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Error estimates for a mixed finite volume method for the p-Laplacian problem

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Abstract In this work we propose and analyze a mixed finite volume method for the p-Laplacian problem which is based on the lowest order Raviart–Thomas element for the vector variable and the $P1$ nonconforming element for the scalar variable. It is shown that this method can be reduced to a $P1$ nonconforming finite element method for the scalar variable only. One can then recover the vector approximation from the computed scalar approximation in a virtually cost-free manner. Optimal a priori error estimates are proved for both approximations by the quasi-norm techniques. We also derive an implicit error estimator of Bank–Weiser type which is based on the local Neumann problems.

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1 Introduction

In this paper we consider the following nonlinear p-Laplacian problem with the Dirichlet boundary condition

$$\begin{cases} -\nabla \cdot (|\nabla u|^{p-2}\nabla u) = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$, $f \in L^{p'}(\Omega)$, and $g \in W^{1-1/p,p}(\partial\Omega)$, with p' being the conjugate exponent of $p \in (1,\infty)$, i.e., $1/p + 1/p' = 1$. This problem is one of the typical examples of

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degenerate nonlinear systems arising from, e.g., power-law materials and quasi-Newtonian flows.

The weak formulation for problem (1.1) reads as follows:
find $u \in W_g^{1,p}(\Omega) := \{v \in W^{1,p}(\Omega) : v|_{\partial\Omega} = g\}$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall v \in W_0^{1,p}(\Omega). \tag{1.2}$$

This is equivalent to the minimization problem

$$J(u) \le J(v) \qquad \forall v \in W_g^{1,p}(\Omega), \tag{1.3}$$

where

$$J(v) = \frac{1}{p}\int_\Omega |\nabla v|^p \, dx - \int_\Omega f v \, dx. \tag{1.4}$$

It is well known (cf. [7,8]) that problem (1.2) is well posed.

Finite element approximations for problem (1.2) have been extensively studied by many authors; see, for example, [7,8,12,19,21] for some previous works on a priori and a posteriori error estimation in the conventional $W^{1,p}(\Omega)$-norm. Sharper a priori error estimates were derived in [3,4,14] by developing the quasi-norm techniques. In [15–17] these techniques were extended to establish improved a posteriori error estimators of residual type for the $P1$ conforming and nonconforming finite element methods.

In some applications it is of primary interest to gain accurate approximations for the vector $\sigma = |\nabla u|^{p-2}\nabla u$. For this purpose, one rewrites problem (1.1) in the mixed form

$$\begin{cases} \sigma - |\nabla u|^{p-2}\nabla u = 0 & \text{in } \Omega, \\ \nabla \cdot \sigma + f = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

The mixed finite element method for this system was studied by Farhloul [11] in the case of $1 < p < 2$.

The goal of this paper is to propose and analyze a mixed finite volume method for the system (1.5) using the quasi-norm techniques of [3,15–17]. This method was first introduced for Poisson's equation (namely, $p = 2$) by Courbet and Croisille [9] who called it the finite volume box method. They considered the lowest order Raviart–Thomas space for the vector variable and the $P1$ nonconforming space for the scalar variable, and discretized the mixed system (1.5) by integrating the equations over each element of a given triangulation, based on the notion of the box method. This method was later extended to more general situations, such as tensor coefficients and quadrilateral grids, by Chou et al. [6].

The main advantages of the above mixed finite volume method can be summarized as follows (cf. [6]):

- Mass is conserved locally on each element.
- Even though the vector variable has continuous normal components, it is possible to decouple it locally from the scalar variable, without using Lagrange multipliers.

– The reduced system for the scalar variable obtained by decoupling the vector variable is exactly the $P1$ nonconforming finite element method with the slightly modified right-hand side, which is much cheaper and easier to implement than the standard mixed finite element method.
– The vector variable can be recovered in a local and inexpensive way from the computed scalar solution.

As we will see later, these good features are carried over to the p-Laplacian problem (1.1) as well. It should be mentioned that a priori elimination of the vector variable not only provides a convenient way of implementation but also a way of deriving error estimates without resort to any theory of saddle point problems, as illustrated in subsequent analysis.

We also derive an implicit error estimator of Bank–Weiser type for our mixed finite volume method by employing the vector approximation with continuous normal components. This error estimator is based on solving the local Neumann problems; see [1,2] for some discussion of this type of error estimators. It is proved that this estimator is equivalent to the true error in the quasi-norm up to higher order terms.

The rest of the paper is organized as follows. In the next section, we introduce the definitions and state some lemmas which will be crucially used in deriving error estimates in quasi-norms. We define our mixed finite volume method for problem (1.1) and establish some main features of it in Section 3. A priori error analysis for this method is carried out in Section 4. Finally, in Section 5, we construct and analyze an implicit error estimator of Bank–Weiser type which is based on the local Neumann problems.

2 Preliminaries

In order to avoid technical difficulties near the boundary, we assume that Ω is a polygonal domain and $g = 0$. It is not difficult to extend the results to more general cases (see, e.g., [16] for details).

Let $\mathcal{T}_h$ be a regular triangulation of Ω into NE triangles such that there is a constant $c > 0$ independent of the mesh size satisfying

$$\frac{h_K}{\rho_K} \le c \qquad \forall K \in \mathcal{T}_h,$$

where h_K is the diameter of K, and ρ_K is the radius of the largest ball contained in K. We set $h = \max_{K \in \mathcal{T}_h} h_K$.

For any subdomain $G \subset \mathbb{R}^2$, we adopt the standard notation $W^{s,p}(G)$ for the Sobolev space on G with norm and seminorm denoted by $\|\cdot\|_{s,p,G}$ and $|\cdot|_{s,p,G}$, and set $H^s(G) := W^{s,2}(G)$. For $G = \Omega$ we simply write $\|\cdot\|_{s,p}$ and $|\cdot|_{s,p}$ instead of $\|\cdot\|_{s,p,G}$ and $|\cdot|_{s,p,G}$. For piecewise smooth functions we define the broken norm

$$\|v\|_{s,p,\mathcal{T}_h} = \left(\sum_{K \in \mathcal{T}_h} \|v\|^p_{s,p,K} \right)^{1/p}$$

and the corresponding seminorm

$$|v|_{s,p,\mathcal{T}_h} = \left(\sum_{K \in \mathcal{T}_h} |v|^p_{s,p,K} \right)^{1/p}.$$

Let us define the lowest order Raviart–Thomas space (cf. [5])

$$\mathcal{RT}_h = \{\tau \in H(\text{div}; \Omega) : \tau|_K \in (P_0(K))^2 \oplus \mathbf{x}P_0(K) \ \ \forall K \in \mathcal{T}_h\}$$

and the $P1$ nonconforming space

$$\begin{aligned} \mathcal{N}_h = \{v \in L^2(\Omega) : v|_K \in P_1(K) \ \ \forall K \in \mathcal{T}_h, \text{ and} \\ v \text{ is continuous at the midpoints of interior edges} \\ \text{and vanishes at the midpoints of boundary edges}\}, \end{aligned}$$

where $\mathbf{x} = (x, y)$, $P_k(K)$ is the space of polynomials on K of total degree at most k, and

$$H(\text{div}; \Omega) = \{\tau \in (L^2(\Omega))^2 : \nabla \cdot \tau \in L^2(\Omega)\}.$$

It is now well known that we have for all $\xi_h \in \mathcal{RT}_h$

$$\nabla \cdot \xi_h|_K \in P_0(K) \qquad \forall K \in \mathcal{T}_h$$

and

$$\xi_h \cdot \nu_K|_e \in P_0(e) \qquad \forall e \subset \partial K,$$

where ν_K denotes the outward unit normal to ∂K, and e is an edge of K. Moreover, $\xi_h|_K$ is completely determined by its normal components $\xi_h \cdot \nu_K$ over the three edges of ∂K.

