/\varepsilon_j)^2\right)\right) \\ &\le \limsup_{j\uparrow\infty} f\left(\mathrm{z}_{h_j}\right) = f(0),\end{aligned}$$

because of (3.22). We remark that by a standard perturbation argument, we also have $-\Delta^+_\infty\varphi(0) \le f(0)$ if $\overline{u} - \varphi$ attains a nonstrict local maximum at $0 \in \Omega$.

Case 2: $\mathbf{p} = \mathbf{0}$, and $\mathbf{M}$ is not strictly negative definite. The assumptions imply that φ has no strict local maximum in $B_\varepsilon\left(\mathrm{z}_{h_j}\right)$. We apply Lemma 3.8 and obtain

$$\begin{aligned}-\Delta^+_\infty\varphi(0) = -\lambda_{\max}(\mathbf{M}) &\le \limsup_{j\uparrow\infty}\left(-\Delta^{\diamond}_{\infty,\mathfrak{h}_j}\varphi\left(\mathrm{z}_{h_j}\right) + C\left(\theta_j^2 + (h_j/\varepsilon_j)^2\right)\right) \\ &\le \limsup_{j\uparrow\infty} f\left(\mathrm{z}_{h_j}\right) = f(0)\end{aligned}$$

because of (3.22).

Case 3: $\mathbf{p} = \mathbf{0}$ and $\mathbf{M}$ is strictly negative definite. Let $0 \neq y \in \mathbb{R}^d$ and define $\varphi_y(x) = \frac{1}{2}(x-y)^\intercal\mathbf{M}(x-y)$. For $|y|$ sufficiently small, we know that $\overline{u} - \varphi_y$ attains a local maximum at $x_y \in B_\delta(0)$. Since $\mathbf{M}$ is strictly negative definite, we have $\varphi(y) = \varphi_y(0) < 0 = \varphi(0) = \varphi_y(y)$ and

$$\overline{u}(x_y) - \varphi_y(x_y) \ge \overline{u}(0) - \varphi_y(0) > \overline{u}(0) - \varphi(0) > \overline{u}(y) - \varphi(y) > \overline{u}(y) - \varphi_y(y).$$

This shows that $x_y \neq y$. Therefore, $\mathrm{D}\varphi_y(x_y) \neq \mathbf{0}$. Consequently, by using the results we have obtained for Case 1, we have

$$-\lambda_{\max}(\mathbf{M}) \le -\Delta^+_\infty\varphi(x_y) \le f(x_y).$$

Letting $y \to 0$, we have $x_y \to 0$ and

$$-\Delta^+_\infty\varphi(0) = -\lambda_{\max}(\mathbf{M}) \le \lim_{y\to 0} f(x_y) = f(0),$$

because of $f \in C(\overline{\Omega})$. □

REMARK 3.10 (consistency). Notice that, in the proof of Lemma 3.9, the case when $\mathbf{p} = \mathbf{0}$ and $\mathbf{M}$ is strictly negative definite required a nonstandard proof. This is due to the rather surprising fact that, in this case our scheme is not consistent. To see this, it suffices to consider the semidiscrete scheme (1.7). Further discretization, i.e., using (2.3), will only add consistency terms depending on $\mathfrak{h}$. Let $\varphi(x) = -\frac{1}{2}|x|^2$ and assume $x = 0$ is an interior point, so that $B_\varepsilon(x) \subset \Omega$. We have

$$\varphi(0) = 0, \qquad \max_{x \in \overline{B}_\varepsilon(0)} \varphi(x) = 0, \qquad \min_{x \in \overline{B}_\varepsilon(0)} \varphi(x) = -\frac{1}{2} \max_{x \in \overline{B}_\varepsilon(0)} |x|^2 = -\frac{1}{2}\varepsilon^2$$

so that

$$-\Delta^\diamond_{\infty,\varepsilon}\varphi(0) = \frac{1}{2} < -\Delta^+_\infty \varphi(0) = 1,$$

independently of the value of ε. The strategy that we follow, and that greatly simplifies that of Armstrong & Smart (2012, Theorem 2.11), is to move away from such a point to one where the consistency issue does not arise.

Remark 3.11 (consistency for Aronsson's example). In the previous remark, we mentioned the possible nonconsistency when $\mathrm{D}\varphi(x) = \mathbf{0}$. Here, we point out that nonconsistency may also appear when $\mathrm{D}\varphi(x) \neq \mathbf{0}$ if φ is not smooth enough. Consider Aronsson's example $\varphi(x, y) = |x|^{4/3} - |y|^{4/3}$, then one could show that

$$\lim_{\varepsilon \to 0^+} \Delta^\diamond_{\infty,\varepsilon}\varphi(x, 0) = \frac{|x|^{-2/3}}{9} \neq 0,$$

where $x \neq 0$ and the semidiscrete operator is defined in (1.7).

3.4 *Boundary behavior and convergence*

To prove convergence of discrete solutions we need, in addition to interior consistency, to control the behavior of discrete solutions $u_\mathfrak{h}$ near $\partial\Omega$. As before, we will assume that we have a sequence of discretization parameters $\{\mathfrak{h}_j = (h_j, \varepsilon_j, \theta_j)\}_{j=1}^\infty$ satisfying that

$$\lim_{j \uparrow \infty} \left[h_j + \varepsilon_j + \theta_j + \frac{h_j}{\varepsilon_j} + \frac{\theta_j}{\varepsilon_j} \right] = 0. \tag{3.25}$$

Let us prove that $\overline{u}$ and $\underline{u}$, defined in (3.21), coincide with g at the boundary $\partial\Omega$.

Lemma 3.12 (boundary behavior). Let all the conditions of Lemma 3.9 be fulfilled. Assume, in addition, that the boundary datum g satisfies Assumptions **BC**.1–2. Let $\{\mathfrak{h}_j\}_{j=1}^\infty$ be a sequence of discretization parameters that satisfies (3.25). The functions $\overline{u}, \underline{u}$, defined as in (3.21) through the family $u_{\mathfrak{h}_j} \in \mathbb{V}_{h_j}$ of discrete solutions, satisfy $\overline{u}(x) = \underline{u}(x) = g(x)$ for all $x \in \partial\Omega$.

Proof. By symmetry, it suffices to prove that $\overline{u}(x) = g(x)$. From the fact that $u_{\mathfrak{h}_j}(\mathrm{z}_{h_j}) = \widetilde{g}(\mathrm{z}_{h_j})$ for all $\mathrm{z}_{h_j} \in \mathcal{N}^b_{h_j,\varepsilon_j}$, we have

$$\overline{u}(x) \geq \limsup_{j \to \infty, \mathrm{z}_{h_j} \to x} \widetilde{g}(\mathrm{z}_{h_j}) = \widetilde{g}(x) = g(x), \quad \forall x \in \partial\Omega.$$

So, it only remains to prove that $\overline{u}(x) \leq g(x)$ for any $x \in \partial\Omega$.

First, we claim that it is sufficient to prove this under the assumption that $\widetilde{g}$ is smooth. To see this, we employ the fact that, for any $\delta > 0$, we have a function $\widetilde{g}_{s,\delta} \in C^\infty(\mathcal{B})$ such that $\|\widetilde{g} - \widetilde{g}_{s,\delta}\|_{L^\infty(\Omega)} \le \delta$. Let the functions $\overline{u}_{s,\delta}$ be associated with the data $(f, g_{s,\delta})$. If we know that $\overline{u}_{s,\delta}(x) \le \widetilde{g}_{s,\delta}(x)$ for $x \in \partial\Omega$, then

$$\overline{u}(x) \le \overline{u}_{s,\delta}(x) + \delta \le \widetilde{g}_{s,\delta}(x) + \delta \le \widetilde{g}(x) + 2\delta,$$

where the first inequality follows from $\widetilde{g} \le \widetilde{g}_{s,\delta} + \delta$ and Remark 3.4. Since δ is arbitrary, this implies that $\overline{u}(x) \le g(x)$ for any $x \in \partial\Omega$.

Now, we can assume that $\widetilde{g}$ is smooth with Lipschitz constant $L > 0$. In the second step, we aim to prove $\overline{u}(x_0) \le g(x_0)$ for any $x_0 \in \partial\Omega$ satisfying the exterior ball condition, i.e., there are $y_0 \in \mathbb{R}^d$ and $0 < R < 1$ such that

$$x_0 \in \partial B_R(y_0), \quad \operatorname{int} B_R(y_0) \cap \Omega = \emptyset.$$

We may write $y_0 = x_0 + R\mathbf{w}$ for some $|\mathbf{w}| = 1$. The exterior ball condition implies that

$$|(x - x_0) - R\mathbf{w}| \ge R \quad \Longrightarrow \quad \mathbf{w}^\mathsf{T}(x - x_0) \le \frac{|x - x_0|^2}{2R}, \tag{3.26}$$

for any $x \in \Omega$. Let $z_0 = x_0 + (R/2)\mathbf{w}$ and consider the following barrier function

$$\varphi(x) = q\left(|x - z_0|\right) = a + b|x - z_0| - \frac{c}{2}|x - z_0|^2, \tag{3.27}$$

where q is a quadratic function to be determined. Fix $c = \sup_{x\in\Omega} f(x) + 1$ and assume that $b > c$, so that $|\mathrm{D}\varphi(\mathrm{z}_{h_j})|$ is bounded away from 0 for all $\mathrm{z}_{h_j} \in \mathcal{N}^I_{h_j,\varepsilon_j}$. We claim that for j large enough,

$$-\Delta^\diamond_{\infty,\mathfrak{h}_j}\varphi\big(\mathrm{z}_{h_j}\big) \ge \sup_{x\in\Omega} f(x) \ge f\big(\mathrm{z}_{h_j}\big),$$

for all $\mathrm{z}_{h_j} \in \mathcal{N}^I_{h_j,\varepsilon_j}$. To see this, notice that a simple calculation leads to

$$-\Delta^\diamond_\infty\varphi\big(\mathrm{z}_{h_j}\big) = c. \tag{3.28}$$

Since φ is smooth away from z_0, we obtain from Lemma 3.7 that

$$\left|\Delta^\diamond_\infty\varphi\big(\mathrm{z}_{h_j}\big) - \Delta^\diamond_{\infty,\mathfrak{h}_j}\varphi\big(\mathrm{z}_{h_j}\big)\right| \le C_R\big(\varepsilon_j + \theta_j + (h_j/\varepsilon_j)^2\big),$$

for any $\mathrm{z}_{h_j} \in \mathcal{N}^I_{h_j,\varepsilon_j}$ where C_R is a constant depending on R. Combining this with (3.28) and using the assumption in (3.25), we see that

$$-\Delta^\diamond_{\infty,\mathfrak{h}_j}\varphi\big(\mathrm{z}_{h_j}\big) \ge -\Delta^\diamond_\infty\varphi\big(\mathrm{z}_{h_j}\big) - C_R\big(\varepsilon_j + \theta_j + (h_j/\varepsilon_j)^2\big) \ge c - 1 \ge \sup_{x\in\Omega} f(x),$$

for j large enough and any $z_{h_j} \in \mathcal{N}^I_{h_j,\varepsilon_j}$. Now we choose the constants a and b to guarantee that

$$\varphi(x) \geq \widetilde{g}(x), \tag{3.29}$$

for any $x \in \Omega$. Since $\widetilde{g}$ is smooth, we let L be the Lipschitz constant of $\widetilde{g}$. It then suffices to require that

$$\varphi(x) \geq \widetilde{g}(x_0) + L|x - x_0| \geq \widetilde{g}(x). \tag{3.30}$$

Let, for some $b_0 > 0$,

$$b = b_0 + c\left(\mathrm{diam}(\Omega) + \frac{R}{2}\right).$$

Then the function $q(t) = a + bt - (c/2)t^2$ satisfies

$$b \geq q'(t) = b - ct \geq b_0,$$

for $t \in [0, \mathrm{diam}(\Omega) + \frac{R}{2}]$, and φ also satisfies $|D\varphi(z_{h_j})| \geq b_0 > 0$ for all $z_{h_j} \in \mathcal{N}^I_{h_j,\varepsilon_j}$. Recall that by (3.26), we have for any $x \in \Omega$

$$\begin{aligned}
|x - z_0|^2 &= |(x_0 - z_0) + (x - x_0)|^2 = |x_0 - z_0|^2 + |x - x_0|^2 + 2(x_0 - z_0)^{\mathsf{T}}(x - x_0) \\
&= \frac{R^2}{4} + |x - x_0|^2 - R\mathbf{w}^{\mathsf{T}}(x - x_0) \geq \frac{R^2}{4} + |x - x_0|^2 - \frac{|x - x_0|^2}{2} = \frac{R^2}{4} + \frac{|x - x_0|^2}{2}.
\end{aligned}$$

This implies that

$$\varphi(x) - \varphi(x_0) = q(|x - z_0|) - q(|x_0 - z_0|) \geq b_0\left(\sqrt{\frac{R^2}{4} + \frac{|x - x_0|^2}{2}} - \frac{R}{2}\right).$$

We choose the constant a such that $\varphi(x_0) = \widetilde{g}(x_0) + L\beta$ for some $\beta > 0$. From the inequality above, a sufficient condition for (3.30) to be accomplished is

$$L\beta + b_0\left(\sqrt{\frac{R^2}{4} + \frac{|x - x_0|^2}{2}} - \frac{R}{2}\right) \geq L|x - x_0|, \tag{3.31}$$

which clearly holds for any x with $|x - x_0| \leq \beta$. In addition, it is enough to have

$$b_0\sqrt{\frac{R^2}{4} + \frac{|x - x_0|^2}{2}} \geq L|x - x_0| + b_0\frac{R}{2}.$$

Simple calculations reveal that it suffices to require

$$b_0 \geq \max\left\{2L,\ \frac{4LR}{|x-x_0|}\right\}.$$

Therefore, if we have

$$b_0 \geq \max\left\{2L,\ \frac{4L}{\beta}\right\}, \tag{3.32}$$

then (3.31) and (3.29) must hold because of $0 < R < 1$.

Once (3.29) is true, by the comparison principle of Theorem 3.2 or Theorem 3.3, we see that $u_{\mathfrak{h}_j}\left(\mathrm{z}_{h_j}\right) \leq \varphi\left(\mathrm{z}_{h_j}\right)$ for any $\mathrm{z}_{h_j} \in \mathcal{N}_{h_j}$ and thus

$$\overline{u}(x) = \limsup_{j\uparrow\infty, \mathcal{N}_{h_j}\ni \mathrm{z}_{h_j}\to x} u_{\mathfrak{h}_j}\left(\mathrm{z}_{h_j}\right) \leq \limsup_{j\uparrow\infty, \mathcal{N}_{h_j}\ni \mathrm{z}_{h_j}\to x} \varphi\left(\mathrm{z}_{h_j}\right) \leq \varphi(x), \quad \forall x \in B_\beta(x_0).$$

This simply shows that

$$\overline{u}(x_0) \leq g(x_0) + L\beta.$$

Since $\beta > 0$ can be chosen arbitrarily small with b_0 chosen afterwards, we have finished the second step, i.e., shown that $\overline{u}(x_0) \leq g(x_0)$ for any $x_0 \in \partial\Omega$ satisfying the exterior ball condition.

In the last step, we consider $\widetilde{g}$ smooth and a point $x \in \partial\Omega$ where the exterior ball condition may not hold. Noticing that from the previous discussions, if the exterior ball condition is satisfied at $x_0 \in \partial\Omega$, then

$$\begin{aligned}\overline{u}(x) &\leq \varphi(x) \leq \varphi(x_0) + b|x-x_0| = g(x_0) + L\beta + b|x-x_0| \\ &\leq g(x) + L|x-x_0| + L\beta + b|x-x_0|.\end{aligned}$$

For any $\delta > 0$, we may choose β such that $L\beta < \delta/2$. Then we choose b_0 satisfying (3.32) and obtain the parameter b. Since $\partial\Omega$ is assumed continuous and compact, there always exist an $x_0 \in \partial\Omega$ satisfying the exterior ball condition with $|x-x_0| < \frac{\delta}{2(L+b)}$. Consequently,

$$\overline{u}(x) \leq g(x) + \delta,$$

for any $\delta > 0$. This proves that $\overline{u}(x) \leq g(x)$ for any $x \in \partial\Omega$ and finishes the proof. □

Gathering all our previous results we can assert convergence of our numerical scheme.

THEOREM 3.13 (convergence). Assume that the right-hand side f satisfies Assumptions **RHS**.1–2. Assume that the boundary datum g satisfies Assumptions **BC**.1–2. Let $\{\mathfrak{h}_j\}_{j=1}^\infty$ be a sequence of discretization parameters that satisfies (3.25); and, if **RHS**.2b holds, it additionally satisfies **M**.1 for all $j \in \mathbb{N}$. In this framework, we have that the sequence $\{u_{\mathfrak{h}_j} \in \mathbb{V}_{h_j}\}_{j=1}^\infty$ of solutions to (2.4) converges uniformly, as $j \uparrow \infty$, to u, the solution of (1.6).

Proof. The proof follows under the framework of Barles & Souganidis (1991). To be specific, since our approximation scheme (2.4) is monotone (Lemma 3.1), stable (Lemma 3.5) and consistent (Lemma 3.7 and Lemma 3.8), the argument in Barles & Souganidis (1991) implies that $\overline{u}$ and $\underline{u}$ are a subsolution and supersolution to the continuous problem (1.6), respectively; see Lemma 3.9.

Notice, in addition, that Lemma 3.12 implies that the Dirichlet boundary condition is attained in the classical sense for $\overline{u}$ and $\underline{u}$. Recall that (1.6) admits a comparison principle for Dirichlet boundary conditions in the classical sense: see Lu & Wang (2008, Theorem 3.3), Armstrong & Smart (2010) and the discussions after Armstrong & Smart (2012, Theorem 2.18), we thus have $\overline{u} \leq \underline{u}$. This yields, since $\overline{u} \geq \underline{u}$ by definition, that $\overline{u} = \underline{u}$ and the uniform convergence of $u_{\mathfrak{h}_j}$ to the solution u. □

4. Rates of convergence

In this section, we prove convergence rates for solutions of (2.4). To be able to do this, additional regularity on the solution of (1.6) must be required. However, we recall that, as we mentioned in Section 1, the regularity theory for (1.6) is far from complete. Nevertheless, we have that, if there is $\alpha \in (0, 1]$ for which $f \in C^{0,\alpha}(\Omega)$ and $g \in C^{0,\alpha}(\partial\Omega)$, then $u \in C^{0,\alpha}(\overline{\Omega})$; see the references mentioned in the Introduction, and Armstrong & Smart (2012, Proposition 6.4) for the case **RHS**.2a, and Propositions 3.11 and 6.7 of Armstrong & Smart (2012) for **RHS**.2b.

In this section, we will tacitly assume that **RHS**.1–2, **BC**.1–2 are valid. We will also assume that, if **RHS**.2b, then the meshes satisfy **M**.1. As a final assumption, we posit that there is $\alpha \in (0, 1]$ such that $f \in C^{0,\alpha}(\overline{\Omega})$, $g \in C^{0,\alpha}(\partial\Omega)$ and $u \in C^{0,\alpha}(\overline{\Omega})$.

