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Existence of classical solutions to a free boundary problem for the *p*-Laplace operator: (I) the exterior convex case

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Abstract. In this paper we prove, under convexity assumptions for the data, the existence of classical solutions for a Bernoulli-type free boundary problem, with the p-Laplacian as the governing operator. The method employed here originates from a pioneering work of A. Beurling where he proves the existence for the harmonic case in the plane, though with no geometrical restrictions.

1. Introduction and the main result

The aim of this paper is twofold; to prove the existence of classical solutions to nonlinear Bernoulli-type free boundary problems, and also to revitalize the celebrated method of sub- and supersolutions of Arne Beurling [Be]. The problem, roughly speaking, is as follows. For a given bounded domain $K \subset \mathbb{R}^N$ ($N \geqq 2$, and K is convex in this paper), one seeks a larger domain Ω such that the gradient of the p-capacitary potential of $\Omega \backslash K$ has a prescribed magnitude on $\partial\Omega$ (the boundary of Ω).

The problem arises in various nonlinear flow laws, and several physical situations, e.g. electrochemical machining and potential flow in fluid mechanics. For application in p-diffusion we refer to [P] and for deformation plasticity see [At-C]. We also refer to the paper of Acker and Meyer [Ac-M] for a good account of applications in general. For $p = 2$ the problem is well studied and there is a vast literature on the subject, treating different aspects. Standard references, relevant to our work for $p = 2$, are [Be], [Ac], [Al-C]; see also [F] and [Fl-R] and the references therein. For $1 < p < \infty$, the problem is more or less in a foster stage, and there are in principle not many known results. A recent paper of Acker and Meyer [Ac-M] treats this problem for starlike domains and with $1 < p < \infty$. They obtain global convergence proof for a particular analytical trial free boundary method for the successive approximation of the classical solution, which they assume to exist.

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It is, by now, well known that if there is a classical solution to the exterior problem (for $1 < p < \infty$) and if K (the data) is convex, then the solution is unique and convex, see [H-Sh], [Ac-M]. In this paper we show that for convex K there does exist a classical convex solution.

Mathematically the problem is formulated as follows: Given a (not necessarily bounded) convex domain $K \subset \mathbb{R}^N$, we look for a function u and a domain Ω $(\supset K)$ satisfying, for a given constant $c > 0$,

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega \backslash \overline{K}, \\ u = 1 & \text{on } \partial K, \\ u = 0 & \text{on } \partial\Omega, \\ |\nabla u| = c & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ_p denotes the p-Laplace operator, i.e., $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. The overdetermined boundary condition $|\nabla u| = c$ is to be understood in the following sense

$$\liminf_{\substack{y \to x \\ y \in \Omega}} |\nabla u(y)| = \limsup_{\substack{y \to x \\ y \in \Omega}} |\nabla u(y)| = c \quad \text{for every } x \in \partial\Omega. \tag{1.2}$$

Theorem 1.1. *Let K be a convex domain, not necessarily bounded or regular. Then there exists a classical solution Ω to the free boundary problem* (1.1) *with $C^{2,\alpha}$ boundary $\partial\Omega$. Moreover if K is bounded then the solution Ω is unique.*

It is noteworthy that the proof of Theorem 1.1 relies mainly on a very simple but fundamental lemma (Lemma 2.2 below, cf. also [H-Sh]).

2. Auxiliary lemmas

In this section we will sum up all auxiliary results used in this paper. First we remark that the usual comparison and maximum principle for elliptic partial differential equations, is one of the basic tools here; see [T]. Standard and new (Lemma 2.2) type of barrier arguments will also be among the fundamental tools.

Lemma 2.1. *Let u be a solution to $\Delta_p u = 0$ in a domain Ω, and introduce the linear elliptic operator L_u defined everywhere, except at critical points of u, by*

$$L_u := |\nabla u|^{p-2}\Delta + (p-2)|\nabla u|^{p-4} \sum_{k,l=1}^{N} \frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_l}\frac{\partial^2}{\partial x_k \partial x_l}. \tag{2.1}$$

Then $L_u(|\nabla u|^p) \geqq 0$ in Ω.

