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SHAPE DERIVATIVE OF DRAG FUNCTIONAL*

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Abstract. In this paper, compressible, stationary Navier–Stokes equations are considered. The model is well-posed, and there exist weak solutions in bounded domains, subject to inhomogeneous boundary conditions. The shape sensitivity analysis is performed for Navier–Stokes boundary value problems in the framework of small perturbations of the so-called *approximate solutions*. The approximate solutions are determined from the Stokes problem, and the small perturbations are given by the unique solutions to the full nonlinear model. The differentiability of small perturbations of the approximate solutions with respect to the coefficients of differential operators implies the shape differentiability of the drag functional. The shape gradient of the drag functional is derived in a form convenient for computations, and an appropriate adjoint state is introduced to this end. The shape derivatives of solutions to the Navier–Stokes equations are given by smooth functions; however, the shape differentiability of the solutions is shown in a weak norm. The proposed method of shape sensitivity analysis is general. The differentiability of solutions to the Navier–Stokes equations with respect to the data leads to the first order necessary conditions for a broad class of optimization problems. The boundary shape gradient as well as the boundary value problems for the shape derivatives of solutions to state equations and the adjoint state equations are obtained in the form of singular limits of volume integrals. This method of shape sensitivity analysis seems to be new and is appropriate for nonlinear problems. It is an important contribution in the field of numerical methods of shape optimization in fluid mechanics.

Key words. compressible Navier–Stokes equations, drag minimization, shape derivative, necessary optimality conditions

AMS subject classifications. Primary, 76N10, 35Q30, 76N25; Secondary, 35Q30, 49J20, 49Q10

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1. Introduction.

In the present paper we derive the boundary form of the shape gradient for the drag functional. In [20] the weak material derivatives are employed to obtain the shape derivative of the drag functional for the compressible Navier–Stokes equations. It is shown that the drag functional is shape differentiable in the sense of [26]. For a smooth obstacle, by the so-called Hadamard representation formula [6] it follows from the results of [26] that the shape gradient of the drag is given by the distribution supported on the boundary of the obstacle. In such a formula for the shape gradient, only the deformations of the obstacle in the normal direction are present. We use the result and identify the density of the shape gradient by a direct approach. This approach is interesting on its own, as it introduces a new and efficient method for the shape sensitivity analysis or shape differentiability of integral functionals as well as of the solutions for nonlinear problems.

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We introduce our method for a simple model problem, replacing in the following example the compressible Navier–Stokes boundary value problem by a nonlinear elliptic equation.

Example. Let us consider the bounded domain Ω with the smooth boundary. Take the state equation in the form

$$-\Delta u(x) = F(x, u(x)) \quad \text{in} \quad \Omega, \qquad u(x) = u_0 \equiv \text{const.} \quad \text{on} \quad \partial\Omega.$$

The shape optimization problem considered here is an optimal choice of the domain Ω within an admissible class to be specified in such a way that the boundary shape functional defined on $\partial\Omega$ is minimized. The functional is selected in the form which looks like the *drag*, i.e.,

$$j(\partial\Omega) = u_0 \int_{\partial\Omega} \nabla u(x) \cdot n(x) dS. \tag{1.1}$$

Using the Gauss formula we can rewrite this functional in the distributed form of a volume integral

$$j(\partial\Omega) \equiv \int_{\Omega} (|\nabla u|^2 - uF(u, x))\, dx. \tag{1.2}$$

Representations of such kind are common in viscous fluid dynamics since they have clear physical meaning. For instance, the gradient part in (1.2) represents the rate of dissipation of the energy. However, functional (1.2) is only weakly lower semicontinuous in the energy space and can be used mostly to assure the existence of solutions for minimization problems. The other approach employed in the paper in order to get a weakly continuous form of the same functional can be described in the following way: Introduce a smooth scalar function $\eta(x)$ such that $\eta(x) \equiv 1$ on $\partial\Omega$ and rewrite the expression for $j(\partial\Omega)$ in the equivalent form

$$J(\Omega) = u_0 \int_{\Omega} (\nabla\eta \cdot \nabla u - \eta F(x, u))\, dx. \tag{1.3}$$

This functional is weakly continuous, and its principal part is linear with respect to the state variable u. The technical difficulty is that the integrand in (1.3) contains an arbitrary function η, while the resulting expression for the shape gradient of (1.1) is independent of η; i.e., we have to eliminate the influence of η on the results of calculations, which is achieved by a singular limit passage in volume integrals to obtain the boundary integral expressions for the shape gradients. Actually, for shape functional (1.3) the shape sensitivity analysis can be performed in the framework of the material derivatives, and therefore in the fixed domain setting. As a result, the shape derivative of $J(\Omega)$ *supported* everywhere in $\overline{\Omega}$ is obtained. In the next step of our derivation, we can perform the limit passage with the velocity perturbations fields and evaluate the singular limits of all integrals; in the limit, formally the perturbations velocity fields are supported on $\partial\Omega$. In this way the shape gradient of $j(\partial\Omega)$ supported on the boundary is identified.

It seems that the proposed method of shape sensitivity analysis is new, and in fact it seems to be the simplest possible in the case of the drag functional evaluated for solutions to compressible Navier–Stokes equations.

Drag functional. We assume that the viscous gas occupies the double-connected domain $\Omega = B\backslash\mathfrak{S}$, where $B \subset \mathbb{R}^3$, is a hold-all domain with the smooth boundary

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$\Sigma = \partial B$, and $\mathfrak{S} \subset B$ is a compact obstacle. The boundary of the obstacle is denoted by $\mathcal{S} := \partial \mathfrak{S}$ for simplicity.

Furthermore, we assume that the velocity of the gas coincides with a given constant vector field $\mathbf{U}$ on the surface Σ. The state variables include the velocity field $\mathbf{u}$ and the gas density ϱ, and satisfy the following equations along with the boundary conditions:

$$\Delta \mathbf{u} + \lambda \nabla \operatorname{div} \mathbf{u} = R\varrho \mathbf{u} \nabla \mathbf{u} + \frac{R}{\epsilon^2} \nabla p(\varrho) \quad \text{in} \quad \Omega, \tag{1.4a}$$

$$\operatorname{div}(\varrho \mathbf{u}) = 0 \quad \text{in} \quad \Omega, \tag{1.4b}$$

$$\mathbf{u} = \mathbf{U} \quad \text{on} \quad \Sigma, \qquad \mathbf{u} = 0 \quad \text{on} \quad \mathcal{S}, \tag{1.4c}$$

$$\varrho = \varrho_0 \quad \text{on} \quad \Sigma_{\text{in}}, \tag{1.4d}$$

where the pressure $p = p(\varrho)$ is a smooth, strictly monotone function of the density, ϵ is the Mach number, R is the Reynolds number, λ is the viscosity ratio, ϱ_0 is a positive constant, and the inlet Σ_{in} and the outlet Σ_{out} are defined by

$$\Sigma_{\text{in}} = \{x \in \Sigma : \mathbf{U} \cdot \mathbf{n} < 0\}, \quad \Sigma_{\text{out}} = \{x \in \Sigma : \mathbf{U} \cdot \mathbf{n} > 0\},$$

respectively. Here $\mathbf{n}$ stands for the outward normal to $\partial\Omega = \Sigma \cup \mathcal{S}$. Boundary value problem (1.4) can be regarded as a mathematical model of viscous gas flow around an airfoil $\mathfrak{S}$ tested in a wind tunnel. In our denotation the stress tensor is equal to

$$\mathbb{T} =: \nabla \mathbf{u} + \nabla \mathbf{u}^\top + (\lambda - 1) \operatorname{div} \mathbf{u} \, \mathbb{I} - \frac{R}{\epsilon^2} p \mathbb{I} \,,$$

and the hydrodynamical force acting on the element dS of the obstacle boundary $\mathcal{S}$ is $-\mathbb{T}\mathbf{n}\, dS$. Hence the hydrodynamical force acting on the body $\mathfrak{S}$ is equal to

$$\mathbf{J}(\mathfrak{S}) =: -\int_{\mathcal{S}} \mathbb{T}\mathbf{n}\, ds = -\int_{\mathcal{S}} \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top + (\lambda - 1) \operatorname{div} \mathbf{u} \mathbb{I} - \frac{R}{\epsilon^2} p \mathbb{I} \right) \mathbf{n} \, dS. \tag{1.5}$$

Note that (1.5) can be identically rewritten in the form of the volume integral. To this end we fix an arbitrary function $\eta \in C^\infty(\Omega)$ such that $\eta = 1$ in an open neighborhood of the obstacle $\mathfrak{S}$ and $\eta = 0$ in a vicinity of Σ. Using the identities

$$\int_{\mathcal{S}} \mathbb{T}\mathbf{n}\, ds = \int_{\Omega} (\eta \operatorname{div} \mathbb{T} + \mathbb{T}\nabla\eta)\, dx, \qquad \operatorname{div} \mathbb{T} = R\varrho \mathbf{u} \nabla \mathbf{u},$$

we obtain

$$\mathbf{J}(\mathfrak{S}) =: -\int_{\Omega} \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top + (\lambda - 1) \operatorname{div} \mathbf{u} \mathbb{I} - \frac{R}{\epsilon^2} p \mathbb{I} \right) \nabla\eta \, dx - R \int_{\Omega} \eta \varrho \mathbf{u} \nabla \mathbf{u} \, dx. \tag{1.6}$$

The value of $\mathbf{J}$ is independent of the choice of the function η. The drag J_D is a work in unit time developed by the component of $\mathbf{J}$ parallel to the airfoil speed $\mathbf{U}$,

$$J_D(\mathfrak{S}) = \mathbf{U} \cdot \mathbf{J}(\mathfrak{S}). \tag{1.7}$$

The minimization of the drag is an important problem of applied aerodynamics. From the mathematical point of view the drag minimization problem is the shape optimization problem for the compressible Navier–Stokes equations. The existence and compactness properties of solutions to the drag minimization problem were established in [9], [10] for the nonstationary case and in [19] for the stationary boundary

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value problem by using the direct methods of calculus of variations. For incompressible Navier–Stokes equations, the existence of nontrivial shape derivatives of solutions and the formulae for the shape derivative of the drag functional and for the adjoint state were obtained in [3], [4], and [22]; see also [23] and [24] for some generalizations. For the general theory of optimization and control problems for incompressible Navier–Stokes equations we refer the reader to [2] and [11]. The growing literature on numerical and applied aspects of the problem is nicely surveyed in [12] and [15]. For compressible Navier–Stokes equations (1.4), the formula for the shape derivative of the drag functional and for an appropriate adjoint state were derived in [20] under the assumptions that the Reynolds and Mach numbers are sufficiently small. The goal of this paper is to simplify the results of this work and give an efficient representation not only for the adjoint state but also of the shape gradient of the drag functional.

In order to formulate the framework for the shape sensitivity analysis, we choose the vector field $\mathbf{T} \in C^2(\mathbb{R}^3)^3$ vanishing in the vicinity of Σ and define the mapping

$$y = x + \varepsilon \mathbf{T}(x), \tag{1.8}$$

which describes the perturbation of the shape of the obstacle. For small ε, the mapping $x \to y$ takes diffeomorphically the flow region Ω onto $\Omega_\varepsilon = B \setminus \mathfrak{S}_\varepsilon$, where the perturbed obstacle $\mathfrak{S}_\varepsilon = y(\mathfrak{S})$. Let $(\bar{\mathbf{u}}_\varepsilon, \bar{\varrho}_\varepsilon)$ be a solution to problem (1.4) in Ω_ε. After substituting $(\bar{\mathbf{u}}_\varepsilon, \bar{\varrho}_\varepsilon)$ into the formula for $\mathbf{J}(\mathfrak{S}_\varepsilon)$, the drag becomes the function of the parameter ε.

It is convenient to reduce such an analysis to the analysis on dependence of solutions with respect to the coefficients of the governing equations. To this end, we introduce the functions $\mathbf{u}_\varepsilon(x)$ and $\varrho_\varepsilon(x)$ defined in the unperturbed domain Ω by the formulae

$$\mathbf{u}_\varepsilon(x) = \mathbf{N}\bar{\mathbf{u}}_\varepsilon(x + \varepsilon\mathbf{T}(x)), \quad \varrho_\varepsilon(x) = \bar{\varrho}_\varepsilon(x + \varepsilon\mathbf{T}(x)),$$

where

$$\mathbf{N}(x) = \det\,(\mathbb{I} + \varepsilon\mathbf{T}'(x))(\mathbb{I} + \varepsilon\mathbf{T}'(x))^{-1} \tag{1.9}$$

is the adjugate matrix of the Jacobi matrix $\mathbb{I} + \varepsilon\mathbf{T}'$. Furthermore, we also use the notation $\mathfrak{g}(x) = \sqrt{\det\,\mathbf{N}}$. Calculations show [20] that for $\mathbf{u}_\varepsilon, \varrho_\varepsilon$, the following boundary value problem is obtained:

$$\Delta\mathbf{u}_\varepsilon + \nabla\Big(\lambda\mathfrak{g}^{-1}\operatorname{div}\mathbf{u}_\varepsilon - \frac{R}{\epsilon^2}p(\varrho_\varepsilon)\Big) = \mathscr{A}(\mathbf{u}_\varepsilon) + R\mathscr{B}(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) \quad \text{in} \quad \Omega, \tag{1.10a}$$

$$\operatorname{div}\big(\varrho_\varepsilon\mathbf{u}_\varepsilon\big) = 0 \quad \text{in} \quad \Omega, \tag{1.10b}$$

$$\mathbf{u}_\varepsilon = \mathbf{N}\mathbf{U} \quad \text{on} \quad \Sigma, \qquad \mathbf{u}_\varepsilon = 0 \quad \text{on} \quad \mathcal{S}, \tag{1.10c}$$

$$\varrho_\varepsilon = \varrho_0 \quad \text{on} \quad \Sigma_{\text{in}}. \tag{1.10d}$$

Here, the linear operator $\mathscr{A}$ and the nonlinear mapping $\mathscr{B}$ are defined in terms of $\mathbf{N}$,

$$\begin{aligned} \mathscr{A}(\mathbf{u}) &= \Delta\mathbf{u} - (\mathbf{N}^\top)^{-1}\operatorname{div}\big(\mathfrak{g}^{-1}\mathbf{N}\mathbf{N}^\top\nabla(\mathbf{N}^{-1}\mathbf{u})\big), \\ \mathscr{B}(\varrho, \mathbf{u}, \mathbf{w}) &= \varrho(\mathbf{N}^\top)^{-1}\big(\mathbf{u}\,\nabla(\mathbf{N}^{-1}\mathbf{w})\big). \end{aligned} \tag{1.11}$$

In the new variables the expression for the force $\mathbf{J}$ reads as

$$\begin{aligned} \mathbf{J} = &-\int_\Omega \big[\mathfrak{g}^{-1}\big(\mathbf{N}^\top\nabla(\mathbf{N}^{-1}\mathbf{u}_\varepsilon) + \nabla(\mathbf{N}^{-1}\mathbf{u}_\varepsilon)^\top\mathbf{N} - \operatorname{div}\mathbf{u}_\varepsilon\big) - q_\varepsilon\mathbb{I}\big]\mathbf{N}^\top\nabla\eta\,dx \\ &- R\int_\Omega \varrho_\varepsilon\mathbf{u}_\varepsilon\nabla(\mathbf{N}^{-1}\mathbf{u}_\varepsilon)\,\eta\,dx, \end{aligned} \tag{1.12}$$

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where *the effective viscous pressure* q is given by

$$q_\varepsilon = \frac{R}{\epsilon^2}\, p(\varrho_\varepsilon) - \lambda \mathfrak{g}^{-1} \operatorname{div} \mathbf{u}_\varepsilon. \tag{1.13}$$

In the proposed framework the question on the existence and representation of the derivative $\partial_\varepsilon J_D(\mathfrak{S}_\varepsilon)$ is reduced to the question on the dependence of solutions to the boundary value problem with respect to the coefficients of governing equations. The preference of such an approach is that the equations are considered in a fixed domain, which makes the calculations more transparent but formally more complicated. Moreover, the obtained results depend on the extension of the mapping $\mathbf{T}(x)$ over $\mathbb{R}^3$ which leads to different formulae for different choices of $\mathbf{T}$. It is worth noting that *the shape derivative depends only on the restriction of the vector field* $\mathbf{T}$ *to* $\mathcal{S}$. *The shape derivative is independent of the extension of* $\mathbf{T}$ *over the flow domain by the Hadamard formula for differentiable shape functionals.* Moreover, expression (1.12) involves an arbitrary function η, while the shape derivative is independent of η.

