Comput Optim Appl (2011) 49:549–566
DOI 10.1007/s10589-009-9308-8

Elliptic control problems with gradient constraints—variational discrete versus piecewise constant controls

Andreas Günther · Michael Hinze

Received: 28 February 2009 / Published online: 5 December 2009
© Springer Science+Business Media, LLC 2009

Abstract We consider an elliptic optimal control problem with pointwise bounds on the gradient of the state. To guarantee the required regularity of the state we include the L^r-norm of the control in our cost functional with $r > d$ $(d = 2, 3)$. We investigate variational discretization of the control problem (Hinze in Comput. Optim. Appl. 30:45–63, 2005) as well as piecewise constant approximations of the control. In both cases we use standard piecewise linear and continuous finite elements for the discretization of the state. Pointwise bounds on the gradient of the discrete state are enforced element-wise. Error bounds for control and state are obtained in two and three space dimensions depending on the value of r.

Keywords Elliptic optimal control problem · Gradient constraints · Error estimates

1 Introduction

Constraints on the gradient of the state play an important role in practical applications where solidification of melts forms a critical process. In order to accelerate the production it is highly desirable to speed up the cooling processes while avoiding damage of the products caused by large material stresses. Cooling frequently is described by systems of partial differential equations involving the temperature as a system variable, so that large (Von Mises) stresses in the optimization can be kept small by imposing pointwise bounds on the gradient of the temperature. Pointwise bounds on the gradient of the state in optimization in general deliver adjoint variables

A. Günther · M. Hinze (✉)
Bereich Optimierung und Approximation, Universität Hamburg, Bundesstr. 55, 20146 Hamburg, Germany
e-mail: michael.hinze@uni-hamburg.de

A. Günther
e-mail: andreas.guenther@uni-hamburg.de

admitting low regularity only. This fact then necessitates the development of tailored discrete concepts which take into account the low regularity of adjoint variables and multipliers involved in the optimality conditions of the underlying optimization problem.

The present work complements the discrete approach to elliptic optimal control problems with gradient constraints presented in [5]. There, variational discretization of the controls is considered combined with the lowest order Raviart-Thomas finite element approximations of a mixed formulation of the state equation. This in particular leads to piecewise constant approximations to the state and the adjoint state, respectively. However, many existing finite element codes use finite elements based on conventional continuous piecewise polynomial Ansatz spaces. This is our motivation to provide numerical analysis for elliptic control problems with gradient constraints also for piecewise polynomial and continuous state approximations. In the present work we besides variational discretization also consider piecewise constant approximations of the controls. In both cases the state is discretized with standard piecewise linear, continuous finite elements. Our main results are stated in Theorems 2.5, 2.8, where we prove

$$\|y - y_h\| \le Ch^{\frac{1}{2}(1-\frac{d}{r})} \quad \text{and} \quad \|u - u_h\|_{L^r} \le Ch^{\frac{1}{r}(1-\frac{d}{r})},$$

for variational discretization as well as for piecewise constant control approximations. Here, y, u and y_h, u_h denote the unique solutions of the optimal control problems (2) and (10), (24), respectively, and $\|\cdot\|$ throughout the paper denotes the L^2-norm on the domain Ω. To the best of the authors knowledge [5] contained the only contribution to finite element error analysis of elliptic optimal control problems with gradient constraints when the present work was submitted. In the meantime the work [14] appeared which presents error bounds similar to ours, but derived by following techniques developed in [4]. For a discussion of literature concerned with optimal control problems in the presence of pointwise bounds on the state we also refer to [12, Chap. 3].

In the presence of gradient constraints variational discretization of the controls automatically leads to globally continuous approximations of the controls, if globally continuous Ansatz functions for the state are used, see relation (15). This is certainly a drawback of the approach, since the optimal control and the associated adjoint state may develop jumps, as the example in Sect. 3 shows. Piecewise constant control approximations here seem to be the better choice. However, the approximation order in both cases is the same, and also the errors in the numerical experiments for both approaches are of similar size, see Tables 1, 2.

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded, convex polyhedral domain with boundary $\partial\Omega$, whose inner dihedral angles at $\partial\Omega$ in the case $d = 3$ are assumed to be smaller than $\frac{3}{4}\pi$. We consider the differential operator $\mathcal{A} := -\Delta$ and associate to it the bilinear form

$$a(y, z) := \int_\Omega \nabla y \cdot \nabla z, \qquad y, z \in H^1(\Omega).$$

With the above assumptions we conclude that there exists some $\bar{r} > d$ such that for a given $f \in L^r(\Omega)$ $(1 < r \leq \bar{r})$ the elliptic boundary value problem

$$\begin{aligned} \mathcal{A}y &= f \quad \text{in } \Omega, \\ y &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

admits a unique solution $y = \mathcal{G}(f) \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ (see [3] for $d = 3$, and [8] for $d = 2$). Furthermore,

$$\|y\|_{W^{2,r}} \leq C\|f\|_{L^r}$$

holds, where $\|\cdot\|_{L^r}$ and $\|\cdot\|_{W^{k,r}}$ denote the usual Lebesgue and Sobolev norms. Moreover, for $f \in W^{-1,r}(\Omega)$ we have $\mathcal{G}(f) \in W_0^{1,r}(\Omega)$ (see [9] for $d = 2$, and [13] for $d = 3$) with

$$\|y\|_{W^{1,r}} \leq C\|f\|_{W^{-1,r}},$$

where the positive constant C is independent of f.

Remark 1.1 We think that our considerations also carry over to more general elliptic operators

$$\mathcal{A}y = -\sum_{i,j=1}^{d} \partial_{x_j}(a_{ij}y_{x_i}) + a_0 y,$$

with sufficiently smooth coefficients a_{ij} and a_0, and to curved domains Ω with sufficiently smooth boundary.

If not specified otherwise, let $d < r < \infty$, $\alpha > 0$ and $y_0 \in L^2(\Omega)$ be given. We now consider the control problem

$$\begin{aligned} &\min_{u \in L^r(\Omega)} J(u) = \frac{1}{2}\int_\Omega |y - y_0|^2 + \frac{\alpha}{r}\int_\Omega |u|^r \\ &\text{subject to} \quad y = \mathcal{G}(u) \text{ and } \nabla y \in \mathcal{K}. \end{aligned} \tag{2}$$

Here,

$$\mathcal{K} = \{\vec{z} \in C^0(\bar{\Omega})^d \mid |\vec{z}(x)| \leq \delta, x \in \bar{\Omega}\}. \tag{3}$$

Since $r > d$ we have $y \in W^{2,r}(\Omega)$ and hence $\nabla y \in C^0(\bar{\Omega})^d$ by a well-known embedding result. We impose the following Slater condition:

$$\exists \hat{u} \in L^r(\Omega) \quad |\nabla\hat{y}(x)| < \delta, \quad x \in \bar{\Omega}, \text{ where } \hat{y} \text{ solves (1) with } u = \hat{u}. \tag{4}$$

Since J is strictly convex and the set of admissible controls and states forms a closed and convex set, problem (2) admits a unique solution u with associated state $\mathcal{G}(u)$.