We immediately note a consequence of this assumed regularity, which will be used repeatedly. For any $\mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^b$, by our definition of $\mathcal{N}_{h,\varepsilon}^b$, there exists $x \in \partial\Omega$ satisfying $|x - \mathrm{z}_h| \leq 2\varepsilon$. Then from the regularity assumptions on g and u, we get

$$|\widetilde{g}(\mathrm{z}_h) - u(\mathrm{z}_h)| \leq |\widetilde{g}(\mathrm{z}_h) - \widetilde{g}(x)| + |u(x) - u(\mathrm{z}_h)| \leq (2\varepsilon)^\alpha |\widetilde{g}|_{C^{0,\alpha}(\overline{\Omega})} + (2\varepsilon)^\alpha |u|_{C^{0,\alpha}(\overline{\Omega})} \lesssim \varepsilon^\alpha, \quad (4.1)$$

where we used **BC**.2. This shows that the boundary condition we enforce for the discrete problem induces an error of no more than $\mathcal{O}(\varepsilon^\alpha)$.

Let us, for convenience, provide here some notation. Given a function $w : \Omega \to \mathbb{R}$, we define its local Lipschitz constant at $x \in \Omega$ as

$$L(w, x) = \lim_{r \downarrow 0} \operatorname{Lip}(w, B_r(x)), \qquad \operatorname{Lip}(w, E) = \sup_{x,y \in E : x \neq y} \frac{|w(x) - w(y)|}{|x - y|}. \quad (4.2)$$

We also define the operators S_ε^+, S_ε^- as

$$S_\varepsilon^+ w(x) = \frac{1}{\varepsilon}\left(\max_{x' \in \overline{B}_\varepsilon(x)} w(x') - w(x)\right), \quad S_\varepsilon^- w(x) = \frac{1}{\varepsilon}\left(w(x) - \min_{x' \in \overline{B}_\varepsilon(x)} w(x')\right). \quad (4.3)$$

We are now ready to prove rates of convergence. We split our discussions depending on how **RHS**.2 is fulfilled.

4.1 *Convergence rates for the inhomogeneous problem*

THEOREM 4.1 (error estimate: inhomogeneous problem). Let u be the viscosity solution of (1.6) and $u_{\mathfrak{h}}$ be the solution of (2.4). Under our running assumptions, suppose that **RHS**.2a holds, and that $\mathfrak{h}$ is sufficiently small, and such that

$$\frac{(\varepsilon\theta)^\alpha + h^\alpha}{\varepsilon^2},$$

is small enough. Then

$$\|u - u_{\mathfrak{h}}\|_{L^\infty(\Omega_h)} \lesssim \varepsilon^\alpha + \frac{\theta^\alpha}{\varepsilon^{2-\alpha}} + \frac{h^\alpha}{\varepsilon^2},$$

where the implied constant depends on the dimension d, the domain Ω, the shape regularity of the mesh $\mathcal{T}_h$, $\min_{x\in\overline{\Omega}} |f(x)|$ and the $C^{0,\alpha}$ norms of the data f, g and the solution u.

Proof. For convenience, we assume that $\min_{x\in\overline{\Omega}} f(x) > 0$. The proof for the case when the right hand is strictly negative follows in a similar way.

Let u be the solution to the continuous problem (1.6). To derive error estimates, we consider the function u^ε defined as

$$u^\varepsilon(x) = \max_{y\in\overline{B}_\varepsilon(x)} u(y), \qquad x \in \Omega^{(\varepsilon)}. \tag{4.4}$$

For $\mathrm{z}_h \in \mathcal{N}^I_{h,\varepsilon}$, let the points $y \in \overline{B}_\varepsilon(\mathrm{z}_h)$ and $y' \in \overline{B}_\varepsilon(y)$ be defined by

$$u(y) = u^\varepsilon(\mathrm{z}_h), \qquad u(y') = u^\varepsilon(y).$$

Since $\mathrm{z}_h \in \mathcal{N}^I_{h,\varepsilon} = \mathcal{N}_h \cap \Omega^{(2\varepsilon)}$, we have $y, y' \in \Omega$.

Noticing that $\mathrm{z}_h \in \overline{B}_\varepsilon(x)$ for any $x \in \overline{B}_\varepsilon(\mathrm{z}_h)$, we have

$$\min_{x\in\overline{B}_\varepsilon(\mathrm{z}_h)} u^\varepsilon(x) \geq u(\mathrm{z}_h)$$

from the definition (4.4) of u^ε. According to Armstrong & Smart (2012, Lemma 5.2), u is locally Lipschitz and

$$S^+_\varepsilon u(\mathrm{z}_h) - \frac{\varepsilon}{2} f^\varepsilon(\mathrm{z}_h) \leq L(u, y) \leq S^+_\varepsilon u(y) + \frac{\varepsilon}{2} f^\varepsilon(y).$$

This implies that

$$\varepsilon S^-_\varepsilon u^\varepsilon(\mathrm{z}_h) = u^\varepsilon(\mathrm{z}_h) - \min_{x\in\overline{B}_\varepsilon(\mathrm{z}_h)} u^\varepsilon(x) \leq u(y) - u(\mathrm{z}_h) = \varepsilon S^+_\varepsilon u(\mathrm{z}_h) \leq \varepsilon \left(L(u, y) + \frac{\varepsilon}{2} f^\varepsilon(\mathrm{z}_h) \right),$$

and

$$\varepsilon S_\varepsilon^+ u^\varepsilon(\mathrm{z}_h) = \max_{x\in\overline{B}_\varepsilon(\mathrm{z}_h)} u^\varepsilon(x) - u^\varepsilon(\mathrm{z}_h) \geq u^\varepsilon(y) - u(y) = \varepsilon S_\varepsilon^+ u(y) \geq \varepsilon\left(L(u,y) - \frac{\varepsilon}{2} f^\varepsilon(y)\right).$$

This implies that

$$S_\varepsilon^- u^\varepsilon(\mathrm{z}_h) - S_\varepsilon^+ u^\varepsilon(\mathrm{z}_h) \leq \frac{\varepsilon}{2}\left(f^\varepsilon(\mathrm{z}_h) + f^\varepsilon(y)\right) \leq \varepsilon f(\mathrm{z}_h) + C\varepsilon^{1+\alpha}|f|_{C^{0,\alpha}(\overline{B}_{2\varepsilon}(\mathrm{z}_h))}. \tag{4.5}$$

Since $\mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h) \subset \overline{B}_\varepsilon(\mathrm{z}_h)$, we have

$$\begin{aligned}
&\left(S_{\mathfrak{h}}^- u^\varepsilon(\mathrm{z}_h) - S_{\mathfrak{h}}^+ u^\varepsilon(\mathrm{z}_h)\right) - \left(S_\varepsilon^- u^\varepsilon(\mathrm{z}_h) - S_\varepsilon^+ u^\varepsilon(\mathrm{z}_h)\right)\\
&= \frac{1}{\varepsilon}\left(\max_{x\in\overline{B}_\varepsilon(\mathrm{z}_h)} u^\varepsilon(x) - \max_{x\in\mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)} u^\varepsilon(x) - \min_{x\in\mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)} u^\varepsilon(x) + \min_{x\in\overline{B}_\varepsilon(\mathrm{z}_h)} u^\varepsilon(x)\right)\\
&\leq \frac{1}{\varepsilon}\left(\max_{x\in\overline{B}_\varepsilon(\mathrm{z}_h)} u^\varepsilon(x) - \max_{x\in\mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)} u^\varepsilon(x)\right).
\end{aligned} \tag{4.6}$$

To control the right-hand side, we notice that

$$\overline{B}_{2\varepsilon}(\mathrm{z}_h) = \bigcup_{y\in\partial\overline{B}_\varepsilon(\mathrm{z}_h)} \overline{B}_\varepsilon(y), \qquad \max_{x\in\overline{B}_\varepsilon(\mathrm{z}_h)} u^\varepsilon(x) = \max_{x\in\overline{B}_{2\varepsilon}(\mathrm{z}_h)} u(x) = \max_{y\in\partial\overline{B}_\varepsilon(\mathrm{z}_h)}\max_{x\in\overline{B}_\varepsilon(y)} u(x),$$

which guarantees that for some $\mathrm{z}_h + \varepsilon\mathbf{v} \in \partial\overline{B}_\varepsilon(\mathrm{z}_h)$ with $|\mathbf{v}| = 1$,

$$\max_{x\in\overline{B}_\varepsilon(\mathrm{z}_h)} u^\varepsilon(x) = \max_{x\in\overline{B}_\varepsilon(\mathrm{z}_h+\varepsilon\mathbf{v})} u(x) = u^\varepsilon(\mathrm{z}_h + \varepsilon\mathbf{v}).$$

By definition of $\mathbb{S}_\theta$, there exists $\mathbf{v}_\theta \in \mathbb{S}_\theta$ such that $|\mathbf{v}_\theta - \mathbf{v}| \leq \theta$. Since $\mathrm{z}_h + \varepsilon\mathbf{v}_\theta \in \mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)$,

$$\max_{x\in\overline{B}_\varepsilon(\mathrm{z}_h)} u^\varepsilon(x) - \max_{x\in\mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)} u^\varepsilon(x) \leq u^\varepsilon(\mathrm{z}_h+\varepsilon\mathbf{v}) - u^\varepsilon(\mathrm{z}_h+\varepsilon\mathbf{v}_\theta) \lesssim (\varepsilon\theta)^\alpha |u|_{C^{0,\alpha}(\overline{B}_{2\varepsilon}(\mathrm{z}_h))}. \tag{4.7}$$

Recall that

$$-\Delta_{\infty,\mathfrak{h}}^\diamond w(\mathrm{z}_h) = -\frac{1}{\varepsilon}\left(S_{\mathfrak{h}}^+ \mathcal{I}_h w(\mathrm{z}_h) - S_{\mathfrak{h}}^- \mathcal{I}_h w(\mathrm{z}_h)\right),$$

we also have

$$-\Delta_{\infty,\mathfrak{h}}^\diamond u^\varepsilon(\mathrm{z}_h) - \frac{1}{\varepsilon}\left(S_{\mathfrak{h}}^- u^\varepsilon(\mathrm{z}_h) - S_{\mathfrak{h}}^+ u^\varepsilon(\mathrm{z}_h)\right) \lesssim \frac{1}{\varepsilon^2}\max_{x\in\overline{B}_\varepsilon(\mathrm{z}_h))}\left(u^\varepsilon(x) - \mathcal{I}_h u^\varepsilon(x)\right) \lesssim \frac{h^\alpha}{\varepsilon^2}|u|_{C^{0,\alpha}(\overline{B}_{3\varepsilon}(\mathrm{z}_h))}.$$

Combine now (4.5)–(4.6)–(4.7) to obtain

$$-\Delta^{\diamond}_{\infty,\mathfrak{h}} u^{\varepsilon}(z_h) \le f(z_h) + C\varepsilon^{\alpha}|f|_{C^{0,\alpha}(\overline{B}_{2\varepsilon}(z_h))} + C\frac{(\varepsilon\theta)^{\alpha}}{\varepsilon^2}|u|_{C^{0,\alpha}(\overline{B}_{2\varepsilon}(z_h))} + C\frac{h^{\alpha}}{\varepsilon^2}|u|_{C^{0,\alpha}(\overline{B}_{3\varepsilon}(z_h))},$$

leading to

$$-\Delta^{\diamond}_{\infty,\mathfrak{h}} u^{\varepsilon}(z_h) \le f(z_h) + C\varepsilon^{\alpha}|f|_{C^{0,\alpha}(\overline{\Omega})} + C\left(\frac{\theta^{\alpha}}{\varepsilon^{2-\alpha}} + \frac{h^{\alpha}}{\varepsilon^2}\right)|u|_{C^{0,\alpha}(\overline{\Omega})}.$$

Choose

$$I_1 = \frac{C}{\min_{x\in\overline{\Omega}} f(x)}\left(\varepsilon^{\alpha}|f|_{C^{0,\alpha}(\overline{\Omega})} + \left(\frac{\theta^{\alpha}}{\varepsilon^{2-\alpha}} + \frac{h^{\alpha}}{\varepsilon^2}\right)|u|_{C^{0,\alpha}(\overline{\Omega})}\right). \tag{4.8}$$

Observe that, for sufficiently small $\mathfrak{h} = (h, \varepsilon, \theta)$, we have $I_1 < 1$. Consequently,

$$\begin{aligned}-\Delta^{\diamond}_{\infty,\mathfrak{h}}[(1-I_1)u^{\varepsilon}](z_h) &\le (1-I_1)\left(f(z_h) + C\varepsilon^{\alpha}|f|_{C^{0,\alpha}(\overline{\Omega})} + C\left(\frac{\theta^{\alpha}}{\varepsilon^{2-\alpha}} + \frac{h^{\alpha}}{\varepsilon^2}\right)|u|_{C^{0,\alpha}(\overline{\Omega})}\right)\\ &\le f(z_h) = -\Delta^{\diamond}_{\infty,\mathfrak{h}} u_{\mathfrak{h}}(z_h).\end{aligned}$$

By the discrete comparison principle of Theorem 3.2, we have, for all $z_h \in \mathcal{N}^I_{h,\varepsilon}$,

$$\begin{aligned}u_{\mathfrak{h}}(z_h) &\ge (1-I_1)u^{\varepsilon}(z_h) + \min_{z_h'\in\mathcal{N}^b_{h,\varepsilon}}[\widetilde{g}(z_h') - (1-I_1)u^{\varepsilon}(z_h')]\\ &\ge u^{\varepsilon}(z_h) - I_1\max_{x\in\overline{\Omega}} u(x) + I_1\min_{z_h'\in\mathcal{N}^b_{h,\varepsilon}} u^{\varepsilon}(z_h') + \min_{z_h'\in\mathcal{N}^b_{h,\varepsilon}}(\widetilde{g}(z_h') - u^{\varepsilon}(z_h'))\\ &\ge u(z_h) - I_1\operatorname{osc}_{\Omega} u - \left(|\widetilde{g}|_{C^{0,\alpha}(\overline{\Omega})} + |u|_{C^{0,\alpha}(\overline{\Omega})}\right)(2\varepsilon)^{\alpha},\end{aligned}$$

where $\operatorname{osc}_{\Omega} u = \max_{x\in\overline{\Omega}} u(x) - \min_{x\in\overline{\Omega}} u(x)$ and we used (4.1) in the derivation. We can obtain a similar upper bound for $u_{\mathfrak{h}}$, and hence

$$\|u - u_{\mathfrak{h}}\|_{L^{\infty}(\Omega_h)} \lesssim \varepsilon^{\alpha} + \frac{\theta^{\alpha}}{\varepsilon^{2-\alpha}} + \frac{h^{\alpha}}{\varepsilon^2}.$$

This proves the result. □

By properly scaling all discretization parameters, the previous result allows us to obtain explicit rates of convergence. The following result is the first explicit rate of convergence for problems involving the normalized ∞-Laplacian.

COROLLARY 4.2 (convergence rates ($f \not\equiv 0$)). Under the same setting and assumptions as in Theorem 4.1, we can choose

$$\varepsilon \approx h^{\frac{\alpha}{2+\alpha}}, \qquad \varepsilon\theta \approx h,$$

to obtain

$$\|u - u_{\mathfrak{h}}\|_{L^\infty(\Omega_h)} \lesssim h^{\frac{\alpha^2}{2+\alpha}}.$$

In particular, if $u \in C^{0,1}(\overline{\Omega})$, we have the error estimate

$$\|u - u_{\mathfrak{h}}\|_{L^\infty(\Omega_h)} \lesssim h^{1/3}.$$

Proof. Use the indicated scalings. □

REMARK 4.3 (improved rates for $u \in C^{1,\alpha}(\overline{\Omega})$). We point out that if, for $\alpha \in (0, 1]$, we have $u \in C^{1,\alpha}(\overline{\Omega})$ then better convergence rates can be derived. Namely,

$$\|u - u_{\mathfrak{h}}\|_{L^\infty(\Omega_h)} \lesssim \varepsilon^\alpha + \frac{\theta^\alpha}{\varepsilon^{2-\alpha}} + \frac{h^{1+\alpha}}{\varepsilon^2}.$$

To see this, it suffices to improve the proof of Theorem 4.1 with the following estimate

$$\max_{x \in \overline{B}_\varepsilon(z_h)} u^\varepsilon(x) - \mathcal{I}_h u^\varepsilon(x) \lesssim h^{1+\alpha} |u|_{C^{1,\alpha}(\overline{B}_{3\varepsilon}(z_h))}. \tag{4.9}$$

Indeed, once (4.9) is shown, in the proof of Theorem 4.1, we have

$$-\Delta^\diamond_{\infty,\mathfrak{h}} u^\varepsilon(z_h) - \frac{1}{\varepsilon}\left(S^-_{\mathfrak{h}} u^\varepsilon(z_h) - S^+_{\mathfrak{h}} u^\varepsilon(z_h)\right) \lesssim \frac{h^{1+\alpha}}{\varepsilon^2} |u|_{C^{1,\alpha}(\overline{B}_{3\varepsilon}(z_h))},$$

which, under the assumptions of Theorem 4.4 and $u \in C^{1,\alpha}(\overline{\Omega})$, implies the claimed rates.

Let us then prove (4.9). Consider a point $x \in B_\varepsilon(z_h)$ and assume that T is a simplex containing x. Thus,

$$\sum_{z'_h \in \mathcal{N}_h \cap T} \widehat{\varphi}_{z'_h}(x) z'_h = x, \quad \mathcal{I}_h u^\varepsilon(x) = \sum_{z'_h \in \mathcal{N}_h \cap T} \widehat{\varphi}_{z'_h}(x)\, u^\varepsilon(z'_h).$$

Let $y \in \overline{B}_\varepsilon(x)$ such that $u(y) = u^\varepsilon(x)$. Since $|y - x| \le \varepsilon$, we have $z'_h + (y - x) \in B_\varepsilon(z'_h)$ and thus

$$u^\varepsilon(z'_h) \ge u(z'_h + (y - x)).$$

Observe now that

$$\sum_{z'_h \in \mathcal{N}_h \cap T} \widehat{\varphi}_{z'_h}(x) \left(z'_h + (y - x)\right) = x + (y - x) = y.$$

This, together with the fact that $u \in C^{1,\alpha}(\overline{\Omega})$, then yields

$$\left| \sum_{z_h' \in \mathcal{N}_h \cap T} \widehat{\varphi}_{z_h'}(x) u(z_h' + (y - x)) - u(y) \right| \lesssim h^{1+\alpha} |u|_{C^{1,\alpha}(\overline{B}_h(y))} \le h^{1+\alpha} |u|_{C^{1,\alpha}(\overline{B}_{3\varepsilon}(z_h))}.$$

Therefore,

$$\mathcal{I}_h u^\varepsilon(x) = \sum_{z_h' \in \mathcal{N}_h \cap T} \widehat{\varphi}_{z_h'}(x)\, u^\varepsilon(z_h') \ge \sum_{z_h' \in \mathcal{N}_h \cap T} \widehat{\varphi}_{z_h'}(x)\, u(z_h' + (y - x)) \ge u(y) - Ch^{1+\alpha} |u|_{C^{1,\alpha}(\overline{B}_{3\varepsilon}(z_h))},$$

which proves (4.9). We comment that in general we only have $u^\varepsilon \in C^{0,1}(\overline{\Omega})$ even if u is smooth. However, intuitively, one can think that u^ε only makes u more convex, so that in one direction we are still able to obtain (4.9) with $h^{1+\alpha}$ instead of h.