The other approach widely used in the shape optimization is a direct analysis of the problem in a variable domain. This leads to the representation of the shape derivative in the form of the integral over $\mathcal{S}$, with the integrand depending only on $\mathbf{T}\cdot\mathbf{n}$, which can be regarded as an infinitesimal shift of $\mathcal{S}$. The goal of this paper is to compare these two approaches and to prove that the expression for shape derivatives in the form of a surface integral is a singular limit of the derivative of functional (1.12) for the perturbation $\mathbf{T}$, which is concentrated near the boundary. Before formulation of the result we recall the expression for the shape derivative of the drag and the formulation of adjoint state equation.

We follow the approach proposed in [18], [20] for resolution of stationary boundary value problems for compressible flows and restrict our considerations to the case where $\lambda \gg 1$, $R \ll 1$, and $\epsilon \ll 1$, which corresponds to almost incompressible flow with low Reynolds number. In such a case a given solution $(\mathbf{u}_0, q_0)$ to the Stokes equations

$$\Delta \mathbf{u}_0 - \nabla q_0 = 0, \quad \operatorname{div} \mathbf{u}_0 = 0 \;\text{ in }\; \Omega, \tag{1.14}$$
$$\mathbf{u}_0 = \mathbf{U} \;\text{ on }\; \Sigma, \quad \mathbf{u}_0 = 0 \;\text{ on }\; \mathcal{S}, \quad \Pi q_0 = q_0 \quad \Pi q =: q - \frac{1}{\operatorname{meas}\Omega}\int_\Omega q\, dx$$

and a constant ϱ_0 can be considered as an approximation for a solution to problem (1.10). Instead of the Mach number we introduce the new parameters

$$\sigma_0 = R/(\lambda\epsilon^2), \quad \sigma = \sigma_0 \rho_0 p'(\rho_0).$$

Hence we can look for particular solutions to problem (1.10) in the form

$$\mathbf{u}_\varepsilon = \mathbf{u}_0 + \mathbf{v}, \quad \varrho_\varepsilon = \varrho_0 + \varphi, \quad q_\varepsilon = q_0 + \lambda\sigma_0 p(\varrho_0) + \pi + \lambda m, \tag{1.15}$$

with the unknown functions $\vartheta = (\mathbf{v}, \pi, \varphi)$ and the unknown constant m. The peculiarity of a stationary boundary problem for compressible Navier–Stokes equations is that mass balance equation (1.10b) degenerates at points where $\mathbf{u}_\varepsilon$ vanishes. The simple algebraic scheme which allows us to cope with this difficulty was proposed in [18]. The basic idea is to consider the effective viscous pressure as a new unknown variable, next to add equality (1.13) to basic system (1.10), and finally, to eliminate the divergence $\mathbf{u}_\varepsilon$ from mass balance equation (1.10b) using relation (1.13). Thus we

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come to the following boundary problem for the triplet $(\mathbf{v}, \pi, \varphi)$:

$$\begin{aligned} \Delta \mathbf{v} - \nabla \pi &= \mathscr{A}(\mathbf{u}_\varepsilon) + R\mathscr{B}(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon) \text{ in } \Omega, \\ \operatorname{div} \mathbf{v} &= \mathfrak{g}\Big(\frac{\sigma}{\varrho_0}\varphi - \Psi - m\Big) \text{ in } \Omega, \\ \mathbf{u}_\varepsilon \cdot \nabla\varphi + \sigma\varphi &= \Psi_1 + m\mathfrak{g}\varrho_\varepsilon \text{ in } \Omega, \\ \mathbf{v} = (\mathbf{N} - \mathbb{I})\mathbf{U} \text{ on } \Sigma, \quad \mathbf{v} = 0 &\text{ on } \mathcal{S}, \quad \varphi = 0 \text{ on } \Sigma_{\text{in}}, \quad \Pi\pi = \pi, \end{aligned} \tag{1.16a}$$

where

$$\Psi_1 = \mathfrak{g}\Big(\varrho\Psi - \frac{\sigma}{\varrho_0}\varphi^2\Big) + \sigma\varphi(1-\mathfrak{g}),\ \Psi = \frac{q_0 + \pi}{\lambda} - \frac{\sigma}{p'(\varrho_0)\varrho_0}H(\varphi),$$
$$H(\varphi) = p(\varrho_0 + \varphi) - p(\varrho_0) - p\prime(\varrho_0)\varphi;$$

the vector field $\mathbf{u}_\varepsilon$ and the function ϱ_ε are given by (1.15). Next it is necessary to specify the constant m. Since the mean value of $\operatorname{div} \mathbf{v}$ is null, the constant m is proportional to the mean value of the quantity $\mathfrak{g}(\psi - \varrho_0^{-1}\sigma\varphi)$. The mean value of the latter quantity involves a large parameter σ, and it is convenient to replace it by a more complicated relation which is independent of the large parameter σ. Following [20] we can introduce such a condition in the form

$$m = \varkappa \int_\Omega (\varrho_0^{-1}\Psi_1\,\zeta - \mathfrak{g}\Psi)\,dx, \quad \varkappa = \Big(\int_\Omega \mathfrak{g}(1 - \zeta - \varrho_0^{-1}\zeta\varphi)\,dx\Big)^{-1}, \tag{1.16b}$$

where the auxiliary function ζ is a solution to the adjoint boundary value problem

$$-\operatorname{div}(\mathbf{u}_\varepsilon\zeta) + \sigma\zeta = \sigma\mathfrak{g} \text{ in } \Omega, \quad \zeta = 0 \text{ on } \Sigma_{\text{out}}. \tag{1.16c}$$

As is shown in [20], relations (1.16) define a system of differential equations and boundary conditions which is equivalent to basic problem (1.10). The proposed algebraic scheme works properly in the range of parameters $R \sim \lambda\epsilon^2$. The results are available for a larger domain in the space of parameters, but in this case the modified equations involve nonlocal operators; see [16], [17], and [21] for details. The following proposition guarantees the existence and uniqueness of solution to boundary value problem (1.16).

For $s \in (0, \infty)$, $r \in (1, \infty)$, denote by $X^{s,r}$ and $Y^{s,r}$ the Banach spaces

$$X^{s,r} = W^{s,r}(\Omega) \cap W^{1,2}(\Omega), \quad Y^{s,r} = W^{s+1,r}(\Omega) \cap W^{2,2}(\Omega).$$

Here $W^{s,r}(\Omega)$ denotes the Sobolev space for an integer s, and the fractional Sobolev–Slobodetski space otherwise. We assume that Ω satisfies the following condition.

Condition 1.1. *The flow domain Ω has a boundary of class C^∞ and it admits the representation $\Omega = B \setminus \mathfrak{S}$, $\mathfrak{S} \Subset B$, in which the hold-all domain B with the boundary $\Sigma = \partial B$ is strictly convex.*

Proposition 1.1. *Let $\mathbf{U}$ be a given vector field, and let the exponents s, r satisfy the inequalities*

$$1/2 < s < 1, \quad 1 < r < 3/(2s-1), \quad sr > 3. \tag{1.17}$$

Then there is σ^*, depending only on s, r, Ω, and $\mathbf{U}$, with the following property: If $\sigma > \sigma^*$, then there exist $\mu_0 > 0$ and $\varepsilon_0 > 0$, depending only on σ, s, r, Ω, and $\mathbf{U}$, such that for all $\mu \in (0, \mu_0)$, $\varepsilon \in [0, \varepsilon_0]$, and

$$R + \lambda^{-1} < \mu, \tag{1.18}$$

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problem (1.16) *has a solution* $(\mathbf{v}, \pi, \varphi, \zeta, m) \in Y^{s,r} \times (X^{s,r})^3 \times \mathbb{R}$. *Moreover, this solution admits the estimates*

$$\|\mathbf{v}\|_{Y^{s,r}} + \|\pi\|_{X^{s,r}} + \|\zeta\|_{X^{s,r}} \le c_0, \quad |m| + \|\varphi\|_{X^{s,r}} \le c_0\mu, \tag{1.19}$$

where the constant c_0 *is independent of* μ *and* ε.

Note that by virtue of (1.17), $(\mathbf{v}, \varphi, \zeta) \in C^{1+\gamma}(\Omega) \times C^\gamma(\Omega)^2$ with the exponent $\gamma = \gamma(r,s) > 0$, and the drag functional $J_D(\mathfrak{S})$ is well defined. Recalling the expressions for $\mathscr{A}$, $\mathscr{B}$, and $\mathfrak{g}$ we conclude that the coefficients of equations (1.16) are analytic functions of the parameter ε. Formal differentiation of both sides of (1.16) leads to the following equations for the variations:

$$(\delta\mathbf{v}, \delta\pi, \delta\varphi, \delta\zeta, \delta m) = \frac{d}{d\varepsilon}(\mathbf{v}, \pi, \varphi, \zeta, m)\Big|_{\varepsilon=0},$$

$$\begin{aligned}
&\Delta\delta\mathbf{v} - \nabla\delta\pi = \mathscr{C}(\delta\varphi, \delta\mathbf{v}) + \mathscr{D}(\mathbf{D}),\\
&\operatorname{div}\,\delta\mathbf{v} = b_{21}\,\delta\varphi + b_{22}\,\delta\pi + b_{23}\,\delta m + b_{20}\,\mathrm{Tr}\,\mathbf{T}',\\
&\mathbf{u}\nabla\delta\varphi + \sigma\delta\varphi = -\delta\mathbf{v}\nabla\varphi + b_{11}\,\delta\varphi + b_{12}\,\delta\pi + b_{13}\,\delta m + b_{10}\,\mathrm{Tr}\,\mathbf{T}',\\
&-\operatorname{div}\,\,(\delta\zeta\,\mathbf{u}) + \sigma\,\delta\zeta = \operatorname{div}\,\,(\zeta\,\delta\mathbf{v}\,) + \sigma\mathrm{Tr}\,\mathbf{T}',\\
&\delta m = \varkappa\int_\Omega (b_{31}\,\delta\varphi + b_{32}\,\delta\pi + b_{34}\,\delta\zeta + b_{30}\,\mathrm{Tr}\,\mathbf{T}')\,dx,\\
&(\mathbb{I} - \Pi)\delta\pi = 0,\\
&\delta\mathbf{v} = 0 \;\text{ on }\; \partial\Omega, \quad \delta\varphi = 0 \;\text{ on }\; \Sigma_{\mathrm{in}}, \quad \delta\zeta = 0 \;\text{ on }\; \Sigma_{\mathrm{out}},
\end{aligned} \tag{1.20}$$

where the matrix $\mathbf{D}$ and the linear forms $\mathscr{C}$, $\mathscr{D}$ are given by the equalities

$$\begin{aligned}
&\mathbf{D} = \frac{d\mathbf{N}}{d\varepsilon}\Big|_{\varepsilon=0} = (\mathrm{Tr}\,\mathbf{T}')\mathbb{I} - \mathbf{T}', \quad \frac{d\mathfrak{g}}{d\varepsilon}\Big|_{\varepsilon=0} = \mathrm{Tr}\,\mathbf{T}' \equiv \operatorname{div}\mathbf{T},\\
&\mathscr{C}(\delta\varphi, \delta\mathbf{v}) = R\delta\varphi\,\mathbf{u}\nabla\mathbf{u} + R\varrho\,\mathbf{u}\nabla\delta\mathbf{v} + R\varrho\,\delta\mathbf{v}\nabla\mathbf{u},\\
&\mathscr{D}(\mathbf{D}) = \operatorname{div}\,\big[\big((\mathrm{Tr}\,\mathbf{T}')\mathbb{I} - \mathbf{D} - \mathbf{D}^\top\big)\nabla\mathbf{u}\big] + \mathbf{D}^\top\Delta\mathbf{u} + \Delta(\mathbf{D}\mathbf{u})\\
&\quad - R\varrho\mathbf{D}^\top(\mathbf{u}\nabla\mathbf{u}) - R\varrho\mathbf{u}\nabla(\mathbf{D}\mathbf{u}),
\end{aligned} \tag{1.21}$$

respectively, and the coefficients b_{ij} are given by the formulae

$$\begin{aligned}
&b_{11} = \Psi + m - \frac{2\sigma}{\varrho_0}\varphi - \frac{\varrho\sigma}{p'(\varrho_0)\varrho_0}H'(\varphi), \quad b_{12} = \frac{\varrho}{\lambda},\\
&b_{13} = \varrho, \quad b_{10} = \varrho(\Psi + m) - \frac{\sigma}{\varrho_0}\varphi^2 - \sigma\varphi,\\
&b_{21} = \frac{\sigma}{\varrho_0} + \frac{\sigma}{p'(\varrho_0)\varrho_0}H'(\varphi), \quad b_{22} = -\frac{1}{\lambda}, \quad b_{23} = -1, \quad b_{20} = \frac{\sigma}{\varrho_0}\varphi - \Psi - m,\\
&b_{31} = \frac{\zeta}{\varrho_0}\left((\Psi + m) - \frac{2\sigma}{\varrho_0}\varphi\right) + \frac{\sigma}{p'(\varrho_0)\varrho_0}\left(1 - \frac{\varrho\zeta}{\varrho_0}\right)H'(\varphi),\\
&b_{32} = \frac{1}{\lambda}\left(\frac{\varrho}{\varrho_0}\zeta - 1\right), \quad b_{34} = \frac{1}{\varrho_0}\left(\varrho(\Psi + m) - \frac{\sigma}{\varrho_0}\varphi^2\right),\\
&b_{30} = \left(\frac{\varrho}{\varrho_0}\zeta - 1\right)(\Psi + m) - \frac{\sigma\varrho}{\varrho_0^2}\varphi\zeta,
\end{aligned} \tag{1.22}$$

in which

$$\mathbf{u} = \mathbf{u}_\varepsilon\Big|_{\varepsilon=0}, \quad \varrho = \varrho_\varepsilon\Big|_{\varepsilon=0}.$$