This lemma is essentially proved, though stated differently, in the papers of Payne and Philippin, [Pa-Ph1] and [Pa-Ph2]. For a proof of this lemma, follow calculations given in these papers and apply Lemma 1, page 391 of [Pa-Ph2] with the particular choices

$\alpha = -1, \beta = 0, \gamma = 1/2$ and $v = 0$, to obtain

$$\Delta|\nabla u|^p + \frac{(p-2)}{|\nabla u|^2} \sum_{k,l=1}^{N} \frac{\partial u}{\partial x_k} \frac{\partial u}{\partial x_l} \frac{\partial^2 |\nabla u|^p}{\partial x_k \partial x_l} \geqq \frac{p(p-1)^2}{4} |\nabla u|^{p-6} \left(\sum_{k=1}^{N} \frac{\partial u}{\partial x_k} \frac{\partial(|\nabla u|^2)}{\partial x_k} \right)^2.$$

Remark (Maximum/comparison principle)**.** By results of Lewis [Le], solutions to the p-Laplacian are $C^{1,\alpha}$ for some $\alpha > 0$. In particular $|\nabla u|$ is C^α in any region where u satisfies the p-Laplace equation

$$\Delta_p u = 0.$$

However the operator L_u, defined above, may fail to have the maximum/comparison principle. The weak maximum principle for the p-Laplace operator is well known and can be find in standard literature in this filed; see [H-K-M], [M-Z], and [D], the latter treats the parabolic case. The operator L_u, despite its linearity, does not fall within the framework of classical linear equations (cf. [G-T]), mainly because the coefficients of L_u may not be bounded or belong to some appropriate Sobolev space.

The same paper by Lewis [Le] shows also that the p-capacitary potential of the ring-shaped region $D_2 \backslash \overline{D}_1$, with convex D_1, D_2 and $D_1 \subset \overline{D}_1 \subset D_2$, is analytic in the region and $0 < |\nabla u| < \infty$ there. It thus remains the problem of avoiding the boundary points. If we wish to make the bounds for $|\nabla u|$ uniform (both from above and below) then we need to stay compactly inside the region. If this is possible then one can use the classical "strong" maximum/comparison principle.

On the other side it can be shown that the weak maximum/comparison principle holds for the linear operator L_u, in the above type domains. Indeed, we first observe that by linearity of L_u we may only consider the weak maximum principle. Next if the function has a strict local maximum point $x^0 \in D_2 \backslash \overline{D}_1$, then near this point the coefficients of L_u are uniformly bounded and analytic. Hence the strong maximum principle can be applied in $B(x^0, r)$ for small enough r to obtain a contradiction.

The above reasoning fails in general when we relax the convexity condition on the domains D_1, D_2. However, one may use some other standard techniques in the theory of the p-Laplacian. One such technique, kindly suggested to us by the referee, is to change the operator slightly by letting

$$(\varepsilon + |\nabla u|^2)^{1/2}$$

replace $|\nabla u|$ in the above and then consider approximate solutions u_ε at all crucial points of the proofs when we need to apply the maximum or comparison principle. We hope to use this technique in the future.

Yet another approach to circumvent the above discussed problem is to change the operator L_u into the form

$$\sum a_{ij} \partial_{ij},$$

where

$$a_{ij} = \delta_{ij} + (p-2)u_i u_j |\nabla u|^{-2}, \quad |a_{ij}| \leqq p-1.$$

This, again, is possible since inside the region the gradient is non-vanishing (according to Lewis [Le]). Hence we can replace L_u with $|\nabla|^{2-p} L_u$, which is exactly the above operator.

For two nested convex sets $D_1 \subset D_2$, and for $x \in \partial D_1$ we denote by $T_{x,a}$ the supporting hyperplane at x with the normal a pointing away from D_1. Obviously, $T_{x,a}$ is not necessarily unique, depending on the geometry of ∂D_1. Now for each $x \in \partial D_1$ there corresponds a point y_x (not necessarily unique) on $\partial D_2 \cap \{z : a \cdot (z-x) > 0\}$ and such that $a \cdot (y_x - x) = \max a \cdot (z-x)$, where the maximum has been taken over all

$$z \in \partial D_2 \cap \{z : a \cdot (z-x) > 0\}.$$

Lemma 2.2. *Let D_1 and D_2 be two nested open convex domains ($D_1 \subset \bar{D}_1 \subset D_2$) and denote by u the p-capacitary potential of $D_2 \backslash D_1$, i.e. the solution of*

$$\begin{cases} \Delta_p u = 0 & \text{in } D_2 \backslash \overline{D_1}, \\ u = c_1 & \text{on } \partial D_1, \\ u = c_2 & \text{on } \partial D_2, \end{cases}$$

where c_1 and c_2 are two given constants $c_1 > c_2 \geqq 0$. Then

$$\limsup_{\substack{z \to x \\ z \in D_2 \backslash \overline{D_1}}} |\nabla u(z)| \geqq \limsup_{\substack{z \to y_x \\ z \in D_2 \backslash \overline{D_1}}} |\nabla u(z)| \quad \forall x \in \partial D_1, \tag{2.2}$$

where y_x is the point indicated in the discussion preceding this lemma.