Let $\pi_h : W_0^{1,p}(\Omega) \to \mathcal{N}_h$ be the well-known interpolation operator defined by

$$\int_e \pi_h u \, ds = \int_e u \, ds$$

for every edge e of $\mathcal{T}_h$. From the standard approximation theory we have the following error estimates: for all $1 \le r \le \infty$, $m = 0, 1$, and $K \in \mathcal{T}_h$,

$$|v - \pi_h v|_{m,r,K} \le C h_K^{1-m} |v|_{1,r,K}. \tag{2.1}$$

Here and in what follows, C will denote a generic positive constant independent of the mesh size h (but dependent on p) which may take different values at different places.

Before closing this section, we state the following lemmas which will be crucially used in deriving error estimates in quasi-norms. Their proofs can be found in [3, 15, 16].

Lemma 2.1 *For all $p > 1$ and $\xi, \eta \in \mathbb{R}^2$, we have*

$$\left|\, |\xi|^{p-2}\xi - |\eta|^{p-2}\eta \,\right| \le C(|\xi| + |\eta|)^{p-2}|\xi - \eta|$$

and

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta) \ge C(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2,$$

where $(\cdot,\cdot)$ denotes the Euclidean inner product in $\mathbb{R}^2$.

Lemma 2.2 *For all $a, \sigma_1, \sigma_2 \ge 0$, $p > 1$, $\theta > 0$, we have*

$$(a + \sigma_1)^{p-2}\sigma_1\sigma_2 \le \theta^{-\gamma}(a + \sigma_1)^{p-2}\sigma_1^2 + \theta(a + \sigma_2)^{p-2}\sigma_2^2,$$

where

$$\gamma = \begin{cases} 1 & 1 < p \le 2,\ \theta \in [1,\infty) \text{ or } 2 < p < \infty,\ \theta \in (0,1), \\ \dfrac{1}{p-1} & 1 < p \le 2,\ \theta \in (0,1) \text{ or } 2 < p < \infty,\ \theta \in [1,\infty). \end{cases}$$

Lemma 2.3 *For all $a, \sigma_1, \sigma_2 \ge 0$, $p > 1$, $\delta > 0$, we have*

$$\sigma_1\sigma_2 \le \delta^{-\beta}(a^{p-1} + \sigma_1)^{p'-2}\sigma_1^2 + \delta(a + \sigma_2)^{p-2}\sigma_2^2,$$

where β is such that $\delta^{-\beta} = \max\{\delta^{-1}, \delta^{-\frac{1}{p-1}}\}$.

Lemma 2.4 *For all $a \ge 0$, $p > 1$ and $\sigma_1, \sigma_2 \in \mathbb{R}^2$, we have*

$$\begin{aligned}&(a + |\sigma_1 + \sigma_2|)^{p-2}|\sigma_1 + \sigma_2|^2 \\ &\quad \le \max(2, 2^{p-1})[(a + |\sigma_1|)^{p-2}|\sigma_1|^2 + (a + |\sigma_2|)^{p-2}|\sigma_2|^2].\end{aligned}$$

Lemma 2.5 *For all $p > 1$ and $\sigma, \sigma_1, \sigma_2 \in \mathbb{R}^2$, we have*

$$\begin{aligned}&(|\sigma_1| + |\sigma_2|)^{p-2}|\sigma_1 - \sigma_2|^2 \\ &\quad \le C(|\sigma| + |\sigma - \sigma_1|)^{p-2}|\sigma - \sigma_1|^2 + C(|\sigma| + |\sigma - \sigma_2|)^{p-2}|\sigma - \sigma_2|^2.\end{aligned}$$

3 Mixed finite volume method

In this section we will present our mixed finite volume method for problem (1.1) and establish some main features of it.

Our mixed finite volume method for problem (1.1) is given as follows: find $(\sigma_h, u_h) \in \mathcal{RT}_h \times \mathcal{N}_h$ such that, for all $K \in \mathcal{T}_h$,

$$\int_K (\sigma_h - |\nabla u_h|^{p-2}\nabla u_h)\, dx = 0, \tag{3.1}$$

$$\int_K (\nabla \cdot \sigma_h + f)\, dx = 0. \tag{3.2}$$

It was shown in [6,9] that the number of equations, $3NE$, is equal to that of unknowns (which are associated with the edges of $\mathcal{T}_h$). The existence and uniqueness of a solution (σ_h, u_h) of this method will be established later in this section.

The most attractive feature of the mixed method (3.1)–(3.2) is that, although σ_h has continuous normal components across the interelement boundaries, it can be decoupled easily to yield a reduced system for u_h only. The following theorem shows that this reduced system for u_h is, in fact, a $P1$ nonconforming finite element method for problem (1.1).

Theorem 3.1 *If* $(\sigma_h, u_h) \in \mathcal{RT}_h \times \mathcal{N}_h$ *is a solution of the mixed finite volume method* (3.1)–(3.2)*, then* u_h *is a solution of the variational problem*

$$\sum_{K\in\mathcal{T}_h} \int_K |\nabla u_h|^{p-2}\nabla u_h \cdot \nabla v_h \, dx = \int_\Omega \bar{f} v_h \, dx \qquad \forall v_h \in \mathcal{N}_h, \tag{3.3}$$

where $\bar{f}$ *is the piecewise average of* f *on* $\mathcal{T}_h$.

Proof It follows directly from (3.2) that

$$\nabla \cdot \sigma_h + \bar{f} = 0. \tag{3.4}$$

Thus, using (3.1) and integration by parts, we obtain for all $v_h \in \mathcal{N}_h$

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \int_K |\nabla u_h|^{p-2}\nabla u_h \cdot \nabla v_h \, dx &= \sum_{K\in\mathcal{T}_h} \int_K \sigma_h \cdot \nabla v_h \, dx \\ &= \sum_{K\in\mathcal{T}_h} \left(\int_{\partial K} \sigma_h \cdot \nu_K \, v_h \, ds + \int_K \bar{f} v_h \, dx \right) = \int_\Omega \bar{f} v_h \, dx,\end{aligned}$$

where the sum of the boundary integrals vanishes due to the continuity properties of σ_h and v_h and the boundary condition of v_h.

Let us next show that one can compute the vector approximation σ_h locally from the computed u_h. For a given edge e_i of $K \in \mathcal{T}_h$ $(i = 1, 2, 3)$, let $\phi_i \in P_1(K)$ be the basis function associated with that edge, that is, $\int_{e_i} \phi_j \, ds = \delta_{ij}|e_i|$, where $|e_i|$ denotes the length of e_i. Then we obtain

$$\begin{aligned}|e_i|(\sigma_h \cdot \nu_K)|_{e_i} &= \int_{\partial K} (\sigma_h \cdot \nu_K)\phi_i \, ds = \int_K \nabla \cdot (\sigma_h \phi_i) \, dx \\ &= \int_K (\sigma_h \cdot \nabla\phi_i + \nabla \cdot \sigma_h \, \phi_i) \, dx,\end{aligned}$$

which gives by (3.1) and (3.4)

$$|e_i|(\sigma_h \cdot \nu_K)|_{e_i} = \int_K |\nabla u_h|^{p-2}\nabla u_h \cdot \nabla\phi_i \, dx - \int_K \bar{f}\phi_i \, dx. \tag{3.5}$$

Note that the right-hand side may be interpreted as a local residual of u_h on K. Alternatively, one can derive the following explicit formula directly from (3.1) and (3.4)

$$\sigma_h|_K = |\nabla u_h|^{p-2}\nabla u_h|_K - \frac{\bar{f}}{2}(\mathbf{x} - \mathbf{x}_K), \tag{3.6}$$

where $\mathbf{x}_K$ denotes the barycenter of the element K. This can be considered as a generalization of Marini's formula [18] to the p-Laplacian problem.