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Formal differentiation of equality (1.12) leads to the following expression for the shape derivative of the drag functional:

$$\frac{d}{d\varepsilon} J_D(\mathfrak{S}_\varepsilon)\Big|_{\varepsilon=0} = L_e(\mathbf{T}) + L_u(\delta\mathbf{v}, \delta\pi, \delta\varphi), \tag{1.23}$$

where the linear forms L_e and L_u are defined by the equalities

$$\begin{aligned} L_e(\mathbf{T}) &= \mathbf{U}\int_\Omega \mathrm{Tr}\ \mathbf{T}'(\nabla\mathbf{u} + \nabla\mathbf{u}^\top - \mathrm{div}\,\mathbf{u}\mathbb{I})\nabla\eta\, dx \\ &- \mathbf{U}\int_\Omega \big[\mathbf{D}^\top\nabla\mathbf{u} + \nabla\mathbf{u}^\top\mathbf{D} - \nabla(\mathbf{D}\mathbf{u}) - \nabla(\mathbf{D}\mathbf{u})^\top\big]\nabla\eta\, dx \\ &- \mathbf{U}\int_\Omega \big[\nabla\mathbf{u} + \nabla\mathbf{u}^\top - \mathrm{div}\,\mathbf{u} - q\mathbb{I}\big]\mathbf{D}^\top\nabla\eta\, dx \\ &+ R\mathbf{U}\int_\Omega \varrho\mathbf{u}\nabla(\mathbf{D}\mathbf{u})\eta\, dx, \end{aligned} \tag{1.24}$$

and

$$L_u(\delta\mathbf{v}, \delta\pi, \delta\varphi) = -\mathbf{U}\int_\Omega \big[(\nabla\delta\mathbf{v} + \nabla\delta\mathbf{v}^\top - \mathrm{div}\ \ \delta\mathbf{v} - \delta\pi)\nabla\eta + \mathscr{C}(\delta\mathbf{v}, \delta\varphi)\eta\big]\, dx. \tag{1.25}$$

The expression for the form L_u can be identically rewritten in terms of the so-called adjoint state. To this end note that, since $\nabla\eta = 0$ on $\partial\Omega$ and $\mathrm{div}\ (\varrho\mathbf{u}) = 0$, we have

$$-\mathbf{U}\int_\Omega [\nabla\delta\mathbf{v} + \nabla\delta\mathbf{v}^\top - \mathrm{div}\ \ \delta\mathbf{v}]\nabla\eta\, dx = \int_\Omega \delta\mathbf{v}\cdot\Delta(\eta\mathbf{U})\, dx$$

and

$$-\mathbf{U}\int_\Omega \rho\eta\mathbf{u}\nabla\delta\mathbf{v}\, dx = \int_\Omega \rho\ (\mathbf{u}\,\nabla\eta)\mathbf{U}\delta\mathbf{v}\, dx.$$

This leads to the integral representation of L_u in $L_2(\Omega)$ scalar product

$$L_u = \int_\Omega (A\cdot\delta\mathbf{v} + B\,\delta\pi + C\,\delta\varphi)\, dx, \tag{1.26}$$

where

$$\begin{aligned} &A = \Delta(\eta\mathbf{U}) - R\eta\varrho\nabla\mathbf{u}\,\mathbf{U} + R\varrho(\mathbf{u}\cdot\nabla\eta)\mathbf{U}, \\ &B = \mathbf{U}\cdot\nabla\eta, \quad C = -R\eta(\mathbf{u}\nabla\mathbf{u})\cdot\mathbf{U}. \end{aligned} \tag{1.27}$$

We define the *adjoint state* $(\mathbf{h}^*, g^*, \varsigma^*, \upsilon^*, l^*)$ as a solution of the following adjoint boundary value problem for system (1.20)

$$\Delta\mathbf{h}^* - \nabla g^* - R\varrho\,\nabla\mathbf{u}\mathbf{h}^* + R\varrho\mathbf{u}\nabla\ \mathbf{h}^* + \varsigma^*\nabla\varphi + \zeta\nabla\upsilon^* = A, \tag{1.28a}$$

$$\mathrm{div}\,\mathbf{h}^* - \Pi(b_{12}\varsigma^* + b_{22}g^* + \varkappa b_{32}l^*) = B, \tag{1.28b}$$

$$-\,\mathrm{div}(\mathbf{u}\varsigma^*) + \sigma\varsigma^* - R(\mathbf{u}\nabla\mathbf{u})\cdot\mathbf{h}^* - b_{21}g^* - b_{11}\varsigma^* - \varkappa b_{31}l^* = C, \tag{1.28c}$$

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$$\mathbf{u}\nabla\upsilon^* + \sigma\upsilon^* - \varkappa b_{34} l^* = 0, \tag{1.28d}$$

$$l^* - \int_\Omega b_{13}\varsigma^*\, dx = 0, \tag{1.28e}$$

$$\mathbf{h}^* = 0 \ \text{ on } \ \partial\Omega, \quad \varsigma^* = 0 \ \text{ on } \ \Sigma_{\mathrm{out}}, \quad \upsilon^* = 0 \ \text{ on } \ \Sigma_{\mathrm{in}}, \quad g^* = \Pi g^*. \tag{1.28f}$$

Using the notion of the adjoint state, we can rewrite the expression for L_u in the form

$$L_u \equiv \int_\Omega \big[\mathrm{Tr}\ \mathbf{T}'\big(b_{10}\varsigma^* + b_{20} g^* + \sigma\upsilon^* + b_{30} l^*\varkappa\big) + \mathscr{D}(\mathbf{D})\mathbf{h}^*\big]\, dx. \tag{1.29}$$

It is important to note that the adjoint state equations are independent of $\mathbf{T}$, but involve an arbitrary function η. Notice also that the value of the shape derivative depends only on the restriction of $\mathbf{T}$ to $\mathcal{S}$, i.e., only on the perturbation of the surface $\mathcal{S}$. In the generic case the perturbation of $\mathcal{S}$ can be taken as

$$\mathcal{S}_\varepsilon = \{x = \omega + \varepsilon f(\omega)\mathbf{n}(\omega), \quad \omega \in \mathcal{S}\}, \tag{1.30}$$

where the normal shift $f(\omega)$ is a smooth function defined on $\mathcal{S}$. This relation shows that the mapping $\mathbf{T}$ satisfies the boundary condition

$$\mathbf{T}(\omega) = f(\omega)\mathbf{n}(\omega) \ \text{ for } \ \omega \in \mathcal{S}.$$

It is reasonable to eliminate η and $\mathbf{T}$ from formulae (1.23) and reformulate the expressions for the forms L_e and L_u only in terms of the normal shift $f(\omega)$. The corresponding result is given by the following theorem, which is the main result of this paper.

THEOREM 1.2. *Let parameters R, λ, σ, flow domain Ω, and exponents s, r meet all requirements of Proposition* 1.1, *and let the perturbed surface $\mathcal{S}_\varepsilon$ be defined by* (1.30) *with $f \in C^\infty(\mathcal{S})$; then*

$$\frac{d}{d\varepsilon} J_D(\mathfrak{S}_\varepsilon)\Big|_{\varepsilon=0} = \int_{\mathcal{S}} f(\omega)\big[b_{10}\varsigma + b_{20}^0 g + \sigma\upsilon + l b_{30} - (\partial_n\mathbf{h}\cdot\mathbf{n})(\partial_n\mathbf{u}\cdot\mathbf{n})\big]\, ds, \tag{1.31}$$

with the adjoint state variables

$$(\mathbf{h}, g, \varsigma, \upsilon, l) \in W^{1+s,r}(\Omega) \times \Pi W^{s,r}(\Omega) \times W^{s,r}(\Omega)^2 \times \mathbb{R}$$

satisfying the equations and boundary conditions

$$\Delta\mathbf{h} - \nabla g - R\varrho\,\nabla\mathbf{u}\mathbf{h} + R\varrho\mathbf{u}\nabla\ \mathbf{h} + \varsigma\nabla\varphi + \zeta\nabla\upsilon = 0, \tag{1.32a}$$

$$\mathrm{div}\,\mathbf{h} - \Pi(b_{12}\varsigma + b_{22} g + \varkappa b_{32} l) = 0, \tag{1.32b}$$

$$-\,\mathrm{div}(\mathbf{u}\varsigma) + \sigma\varsigma - R(\mathbf{u}\nabla\mathbf{u})\cdot\mathbf{h} - b_{21} g - b_{11}\varsigma - \varkappa b_{31} l = 0, \tag{1.32c}$$

$$\mathbf{u}\nabla\upsilon + \sigma\upsilon - \varkappa b_{34} l = 0, \tag{1.32d}$$

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$$l - \int_\Omega b_{13}\varsigma\, dx = 0, \tag{1.32e}$$

(1.32f)

$$\mathbf{h} = 0 \quad on \quad \Sigma, \quad \mathbf{h} = -\mathbf{U} \quad on \quad \mathcal{S}, \quad \varsigma = 0 \quad on \quad \Sigma_{\text{out}}, \quad \upsilon = 0 \quad on \quad \Sigma_{\text{in}}, \quad g = \Pi g.$$

Remark 1.1. It is easily seen that the solutions to (1.28) and (1.32) are connected by the simple relation

$$(\mathbf{h}^*, g^*, \varsigma^*, \upsilon^*, l^*) = (\eta\mathbf{U}, 0, 0, 0, 0) + (\mathbf{h}, g, \varsigma, \upsilon, l). \tag{1.33}$$

Hence the dependence of the adjoint state on η is additive, while the dependence of J_D on η is multiplicative.

Remark 1.2. If a solution to problem (1.32) admits continuous derivatives, then $\mathbf{u}\nabla\varsigma$, $\mathbf{u}\nabla\upsilon$ vanish on $\mathcal{S}$, and (1.32c), (1.32d) give simple algebraic relations for boundary values of the functions g, ς, and υ. Using these relations we can eliminate ς and υ from formula (1.31).

It follows from (1.23) that we can split the expression for the shape derivative into the *geometrical* part L_e and *dynamical* part L_u. The geometrical part depends only on the normal component on the boundary of the shape perturbation and on the state variables $\mathbf{u}$ and ϱ. The evaluation of the dynamical part is more complicated and requires the solution of the boundary value problem for adjoint state equations. Theorem 1.2 shows that the geometrical part of the shape derivative vanishes, and therefore cannot be used in the numerical process of the shape optimization in our framework.

The remaining part of the paper is devoted to the proof of Theorem 1.2. In section 2 we prove the existence of solutions to problems (1.28) and (1.32) with continuously differentiable vector fields $\mathbf{h}^*$ and $\mathbf{h}$. In the next section we discuss the properties of the normal coordinates in a neighborhood of $\mathcal{S}$. In section 4 we derive the formulae for the singular limits of integrals of functions concentrated near $\mathcal{S}$. Finally, in section 5 the proof of Theorem 1.2 is completed. □

2. Estimates of solutions to adjoint state equations.

In this section we prove the well-posedness and derive a priori estimates for solutions of the adjoint state equations.

Function spaces. First we recall some basic facts from the theory of Sobolev–Slobodetsky spaces, which can be found in [1] and [25]. Let Ω be the whole space $\mathbb{R}^d$, $d \geq 1$, or a bounded domain in $\mathbb{R}^d$ with the boundary $\partial\Omega$ of class C^1. For an integer $l \geq 0$ and for an exponent $r \in [1, \infty)$, we denote by $W^{l,r}(\Omega)$ the Sobolev space endowed with the norm $\|u\|_{W^{l,r}(\Omega)} = \sup_{|\alpha|\leq l} \|\partial^\alpha u\|_{L^r(\Omega)}$. For real $0 < s < 1$, the fractional Sobolev space $W^{s,r}(\Omega)$ is obtained by the real interpolation method between $L^r(\Omega)$ and $W^{1,r}(\Omega)$ and consists of all measurable functions with the finite norm

$$\|u\|_{W^{s,r}(\Omega)} = \|u\|_{L^r(\Omega)} + |u|_{s,r,\Omega},$$

where

$$|u|^r_{s,r,\Omega} = \int_{\Omega\times\Omega} |x - y|^{-d-rs} |u(x) - u(y)|^r \, dxdy. \tag{2.1}$$

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In the general case, the Sobolev space $W^{l+s,r}(\Omega)$ is defined as the space of measurable functions with the finite norm $\|u\|_{W^{l+s,r}(\Omega)} = \|u\|_{W^{l,r}(\Omega)} + \sup_{|\alpha|=l} \|\partial^\alpha u\|_{W^{s,r}(\Omega)}$.

Furthermore, the notation $W_0^{s,r}(\Omega)$, $0 \le s \le 1$, stands for the closed subspace of $W^{s,r}(\mathbb{R}^d)$, which consists of all functions $u \in W^{s,r}(\mathbb{R}^d)$ vanishing outside of Ω. We will identify functions of $W_0^{s,r}(\Omega)$ with their restriction to Ω. Recall that $W^{s,r}(\Omega) = W_0^{s,r}(\Omega)$ for $sr < 1$. It is important to note that $C_0^\infty(\Omega)$ is dense in $W_0^{s,r}(\Omega)$, but the space $W_0^{s,r}(\Omega)$ coincides with the completion of $C_0^\infty(\Omega)$ in the norm of $W^{s,r}(\Omega)$ if and only if $s \ne 1/r +$ integer.