Proof. Fix $x \in \partial D_1$. If ∂D_1 is not C^1 at X, then one may construct (see e.g. [K]) a barrier $C^{1,\alpha}$ function $v \geqq u \geqq 0$ locally near x, $v(x) = 0$, and $|\nabla v|(x) = 0$. In particular $|\nabla u|(x) = 0$, which is the desired result. Similarly if ∂D_2 is not C^1 at y_x then one may construct (see e.g. [K]) a barrier function v with

$$v(y_x) = u(y_x) \geqq v \geqq u$$

locally near y_x, and $|\nabla v(y_x)| = \infty$. Hence $|\nabla u|(y_x) = \infty$. Which again gives the desired result.

From the above discussion it follows that we may assume x and y_x to be regular points of the boundary. Next, using rotation and translation invariance we may assume that $\{x_1 = 0\}$ is a supporting plane to D_1 at x, and $D_1 \subset \{x_1 < 0\}$. Set now

$$D_2' = D_2 \cap \{x_1 > 0\}$$

and introduce a new function $v = u + \alpha x_1$ where

$$\alpha = \limsup_{\substack{z \to y_x \\ z \in D_2 \backslash \overline{D_1}}} |\nabla u(z)| - \varepsilon$$

with ε a small positive number.

Since the second derivatives of v and u coincide, we have

$$L_u v = L_u u = 0 \quad \text{in } D_2'.$$

Therefore v has its maximum on the boundary of D_2', and by the definition at either of the points x and y_x. The latter is excluded since

$$0 \leqq \limsup \frac{\partial v}{\partial x_1}(y_x) = -\limsup |\nabla u(y_x)| + \alpha = -\varepsilon < 0.$$

Hence the maximum of v is achieved at x and we have

$$0 \leqq -\limsup \frac{\partial v}{\partial x_1}(x) = \limsup |\nabla u(x)| - \alpha. \tag{2.3}$$

Since (2.3) holds for all $\varepsilon > 0$ the lemma follows. □

A proof of the above lemma can be given, based on the comparison principle for the p-Laplacian; see [H-Sh].

Lemma 2.3. *Let D_1, D_2, and u be as in Lemma 2.2, and suppose there is $r_0 > 0$ such that for each $x \in \partial D_1$ there exists $z_x \in D_1$ with the property that $x \in \partial B(z_x, r_0) \cap \partial D_1$ (the uniform interior ball property for D_1). Denote also by $d_0 = \min_{x \in \partial D_2} d(x, D_1)$. Then there is a constant $M = M(r_0, d_0, N)$ such that*

$$|\nabla u| \leqq M \quad \textit{in } D_2 \backslash \bar{D}_1.$$

Proof. By Lemma 2.1 it suffices to show that $|\nabla u| \leqq M$ on $\partial D_1 \cup \partial D_2$. We divide the proof into two cases.

Case (i): Estimates on ∂D_1. Fix $x \in \partial D_1$, and let $r_1 = r_0 + d_0$. Define

$$D = B(z_x, r_1) \backslash B(z_x, r_0)$$

and let u_0 denote the capacitary potential of D. By the comparison principle $u_0 \leqq u$. Since also $u(x) = u_0(x)$ we may conclude

$$C(r_0, d_0, N) \geqq |\nabla u_0(x)| \geqq |\nabla u(x)|, \tag{2.4}$$

where $C(r_0, d_0, N)$ is a fixed constant depending only on r_0, d_0 and the space dimension N; see Section 4 for an explicit expression of u_0.