The KKT system of problem (2) is obtained with the help of [2, Corollary 1]. There holds

Theorem 1.2 *An element $u \in L^r(\Omega)$ is a solution of (2) if and only if there exist $\vec{\mu} \in \mathcal{M}(\bar{\Omega})^d$ and $p \in L^t(\Omega)$ $(t < \frac{d}{d-1})$ such that*

$$\int_\Omega p\mathcal{A}z - \int_\Omega (y - y_0)z - \int_{\bar{\Omega}} \nabla z \cdot d\vec{\mu} = 0 \quad \forall z \in W^{2,t'}(\Omega) \cap W_0^{1,t'}(\Omega), \tag{5}$$

$$p + \alpha |u|^{r-2} u = 0 \quad \text{in } \Omega, \tag{6}$$

$$\int_{\bar{\Omega}} (\vec{z} - \nabla y) \cdot d\vec{\mu} \leq 0 \quad \forall \vec{z} \in \mathcal{K}. \tag{7}$$

Here, y is the solution of (1), $\frac{1}{t} + \frac{1}{t'} = 1$, and $\mathcal{M}(\bar{\Omega})$ denotes the space of regular Borel measures.

Remark 1.3 In [2, Lemma 1] Casas and Fernandéz show that the vector valued measure $\vec{\mu}$ appearing in Theorem 1.2 admits the representation

$$\vec{\mu} = \frac{1}{\delta} \nabla y \mu, \tag{8}$$

where $\mu \in \mathcal{M}(\bar{\Omega})$ is a nonnegative measure that is concentrated in the set $\{x \in \bar{\Omega} \,|\, |\nabla y(x)| = \delta\}$.

2 Finite element discretization

We sketch an approach from [12, Sect. 3.3.2] which uses classical piecewise linear, continuous approximations of the states. In [5] Deckelnick, Günther and Hinze present a finite element approximation to problem (2) which uses mixed finite element approximations for the states.

Let $\mathcal{T}_h$ denote a quasi-uniform triangulation of Ω with maximum mesh size $h := \max_{T \in \mathcal{T}_h} \operatorname{diam}(T)$. Let us recall the definition of the space of linear finite elements,

$$X_h := \{v_h \in C^0(\bar{\Omega}) \mid v_h \text{ is a linear polynomial on each } T \in \mathcal{T}_h\}$$

and let $X_{h0} := X_h \cap H_0^1(\Omega)$. Furthermore let us recall the definition of the discrete approximation of the operator $\mathcal{G}$. For a given function $v \in L^2(\Omega)$ we denote by $z_h = \mathcal{G}_h(v) \in X_{h0}$ the solution of

$$a(z_h, v_h) = \int_\Omega v v_h \quad \text{for all } v_h \in X_{h0}.$$

It is well-known that for all $v \in L^r(\Omega)$ by an embedding theorem the corresponding state $\mathcal{G}(v)$ is in $W^{1,\infty}(\Omega)$, where we recall $r > d$. Furthermore, using [10, (1.2)] and [1, (4.4.29)]

$$\begin{aligned} \|\mathcal{G}(v) - \mathcal{G}_h(v)\|_{W^{1,\infty}} &\leq C \inf_{z_h \in X_{h0}} \|\mathcal{G}(v) - z_h\|_{W^{1,\infty}} \\ &\leq C h^{1-\frac{d}{r}} \|\mathcal{G}(v)\|_{W^{2,r}} \leq C h^{1-\frac{d}{r}} \|v\|_{L^r}. \end{aligned} \tag{9}$$

For each $T \in \mathcal{T}_h$ let $\vec{z}_T \in \mathbb{R}^d$ denote constant vectors. We define

$$\mathcal{K}_h := \{\vec{z}_h : \Omega \to \mathbb{R}^d | \vec{z}_{h|T} = \vec{z}_T \text{ on } T \text{ and } |\vec{z}_{h|T}| \le \delta, T \in \mathcal{T}_h\}.$$

Let us first consider variational discretization of problem (2) which reads:

$$\min_{u \in L^r(\Omega)} J_h(u) := \frac{1}{2}\int_\Omega |y_h - y_0|^2 + \frac{\alpha}{r}\int_\Omega |u|^r$$
$$\text{subject to} \quad y_h = \mathcal{G}_h(u) \text{ and } \nabla y_h \in \mathcal{K}_h. \tag{10}$$

Now (9) implies that $\hat{y}_h := \mathcal{G}_h(\hat{u})$ satisfies the Slater condition

$$|\nabla \hat{y}_h(x)| < \delta \quad \text{for all } x \in \bar{\Omega}, \tag{11}$$

and for $0 < h \le h_0$ with $h_0 > 0$ small enough. This delivers

Lemma 2.1 *Problem* (10) *admits a unique solution $u_h \in L^r(\Omega)$. There exist $\vec{\mu}_T \in \mathbb{R}^d$, $T \in \mathcal{T}_h$ and $p_h \in X_{h0}$ such that with $y_h = \mathcal{G}_h(u_h)$ we have*

$$a(v_h, p_h) - \int_\Omega (y_h - y_0)v_h - \sum_{T \in \mathcal{T}_h} \nabla v_{h|T} \cdot \vec{\mu}_T = 0 \quad \forall v_h \in X_{h0}, \tag{12}$$

$$p_h + \alpha|u_h|^{r-2}u_h = 0 \quad \textit{in } \Omega, \tag{13}$$

$$\sum_{T \in \mathcal{T}_h} \left(\vec{z}_T - \nabla y_{h|T}\right) \cdot \vec{\mu}_T \le 0 \quad \forall \vec{z}_h \in \mathcal{K}_h. \tag{14}$$

In problem (10) we apply variational discretization of [11]. From (13) we infer for the discrete optimal control

$$u_h = -\alpha^{-\frac{1}{r-1}}|p_h|^{\frac{2-r}{r-1}}p_h. \tag{15}$$

Furthermore, according to (8) we have the following representation of the discrete multipliers.