The following theorem is a converse statement of the above results.

Theorem 3.2 *Let* $u_h \in \mathcal{N}_h$ *be a solution of the variational problem* (3.3) *and let* σ_h *be defined on each element* K *by* (3.5)*. Then we have* $\sigma_h \in \mathcal{RT}_h$*, and* (σ_h, u_h) *is a solution of the mixed finite volume method* (3.1)–(3.2).

Proof For a given interior edge $e = \partial K_1 \cap \partial K_2$, let $\phi_e \in \mathcal{N}_h$ be the basis function associated with e. By taking $v_h = \phi_e$ in (3.3), we easily see that

$$|e|(\sigma_h \cdot \nu_{K_1}|_e + \sigma_h \cdot \nu_{K_2}|_e) = 0,$$

which implies that $\sigma_h \in \mathcal{RT}_h$. Now, summing over $i = 1, 2, 3$ in (3.5) and using the fact that $\sum_{i=1}^{3} \phi_i \equiv 1$, we deduce that σ_h satisfies the local conservation law (3.2). Finally, using integration by parts and then (3.5), we obtain

$$\begin{aligned}\int_K \sigma_h \cdot \nabla\phi_i \, dx &= |e_i|(\sigma_h \cdot \nu_K)|_{e_i} - \int_K \nabla \cdot \sigma_h \, \phi_i \, dx \\ &= \int_K |\nabla u_h|^{p-2} \nabla u_h \cdot \nabla\phi_i \, dx,\end{aligned}$$

which yields (3.1). This completes the proof.

Note that the previous two theorems establish the unique solvability of the mixed finite volume method (3.1)–(3.2). We state this in the following theorem, and in addition, prove some uniform a priori bound for (σ_h, u_h).

Theorem 3.3 *There exists a unique solution* $(\sigma_h, u_h) \in \mathcal{RT}_h \times \mathcal{N}_h$ *of the mixed finite volume method* (3.1)–(3.2) *which satisfies*

$$\|\sigma_h\|_{0,p'} + |u_h|_{1,p,\mathcal{T}_h} \le C\big(\|f\|_{0,p'} + \|f\|_{0,p'}^{1/(p-1)}\big). \tag{3.7}$$

Proof By taking $v_h = u_h$ in (3.3) it is easy to see that

$$|u_h|_{1,p,\mathcal{T}_h} \le C\|f\|_{0,p'}^{1/(p-1)}.$$

By (3.6) we also obtain

$$\|\sigma_h\|_{0,p'} \le |u_h|_{1,p,\mathcal{T}_h}^{p-1} + Ch\|f\|_{0,p'}.$$

Thus the desired assertion follows.

4 A priori error estimates

To derive the error estimates for the mixed finite volume method defined in the previous section, we introduce the quasi-norms for $v \in W^{1,p}(\Omega) + \mathcal{N}_h$

$$|v|_{(u,p)}^2 := \sum_{K \in \mathcal{T}_h} \int_K (|\nabla u| + |\nabla v|)^{p-2} |\nabla v|^2 \, dx$$

and for $\tau \in (L^{p'}(\Omega))^2$

$$\|\tau\|_{(\sigma,p')}^2 := \int_\Omega (|\sigma| + |\tau|)^{p'-2} |\tau|^2 \, dx,$$

where (σ, u) is the solution of the mixed system (1.5). Note that we have $|\sigma| = |\nabla u|^{p-1}$.

The following relationship between these quasi-norms and the standard Sobolev norms can be found in [3,15].

Proposition 4.1 *Let $v \in W^{1,p}(\Omega) + \mathcal{N}_h$. Then we have for $1 < p < 2$*

$$|v|^2_{(u,p)} \leq |v|^p_{1,p,\mathcal{T}_h} \leq |v|^p_{(u,p)} D(u,v), \tag{4.1}$$

and for $2 < p < \infty$

$$|v|^p_{1,p,\mathcal{T}_h} \leq |v|^2_{(u,p)} \leq |v|^2_{1,p,\mathcal{T}_h} D(u,v), \tag{4.2}$$

where we set

$$D(u,v) := \left(\sum_{K \in \mathcal{T}_h} \int_K (|\nabla u| + |\nabla v|)^p \, dx \right)^r \leq C(|u|_{1,p} + |v|_{1,p,\mathcal{T}_h})^{rp}$$

with $r = \frac{2-p}{2}$ for $1 < p < 2$ and $r = \frac{p-2}{p}$ for $2 < p < \infty$. Furthermore, the following triangle inequality holds for any v_1, v_2

$$|v_1 + v_2|_{(u,p)} \leq C(|v_1|_{(u,p)} + |v_2|_{(u,p)}) \tag{4.3}$$

with $C = \max(2, 2^{p-1})^{1/2}$. Similar results hold as well for the quasi-norm $\|\tau\|_{(\sigma,p')}$.

We also note the following inequality which will be used frequently in the subsequent analysis: for all $p > 1$ and $\sigma_1, \sigma_2 \in \mathbb{R}^2$,

$$C_1(|\sigma_1| + |\sigma_2|)^{p-2} \leq (|\sigma_1| + |\sigma_1 - \sigma_2|)^{p-2} \leq C_2(|\sigma_1| + |\sigma_2|)^{p-2}.$$

This follows easily from the triangle inequality.

Now we are ready to derive the following error estimate for the $P1$ nonconforming finite element method (3.3), which is a slight modification of Theorem 4.1 of [15].

Theorem 4.1 *Let u and u_h be the solutions of* (1.2) *and* (3.3)*, respectively. Then we have for all $v_h \in \mathcal{N}_h$*

$$|u - u_h|^2_{(u,p)} \leq C(|u - v_h|^2_{(u,p)} + E + \epsilon), \tag{4.4}$$

where

$$E = \sum_{K \in \mathcal{T}_h} h_K \int_{\partial K} (|\nabla u| + |\nabla (u - v_h)|)^{p-2} |\nabla (u - v_h)|^2 \, dx, \tag{4.5}$$

$$\epsilon = \sum_{K \in \mathcal{T}_h} \int_K (|\nabla u_h|^{p-1} + h_K |f - \bar{f}|)^{p'-2} h_K^2 |f - \bar{f}|^2 \, dx. \tag{4.6}$$

Proof Let u_h^* be the solution of the variational problem

$$\sum_{K \in \mathcal{T}_h} \int_K |\nabla u_h^*|^{p-2} \nabla u_h^* \cdot \nabla v_h \, dx = \int_\Omega f v_h \, dx \qquad \forall v_h \in \mathcal{N}_h.$$

It was shown in [15] that

$$|u - u_h^*|^2_{(u,p)} \leq C(|u - v_h|^2_{(u,p)} + E).$$

Since we have by Lemma 2.5

$$|u - u_h|^2_{(u,p)} \le C(|u - u_h^*|^2_{(u,p)} + |u_h^* - u_h|^2_{(u_h,p)}),$$

it suffices to prove that

$$|u_h^* - u_h|^2_{(u_h,p)} \le \epsilon.$$

Note that we have by Lemma 2.1

$$|u_h^* - u_h|^2_{(u_h,p)} \le C \int_\Omega (f - \bar{f})(u_h^* - u_h)\, dx.$$

Now, by using the estimate

$$\|(u_h^* - u_h) - \overline{(u_h^* - u_h)}\|_{0,\infty,K} \le C h_K |\nabla(u_h^* - u_h)|$$

and then Lemma 2.3, it follows that

$$\begin{aligned}
&\int_\Omega (f - \bar{f})(u_h^* - u_h)\, dx \\
&\le C \sum_{K \in \mathcal{T}_h} \int_K |f - \bar{f}| \cdot h_K |\nabla(u_h^* - u_h)|\, dx \\
&\le C\delta^{-\beta} \sum_{K \in \mathcal{T}_h} \int_K (|\nabla u_h|^{p-1} + h_K |f - \bar{f}|)^{p'-2} h_K^2 |f - \bar{f}|^2\, dx \\
&\quad + C\delta \sum_{K \in \mathcal{T}_h} \int_K (|\nabla u_h| + |\nabla(u_h^* - u_h)|)^{p-2} |\nabla(u_h^* - u_h)|^2\, dx \\
&= C\delta^{-\beta}\epsilon + C\delta |u_h^* - u_h|^2_{(u_h,p)}.
\end{aligned}$$

Thus the proof is completed by choosing $\delta > 0$ sufficiently small.