Case (ii): Estimates on ∂D_2. We assume without loss of generality $c_1 = 1$ and $c_2 = 0$. Fix $x \in \partial D_2$ and consider a supporting plane T_x to ∂D_2 at x (T_x is not necessarily unique). By rotation and translation we assume $T_x = \{x_1 = 0\}$ and $D_2 \subset \{x_1 > 0\}$. Let now $\{x_1 = d\}$ be a supporting plane to D_1 with the property $D_1 \subset \{x_1 > d\}$. Define now $D = D_2 \cap \{0 < x_1 < d\}$ and $w = x_1/d$. Then by comparison principle $w \geqq u$ in D. Since also $w(x) = u(x)$ we conclude that

$$\frac{1}{d_0} \geqq \frac{1}{d} = |\nabla w(x)| \geqq |\nabla u(x)|. \tag{2.5}$$

Now (2.4)–(2.5) and Lemma 2.1 give the desired result. $\square$

3. A. Beurling's technique

3.1. Sub- and supersolutions. Following A. Beurling [Be], we introduce different classes of open subsets by

$$\mathscr{C} = \{\Omega \text{ convex bounded open subset of } \mathbb{R}^N, K \subset \Omega\}.$$

For $\Omega \in \mathscr{C}$, u_Ω will denote the p-capacitary potential related to $\Omega \backslash K$. Let us remark that since we work with convex domains, u_Ω satisfies the Dirichlet boundary condition in a classical sense

$$\lim_{\substack{y \to x \\ y \in \Omega}} u_\Omega(y) = 0 \quad \forall x \in \partial\Omega. \tag{3.1}$$

The claim (3.1), besides being classical is very elementary to prove, using e.g. linear or spherical barriers; cf. the proofs in the previous section. Next we define three classes of domains:

$$\mathscr{A} = \left\{ \Omega \in \mathscr{C} : \liminf_{\substack{y \to x \\ y \in \Omega}} |\nabla u_\Omega(y)| \geqq c \quad \forall x \in \partial\Omega \right\} \quad \text{(subsolutions)},$$

$$\mathscr{A}_0 = \left\{ \Omega \in \mathscr{C} : \liminf_{\substack{y \to x \\ y \in \Omega}} |\nabla u_\Omega(y)| > c \quad \forall x \in \partial\Omega \right\},$$

$$\mathscr{B} = \left\{ \Omega \in \mathscr{C} : \limsup_{\substack{y \to x \\ y \in \Omega}} |\nabla u_\Omega(y)| \leqq c \quad \forall x \in \partial\Omega \right\} \quad \text{(supersolutions)}.$$

The objective will be to prove $\mathscr{A} \cap \mathscr{B} \neq \emptyset$.

3.2. Stability results for the class $\mathscr{B}$. First we show that the class $\mathscr{B}$ is closed under intersection.

Proposition 3.1. *Let Ω_1, Ω_2 be in $\mathscr{B}$. Then $\Omega_1 \cap \Omega_2 \in \mathscr{B}$.*

Proof. As the intersection of two convex domains is convex, we need to prove the condition on the gradient for $u_{\Omega_1 \cap \Omega_2}$ at the boundary of $\Omega_1 \cap \Omega_2$. By comparison principle, $u_{\Omega_1 \cap \Omega_2} \leqq \min(u_{\Omega_1}, u_{\Omega_2})$, which implies that for $x \in \partial(\Omega_1 \cap \Omega_2) \subset \partial\Omega_1 \cup \partial\Omega_2$ we have

$$\begin{cases} u_{\Omega_1\cap\Omega_2}(x) = u_{\Omega_1}(x) = 0 \quad \text{and} \quad \limsup\limits_{\substack{y\to x\\ y\in\Omega_1\cap\Omega_2}} |\nabla u_{\Omega_1\cap\Omega_2}(y)| \leqq \limsup\limits_{\substack{y\to x\\ y\in\Omega_1}} |\nabla u_{\Omega_1}(y)| \leqq c \\ \qquad\qquad \text{or} \\ u_{\Omega_1\cap\Omega_2}(x) = u_{\Omega_2}(x) = 0 \quad \text{and} \quad \limsup\limits_{\substack{y\to x\\ y\in\Omega_1\cap\Omega_2}} |\nabla u_{\Omega_1\cap\Omega_2}(y)| \leqq \limsup\limits_{\substack{y\to x\\ y\in\Omega_2}} |\nabla u_{\Omega_2}(y)| \leqq c \end{cases}$$

depending on whether $x \in \partial\Omega_1$ or $x \in \partial\Omega_2$. $\square$

Now, the technical and more difficult point is to prove that $\mathscr{B}$ is stable, in some sense, for decreasing sequences of convex domains. Indeed, our aim is to construct a solution to the free boundary problem, by taking a minimal element (for inclusion) in the class $\mathscr{B}$. So, we need some stability of $\mathscr{B}$ by the constructing process that we are going to use.