4.2 *Convergence rates for the homogeneous problem*

Let us now obtain convergence rates in the case that Assumption **RHS**.2b holds. First of all, to understand why different arguments are needed in this case, consider two problems of the form (1.6), with the same boundary condition, but different right-hand sides:

$$-\Delta_\infty^\diamond u_1(x) = f_1(x), \qquad -\Delta_\infty^\diamond u_2(x) = f_2(x), \qquad \forall x \in \Omega.$$

If $f_1(x), f_2(x) \ge f_0 > 0$ for all $x \in \overline{\Omega}$, then we have the stability result

$$\|u_1 - u_2\|_{L^\infty(\Omega)} \lesssim \frac{1}{f_0} \|f_1 - f_2\|_{L^\infty(\Omega)};$$

see Armstrong & Smart (2012, Proposition 6.3). However, in general, if f_1, f_2 are not bounded away from zero, it is not possible to have a stability estimate of the form

$$\|u_1 - u_2\|_{L^\infty(\Omega)} \lesssim \|f_1 - f_2\|_{L^\infty(\Omega)}.$$

In fact, if this could be proved, then the error estimates that we prove below can be improved. The optimal case would also give $\mathcal{O}(h^{1/3})$.

Theorem 4.4 (error estimate: homogeneous problem). Let u be the viscosity solution of (1.6) and $u_{\mathfrak{h}}$ be the solution of (2.4). Under our running assumptions suppose that **RHS**.2b and, for all $h > 0$, **M**.1 holds. If $\mathfrak{h}$ is sufficiently small, such that $\varepsilon - \varepsilon\theta - h > 0$, and

$$(2h + \varepsilon\theta)/\varepsilon^2,$$

can be made sufficiently small, then we have

$$\|u - u_{\mathfrak{h}}\|_{L^\infty(\Omega_h)} \lesssim \varepsilon^\alpha + \frac{\sqrt{2h + \varepsilon\theta}}{\varepsilon},$$

where the implied constant depends only on the dimension d, the domain Ω, the shape regularity of the family $\{\mathcal{T}_h\}_{h>0}$ and the $C^{0,\alpha}$ norms of the data g and the solution u.

Proof. Similar to the proof of Theorem 4.1, we aim to construct a discrete subsolution. To achieve this, first, we employ the approximation introduced in Crandall *et al.* (2007, Theorem 2.1); see also the proof of Theorem 2.19 in Armstrong & Smart (2012). This result asserts, for $\gamma > 0$, the existence of $v_\gamma \in C(\overline{\Omega}) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ with the following properties:

$$-\Delta^{\diamond}_{\infty} v_\gamma \le 0, \text{ in } \Omega, \quad v_\gamma = g, \text{ on } \partial\Omega, \quad L(v_\gamma, \cdot) \ge \gamma, \text{ in } \Omega, \quad u - \gamma\,\mathrm{diam}(\Omega) \le v_\gamma \le u, \text{ in } \Omega,$$

where $L(v_\gamma, \cdot)$ is the local Lipschitz constant defined in (4.2). Let $m = \min_{x\in\overline{\Omega}} u(x)$, and $M = \max_{x\in\overline{\Omega}} u(x)$. We have

$$m_\gamma = m - \gamma\,\mathrm{diam}(\Omega) \le v_\gamma \le M.$$

Define the function

$$v_\gamma^{\varepsilon+h}(x) = \max_{y\in\overline{B}_{\varepsilon+h}(x)} v_\gamma(y), \quad x \in \Omega^{(\varepsilon+h)},$$

and a discrete subsolution of the form $u_{\mathfrak{h}}^- = q(v_\gamma^{\varepsilon+h})$, where $q(t) = t + a(t-m_\gamma)^2$ is a quadratic function with a parameter $a > 0$ to be determined. Nevertheless, since $a > 0$, this implies that $q'(t) \ge 0$ for $t \in [m_\gamma, M]$. Consider a vertex $\mathrm{z}_h \in \mathcal{N}^I_{h,\varepsilon}$, and let $y \in \overline{B}_{\varepsilon+h}(\mathrm{z}_h,)$ be such that

$$v_\gamma(y) = v_\gamma^{\varepsilon+h}(\mathrm{z}_h) = \max_{x\in\overline{B}_{\varepsilon+h}(\mathrm{z}_h)} v_\gamma(x).$$

Recall that

$$S^-_{\mathfrak{h}}\mathcal{I}_h w(\mathrm{z}_h) = \frac{1}{\varepsilon}\left(w(\mathrm{z}_h) - \min_{x\in\mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)} \mathcal{I}_h w(x)\right).$$

Consider a point $x \in \mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)$ and an element $T \ni x$. Then for any vertex $\mathrm{z}'_h \in T \cap \mathcal{N}_h$, we have

$$|\mathrm{z}_h - \mathrm{z}'_h| \le |\mathrm{z}_h - x| + |x - \mathrm{z}'_h| \le \varepsilon + h,$$

which implies that

$$v_\gamma^{\varepsilon+h}(\mathrm{z}'_h) \ge v_\gamma(\mathrm{z}_h).$$

Using the monotonicity of the quadratic function q, we see that

$$S^-_{\mathfrak{h}}\mathcal{I}_h u^-_{\mathfrak{h}}(\mathrm{z}_h) = \frac{1}{\varepsilon}\left(q\big(v_\gamma^{\varepsilon+h}(\mathrm{z}_h)\big) - \min_{x\in\mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)} \mathcal{I}_h q\big(v_\gamma^{\varepsilon+h}(x)\big)\right) \le \frac{1}{\varepsilon}\big(q(v_\gamma(y)) - q(v_\gamma(\mathrm{z}_h))\big).$$

For convenience, let $b = S^+_{\varepsilon+h} v_\gamma(\mathrm{z}_h)$ where the operator is defined in (4.3), then

$$v_\gamma(y) = v_\gamma(\mathrm{z}_h) + (\varepsilon + h)b,$$

and thus

$$\varepsilon S^-_{\mathfrak{h}} \mathcal{I}_h u^-_{\mathfrak{h}}(\mathrm{z}_h) \le q(v_\gamma(y)) - q(v_\gamma(y) - (\varepsilon + h)b). \tag{4.10}$$

By Armstrong & Smart (2012, Lemma 5.2), we have

$$S^+_\delta v_\gamma(y) \ge L(v_\gamma, y) \ge b = S^+_{\varepsilon+h} v_\gamma(\mathrm{z}_h) \ge L(v_\gamma, \mathrm{z}_h) \ge \gamma,$$

for any $\delta > 0$. Let $\delta = \varepsilon - \varepsilon\theta - h > 0$ and $y' \in \overline{B}_\delta(y)$ such that

$$v_\gamma(y') = \max_{x \in \overline{B}_\delta(y)} v_\gamma(x) = v_\gamma(y) + \delta S^+_\delta v_\gamma(y) \ge v_\gamma(y) + \delta b.$$

Since

$$|y' - \mathrm{z}_h| \le |y' - y| + |y - \mathrm{z}_h| \le \varepsilon - \varepsilon\theta - h + \varepsilon + h = 2\varepsilon - \varepsilon\theta,$$

we claim that there exists $\mathbf{v}_\theta \in \mathbb{S}_\theta$ such that $x' = \mathrm{z}_h + \varepsilon \mathbf{v}_\theta \in \mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)$ satisfies $|x' - y'| \le \varepsilon$. To see this, let $\mathbf{v} \in \mathbb{S}$ be such that $y' - \mathrm{z}_h = |y' - \mathrm{z}_h|\mathbf{v}$ and choose $\mathbf{v}_\theta \in \mathbb{S}_\theta$ such that $|\mathbf{v}_\theta - \mathbf{v}| \le \theta$. If $|y' - \mathrm{z}_h| \ge \varepsilon\theta$, then we have

$$|x' - y'| \le |x' - (\mathrm{z}_h + \varepsilon\mathbf{v})| + |(\mathrm{z}_h + \varepsilon\mathbf{v}) - y'| \le \varepsilon|\mathbf{v}_\theta - \mathbf{v}| + |\varepsilon - |y' - \mathrm{z}_h|| \le \varepsilon\theta + (\varepsilon - \varepsilon\theta) = \varepsilon.$$

On the other hand, if $|y' - \mathrm{z}_h| < \varepsilon\theta$, then

$$\begin{aligned} |x' - y'| &\le |x' - (\mathrm{z}_h + |y' - \mathrm{z}_h|\mathbf{v}_\theta)| + |(\mathrm{z}_h + |y' - \mathrm{z}_h|\mathbf{v}_\theta) - y'| \\ &\le |\varepsilon - |y' - \mathrm{z}_h|| + |y' - \mathrm{z}_h||\mathbf{v}_\theta - \mathbf{v}| \le \varepsilon - |y' - \mathrm{z}_h| + |y' - \mathrm{z}_h|\theta \le \varepsilon, \end{aligned}$$

because of $\theta \le 1$.

Let now $T \in \mathcal{T}_h$ be such that $x' \in T$. Then for any vertex $\mathrm{z}'_h \in T$,

$$|y' - \mathrm{z}'_h| \le |y' - x'| + |x' - \mathrm{z}'_h| \le \varepsilon + h \le \varepsilon + h.$$

This implies that

$$v^{\varepsilon+h}_\gamma(\mathrm{z}'_h) \ge v_\gamma(y') \ge v_\gamma(y) + \delta b.$$

By the monotonicity of q, we also have

$$q(v^{\varepsilon+h}_\gamma(\mathrm{z}'_h)) \ge q(v_\gamma(y) + \delta b),$$

which implies that

$$u_{\mathfrak{h}}^-(x') = \mathcal{I}_h q\big(v_\gamma^{\varepsilon+h}(x')\big) \geq q(v_\gamma(y) + \delta b).$$

Hence,

$$\varepsilon S_{\mathfrak{h}}^+ \mathcal{I}_h u_{\mathfrak{h}}^-(\mathrm{z}_h) \geq u_{\mathfrak{h}}^-(x') - u_{\mathfrak{h}}^-(\mathrm{z}_h) \geq q(v_\gamma(y) + \delta b) - q(v_\gamma(y)). \tag{4.11}$$

Let us now choose the parameters a and γ to guarantee that

$$-\Delta_{\infty,\mathfrak{h}}^\diamond u_{\mathfrak{h}}^-(\mathrm{z}_h) = -\frac{1}{\varepsilon}\big(S_{\mathfrak{h}}^+ \mathcal{I}_h u_{\mathfrak{h}}^-(\mathrm{z}_h) - S_{\mathfrak{h}}^- \mathcal{I}_h u_{\mathfrak{h}}^-(\mathrm{z}_h)\big) \leq 0,$$

i.e., $S_{\mathfrak{h}}^+ \mathcal{I}_h u_{\mathfrak{h}}^-(\mathrm{z}_h) \geq S_{\mathfrak{h}}^- \mathcal{I}_h u_{\mathfrak{h}}^-(\mathrm{z}_h)$. Upon combining (4.10) and (4.11), we see that, to achieve this, it suffices to ensure

$$q(v_\gamma(y) + \delta b) - q(v_\gamma(y)) \geq q(v_\gamma(y)) - q(v_\gamma(y) - (\varepsilon + h)b).$$

Since

$$\begin{aligned}
q(v_\gamma(y) + \delta b) - q(v_\gamma(y)) &= \delta b(1 + 2a(v_\gamma(y) - m_\gamma)) + a(\delta b)^2, \\
q(v_\gamma(y)) - q(v_\gamma(y) - (\varepsilon + h)b) &= (\varepsilon + h)b(1 + 2a(v_\gamma(y) - m_\gamma)) - a((\varepsilon + h)b)^2,
\end{aligned}$$

it is enough to require

$$\begin{aligned}
0 &\leq a(\delta b)^2 + a((\varepsilon + h)b)^2 - (\varepsilon + h)b(1 + 2a(v_\gamma(y) - m_\gamma)) + \delta b(1 + 2a(v_\gamma(y) - m_\gamma)) \\
&= ab^2(\delta^2 + (\varepsilon + h)^2) - (2h + \varepsilon\theta)b(1 + 2a(v_\gamma(y) - m_\gamma)).
\end{aligned}$$

Since $b \geq \gamma$ and

$$ab(\delta^2 + (\varepsilon + h)^2) - (2h + \varepsilon\theta)(1 + 2a(v_\gamma(y) - m_\gamma)) \geq ab\varepsilon^2 - (2h + \varepsilon\theta)(1 + 2a(M - m + \gamma)),$$

then we always have $-\Delta_{\infty,\mathfrak{h}}^\diamond u_{\mathfrak{h}}^-(\mathrm{z}_h) \leq 0$ provided that

$$\frac{1}{3}a\gamma\varepsilon^2 \geq 2h + \varepsilon\theta, \quad \frac{1}{3}a\gamma\varepsilon^2 \geq (2h + \varepsilon\theta)2a(M - m), \quad \frac{1}{3}a\gamma\varepsilon^2 \geq (2h + \varepsilon\theta)2a\gamma. \tag{4.12}$$

This can be achieved by requiring that $\beta = (2h + \varepsilon\theta)/\varepsilon^2$ is small enough and satisfies

$$2(M - m)\sqrt{3\beta} \leq 1, \quad \beta \leq 1/6.$$

Choosing $a = \gamma = \sqrt{3\beta}$ guarantees (4.12) and thus $-\Delta^{\diamond}_{\infty,\mathfrak{h}} u^-_{\mathfrak{h}}(\mathrm{z}_h) \le 0$. Consequently, $u^-_{\mathfrak{h}}$ is a subsolution and we can apply the comparison principle of Theorem 3.3 to obtain

$$u_{\mathfrak{h}}(\mathrm{z}_h) \ge u^-_{\mathfrak{h}}(\mathrm{z}_h) + \min_{\mathrm{z}'_h \in \mathcal{N}^b_{h,\varepsilon}} (u_{\mathfrak{h}}(\mathrm{z}'_h) - u^-_{\mathfrak{h}}(\mathrm{z}'_h)), \quad \forall \mathrm{z}_h \in \mathcal{N}^I_{h,\varepsilon}. \tag{4.13}$$

Notice that, for any vertex $\mathrm{z}_h \in \mathcal{N}_h$,

$$\begin{aligned}
|u^-_{\mathfrak{h}}(\mathrm{z}_h) - u(\mathrm{z}_h)| &\le \left|q\big(v^{\varepsilon+h}_\gamma(\mathrm{z}_h)\big) - v^{\varepsilon+h}_\gamma(\mathrm{z}_h)\right| + \left|v^{\varepsilon+h}_\gamma(\mathrm{z}_h) - u^{\varepsilon+h}(\mathrm{z}_h)\right| + \left|u^{\varepsilon+h}(\mathrm{z}_h) - u(\mathrm{z}_h)\right| \\
&\le a(M-m+\gamma)^2 + \gamma\,\mathrm{diam}(\Omega) + (\varepsilon+h)^\alpha |u|_{C^{0,\alpha}(\overline{\Omega})} \lesssim a(M-m)^2 + \gamma\,\mathrm{diam}(\Omega) + (\varepsilon+h)^\alpha |u|_{C^{0,\alpha}(\overline{\Omega})} \\
&\lesssim \big((M-m)^2 + \mathrm{diam}(\Omega)\big)\sqrt{\beta} + \varepsilon^\alpha |u|_{C^{0,\alpha}(\overline{\Omega})}.
\end{aligned}$$

Now, if $\mathrm{z}'_h \in \mathcal{N}^b_{h,\varepsilon}$, we recall (4.1) to obtain

$$\begin{aligned}
\left|u_{\mathfrak{h}}(\mathrm{z}'_h) - u^-_{\mathfrak{h}}(\mathrm{z}'_h)\right| &= \left|\widetilde{g}(\mathrm{z}'_h) - u^-_{\mathfrak{h}}(\mathrm{z}'_h)\right| \le \left|\widetilde{g}(\mathrm{z}'_h) - u(\mathrm{z}'_h)\right| + \left|u(\mathrm{z}'_h) - u^-_{\mathfrak{h}}(\mathrm{z}'_h)\right| \\
&\lesssim \big((M-m)^2 + \mathrm{diam}(\Omega)\big)\sqrt{\beta} + \varepsilon^\alpha \big(|\widetilde{g}_\varepsilon|_{C^{0,\alpha}(\overline{\Omega})} + |u|_{C^{0,\alpha}(\overline{\Omega})}\big).
\end{aligned}$$

Plugging the inequalities above into (4.13) implies

$$u_{\mathfrak{h}}(\mathrm{z}_h) \ge u(\mathrm{z}_h) - C\big((M-m)^2 + \mathrm{diam}(\Omega)\big)\sqrt{\beta} - C\varepsilon^\alpha \big(|\widetilde{g}_\varepsilon|_{C^{0,\alpha}(\overline{\Omega})} + |u|_{C^{0,\alpha}(\overline{\Omega})}\big).$$

We can obtain a similar upper bound for $u_{\mathfrak{h}}$, and hence we conclude that

$$\|u - u_{\mathfrak{h}}\|_{L^\infty(\Omega_h)} \lesssim \quad \varepsilon^\alpha \big(|g|_{C^{0,\alpha}(\overline{\Omega})} + |u|_{C^{0,\alpha}(\overline{\Omega})}\big) + \frac{\sqrt{2h+\varepsilon\theta}}{\varepsilon}\left(\left(\max_{x\in\overline{\Omega}} u(x) - \min_{x\in\overline{\Omega}} u(x)\right)^2 + \mathrm{diam}(\Omega)\right),$$

as we intended to show. □

Once again, we can properly scale the parameters to obtain explicit rates of convergence.

Corollary 4.5 (convergence rates ($f \equiv 0$)). Under the same assumptions of Theorem 4.4, we can choose

$$\varepsilon \approx h^{\frac{1}{2(\alpha+1)}}, \qquad \varepsilon\theta \approx h,$$

to obtain

$$\|u - u_{\mathfrak{h}}\|_{L^\infty(\Omega_h)} \lesssim h^{\frac{\alpha}{2(\alpha+1)}}.$$

In particular, if $u \in C^{0,1}(\overline{\Omega})$, we have the error estimate

$$\|u - u_{\mathfrak{h}}\|_{L^\infty(\Omega_h)} \lesssim h^{1/4}.$$

Proof. It immediately follows from Theorem 4.4 when using the prescribed scalings. □

REMARK 4.6 (nonvanishing gradient). We point out that if we have $\inf_{x\in\Omega} |Du(x)| = \delta > 0$, then the PDE is not degenerate and the error estimate of Theorem 4.4 can be improved. To be more specific, in this setting, the proof of Theorem 4.4 does not need to introduce the perturbation v_γ as we automatically have $L(u, \cdot) \geq \delta$. Consequently, one just needs to choose $a = 3\beta/\delta$ to guarantee (4.12) and this implies the estimate

$$\|u - u_{\mathfrak{h}}\|_{L^\infty(\Omega_h)} \lesssim \varepsilon^\alpha + \frac{\theta^\alpha}{\varepsilon^{2-\alpha}} + \frac{h^\alpha}{\varepsilon^2},$$

which coincides with the one obtained in Theorem 4.1. The error estimates can also be improved for some specific problems, where the dependence on the right-hand side f can be better quantified; see Smart (2010, Proposition 3.1.4).