The consistency error term E was shown to be optimal in [15] (with $v_h = \pi_h u$) for sufficiently smooth u: if $u \in C^{2,2/p-1}(\overline{\Omega}) \cap W^{3,1}(\Omega)$ for $1 < p < 2$ and $u \in W^{1,\infty}(\Omega) \cap H^2(\Omega)$ for $2 < p < \infty$, then we have

$$E \le Ch^2. \tag{4.7}$$

The second term ϵ is the one which typically arises from a posteriori error analysis for the p-Laplacian problem (cf. [15–17]), and can be also shown to be of order $O(h^2)$ under suitable conditions on f. For $1 < p < 2$ we can obtain by Hölder's inequality and (3.7)

$$\epsilon \le Ch^2 \|f - \bar{f}\|^2_{0,p'}. \tag{4.8}$$

For $2 < p < \infty$ it is trivial to verify that

$$\epsilon \le Ch^{p'} \|f - \bar{f}\|^{p'}_{0,p'}. \tag{4.9}$$

Therefore, it follows that $\epsilon = O(h^2)$ if $f \in L^{p'}(\Omega)$ for $1 < p < 2$ and $f|_K \in W^{1-2/p,p'}(K)$ for $2 < p < \infty$.

Combining these results with Theorem 4.1, we arrive at the following error estimates for $u - u_h$ in the quasi-norm.

Theorem 4.2 *Let u and u_h be the solutions of* (1.2) *and* (3.3)*, respectively. Suppose that $f \in L^{p'}(\Omega)$ for $1 < p < 2$ and $f|_K \in W^{1-2/p,p'}(K)$ for $2 < p < \infty$. Then we have for $1 < p < 2$*

$$|u - u_h|_{(u,p)} \le \begin{cases} Ch^{p/2} & \text{if } u \in W^{2,p}(\Omega), \\ Ch & \text{if } u \in C^{2,2/p-1}(\overline{\Omega}) \cap W^{3,1}(\Omega), \end{cases}$$

and for $2 < p < \infty$

$$|u - u_h|_{(u,p)} \le Ch^{s/2} \qquad \text{if } u \in W^{1,\infty}(\Omega) \cap W^{2,s}(\Omega),\ s \in [1,2].$$

Proof Take $v_h = \pi_h u$ and proceed as in [3] to estimate $|u - v_h|_{(u,p)}$ under the stated regularity condition on u.

From Proposition 4.1 and Theorem 4.2 one can readily derive the error estimates for $u - u_h$ in the $W^{1,p}(\Omega)$-norm. We refer to [3,13–15] for full details concerning these error estimates and the achievability of regularity requirements on u stated above.

Now we turn to the error estimates for the vector approximation.

Theorem 4.3 *Let (σ_h, u_h) be the solution of the mixed finite volume method* (3.1)–(3.2)*. Then we have*

$$\|\sigma - \sigma_h\|^2_{(\sigma,p')} \le C(|u - u_h|^2_{(u,p)} + \epsilon), \tag{4.10}$$

where ϵ was defined in Theorem 4.1.

Proof Letting

$$D_1 = |\nabla u|^{p-2}\nabla u - |\nabla u_h|^{p-2}\nabla u_h$$

and

$$D_2 = |\nabla u_h|^{p-2}\nabla u_h - \sigma_h,$$

we obtain by Lemma 2.5

$$\begin{aligned} \|\sigma - \sigma_h\|^2_{(\sigma,p')} &\le C \sum_{K \in \mathcal{T}_h} \int_K (|\nabla u_h|^{p-1} + |D_1|)^{p'-2}|D_1|^2\,dx \\ &\quad + C \sum_{K \in \mathcal{T}_h} \int_K (|\nabla u_h|^{p-1} + |D_2|)^{p'-2}|D_2|^2\,dx \\ &:= I_1 + I_2. \end{aligned}$$

Let us first consider the term I_1. Since we have by Lemma 2.1

$$D_1 \le C(|\nabla u| + |\nabla u_h|)^{p-2}|\nabla(u - u_h)| \le C(|\nabla u| + |\nabla u_h|)^{p-1},$$

it follows for $1 < p < 2$ that

$$\begin{aligned} &(|\nabla u_h|^{p-1} + |D_1|)^{p'-2}|D_1|^2 \\ &\quad \le C(|\nabla u| + |\nabla u_h|)^{(p-1)(p'-2)}(|\nabla u| + |\nabla u_h|)^{2(p-2)}|\nabla(u - u_h)|^2 \\ &\quad = C(|\nabla u| + |\nabla u_h|)^{p-2}|\nabla(u - u_h)|^2, \end{aligned}$$

which immediately yields

$$I_1 \leq C|u - u_h|^2_{(u,p)}.$$

For $2 < p < \infty$, we obtain

$$\begin{aligned}
&|\nabla u_h|^{p-1} + (|\nabla u| + |\nabla u_h|)^{p-2}|\nabla(u - u_h)| \\
&\quad \geq |\nabla u_h|^{p-1} + |\nabla(u - u_h)|^{p-1} \geq C(|\nabla u| + |\nabla u_h|)^{p-1}.
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
&(|\nabla u_h|^{p-1} + |D_1|)^{p'-2}|D_1|^2 \\
&\quad \leq C(|\nabla u_h|^{p-1} + (|\nabla u| + |\nabla u_h|)^{p-2}|\nabla(u - u_h)|)^{p'-2} \\
&\qquad\qquad \times(|\nabla u| + |\nabla u_h|)^{2(p-2)}|\nabla(u - u_h)|^2 \\
&\quad \leq C(|\nabla u| + |\nabla u_h|)^{(p-1)(p'-2)}(|\nabla u| + |\nabla u_h|)^{2(p-2)}|\nabla(u - u_h)|^2 \\
&\quad = C(|\nabla u| + |\nabla u_h|)^{p-2}|\nabla(u - u_h)|^2,
\end{aligned}$$

where, in the first inequality, we have used the fact that the function $\varphi(x) = (a + x)^{p-2}x^2$ is increasing on $[0, \infty)$ for all $p > 1$ and $a \geq 0$. This again yields

$$I_1 \leq C|u - u_h|^2_{(u,p)}.$$

To estimate the other term I_2, we note by (3.6) that $|D_2| \leq Ch_K|\bar{f}|$, which implies

$$I_2 \leq C \sum_{K\in\mathcal{T}_h} \int_K (|\nabla u_h|^{p-1} + h_K|\bar{f}|)^{p'-2}h_K^2|\bar{f}|^2\,dx.$$

Let us point out that the right-hand side has the same form as the element residual part of the residual-type error estimators given in [15]. Following the proof of Theorem 5.2 there, one can easily derive

$$\sum_{K\in\mathcal{T}_h} \int_K (|\nabla u_h|^{p-1} + h_K|\bar{f}|)^{p'-2}h_K^2|\bar{f}|^2\,dx \leq C(|u - u_h|^2_{(u,p)} + \epsilon).$$

This completes the proof. $\square$

Using the results of Proposition 4.1 and Theorems 4.2–4.3 , we obtain the following error estimates for the vector approximation.