Theorem 3.2. *Let $\Omega_1 \supset \Omega_2 \supset \cdots$ be a decreasing sequence of convex domains in $\mathscr{B}$, and suppose $\Omega = \overset{\circ}{\overline{\bigcap \Omega_n}}$ (the interior of the closure) belongs to $\mathscr{C}$. Then $\Omega \in \mathscr{B}$.*

Proof. Let u_n denote the p-capacitary potential of Ω_n. Then $0 \leqq u_n \leqq 1$, and it is a decreasing sequence. Hence we can use standard compactness argument to deduce that on every compact subset of Ω, u_n converges to a p-harmonic function u, in $C^{1,\alpha}$-norm.

Now by Lemma 2.3, there exists $M > 0$ such that

$$|\nabla u_n(x)| \leqq M \quad \forall x \in \Omega_n. \tag{3.2}$$

Hence by (3.2), $u_n \to u$ in $C^\beta_{\text{loc}}(\mathbb{R}^N)$, for any $\beta < 1$, with $u \in C^\beta_{\text{loc}}(\mathbb{R}^N)$. To show that $u = u_\Omega$ we need to prove $u = 0$ on $\mathbb{R}^N\backslash\Omega$ and $u > 0$ in Ω. The latter follows by the minimum principle since u is p-harmonic in Ω. Let now $x \notin \Omega$. Then obviously $d(x, \mathbb{R}^N\backslash\Omega_n) \to 0$ as $n \to \infty$. Now by Lemma 2.3

$$u_n(x) \leqq d(x, \mathbb{R}^N\backslash\Omega_n)\sup|\nabla u_n| \leqq d(x, \mathbb{R}^N\backslash\Omega_n)M \to 0 \quad \text{as } n \to \infty,$$

implying $u(x) = 0$. It thus follows that $u = u_\Omega$ is the p-capacitary potential of $\Omega\backslash K$ in a classical sense.

Next we prove $\Omega \in \mathscr{B}$. Define $M = \max\limits_n \left(\sup\limits_{\Omega_n} |\nabla u_n|\right) < \infty$ (by Lemma 2.3; observe that $M = M(r_0, d_0)$ and in our case here $\Omega \subset \Omega_n$ for all n, implying $d_0 > 0$). Let also $0 < \delta_n \searrow 0$ be small enough such that

$$\phi_n := \frac{|\nabla u_n|^p - c^p}{M^p} - \frac{1}{n} \leqq 0 \quad \text{on } \{u_n = \delta_n\}.$$

Now set

$$v_n = \frac{(u_n - \delta_n)_+}{1 - 2\delta_n} \quad \text{in } D_n := \{u_n > \delta_n\}\backslash\{u_n < 1 - \delta_n\}.$$

Observe that L_{u_n} is uniformly elliptic in D_n, and it has actually bounded analytic coefficients there. (See the remark preceding Lemma 2.2.) Using

$$L_{u_n}(\phi_n) = \frac{1}{M^p} L_{u_n}(|\nabla u_n|^p) \geqq 0 = L_{u_n}(v_n) \quad \text{in } D_n,$$

and $\phi_n \leqq v_n$ on ∂D_n we may invoke Lemma 2.1 and the comparison principle to conclude

$$\phi_n \leqq v_n \quad \text{in } D_n. \tag{3.3}$$

Now fix $\varepsilon > 0$. Then there exists a small neighborhood U_ε of $\partial\Omega$ such that

$$u_n \leqq \varepsilon \quad \text{in } U_\varepsilon. \tag{3.4}$$

Hence for any compact set $U' \subset U_\varepsilon \cap \Omega$

$$\phi_n \to \frac{|\nabla u|^p - c^p}{M^p},$$

uniformly in U'. This in conjunction with (3.3)–(3.4) implies

$$\frac{|\nabla u|^p - c^p}{M^p} \leqq u \leqq \varepsilon \quad \text{in } U'.$$

Letting $\varepsilon \to 0$ we'll have the desired result. □

3.3. Construction and properties of the minimal element in $\mathscr{B}$. In this section, we will first construct a domain which will be a good candidate as a solution of the free boundary problem. This domain is obtained as the minimal element (minimal for the inclusion) in the class $\mathscr{B}$. Then we will use this minimality to prove that it is a solution of the free boundary problem.