REMARK 4.7 (improved rates for $u \in C^{1,\alpha}(\overline{\Omega})$). Arguing as in Remark 4.3, improved rates of convergence could be obtained in the case there is $\alpha \in (0, 1]$ for which $u \in C^{1,\alpha}(\overline{\Omega})$. Since, at the time of this writing, no up to the boundary $C^{1,\alpha}$ estimates are available for the solution of this problem (see Savin, 2005; Evans & Savin, 2008; Crasta & Fragalà, 2016, for the state of the art), we shall not dwell on this further.

5. Extensions and variations

In this section, we consider some possible variations on the type of problems that our approach can handle. To avoid unnecessary repetition, and to keep the presentation short, we will mostly state the main results and sketch their proofs, as many of the arguments are repetitions or slight variants of what we have presented before.

5.1 *An alternative way to handle boundary conditions*

Thus far, our discussion assumes that an extension of the boundary data g is available to us. In this section, we explore how to circumvent this, at the expense of having a variable coarse scale, which may somewhat complicate the implementation. We shall, locally for this section, overload the notation and say that $\mathcal{N}_h^b = \mathcal{N}_h \cap \partial\Omega_h$. Correspondingly, the set of interior nodes is $\mathcal{N}_h^I = \mathcal{N}_h \cap \Omega_h$. Thus, $g(\mathrm{z}_h)$ is well-defined for $\mathrm{z}_h \in \mathcal{N}_h^b$. Since $\mathrm{z}_h + \varepsilon \mathbf{v}_\theta$ might be outside Ω_h for points $\mathrm{z}_h \in \mathcal{N}_h^I$ satisfying $\mathrm{dist}(\mathrm{z}_h, \partial\Omega_h) < \varepsilon$, the discrete operator in (2.3) might not be well-defined. For this reason, for every interior node $\mathrm{z}_h \in \mathcal{N}_h^I$, we must define the local coarse scale as

$$\varepsilon(\mathrm{z}_h) = \max\{t \leq \varepsilon : \mathrm{z}_h + t\mathbf{v}_\theta \in \overline{\Omega}_h \ \forall \mathbf{v}_\theta \in \mathbb{S}_\theta\}.$$

We immediately point out that

$$\varepsilon(\mathrm{z}_h) \geq \min\{\varepsilon, \mathrm{dist}(\mathrm{z}_h, \partial\Omega_h)\}.$$

In addition, from this definition, it follows that $z_h + \varepsilon(z_h)\mathbf{v}_\theta \in \overline{\Omega}_h$ for any $\mathbf{v}_\theta \in \mathbb{S}_\theta$. We replace the fixed coarse scale ε with $\varepsilon(z_h)$ in the definitions of $\mathcal{N}_\mathfrak{h}$ and $S^\pm_\mathfrak{h} w$ to define, in a local abuse of notation,

$$\mathcal{N}_\mathfrak{h}(z_h) = \{z_h\} \cup \{z_h + \varepsilon(z_h)\mathbf{v}_\theta : \mathbf{v}_\theta \in \mathbb{S}_\theta\},$$
$$S^+_\mathfrak{h} w(z_h) = \frac{1}{\varepsilon(z_h)}\left(\max_{x\in\mathcal{N}_\mathfrak{h}(z_h)} w(x) - w(z_h)\right), \qquad S^-_\mathfrak{h} w(z_h) = \frac{1}{\varepsilon(z_h)}\left(w(z_h) - \min_{x\in\mathcal{N}_\mathfrak{h}(z_h)} w(x)\right).$$

Therefore, the discrete operator now reads

$$-\Delta^\diamond_{\infty,\mathfrak{h}} w(z_h) = -\frac{1}{\varepsilon(z_h)}\left(S^+_\mathfrak{h}\mathcal{I}_h w(z_h) - S^-_\mathfrak{h}\mathcal{I}_h w(z_h)\right), \qquad \forall z_h \in \mathcal{N}^I_h, \tag{5.1}$$

and the discrete solution $u_\mathfrak{h} \in \mathbb{V}_h$ satisfies

$$-\Delta^\diamond_{\infty,\mathfrak{h}} u_\mathfrak{h}(z_h) = f(z_h), \quad \forall z_h \in \mathcal{N}^I_h, \qquad u_\mathfrak{h}(z_h) = g(z_h), \quad \forall z_h \in \mathcal{N}^b_h. \tag{5.2}$$

One notices that for the nodes that satisfy $z_h \in \mathcal{N}^I_{h,\varepsilon}$, as defined in Section 2.2, the operator remains the same, and thus the modifications are restricted to a region close to the boundary. This new scheme might offer an advantage in the sense that it does not require us to have a layer of boundary nodes and does not need an extension $\widetilde{g}$ of g. However, it also adds some difficulties in the implementation and analysis. Specifically, with regards to implementation, the search directions needed to compute $S^\pm_\mathfrak{h}$ need to change at every point. With regard to analysis, at least with our methodologies, we obtain slightly weaker results compared to the original scheme in Section 2.3; see our discussion below regarding rates of convergence.

It is easy to check that Lemma 3.1, Theorem 3.2, Theorem 3.3, Lemma 3.5 and Lemma 3.6 still follow from the same arguments. In Lemma 3.9, for any point $x \in \Omega$, as $\mathfrak{h}_j \to (0,0,0)$, the changes we make near the boundary would have no impact at nodes close to x. Consequently, Lemma 3.9 is also valid for the modified scheme.

We next prove Lemma 3.12 for the modified scheme.

Proof of Lemma 3.12 *for the modified scheme.* Following from the same arguments as in the original proof, it only remains to prove, under the assumption that the boundary condition g is smooth, that $\overline{u}(x_0) \le g(x_0)$ for any $x_0 \in \partial\Omega$. Let L be the Lipschitz constant of g. If the exterior ball condition is satisfied at x_0, then we consider $\varphi(x) = a + b|x - x_0| - \frac{c}{2}|x - x_0|^2$ and

$$\widetilde{\varphi}\left(z_{h_j}\right) = \begin{cases} \varphi\left(z_{h_j}\right) + 2b\varepsilon_j + 2C_\sigma b h_j + c\varepsilon_j^2, & z_{h_j} \in \mathcal{N}^I_{h_j}, \\ \varphi\left(z_{h_j}\right), & z_{h_j} \in \mathcal{N}^b_{h_j}, \end{cases} \tag{5.3}$$

where $C_\sigma > 0$ is a constant that depends on the shape regularity of the mesh $\mathcal{T}_h$. We claim that for j large enough,

$$-\Delta^\diamond_{\infty,\mathfrak{h}_j}\widetilde{\varphi}\left(z_{h_j}\right) \ge \sup_{x\in\Omega} f(x) \ge f\left(z_{h_j}\right), \tag{5.4}$$

for all $z_{h_j} \in \mathcal{N}_{h_j}^I$ provided that

$$c = \sup_{x\in\Omega} f(x) + 1, \qquad b = b_0 + c\left(\operatorname{diam}(\Omega) + \frac{R}{2}\right).$$

If $\varepsilon_j\left(z_{h_j}\right) = \varepsilon_j$, (5.4) follows immediately from the original proof.

Consider now the case when $\varepsilon_j\left(z_{h_j}\right) < \varepsilon_j$. In this scenario, there exists $\mathbf{v}_\theta \in \mathbb{S}_\theta$ such that

$$z_{h_j} + \varepsilon_j\left(z_{h_j}\right)\mathbf{v}_\theta \in \mathcal{N}_{h_j}^b.$$

Since $|\varphi|_{C^{0,1}(\overline{\Omega})} \le b$, we have

$$\left|\mathcal{I}_{h_j}\varphi(x) - \varphi(x)\right| \le C_\sigma b h_j$$

for a constant C_σ that depends only on the shape regularity of the mesh $\mathcal{T}_h$. This implies that

$$\min_{x\in\mathcal{N}_{\mathfrak{h}_j}(z_{h_j})} \mathcal{I}_{h_j}\widetilde{\varphi}(x) \le \mathcal{I}_{h_j}\widetilde{\varphi}\left(z_{h_j} + \varepsilon_j\left(z_{h_j}\right)\mathbf{v}_\theta\right) = \mathcal{I}_{h_j}\varphi\left(z_{h_j} + \varepsilon_j\left(z_{h_j}\right)\mathbf{v}_\theta\right) \le \varphi\left(z_{h_j}\right) + b\varepsilon_j + C_\sigma b h_j$$

and

$$\max_{x\in\mathcal{N}_{\mathfrak{h}_j}(z_{h_j})} \mathcal{I}_{h_j}\widetilde{\varphi}(x) \le \max_{x\in\mathcal{N}_{\mathfrak{h}_j}(z_{h_j})} \mathcal{I}_{h_j}\varphi(x) + 2b\varepsilon_j + 2C_\sigma b h_j + c\varepsilon_j^2 \le \varphi\left(z_{h_j}\right) + 3b\varepsilon_j + 3C_\sigma b h_j + c\varepsilon_j^2.$$

Therefore,

$$-\Delta_{\infty,\mathfrak{h}_j}^{\diamond}\widetilde{\varphi}\left(z_{h_j}\right) \ge \frac{c\varepsilon_j^2}{\varepsilon_j\left(z_{h_j}\right)^2} \ge c,$$

which proves the claim (5.4). Since for fixed b,

$$\lim_{j\to\infty} 2C_\sigma b\varepsilon_j + c\varepsilon_j^2 = 0,$$

the same procedure in the original proof implies that $\overline{u}(x_0) \le g(x_0)$ for any $x_0 \in \partial\Omega$ satisfying the exterior ball condition. Finally, arguing in the same way, we can use this to prove $\overline{u}(x_0) \le g(x_0)$ for any $x_0 \in \partial\Omega$ and this finishes the proof. □

Now, we can use the same argument to combine the results for the modified scheme together and prove its convergence as in Theorem 3.13.

As for the error analysis, we shall assume that:

D.1. The domain Ω has a Lipschitz boundary.

With this at hand, one notices that the discrete solution $u_{\mathfrak{h}}$ to the modified scheme (5.2) is also the solution to (2.4) where $\widetilde{g}$ is replaced by $u_{\mathfrak{h}}$. Owing to Remark 3.4, this implies that

$$\max_{z_h \in \mathcal{N}_h} |u_{\mathfrak{h}}(z_h) - u'_{\mathfrak{h}}(z_h)| \leq \max_{z_h \in \mathcal{N}^b_{h,\varepsilon}} |u_{\mathfrak{h}}(z_h) - \widetilde{g}(z_h)|,$$

where $u'_{\mathfrak{h}}$ is the discrete solution to the original scheme (2.4). Therefore, to obtain error estimates, one only needs to control

$$\max_{z_h \in \mathcal{N}^b_{h,\varepsilon}} |u_{\mathfrak{h}}(z_h) - \widetilde{g}(z_h)|.$$

Since $\Omega^{(h)} \subset \Omega_h$, for any $z'_h \in \mathcal{N}^b_{h,\varepsilon}$, there exists $x_0 \in \mathcal{N}^b_h$ such that

$$|x_0 - z'_h| \leq 2\varepsilon + 2h \leq 4\varepsilon,$$

provided that h is sufficiently small. For $g \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0, 1]$, we consider the following barrier function

$$\varphi(x) = q\left(|x - x_0|\right) = a + b|x - x_0|^\alpha - \frac{c}{2}|x - x_0|^2, \tag{5.5}$$

where a, b, c are to be determined. This function is similar to the barrier function in the proof of Lemma 3.12, but centered at $x_0 \in \mathcal{N}^b_h$ instead of $z_0 \notin \overline{\Omega}$. As a consequence, this barrier function φ is not smooth in Ω. Let

$$a = g(x_0),\ b = |g|_{C^{0,\alpha}(\partial\Omega)} + \frac{c}{\alpha}\operatorname{diam}(\Omega)^{2-\alpha},\ c = \sup_{x\in\Omega} f(x) + 1.$$

For $t \in [0, \operatorname{diam}(\Omega)]$, we have

$$q'(t) = b\alpha t^{\alpha-1} - ct \geq 0, \quad |g|_{C^{0,\alpha}(\partial\Omega)} t^\alpha \leq q(t) - q(0) \leq bt^\alpha$$

and thus for any $x \in \mathcal{N}^b_h$

$$\varphi(x) \geq \varphi(x_0) + |g|_{C^{0,\alpha}(\partial\Omega)}|x - x_0|^\alpha = g(x_0) + |g|_{C^{0,\alpha}(\partial\Omega)}|x - x_0|^\alpha \geq g(x). \tag{5.6}$$

Similar to (5.3), we let

$$\widetilde{\varphi}(z_h) = \begin{cases} \varphi(z_h) + 2b\varepsilon^\alpha + 2C_\sigma bh^\alpha + c\varepsilon^2, & z_h \in \mathcal{N}^I_h, \\ \varphi(z_h), & z_h \in \mathcal{N}^b_h, \end{cases}$$

where $C_\sigma > 0$ is a constant that depends on the shape regularity of the mesh $\mathcal{T}_h$, satisfying

$$\left|\mathcal{I}_{h_j}\varphi(x) - \varphi(x)\right| \leq C_\sigma h^\alpha |\varphi|_{C^{0,\alpha}(\overline{\Omega})} \leq C_\sigma bh^\alpha. \tag{5.7}$$

We claim that for $((\varepsilon\theta)^\alpha + h^\alpha)/\varepsilon^2$ small enough,

$$-\Delta^{\diamond}_{\infty,\mathfrak{h}}\widetilde{\varphi}(z_h) \geq \sup_{x\in\Omega} f(x) \geq f(z_h),$$

for any $z_h \in \mathcal{N}_h^I$ and thus $\widetilde{\varphi}$ is a discrete supersolution. To this aim, we first argue in the same way as (5.3) to prove that

$$-\Delta^{\diamond}_{\infty,\mathfrak{h}}\widetilde{\varphi}(z_h) \geq c \geq \sup_{x\in\Omega} f(x)$$

for any z_h with $\varepsilon(z_h) < \varepsilon$. For z_h with $\varepsilon(z_h) = \varepsilon$, we first observe that, since **D**.1 holds, then, for h sufficiently small, Ω_h is uniformly Lipschitz. The definition of $\varepsilon(z_h)$ then implies that

$$|z_h - x_0| \geq \varepsilon - C\varepsilon\theta \tag{5.8}$$

for some constant $C > 0$ depending on the Lipschitz constant of Ω_h. Since $q'(t) \geq 0$, we have

$$\max_{x\in\mathcal{N}_{\mathfrak{h}}(z_h)} \varphi(x) = q\left(\max_{x\in\mathcal{N}_{\mathfrak{h}}(z_h)} |x - x_0|\right).$$

By the definition $\mathbb{S}_\theta$, there exists $\mathbf{w}_\theta \in \mathbb{S}_\theta$ such that

$$\left|\frac{z_h - x_0}{|z_h - x_0|} - \mathbf{w}_\theta\right| \leq \theta \leq 1,$$

implying that $|z_h - x_0 + \varepsilon\mathbf{w}_\theta| > |z_h - x_0|$. Furthermore, it is easy to see that

$$\arg\max_{\mathbf{v}\in\mathbb{S}_\theta} |z_h - x_0 + \varepsilon\mathbf{v}| = \arg\min_{\mathbf{v}\in\mathbb{S}_\theta}\left|\frac{z_h - x_0}{|z_h - x_0|} - \mathbf{v}\right|.$$

Therefore, we can choose $\mathbf{v}_\theta \in \mathbb{S}_\theta$ such that

$$e_\theta = \left|\frac{z_h - x_0}{|z_h - x_0|} - \mathbf{v}_\theta\right| \leq \theta, \quad \varphi(z_h + \varepsilon\mathbf{v}_\theta) = \max_{x\in\mathcal{N}_{\mathfrak{h}}(z_h)} \varphi(x).$$

Using the expression of φ, we obtain

$$\begin{aligned}
\frac{1}{\varepsilon^2}\left(2\varphi(z_h) - \max_{x\in\mathcal{N}_{\mathfrak{h}}(z_h)}\varphi(x) - \min_{x\in\mathcal{N}_{\mathfrak{h}}(z_h)}\varphi(x)\right) &\geq \frac{1}{\varepsilon^2}\left(2\varphi(z_h) - \varphi(z_h + \varepsilon\mathbf{v}_\theta) - \varphi(z_h + \varepsilon\mathbf{v}_\theta)\right) \\
&= c + \frac{b}{\varepsilon^2}\left(2|z_h - x_0|^\alpha - |z_h + \varepsilon\mathbf{v}_\theta - x_0|^\alpha - |z_h + \varepsilon\mathbf{v}_\theta - x_0|^\alpha\right).
\end{aligned} \tag{5.9}$$

Since

$$
\begin{aligned}
|z_h + \varepsilon \mathbf{v}_\theta - x_0| &= \sqrt{|z_h - x_0|^2 + \varepsilon^2 + 2\varepsilon (z_h - x_0)^\intercal \mathbf{v}_\theta} \\
&= \sqrt{|z_h - x_0|^2 + \varepsilon^2 + \varepsilon |z_h - x_0| \left(2 - \left|\frac{z_h - x_0}{|z_h - x_0|} - \mathbf{v}_\theta\right|^2\right)} = \sqrt{(|z_h - x_0| + \varepsilon)^2 - \varepsilon |z_h - x_0| e_\theta^2}, \\
|z_h - \varepsilon \mathbf{v}_\theta - x_0| &= \sqrt{(|z_h - x_0| - \varepsilon)^2 + \varepsilon |z_h - x_0| e_\theta^2},
\end{aligned}
$$

we derive from the concavity, for $t \geq 0$, of $t \mapsto t^{\alpha/2}$ that

$$
\begin{aligned}
|z_h + \varepsilon \mathbf{v}_\theta - x_0|^\alpha &\leq (|z_h - x_0| + \varepsilon)^\alpha - \frac{\alpha}{2} (|z_h - x_0| + \varepsilon)^{\alpha - 2} \varepsilon |z_h - x_0| e_\theta^2, \\
|z_h - \varepsilon \mathbf{v}_\theta - x_0|^\alpha &\leq \big||z_h - x_0| - \varepsilon\big|^\alpha + \frac{\alpha}{2} \big||z_h - x_0| - \varepsilon\big|^{\alpha - 2} \varepsilon |z_h - x_0| e_\theta^2.
\end{aligned}
$$

This implies that for z_h satisfying $|z_h - x_0| \geq 2\varepsilon$, we have

$$
\begin{aligned}
&2|z_h - x_0|^\alpha - |z_h + \varepsilon \mathbf{v}_\theta - x_0|^\alpha - |z_h + \varepsilon \mathbf{v}_\theta - x_0|^\alpha \\
&\geq 2|z_h - x_0|^\alpha - (|z_h - x_0| + \varepsilon)^\alpha - (|z_h - x_0| - \varepsilon)^\alpha \\
&\quad - \frac{\alpha}{2} \varepsilon |z_h - x_0| e_\theta^2 \left(\left(|z_h - x_0| - \varepsilon\right)^{\alpha - 2} - \left(|z_h - x_0| + \varepsilon\right)^{\alpha - 2}\right) \\
&\geq -\frac{\alpha}{2} \varepsilon |z_h - x_0| e_\theta^2 \left(\left(|z_h - x_0| - \varepsilon\right)^{\alpha - 2} - \left(|z_h - x_0| + \varepsilon\right)^{\alpha - 2}\right) \\
&\geq -\frac{\alpha}{2} \varepsilon |z_h - x_0| \theta^2 (2 - \alpha) \left(|z_h - x_0| - \varepsilon\right)^{\alpha - 3} (2\varepsilon) \\
&\geq -\varepsilon^2 |z_h - x_0| \theta^2 \left(|z_h - x_0| - \varepsilon\right)^{\alpha - 3} \geq -2\theta^2 \varepsilon^\alpha.
\end{aligned}
$$