Theorem 4.4 *Let* (σ, u) *and* (σ_h, u_h) *be the solutions of the mixed system* (1.5) *and the mixed finite volume method* (3.1)–(3.2)*, respectively. Suppose that* $f \in L^{p'}(\Omega)$ *for* $1 < p < 2$ *and* $f|_K \in W^{1-2/p,p'}(K)$ *for* $2 < p < \infty$*. Then we have for* $1 < p < 2$

$$\|\sigma - \sigma_h\|_{(\sigma,p')} \leq \begin{cases} Ch^{p/2} & \text{if } u \in W^{2,p}(\Omega), \\ Ch & \text{if } u \in C^{2,2/p-1}(\overline{\Omega}) \cap W^{3,1}(\Omega), \end{cases}$$

and for $2 < p < \infty$

$$\|\sigma - \sigma_h\|_{(\sigma,p')} \le Ch^{s/2} \qquad \text{if } u \in W^{1,\infty}(\Omega) \cap W^{2,s}(\Omega),\ s \in [1,2].$$

Moreover, the following $L^{p'}$*-error estimates hold: we have for* $1 < p < 2$

$$\|\sigma - \sigma_h\|_{0,p'} \le \begin{cases} Ch^{p-1} & \text{if } u \in W^{2,p}(\Omega), \\ Ch^{2/p'} & \text{if } u \in C^{2,2/p-1}(\overline{\Omega}) \cap W^{3,1}(\Omega), \end{cases}$$

and for $2 < p < \infty$

$$\|\sigma - \sigma_h\|_{0,p'} \le Ch^{s/2} \qquad \text{if } u \in W^{1,\infty}(\Omega) \cap W^{2,s}(\Omega),\ s \in [1,2].$$

Remark 4.1 In the case of $1 < p < 2$, Farhloul [11] obtained the following error estimate for the mixed finite element method of problem (1.1)

$$\|\sigma - \sigma_h\|_{0,p'} \le Ch^{2/p'},$$

assuming that $\sigma \in (W^{1,p'}(\Omega))^2$ and $u \in H^1(\Omega)$. He did not give any result for $2 < p < \infty$ due to some technical difficulties in establishing the inf-sup condition. Our mixed finite volume method leads to the explicit formula (3.6) for σ_h in terms of u_h, and thereby we can avoid all difficulties related to the inf-sup condition. Moreover, the regularity assumption is focused on the scalar u rather than the vector σ.

5 Implicit error estimator

In this section we derive an implicit error estimator of Bank-Weiser type for the mixed finite volume method (3.1)–(3.2) which is based on solving the p-Laplacian problem locally on each $K \in \mathcal{T}_h$ subject to the Neumann boundary condition given by the vector approximation σ_h.

Before going further, let us point out that, by the estimates (4.8)–(4.9), we obtain $\epsilon = o(h^2)$ if $f|_K \in W^{\alpha,p'}(K)$, where $\alpha > 0$ for $1 < p < 2$ and $\alpha > 1 - 2/p$ for $2 < p < \infty$. Therefore, Theorem 4.3 implies that any upper bound for $|u - u_h|_{(u,p)}$ also yields an upper bound for $\|\sigma - \sigma_h\|_{(\sigma,p')}$ up to a higher order term.

Now we present an implicit error estimator for $|u - u_h|_{(u,p)}$. To this end we define $\phi_K \in W^{1,p}(K)$ to be a solution of the local Neumann problem

$$\int_K |\nabla \phi_K|^{p-2} \nabla \phi_K \cdot \nabla v \, dx = \int_{\partial K} \sigma_h \cdot \nu_K \, v \, ds + \int_K f v \, dx \tag{5.1}$$

for all $v \in W^{1,p}(K)$, and let $\phi|_K = \phi_K$. From the local conservation law (3.2), it follows that such a solution exists and is unique up to an additive constant (which does not affect the subsequent results).

We also need an auxiliary function $\tilde{u}_h \in W_0^{1,p}(\Omega)$ which is defined to be the continuous piecewise linear function on $\mathcal{T}_h$ constructed by averaging the values of u_h at the interior vertices of $\mathcal{T}_h$, i.e.,

$$\tilde{u}_h(z) := \sum_{K \in \omega_z} \alpha_{z,K} u_h|_K(z). \tag{5.2}$$

Here z is a given interior vertex of $\mathcal{T}_h$, $\omega_z = \{K_j\}_{j=1}^{n_z}$ is the set of all elements $K \in \mathcal{T}_h$ such that z is a vertex of K, and the weights $\{\alpha_{z,K}\}$ are chosen such that

$$\sum_{K \in \omega_z} \alpha_{z,K} = 1 \qquad \text{and} \qquad 0 \le \alpha_{z,K} \le 1. \tag{5.3}$$

For example, one can choose the simple averaging

$$\tilde{u}_h(z) := \frac{1}{n_z} \sum_{K \in \omega_z} u_h|_K(z), \tag{5.4}$$

or the weighted averaging

$$\tilde{u}_h(z) := \frac{\sum_{K \in \omega_z} |K|\, u_h|_K(z)}{\sum_{K \in \omega_z} |K|}. \tag{5.5}$$

This kind of auxiliary functions are also utilized in [10,20] to derive some a posteriori error estimators for nonconforming finite element methods.

With the above definitions, we define our error estimator by

$$\eta^2 := \sum_{K \in \mathcal{T}_h} \int_K (|\nabla \phi_K|^{p-2} \nabla \phi_K - |\nabla \tilde{u}_h|^{p-2} \nabla \tilde{u}_h) \cdot \nabla(\phi_K - \tilde{u}_h)\, dx, \tag{5.6}$$

or equivalently (by Lemma 2.1)

$$\eta^2 := |\phi - \tilde{u}_h|^2_{(\tilde{u}_h, p)}. \tag{5.7}$$

We prefer the former definition for η^2 over the latter one, since we obtain by definition (5.1)

$$\begin{aligned} \eta^2 = \sum_{K \in \mathcal{T}_h} &\left(\int_{\partial K} \sigma_h \cdot \nu_K\, (\phi_K - \tilde{u}_h)\, ds + \int_K f(\phi_K - \tilde{u}_h)\, dx \right) \\ &- \sum_{K \in \mathcal{T}_h} \int_K |\nabla \tilde{u}_h|^{p-2} \nabla \tilde{u}_h \cdot \nabla(\phi_K - \tilde{u}_h)\, dx. \end{aligned}$$

Note that this is computationally more convenient, since there is no need to compute $(p-2)$-th powers of nonconstant functions.

Now let us prove that η provides an upper bound for $|u - u_h|_{(u,p)}$.

Theorem 5.1 *Let u and u_h be the solutions of* (1.2) *and* (3.3)*, respectively. Then we have*

$$|u - u_h|^2_{(u,p)} \le C(\eta^2 + \epsilon), \tag{5.8}$$

where ϵ was defined in Theorem 4.1.