Proposition 3.3. *Assume that there exist two domains $\Omega_0 \in \mathscr{A}_0$ and $\Omega_1 \in \mathscr{B}$, with $\Omega_0 \subset \Omega_1$, and set*

$$\mathscr{S} := \{D \in \mathscr{B}, \textit{ with } \Omega_0 \subset D\}.$$

Then there exists a domain Ω in the class $\mathscr{S}$ which is minimal for the inclusion (i.e. $\tilde\Omega \in \mathscr{S}$ and $\tilde\Omega \subset \Omega \Rightarrow \tilde\Omega = \Omega$).

Proof. Let I be the intersection of all domains in the class $\mathscr{S}$ and set $\Omega = \mathring{\bar{I}}$ (the interior of the closure, which is still convex). To prove $\Omega \in \mathscr{B}$, we select a sequence of domains $\{D_n\}_{n=1}^\infty$ in $\mathscr{S}$ such that $\bigcap_{n \geqq 1} D_n = I$ and we consider the sequence of domains $\{\Omega_n\}_{n=1}^\infty$ defined by $\Omega_1 = D_1$ and $\Omega_{n+1} = \Omega_n \cap D_{n+1}$ ($n \geqq 2$). By Proposition 3.1 each Ω_n is convex and belongs to $\mathscr{B}$. Hence Theorem 3.2 gives the desired result. □

In the sequel, we always denote by Ω the minimal element in the class $\mathscr{S}$ defined in the previous proposition.

Definition 3.1 (Extremal points). A point $x \in \partial\Omega$ is said to be extremal if there exists a supporting plane to Ω touching $\partial\Omega$ at x only. We denote the set of all extremal points of Ω by E_Ω.

The first step consists in proving that, for a minimal element, we have not only that the lim sup of the gradient is smaller than c on the exterior boundary, but that it is exactly equal to c on the set $\overline{E}_\Omega$ (closure).

Lemma 3.4. *Let $x \in \overline{E}_\Omega$. Then*

$$\limsup_{\substack{y \to x \\ y \in \Omega}} |\nabla u_\Omega(y)| = c.$$

Proof. Suppose the conclusion of the lemma fails. Then there exists $y_0 \in \overline{E}_\Omega$ such that

$$\limsup_{\substack{y \to y_0 \\ y \in \Omega}} |\nabla u_\Omega(y)| = c(1 - 4\alpha) \quad \text{with } \alpha > 0.$$

Hence for some small neighborhood $\mathscr{V}$ of y_0 there holds

$$|\nabla u_\Omega(x)| \leqq c(1-\alpha) \quad \forall x \in \mathscr{V}. \tag{3.5}$$

Obviously we may assume $y_0 \in E_\Omega$. Let us fix a hyperplane T_d with $\operatorname{dist}(y_0, T_d) = d$ and such that $T_d \cap \Omega \subset \mathscr{V}$. This is possible due to the extremal property of y_0.

By rotation and translation, we assume y_0 is the origin and $T_d = \{x_1 = d\}$. Let now $T_\delta = \{x_1 = \delta\}$ and set $\Omega_\delta = \Omega \backslash \{x_1 \leqq \delta\}$ (see Figure 1). Then by comparison principle the p-capacitary potential u_δ of Ω_δ satisfies

$$0 \leqq u_\delta \leqq u_\Omega \quad \text{in } \Omega_\delta, \tag{3.6}$$

which implies

$$\limsup |\nabla u_\delta| \leqq \limsup |\nabla u_\Omega| \leqq c \quad \text{on } \partial\Omega \cap \partial\Omega_\delta.$$

Observe that, by barrier arguments, $|\nabla u_\delta|$ becomes zero on part of $\partial\Omega \cap \partial\Omega_\delta$ that is not C^1. Now by (3.5) and (3.6)

$$\max_{T_d} u_\delta \leqq \max_{T_d} u_\Omega \leqq d \sup_{\{0 \leqq x_1 \leqq d\}} |\nabla u_\Omega| \leqq c(1-\alpha)d. \tag{3.7}$$

Define

$$v := u_\delta + \frac{c(1-\alpha)d}{d-\delta}(d - x_1).$$

Since the second derivatives of v and u_δ coincide, we have

$$L_{u_\delta} v = L_{u_\delta} u_\delta = 0, \quad \Omega_\delta \cap \{x_1 < d\}.$$