In addition, for $|z_h - x_0| < 2\varepsilon$, we have

$$
\sqrt{|z_h + \varepsilon \mathbf{v}_\theta - x_0|} \leq |z_h - x_0| + \varepsilon, \ \sqrt{z_h - \varepsilon \mathbf{v}_\theta - x_0|} \leq \big||z_h - x_0| - \varepsilon\big| + \sqrt{\varepsilon |z_h - x_0|} e_\theta.
$$

Therefore, if $\varepsilon \leq |z_h - x_0| < 2\varepsilon$, using the concavity, for $t \geq 0$, of $t \mapsto t^\alpha$, we have

$$
\begin{aligned}
&2|z_h - x_0|^\alpha - |z_h + \varepsilon \mathbf{v}_\theta - x_0|^\alpha - |z_h + \varepsilon \mathbf{v}_\theta - x_0|^\alpha \\
&\geq 2|z_h - x_0|^\alpha - (|z_h - x_0| + \varepsilon)^\alpha - (|z_h - x_0| - \varepsilon)^\alpha - \left(\varepsilon |z_h - x_0|\right)^{\alpha/2} e_\theta^\alpha \\
&\geq - \left(\varepsilon |z_h - x_0|\right)^{\alpha/2} e_\theta^\alpha \geq -\left(2\varepsilon^2\right)^{\alpha/2} \theta^\alpha \geq -\sqrt{2} (\varepsilon \theta)^\alpha.
\end{aligned}
$$

And, if $|z_h - x_0| < \varepsilon$, we recall (5.8) and obtain

$$
|z_h + \varepsilon \mathbf{v}_\theta - x_0| \leq 2\varepsilon, \ |z_h - \varepsilon \mathbf{v}_\theta - x_0| \leq \big||z_h - x_0| - \varepsilon\big| + \sqrt{\varepsilon |z_h - x_0|} e_\theta \leq C\varepsilon\theta + \varepsilon\theta,
$$

and thus

$$\begin{aligned} &2|\mathrm{z}_h - x_0|^\alpha - |\mathrm{z}_h + \varepsilon \mathbf{v}_\theta - x_0|^\alpha - |\mathrm{z}_h + \varepsilon \mathbf{v}_\theta - x_0|^\alpha \\ &\quad \geq 2((1-C\theta)\varepsilon)^\alpha - (2\varepsilon)^\alpha - (C+1)^\alpha(\varepsilon\theta)^\alpha \gtrsim -(\varepsilon\theta)^\alpha. \end{aligned}$$

Combining the results together, we have that for all z_h that

$$2|\mathrm{z}_h - x_0|^\alpha - |\mathrm{z}_h + \varepsilon \mathbf{v}_\theta - x_0|^\alpha - |\mathrm{z}_h + \varepsilon \mathbf{v}_\theta - x_0|^\alpha \gtrsim -(\varepsilon\theta)^\alpha$$

and thus in (5.9)

$$\frac{1}{\varepsilon^2}\left(2\varphi(\mathrm{z}_h) - \max_{x\in\mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)} \varphi(x) - \min_{x\in\mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)} \varphi(x)\right) \geq c - C\frac{b}{\varepsilon^2}(\varepsilon\theta)^\alpha, \tag{5.10}$$

for some constant $C > 0$ independent of $\mathfrak{h}$. We now combine this estimate with (5.7) to obtain that

$$-\Delta^\diamond_{\infty,\mathfrak{h}}\varphi(\mathrm{z}_h) = \frac{1}{\varepsilon^2}\left(2\varphi(\mathrm{z}_h) - \max_{x\in\mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)} \mathcal{I}_h\varphi(x) - \min_{x\in\mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)} \mathcal{I}_h\varphi(x)\right) \geq c - C\frac{b}{\varepsilon^2}(\varepsilon\theta)^\alpha - \frac{2C_\sigma b h^\alpha}{\varepsilon^2},$$

which implies that for $((\varepsilon\theta)^\alpha + h^\alpha)/\varepsilon^2$ small enough

$$-\Delta^\diamond_{\infty,\mathfrak{h}}\varphi(\mathrm{z}_h) \geq c - 1 = \sup_{x\in\Omega} f(x).$$

This finishes the proof of $-\Delta^\diamond_{\infty,\mathfrak{h}}\widetilde{\varphi}(\mathrm{z}_h) \geq f(\mathrm{z}_h)$ for any $\mathrm{z}_h \in \mathcal{N}_h^I$ because of $-\Delta^\diamond_{\infty,\mathfrak{h}}\widetilde{\varphi}(\mathrm{z}_h) \geq -\Delta^\diamond_{\infty,\mathfrak{h}}\varphi(\mathrm{z}_h)$. Thanks to (5.6), we have $\widetilde{\varphi}(\mathrm{z}_h) \geq g(\mathrm{z}_h)$ for any $\mathrm{z}_h \in \mathcal{N}_h^b$ as well, leading to $u_{\mathfrak{h}} \leq \mathcal{I}_h\widetilde{\varphi}$ because of comparison principles. Consequently,

$$\begin{aligned} u_{\mathfrak{h}}(\mathrm{z}_h') &\leq g(x_0) + b|x_0 - \mathrm{z}_h'|^\alpha + 2b\varepsilon^\alpha + 2C_\sigma b h^\alpha + c\varepsilon^2 \\ &\leq g(x_0) + \left(|g|_{C^{0,\alpha}(\partial\Omega)} + \frac{c}{\alpha}\operatorname{diam}(\Omega)^{2-\alpha}\right)(4\varepsilon)^\alpha + 2b\varepsilon^\alpha + 2C_\sigma b h^\alpha + c\varepsilon^2, \end{aligned}$$

which implies

$$u_{\mathfrak{h}}(\mathrm{z}_h') - g(\mathrm{z}_h') \lesssim \varepsilon^\alpha + h^\alpha \lesssim \varepsilon^\alpha.$$

By symmetry, we can also prove the other direction to obtain that

$$\max_{\mathrm{z}_h\in\mathcal{N}_{h,\varepsilon}^b} |u_{\mathfrak{h}}(\mathrm{z}_h) - \widetilde{g}(\mathrm{z}_h)| \lesssim \varepsilon^\alpha.$$

Now, following the proofs of Theorem 5.9 and Theorem 5.8, we finally obtain the error estimates for the modified scheme (5.2).

THEOREM 5.1 (error estimate: inhomogeneous problem and modified scheme). Let the function u be the viscosity solution of (1.6) and $u_{\mathfrak{h}}$ be the solution of (5.2). Under our running assumptions suppose that **RHS**.2a and **D**.1 hold, that $\mathfrak{h}$ is sufficiently small, and such that

$$\frac{(\varepsilon\theta)^\alpha + h^\alpha}{\varepsilon^2},$$

is small enough. Then

$$\|u - u_{\mathfrak{h}}\|_{L^\infty(\Omega_h)} \lesssim \varepsilon^\alpha + \frac{\theta^\alpha}{\varepsilon^{2-\alpha}} + \frac{h^\alpha}{\varepsilon^2},$$

where the implied constant depends on the dimension d, the domain Ω, the shape regularity of the mesh $\mathcal{T}_h$, $\min_{x\in\overline{\Omega}} |f(x)|$ and the $C^{0,\alpha}$ norms of the data f, g and the solution u.

THEOREM 5.2 (error estimate: homogeneous problem and modified scheme). Let u be the viscosity solution of (1.6) and $u_{\mathfrak{h}}$ be the solution of (5.2). Under our running assumptions suppose that **RHS**.2b, **D**.1 and, for all $h > 0$, **M**.1 hold. If $\mathfrak{h}$ is sufficiently small, such that $\varepsilon - \varepsilon\theta - h > 0$, and

$$(2h + \varepsilon\theta)/\varepsilon^2,$$

can be made sufficiently small, then we have

$$\|u - u_{\mathfrak{h}}\|_{L^\infty(\Omega_h)} \lesssim \varepsilon^\alpha + \frac{\sqrt{2h + \varepsilon\theta}}{\varepsilon},$$

where the implied constant depends only on the dimension d, the domain Ω, the shape regularity of the family $\{\mathcal{T}_h\}_{h>0}$ and the $C^{0,\alpha}$ norms of the data g and the solution u.

5.2 *Obstacle problems*

In Manfredi *et al.* (2015), see also Blanc & Rossi (2019, Section 5.2), the following variant of the game described in Section 1.2 was proposed and analyzed. In addition to the already stated rules, Player I has the option, after every move, to stop the game. In this case, Player II will pay Player I the amount

$$Q = \chi(x_{k+1}).$$

Here $\chi : \overline{\Omega} \to \mathbb{R}$ is a given function with $\chi < g$ on $\partial\Omega$. It is shown that, in this case, the game also has a value u^ε that satisfies

$$u^\varepsilon(x) \geq \chi(x), \qquad 2u^\varepsilon(x) - \left(\sup_{y\in\overline{B}_\varepsilon(x)\cap\bar{\Omega}} u^\varepsilon(y) + \inf_{y\in\overline{B}_\varepsilon(x)\cap\bar{\Omega}} u^\varepsilon(y) \right) \geq \varepsilon^2 f(x),$$

and, if the first inequality is strict, the second is an equality. In the limit $\varepsilon \downarrow 0$, we obtain the following obstacle problem

$$\mathbb{T}[u;f,\chi](x) = 0, \text{ in } \Omega, \qquad u = g, \text{ on } \partial\Omega; \tag{5.11}$$

where

$$\mathbb{T}[w;f,\chi](x) = \min\left\{-\Delta_\infty^\diamond w(x) - f(x), w(x) - \chi(x)\right\}.$$

For convenience of presentation, given $w \in C(\overline{\Omega})$, we define the coincidence, noncoincidence sets and free boundary to be

$$\mathcal{C}(w) = \{x \in \Omega : w(x) = \chi(x)\}, \quad \Omega^+(w) = \Omega \setminus \mathcal{C}(w), \quad \Gamma(w) = \partial\Omega^+(w) \cap \Omega.$$

Notice that, if u is sufficiently smooth, (5.11) is equivalent to the following complementarity conditions: $u(x) \geq \chi(x)$ for all $x \in \Omega$, and,

$$-\Delta_\infty^\diamond u(x) = f(x), \text{ in } \Omega^+(u), \quad -\Delta_\infty^\diamond u(x) \geq f(x), \text{ in } \mathcal{C}(u).$$

To analyze the problem, we introduce an assumption on the obstacle.

OBS.1. The obstacle $\chi \in C(\overline{\Omega})$ is such that $\chi < g$ on $\partial\Omega$.

Under assumptions **RHS**.1, **BC**.1 and **OBS**.1, the existence of viscosity solutions to (5.11) was established in Manfredi *et al.* (2015). In addition, if **RHS**.2, then uniqueness is guaranteed. Finally, it is shown that if $f \in C^{0,1}(\overline{\Omega})$, $g \in C^{0,1}(\partial\Omega)$ and $\chi \in C^2(\overline{\Omega})$, then $u \in C^{0,1}(\overline{\Omega})$. Further properties of the solution to (5.11) were explored in Rossi *et al.* (2015).

5.2.1 *Comparison principle.* In addition to the properties described above, to analyze numerical schemes, we will need a comparison principle for semicontinuous sub- and supersolutions of (5.11). Since we were not able to locate one in the form that is suited for our purposes, we present one here.

We first point out that a comparison principle with strict inequality, like the one we give below, follows directly from Crandall *et al.* (1992, Section 5.C).

LEMMA 5.3 (strict comparison). Let $\delta_w, \delta_v \in \mathbb{R}$. Assume that $w \in \mathrm{USC}(\overline{\Omega})$ and $v \in \mathrm{LSC}(\overline{\Omega})$ satisfy, in the viscosity sense,

$$\mathbb{T}[w;f,\chi] \leq \delta_w < \delta_v \leq \mathbb{T}[v;f,\chi] \quad \text{in } \Omega.$$

Further assume that f, χ satisfy Assumptions **RHS**.1 and **OBS**.1. If $w \leq v$ on $\partial\Omega$, then

$$w \leq v \quad \text{in } \Omega.$$

With the aid of Lemma 5.3, we can now prove a comparison principle.

THEOREM 5.4 (comparison). Assume that $w \in \mathrm{USC}(\overline{\Omega})$, $v \in \mathrm{LSC}(\overline{\Omega})$ satisfy, in the viscosity sense,

$$\mathbb{T}[w;f,\chi] \leq 0 \leq \mathbb{T}[v;f,\chi] \quad \text{in } \Omega.$$

Assume, in addition, that f, χ satisfy Assumptions **RHS**.1–2 and **OBS**.1. If $w \leq v$ on $\partial\Omega$, then

$$w \leq v \quad \text{in } \Omega.$$

Proof. Based on how Assumption **RHS**.2 is satisfied, we split the proof into three cases.

Case 1: $\sup\{f(x) : x \in \Omega\} \leq -f_0 < 0$. Since $w \in \mathrm{USC}(\overline{\Omega})$, we have that w is bounded from above. Let M be an upper bound of w. For any $\beta > 0$, consider the function

$$w_\beta(x) = (1+\beta)w(x) - (M+1)\beta \leq w(x) - \beta.$$

From $\mathbb{T}[w;f,\chi] \leq 0$, we argue that w_β satisfies, in the viscosity sense,

$$\mathbb{T}[w_\beta;f,\chi] \leq \max\{-\beta f_0, -\beta\} < 0.$$

Let us, at least formally, provide an explanation. If $x \in \mathcal{C}(w)$, we have

$$\mathbb{T}[w_\beta;f,\chi](x) \leq w_\beta(x) - \chi(x) \leq w(x) - \beta - \chi(x) = -\beta.$$

On the other hand, for $x \in \Omega^+(w)$, we have

$$\mathbb{T}[w_\beta;f,\chi](x) \leq -\Delta_\infty^\diamond w_\beta(x) - f(x) = -(1+\beta)\Delta_\infty^\diamond w(x) - f(x) = -\beta\Delta_\infty^\diamond w(x) \leq -\beta f_0.$$

Now, since we have $\mathbb{T}[w_\beta;f,\chi] \leq \max\{-\beta f_0, -\beta\} < 0 \leq \mathbb{T}[v;f,\chi]$, we can apply Lemma 5.3 to obtain that

$$w_\beta(x) = (1+\beta)w(x) - (M+1)\beta \leq v(x), \qquad \forall x \in \Omega, \quad \forall \beta > 0.$$

Letting $\beta \downarrow 0$ finishes the proof.

Case 2: $\inf\{f(x) : x \in \Omega\} \geq f_0 > 0$. Similar to the Case 1, let m be a lower bound of v and, for $\beta > 0$, define the function

$$v_\beta(x) = (1+\beta)v(x) - (m-1)\beta \geq v(x) + \beta.$$

It is easy to verify that

$$\mathbb{T}[v_\beta;f,\chi] \geq \min\{\beta f_0, \beta\} > 0$$

in the viscosity sense. Thus, we use Lemma 5.3 to arrive at

$$v_\beta(x) = (1+\beta)v(x) - (m-1) \geq w(x), \qquad \forall x \in \Omega, \quad \forall \beta > 0.$$

Letting $\beta \downarrow 0$ finishes the proof.

Case 3: $f \equiv 0$. As usual, this case is more delicate. Let $M = \max_{x\in\overline{\Omega}} \chi(x)$, we first claim that it is enough to prove the result in the case that v satisfies $v(x) \leq M$. To see this, define

$$\check{v}(x) = \min\{v(x), M\} \leq M.$$

This function verifies $\mathbb{T}[\check{v};0,\chi] \geq 0$ if $\mathbb{T}[v;0,\chi] \geq 0$. Consequently, if we know the comparison holds between w and $\check{v}$, then we can conclude that $w \leq \check{v} \leq v$.

Let us assume then that v is bounded from above. Owing to the fact that $v \in \mathrm{LSC}(\overline{\Omega})$, it is also bounded from below and as a consequence $v \in L^\infty(\Omega)$. Now, since $0 \leq \mathbb{T}[v; 0, \chi]$, we must have that $v(x) \geq \chi(x)$ and $-\Delta_\infty^\diamond v(x) \geq 0$ in the viscosity sense. In other words, we have that v is bounded and infinity superharmonic. This implies that $v \in \mathrm{LSC}(\overline{\Omega}) \cap C(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$; see Lindqvist & Manfredi (1997, Lemma 4.1).

We now wish to construct, in the spirit of Crandall *et al.* (2007, Theorem 2.1), an approximation to this function that is not flat, i.e., its local Lipschitz constant (defined in (4.2)) is bounded from below. The main difficulty here, and the reason why we cannot simply invoke Crandall *et al.* (2007, Theorem 2.1) is that v may not be continuous near the boundary. Nevertheless, we claim that, for any $\gamma > 0$, there exists $v_\gamma \in \mathrm{LSC}(\overline{\Omega}) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$, which satisfies following properties:

$$-\Delta_\infty^\diamond v_\gamma \geq 0, \text{ in } \Omega, \quad v_\gamma \geq v, \text{ on } \partial\Omega, \quad L(v_\gamma, \cdot) \geq \gamma, \text{ in } \Omega, \quad v \leq v_\gamma \leq v + C\gamma, \text{ in } \Omega, \tag{5.12}$$

where the constant $C > 0$ depends only on Ω.

Let us sketch the construction of this v_γ. For $\gamma > 0$, define

$$V_\gamma = \{x \in \Omega : L(v, x) < \gamma\},$$

which, by definition, is open. Write $V_\gamma = \bigcup_i \mathcal{O}_i$ where each $\mathcal{O}_i$ is a connected component of V_γ. On each $\mathcal{O}_i$, the function v is uniformly Lipschitz and, consequently, we can define

$$\widetilde{v}_i(x) = \lim_{\mathcal{O}_i \ni y \to x} v(y), \qquad x \in \overline{\mathcal{O}_i}.$$

Clearly, $\widetilde{v}_i = v$ in $\overline{\mathcal{O}_i} \cap \Omega$, and for $x \in \overline{\mathcal{O}_i} \cap \partial\Omega$, we have $\widetilde{v}_i(x) \geq v(x)$ because of $v \in \mathrm{LSC}(\overline{\Omega})$. We then define the function v_γ through

$$v_\gamma(x) = \begin{cases} \inf_{y \in \partial\mathcal{O}_i} \left[\widetilde{v}_i(y) + \gamma \operatorname{dist}_{\mathcal{O}_i}(x, y)\right], & x \in \mathcal{O}_i, \\ v(x), & x \in \overline{\Omega} \setminus V_\gamma, \end{cases}$$

where by $\operatorname{dist}_{\mathcal{O}_i}(x, y)$ we denote the path distance, relative to $\mathcal{O}_i$, between the points x and y. Clearly, $L(v_\gamma, x) \geq \gamma$ for all $x \in \Omega$. Following the proof in Crandall *et al.* (2007), one can verify all remaining properties in (5.12).