Proof We note that, for any $v \in W_0^{1,p}(\Omega)$,

$$\begin{aligned}\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla \tilde{u}_h|^{p-2}\nabla \tilde{u}_h)\cdot \nabla v\,dx &= \int_\Omega f v\,dx - \int_\Omega |\nabla \tilde{u}_h|^{p-2}\nabla \tilde{u}_h \cdot \nabla v\,dx \\ &= \sum_{K\in\mathcal{T}_h}\int_K (|\nabla \phi_K|^{p-2}\nabla\phi_K - |\nabla \tilde{u}_h|^{p-2}\nabla \tilde{u}_h)\cdot\nabla v\,dx,\end{aligned}$$

where the last equality follows from the fact that σ_h has continuous normal components across the interelement boundaries. By taking $v = u - \tilde{u}_h$ and using Lemmas 2.1–2.2, we obtain

$$\begin{aligned}|u-\tilde{u}_h|^2_{(\tilde{u}_h,p)} &\le C\sum_{K\in\mathcal{T}_h}\int_K (|\nabla\tilde{u}_h| + |\nabla(\phi_K-\tilde{u}_h)|)^{p-2}|\nabla(\phi_K-\tilde{u}_h)| \\ &\qquad\qquad\times|\nabla(u-\tilde{u}_h)|\,dx \\ &\le C\theta^{-\gamma}|\phi-\tilde{u}_h|^2_{(\tilde{u}_h,p)} + C\theta|u-\tilde{u}_h|^2_{(\tilde{u}_h,p)}.\end{aligned}$$

It then follows by choosing $\theta > 0$ sufficiently small that

$$|u-\tilde{u}_h|^2_{(\tilde{u}_h,p)} \le C|\phi-\tilde{u}_h|^2_{(\tilde{u}_h,p)} \le C\eta^2.$$

On the other hand, we obtain by using (3.1), (3.4) and taking $v = u_h - \tilde{u}_h$ in (5.1)

$$\begin{aligned}\int_K (|\nabla u_h|^{p-2}\nabla u_h - |\nabla\tilde{u}_h|^{p-2}\nabla\tilde{u}_h)\cdot\nabla(u_h-\tilde{u}_h)\,dx &= \int_K (\sigma_h - |\nabla\tilde{u}_h|^{p-2}\nabla\tilde{u}_h)\cdot\nabla(u_h-\tilde{u}_h)\,dx \\ &= \int_K (|\nabla\phi_K|^{p-2}\nabla\phi_K - |\nabla\tilde{u}_h|^{p-2}\nabla\tilde{u}_h)\cdot\nabla(u_h-\tilde{u}_h)\,dx \\ &\qquad - \int_K (f-\bar{f})(u_h-\tilde{u}_h)\,dx.\end{aligned}$$

By Lemmas 2.1–2.3 and the estimate

$$\|(v_h-u_h) - \overline{(v_h-u_h)}\|_{0,\infty,K} \le Ch_K|\nabla(v_h-u_h)|,$$

it follows that

$$\begin{aligned}|u_h-\tilde{u}_h|^2_{(\tilde{u}_h,p)} &\le C\sum_{K\in\mathcal{T}_h}\int_K (|\nabla\tilde{u}_h| + |\nabla(\phi_K-\tilde{u}_h)|)^{p-2}|\nabla(\phi_K-\tilde{u}_h)| \\ &\qquad\qquad\times|\nabla(u_h-\tilde{u}_h)|\,dx \\ &\qquad + C\sum_{K\in\mathcal{T}_h}\int_K |f-\bar{f}|\cdot h_K|\nabla(u_h-\tilde{u}_h)|\,dx \\ &\le C\theta^{-\gamma}|\phi-\tilde{u}_h|^2_{(\tilde{u}_h,p)} + C\theta|u_h-\tilde{u}_h|^2_{(\tilde{u}_h,p)} \\ &\qquad + C\delta^{-\beta}\epsilon + C\delta|u_h-\tilde{u}_h|^2_{(\tilde{u}_h,p)},\end{aligned}$$

which yields by choosing $\theta > 0$ and $\delta > 0$ sufficiently small

$$|u_h - \tilde{u}_h|^2_{(\tilde{u}_h,p)} \le C(|\phi - \tilde{u}_h|^2_{(\tilde{u}_h,p)} + \epsilon) \le C(\eta^2 + \epsilon).$$

Collecting all the results and using Lemma 2.5, we finally obtain

$$\begin{aligned} |u - u_h|^2_{(u,p)} &\le C(|u - \tilde{u}_h|^2_{(\tilde{u}_h,p)} + |u_h - \tilde{u}_h|^2_{(\tilde{u}_h,p)}) \\ &\le C(\eta^2 + \epsilon). \end{aligned}$$

This completes the proof. □

Remark 5.1 Theorem 5.1 is valid regardless of the choice of $\tilde{u}_h$, as long as it is a continuous piecewise linear function on $\mathcal{T}_h$. However, in order for the estimator η to be an efficient estimator for the true error, $\tilde{u}_h$ should be chosen sufficiently close to the solution u_h, and in addition, should be easy to compute from u_h. Averaging of the values of u_h at the interior vertices provides a good choice of $\tilde{u}_h$ satisfying these requirements.

We next establish that η provides a lower bound for $|u - u_h|_{(u,p)}$ up to some higher order terms. This is done by comparing η with the second error estimator given in [15]. To do this, we introduce some further notation. Given an element $K \in \mathcal{T}_h$, let $O(K)$ be the collection of all $K' \in \mathcal{T}_h$ such that $\overline{K}' \cap \overline{K} \neq \emptyset$, and $L(K)$ the collection of all the edges of $K' \in O(K)$. Let A_e be the jump of the tangential derivative of u_h on an edge e, and set

$$|\hat{A}_K| = \max_{e \in L(K)} |A_e|.$$

Then, following the proof of Theorem 6.2 in [15], one can derive

$$\bar{\eta}_1^2 + \hat{\eta}_2^2 \le C\Big(|u - u_h|^2_{(u,p)} + \epsilon + \inf_{v_h \in S_h^1(k)} |u - v_h|^2_{(u,p)}\Big), \tag{5.9}$$

where ϵ was defined in Theorem 4.1, and

$$\bar{\eta}_1^2 := \sum_{K \in \mathcal{T}_h} \int_K (|\nabla u_h|^{p-1} + h_K|\bar{f}|)^{p'-2} h_K^2 |\bar{f}|^2 \, dx, \tag{5.10}$$

$$\hat{\eta}_2^2 := \sum_{K \in \mathcal{T}_h} \int_K (|\nabla u_h| + |\hat{A}_K|)^{p-2} |\hat{A}_K|^2 \, dx, \tag{5.11}$$

and

$$S_h^1(k) := \{v \in C^1(\overline{\Omega}) : v|_K \in P_k(K) \ \ \forall K \in \mathcal{T}_h, \text{ and } v|_{\partial\Omega} = 0\}.$$

We also need the following result which is similar to those given in [17].

Lemma 5.1 *Let $K \in \mathcal{T}_h$ and $p > 1$. Then we have for all $v \in W^{1,p}(K)$*

$$\begin{aligned} &\int_K (|\nabla u_h| + h_K^{-1}|v - \bar{v}|)^{p-2} h_K^{-2} |v - \bar{v}|^2 \, dx \\ &\quad \le C \int_K (|\nabla u_h| + |\nabla v|)^{p-2} |\nabla v|^2 \, dx, \end{aligned} \tag{5.12}$$

where $\bar{v}$ is the average of v over K.

Proof We use the standard scaling argument and the quasi-norm version of the Poincare-type inequality proven in [17]. Letting $\hat{K}$ be the reference element and applying Lemma 3.1 of [17] (with $f(v) = \bar{v}$), we obtain

$$\begin{aligned}\int_K &(|\nabla u_h| + h_K^{-1}|v - \bar{v}|)^{p-2} h_K^{-2}|v - \bar{v}|^2\, dx\\ &\le C h_K^{2-p} \int_{\hat{K}} (h_K|\nabla u_h| + |v - \bar{v}|)^{p-2}|v - \bar{v}|^2\, d\hat{x}\\ &\le C h_K^{2-p} \int_{\hat{K}} (h_K|\nabla u_h| + |\nabla v|)^{p-2}|\nabla v|^2\, d\hat{x},\end{aligned}$$

where C depends only on p, $\hat{K}$ and the shape-regularity constant of K. Since we have $|\nabla v|_{\hat{K}}(\hat{x})| \le C h_K |\nabla v|_K(x)|$, and $\varphi(x) = (a + x)^{p-2}x^2$ is increasing on $[0, \infty)$ for all $p > 1$ and $a \ge 0$, it follows that

$$\begin{aligned}\int_{\hat{K}} &(h_K|\nabla u_h| + |\nabla v|)^{p-2}|\nabla v|^2\, d\hat{x}\\ &\le C h_K^{p-2} \int_K (|\nabla u_h| + |\nabla v|)^{p-2}|\nabla v|^2\, dx,\end{aligned}$$

which gives the desired estimate. □

Now we are ready to prove the second main result of this section.