Lemma 2.2 *Let u_h denote the unique solution of* (10) *with corresponding state $y_h = \mathcal{G}_h(u_h)$ and multiplier $(\vec{\mu}_T)_{T \in \mathcal{T}_h}$. Then there holds*

$$\vec{\mu}_T = |\vec{\mu}_T|\frac{1}{\delta}\nabla y_{h|T} \quad \textit{for all } T \in \mathcal{T}_h. \tag{16}$$

Proof Fix $T \in \mathcal{T}_h$. The assertion is clear if $\vec{\mu}_T = 0$. Suppose that $\vec{\mu}_T \ne 0$ and define $\vec{z}_h : \bar{\Omega} \to \mathbb{R}^d$ by

$$\vec{z}_{h|\tilde{T}} := \begin{cases} \nabla y_{h|T}, & \tilde{T} \ne T, \\ \delta\frac{\vec{\mu}_T}{|\vec{\mu}_T|}, & \tilde{T} = T. \end{cases}$$

Clearly, $\vec{z}_h \in \mathcal{K}_h$ so that (14) implies

$$\vec{\mu}_T \cdot \left(\delta \frac{\vec{\mu}_T}{|\vec{\mu}_T|} - \nabla y_{h|T}\right) \le 0,$$

and therefore, since $(\nabla y_{h|T})_{T \in \mathcal{T}_h} \in \mathcal{K}_h$,

$$\delta |\vec{\mu}_T| \le \vec{\mu}_T \cdot \nabla y_{h|T} \le \delta |\vec{\mu}_T|.$$

Hence we obtain $\frac{\vec{\mu}_T}{|\vec{\mu}_T|} = \frac{1}{\delta} \nabla y_{h|T}$ and the lemma is proved. □

As a consequence of Lemma 2.2 we immediately infer that

$$|\vec{\mu}_T| = \vec{\mu}_T \cdot \frac{1}{\delta} \nabla y_{h|T} \quad \text{for all } T \in \mathcal{T}_h. \tag{17}$$

We now use (17) in order to derive an important a priori estimate.

Lemma 2.3 *Let $u_h \in L^r(\Omega)$ be the optimal solution of* (10) *with corresponding state $y_h \in X_{h0}$ and adjoint variables $p_h \in X_{h0}$, $\vec{\mu}_T \in \mathbb{R}^d$, $T \in \mathcal{T}_h$. Then there exists $h_0 > 0$ such that*

$$\|y_h\|, \|u_h\|_{L^r}, \|p_h\|_{L^{\frac{r}{r-1}}}, \sum_{T \in \mathcal{T}_h} |\vec{\mu}_T| \le C \quad \textit{for all } 0 < h \le h_0.$$

Proof Combining (17) with (11) we deduce

$$\vec{\mu}_T \cdot (\nabla y_{h|T} - \nabla \hat{y}_{h|T}) \ge \delta |\vec{\mu}_T| - (1-\epsilon)\delta |\vec{\mu}_T| = \epsilon \delta |\vec{\mu}_T|.$$

Choosing $v_h = y_h - \hat{y}_h$ in (12) and using the definition of $\mathcal{G}_h$ together with (13) we hence obtain

$$\begin{aligned}
\epsilon\delta \sum_{T \in \mathcal{T}_h} |\vec{\mu}_T| &\le \sum_{T \in \mathcal{T}_h} \vec{\mu}_T \cdot (\nabla y_{h|T} - \nabla \hat{y}_{h|T}) \\
&= a(y_h - \hat{y}_h, p_h) - \int_\Omega (y_h - y_0)(y_h - \hat{y}_h) \\
&= \int_\Omega (u_h - \hat{u}) p_h - \int_\Omega (y_h - y_0)(y_h - \hat{y}_h) \\
&= -\alpha \int_\Omega |u_h|^r + \alpha \int_\Omega |u_h|^{r-2} u_h \hat{u} - \int_\Omega y_h^2 + \int_\Omega y_h (y_0 + \hat{y}_h) - \int_\Omega y_0 \hat{y}_h \\
&\le -\alpha \int_\Omega |u_h|^r + \alpha \left\| u_h^{r-1} \right\|_{L^{\frac{r}{r-1}}} \|\hat{u}\|_{L^r} - \frac{1}{2} \int_\Omega y_h^2 + \frac{1}{2} \int_\Omega y_0^2 + \frac{1}{2} \int_\Omega \hat{y}_h^2 \\
&\le -\frac{\alpha}{2} \int_\Omega |u_h|^r - \frac{1}{2} \int_\Omega |y_h|^2 + C\big(1 + \|y_0\|^2 + \|\hat{u}\|_{L^r}^r\big),
\end{aligned}$$

where we have used $y_h(y_0 + \hat{y}_h) \le \frac{1}{2} y_h^2 + \frac{1}{2}(y_0 + \hat{y}_h)^2$. This implies the bounds on y_h, u_h and $\vec{\mu}_T$. The bound on p_h follows from (13). □

Remark 2.4 For the measure $\vec{\mu}_h \in \mathcal{M}(\bar{\Omega})^d$ defined by

$$\int_{\bar{\Omega}} \vec{f} \cdot d\vec{\mu}_h := \sum_{T \in \mathcal{T}_h} \int_T \vec{f} dx \cdot \vec{\mu}_T \quad \text{for all } \vec{f} \in C^0(\bar{\Omega})^d,$$

it follows immediately that

$$\|\vec{\mu}_h\|_{\mathcal{M}(\bar{\Omega})^d} \le C.$$

Now we are in the position to prove the following error estimates.

Theorem 2.5 *Let u and u_h be the solutions of (2) and (10) respectively. Then there exists $h_1 \le h_0$ such that*

$$\|y - y_h\| \le C h^{\frac{1}{2}(1-\frac{d}{r})} \quad \textit{and} \quad \|u - u_h\|_{L^r} \le C h^{\frac{1}{r}(1-\frac{d}{r})}$$

for all $0 < h \le h_1$.

Proof Let us introduce $y^h := \mathcal{G}(u_h) \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$, and $\tilde{y}_h := \mathcal{G}_h(u)$. In view of Lemma 2.3 and (9) we have

$$\|y^h - y_h\|_{W^{1,\infty}} \le C h^{1-\frac{d}{r}} \|u_h\|_{L^r} \le C h^{1-\frac{d}{r}}. \tag{18}$$

Let us now turn to the actual error estimate. To begin, we recall that for $r \ge 2$ there exists $\theta_r > 0$ such that

$$(|a|^{r-2}a - |b|^{r-2}b)(a - b) \ge \theta_r |a - b|^r \quad \forall a, b \in \mathbb{R}.$$

Hence, using (6) and (13),

$$\begin{aligned}
\alpha\theta_r \int_\Omega |u - u_h|^r &\le \alpha \int_\Omega \big(|u|^{r-2}u - |u_h|^{r-2}u_h\big)(u - u_h) \\
&= \underbrace{\int_\Omega p(u_h - u)}_{=:(1)} + \underbrace{\int_\Omega p_h(u - u_h)}_{=:(2)}.
\end{aligned}$$

Recalling (5) we have

$$\begin{aligned}
(1) &= \int_\Omega p(\mathcal{A}y^h - \mathcal{A}y) \\
&= \int_\Omega (y - y_0)(y^h - y) + \int_{\bar{\Omega}} (\nabla y^h - \nabla y) \cdot d\vec{\mu} \\
&= \int_\Omega (y - y_0)(y^h - y) + \underbrace{\int_{\bar{\Omega}} (P_\delta(\nabla y^h) - \nabla y) \cdot d\vec{\mu}}_{\le 0} + \int_{\bar{\Omega}} (\nabla y^h - P_\delta(\nabla y^h)) \cdot d\vec{\mu}
\end{aligned}$$

where P_δ denotes the orthogonal projection onto $\bar{B}_\delta(0) = \{x \in \mathbb{R}^d \,|\, |x| \le \delta\}$. Note that

$$|P_\delta(x) - P_\delta(\tilde{x})| \le |x - \tilde{x}| \quad \forall x, \tilde{x} \in \mathbb{R}^d. \tag{19}$$