Once we have v_γ, for any $a > 0$, we define the quadratic function

$$q_a(t) = a\left(\|v_\gamma\|^2_{L^\infty(\Omega)} + 1\right) + t - at^2.$$

It is possible to choose a small enough, so that $q_a'(t) = 1 - 2at > 0$ for all $t \in [0, M + C\gamma]$. Consider now the function $q_a(v_\gamma) \in \mathrm{LSC}(\overline{\Omega})$. We have that $q_a(v_\gamma) \geq v_\gamma$, and that

$$\mathbb{T}[q_a(v_\gamma); 0, \chi] \geq \min\{2a\gamma^2, a\} > 0.$$

To see this, let us do a formal calculation, which can be justified with the usual arguments. If v_γ is smooth and with $|\mathrm{D}v_\gamma(x)| = L(v_\gamma, x) \geq \gamma$, we have that

$$\mathrm{D}q_a(v_\gamma) = q_a'(v_\gamma)\mathrm{D}v_\gamma, \quad \mathrm{D}^2 q_a(v_\gamma) = q_a'(v_\gamma)\mathrm{D}^2 v_\gamma + q_a''(v_\gamma)\mathrm{D}v_\gamma \otimes \mathrm{D}v_\gamma,$$

and consequently

$$-\Delta_\infty^\diamond q_a(v_\gamma) = -q_a'(v_\gamma)\Delta_\infty^\diamond v_\gamma - q_a''(v_\gamma)|\mathrm{D}v_\gamma|^2 \geq -q_a''(v_\gamma)|\mathrm{D}v_\gamma|^2 \geq 2a|\mathrm{D}v_\gamma|^2 \geq 2a\gamma^2,$$

where we used $q_a'(v_\gamma) > 0$ and $-\Delta_\infty^\diamond v_\gamma \geq 0$. Therefore, we can apply Lemma 5.3 between w and $q_a(v_\gamma)$ to conclude that, for $a > 0$, but small enough,

$$w(x) \leq q_a(v_\gamma) = a\big(\|v_\gamma\|^2_{L^\infty(\Omega)} + 1\big) + v_\gamma(x) - a v_\gamma(x)^2.$$

Letting $a \downarrow 0$ implies that

$$w(x) \leq v_\gamma(x),$$

for any $\gamma > 0$. Finally, taking $\gamma \downarrow 0$ finishes our proof. □

5.2.2 *The numerical scheme.* Following the notation and constructions of Section 2, the discretization simply reads

$$\mathbb{T}_\mathfrak{h}[u_\mathfrak{h}; f, \chi](\mathrm{z}_h) = 0, \quad \forall \mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^I, \qquad u_\mathfrak{h}(\mathrm{z}_h) = \widetilde{g}(\mathrm{z}_h), \quad \forall \mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^b, \tag{5.13}$$

where

$$\mathbb{T}_\mathfrak{h}[u_\mathfrak{h}; f, \chi](\mathrm{z}_h) = \min\big\{ -\Delta_{\infty,\mathfrak{h}}^\diamond u_\mathfrak{h}(\mathrm{z}_h) - f(\mathrm{z}_h), u_\mathfrak{h}(\mathrm{z}_h) - \chi(\mathrm{z}_h)\big\}, \quad \mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^I. \tag{5.14}$$

Let us now prove a discrete comparison principle for this scheme.

THEOREM 5.5 (discrete comparison). Assume that the obstacle χ satisfies **OBS**.1, and the right-hand side f satisfies **RHS**.2. Moreover, assume that if **RHS**.2b holds, then the mesh $\mathcal{T}_h$ satisfies **M**.1. Let $w_h, v_h \in \mathbb{V}_h$ be such that

$$\mathbb{T}_\mathfrak{h}[w_h; f, \chi](\mathrm{z}_h) \leq 0 \leq \mathbb{T}_\mathfrak{h}[v_h; f, \chi](\mathrm{z}_h), \quad \forall \mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^I, \tag{5.15}$$

and $w_h(\mathrm{z}_h) \leq v_h(\mathrm{z}_h)$ for any $\mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^b$. Then we have

$$w_h(\mathrm{z}_h) \leq v_h(\mathrm{z}_h) \quad \forall \mathrm{z}_h \in \mathcal{N}_h.$$

Proof. The proof proceeds in a similar way to Theorem 3.2 or Theorem 3.3, depending on how **RHS**.2 is satisfied. We assume, for the sake of contradiction, that

$$m = \max_{z_h \in \mathcal{N}_h} \left[w_h(z_h) - v_h(z_h) \right] = \max_{z_h \in \mathcal{N}_{h,\varepsilon}^I} \left[w_h(z_h) - v_h(z_h) \right] > 0.$$

Consider the set of points

$$E = \{ z_h \in \mathcal{N}_h : (w_h - v_h)(z_h) = m \} \subset \mathcal{N}_{h,\varepsilon}^I.$$

Clearly, the set E is nonempty because of the definition of m. For any $z_h \in E$, we have

$$w_h(z_h) - v_h(z_h) \geq w_h(z_h') - v_h(z_h') \quad \forall z_h' \in \mathcal{N}_h. \tag{5.16}$$

By Lemma 3.1, this implies that

$$S_{\mathfrak{h}}^+ w_h(z_h) \leq S_{\mathfrak{h}}^+ v_h(z_h), \qquad S_{\mathfrak{h}}^- w_h(z_h) \geq S_{\mathfrak{h}}^- v_h(z_h),$$

and thus $\mathbb{T}_{\mathfrak{h}}[w_h; f, \chi](z_h) \geq \mathbb{T}_{\mathfrak{h}}[v_h; f, \chi](z_h)$. Combining with (5.15), we see that

$$\mathbb{T}_{\mathfrak{h}}[w_h; f, \chi](z_h) = \mathbb{T}_{\mathfrak{h}}[v_h; f, \chi](z_h).$$

Now we claim that

$$\mathbb{T}_{\mathfrak{h}}[w_h; f, \chi](z_h) = -\Delta_{\infty,\mathfrak{h}}^{\diamond} w_{\mathfrak{h}}(z_h) - f(z_h). \tag{5.17}$$

If this is not the case, then

$$\mathbb{T}_{\mathfrak{h}}[w_h; f, \chi](z_h) = w_h(z_h) - \chi(z_h) = v_h(z_h) + m - \chi(z_h) > v_h(z_h) - \chi(z_h) \geq \mathbb{T}_{\mathfrak{h}}[v_h; f, \chi](z_h),$$

which is a contradiction. From (5.17) then we have

$$-\Delta_{\infty,\mathfrak{h}}^{\diamond} w_h(z_h) \leq f(z_h) \leq -\Delta_{\infty,\mathfrak{h}}^{\diamond} v_h(z_h), \quad \forall z_h \in E.$$

Applying Theorem 3.2 or Theorem 3.3 where $\mathcal{N}_{h,\varepsilon}^I$ is our set E here we obtain the contradiction

$$m \leq \max_{z_h \in \mathcal{N}_h} \left[w_h(z_h) - v_h(z_h) \right] = \max_{z_h \in \mathcal{N}_h \setminus E} \left[w_h(z_h) - v_h(z_h) \right] < m.$$

We point out that the Assumption **M**.1 required in Theorem 3.3 still holds since $E \subset \mathcal{N}_{h,\varepsilon}^I$. This finishes our proof. □

Similar to the discussions in Section 3.2, using the discrete comparison principle, we obtain the following properties for the obstacle problem.

LEMMA 5.6 (existence, uniqueness, stability). Assume **RHS**.1–2, **BC**.1–2 and **OBS**.1. Moreover, if **RHS**.2b holds, suppose that for all $\mathfrak{h}$ we have **M**.1. For any choice of parameters $\mathfrak{h}$, there exists a unique $u_{\mathfrak{h}} \in \mathbb{V}_h$ that solves (5.13). Moreover, this solution satisfies

$$\|u_{\mathfrak{h}}\|_{L^\infty(\Omega_h)} \leq C\|f\|_{L^\infty(\Omega_h)} + \|\chi\|_{L^\infty(\Omega_h)} + \max_{z_h \in \mathcal{N}_{h,\varepsilon}^b} |\widetilde{g}(z_h)|. \tag{5.18}$$

Proof. We can prove the stability result (5.18), which is an analogue of Lemma 3.1:

- The lower bound of $u_{\mathfrak{h}}$ is automatically given by χ.
- The upper bound of $u_{\mathfrak{h}}$ is obtained using the discrete comparison principle of Theorem 5.5 and the barrier function $\varphi(x) = \mathbf{p}^\intercal x - \frac{1}{2}|x|^2 + A$ constructed in the proof of Lemma 3.1. The only modification needed here is to choose the constant A large enough so that $\varphi \geq \chi$.

The existence and uniqueness then follows from the discrete comparison principle and the stability result as in Lemma 3.6. □

The consistency properties proved in Lemma 3.7 and Lemma 3.8 imply the corresponding consistency results of the operator $\mathbb{T}_{\mathfrak{h}}$. Based on this, we have the following convergence result.

THEOREM 5.7 (convergence). Assume that the right-hand side f satisfies Assumptions **RHS**.1–2. Assume that the boundary datum g satisfies Assumptions **BC**.1–2. Assume the obstacle χ satisfies Assumption **OBS**.1. Let $\{\mathfrak{h}_j\}_{j=1}^\infty$ be a sequence of discretization parameters that satisfies (3.25); and, if **RHS**.2b holds, it additionally satisfies **M**.1 for all $j \in \mathbb{N}$. In this framework, we have that the sequence $\{u_{\mathfrak{h}_j} \in \mathbb{V}_{h_j}\}_{j=1}^\infty$ of solutions to (5.13) converges uniformly, as $j \uparrow \infty$, to u, the solution of (5.11).

Proof. We could almost repeat the proofs of Lemma 3.12 and Theorem 3.13. The only minor modification needed is in the proof of the boundary behavior $\overline{u}(x) = g(x)$ for all $x \in \partial\Omega$. To be more specific, in the construction of the discrete supersolution φ, one has to further require that $\varphi(x) \geq \chi(x)$ for any $x \in \Omega$. To this aim, we use a similar argument as in the proof of Lemma 3.12 and assume without loss of generality that χ is smooth. Then the same calculations we did to guarantee (3.29) give a similar sufficient condition needed for φ and thus concludes the proof. □

We obtain the following error estimates as analogues of Theorem 4.1 and Theorem 4.4.

THEOREM 5.8 (error estimate: inhomogeneous problem). Let u be the viscosity solution of (5.11) and $u_{\mathfrak{h}}$ be the solution of (5.13). Under our running assumptions, suppose that **RHS**.2a holds, and that $\mathfrak{h}$ is sufficiently small, and such that

$$\frac{(\varepsilon\theta)^\alpha + h^\alpha}{\varepsilon^2},$$

is small enough. Then

$$\|u - u_{\mathfrak{h}}\|_{L^\infty(\Omega_h)} \lesssim \varepsilon^\alpha + \frac{\theta^\alpha}{\varepsilon^{2-\alpha}} + \frac{h^\alpha}{\varepsilon^2},$$

where the implied constant depends on the dimension d, the domain Ω, the shape regularity of the mesh $\mathcal{T}_h$, $\min_{x\in\overline{\Omega}} |f(x)|$ and the $C^{0,\alpha}$ norms of the data f, g, χ and the solution u.

Proof. For convenience, we assume that $\min_{x\in\overline{\Omega}} f(x) > 0$. To get a lower bound for $u_{\mathfrak{h}}$, we consider

$$u_h^- = (1-I_1)u^\varepsilon + I_1 \min_{z_h\in\mathcal{N}_h} u(z_h) - (2\varepsilon)^\alpha |\widetilde{g}|_{C^{0,\alpha}(\overline{\Omega})} - (3\varepsilon)^\alpha |u|_{C^{0,\alpha}(\overline{\Omega})} - \varepsilon^\alpha |\chi|_{C^{0,\alpha}(\overline{\Omega})},$$

where u^ε is defined in (4.4), and I_1 is defined in (4.8). To show that u_h^- is a discrete subsolution in the sense

$$\mathbb{T}_{\mathfrak{h}}[u_h^-;f,\chi](z_h) \le 0, \quad \forall z_h \in \mathcal{N}_{h,\varepsilon}^I, \qquad u_h^-(z_h) \le \widetilde{g}(z_h), \quad \forall z_h \in \mathcal{N}_{h,\varepsilon}^b,$$

we discuss two different scenarios based on the distance between z_h and $\mathcal{C}(u)$.

If $B_{2\varepsilon}(z_h) \subset \Omega^+(u)$, then $-\Delta_\infty^\diamond u(x) \le f(x)$ for any $x \in B_{2\varepsilon}(z_h)$. The proof of Theorem 4.1 immediately implies that

$$\mathbb{T}_{\mathfrak{h}}[u_h^-;f,\chi](z_h) \le -\Delta_{\infty,\mathfrak{h}}^\diamond u_h^-(z_h) - f(z_h) \le 0.$$

If $B_{2\varepsilon}(z_h) \cap \mathcal{C}(u) \ne \emptyset$, then there exists $z \in B_{2\varepsilon}(z_h)$ such that $u(z) = \chi(z)$. Hence,

$$\begin{aligned} u^\varepsilon(z_h) - \chi(z_h) &\le u(y) - u(z) + \chi(z) - \chi(z_h) \le |y-z|^\alpha |u|_{C^{0,\alpha}(\overline{\Omega})} + |z-z_h||\chi|_{C^{0,\alpha}(\overline{\Omega})} \\ &\le (2\varepsilon)^\alpha |u|_{C^{0,\alpha}(\overline{\Omega})} + \varepsilon^\alpha |\chi|_{C^{0,\alpha}(\overline{\Omega})}, \end{aligned}$$

where $y \in \overline{B}_\varepsilon(z_h)$ such that $u(y) = u^\varepsilon(z_h)$. From this, we simply see that

$$\mathbb{T}_{\mathfrak{h}}[u_h^-;f,\chi](z_h) \le u_h^-(z_h) - \chi(z_h) \le u^\varepsilon(z_h) - (2\varepsilon)^\alpha |u|_{C^{0,\alpha}(\overline{\Omega})} - \varepsilon^\alpha |\chi|_{C^{0,\alpha}(\overline{\Omega})} - \chi(z_h) \le 0.$$

To show the inequality on the boundary nodes, we recall (4.1) and get

$$u_h^-(z_h) - \widetilde{g}(z_h) \le u^\varepsilon(z_h) - \widetilde{g}(z_h) - (2\varepsilon)^\alpha |\widetilde{g}|_{C^{0,\alpha}(\overline{\Omega})} - (3\varepsilon)^\alpha |u|_{C^{0,\alpha}(\overline{\Omega})} \le 0.$$

The discussions above prove that u_h^- is a discrete subsolution and thus gives a lower bound of $u_{\mathfrak{h}}$ by Theorem 5.5.

To obtain the upper bound of u_h, we consider

$$u_h^+ = \frac{1}{1-I_1} u_\varepsilon - \frac{I_1}{1-I_1} \min_{z_h\in\mathcal{N}_h} u(z_h) + (2\varepsilon)^\alpha |\widetilde{g}_\varepsilon|_{C^{0,\alpha}(\overline{\Omega})} + (3\varepsilon)^\alpha |u|_{C^{0,\alpha}(\overline{\Omega})},$$

where, in full analogy to (4.4), we defined $u_\varepsilon(x) = \min_{y\in\overline{B}_\varepsilon(x)} u(y)$. A similar calculation shows that $u_h^+(z_h) \ge \widetilde{g}_\varepsilon(z_h)$ for any $z_h \in \mathcal{N}_{h,\varepsilon}^b$. From the facts that $-\Delta_\infty^\diamond u(x) \ge f(x)$ and $u(x) \ge \chi(x)$ for any $x \in \Omega$, we similarly derive that

$$-\Delta_{\infty,\mathfrak{h}}^\diamond u_h^+(z_h) - f(z_h) \ge 0, \quad u_h^+(z_h) - \chi(z_h) \ge 0$$

for any $z_h \in \mathcal{N}_{h,\varepsilon}^I$. These inequalities prove that u_h^+ is a discrete supersolution and thus it provides an upper bound for $u_{\mathfrak{h}}$. Combining the lower and upper bound together, we have

$$\|u - u_{\mathfrak{h}}\|_{L^\infty(\Omega_h)} \lesssim \varepsilon^\alpha + \frac{\theta^\alpha}{\varepsilon^{2-\alpha}} + \frac{h^\alpha}{\varepsilon^2}.$$

□

THEOREM 5.9 (error estimate: homogeneous problem). Let u be the viscosity solution of (5.11) and $u_{\mathfrak{h}}$ be the solution of (5.13). Under our running assumptions suppose that **RHS**.2b and, for all $h > 0$, **M**.1 holds. If $\mathfrak{h}$ is sufficiently small, such that $\varepsilon - \varepsilon\theta - h > 0$, and

$$(2h + \varepsilon\theta)/\varepsilon^2,$$

can be made sufficiently small, then we have

$$\|u - u_{\mathfrak{h}}\|_{L^\infty(\Omega_h)} \lesssim \varepsilon^\alpha + \frac{\sqrt{2h + \varepsilon\theta}}{\varepsilon},$$

where the implied constant depends on the dimension d, the domain Ω, the shape regularity of the mesh $\mathcal{T}_h$, and the $C^{0,\alpha}$ norms of the data g, χ and the solution u.

Proof. We have already shown in the proof of Theorem 5.8 how to modify the error estimates in Section 4 to obtain the error estimates for the obstacle problem. One different thing here is that in the construction of the discrete subsolution u_h^-, one needs to employ the approximation in $\Omega^+(u)$ instead of all of Ω. To be more specific, we introduce $v_\gamma \in C(\overline{\Omega}) \cap W^{1,\infty}_{\text{loc}}(\Omega^+(u))$ with the following properties:

$$\begin{aligned} -\Delta_\infty^\diamond v_\gamma &\le 0, \text{ in } \Omega^+(u), & v_\gamma &= u, \text{ on } \partial\Omega^+(u), \\ L(v_\gamma, \cdot) &\ge \gamma, \text{ in } \Omega^+(u), & u - \gamma\,\mathrm{diam}(\Omega) &\le v_\gamma \le u, \text{ in } \Omega^+(u). \end{aligned}$$

This is because we only have $-\Delta_\infty^\diamond u \le 0$ in $\Omega^+(u)$. However, since $-\Delta_\infty^\diamond u \ge 0$ in Ω, the approximation needed in the construction of the supersolution is still in Ω as before. Repeating some arguments in the proof of Theorem 4.4 and Theorem 5.8, we know that

$$\begin{aligned} u_h^- &= q(v_\gamma^{\varepsilon+h}) - C\big((M-m)^2 + \mathrm{diam}(\Omega)\big)\sqrt{\beta} \\ &\quad - C\varepsilon^\alpha\big(|\widetilde{g}|_{C^{0,\alpha}(\overline{\Omega})} + |u|_{C^{0,\alpha}(\overline{\Omega})} + |\chi|_{C^{0,\alpha}(\overline{\Omega})}\big) \end{aligned}$$

is a discrete subsolution where the definitions of m, M, q, β can be found in the proof of Theorem 4.4. This gives a lower bound of $u_{\mathfrak{h}}$. Combining it with the upper bound obtained in a similar fashion as before proves the desired error estimate. □

Rates of convergence, analogous to those of Corollary 4.2 and Corollary 4.5, can also be obtained. We omit the details.