Theorem 5.2 *Let u and u_h be the solutions of* (1.2) *and* (3.3)*, respectively. Then we have*

$$\eta^2 \le C(\bar{\eta}_1^2 + \hat{\eta}_2^2 + \epsilon) \tag{5.13}$$

and

$$\eta^2 \le C\Big(|u - u_h|^2_{(u,p)} + \epsilon + \inf_{v_h \in S_h^1(k)} |u - v_h|^2_{(u,p)}\Big), \tag{5.14}$$

where $\bar{\eta}_1$ and $\hat{\eta}_2$ were defined above, and ϵ was defined in Theorem 4.1.

Proof In light of the estimate (5.9), it suffices to prove the first assertion. By Lemma 2.5 we can divide η^2 into the two parts

$$\eta^2 \le C(|\phi - u_h|^2_{(u_h,p)} + |u_h - \tilde{u}_h|^2_{(u_h,p)}).$$

Let us estimate the first term. It follows from (3.6) and Lemma 2.3 that

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} &\int_K (|\nabla\phi_K|^{p-2}\nabla\phi_K - |\nabla u_h|^{p-2}\nabla u_h)\cdot\nabla(\phi_K - u_h)\, dx\\ &= \sum_{K\in\mathcal{T}_h} \int_K (\sigma_h - |\nabla u_h|^{p-2}\nabla u_h)\cdot\nabla(\phi_K - u_h)\, dx\\ &\qquad + \sum_{K\in\mathcal{T}_h} \int_K (\nabla\cdot\sigma_h + f)(\phi_K - u_h)\, dx\end{aligned}$$

$$
\begin{aligned}
&\le C \sum_{K\in\mathcal{T}_h} \int_K h_K |\bar f|\, |\nabla(\phi_K - u_h)|\, dx \\
&\qquad\qquad + C \sum_{K\in\mathcal{T}_h} \int_K |f - \bar f|\, |(\phi_K - u_h) - \overline{(\phi_K - u_h)}|\, dx \\
&\le C\delta^{-\beta}(\bar\eta_1^2 + \epsilon) + C\delta|\phi - u_h|^2_{(u_h,p)} \\
&\quad + C\delta \sum_{K\in\mathcal{T}_h} \int_K (|\nabla u_h| + h_K^{-1}|(\phi_K - u_h) - \overline{(\phi_K - u_h)}|)^{p-2} \\
&\qquad\qquad\qquad \times h_K^{-2}|(\phi_K - u_h) - \overline{(\phi_K - u_h)}|^2\, dx.
\end{aligned}
$$

From Lemma 5.1 it follows that the last term can be bounded above by $C\delta|\phi - u_h|^2_{(u_h,p)}$, and so we conclude that

$$
|\phi - u_h|^2_{(u_h,p)} \le C(\bar\eta_1^2 + \epsilon)
$$

by taking $\delta > 0$ to be sufficiently small.

Now it remains to estimate the second term $|u_h - \tilde u_h|^2_{(u_h,p)}$. This will be done by proving that

$$
|u_h - \tilde u_h|^2_{(u_h,p)} \le C\hat\eta_2^2.
$$

Recall that $\varphi(x) = (a + x)^{p-2}x^2$ is increasing on $[0, \infty)$ for all $p > 1$ and $a \ge 0$. Thus it suffices to prove

$$
|\nabla(u_h - \tilde u_h)|_K| \le C|\hat A_K|. \tag{5.15}
$$

To this end we first note that, for any element K with vertices z_1, z_2, z_3,

$$
|\nabla(u_h - \tilde u_h)|_K| \le Ch_K^{-1} \max_{1\le i\le 3} |(u_h - \tilde u_h)|_K(z_i)|. \tag{5.16}
$$

This follows, for example, from the integral identity

$$
\int_K \nabla(u_h - \tilde u_h)\, dx = \int_{\partial K} (u_h - \tilde u_h)\nu_K\, ds.
$$

Now fix $z = z_i$, and label the elements $\omega_z = \{K_j\}_{j=1}^{n_z}$ in such a way that $K_1 = K$, and K_j and K_{j+1} are adjacent elements. From the definition (5.2) of $\tilde u_h$ and the fact that n_z is uniformly bounded, one can deduce that

$$
\begin{aligned}
|(u_h - \tilde u_h)|_K(z)| &\le \sum_{j=2}^{n_z} \alpha_{z,K_j} |u_h|_{K_1}(z) - u_h|_{K_j}(z)| \qquad (5.17)\\
&\le \sum_{j=2}^{n_z} \alpha_{z,K_j} \left(\sum_{k=1}^{j-1} |u_h|_{K_k}(z) - u_h|_{K_{k+1}}(z)| \right) \\
&\le C \max_{e\in L(z)} |[u_h]_e(z)|,
\end{aligned}
$$

where $L(z)$ is the set of all edges whose one endpoint is z, and $[u_h]_e$ denotes the jump of $[u_h]$ on an edge e. On the other hand, since $[u_h]_e$ is a linear function which vanishes at the midpoint of e, it follows that, for any edge e with endpoints z_1, z_2,

$$|A_e| = \frac{2}{|e|}|[u_h]_e(z_1)| = \frac{2}{|e|}|[u_h]_e(z_2)|. \tag{5.18}$$

Therefore, the desired estimate (5.15) is proved by combining (5.16)–(5.18).

Remark 5.2 As noted at the beginning of this section, the term ϵ is of higher order $o(h^2)$ for sufficiently smooth f. It was also shown in [15] that

$$\inf_{v_h \in S_h^1(k)} |u - v_h|^2_{(u,p)} = o(h^2)$$

under the realistic regularity condition that $u \in C^{2,2/p-1}(\overline{\Omega}) \cap W^{3,1}(\Omega)$ for $1 < p < 2$ and $u \in W^{1,\infty}(\Omega) \cap H^2(\Omega)$ for $2 < p < \infty$ which ensures optimal convergence for $|u - u_h|^2_{(u,p)}$ (see Theorem 4.2).

Remark 5.3 It is impractical to solve exactly the local Neumann problems (5.1) which are posed in infinite-dimensional spaces. In practice, we solve them by using high order finite element methods, as in [1]: let $\phi_K^l \in P_l(K)$ be a solution of the local Neumann problem

$$\int_K |\nabla \phi_K^l|^{p-2} \nabla \phi_K^l \cdot \nabla v \, dx = \int_{\partial K} \sigma_h \cdot \nu_K \, v \, ds + \int_K f v \, dx \tag{5.19}$$

for all $v \in P_l(K)$, and let $\phi^l|_K = \phi_K^l$. Then it is not difficult to show that the lower bound is unchanged, whereas we obtain the new upper bound

$$|u - u_h|^2_{(u,p)} \le C\Big(\eta^2 + \epsilon + \inf_{v_h \in S_h^0(l)} |u - v_h|^2_{(u,p)}\Big),$$

where

$$S_h^0(l) := \{v \in C^0(\overline{\Omega}) : v|_K \in P_l(K) \ \ \forall K \in \mathcal{T}_h, \ \text{and } v|_{\partial\Omega} = 0\}.$$

The new extra term $\inf_{v_h \in S_h^0(l)} |u - v_h|^2_{(u,p)}$ is also of higher order $o(h^2)$ under the same regularity condition stated in Remark 5.2 (see [15, 16]).