Since $x \mapsto P_\delta(\nabla y^h(x)) \in \mathcal{K}$ we infer from (7)

$$(1) \le \int_\Omega (y - y_0)(y^h - y) + \max_{x \in \bar{\Omega}} |\nabla y^h(x) - P_\delta(\nabla y^h(x))| \|\vec{\mu}\|_{\mathcal{M}(\bar{\Omega})^d}. \tag{20}$$

Let $x \in \bar{\Omega}$, say $x \in T$ for some $T \in \mathcal{T}_h$. Since u_h is feasible for (10) we have that $\nabla y_{h|T} \in \bar{B}_\delta(0)$ so that (19) together with (18) implies

$$\begin{aligned} \left|\nabla y^h(x) - P_\delta(\nabla y^h(x))\right| &\le \left|\nabla y^h(x) - \nabla y_{h|T}\right| + \left|P_\delta(\nabla y^h(x)) - P_\delta(\nabla y_{h|T})\right| \\ &\le 2\left|\nabla y^h(x) - \nabla y_{h|T}\right| \le C h^{1-\frac{d}{r}} \|u_h\|_{L^r}. \end{aligned} \tag{21}$$

Thus

$$(1) \le \int_\Omega (y - y_0)(y^h - y) + C h^{1-\frac{d}{r}}. \tag{22}$$

Similarly,

$$\begin{aligned} (2) = a(\tilde{y}_h - y_h, p_h) &= \int_\Omega (y_h - y_0)(\tilde{y}_h - y_h) + \sum_{T \in \mathcal{T}_h} (\nabla \tilde{y}_{h|T} - \nabla y_{h|T}) \cdot \vec{\mu}_T \\ &= \int_\Omega (y_h - y_0)(\tilde{y}_h - y_h) + \sum_{T \in \mathcal{T}_h} (\nabla \tilde{y}_{h|T} - P_\delta(\nabla \tilde{y}_{h|T})) \cdot \vec{\mu}_T \\ &\quad + \underbrace{\sum_{T \in \mathcal{T}_h} (P_\delta(\nabla \tilde{y}_{h|T}) - \nabla y_{h|T}) \cdot \vec{\mu}_T}_{\le 0} \\ &\le \int_\Omega (y_h - y_0)(\tilde{y}_h - y_h) + \sum_{T \in \mathcal{T}_h} (\nabla \tilde{y}_{h|T} - \nabla y(x_T)) \cdot \vec{\mu}_T \\ &\quad + \sum_{T \in \mathcal{T}_h} (P_\delta(\nabla y(x_T)) - P_\delta(\nabla \tilde{y}_{h|T})) \cdot \vec{\mu}_T, \end{aligned}$$

where $x_T \in T$, so that $(\nabla y(x_T))_{T \in \mathcal{T}_h} \in \mathcal{K}_h$. We infer from Lemma 2.3 and (9)

$$\begin{aligned} (2) &\le \int_\Omega (y_h - y_0)(\tilde{y}_h - y_h) + 2 \max_{T \in \mathcal{T}_h} |\nabla \tilde{y}_{h|T} - \nabla y(x_T)| \sum_{T \in \mathcal{T}_h} |\vec{\mu}_T| \\ &\le \int_\Omega (y_h - y_0)(\tilde{y}_h - y_h) + C h^{1-\frac{d}{r}} \|u\|_{L^r}. \end{aligned} \tag{23}$$

Combining (1) and (2) we finally obtain

$$
\begin{aligned}
\alpha\theta_r \int_\Omega |u-u_h|^r &\le \int_\Omega (y-y_0)(y^h-y) + \int_\Omega (y_h-y_0)(\tilde{y}_h-y_h) + Ch^{1-\frac{d}{r}} \\
&= -\int_\Omega |y-y_h|^2 \\
&\quad + \int_\Omega ((y_0-y_h)(y-\tilde{y}_h) + (y-y_0)(y^h-y_h)) + Ch^{1-\frac{d}{r}} \\
&\le -\int_\Omega |y-y_h|^2 + C(\|y-\tilde{y}_h\| + \|y^h-y_h\|) + Ch^{1-\frac{d}{r}} \\
&\le -\int_\Omega |y-y_h|^2 + Ch(\|u\| + \|u_h\|) + Ch^{1-\frac{d}{r}}
\end{aligned}
$$

and the result follows. □

Remark 2.6 Theorem 2.5 suggests to use the coupling $r=2d$ to obtain the best convergence order for the control error. This would deliver errors of magnitude $O(h^{1/8})$ for $d=2$ and of magnitude $O(h^{1/12})$ for $d=3$. We note that our numerical results for $d=2$ deliver $O(h^{1/4})$. However, presently we are not able to prove this result for the control problems (2), (10).

2.1 Piecewise constant controls

Let us now consider the following optimal control problem with piecewise constant controls as discretization of problem (2);

$$
\begin{aligned}
&\min_{u_h\in U_h} J_h(u_h) := \frac{1}{2}\int_\Omega |y_h-y_0|^2 + \frac{\alpha}{r}\int_\Omega |u_h|^r \\
&\text{subject to} \quad y_h = \mathcal{G}_h(u_h) \text{ and } \nabla y_h \in \mathcal{K}_h,
\end{aligned} \tag{24}
$$

where $U_h := \{v_h \in L^r(\Omega) | v_{h|T} \in \mathbb{R} \text{ for all } T \in \mathcal{T}_h\}$. It is not difficult to prove that this problem admits a unique solution $u_h \in U_h$. Our finite element error analysis for this problem is based on approximation properties of the orthogonal L^2-projection $Q_h : L^2(\Omega) \to U_h$ defined by

$$
(Q_h v)(x) := ⨍_T v = \frac{1}{|T|}\int_T v \quad \text{for all } v \in L^2(\Omega), x \in T.
$$