5.3 *Symmetric Finsler norms*

Let us consider one last variation of the game described in Section 1.2. Namely, the points where the token can be moved to do not need to constitute an (Euclidean) ball. Indeed, we can more generally consider rescaled translations of a convex set B with $0 \in B$. In this case, the game value should satisfy

$$2u^{\varepsilon}(x) - \left(\sup_{y \in x+\varepsilon B} u^{\varepsilon}(y) + \inf_{y \in x+\varepsilon B} u^{\varepsilon}(y) \right) = \varepsilon^2 f(x), \tag{5.19}$$

where

$$x + \varepsilon B = \{x + \varepsilon \mathbf{v} : \mathbf{v} \in B\}.$$

It is then of interest to understand what is the ensuing differential equation, and its properties.

To proceed further with the discussion of what the limiting differential equation is, we must restrict the class of admissible convex sets to those that can be characterized as the unit ball of a (symmetric) Finsler–Minkowski norm.

Definition 5.10 (Minkowski norm). A (symmetric) Finsler–Minkowski norm on $\mathbb{R}^d$ is a function $F : \mathbb{R}^d \to \mathbb{R}$ that satisfies

- **Smoothness**: $F \in C^2(\mathbb{R}^d \setminus \{\mathbf{0}\})$.
- **Absolute homogeneity**: For all $\mathbf{v} \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$, we have $F(\lambda \mathbf{v}) = |\lambda| F(\mathbf{v})$.
- **Strong convexity:** For all $\mathbf{0} \neq \mathbf{v} \in \mathbb{R}^d$ the matrix $\mathrm{D}^2 F(\mathbf{v})$ is positive definite.

Remark 5.11 (symmetry). In Definition 5.10, the qualifier symmetric comes from the fact that, as a consequence of absolute homogeneity, we have $F(-\mathbf{v}) = F(\mathbf{v})$. A general Minkowski norm needs not to satisfy absolute homogeneity, but only positive homogeneity, i.e., $F(t\mathbf{v}) = tF(\mathbf{v})$ for all $\mathbf{v} \in \mathbb{R}^d$ and $t > 0$. Notice that, in this case, the function F is not necessarily symmetric. While to a certain extent it is possible, see Guo *et al.* (2016); Biset *et al.* (2020); Mebrate & Mohammed (2020a); Mebrate & Mohammed (2020b); Mebrate & Mohammed (2021), to develop a theory for nonsymmetric Finsler norms, we will not consider this case here. The reason for this restriction is detailed below.

We refer the reader to Bao *et al.* (2000) for a thorough treatise regarding Finsler structures and Finsler geometry. Here we will just mention, in addition to Definition 5.10, the so-called dual function of F, which is given by

$$F^*(\mathbf{w}) = \sup_{\mathbf{0} \neq \mathbf{v} \in \mathbb{R}^d} \frac{\mathbf{w}^{\mathsf{T}} \mathbf{v}}{F(\mathbf{v})}.$$

Notice that the dual is also a symmetric Finsler–Minkowski norm. Moreover, by positive homogeneity,

$$\mathbf{v}^{\mathsf{T}} \mathrm{D}F(\mathbf{v}) = F(\mathbf{v}), \qquad \forall \mathbf{v} \in \mathbb{R}^d.$$

From this, several important relations between F and its dual F^* follow; see, for instance, Mebrate & Mohammed (2020a, Lemma 3.5).

With these definitions at hand, we can now describe the equation that ensues from (5.19) in, at least, a particular case. Given a symmetric Finsler–Minkowski norm F, assume that the set of admissible moves is given by

$$B_F = \left\{ \mathbf{v} \in \mathbb{R}^d : F^*(\mathbf{v}) \leq 1 \right\}, \tag{5.20}$$

where F^* is the dual of F. Then, as Mebrate & Mohammed (2021, Theorem 1.3 and Corollary 1.4) show, equation (5.19), in the formal limit $\varepsilon \downarrow 0$, becomes

$$-\Delta^{\diamond}_{F,\infty} u = f, \text{ in } \Omega, \qquad u_{|\partial\Omega} = g. \tag{5.21}$$

The operator $-\Delta^{\diamond}_{F,\infty}$, called for obvious reasons the (normalized) Finsler infinity Laplacian, is

$$-\Delta^{\diamond}_{F,\infty}\varphi(x) = -\mathrm{D}F(\mathrm{D}\varphi(x)) \otimes \mathrm{D}F(\mathrm{D}\varphi(x)) : \mathrm{D}^2\varphi(x).$$

Notice that, once again, this operator is singular in the case that $\mathrm{D}\varphi(x) = \mathbf{0}$. For completeness, we present the upper and lower semicontinuous envelopes of this operator:

$$\Delta^{+}_{F,\infty}\varphi(x) = \begin{cases} \mathrm{D}F(\mathrm{D}\varphi(x)) \otimes \mathrm{D}F(\mathrm{D}\varphi(x)) : \mathrm{D}^2\varphi(x), & \mathrm{D}\varphi(x) \neq \mathbf{0}, \\ \max\left\{ \mathbf{v}^\intercal \mathrm{D}^2\varphi(x)\mathbf{v} : F^*(\mathbf{v}) = 1 \right\}, & \mathrm{D}\varphi(x) = \mathbf{0}, \end{cases}$$

$$\Delta^{-}_{F,\infty}\varphi(x) = \begin{cases} \mathrm{D}F(\mathrm{D}\varphi(x)) \otimes \mathrm{D}F(\mathrm{D}\varphi(x)) : \mathrm{D}^2\varphi(x), & \mathrm{D}\varphi(x) \neq \mathbf{0}, \\ \min\left\{ \mathbf{v}^\intercal \mathrm{D}^2\varphi(x)\mathbf{v} : F^*(\mathbf{v}) = 1 \right\}, & \mathrm{D}\varphi(x) = \mathbf{0}. \end{cases}$$

Problem (5.21) has been studied in Guo *et al.* (2016); Biset *et al.* (2020); Mebrate & Mohammed (2020a); Mebrate & Mohammed (2020b); Mebrate & Mohammed (2021) where, under assumptions **RHS**.1–2 and **BC**.1, several results have been obtained. In particular, a comparison principle for semicontinuous functions is obtained in (Mebrate & Mohammed, 2020a, Theorem 6.4). Existence of solutions is shown in (Mebrate & Mohammed, 2020b, Theorem 6.1). Uniqueness, under assumption **RHS**.2 is also obtained. In addition to the game theoretic interpretation of (5.21), we comment that a Finsler variant of the minimization problem of Section 1.1 was studied in Guo *et al.* (2016). Namely, the authors study the minimal Lipschitz extension problem, where the Lipschitz constant is measured with respect to the Finsler norm F, i.e.,

$$\mathrm{Lip}_F(w, E) = \sup\left\{ \frac{|w(x) - w(y)|}{F(x-y)} : x, y \in E, x \neq y \right\}.$$

Reference Guo *et al.* (2016) shows the existence and uniqueness of solutions. A comparison principle for continuous functions is also obtained. Interestingly, the proofs of all the results mentioned here follow by adapting the techniques of Armstrong & Smart (2012). One needs to use, as semidiscrete scheme, relation (5.19) with B given as in (5.20). Finally, some applications of (5.21) to global geometry are explored in Araújo *et al.* (2021).

It is at this point that we are able to describe why we must assume that our Finsler–Minkowski norm is symmetric. The reason is because one of the main tools in the analysis is a comparison principle with quadratic Finsler cones, which, in the nonsymmetric case, only holds with a restriction on the coefficients that define the cone; see Mebrate & Mohammed (2020b, Theorem 4.1).

However, as Mebrate & Mohammed (2020b, Remark 4.3 item (2)) shows, such a restriction is not needed for the symmetric case. This comparison principle, essentially, allows one to translate all the results of Armstrong & Smart (2012) to the present case.

The investigations detailed above motivate us to present a numerical scheme for (5.21). To do so, we follow the ideas of Section 2.3. We need then to discretize ∂B_F, given in (5.20). In full analogy to $\mathbb{S}_\theta$, we let $\mathbb{S}_{F,\theta} = \{\mathbf{v}_\theta\}$ be a finite and symmetric set of vectors such that

$$\mathbf{v}_\theta \in \mathbb{S}_{F,\theta} \quad \Longrightarrow \quad F^*(\mathbf{v}_\theta) = 1,$$

and, for every $\mathbf{v} \in \mathbb{R}^d$ such that $F^*(\mathbf{v}) = 1$, there is $\mathbf{v}_\theta \in \mathbb{S}_{F,\theta}$ with

$$F^*(\mathbf{v} - \mathbf{v}_\theta) \leq \theta.$$

For $z_h \in \mathcal{N}^I_{h,\varepsilon}$, we then define

$$\mathcal{N}_{F,\mathfrak{h}}(z_h) = \{z_h\} \cup \left\{z_h + \varepsilon \mathbf{v}_\theta : \mathbf{v}_\theta \in \mathbb{S}_{F,\theta}\right\},$$

and, for $w \in C(\overline{\Omega})$,

$$S^+_{F,\mathfrak{h}} w(z_h) = \frac{1}{\varepsilon}\left(\max_{x \in \mathcal{N}_{F,\mathfrak{h}}(z_h)} w(x) - w(z_h)\right), \quad S^-_{F,\mathfrak{h}} w(z_h) = \frac{1}{\varepsilon}\left(w(z_h) - \min_{x \in \mathcal{N}_{F,\mathfrak{h}}(z_h)} w(x)\right).$$

Our fully discrete Finsler infinity Laplacian is defined, for $w \in C(\overline{\Omega})$, by

$$-\Delta^\diamond_{F,\infty,\mathfrak{h}} w(z_h) = -\frac{1}{\varepsilon}\left(S^+_{F,\mathfrak{h}} \mathcal{I}_h w(z_h) - S^-_{F,\mathfrak{h}} \mathcal{I}_h w(z_h)\right), \qquad \forall z_h \in \mathcal{N}^I_{h,\varepsilon}. \tag{5.22}$$

Under Assumption **BC**.2, the numerical scheme to approximate the solution to (5.21) is: find $u_\mathfrak{h} \in \mathbb{V}_h$ such that

$$-\Delta^\diamond_{F,\infty,\mathfrak{h}} u_\mathfrak{h}(z_h) = f(z_h), \quad \forall z_h \in \mathcal{N}^I_{h,\varepsilon}, \qquad u_\mathfrak{h}(z_h) = \widetilde{g}(z_h), \quad \forall z_h \in \mathcal{N}^b_{h,\varepsilon}. \tag{5.23}$$

We will not dwell on the properties of solutions to (5.23), as these merely repeat all of the arguments we presented in Section 3. We only comment that, during the analysis, one can use in several places the fact that F and F^* are indeed norms on $\mathbb{R}^d$. Norm equivalence with the standard Euclidean norm is useful in obtaining many estimates. In short, the proof of convergence repeats Theorem 3.13.

6. Solution schemes

Let us now make some comments regarding the practical solution of (2.4). Since this involves at every point computing maxima and minima, it is only natural then to, as a first attempt, apply one of the variants of Howard's algorithm presented in Bokanowski *et al.* (2009), or equivalently active set strategies Hintermüller (2002). All these approaches are related to the fact that computing maxima and minima are slantly differentiable operations (Hintermüller, 2002, Lemma 3.1) and, thus, a semismooth Newton method can be formulated. If this is to converge we then have, at least locally, that this convergence is superlinear (Neilan *et al.*, 2017, Theorem 5.31).

However, convergence of such schemes can only be guaranteed if, at every iteration step $n \in \mathbb{N}_0$, the ensuing matrices are nonsingular and monotone. While monotonicity is not an issue, for the problem at hand, it is possible to end with singular matrices. The reason behind this is as follows. At each step, one needs to solve a linear system of equations, which reads

$$2u_{\mathfrak{h}}^{n+1}(\mathrm{z}_h) - \sum_{\mathrm{z}'_h \in \sigma_+^n(\mathrm{z}_h)} \alpha^n_{\mathrm{z}'_h} u_{\mathfrak{h}}^{n+1}(\mathrm{z}'_h) - \sum_{\mathrm{z}'_h \in \sigma_-^n(\mathrm{z}_h)} \beta^n_{\mathrm{z}'_h} u_{\mathfrak{h}}^{n+1}(\mathrm{z}'_h) = \varepsilon^2 f(\mathrm{z}_h), \qquad \mathrm{z}_h \in \mathcal{N}^I_{h,\varepsilon},$$

$$u_{\mathfrak{h}}^{n+1}(\mathrm{z}_h) = \widetilde{g}(\mathrm{z}_h), \qquad \mathrm{z}_h \in \mathcal{N}^b_{h,\varepsilon}.$$

Here $\sigma_+^n(\mathrm{z}_h), \sigma_-^n(\mathrm{z}_h) \subset \mathcal{N}_h$ are such that

$$u_{\mathfrak{h}}^n(x^+) = \sum_{\mathrm{z}'_h \in \sigma_+^n(\mathrm{z}_h)} \alpha^n_{\mathrm{z}'_h} u_{\mathfrak{h}}^n(\mathrm{z}'_h) = \max_{x \in \mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)} u_{\mathfrak{h}}^n(x),$$

i.e., $\widehat{\varphi}_{\mathrm{z}'_h}(x^+) = \alpha^n_{\mathrm{z}'_h}$ for $\mathrm{z}'_h \in \widetilde{\mathcal{N}}_{\mathfrak{h}}(\mathrm{z}_h)$ and $x^+ \in \overline{\Omega}$ is the point where this local maximum is attained. A similar characterization can be made for σ_-^n. These considerations show that

$$\alpha^n_{\mathrm{z}'_h}, \beta^n_{\mathrm{z}'_h} \in [0,1], \qquad \sum_{\mathrm{z}'_h \in \sigma_+^n(\mathrm{z}_h)} \alpha^n_{\mathrm{z}'_h} = \sum_{\mathrm{z}'_h \in \sigma_+^n(\mathrm{z}_h)} \beta^n_{\mathrm{z}'_h} = 1.$$

In other words, the ensuing system matrix is diagonally dominant, but not strictly diagonally dominant, and thus it may be singular. To conclude nonsingularity, the usual argument in this scenario involves invoking boundary nodes: for $\mathrm{z}_h \in \mathcal{N}^I_{h,\varepsilon}$ that is close to $\partial\Omega$, the stencil must contain boundary nodes. However, there is no guarantee that this will be the case in our setting. As a consequence, we cannot guarantee that the matrices will be nonsingular.

To circumvent this difficulty, we, instead, propose the fixed point iteration presented in Algorithm 1. As the following result shows, this fixed point iteration is globally convergent.

Algorithm 1: Fixed point iteration to solve (2.4).

Input: The initial guess: $u^0_{\mathfrak{h}} \in \mathbb{V}_h$ such that $u^0_{\mathfrak{h}} = \widetilde{g}$ in $\mathcal{N}^b_{h,\varepsilon}$.
Output: $u_{\mathfrak{h}}$ the solution to (2.4).
$n = 0$;
repeat
  foreach $\mathrm{z}_h \in \mathcal{N}^I_{h,\varepsilon}$ **do**

$$u_{\mathfrak{h}}^{n+1}(\mathrm{z}_h) = \frac{1}{2}\left[\varepsilon^2 f(\mathrm{z}_h) + \max_{x \in \mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)} u^n_{\mathfrak{h}}(x) + \min_{x \in \mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)} u^n_{\mathfrak{h}}(x)\right];$$

  end
  foreach $\mathrm{z}_h \in \mathcal{N}^b_{h,\varepsilon}$ **do**

$$u_{\mathfrak{h}}^{n+1}(\mathrm{z}_h) = \widetilde{g}(\mathrm{z}_h);$$

  end
  $n \to n+1$;
until *false*;

THEOREM 6.1 (convergence). For every $u_{\mathfrak{h}}^0 \in \mathbb{V}_h$ such that $u_{\mathfrak{h}}^0 = \widetilde{g}$ in $\mathcal{N}_{h,\varepsilon}^b$, the sequence $\{u_{\mathfrak{h}}^n\}_{n\in\mathbb{N}_0}$ generated by Algorithm 1 converges to $u_{\mathfrak{h}}$, the solution to (2.4).

Proof. We divide the proof in three steps.

First, let us assume that $u_{\mathfrak{h}}^0$ is a supersolution, i.e.,

$$-\Delta_{\infty,\mathfrak{h}}^{\diamond} u_h^0(\mathrm{z}_h) \geq f(\mathrm{z}_h), \quad \forall \mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^I, \qquad u_h^0(\mathrm{z}_h) = \widetilde{g}(\mathrm{z}_h), \quad \forall \mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^b.$$

Then, by induction, we see that, for all $n \in \mathbb{N}_0$, we have $u_{\mathfrak{h}} \leq u_{\mathfrak{h}}^{n+1} \leq u_{\mathfrak{h}}^n$, and that $\{u_{\mathfrak{h}}^n\}_{n\in\mathbb{N}_0}$ is a family of supersolutions. We observe first that if $u_{\mathfrak{h}}^n$ is a supersolution, by comparison we must have $u_{\mathfrak{h}} \leq u_{\mathfrak{h}}^n$. Next, since $u_{\mathfrak{h}}^n$ is a supersolution

$$u_{\mathfrak{h}}^n(\mathrm{z}_h) \geq \frac{1}{2}\left[\varepsilon^2 f(\mathrm{z}_h) + \max_{x\in\mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)} u_{\mathfrak{h}}^n(x) + \min_{x\in\mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)} u_{\mathfrak{h}}^n(x)\right] = u_{\mathfrak{h}}^{n+1}(\mathrm{z}_h), \qquad \forall \mathrm{z}_h \in \mathcal{N}_{h,\varepsilon}^I,$$

with equality on $\mathcal{N}_{h,\varepsilon}^b$. Notice that this implies that

$$\begin{aligned} u_{\mathfrak{h}}^{n+1}(\mathrm{z}_h) &= \frac{1}{2}\left[\varepsilon^2 f(\mathrm{z}_h) + \max_{x\in\mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)} u_{\mathfrak{h}}^n(x) + \min_{x\in\mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)} u_{\mathfrak{h}}^n(x)\right] \\ &\geq \frac{1}{2}\left[\varepsilon^2 f(\mathrm{z}_h) + \max_{x\in\mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)} u_{\mathfrak{h}}^{n+1}(x) + \min_{x\in\mathcal{N}_{\mathfrak{h}}(\mathrm{z}_h)} u_{\mathfrak{h}}^{n+1}(x)\right], \end{aligned}$$

so that $-\Delta_{\infty,\mathfrak{h}}^{\diamond} u_{\mathfrak{h}}^{n+1}(\mathrm{z}_h) \geq f(\mathrm{z}_h)$, and $u_{\mathfrak{h}}^{n+1}$ is a supersolution as well. Consequently, $u_{\mathfrak{h}}^{n+1} \geq u_{\mathfrak{h}}$. Finally, a standard Perron-like reasoning involving monotonicity and comparison yields then that the limit must be a solution.