6 Numerical results

Some numerical experiments are performed on the unit square $\Omega = (0, 1)^2$ in order to confirm the theoretical results obtained in the previous sections. Given a triangulation of Ω, we use the Polak-Ribiére conjugate gradient method to compute the scalar solution u_h of problem (3.3), with $\bar{f}$ being computed by the one-point quadrature on each element. The vector solution σ_h is then immediately obtained from the formula (3.6).

In the examples below we report the errors in both the standard Sobolev norms and the quasi-norms. To compute the integrals in the approximation errors and the estimator η given by (5.6), a sufficiently accurate numerical quadrature is adopted. We choose the quadratic polynomial space $P_2(K)$ for the local problem (5.1) and the weighted average (5.5) for $\tilde{u}_h$.

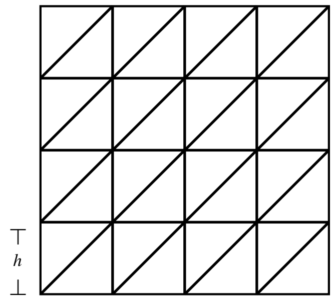

Fig. 6.1 Uniform mesh of right isosceles triangles of size h

Table 6.1 Scalar errors for Example 6.1

h	$\lvert u-u_h\rvert_{1,p}$	$\lvert u-u_h\rvert_{(u,p)}$	η
1/16	0.011398	0.015827	0.016885
1/32	0.005708	0.008004	0.008407
1/64	0.002855	0.004026	0.004184
1/128	0.001428	0.002019	0.002086
1/256	0.000714	0.001011	0.001041
order	0.999	0.992	1.005

Table 6.2 Vector errors for Example 6.1

h	$\lVert\sigma-\sigma_h\rVert_{0,p'}$	$\lVert\sigma-\sigma_h\rVert_{(\sigma,p')}$	$\lVert\sigma-\bar\sigma_h\rVert_{0,p'}$	$\lVert\sigma-\bar\sigma_h\rVert_{(\sigma,p')}$
1/16	0.011382	0.006877	0.019627	0.011662
1/32	0.005937	0.003438	0.010228	0.005780
1/64	0.003067	0.001717	0.005281	0.002874
1/128	0.001574	0.000857	0.002709	0.001432
1/256	0.000801	0.000428	0.001381	0.000715
order	0.957	1.001	0.957	1.006

Example 6.1 We consider the radially symmetric problem from [3, 16] whose solution is

$$u(x,y)=\frac{p-1}{p-s}\left(\frac{1}{2-s}\right)^{\frac{1}{p-1}}(1-r^{\frac{p-s}{p-1}}),\qquad f(x,y)=r^{-s}$$

with $p=1.5$, $s=0.6$ and $r=(x^2+y^2)^{1/2}$. The triangulation consists of right isosceles triangles of equal size h, like shown in Figure 6.1.

We report the numerical results in Tables 6.1–6.2, where it is found that all errors are nearly of the order $O(h)$ and the error estimator η shows a very good performance. In particular, the order of convergence for $\|\sigma-\sigma_h\|_{0,p'}$ is better than the suboptimal $2/p'=2/3$ predicted in Theorem 4.4. It is also noteworthy that σ_h produces a better approximation to σ than the simple flux $\bar\sigma_h=|\nabla u_h|^{p-2}\nabla u_h$.

Example 6.2 In this example we take $p=4.0$ and $s=0.6$ for the solution of Example 6.1 with $r=((x+1)^2+(y+1)^2)^{1/2}$. Thus u and σ are infinitely

Table 6.3 Scalar errors for Example 6.2

h	$\|u - u_h\|_{1,p}$	$\|u - u_h\|_{(u,p)}$	η
1/16	0.007214	0.005447	0.006593
1/32	0.003608	0.002712	0.003275
1/64	0.001804	0.001353	0.001631
1/128	0.000903	0.000677	0.000815
1/256	0.000452	0.000338	0.000405
order	0.999	1.002	1.005

Table 6.4 Vector errors for Example 6.2

h	$\\|\sigma - \sigma_h\\|_{0,p'}$	$\\|\sigma - \sigma_h\\|_{(\sigma,p')}$	$\\|\sigma - \bar{\sigma}_h\\|_{0,p'}$	$\\|\sigma - \bar{\sigma}_h\\|_{(\sigma,p')}$
1/16	0.003421	0.003680	0.005817	0.006353
1/32	0.001710	0.001840	0.002908	0.003181
1/64	0.000855	0.000921	0.001454	0.001592
1/128	0.000428	0.000461	0.000727	0.000797
1/256	0.000211	0.000227	0.000362	0.000396
order	1.003	1.003	1.001	1.000

smooth, and as a result, Tables 6.3–6.4 show even better results than the previous ones.

Example 6.3 In the third example we consider the solution from [16]

$$u(x, y) = (x^2 + y^2)^{(p-2)/2(p-1)}, \qquad f = 0$$

with $p = 4.0$. Note that u has a strong singularity at the origin (0, 0).

Starting with the same initial mesh of size $h = 1/4$ in Figure 6.1, we perform both the uniform and adaptive mesh refinement for comparison of numerical results. For adaptive mesh refinement we follow the simple strategy of Verfürth [22]: mark the element K if

$$\eta_K \geq 0.5 \max_{K' \in \mathcal{T}_h} \eta_{K'}$$

with the local estimator η_K given by

$$\eta_K^2 = \int_K (|\nabla \phi_K|^{p-2} \nabla \phi_K - |\nabla \tilde{u}_h|^{p-2} \nabla \tilde{u}_h) \cdot \nabla(\phi_K - \tilde{u}_h)\, dx,$$

and further refine adjacent elements to avoid hanging nodes.

Numerical results are plotted in Figure 6.2 which displays the convergence behavior of the errors $|u - u_h|_{1,p}$ and $\|\sigma - \sigma_h\|_{0,p'}$ as the mesh is refined uniformly or adaptively. We calculate the order of convergence γ by the least-squares fitting, assuming that the error behaves like $O(N^{-\gamma})$ with N being the number of edges. It is observed that the adaptive mesh refinement almost achieves the optimal value $1/2$.

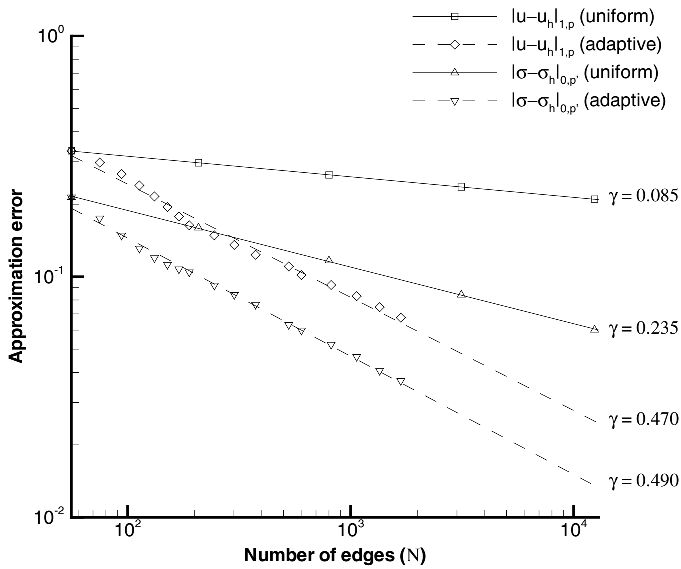

Fig. 6.2 Scalar and vector errors for Example 6.3. The errors are like $O(N^{-\gamma})$.

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