For $v \in L^r(\Omega)$ we have the stability estimate

$$
\begin{aligned}
\|Q_h v\|_{L^r} = \left(\sum_{T\in\mathcal{T}_h} |T|^{1-r}\left|\int_T v\right|^r\right)^{\frac{1}{r}} &\le \left(\sum_{T\in\mathcal{T}_h} |T|^{1-r} |\|1\cdot v\|_{L^1(T)}|^r\right)^{\frac{1}{r}} \\
&\le \left(\sum_{T\in\mathcal{T}_h} \|v\|^r_{L^r(T)}\right)^{\frac{1}{r}} = \|v\|_{L^r},
\end{aligned} \tag{25}
$$

and for $\phi \in W^{1,r}(\Omega)$ the approximation property

$$\|\phi - Q_h\phi\|_{L^r} \le Ch^l \|\phi\|_{W^{l,r}}, \quad 0 \le l \le 1, \tag{26}$$

holds, see [6, Proposition 1.135]. Furthermore,

$$\|\mathcal{G}(v) - \mathcal{G}_h(Q_h v)\|_{W^{1,\infty}} \le \underbrace{\|\mathcal{G}(v) - \mathcal{G}(Q_h v)\|_{W^{1,\infty}}}_{:=(1)} + \underbrace{\|\mathcal{G}(Q_h v) - \mathcal{G}_h(Q_h v)\|_{W^{1,\infty}}}_{:=(2)}.$$

Now, for $v \in L^r(\Omega)$, by (9) and (25) there holds

$$(2) \le Ch^{1-\frac{d}{r}} \|Q_h v\|_{L^r} \le Ch^{1-\frac{d}{r}} \|v\|_{L^r}.$$

Furthermore

$$\begin{aligned}\|\nabla\mathcal{G}(v - Q_h v)\|_{L^\infty} &\le C\|\nabla\mathcal{G}(v - Q_h v)\|_{L^r}^{\beta} |\nabla\mathcal{G}(v - Q_h v)|_{W^{1,r}}^{1-\beta}\\ &\le C\|v - Q_h v\|_{W^{-1,r}}^{\beta} \|v - Q_h v\|_{L^r}^{1-\beta},\end{aligned}$$

where we have used the Lyapunov inequality [7, Theorem 10.1] with $0 < \beta := 1 - \frac{d}{r} < 1$. Now, for $w \in W^{1,r'}(\Omega)$ with $\frac{1}{r} + \frac{1}{r'} = 1$ we have

$$\begin{aligned}\int_\Omega (v - Q_h v)w = \int_\Omega (v - Q_h v)(w - Q_h w) &\le \|v - Q_h v\|_{L^r} \|w - Q_h w\|_{L^{r'}}\\ &\le Ch\|v - Q_h v\|_{L^r} \|w\|_{W^{1,r'}}.\end{aligned}$$

This yields

$$\|v - Q_h v\|_{W^{-1,r}} = \sup_{0 \ne w \in W^{1,r'}(\Omega)} \frac{\int_\Omega (v - Q_h v)w}{\|w\|_{W^{1,r'}}} \le Ch\|v\|_{L^r},$$

so that we obtain again by (26)

$$\|\nabla\mathcal{G}(v - Q_h v)\|_{L^\infty} \le Ch^{1-\frac{d}{r}} \|v\|_{L^r}.$$

Hence (1) can also be estimated by

$$(1) = \|\mathcal{G}(v - Q_h v)\|_{W^{1,\infty}} \le C\|\nabla\mathcal{G}(v - Q_h v)\|_{L^\infty} \le Ch^{1-\frac{d}{r}} \|v\|_{L^r}.$$

Finally we conclude

$$\|\mathcal{G}(v) - \mathcal{G}_h(Q_h v)\|_{W^{1,\infty}} \le Ch^{1-\frac{d}{r}} \|v\|_{L^r}. \tag{27}$$

Thus, with $v := \hat{u} \in L^r(\Omega)$ we have that for $h > 0$ small enough the function $\hat{y}_h := \mathcal{G}_h(Q_h v)$ satisfies the Slater condition (11). For the optimal control problem (24) the result of Lemma 2.1 is valid if we replace (13) by

$$\int_\Omega (p_h + \alpha|u_h|^{r-2} u_h) v_h = 0 \quad \forall v_h \in U_h. \tag{28}$$

Furthermore Lemma 2.2 holds accordingly and the analogon to Lemma 2.3 reads

Lemma 2.7 *Let $u_h \in U_h$ be the optimal solution of* (24) *with corresponding state $y_h \in X_{h0}$ and adjoint variables $p_h \in X_{h0}$, $\vec{\mu}_T$, $T \in \mathcal{T}_h$. Then there exists $h_0 > 0$ such that*

$$\|y_h\|, \|u_h\|_{L^r}, \sum_{T\in\mathcal{T}_h} |\vec{\mu}_T| \le C \quad \textit{for all } 0 < h \le h_0$$

holds.

Proof Since $0 \le J_h(u_h) \le J_h(Q_h\hat{u}) \le C$ uniformly in h we have

$$\|y_h\|, \|u_h\|_{L^r} \le C \quad \text{for all } 0 < h \le h_0.$$

We continue with the estimate

$$\begin{aligned}
\vec{\mu}_T \cdot (\nabla y_{h|T} - \nabla \hat{y}_{h|T}) &= \delta|\vec{\mu}_T| - |\vec{\mu}_T| \frac{1}{\delta} \nabla y_{h|T} \cdot \nabla \hat{y}_{h|T} \\
&\ge \delta|\vec{\mu}_T| - |\vec{\mu}_T||\nabla \hat{y}_{h|T}| \\
&\ge \delta|\vec{\mu}_T| - \left(\delta - \frac{\varepsilon}{4}\right)|\vec{\mu}_T| = \frac{\varepsilon}{4}|\vec{\mu}_T|,
\end{aligned}$$

for some $\varepsilon > 0$. Choosing $v_h = y_h - \hat{y}_h$ in (12) and using the definition of $\mathcal{G}_h$ together with (28) we hence obtain

$$\begin{aligned}
\frac{\varepsilon}{4} \sum_{T\in\mathcal{T}_h} |\vec{\mu}_T| &\le \sum_{T\in\mathcal{T}_h} \vec{\mu}_T \cdot (\nabla y_{h|T} - \nabla \hat{y}_{h|T}) \\
&= a(y_h - \hat{y}_h, p_h) - \int_\Omega (y_h - y_0)(y_h - \hat{y}_h) \\
&= \int_\Omega (u_h - Q_h v) p_h - \int_\Omega y_h^2 + \int_\Omega y_h(y_0 + \hat{y}_h) - \int_\Omega y_0 \hat{y}_h \\
&\le -\alpha \int_\Omega |u_h|^{r-2} u_h (u_h - Q_h v) - \frac{1}{2}\int_\Omega y_h^2 + \frac{1}{2}\int_\Omega y_0^2 + \frac{1}{2}\int_\Omega \hat{y}_h^2 \\
&\le -\alpha \int_\Omega |u_h|^r + \alpha \int_\Omega |u_h|^{r-2} u_h Q_h v + C \int_\Omega \left(y_0^2 + \hat{y}_h^2\right) \\
&\le \alpha \left\|u_h^{r-1}\right\|_{L^{\frac{r}{r-1}}} \|Q_h v\|_{L^r} + C \int_\Omega \left(y_0^2 + \hat{y}_h^2\right) \\
&= \alpha \|u_h\|_{L^r}^{r-1} \|Q_h v\|_{L^r} + C \int_\Omega \left(y_0^2 + \hat{y}_h^2\right) \\
&\le C(\|Q_h v\|_{L^r} + \|y_0\|^2 + \|\hat{y}_h\|^2),
\end{aligned}$$

where we again have used $y_h(y_0 + \hat{y}_h) \le \frac{1}{2} y_h^2 + \frac{1}{2}(y_0 + \hat{y}_h)^2$. This implies the bound on $\vec{\mu}_T$. □

Theorem 2.8 *Let u and u_h be the solutions of (2) and (24) respectively. Then there exists $h_1 \le h_0$ such that*

$$\|y - y_h\| \le Ch^{\frac{1}{2}(1-\frac{d}{r})} \quad \textit{and} \quad \|u - u_h\|_{L^r} \le Ch^{\frac{1}{r}(1-\frac{d}{r})}$$

for all $0 < h \le h_1$.