For all $0 < s < 1$ and $1 < r < \infty$, we denote by $\mathcal{W}_0^{s,r}(\Omega)$ the interpolation space $[\mathcal{W}_0^{0,r}(\Omega), \mathcal{W}_0^{1,r}(\Omega)]_{s,r}$ endowed with one of the equivalent norms defined by the real interpolation method. In other words, $\mathcal{W}_0^{s,r}(\Omega)$ is obtained by real interpolation of the subspaces $\mathcal{W}_0^{1,r}(\Omega) = W_0^{1,r}(\Omega) \subset W^{1,r}(\Omega)$ and $\mathcal{W}_0^{0,r}(\Omega) = L^r(\Omega)$. Generally speaking, $\mathcal{W}_0^{s,r}(\Omega)$ is not a subspace of $W^{s,r}(\Omega)$, and its norm is stronger than the norm of $W^{s,r}(\Omega)$. The question of the interpolation of subspaces is one of the difficult questions of interpolation theory. In our particular case, the following result was obtained in [8] and [13]; see also [14] for a complete account of the theory:

$$\mathcal{W}_0^{s,r}(\Omega) = W_0^{s,r}(\Omega) \quad \text{for} \quad s \ne 1/r + \text{ integer}. \tag{2.2}$$

The other application of results [13], [14] is the interpolation of subspaces of finite codimension. Recall denotation (1.14) for the projection Π. We have for $s \in (0,1)$

$$\Pi W^{s,r}(\Omega) = [\Pi L^r(\Omega), \Pi W_0^{1,r}(\Omega)]_{s,r}. \tag{2.3}$$

Embedding theorems. For $sr > d$ and $0 \le \alpha < s - r/d$, the embedding $W^{s,r}(\Omega) \hookrightarrow C^\alpha(\Omega)$ is continuous and compact. Moreover, for $sr > d$, and all $u, v \in W^{s,r}(\Omega)$,

$$\|uv\|_{W^{s,r}(\Omega)} \le c(r,s)\|u\|_{W^{s,r}(\Omega)}\|v\|_{W^{s,r}(\Omega)}. \tag{2.4}$$

If $sr < d$ and $t^{-1} = r^{-1} - d^{-1}s$, then the embedding $W^{s,r}(\Omega) \hookrightarrow L^t(\Omega)$ is continuous [1, Thm. 7.57]. We have also (see [1, Thm. 7.58]) for $\alpha < s$, $(s-\alpha)r < d$, and $\beta^{-1} \ge r^{-1} - d^{-1}(s-\alpha)$,

$$\|u\|_{W^{\alpha,\beta}(\Omega)} \le c(r,s,\alpha,\beta,\Omega)\|u\|_{W^{s,r}(\Omega)}. \tag{2.5}$$

Duality. Let $s \in [0,1]$. We define

$$\langle u, v\rangle = \int_\Omega u\, v\, dx \tag{2.6}$$

for all functions such that the right-hand side make sense. For $r \in (1,\infty)$, each element $v \in L^r(\Omega)$ determines the functional $\mathcal{L}_v$ of $(W_0^{s,r'}(\Omega))'$, $r' = r/(r-1)$, by the identity $\mathcal{L}_v(u) = \langle u, v\rangle$. We introduce the $(-s,r)$-norm of an element $v \in L^r(\Omega)$ to be, by definition, the norm of the functional $\mathcal{L}_v$, that is,

$$\|v\|_{W^{-s,r}(\Omega)} = \sup_{\substack{u \in W_0^{s,r'}(\Omega) \\ \|u\|_{W_0^{s,r'}(\Omega)}=1}} |\langle u, v\rangle|. \tag{2.7}$$

Let $W^{-s,r}(\Omega)$ denote the completion of the space $L^r(\Omega)$ with respect to the $(-s,r)$-norm. The space $W^{-s,r}(\Omega)$ is topologically and algebraically isomorphic to $(W_0^{s,r'}(\Omega))'$.

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Similarly, we can define the dual space $\mathcal{W}^{-s,r}(\Omega)$ as the completion of $L^r(\Omega)$ in the norm

$$\|v\|_{\mathcal{W}^{-s,r}(\Omega)} = \sup_{\substack{u\in\mathcal{W}_0^{s,r'}(\Omega)\\ \|u\|_{\mathcal{W}_0^{s,r'}(\Omega)}=1}} |\langle u, v\rangle|. \tag{2.8}$$

The space $\mathcal{W}^{-s,r}(\Omega)$ is topologically and algebraically isomorphic to $(\mathcal{W}_0^{s,r'}(\Omega))'$. Moreover, we can identify $\mathcal{W}^{-s,r}(\Omega)$ with the interpolation space $[L^r(\Omega), W_0^{-1,r}(\Omega)]_{s,r}$; see [5]. Since $s-1/r'$ is not an integer for a noninteger $-s-1/r$, it follows from (2.2) that

$$\mathcal{W}^{-s,r}(\Omega) = W^{-s,r}(\Omega) \quad \text{for} -s \neq 1/r \ + \ \text{integer}. \tag{2.9}$$

It is well known that the operator $\nabla : W^{s,r}(\Omega) \mapsto W^{s-1,r}(\Omega)$ is continuous for $s = 0, 1$. It follows from this that $\nabla : W^{s,r}(\Omega) \mapsto \mathcal{W}^{s-1,r}(\Omega)$ is continuous for all $s \in (0, 1)$, and hence

$$\|\nabla u\|_{W^{s-1,r}(\Omega)} \leq c(s,r)\|u\|_{W^{s,r}(\Omega)} \quad \text{for all} \quad s \neq 1/r + \text{integer}, \quad s \in [0,1]. \tag{2.10}$$

Note that in this case $s-1 = -|s-1| \neq 1/r + \text{integer}$, and $\mathcal{W}^{s-1,r}(\Omega) = W^{s-1,r}(\Omega)$.

The above results lead to the following lemmas, which will be used in the proof of the solvability of the adjoint state equations.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\partial\Omega$ of class C^2, let $(F,G) \in W^{s-1,r}(\Omega) \times \Pi W^{s,r}(\Omega)$, and let*

$$s \in [0,1], \quad r \in (1,\infty), \quad s \neq 1/r \ + \ \text{integer}.$$

Then the boundary value problem

$$\begin{aligned} \Delta \mathbf{v} - \nabla\pi = F, \quad \operatorname{div}\mathbf{v} = \Pi G \quad &\text{in } \Omega,\\ \mathbf{v} = 0 \quad \text{on } \partial\Omega, \quad \Pi\pi = \pi& \end{aligned} \tag{2.11}$$

has a unique solution $(\mathbf{v},\pi) \in W^{s+1,r}(\Omega) \times \Pi W^{s,r}(\Omega)$ such that

$$\|\mathbf{v}\|_{W^{s+1,r}(\Omega)} + \|\pi\|_{W^{s,r}(\Omega)} \leq c(\Omega,r,s)(\|F\|_{W^{s-1,r}(\Omega)} + \|G\|_{W^{s,r}(\Omega)}). \tag{2.12}$$

Proof. For $s = 0, 1$, the lemma is a classical result of the theory of Stokes equations (see [7]). Hence the mapping $(F,G) \to (\mathbf{v},\pi)$ defines the continuous operator from $W^{s-1,r} \times \Pi W^{s,r}(\Omega)$ to $W^{s+1,r} \times \Pi W^{s,r}$, $s = 0, 1$. By the main theorem of the interpolation theory, this operator is continuous from $\mathcal{W}^{s-1,r} \times \Pi W^{s,r}(\Omega)$ to $W^{s+1,r} \times \Pi W^{s,r}$ for all $s \in (0,1)$. In particular, we have

$$\|\mathbf{v}\|_{W^{s+1,r}(\Omega)} + \|\pi\|_{W^{s,r}(\Omega)} \leq c(\Omega,r,s)(\|F\|_{\mathcal{W}^{s-1,r}(\Omega)} + \|G\|_{W^{s,r}(\Omega)}),$$

which, along with (2.9), gives (2.12). ∎

The next lemma gives the multiplicative estimate in fractional Sobolev space. Assume that exponents s, s' and r, r' satisfy the conditions

$$s' + s = 1, \quad \frac{1}{r} + \frac{1}{r'} = 1, \quad s, s' \in (0,1), \quad r, r' \in (1,\infty). \tag{2.13}$$

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LEMMA 2.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with the Lipschitz boundary. Furthermore, assume that exponents s, s' and r, r' satisfy* (2.13) *and the conditions*

$$sr > d, \quad 1/2 < s, \quad s \neq 1/r + \text{ integer}. \tag{2.14}$$

Then there is a constant c, depending only on r, s, and Ω, such that for all $\varsigma \in W^{s,r}(\Omega)$ and $u \in W^{s',r'}(\Omega)$,

$$\|u\varsigma\|_{W^{s',r'}(\Omega)} \leq c\|u\|_{W^{s',r'}(\Omega)}\|\varsigma\|_{W^{s,r}(\Omega)}. \tag{2.15}$$

If $u \in W_0^{s',r'}(\Omega)$, then $\varsigma u \in W_0^{s',r'}(\Omega)$. Moreover, if in addition $sr \neq d + \text{integer}$ and $\varphi \in W^{s,r}(\Omega)$, then $\varsigma\nabla\varphi \in W^{s-1,r}(\Omega)$ and

$$\|\varsigma\nabla\varphi\|_{W^{s-1,r}(\Omega)} \leq c\|\varphi\|_{W^{s,r}(\Omega)}\|\varsigma\|_{W^{s,r}(\Omega)}. \tag{2.16}$$

Proof. Since the embedding $W^{s,r}(\Omega) \hookrightarrow C(\Omega)$ is continuous, we have

$$\begin{aligned}
&\int_{\Omega\times\Omega} |x-y|^{-d-r's'}|\varsigma(x)u(x) - \varsigma(y)u(y)|^{r'}\,dxdy \\
&\leq \int_{\Omega\times\Omega} |x-y|^{-d-r's'}|u(x)|^{r'}|\varsigma(x) - \varsigma(y)|^{r'}\,dxdy \\
&+ \|\varsigma\|^{r'}_{W^{s,r}(\Omega)} \int_{\Omega\times\Omega} |x-y|^{-d-r's'}|u(x) - u(y)|^{r'}\,dxdy \\
&\leq \int_{\Omega\times\Omega} |x-y|^{-d-r's'}|u(x)|^{r'}|\varsigma(x) - \varsigma(y)|^{r'}\,dxdy + c\big(\|\varsigma\|_{W^{s,r}(\Omega)}\|u\|_{W^{s',r'}(\Omega)}\big)^{r'}.
\end{aligned} \tag{2.17}$$

Now set

$$t^{-1} = (r')^{-1} - d^{-1}s', \quad \beta = d(s')^{-1}, \quad \alpha^{-1} = 1 - r't^{-1} = r'\beta^{-1}, \quad t, \beta, \alpha \in (1, \infty).$$

By virtue of (2.14) there is $\delta > 0$ satisfying the inequalities

$$s > 1 - s + \delta \equiv s' + \delta \quad \text{and} \quad d - rs + r\delta \leq 0. \tag{2.18}$$

Fix such a δ and set

$$M(x) = \int_\Omega |x-y|^{-d-r's'}|\varsigma(x)-\varsigma(y)|^{r'}\,dy, \quad N(x) = \int_\Omega |x-y|^{-d-\beta(s'+\delta)}|\varsigma(x)-\varsigma(y)|^{\beta}\,dy.$$

Since the embedding $W^{s',r'}(\Omega) \hookrightarrow L^t(\Omega)$ is continuous, we have from the Hölder inequality

$$\begin{aligned}
&\int_{\Omega\times\Omega} |x-y|^{-d-r's'}|u(x)|^{r'}|\varsigma(x) - \varsigma(y)|^{r'}\,dxdy \\
&\quad = \int_\Omega |u(x)|^{r'}M(x)\,dx \leq \|u\|^{r'}_{L^t(\Omega)}\|M\|_{L^\alpha(\Omega)} \leq c\|u\|^{r'}_{W^{s',r'}(\Omega)}\|M\|_{L^\alpha(\Omega)}.
\end{aligned} \tag{2.19}$$

Next, note that

$$|x-y|^{-d-r's'}|\varsigma(x) - \varsigma(y)|^{r'} = (|x-y|^{-d-\beta(s'+\delta)}|\varsigma(x) - \varsigma(y)|^{\beta})^{r'/\beta}|x-y|^{-\gamma},$$

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where

$$\gamma = d + s'r' - \frac{r'}{\beta}[d + (s' + \delta)\beta] = d\frac{\beta - r'}{\beta} - r'\delta \quad \text{and} \quad \gamma\frac{\beta}{\beta - r'} < d.$$

From this and the Hölder inequality we conclude that

$$M \le N^{r'/\beta}\Big(\int_\Omega |x-y|^{-\gamma\frac{\beta}{\beta-r'}}\,dy\Big)^{(\beta-r')/\beta} \le c(\delta,\Omega)N^{r'/\beta}$$

and $M^\alpha \le N$. From this and (2.19) we obtain

$$\begin{aligned}
&\int_{\Omega\times\Omega} |x-y|^{-d-r's'}|u(x)|^{r'}|\varsigma(x)-\varsigma(y)|^{r'}\,dxdy \le c\|u\|^{r'}_{W^{s',r'}(\Omega)}\Big(\int_\Omega N\,dx\Big)^{1/\alpha}\\
&= c\|u\|^{r'}_{W^{s',r'}(\Omega)}\Big(\int_{\Omega\times\Omega} |x-y|^{-d-\beta(s'+\delta)}|\varsigma(x)-\varsigma(y)|^\beta\,dy\Big)^{r'/\beta}\\
&\le c\|u\|^{r'}_{W^{s',r'}(\Omega)}\|\varsigma\|^{r'}_{W^{s'+\delta,\beta}(\Omega)}.
\end{aligned} \tag{2.20}$$

Notice that by virtue of the identity $s' = 1 - s$ and inequalities (2.18), we have $s > s' + \delta$. Let us prove that the embedding $W^{s,r}(\Omega) \hookrightarrow W^{s'+\delta,\beta}(\Omega)$ is bounded. If $r(s - s' - \delta) \ge d$, it is bounded for all $\beta < \infty$. For $r(s - s' - \delta) < d$ it is bounded if and only if β satisfies the inequality

$$\frac{1}{\beta} \equiv \frac{s'}{d} \ge \frac{1}{r} - \frac{2s - 1 - \delta}{d},$$

which obviously follows from (2.18). Thus we get $\|\varsigma\|_{W^{1-s+\delta,\beta}(\Omega)} \le \|\varsigma\|_{W^{s,r}(\Omega)}$. Combining this result with (2.17) and (2.20), we finally obtain

$$\int_{\Omega\times\Omega} |x-y|^{-d-r's'}|\varsigma(x)u(x) - \varsigma(y)u(y)|^{r'}\,dxdy \le c\|u\|^{r'}_{W^{s',r'}(\Omega)}\|\varsigma\|^{r'}_{W^{s,r}(\Omega)},$$

which yields (2.15).