Next, we assume that $u_{\mathfrak{h}}^0$ is a subsolution. In a similar manner, we obtain that the sequence $\{u_{\mathfrak{h}}^n\}_{n\in\mathbb{N}_0}$ is a family of subsolutions, that $u_{\mathfrak{h}}^n \leq u_{\mathfrak{h}}^{n+1} \leq u_{\mathfrak{h}}$, and that $u_{\mathfrak{h}}^n \to u_{\mathfrak{h}}$.

Consider now the general case, i.e., we only assume that $u_{\mathfrak{h}}^0 = \widetilde{g}$ in $\mathcal{N}_{h,\varepsilon}^b$. With an argument similar to that of the proof of Lemma 3.5 we can construct $w_h^{\pm} \in \mathbb{V}_h$ such that $w_h^{\pm} = \widetilde{g}$ in $\mathcal{N}_{h,\varepsilon}^b$,

$$w_h^- \leq u_{\mathfrak{h}}^0 \leq w_h^+,$$

w_h^- is a subsolution and w_h^+ is a supersolution. The monotonicity arguments of the previous two steps show that

$$w_h^{-,n} \leq u_{\mathfrak{h}}^n \leq w_h^{+,n},$$

where $\{w_h^{\pm,n}\}_{n\in\mathbb{N}_0}$ are the sequences obtained by applying Algorithm 1 with initial guess $w_h^{\pm,0} = w_h^{\pm}$, respectively. The previous steps then show that $w_h^{\pm,n} \to u_{\mathfrak{h}}$, and this shows convergence in the general case. □

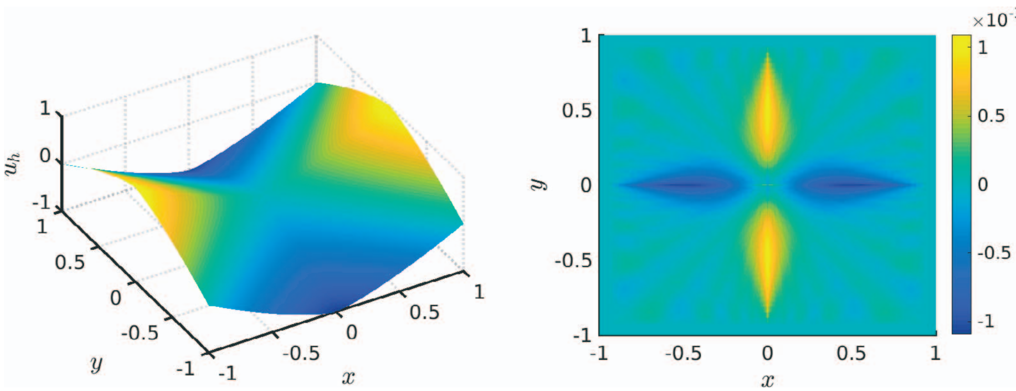

FIG. 1. Example 7.1. Left: plot of $u_{\mathfrak{h}}$. Right: plot of $u_{\mathfrak{h}} - u$.

7. Numerical experiments

In this section, we present some simple numerical examples to validate our analysis. All the computations were done with an in-house code that was written in MATLAB©. In practice, we choose a tolerance TOL > 0 and stop the iteration in Algorithm 1 when

$$\left\| \left\{ -\Delta^{\diamond}_{\infty,\mathfrak{h}} u^n_{\mathfrak{h}}(z_h) - f(z_h) \right\}_{z_h \in \mathcal{N}^I_{h,\varepsilon}} \right\|_{\ell^\infty} < \mathrm{TOL}.$$

EXAMPLE 7.1 ($f \equiv 0$, Aronsson's example). Let $\Omega = (-1,1)^2, f \equiv 0$ and $u(x,y) = |x|^{4/3} - |y|^{4/3}$. Notice that, in this example, the right-hand side f is smooth and $u \in C^{1,1/3}(\overline{\Omega})$.

We first wish to study the error induced by the discretization of the operator. To do so, we use an exact boundary condition, i.e., $\widetilde{g} = u$. Figure 1 shows the computed solution $u_{\mathfrak{h}}$, together with the error $u_{\mathfrak{h}} - u$ for $h = 2^{-8}, \varepsilon = 2^{-4.75}$ and $\theta = 2^{-3.25}$. From the pictures, we see that a larger error appears near the coordinate axes, where the solution u is not smooth. One can also observe the radial pattern of the error, which might be a result from the discretization, $\mathbb{S}_\theta$, of the unit sphere $\mathbb{S}$.

We now turn our attention to rates of convergence. We compute the approximate solution for a sequence of discretization parameters $\mathfrak{h} = (h, \varepsilon, \theta)$, satisfying

$$\varepsilon = C_\varepsilon h^\beta, \quad \theta = C_\theta \frac{h}{\varepsilon}, \tag{7.1}$$

where the positive constants C_ε, C_θ and the power β are chosen manually. We set $C_\theta = 1$ and display, in Fig. 2, plots of the L^∞ errors vs. meshsizes h for $\beta \in \left\{0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}$. The corresponding values of C_ε are chosen to guarantee that when $h = 2^{-5}$ we have $\varepsilon = 2^{-4}$ for all the choices of β. From the plot, we observe that the errors get stuck for small h when $\beta = 0$ or 1. This is consistent with our theory because in either case the consistency error of our discretization may not tend to zero as $h \to 0$; see Lemma 3.7. For $\beta \in \left\{\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right\}$, we also measure the convergence orders using a least squares fit for the errors of $h = 2^{-7}, 2^{-8}, 2^{-9}$. The orders are about 1.03, 1.57, 1.45, 1.06, respectively, which are much better than the theoretical ones $\min\left\{\beta, \frac{1}{2} - \beta\right\}$ from Theorem 4.4 and the improved rates $\min\{\beta, 1 - 2\beta\}$ from

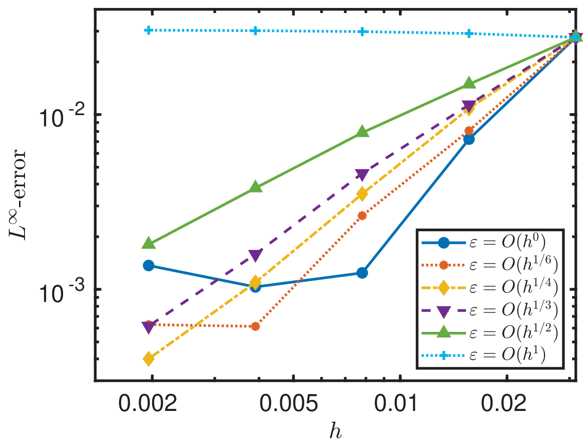

FIG. 2. Example 7.1. Experimental rates of convergence for L^∞ errors with $\beta \in \{0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\}$.

Remark 4.6. This may be caused by the fact that, although $u \in C^{1,1/3}(\overline{\Omega})$ only, it is smooth away from the coordinate axes.

We next investigate the error induced by the extension of the boundary datum $\widetilde{g}$. We define

$$\widetilde{g}(x,y) = g(T(x,y)), \qquad T(x,y) = \frac{(x,y)}{\|(x,y)\|_{\ell^\infty}} \in \partial\Omega.$$

Clearly, this choice of function $\widetilde{g}$ satisfies Assumption **BC**.2. For this inexact boundary condition, in Fig. 3, we plot the error $u_{\mathfrak{h}} - u$ for $h = 2^{-8}, \varepsilon = 2^{-4.75}$ and $\theta = 2^{-3.25}$. Due to the inexact boundary, the error is larger than the one shown in Fig. 1 (left). We also plot the L^∞ errors for the same choices of parameters with $\beta \in \{0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\}$. We observe again that, in accordance to the theory, the solutions do not seem to converge for $\beta = 0$ or 1. For $\beta \in \{\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\}$, we measure the convergence orders using least square fit for the errors of $h = 2^{-7}, 2^{-8}, 2^{-9}$. The orders are about $0.13, 0.25, 0.34, 0.45$, respectively, which are worse than the ones obtained for exact boundary, and are close to the improved rates $\min\{\beta, 1-2\beta\}$ in Remark 4.6 for $\beta = \frac{1}{6}, \frac{1}{4}, \frac{1}{3}$. Notice that, in fact, Theorem 4.4 or Remark 4.6 does not guarantee convergence for $\beta = \frac{1}{2}$.

Finally, we use the modified scheme (5.2) introduced in Section 5.1. We present the error $u_{\mathfrak{h}} - u$ for $h = 2^{-8}, \varepsilon = 2^{-4.75}$ and $\theta = 2^{-3.25}$ in Fig. 4 (left). The plot is similar to the one computed using exact boundary data in Fig. 1 (right). In fact, for the displayed plots, L^∞-error for the modified scheme is 1.212×10^{-3}, which is very close to the L^∞-error 1.095×10^{-3} for the original scheme using exact boundary data. Let $\mathcal{N}^b_{h,\varepsilon}$ denote the set of boundary nodes if we use the original scheme, the error near boundary, i.e., $\max_{z_h \in \mathcal{N}^b_{h,\varepsilon}} |u_{\mathfrak{h}}(z_h) - u(z_h)|$, for the modified scheme is 4.547×10^{-4}. This is smaller than the L^∞-error 1.212×10^{-3} over the whole domain, which might indicate that experimentally the variable ε for vertices near the boundary does not worsen the convergence rates. The convergence rates under the same choices of parameters are also presented in Fig. 4 (right). It seems that the modified scheme converges for $\beta \in \{\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\}$, and the least squares fit orders computed the using data for $h = 2^{-7}, 2^{-8}, 2^{-9}$ are about $1.13, 1.65, 1.47, 1.08$, respectively. These rates are close to the experimental rates using exact boundary data, and are also better than the improved rates $\min\{\beta, 1-2\beta\}$ in Remark 4.6.

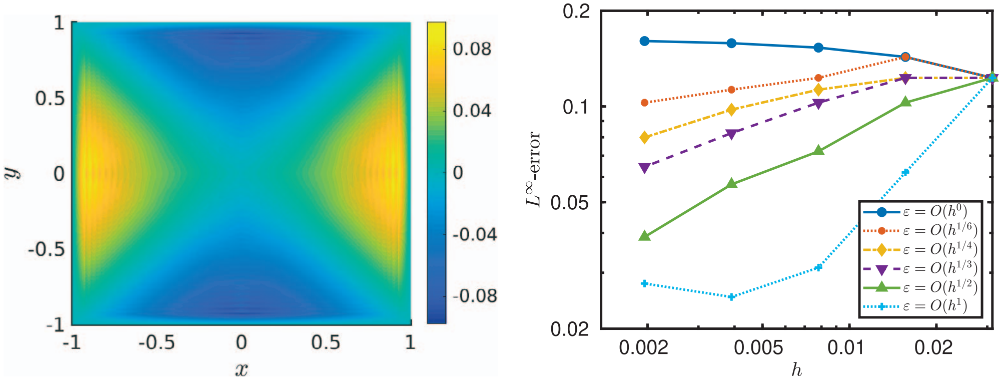

FIG. 3. Example 7.1 with inexact boundary condition $\widetilde{g}$. Left: plot of $u_{\mathfrak{h}} - u$. Right: experimental rates of convergence for L^∞ errors with $\beta \in \left\{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}$.

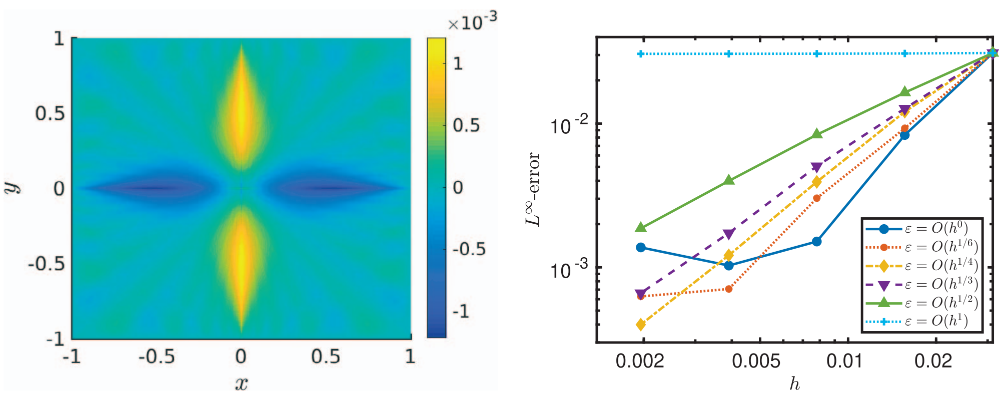

FIG. 4. Example 7.1. Left: plot of $u_{\mathfrak{h}} - u$ for the modified scheme. Right: experimental rates of convergence for L^∞ errors with $\beta \in \left\{0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}$ for the modified scheme.

EXAMPLE 7.2 ($f > 0$). Let $\Omega = B_1$ be the unit ball, $f \equiv 1$ and $u(x, y) = (1 - x^2)/2$. In this example, we have a smooth right-hand side f and solution u.

We choose $\widetilde{g} = u$ and compute the numerical solution $u_{\mathfrak{h}}$ for $h = 2^{-8}$, $\varepsilon = 2^{-4.75}$ and $\theta = 2^{-3.25}$. The error $u_{\mathfrak{h}} - u$ is displayed in Fig. 5 (left), where the largest error appears near the y-axis. To measure the orders of convergence of the L^∞ error, we let $\mathfrak{h} = (h, \varepsilon, \theta)$ satisfy (7.1) with $\beta \in \left\{0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}$, $C_\theta = 1$ and C_ε such that $\varepsilon = 2^{-4}$ when $h = 2^{-5}$. We observe again that the solutions do not seem to converge for $\beta = 0$ or 1 as expected from Theorem 4.1. For $\beta \in \left\{\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right\}$, the convergence orders measured for errors of $h = 2^{-7}, 2^{-8}, 2^{-9}$ using least squares are about 0.15, 0.24, 0.44, 0.59, respectively. The orders for $\beta \in \left\{\frac{1}{3}, \frac{1}{2}\right\}$ are better than the theoretical ones $\min\{\beta, 1 - 2\beta\}$ in Theorem 4.1. In fact, Theorem 4.1 does not guarantee convergence for $\beta = \frac{1}{2}$. The better rates might result from the fact that the solution u is smooth, but our theory only uses the fact that $u \in C^{0,1}(\overline{\Omega})$.

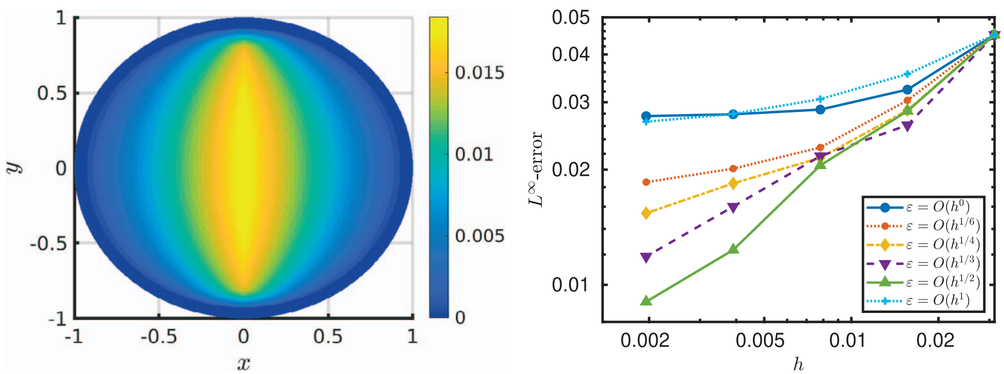

FIG. 5. Example 7.2 with exact boundary condition $\widetilde{g} = u$. Left: plot of $u_{\mathfrak{h}} - u$. Right: experimental rates of convergence for L^∞ errors with $\beta \in \left\{0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}$.

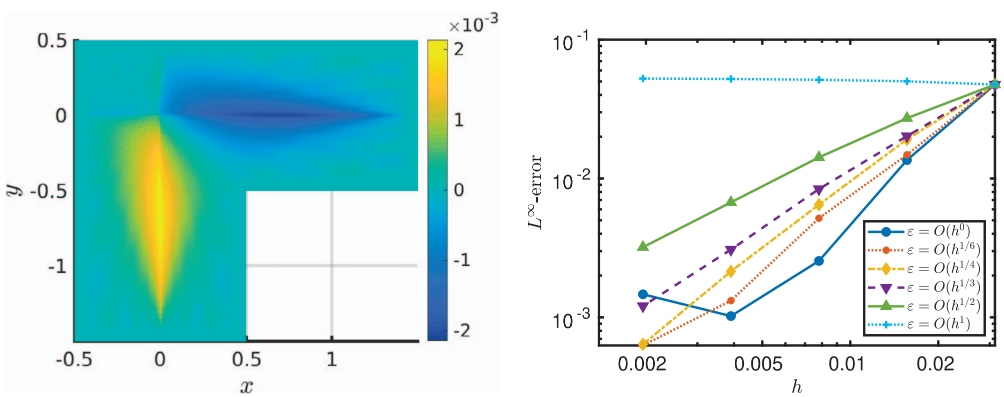

FIG. 6. Example 7.3 with exact boundary condition $\widetilde{g} = u$. Left: plot of $u_{\mathfrak{h}} - u$. Right: experimental rates of convergence for L^∞ errors with $\beta \in \left\{0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}$.

EXAMPLE 7.3 ($f \equiv 0$, Aronsson's example in an L-shaped domain). Here, as in Example 7.1, we consider $f \equiv 0$ and $u(x, y) = |x|^{4/3} - |y|^{4/3}$, but we change the domain to an L-shaped domain $\Omega = (-0.5, 1.5) \times (-1.5, 0.5) \setminus [0.5, 1.5) \times (-1.5, -0.5]$. We point out that in this example f is smooth, $u \in C^{1,1/3}(\overline{\Omega})$, and the domain Ω is not convex.

We choose $\widetilde{g} = u$ and compute the numerical solution $u_{\mathfrak{h}}$ for $h = 2^{-8}, \varepsilon = 2^{-4.75}$ and $\theta = 2^{-3.25}$. The error $u_{\mathfrak{h}} - u$ is presented in Fig. 6 (left). We also measure the orders of convergence of the L^∞-error by considering parameters $\mathfrak{h} = (h, \varepsilon, \theta)$, the same as Example 7.1, and plot the errors in Fig. 6 (right). We see that the solution converges for $\beta \in \left\{\frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right\}$. Using least squares fit for the results obtained for $h = 2^{-7}, 2^{-8}, 2^{-8}$ gives us the experimental order of convergence rates for the L^∞-error in terms of h are about $1.52, 1.67, 1.40, 1.07$ for $\beta = \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$, respectively, which are much higher compared to the predicted general rates $\min\left\{\beta, \frac{1}{2} - \beta\right\}$ in Theorem 4.4 and the improved rates $\min\{\beta, 1 - 2\beta\}$ from Remark 4.6. This may be caused by the fact that, although $u \in C^{1,1/3}(\overline{\Omega})$ only, it is smooth away from the coordinate axes.

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Data Availability

The source code, and produced data, for the numerical experiments is available upon reasonable request..

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