Proof Let us introduce $y^h := \mathcal{G}(u_h) \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$, and $\tilde{y}_h := \mathcal{G}_h(Q_h u)$. In view (9) we have

$$\|y^h - y_h\|_{W^{1,\infty}} \le Ch^{1-\frac{d}{r}} \|u_h\|_{L^r} \le Ch^{1-\frac{d}{r}}.$$

Let us now turn to the actual error estimate. Using (6) and (28) we have

$$\begin{aligned}
\alpha\theta_r \int_\Omega |u - u_h|^r &\le \alpha \int_\Omega \big(|u|^{r-2}u - |u_h|^{r-2}u_h\big)(u - u_h) \\
&= \underbrace{\int_\Omega p(u_h - u)}_{=:(1)} + \underbrace{\int_\Omega p_h(Q_h u - u_h)}_{=:(2)} \underbrace{- \alpha \int_\Omega \underbrace{|u_h|^{r-2}u_h}_{\in U_h} \underbrace{(u - Q_h u)}_{\in U_h^\perp}}_{=0}.
\end{aligned}$$

To estimate the terms (1) and (2) we follow the lines of the proof of Theorem 2.5 and obtain

$$(1) \le \int_\Omega (y - y_0)(y^h - y) + Ch^{1-\frac{d}{r}}, \tag{29}$$

as well as

$$\begin{aligned}
(2) &\le \int_\Omega (y_h - y_0)(\tilde{y}_h - y_h) + 2 \max_{T \in \mathcal{T}_h} |\nabla \tilde{y}_{h|T} - \nabla y(x_T)| \sum_{T \in \mathcal{T}_h} |\vec{\mu}_T| \\
&\le \int_\Omega (y_h - y_0)(\tilde{y}_h - y_h) + C\|\nabla(\tilde{y}_h - y)\|_{L^\infty}.
\end{aligned} \tag{30}$$

As in inequality (27) with $v := u$ we estimate

$$\|\nabla(\tilde{y}_h - y)\|_{L^\infty} = \|\nabla(\mathcal{G}_h(Q_h u) - \mathcal{G}(u))\|_{L^\infty} \le Ch^{1-\frac{d}{r}}$$

and thus

$$(2) \le \int_\Omega (y_h - y_0)(\tilde{y}_h - y_h) + Ch^{1-\frac{d}{r}}.$$

Combining (1) and (2) we finally obtain

$$\alpha\theta_r \int_\Omega |u - u_h|^r + \int_\Omega |y - y_h|^2 \le Ch\big(\|u\| + \|u_h\|\big) + Ch^{1-\frac{d}{r}}$$

and the result follows. □

3 A numerical experiment with pointwise constraints on the gradient

We now consider the finite element approximation of problem (2) with the following data. We consider (2) with the choices $\Omega = B_2(0) \subset \mathbb{R}^2$, $\alpha = 1$,

$$\mathcal{K} = \left\{ \vec{z} \in C^0(\bar{\Omega})^2 | |\vec{z}(x)| \leq \frac{1}{2}, x \in \bar{\Omega} \right\}$$

as well as

$$y_0(x) := \begin{cases} \frac{1}{4} + \frac{1}{2}\log 2 - \frac{1}{4}|x|^2, & 0 \leq |x| \leq 1, \\ \frac{1}{2}\log 2 - \frac{1}{2}\log|x|, & 1 < |x| \leq 2. \end{cases}$$

In the state equation we allow an additional right hand side f, i.e. we consider the problem

$$\begin{aligned} -\Delta y &= f + u \quad &&\text{in } \Omega \\ y &= 0 &&\text{on } \partial\Omega, \end{aligned}$$

where

$$f(x) := \begin{cases} 2, & 0 \leq |x| \leq 1, \\ 0, & 1 < |x| \leq 2. \end{cases}$$

The optimization problem then has the unique solution

$$u(x) = \begin{cases} -1, & 0 \leq |x| \leq 1, \\ 0, & 1 < |x| \leq 2 \end{cases}$$

with corresponding state $y \equiv y_0$. We note that we obtain equality in (6), i.e. $p = -u$ for all $r > d$. For all numerical computations we take $r = 4$. Furthermore, the action of the measure $\vec{\mu}$ applied to a vectorfield $\vec{\phi} \in C^0(\bar{\Omega})^2$ is given by

$$\int_{\bar{\Omega}} \vec{\phi} \cdot d\vec{\mu} = -\int_{\partial B_1(0)} x \cdot \vec{\phi}\, dS.$$

3.1 Variational discretization

We solve problem (10), where we essentially make use of the structure of u_h in terms of (15). Figure 1 illustrates the optimal solution u_h and corresponding adjoint state p_h on a mesh consisting of $nt = 512$ triangles. We note that due to relation (15) the variational control has to be a continuous function. The exact control however has a jump. We conclude that variational discretization combined with piecewise linear and continuous finite elements for the state approximation is not ideally suited to approximate control problems with gradient constraints on the state. To illustrate this fact we in Table 1 present some numerical computations for up to $nt = 512$ elements.