Now assume that $u \in W^{s',r'}_0(\Omega)$. Recall (see [25]) that a function $u \in W^{s',r'}_0(\Omega)$ if and only if $u \in W^{s',r'}(\Omega)$ and $\operatorname{dist}(x,\partial\Omega)^{-s'} \in L^{r'}(\Omega)$. Since the function ς is bounded, the function $u\varsigma$ belongs to $W^{s',r'}_0(\Omega)$ for $u \in W^{s',r'}_0(\Omega)$. Moreover, since by virtue of (2.14) $s' \ne r'^{-1}$ + integer, we have

$$\|u\varsigma\|_{W^{s',r'}_0(\Omega)} \le c\|u\|_{W^{s',r'}_0(\Omega)}\|\varsigma\|_{W^{s,r}(\Omega)}.$$

Finally, notice that condition (2.14) yields the relation $-s' \ne 1/r +$ integer. Hence the identity $\langle u, \varsigma\nabla\varphi\rangle = \langle u\varsigma, \nabla\varphi\rangle$ determines a continuous linear functional on $W^{-s',r}(\Omega)$ and the lemma follows. ▯

Transport equation. The following lemma is a particular case of the general results on solvability of transport equations in fractional Sobolev spaces; see [20] and [21]. It concerns the existence and uniqueness of solutions to the following boundary value problems for the first order differential equations

$$\mathscr{L}\varphi := \mathbf{u}\nabla\varphi + \sigma\varphi = f \quad \text{in } \Omega, \qquad \varphi = 0 \quad \text{on } \Sigma_{\text{in}}, \tag{2.21}$$

$$\mathscr{L}^*\varphi^* := -\operatorname{div}(\varphi^*\mathbf{u}) + \sigma\varphi^* = f \quad \text{in } \Omega, \qquad \varphi^* = 0 \quad \text{on } \Sigma_{\text{out}}. \tag{2.22}$$

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LEMMA 2.3. *Let a vector field* $\mathbf{u} \in C^1(\Omega)$ *satisfy the boundary condition*

$$\mathbf{u} = \mathbf{U} \quad \text{on} \quad \Sigma, \quad \mathbf{u} = 0 \quad \text{on} \quad \mathcal{S}, \tag{2.23}$$

where $\mathbf{U}$ *is a given constant vector field. Furthermore, assume that the exponents* s, r *satisfy the inequalities* (1.17)*. Then there are positive constants* σ^* *and* δ^**, depending only on* Σ*,* $\mathbf{U}$*,* s*,* r*, and* $\|\mathbf{u}\|_{C^1(\Omega)}$*, with the following property. If* $\|\operatorname{div}\mathbf{u}\|_{W^{s,r}(\Omega)} + \|\operatorname{div}\mathbf{u}\|_{L^\infty(\Omega)} \le \delta^*$*,* $\sigma > \sigma^*$*, then for any* $f \in W^{s,r}(\Omega)$*, each of problems* (2.21) *and* (2.22) *has a unique solution* $\varphi, \varphi^* \in W^{s,r}(\Omega)$*, which admits the estimates*

$$\|\varphi\|_{W^{s,r}(\Omega} \le C\|f\|_{W^{s,r}(\Omega)}, \quad \|\varphi^*\|_{W^{s,r}(\Omega)} \le C\|f\|_{W^{s,r}(\Omega)}. \tag{2.24}$$

The constant C *depends only on* $\|\mathbf{u}\|_{C^1(\Omega)}$*,* r, s σ*,* $\mathbf{U}$*, and* Ω*.*

Solvability of adjoint state equations. Let us consider the boundary value problem

$$\Delta\mathbf{h} - \nabla g - R\varrho\,\nabla\mathbf{u}\mathbf{h} + R\varrho\mathbf{u}\nabla\ \mathbf{h} + \varsigma\nabla\varphi + \zeta\nabla\upsilon = \mathbf{F}, \tag{2.25a}$$

$$\operatorname{div}\mathbf{h} - \Pi(b_{12}\varsigma + b_{22}g + \kappa b_{32}l) = G, \tag{2.25b}$$

$$-\operatorname{div}(\mathbf{u}\varsigma) + \sigma\varsigma - R(\mathbf{u}\nabla\mathbf{u})\cdot\mathbf{h} - b_{21}g - b_{11}\varsigma - \varkappa b_{31}l = P, \tag{2.25c}$$

$$\mathbf{u}\nabla\upsilon + \sigma\upsilon - \varkappa b_{34}l = Q, \tag{2.25d}$$

$$l - \int_\Omega b_{13}\varsigma\,dx = n, \tag{2.25e}$$

$$\mathbf{h} = 0 \text{ on } \partial B, \quad \mathbf{h} = \mathbf{a} \quad \text{on} \quad \mathcal{S}, \quad \varsigma = 0 \text{ on } \Sigma_{\text{out}}, \quad \upsilon = 0 \text{ on } \Sigma_{\text{in}}, \quad g = \Pi g. \tag{2.25f}$$

The following proposition is the main result of this section.

PROPOSITION 2.4. *Assume that the domain* Ω *satisfies Condition* 1.1*, and that the exponents* s, r *satisfy conditions* (1.17) *and inequalities* $s \ne 1/r +$ *integer. Furthermore, assume that a solution* $(\mathbf{v}, \pi, \varphi, \varsigma, m)$ *to problem* (1.16) *with* $\varepsilon = 0$ *meets all requirements of Proposition* 1.1*. Then there exist* $\mu > 0$ *and* c_0*, depending only on* $\mathbf{U}$*,* Ω*,* s*, and* r*, such that for all*

$$R + \lambda^{-1} \le \mu \tag{2.26}$$

and $\mathbf{F} \in W^{s-1,r}(\Omega)$*,* $G \in \Pi W^{s,r}(\Omega)$ *,* $P, Q \in W^{s,r}(\Omega)$*,* $n \in \mathbb{R}$*,* $\mathbf{a} \in \mathbb{R}^d$*, problem* (2.25) *has the only solution which admits the estimate*

$$\begin{aligned} &\|\mathbf{h}\|_{W^{1+s,r}(\Omega)} + \|g\|_{W^{s,r}(\Omega)} + \|\varsigma\|_{W^{s,r}(\Omega)} + \|\upsilon\|_{W^{s,r}(\Omega)} + |l| \\ &\le c_0(\|\mathbf{F}\|_{W^{s-1,r}(\Omega)} + \|G\|_{W^{s,r}(\Omega)} + \|P\|_{W^{s,r}(\Omega)} + \|Q\|_{W^{s,r}(\Omega)} + |n| + |\mathbf{a}|). \end{aligned} \tag{2.27}$$

.

Proof. It suffices to prove the proposition for $\mathbf{a} = 0$. We begin with the observation that by virtue of inequalities (1.17), the spaces $W^{s,r}(\Omega)$ and $W^{1+s,r}(\Omega)$ are the Banach algebras, and the embeddings $X^{s,r} \hookrightarrow W^{s,r}(\Omega)$, $Y^{s,r} \hookrightarrow W^{s,r}(\Omega)$ are continuous. It

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follows from this, inequalities (1.19), (2.26), and formulae (1.22) that coefficients b_{ij} admit the estimates

$$\|b_{13}\|_{W^{s,r}(\Omega)} + \|b_{21}\|_{W^{s,r}(\Omega)} + \|b_{23}\|_{W^{s,r}(\Omega)} \le c(s,r,\sigma,\Omega) \tag{2.28}$$

and

$$\begin{aligned} &\|b_{11}\|_{W^{s,r}(\Omega)} + \|b_{12}\|_{W^{s,r}(\Omega)} + \|b_{10}\|_{W^{s,r}(\Omega)} \\ &+ \|b_{22}\|_{W^{s,r}(\Omega)} + \|b_{20}\|_{W^{s,r}(\Omega)} + \|b_{3i}\|_{W^{s,r}(\Omega)} \le c(s,r,\sigma,\Omega)\mu. \end{aligned} \tag{2.29}$$

Let us consider the truncated problem

$$\Delta \mathbf{h} - \nabla g + \zeta \nabla \upsilon = \mathbf{F}, \quad \operatorname{div} \mathbf{h} = G, \tag{2.30a}$$

$$-\operatorname{div}(\mathbf{u}\varsigma) + \sigma\varsigma - b_{21} g = P, \tag{2.30b}$$

$$\mathbf{u}\nabla\upsilon + \sigma\upsilon = Q, \tag{2.30c}$$

$$l - \int_\Omega b_{13}\varsigma \, dx = n, \tag{2.30d}$$

$$\mathbf{h} = 0 \ \text{ on } \ \partial\Omega, \quad \varsigma = 0 \ \text{ on } \ \Sigma_{\text{out}}, \quad \upsilon = 0 \ \text{ on } \ \Sigma_{\text{in}}, \quad g = \Pi g. \tag{2.30e}$$

This system is of a triangular nature. Using Lemma 2.3 we conclude that in the assumptions of Proposition 1.1 we can choose σ^*, depending only on Ω, s, r, and $\mathbf{U}$, so that for every fixed $\sigma \ge \sigma^*$, equation (2.30c) has the only solution satisfying boundary condition (2.30e) and the inequality

$$\|\upsilon\|_{W^{s,r}(\Omega)} \le c\|Q\|_{W^{s,r}(\Omega)}. \tag{2.31}$$

Next, using estimate (1.19) and Lemma 2.2 we conclude that

$$\|\zeta\nabla\upsilon\|_{W^{s-1,r}(\Omega)} \le c\|\upsilon\|_{W^{s,r}(\Omega)} \le c\|Q\|_{W^{s,r}(\Omega)}.$$

Combining this result with Lemma 2.1, we obtain that equations (2.30a) have a unique solution $(\mathbf{h}, g) \in W^{s+1,r}(\Omega) \times \Pi W^{s,r}(\Omega)$ satisfying boundary condition (2.30e) and the inequality

$$\|\mathbf{h}\|_{W^{s+1,r}(\Omega)} + \|g\|_{W^{s,r}(\Omega)} \le c(\|Q\|_{W^{s,r}(\Omega)} + \|G\|_{W^{s,r}(\Omega)} + \|\mathbf{F}\|_{W^{s-1,r}(\Omega)}). \tag{2.32}$$

Applying Lemma 2.3 and using estimates (2.28), we obtain that (2.30b) and (2.30d) have solutions satisfying the inequalities

$$\|\varsigma\|_{W^{s,r}(\Omega)} + |l| \le c(\|P\|_{W^{s,r}(\Omega)} + \|Q\|_{W^{s,r}(\Omega)} + \|G\|_{W^{s,r}(\Omega)} + \|\mathbf{F}\|_{W^{s-1,r}(\Omega)} + |n|). \tag{2.33}$$

It follows from this that the mapping $(\mathbf{F}, G, P, Q, n) \to (\mathbf{h}, g, \varsigma, \upsilon, l)$ determined by (2.30) determines the continuous operator $\mathfrak{A} : W^{s-1,r}(\Omega) \times \Pi W^{s,r}(\Omega) \times W^{s,r}(\Omega)^2 \times \mathbb{R} \mapsto W^{s+1,r}(\Omega) \times \Pi W^{s,r}(\Omega) \times W^{s,r}(\Omega)^2 \times \mathbb{R}$. Let us consider the operator $\mathfrak{B} = (\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4)$ defined by the equalities

$$\begin{aligned} \mathfrak{B}_1(\mathbf{h}, g, \varsigma, \upsilon, l) &= -R\varrho\,\nabla\mathbf{u}\mathbf{h} + R\varrho\mathbf{u}\nabla\ \mathbf{h} + \varsigma\nabla\varphi, \\ \mathfrak{B}_2(\mathbf{h}, g, \varsigma, \upsilon, l) &= -\Pi(b_{12}\varsigma + b_{22}g + \kappa b_{32} l), \\ \mathfrak{B}_3(\mathbf{h}, g, \varsigma, \upsilon, l) = -R(\mathbf{u}\nabla\mathbf{u})\cdot\mathbf{h} - b_{11}\varsigma - \varkappa b_{31} l, &\quad \mathfrak{B}_4(\mathbf{h}, g, \varsigma, \upsilon, l) = 0. \end{aligned}$$

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It follows from estimate (1.19) and Lemma 2.2 that

$$\|\varsigma\nabla\varphi\|_{W^{s-1,r}(\Omega)} \le c\|\varsigma\|_{W^{s,r}(\Omega)}\|\varphi\|_{W^{s,r}(\Omega)} \le c\mu\|\varsigma\|_{W^{s,r}(\Omega)}.$$

Combining this result with estimates (1.19), (2.29), we obtain that the norm of the operator $\mathfrak{B} : W^{s-1,r}(\Omega) \times \Pi W^{s,r}(\Omega) \times W^{s,r}(\Omega)^2 \times \mathbb{R} \mapsto W^{s+1,r}(\Omega) \times \Pi W^{s,r}(\Omega) \times W^{s,r}(\Omega)^2 \times \mathbb{R}$ does not exceed $c\mu$. Since equations (2.25) are equivalent to the operator equation

$$(\mathbb{I} + \mathfrak{A}\mathfrak{B})(\mathbf{h}, g, \varsigma, \upsilon, l) = \mathfrak{A}(\mathbf{F}, G, P, Q, n),$$

the assertion of the lemma results from the contraction mapping principle. ∎

3. Normal coordinates. For any point $\omega \in \mathcal{S}$ denote by $\mathbf{n}$ the unique outward normal vector to $\mathcal{S}$ at point ω. Since $\mathcal{S}$ is a smooth manifold, there exist a neighborhood $\mathcal{O}$ of $\mathcal{S}$ and a number $a > 0$ so that the mapping $(\omega, t_3) \to \omega + t_3\mathbf{n}(\omega)$ takes diffeomorphically the set $\mathcal{S} \times [-a, a] \to \mathcal{O}$.