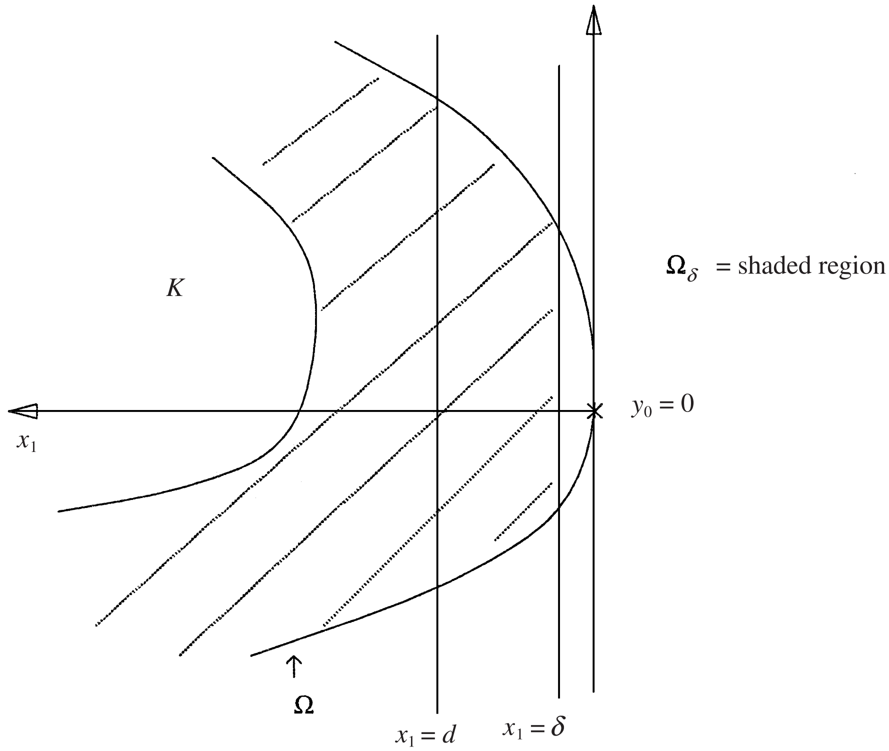

Figure 1

Therefore in $\Omega_\delta \cap \{x_1 < d\}$, v takes its maximum on the boundary (see the remark preceding Lemma 2.2). By inspection and (3.7), it is also not hard to see that on

$$\partial(\Omega_\delta \cap \{x_1 < d\}) \subset T_d \cup T_\delta \cup (\partial\Omega \cap \{\delta < x_1 < d\}),$$

$v \leqq c(1-\alpha)d$, with equality on T_δ. Hence $\dfrac{\partial v}{\partial x_1} \leqq 0$ on T_δ, i.e.,

$$|\nabla u_\delta| - \frac{c(1-\alpha)d}{d-\delta} \leqq 0 \quad \text{on } T_\delta,$$

and if we choose δ such that $\delta \leqq \alpha d$ we have $|\nabla u_\delta| \leqq c$ on T_δ whence $\Omega_\delta \in \mathscr{B}$. This finishes the proof. $\square$

A proof of the above lemma can be given, based on the comparison principle for the p-Laplacian. One needs to compare the p-harmonic functions u_δ and

$$c(1-\alpha)d - \frac{c(1-\alpha)d}{d-\delta}(d - x_1).$$

Lemma 3.5. *Let Ω be the minimal element in the class $\mathscr{B}$ and u_Ω its capacitary potential. Then*

$$|\nabla u_\Omega(x)| \geqq c \quad \forall x \in \Omega \backslash K.$$

Proof. Let us consider a level set $\mathscr{L}_\alpha = \{x : u_\Omega(x) > \alpha\}$, with $0 < \alpha < 1$ and a given point x on $\partial \mathscr{L}_\alpha$. According to [Le], this level set is convex and therefore we can apply Lemma 2.2 with $D_1 = \mathscr{L}_\alpha$ and $D_2 = \Omega$. Let T be the tangent hyperplane to $\mathscr{L}_\alpha$ at x, elementary geometry shows that among all points $y \in \partial\Omega$ with largest distance to x in the direction orthogonal to T (outward normal direction to $\mathscr{L}_\alpha$), there exists at least one point in $\bar{E}_\Omega$. So, according to Lemma 2.2 and Lemma 3.4, we have

$$|\nabla u_\Omega(x)| \geqq \limsup_{\substack{z \to y \\ z \in \Omega}} |\nabla u_\Omega(z)| = c.$$

The lemma is proved. □

Now the following theorem is a consequence of Lemma 3.5 and Proposition 3.3.