Led by the findings of [5] we think that variational discretization combined with the lowest order Raviart-Thomas finite element as state approximations in a mixed formulation of the state equation seems to be a more appropriate choice. However, many existing finite element codes use standard finite elements, so that there exists a

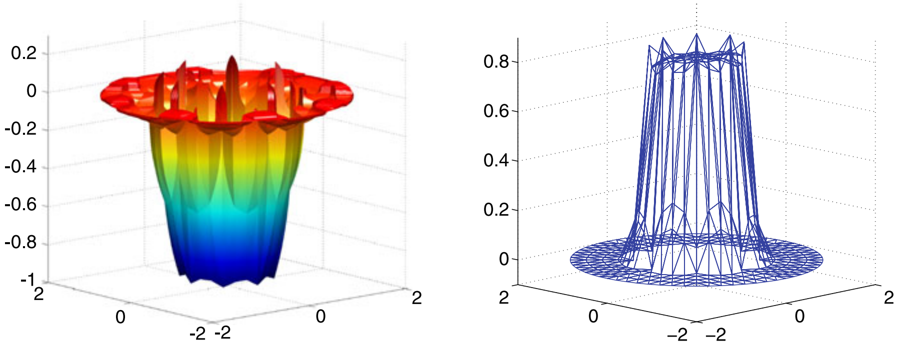


Fig. 1 Control (*left*) and adjoint state (*right*) (variational discretization)

Table 1 Errors (*top*) and EOCs for the numerical example (variational discretization)

nt	$\|u-u_h\|_{L^4(\Omega)}$	$\|u-u_h\|$	$\|y-y_h\|$
32	$8.34633 \cdot 10^{-1}$	1.36003	$2.20346 \cdot 10^{-1}$
128	$5.88566 \cdot 10^{-1}$	$9.04770 \cdot 10^{-1}$	$7.97200 \cdot 10^{-2}$
512	$4.84191 \cdot 10^{-1}$	$5.82014 \cdot 10^{-1}$	$3.52102 \cdot 10^{-2}$
	0.54884	0.64041	1.59745
	0.29263	0.66136	1.22499

demand in these approximation approaches also in optimization of elliptic PDEs in the presence of gradient constraints on the state. Therefore, in the present work we also investigate piecewise constant control approximations combined with piecewise linear, continuous approximations of the state.

3.2 Piecewise constant controls

We use piecewise constant, discontinuous Ansatz functions for the control u_h. For the numerical solution we use the routine `fmincon` contained in the MATLAB Optimization Toolbox. The state equation is approximated with piecewise linear, continuous finite elements on quasi-uniform triangulations $\mathcal{T}_h$ of $B_2(0)$. The gradient constraints are required element-wise. The resulting discretized optimization problem then reads

$$\min_{u_h \in U_h} J_h(u_h) = \frac{1}{2}\|y_h - y_0\|^2 + \frac{\alpha}{r}\|u_h\|_{L^r}^r$$

$$\text{subject to} \quad y_h = \mathcal{G}_h(u_h) \text{ and } |\nabla y_{h|T}| \le \delta = \frac{1}{2} \quad \forall T \in \mathcal{T}_h.$$

In Figs. 2, 3 we present the numerical approximations u_h, y_h, and μ_h on a grid containing $nt = 8192$ triangles, where μ_h is obtained by $\vec{\mu}_h$ according to relation (17). Figure 3 clearly shows that the support of μ_h is concentrated at $|x| = 1$.

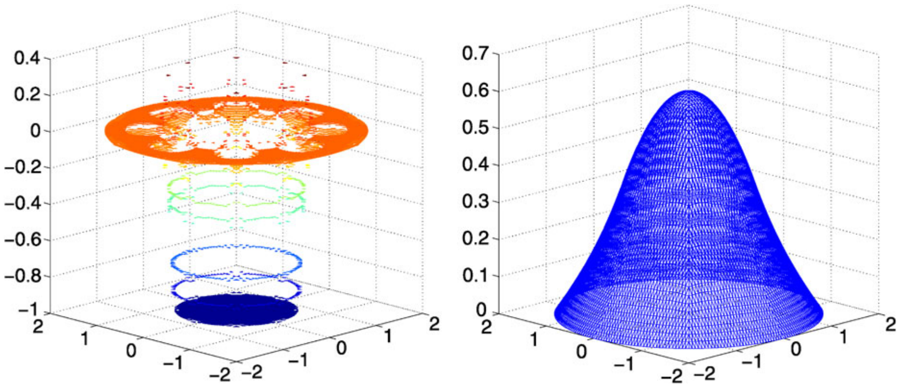
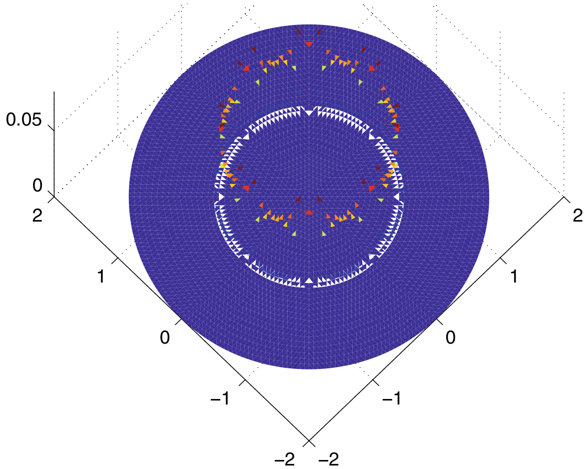


Fig. 2 Control (*left*) and state (*right*) (piecewise constant controls)

Fig. 3 Discrete multiplier (piecewise constant controls)



In Table 2 we document the experimental order of convergence. The controls show an approximation behavior which is slightly better than that predicted by Theorem 2.8. However, this may be caused by the fact that $\|u\|_{L^\infty}, \|u_h\|_{L^\infty} \le C$ uniformly in h. The L^2-norm of the state seems to converge at least with linear order. This can be explained by the high regularity of the exact solution. In the last column we display the values of $\sum_{T\in\mathcal{T}_h} |\vec{\mu}_T|$. These values are expected to converge to 2π as $h \to 0$, since this gives the value of μ applied to the function which is identically equal to 1 on $\bar{\Omega}$.

In order to motivate the convergence behavior of $\|u - u_h\|$ we briefly consider

Table 2 Errors (*top*), EOCs and multiplier approximation for the numerical example (piecewise constant controls)

nt	$\|u-u_h\|_{L^4}$	$\|u-u_h\|$	$\|y-y_h\|$	$\sum_{T\in\mathcal{T}_h} \|\vec{\mu}_T\|$
32	$8.34550 \cdot 10^{-1}$	1.37619	$2.30207 \cdot 10^{-1}$	0
128	$5.41825 \cdot 10^{-1}$	$8.45567 \cdot 10^{-1}$	$8.11347 \cdot 10^{-2}$	2.497502
512	$4.57207 \cdot 10^{-1}$	$6.03292 \cdot 10^{-1}$	$3.26818 \cdot 10^{-2}$	4.216741
2048	$3.63216 \cdot 10^{-1}$	$4.11190 \cdot 10^{-1}$	$1.33259 \cdot 10^{-2}$	5.213440
8192	$2.95328 \cdot 10^{-1}$	$2.74811 \cdot 10^{-1}$	$5.27703 \cdot 10^{-3}$	5.739806
	0.67870	0.76530	1.63860	
	0.25455	0.50609	1.36307	
	0.33810	0.56318	1.31796	
	0.30116	0.58653	1.34830	