Let us calculate the derivatives and Jacobian of this mapping. To this end we fix an arbitrary point $\omega_0 \in \mathcal{S}$. In some neighborhood $\mathcal{O}_0$ of this point, the surface $\mathcal{S}$ admits the parametric representation

$$\mathcal{S} \cap \mathcal{O}_0 : \omega = \mathbf{r}(t_1, t_2), \quad |t_i| \le a, \quad \mathbf{r}(0,0) = \omega_0, \tag{3.1}$$

where the vector-valued function $\mathbf{r}(t_1, t_2)$ belongs to the class C^∞ in the rectangular

$$Q = \{(t_1, t_2) : \ |t_i| \le a, \ i = 1, 2\}$$

for an appropriate choice of a independent of ω_0. We denote

$$\mathbf{t} = (t_1, t_2, t_3), \quad \bar{t} = (t_1, t_2) \ \text{ so that } \ \mathbf{t} = (\bar{t}, t_3)$$

and introduce the moving frame

$$\mathbf{e}_i(\bar{t}) = \partial_{t_i}\mathbf{r}(\bar{t}), \quad i = 1, 2, \quad \mathbf{e}_3(\bar{t}) = \mathbf{n}(\mathbf{r}(\bar{t})) = \frac{1}{|\mathbf{e}_1 \times \mathbf{e}_2|}\mathbf{e}_1 \times \mathbf{e}_2.$$

Thus the mapping

$$X : \mathbf{t} \mapsto \mathbf{r}(\bar{t}) + t_3\mathbf{e}_3(\bar{t}) \equiv \omega + t_3\mathbf{n}(\omega)$$

takes diffeomorphically

$$Q \times [-a, a] \mapsto \mathcal{O}_0 \quad \text{and} \quad Q \times [-a, 0] \mapsto \mathcal{O}_0 \cap \Omega.$$

Recall the denotations

$$g_{ij} = \mathbf{e}_i(\bar{t}) \cdot \mathbf{e}_j(\bar{t}), \quad b_{ij} = \mathbf{e}_3(\bar{t}) \cdot \partial^2_{t_i\, t_j}\mathbf{r}(\bar{t}), \quad i, j = 1, 2,$$

for the coefficients of the first and second fundamental forms of the surface $\mathcal{S}$, and set

$$g = \det\,(g_{ij}), \quad (g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}.$$

In this notation, the mean and Gauss curvatures of $\mathcal{S}$ are determined by the equalities

$$H = \frac{1}{2}(b_1^1 + b_2^2), \quad K = b_1^1 b_2^2 - b_2^1 b_1^2, \ \text{ where } \ b_j^i = g^{i\alpha}b_{\alpha j}.$$

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Using the Weigharten equations

$$\partial_{t_j} \mathbf{e}_3 \; = \; -b^i_j \, \mathbf{e}_i \quad \text{with} \;\; i, j = 1, 2,$$

we obtain the expression for the columns of the Jacobi matrix $X'(\mathbf{t})$,

$$\partial_{t_j} X = \mathbf{e}_j - t_3(b^1_j \mathbf{e}_1 + b^2_j \mathbf{e}_2), \;\; j = 1, 2, \quad \partial_{t_3} X = \mathbf{e}_3. \tag{3.2}$$

In order to obtain the inverse of $X'(\mathbf{t})$ we use the identity

$$\begin{aligned} &(X'^{-1})^\top \equiv \big[\nabla_x t_1(X(\mathbf{t})), \nabla_x t_2(X(\mathbf{t})), \nabla_x t_3(X(\mathbf{t}))\big] \\ &= (\det\, X')^{-1}\big[\partial_{t_2} X \times \partial_{t_3} X,\, \partial_{t_3} X \times \partial_{t_1} X,\, \partial_{t_1} X \times \partial_{t_2} X, \,\big]. \end{aligned} \tag{3.3}$$

Next, note that for $j = 1, 2$,

$$\mathbf{e}_j \times \mathbf{e}_3 = g^{-1/2}\mathbf{e}_j \times (\mathbf{e}_1 \times \mathbf{e}_2) = g^{-1/2}\big((\mathbf{e}_j \cdot \mathbf{e}_2)\mathbf{e}_1 - (\mathbf{e}_j \cdot \mathbf{e}_1)\mathbf{e}_2\big) = g^{-1/2}(g_{j2}\mathbf{e}_1 - g_{j1}\mathbf{e}_2)$$

and $\mathbf{e}_1 \times \mathbf{e}_2 = g^{1/2}\mathbf{e}_3$. From this and (3.2), (3.3) we obtain

$$\begin{aligned} &\partial_{t_j} X \times \partial_{t_3} X = g^{-1/2}(g_{j2}\mathbf{e}_1 - g_{j1}\mathbf{e}_2) \\ -t_3 g^{-1/2}((2b_{j2} &+ b_{j1}(g^{11}g_{12} + g^{21}g_{22}))\mathbf{e}_1 - (2b_{1j} + b_{2j}(g^{12}g_{11} + g^{22}g_{21}))\mathbf{e}_2). \end{aligned}$$

and

$$\partial_{t_1} X \times \partial_{t_2} X = g^{1/2}(1 - 2Ht_3 + Kt_3^2)\mathbf{e}_3.$$

In particular we have the following expression for the Jacobian:

$$\det\, X' \equiv (\partial_{t_1} X \times \partial_{t_2} X) \cdot \partial_{t_3} X = g^{1/2}(1 - 2Ht_3 + Kt_3^2). \tag{3.4}$$

We also get the following expression along the lines of the matrix $(X')^{-1}$ (the columns of the transposed matrix $(X'^{-1})^\top$):

$$\begin{aligned} &(\nabla_x t_1)(X(\mathbf{t})) = g^{-1}(1 - 2Ht_3 + Kt_3^2)^{-1}\big\{g_{22}\mathbf{e}_1 - g_{21}\mathbf{e}_2 - \\ &t_3((2b_{22} + b_{21}(g^{11}g_{12} + g^{21}g_{22}))\mathbf{e}_1 - (2b_{12} + b_{22}(g^{12}g_{11} + g^{22}g_{21}))\mathbf{e}_2)\big\}, \\ &(\nabla_x t_2)(X(\mathbf{t})) = -g^{-1}(1 - 2Ht_3 + Kt_3^2)^{-1}\big\{(g_{12}\mathbf{e}_1 - g_{11}\mathbf{e}_2) - \\ &t_3((2b_{12} + b_{11}(g^{11}g_{12} + g^{21}g_{22}))\mathbf{e}_1 - (2b_{11} + b_{21}(g^{12}g_{11} + g^{22}g_{21}))\mathbf{e}_2)\big\}, \\ &(\nabla_x t_3)(X(\mathbf{t})) = \mathbf{e}_3. \end{aligned} \tag{3.5}$$

Conformal coordinates. The formulae can be essentially simplified if the parametrization $x = \mathbf{r}(\bar{t})$ is conformal. This means that

$$g_{12} = 0, \quad g_{11} = g_{22} = g^{1/2}.$$

In this case the expression for the lines of the inverse Jacobi becomes

$$\begin{aligned} &(\nabla_x t_1)(X(\mathbf{t})) = g^{-1}(1 + 2Ht_3 + Kt_3^2)^{-1}\big\{g^{1/2}\mathbf{e}_1 - t_3(b_{22}\mathbf{e}_1 - b_{21}\mathbf{e}_2)\big\}, \\ &(\nabla_x t_2)(X(\mathbf{t})) = g^{-1}(1 + 2Ht_3 + Kt_3^2)^{-1}\big\{g^{1/2}\mathbf{e}_2 - t_3(b_{11}\mathbf{e}_2 - b_{12}\mathbf{e}_1)\big\}, \\ &(\nabla_x t_3)(X(\mathbf{t})) = \mathbf{e}_3. \end{aligned} \tag{3.6}$$

In particular, we have the following expression for the Jacobi matrix and its inverse at the sheet $\mathcal{S} \cap \mathcal{O}_0$:

$$X'(\mathbf{t}) = \big[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\big], \quad (X'^{-1})^\top(\mathbf{t}) = \big[g^{-1/2}\mathbf{e}_1, g^{-1/2}\mathbf{e}_2, \mathbf{e}_3\big]. \tag{3.7}$$

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3.1. Surface perturbations. Choose an arbitrary function $f \in C^\infty(\mathcal{S})$. Without any loss of generality we can assume that f is extended over $\mathbb{R}^3$ and the extension belongs to the class $C^\infty(\mathbb{R}^3)$. Let us consider the immersion

$$\mathcal{S}_\tau : x = \omega + \tau f(\omega)\mathbf{n}(\omega), \ \ \omega \in \mathcal{S}, \tag{3.8}$$

where $\tau \in [0, a]$ is a small parameter. For all small τ this immersion is the embedding and we obtain the family of smooth surfaces $\mathcal{S}_\tau$, which can be regarded as a normal perturbation of $\mathcal{S}$ generated by the normal shift $f\mathbf{n}$. Now our task is to extend the mapping (3.8) over $\mathbb{R}^3$. To this end fix a function $\chi \in C_0^\infty(\mathbb{R})$ satisfying the relations

$$\chi(s) = 1 \quad \text{for} \quad |s| \le 1/2, \qquad \chi(t) = 0 \quad \text{for} \quad |s| \ge 1.$$

Introduce a vector field $\mathbf{T}_\tau(x)$ defined in the neighborhood $\mathcal{O}$ of the surface ∂S by the relations

$$\mathbf{T}_\tau(x) = \mathbf{T}_\tau(\omega + t_3\mathbf{n}(\omega)) = f(\omega)\chi(t_3/\tau)\mathbf{n}(\omega), \quad \text{where} \ \ t_3 = t_3(x) \equiv \text{dist } (x, \mathcal{S}). \tag{3.9}$$

Since the vector field $\mathbf{T}_\tau$ vanishes for $t_3 \ge a$, and is extended by 0 over $\mathbb{R}^3$, it belongs to the class $C^\infty(\mathbb{R}^3)$. Hence for all small ε, the mapping $x \to x + \varepsilon\mathbf{T}_\tau(x)$ determines the diffeomorphism of $\mathbb{R}^3$. Moreover, this diffeomorphism is equal to the identical mapping outside of the τ-neighborhood of $\mathcal{S}$.

Now we consider in more detail the structure of the derivative of $\mathbf{T}'_\tau$ and of the related matrix $\mathbf{D}_\tau = \big(\text{Tr } \mathbf{T}'_\tau\big)\mathbb{I} - \mathbf{T}'_\tau$. In order to formulate the corresponding result it is convenient to introduce the notation $\mathbf{M}(\mathbf{t})$ for the Jacobi matrix $X'(\mathbf{t})$.

Lemma 3.1. *Under the above assumptions,*

$$\begin{aligned} \mathbf{T}'_\tau(x) = {} & \frac{1}{\tau}\chi'(t_3/\tau)f(\omega)\mathbf{n}(\omega) \otimes \mathbf{n}(\omega) \\ & + \chi(t_3/\tau)\mathbf{n}(\omega) \otimes \nabla_{tan} f(\omega) + \chi(t_3/\tau)\mathbf{S}, \end{aligned} \tag{3.10}$$

where $t_3 = t_3(x)$, the tangential gradient

$$\nabla_{tan} f(\omega) = \big(\mathbb{I} - \mathbf{n}(\omega) \otimes \mathbf{n}(\omega)\big)\nabla f(\omega)|_{t_3=0}$$

stands for the orthogonal projection of the gradient ∇f onto the tangent plane to $\mathcal{S}$, and the elements $\mathbf{S}_{ij}$ of the matrix $\mathbf{S}$ are defined in the local coordinates $\mathbf{t}$ by the equalities

$$\mathbf{S}_{ij} = \chi(t_3/\tau)f(\omega)\sum_{\alpha=1,2} W_j^\alpha(\omega)e_{i\alpha}, \quad W_j^\alpha(\bar{t}) = -b_\beta^\alpha(\mathbf{M}^{-1})_{\beta j},$$

where $e_{i\alpha}$ is the ith component of the vector $\mathbf{e}_\alpha$. Moreover, we have

$$\text{Tr } \mathbf{T}'_\tau = \frac{1}{\tau}\chi'(t_3/\tau)f(\omega) \ - 2\chi f(t_3/\tau)H(\omega), \tag{3.11}$$

where $H(\omega)$ is the mean curvature of $\mathcal{S}$. The matrix $\mathbf{D}_\tau$ admits the representation

$$\begin{aligned} \mathbf{D}_\tau = {} & \left(\frac{1}{\tau}\chi'(t_3/\tau)f(\omega) \ - 2\chi f(t_3/\tau)H(\omega)\right) \ \mathbb{I} \\ & -\frac{1}{\tau}\chi'(t_3/\tau)f(\omega) \ \mathbf{n}(\omega) \otimes \mathbf{n}(\omega) - \chi(t_3/\tau) \ \mathbf{n}(\omega) \otimes \nabla_{tan} f(\omega) - \chi(t_3/\tau)\mathbf{S}. \end{aligned} \tag{3.12}$$

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The matrices $\mathbf{T}'_\tau$ *and* $\mathbf{D}_\tau$ *satisfy the boundary conditions*

(3.13)
$$\mathbf{T}'_\tau = \mathbf{n}(\omega) \otimes \nabla_{tan} f(\omega) + \mathbf{S}, \qquad \mathbf{D}_\tau = -2fH\mathbb{I} - \mathbf{n}(\omega) \otimes \nabla_{tan} f(\omega) - \mathbf{S} \quad on \quad \mathcal{S}.$$

Proof. By virtue of the partition of unity approach, it suffices to prove the lemma in the neighborhood $\mathcal{O}_0$ of an arbitrary point $\omega_0 \in \mathcal{S}$. Let $\mathbf{t} \in Q \times [-a, a]$ be a local coordinate system in this neighborhood such that $\bar{t}$ is a local conformal system of coordinates on the manifold $\mathcal{S} \cap \mathcal{O}_0$. In this system of coordinates we have

$$\partial_{x_i} \mathbf{T}_\tau = \frac{1}{\tau}\chi' f \; \partial_{x_i} t_3 \; \mathbf{n} + \chi \partial_{r_p} f \; \partial_{t_q} (\mathbf{r})_p \; \partial_{x_i} t_q \; \mathbf{n} + \chi f \; \partial_{x_i} t_p \; \partial_{t_p} \mathbf{n}, \tag{3.14}$$

where $(\mathbf{r})_p$ is the pth component of the vector-valued function $\mathbf{r}(\bar{t})$. Noting that $\nabla_x t_3 = \mathbf{e}_3(\mathbf{t}) = \mathbf{n}(x(\mathbf{t}))$, we arrive at the first basic identity

$$\frac{1}{\tau}\chi' f \; \partial_{x_i} t_3 \; \mathbf{n} = \frac{1}{\tau}\chi' f n_i \; \mathbf{n}. \tag{3.15}$$

Next we have

$$\partial_{r_p} f \; \partial_{t_q} (\mathbf{r})_p \; \partial_{x_i} t_q \equiv \sum_{q=1,2} \partial_{r_p} f \; \partial_{t_q} (\mathbf{r})_p \; \partial_{x_i} t_q$$
$$\equiv \sum_{q=1,2} \partial_{r_p} f \; M_{pq} N_{qi} = \partial_{r_i} f - M_{p3} N_{3i} \partial_{r_p} f = \partial_{r_i} f - n_p n_i \; \partial_{r_p} f,$$

which leads to the second identity

$$\chi \partial_{r_p} f \; \partial_{t_q} (\mathbf{r})_p \; \partial_{x_i} t_q \; \mathbf{n} = (\nabla_{tan} f)_i \, \mathbf{n} \chi, \tag{3.16}$$

where $(\nabla_{tan} f)_i$ is the ith component of the vector $\nabla_{tan} f$. Using the Weigharten equation, we also get the third identity

$$\chi f \; \partial_{x_i} t_p \; \partial_{t_p} \mathbf{n} = -\chi f \sum_{\alpha,\beta=1,2} b^\alpha_\beta (\mathbf{M}^{-1})_{\beta i} \mathbf{e}_\alpha =: \chi f \sum_{\alpha,\beta=1,2} W^\alpha_i \mathbf{e}_\alpha. \tag{3.17}$$

Combining (3.14)–(3.17), we finally obtain (3.10). Since $e_{i,\alpha} = M_{i\alpha}$, we have

$$\text{Tr } \mathbf{S} = \chi f \sum_{i=1,2,3} \sum_{\alpha=1,2} W^\alpha_i e_{i\alpha} = -\chi f \sum_{\alpha=1,2} b^\alpha_\beta (\mathbf{M}^{-1})_{\beta i} \mathbf{M}_{i\alpha} = -\chi f b^\alpha_\alpha = -2\chi f H,$$

which along with (3.10) and the identity $\nabla_{tan} f \cdot \mathbf{n} = 0$ leads to the representation (3.11). Formula (3.12) is an obvious consequence of (3.10) and (3.11). It remains to note that since $\chi' = 0$ and $\chi = 1$ on $\mathcal{S}$, boundary conditions (3.13) follow from (3.11) and (3.12). ∎

4. Limiting relations. In this section we establish some auxiliary facts on the singular limits of integrals, with the integrands depending on the derivatives of $\mathbf{T}_\tau$.