Theorem 3.6. *Assume there exist a subsolution $\Omega_0 \in \mathscr{A}_0$ and a supersolution $\Omega_1 \in \mathscr{B}$ with $\Omega_0 \subset \Omega_1$. Then there exists a solution to the free boundary problem* (1.1) *in a strong sense*, *i.e.* (1.2).

4. Proof of Theorem 1.1

4.1. The bounded regular case. According to Theorem 3.6, to prove the existence of a classical solution, it is enough to exhibit a strict subsolution and a supersolution.

Supersolution: Choose $R_0 > 0$ large enough such that $B_{R_0} \supset K$. Then for $R > R_0$ the p-capacitary potential of $B_R \backslash B_{R_0}$ can be written as

$$u_0(x) = \begin{cases} \dfrac{|x|^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}}}{R_0^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}}} & \text{for } p \neq N, \\[2ex] \dfrac{\log|x| - \log R}{\log R_0 - \log R} & \text{for } p = N. \end{cases}$$

In particular $|\nabla u_0| \leqq c$ on ∂B_R for large enough R.

Now, by the comparison principle, the p-capacitary potential u_R of $B_R \backslash K$ satisfies $u_R \leqq u_0$. This in turn gives $|\nabla u_R| \leqq |\nabla u_0| \leqq c$ on ∂B_R for R large enough.

Subsolution: Let B_R and u_R be as above. Obviously (cf. Lemma 2.2) there exists a small neighborhood U of ∂K such that

$$|\nabla u_R(x)| \geqq \alpha > 0 \quad \text{for all } x \in U \backslash K,$$

and some $\alpha > 0$. Let us choose a level set of u_R: $\mathscr{L}_\varepsilon = \{x : u_R(x) > 1 - \varepsilon\}$ in this neighborhood. Then the p-capacitary potential of $\mathscr{L}_\varepsilon$ is given by

$$u_{\mathscr{L}_\varepsilon} = \frac{u_R - (1 - \varepsilon)}{\varepsilon}.$$

Therefore on $\partial\mathscr{L}_\varepsilon$, $|\nabla u_{\mathscr{L}_\varepsilon}| = \dfrac{|\nabla u_R|}{\varepsilon} \geqq \dfrac{\alpha}{\varepsilon} > c$ for ε small enough. This gives the desired subsolution.

Now the $C^{2,\alpha}$ regularity of $\partial\Omega$ is a consequence of a result of A. L. Vogel (see [V], Theorem 1). At last, uniqueness of the solution in the case where K is convex is a classical consequence of the Lavrentiev principle, see e.g. [Ac-M] or [H-Sh]. $\square$

4.2. The unbounded and irregular case. The changes that will occur in the arguments are as follows.

Lemma 2.3 is not valid anymore, and we replace the gradient estimate on $D_2 \backslash D_1$ by a Hölder estimate, since convexity will permit us to use conical barriers (see [M], cf. also [T]). The proof of Theorem 3.2 goes thus as usual and we obtain convergence in C^β-norm. Next one obtains (3.2) in $\Omega_n \backslash K'$ for some $K \subset K' \subset \Omega$, due to the convergence of u_n in $C^{1,\alpha}$ norm locally in $\Omega \backslash K$. The rest of the paper works as usual, without any changes.

As to the unbounded case, we consider a sequence $K_s = K \cap B(0, s)$, with $s \to \infty$ and for each s we obtain a solution u_s to the free boundary problem (1.1). Now one repeats all the arguments again to show that the increasing family of the functions u_s converges to a solution u with the desired property. More precisely one starts with Theorem 3.2, and uses linear and conical barriers as mentioned in Lemma 2.2 and in the above discussion. Next a similar argument as in the proof of Lemma 3.4 works and gives the equality of the desired boundary value for the modulus of the gradient of u_s at extremal points. Finally, the proof of Lemma 3.5 will follow, due to the observation that the limit function u, of the capacitary potentials u_s, have convex level sets, since each u_s does.

The uniqueness for bounded irregular case is the same as that of regular K. However, the Lavrentiev principle does not work in the unbounded case. We also believe that there is no uniqueness in unbounded case. However, this problem remains open as we know of no technique for comparing two solutions with unbounded support. Observe that also when the solutions are unbounded the functions may also become unbounded. In fact their behavior near the infinity is like that of the hyperplane solutions $\max(x_1, 0)$ in some suitable coordinates. This discussion is, however, out of the scope of this paper.

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