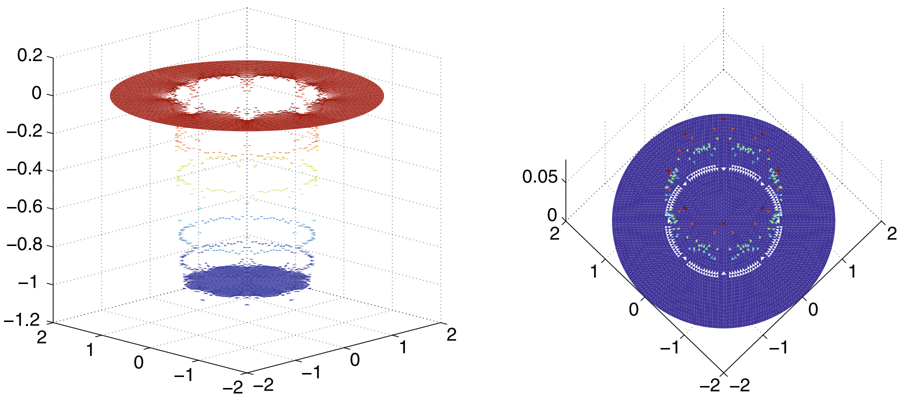


Fig. 4 Control (*left*) and multiplier (*right*) (Tychonov regularization)

3.3 Tychonov regularization

Since $u \in L^r(\Omega)$ with $r > d \geq 2$ we may also penalize with the L^2-norm of the control. The corresponding optimal control problem reads

$$\min_{u_h \in U_h} J_h(u_h) = \frac{1}{2}\|y_h - y_0\|^2 + \frac{\alpha}{2}\|u_h\|^2 + \frac{\alpha}{r}\|u_h\|_{L^r}^r$$

$$\text{subject to} \quad y_h = \mathcal{G}_h(u_h) \text{ and } |\nabla y_{h|T}| \leq \delta = \frac{1}{2} \quad \forall T \in \mathcal{T}_h.$$

An analytic solution can be obtained by adapting the constants in our example. Since the variational equality for the control for this control problem reads

$$\int_\Omega (p_h + \alpha(u_h + |u_h|^{r-2}u_h))v_h = 0 \quad \text{for all } v_h \in U_h$$

Table 3 Errors (*top*), EOCs and multiplier approximation for the numerical example (Tychonov regularization)

nt	$\|u-u_h\|_{L^4}$	$\|u-u_h\|$	$\|y-y_h\|$	$\sum_{T\in\mathcal{T}_h}\|\vec{\mu}_T\|$
32	$8.63533\cdot 10^{-1}$	1.22454	$3.83556\cdot 10^{-1}$	0.923216
128	$5.30078\cdot 10^{-1}$	$7.72724\cdot 10^{-1}$	$1.14305\cdot 10^{-1}$	3.656823
512	$4.25213\cdot 10^{-1}$	$5.03372\cdot 10^{-1}$	$4.94054\cdot 10^{-2}$	4.957956
2048	$3.52524\cdot 10^{-1}$	$3.48416\cdot 10^{-1}$	$2.13540\cdot 10^{-2}$	5.602883
8192	$2.89696\cdot 10^{-1}$	$2.41345\cdot 10^{-1}$	$9.58600\cdot 10^{-3}$	5.940486
	0.76678	0.72339	1.90217	
	0.33044	0.64248	1.25741	
	0.27542	0.54054	1.23233	
	0.28570	0.53442	1.16576	

we have a solution for the same data as before except for $\alpha = 0.5$. An analysis along the lines of Theorems 2.5 and 2.8 now shows that we also get

$$\|u-u_h\| \le Ch^{\frac{1}{2}(1-\frac{d}{r})},$$

with $C = C(\|u\|_{L^r}, \|u_h\|_{L^r})$. Since in the present example we have $u \in L^\infty(\Omega)$ and that $\|u_h\|_{L^\infty}$ is uniformly bounded in h we expect the error behavior $\|u-u_h\| \sim O(h^{\frac{1}{2}-\epsilon})$ for $h \to 0$. In Fig. 4 we present the numerical approximations u_h and μ_h on a grid containing $nt = 8192$ triangles. In Table 3 we investigate the experimental order of convergence for different error functionals. All convergence orders are in the same range as those obtained in the case without Tychonov regularization and piecewise constant controls. We observe that the control does not oscillate that much along $\partial B_1(0)$ as in the unregularized case.

Acknowledgements The authors thank Klaus Deckelnick from the University of Magdeburg for many fruitful discussions and the collaboration on the subject of this work. The authors also gratefully acknowledge support of the DFG Priority Program 1253 through grants DFG06-381 and DFG06-382.

References

1. Brenner, S.C., Scott, R.L.: The Mathematical Theory of Finite Element Methods, 3rd edn. Texts in Applied Mathematics, vol. 15. Springer, New York (2008)
2. Casas, E., Fernández, L.: Optimal control of semilinear elliptic equations with pointwise constraints on the gradient of the state. Appl. Math. Optim. **27**, 35–56 (1993)
3. Dauge, M.: Neumann and mixed problems on curvilinear polyhedra. Integr. Equ. Oper. Theory **15**(2), 227–261 (1992)
4. Deckelnick, K., Hinze, M.: Convergence of a finite element approximation to a state constrained elliptic control problem. SIAM J. Numer. Anal. **45**, 1937–1953 (2007)
5. Deckelnick, K., Günther, A., Hinze, M.: Finite element approximation of elliptic control problems with constraints on the gradient. Numer. Math. **111**, 335–350 (2009)
6. Ern, A., Guermond, J.-L.: Theory and Practice of Finite Elements. Appl. Math. Sci., vol. 159. Springer, Berlin (2004)
7. Friedman, A.: Partial Differential Equations. Holt, Rinehart & Winston, New York (1969)

8. Grisvard, P.: Singularities in Boundary Value Problems. Recherches en Mathématiques Appliquées (Research in Applied Mathematics), vol. 22. Masson, Paris (1992)
9. Gröger, K.: A $W^{1,p}$-estimate for solutions to mixed boundary value problems for second order elliptic differential equations. Math. Ann. **283**, 679–687 (1989)
10. Guzmán, J., Leykekhman, D., Roßmann, J., Schatz, A.H.: Hölder estimates for Green's functions on convex polyhedral domains and their applications to finite element methods. Numer. Math. **112**, 221–243 (2009)
11. Hinze, M.: A variational discretization concept in control constrained optimization: the linear-quadratic case. Comput. Optim. Appl. **30**, 45–63 (2005)
12. Hinze, M., Pinnau, R., Ulbrich, M., Ulbrich, S.: Optimization with PDE Constraints. Mathematical Modelling: Theory and Applications, vol. 23. Springer, Berlin (2009)
13. Jerison, D., Kenig, C.E.: The inhomogeneous Dirichlet problem in Lipschitz domains. J. Funct. Anal. **130**(1), 161–219 (1995)
14. Ortner, C., Wollner, W.: A priori error estimates for optimal control problems with pointwise constraints on the gradient of the state. DFG Schwerpunktprogramm 1253, Preprint No. SPP1253-071 (2009)