LEMMA 4.1. *For any* $\psi \in C(\Omega)$,

$$\begin{aligned} &\lim_{\tau \searrow 0} \int_\Omega \psi(x) \mathbf{T}'_\tau(x) \, dx = \int_{\mathcal{S}} \psi(\omega) f(\omega) \mathbf{n} \otimes \mathbf{n} \, dS, \\ &\lim_{\tau \searrow 0} \int_\Omega \psi(x) \, Tr \; \mathbf{T}'_\tau(x) \, dx = \int_{\mathcal{S}} \psi(\omega) f(\omega) \, dS. \end{aligned} \tag{4.1}$$

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Proof. Using the partition of unity, we can assume that the support spt ψ is concentrated in a neighborhood of the point $\omega_0 \in \mathcal{S}$, and hence

$$\int_\Omega \psi(x)\mathbf{T}'_\tau(x)\,dx = \int_Q \int_{-\tau}^0 \tilde{\psi}(\mathbf{t})\mathbf{T}'_\tau(X(\mathbf{t}))\det X'(\mathbf{t})\,d\mathbf{t},$$

where $\tilde{\psi}(\mathbf{t}) = \psi(X(\mathbf{t}))$ and $Q = [-a, a]^2$. Using identity (3.10) and formula (3.4), we obtain

$$\mathbf{T}'_\tau(X(\mathbf{t}))\ \ \det X'(\mathbf{t}) = \frac{1}{\tau}\chi'(t_3/\tau)f(\omega(\bar{t}))\ \mathbf{e}_3(\bar{t})\otimes\mathbf{e}_3(\bar{t})\ g^{1/2}(\bar{t}) + \mathbf{R}(\mathbf{t}),$$

where

$$\mathbf{R}(\mathbf{t}) = \frac{t_3}{\tau}\chi'(t_3/\tau)f(\omega(\bar{t}))\ \mathbf{e}_3(\bar{t})\otimes\mathbf{e}_3(\bar{t})\ (-2H + t_3K)g^{1/2}$$
$$+\Big(\chi(t_3/\tau)\ \mathbf{e}_3(\mathbf{t})\otimes\nabla_{tan}f(\omega(\bar{t})) + \chi(t_3/\tau)\mathbf{S}(\mathbf{t})\Big)g^{1/2}(1 - 2t_3H + t_3^2K).$$

Since the functions f, ∇f and the elements of matrix $\mathbf{S}$ are uniformly bounded in $Q\times[-\tau, 0]$, we have

$$\int_Q\int_{-\tau}^0 |\tilde{\psi}(\mathbf{t})\mathbf{R}(\mathbf{t})|\,d\mathbf{t} \le c\tau \to 0 \quad \text{as} \quad \tau\to 0.$$

Thus we get

$$\lim_{\tau\searrow 0}\int_\Omega \psi(x)\mathbf{T}'_\tau(x)\,dx = \lim_{\tau\searrow 0}\frac{1}{\tau}\int_Q\int_{-\tau}^0 \tilde{\psi}(\mathbf{t})\chi'(t_3/\tau)f(\omega(\bar{t}))\mathbf{e}_3(\bar{t})\otimes\mathbf{e}_3(\bar{t})g^{1/2}(\bar{t})\,d\mathbf{t}.$$

Noting that

$$\lim_{\tau\searrow 0}\frac{1}{\tau}\int_Q\int_{-\tau}^0 [\tilde{\psi}(\mathbf{t}) - \tilde{\psi}(\bar{t}, 0)]\chi'(t_3/\tau)f(\omega(\bar{t}))\mathbf{e}_3(\bar{t})\otimes\mathbf{e}_3(\bar{t})g^{1/2}(\bar{t})d\mathbf{t} = 0$$

and using the relations $\mathbf{e}_3(\bar{t}) = \mathbf{n}(\omega(\bar{t}))$ and $g^{1/2}dt_1dt_2 = dS$, we obtain

$$\lim_{\tau\searrow 0}\int_\Omega \psi(x)\mathbf{T}'_\tau(x)\,dx = \lim_{\tau\searrow 0}\frac{1}{\tau}\int_Q\int_{-\tau}^0 \tilde{\psi}(\bar{t}, 0)\chi'(t_3/\tau)f(\omega(\bar{t}))\ \mathbf{e}_3(\bar{t})\otimes\mathbf{e}_3(\bar{t})\ g^{1/2}(\bar{t})\,d\mathbf{t}$$
$$= \Big(\lim_{\tau\searrow 0}\frac{1}{\tau}\int_{-\tau}^0 \chi'(t_3/\tau)dt_3\Big)\ \int_Q \tilde{\psi}(\bar{t}, 0)f(\omega(\bar{t}))\ \mathbf{e}_3(\bar{t})\otimes\mathbf{e}_3(\bar{t})\ g^{1/2}(\bar{t})\,dt_1dt_2$$
$$= \int_Q \tilde{\psi}(\bar{t}, 0)f(\omega(\bar{t}))\mathbf{e}_3(\bar{t})\otimes\mathbf{e}_3(\bar{t})\ g^{1/2}(\bar{t})\,dt_1dt_2 = \int_{\mathcal{S}} \psi(\omega)f(\omega)\mathbf{n}(\omega)\otimes\mathbf{n}(\omega)\,dS,$$

which yields the first equality in (4.1). The second equality in (4.1) obviously follows from the first. ∎

The next lemma establishes the limiting relation for the integral with the integrand depending on the derivatives of mapping $\mathbf{D}_\tau$.

LEMMA 4.2. *Let the vector fields $\mathbf{h}, \mathbf{u} : \Omega \mapsto \mathbb{R}^3$ belong to the class $C^1(\Omega)^3$ and vanish on $\mathcal{S}$. Then*

$$\lim_{\tau\searrow 0}\int_\Omega \partial_{x_i}\mathbf{h}(x)\cdot(\partial_{x_i}\mathbf{D}_\tau)\mathbf{u}\,dx = -\int_{\mathcal{S}} f(\omega)\partial_n\mathbf{h}(\omega)\cdot(\mathbb{I} - \mathbf{n}\otimes\mathbf{n})\partial_n\mathbf{u}(\omega)\,dS. \tag{4.2}$$

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Proof. Using the partition of unity, we can assume that the support spt $\mathbf{h}$ is concentrated in a neighborhood of point $\omega_0 \in \mathcal{S}$, which gives

$$\int_\Omega \partial_{x_i}\mathbf{h}(x)\,\cdot(\partial_{x_i}\mathbf{D}_\tau)\,\mathbf{u}\,dx = \int_Q\int_{-\tau}^0 \partial_{x_i}\mathbf{h}(X(\mathbf{t}))\,\cdot\big[\partial_{x_i}\mathbf{D}_\tau(X(\mathbf{t}))\big]\mathbf{u}(X(\mathbf{t}))\mathrm{det}X'(\mathbf{t})\,d\mathbf{t}.$$

Notice that

$$\partial_{x_i}\mathbf{h}(X(\mathbf{t}))\cdot\big[\partial_{x_i}\mathbf{D}_\tau(X(\mathbf{t}))\big]\,\mathbf{u}(X(\mathbf{t})) = G^{pq}\partial_{t_p}\tilde{\mathbf{h}}(\mathbf{t})\cdot\big[\partial_{t_q}\mathbf{D}_\tau(X(\mathbf{t}))\big]\,\tilde{\mathbf{u}}(\mathbf{t}),$$

where

$$\tilde{\mathbf{h}}(\mathbf{t}) = \mathbf{h}(X(\mathbf{t})),\quad \tilde{\mathbf{u}}(\mathbf{t}) = \mathbf{u}(X(\mathbf{t})),\quad G^{pq} = (\nabla_X t_p)(X(\mathbf{t}))\cdot(\nabla_X t_q)(X(\mathbf{t})).$$

Thus we get

$$\int_\Omega \partial_{x_i}\mathbf{h}(x)\cdot(\partial_{x_i}\mathbf{D}_\tau)\,\mathbf{u}\,dx = \int_Q\int_{-\tau}^0 G^{pq}\partial_{t_p}\tilde{\mathbf{h}}(\mathbf{t})\cdot\big[\partial_{t_q}\mathbf{D}_\tau(X(\mathbf{t}))\big]\,\tilde{\mathbf{u}}(\mathbf{t})\mathrm{det}X'(\mathbf{t})\,d\mathbf{t}.$$

Since $\mathbf{u}$ vanishes at $\mathcal{S}$, the Taylor formula gives

$$\tilde{\mathbf{u}}(\mathbf{t}) \equiv \mathbf{u}(\omega(\overline{t})) + t_3\mathbf{n}(\omega(\overline{t})) = \partial_n\mathbf{u}(\omega(\overline{t}))t_3 + r(\omega, t_3),$$

where

$$\frac{1}{t_3}r(\omega,t_3)\to 0 \ \text{ as } \ t_3\to 0 \ \text{ uniformly with respect to } \ \omega\in\mathcal{S}.$$

Since the first derivatives of $\tilde{\mathbf{h}}$ and the functions G^{pq} are uniformly bounded and since $|\partial_{t_q}\mathbf{D}_\tau| \le c\tau^{-2}$, we have

$$\int_Q\int_{-\tau}^0 \big|\,G^{pq}\partial_{t_p}\tilde{\mathbf{h}}(\mathbf{t})\cdot\partial_{t_q}\mathbf{D}_\tau(X(\mathbf{t}))\,r(\mathbf{t})|\mathrm{det}X'(\mathbf{t})\,d\mathbf{t} \le c\tau^{-2}\int_Q\int_{-\tau}^0 |r(\mathbf{t})|\,d\mathbf{t}\to 0$$

as $\tau\to 0$. Thus we get

$$\begin{aligned}&\lim_{\tau\searrow 0}\int_\Omega \partial_{x_i}\mathbf{h}(x)\cdot\partial_{x_i}\mathbf{D}_\tau\mathbf{u}\,dx\\ &= \lim_{\tau\searrow 0}\int_Q\int_{-\tau}^0 t_3\,G^{pq}\,\partial_{t_p}\tilde{\mathbf{h}}(\mathbf{t})\cdot\partial_{t_q}\mathbf{D}_\tau(X(\mathbf{t}))\,\partial_n\mathbf{u}(\omega(\overline{t}))\ \ \mathrm{det}X'(\mathbf{t})\,dt_1dt_2dt_3.\end{aligned}$$

Next it follows from the representation (3.12) that

$$|\partial_{t_q}\mathbf{D}_\tau| \le c\tau^{-1} \ \text{ for } \ q=1,2$$

and

$$\partial_{t_3}\mathbf{D}_\tau = \frac{1}{\tau^2}\chi''(t_3/\tau)f(\omega(\overline{t}))(\mathbb{I}-\mathbf{n}\otimes\mathbf{n}) + R_1 \ \text{ with } \ |R_1(\mathbf{t})|\le c\tau^{-1}.$$

It follows from this that

$$\begin{aligned}&t_3G^{pq}\partial_{t_p}\tilde{\mathbf{h}}(\mathbf{t})\cdot\big[\partial_{t_q}\mathbf{D}_\tau(X(\mathbf{t}))\big]\,\partial_n\mathbf{u}(\omega(\overline{t}))\mathrm{det}X'(\mathbf{t})\\ &= \frac{t_3}{\tau^2}\chi''(t_3/\tau)f(\omega(\overline{t}))G^{p3}\partial_{t_p}\tilde{\mathbf{h}}(\mathbf{t})\cdot(\mathbb{I}-\mathbf{n}\otimes\mathbf{n})\partial_n\mathbf{u}(\omega(\overline{t}))\mathrm{det}X'(\mathbf{t}) + R_2,\end{aligned}$$

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where

$$|R_2| \le c|t_3|/\tau \le c.$$

Using formula (3.4) for the Jacobian, we obtain

$$\begin{aligned}&t_3 G^{pq}\partial_{t_p}\tilde{\mathbf{h}}(\mathbf{t})\cdot\big[\partial_{t_q}\mathbf{D}_\tau(X(\mathbf{t}))\big]\,\partial_n\mathbf{u}(\omega(\bar t))\ \ \det X'(\mathbf{t})\\ &\quad=\frac{t_3}{\tau^2}\chi''(t_3/\tau)f(\omega(\bar t))G^{p3}\partial_{t_p}\tilde{\mathbf{h}}(\mathbf{t})\cdot(\mathbb{I}-\mathbf{n}\otimes\mathbf{n})\partial_n\mathbf{u}(\omega(\bar t))g^{1/2}+R_3,\end{aligned}$$

where $|R_3|\le c$. From this we conclude that

$$\begin{aligned}&\lim_{\tau\searrow 0}\int_\Omega \partial_{x_i}\mathbf{h}(x)\cdot(\partial_{x_i}\mathbf{D}_\tau)\mathbf{u}\,dx\\ &=\lim_{\tau\searrow 0}\frac{1}{\tau^2}\int_Q\int_{-\tau}^0 t_3\chi''(t_3/\tau)f(\omega(\bar t))\,G^{p3}\,\partial_{t_p}\tilde{\mathbf{h}}(\mathbf{t})\cdot(\mathbb{I}-\mathbf{n}\otimes\mathbf{n})\partial_n\mathbf{u}(\omega(\bar t))g^{1/2}\,d\mathbf{t}.\end{aligned}$$

Since the vector field $\tilde{\mathbf{h}}$ vanishes at $t_3=0$ we have for $\alpha=1,2$

$$\partial_{t_\alpha}\tilde{\mathbf{h}}=r_\alpha(\mathbf{t}),\quad \partial_{t_3}\tilde{\mathbf{h}}(\mathbf{t})-\partial_{t_3}\tilde{\mathbf{h}}(\bar t,0)=r_3(\mathbf{t}),$$

where $r_i(\mathbf{t})\to 0$ as $t_3\to 0$ uniformly in $\bar t\in Q$. In particular we have

$$\begin{aligned}&\frac{1}{\tau^2}\Big|\int_Q\int_{-\tau}^0 t_3\chi''(t_3/\tau)f(\omega(\bar t))G^{p3}r_p(\mathbf{t})\cdot(\mathbb{I}-\mathbf{n}\otimes\mathbf{n})\partial_n\mathbf{u}(\omega(\bar t))g^{1/2}\,d\mathbf{t}\Big|\\ &\le\lim_{\tau\searrow 0}\frac{1}{\tau^2}\int_Q\int_{-\tau}^0 t_3\chi''(t_3/\tau)f(\omega(\bar t))\,G^{33}\,\partial_{t_3}\tilde{\mathbf{h}}(\bar t,0)\cdot(\mathbb{I}-\mathbf{n}\otimes\mathbf{n})\partial_n\mathbf{u}(\omega(\bar t))g^{1/2}\,d\mathbf{t}\\ &\qquad+\frac{c}{\tau^2}\int_Q\int_{-\tau}^0|t_3|(|r_1|+|r_2|+|r_3|)\,d\mathbf{t}\\ &\qquad\to\lim_{\tau\searrow 0}\frac{1}{\tau^2}\int_Q\int_{-\tau}^0 t_3\chi''(t_3/\tau)f(\omega(\bar t))\,G^{33}\,\partial_{t_3}\tilde{\mathbf{h}}(\bar t,0)\\ &\qquad\cdot(\mathbb{I}-\mathbf{n}\otimes\mathbf{n})\partial_n\mathbf{u}(\omega(\bar t))g^{1/2}\,d\mathbf{t}\quad\text{as}\quad\tau\to 0.\end{aligned}$$

Combining the obtained results, we arrive at the equality

$$\begin{aligned}&\lim_{\tau\searrow 0}\int_\Omega\partial_{x_i}\mathbf{h}(x)\cdot\partial_{x_i}\mathbf{D}\mathbf{u}\,dx\\ &=\lim_{\tau\searrow 0}\frac{1}{\tau^2}\int_Q\int_{-\tau}^0 t_3\chi''(t_3/\tau)f(\omega(\bar t))\,G^{33}\,\partial_{t_3}\tilde{\mathbf{h}}(\bar t,0)\cdot(\mathbb{I}-\mathbf{n}\otimes\mathbf{n})\partial_n\mathbf{u}(\omega(\bar t))g^{1/2}\,d\mathbf{t}.\end{aligned}$$

Recalling that

$$\partial_{t_3}\tilde{\mathbf{h}}(\bar t,0)=\partial_n\mathbf{h}(\omega(\bar t))\text{ and }G^{33}=\mathbf{e}_3\cdot\mathbf{e}_3=1$$

and using the identity

$$\frac{1}{\tau^2}\int_{-\tau}^0 t_3\chi''(t_3/\tau)\,dt_3=-1$$

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we finally obtain

$$\begin{aligned}
&\lim_{\tau\searrow 0}\int_\Omega \partial_{x_i}\mathbf{h}(x)\cdot(\partial_{x_i}\mathbf{D}_\tau)\mathbf{u}\,dx\\
&=\lim_{\tau\searrow 0}\frac{1}{\tau^2}\int_Q\int_{-\tau}^0 t_3\chi''(t_3/\tau)f(\omega(\bar t))\partial_n\mathbf{h}(\omega(\bar t))\cdot(\mathbb{I}-\mathbf{n}\otimes\mathbf{n})\partial_n\mathbf{u}(\omega(\bar t))g^{1/2}\,d\mathbf{t}\\
&=\Big(\lim_{\tau\searrow 0}\frac{1}{\tau^2}\int_{-\tau}^0 t_3\chi''(t_3/\tau)\,dt\Big)\int_Q f(\omega(\bar t))\partial_n\mathbf{h}(\omega(\bar t))\cdot(\mathbb{I}-\mathbf{n}\otimes\mathbf{n})\partial_n\mathbf{u}(\omega(\bar t))g^{1/2}\,dt_1dt_2\\
&=-\int_Q f(\omega(\bar t))\partial_n\mathbf{h}(\omega(\bar t))\cdot(\mathbb{I}-\mathbf{n}\otimes\mathbf{n})\partial_n\mathbf{u}(\omega(\bar t))g^{1/2}\,dt_1dt_2\\
&=-\int_{\mathcal{S}} f(\omega)\partial_n\mathbf{h}(\omega)\cdot(\mathbb{I}-\mathbf{n}\otimes\mathbf{n})\partial_n\mathbf{u}(\omega)\,dS,
\end{aligned}$$

which yields (4.2), and the proof of the lemma is completed. ∎

5. Proof of Theorem 1.2. Fix an arbitrary function $\eta\in C^\infty(\Omega)$ satisfying the boundary conditions $\eta=0$ on Σ, and assume that $\eta=1$ in a neighborhood of $\mathcal{S}$. Next choose the perturbation field $\mathbf{T}_\tau$ in the form (3.9). Notice that the derivatives of the function η vanish in a neighborhood of $\mathcal{S}$ and that the vector field $\mathbf{u}$ belongs to $C^1(\Omega)$. Substituting $\mathbf{T}_\tau$ into (1.24), integrating by parts, and applying Lemma 4.1, we obtain

$$\lim_{\tau\searrow 0}L_e(\mathbf{T}_\tau)=0. \tag{5.1}$$

Since the function η is infinitely differentiable, it implies that the right-hand sides (A,B,C) of adjoint state equations (1.28) belong to $W^{1+s,r}(\Omega)\times\Pi W^{s,r}(\Omega)\times W^{s,r}(\Omega)$; thus from Proposition 2.4 it follows that $(h^*,g^*,\varsigma^*,\upsilon^*,l^*)\in W^{1+s,r}(\Omega)\times\Pi W^{s,r}(\Omega)\times W^{s,r}(\Omega)\times\mathbb{R}$. Applying Proposition 2.4 to problem (1.32) we conclude that $(h,g,\varsigma,\upsilon,l)\in W^{1+s,r}(\Omega)\times\Pi W^{s,r}(\Omega)\times W^{s,r}(\Omega)\times\mathbb{R}$.

It follows from this that the functions g^*, ς^*, and υ^* are continuous. From this and Lemma 4.1 we obtain

$$\lim_{\tau\searrow 0}\int_\Omega \mathrm{Tr}\,\mathbf{T}'_\tau\big(b_{10}\varsigma^*+b_{20}g^*+\sigma\upsilon^*+l^*b_{30}\big)\,dx=\int_{\mathcal{S}} f(\omega)\big(b_{10}\varsigma^*+b_{20}g^*+\sigma\upsilon^*+l^*b_{30}\big)\mathbf{n}\otimes\mathbf{n}\,dS. \tag{5.2}$$

Next we have

$$\begin{aligned}
\int_\Omega \mathscr{D}(\mathbf{T}_\tau)\cdot\mathbf{h}^*\,dx&=\int_\Omega \mathrm{div}[(\mathrm{Tr}\,\mathbf{T}'_\tau\mathbb{I}-\mathbf{D}_\tau-\mathbf{D}_\tau^\top)\nabla u]\cdot h^*\,dx\\
&\quad+\int_\Omega(\mathbf{D}_\tau^\top\Delta\mathbf{u}+\Delta(\mathbf{D}_\tau\mathbf{u}))\cdot\mathbf{h}^*\,dx\\
-R\int_\Omega\varrho\big[\mathbf{D}_\tau^\top(\mathbf{u}\nabla\mathbf{u})+\mathbf{u}\nabla(\mathbf{D}_\tau\mathbf{u})\big]\cdot\mathbf{h}^*\,dx&=I_1(\tau)+I_2(\tau)+I_3(\tau).
\end{aligned}\tag{5.3}$$

Recall that by virtue of the definition of the spaces $Y^{s,r}$, we have $\mathbf{u}\in Y^{s,r}\subset W^{2,2}(\Omega)$ and $\varrho\in X^{s,r}\subset W^{1,2}(\Omega)$. On the other hand, the vector field $\mathbf{h}^*$ has continuous derivatives in Ω. Hence we can integrate by parts to obtain the identity

$$I_3(\tau)=R\int_\Omega\big[\varrho\mathbf{D}_\tau^\top(\mathbf{u}\nabla\mathbf{u})\cdot\mathbf{h}^*-\varrho(\mathbf{u}\nabla\mathbf{h}^*)\cdot(\mathbf{D}_\tau\mathbf{u})\big]\,dx. \tag{5.4}$$

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Here we use the equality $\operatorname{div}(\varrho\mathbf{u}) = 0$. Since ϱ, $\nabla\mathbf{u}$, $\nabla\mathbf{h}^*$ are continuous and $\mathbf{u}$, $\mathbf{h}^*$ vanish at $\mathcal{S}$, we can apply Lemma 4.1 to obtain $\lim_{\tau\searrow 0} I_3(\tau) = 0$. Next introduce the matrix

$$V = \mathrm{Tr}\ \mathbf{T}'_\tau\mathbb{I} - \mathbf{D}_\tau - \mathbf{D}_\tau^\top.$$

Note that

$$(V\nabla\mathbf{u})_{ij} = V_{ik}\partial_{x_k}u_j, \quad [\operatorname{div}(V\nabla\mathbf{u})]_j = \partial_{x_i}\Big(V_{ik}\partial_{x_k}u_j\Big).$$

Thus we get

$$I_1(\tau) \equiv \int_\Omega \operatorname{div}(V\nabla\mathbf{u})\cdot\mathbf{h}^* = -\int_\Omega V_{ik}\partial_{x_k}u_j\ \partial_{x_i}h_j^*\,dx.$$

It follows from Lemma 4.1 and the equality $\mathbf{D}_\tau = \mathrm{Tr}\ \mathbf{T}_\tau\mathbb{I} - \mathbf{T}'_\tau$ that for any $\psi \in C(\Omega)$,

$$\lim_{\tau\searrow 0}\int_\Omega V_{ik}\psi\,dx = \int_{\mathcal{S}} f(\omega)(-\delta_{ik} + 2n_i\ n_k)\psi(\omega)\,ds.$$

Since $\nabla\mathbf{u}$ and $\nabla\mathbf{h}^*$ are continuous, we conclude from this that

$$\lim_{\tau\searrow 0} I_1(\tau) = \int_{\mathcal{S}} f(\omega)\partial_{x_i}u_j\ \partial_{x_i}h_j^*\,dS - 2\int_{\mathcal{S}} f(\omega)n_k\partial_{x_k}u_j\ n_i\partial_{x_i}h_j^*\,dS.$$

Recalling that $\mathbf{u}$ and $\mathbf{h}^*$ vanish on $\mathcal{S}$, we obtain

$$\partial_{x_i}u_j\ \partial_{x_i}h_j^* = \partial_n\mathbf{u}\cdot\partial_n\mathbf{h}^*, \quad n_k\partial_{x_k}u_j\ n_i\partial_{x_i}h_j^* = \partial_n\mathbf{u}\cdot\partial_n\mathbf{h}^*,$$

which gives

$$\lim_{\tau\searrow 0} I_1(\tau) = -\int_{\mathcal{S}} f(\omega)\,\partial_n\mathbf{u}\cdot\partial_n\mathbf{h}^*\,dS.$$

Next, integrating by parts we obtain

$$I_2(\tau) = -\int_\Omega \Big[(\mathbf{D}_\tau + \mathbf{D}_\tau^\top)\partial_{x_i}\mathbf{u}\cdot\partial_{x_i}\mathbf{h}^* + \partial_{x_i}\mathbf{u}\cdot(\partial_{x_i}\mathbf{D}_\tau)\mathbf{h}^* + \partial_{x_i}\mathbf{h}^*\cdot(\partial_{x_i}\mathbf{D}_\tau)\mathbf{u}\Big]\,dx.$$

Applying Lemmas 4.1 and 4.2, we arrive at the equality

$$\begin{aligned}\lim_{\tau\searrow 0} I_2(\tau) = &-2\int_{\mathcal{S}} f(\omega)\Big[\partial_{x_i}\mathbf{u}\cdot\partial_{x_i}\mathbf{h}^* - \big(\mathbf{n}\cdot\partial_{x_i}\mathbf{u}\big)\big(\mathbf{n}\cdot\partial_{x_i}\mathbf{h}^*\big)\Big]\,dS\\ &+2\int_{\mathcal{S}} f(\omega)\Big[\partial_n\mathbf{u}\cdot\partial_n\mathbf{h}^* - \big(\mathbf{n}\cdot\partial_n\mathbf{u}\big)\big(\mathbf{n}\cdot\partial_n\mathbf{h}^*\big)\Big]\,dS.\end{aligned}$$

Since $\mathbf{u}$ and $\mathbf{h}^*$ vanish on $\mathcal{S}$, we have $\partial_{x_i}\mathbf{u} = n_i\partial_n\mathbf{u}$ and $\partial_{x_i}\mathbf{h}^* = n_i\partial_n\mathbf{h}^*$, which gives

$$\partial_{x_i}\mathbf{u}\cdot\partial_{x_i}\mathbf{h}^* - \big(\mathbf{n}\cdot\partial_{x_i}\mathbf{u}\big)\big(\mathbf{n}\cdot\partial_{x_i}\mathbf{h}^*\big) = \partial_n\mathbf{u}\cdot\partial_n\mathbf{h}^* - \big(\mathbf{n}\cdot\partial_n\mathbf{u}\big)\big(\mathbf{n}\cdot\partial_n\mathbf{h}^*\big).$$

Thus we get $\lim_{\tau\searrow 0} I_2(\tau) = 0$. Combining (1.23) and (1.29), we arrive at the following expression for the shape derivative of the drag functional:

$$\frac{d}{d\varepsilon}J_D(\mathfrak{S}_\varepsilon)\Big|_{\varepsilon=0} = L_e(\mathbf{T}_\tau) + \int_\Omega \mathrm{Tr}\ \mathbf{T}'_\tau\big(b_{10}\varsigma^* + b_{20}g^* + \sigma\upsilon^* + l^*b_{30}\big)\,dx + I_1(\tau) + I_2(\tau) + I_3(\tau).$$

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Letting $\tau \searrow 0$, we finally obtain

$$\frac{d}{d\varepsilon} J_D(\mathfrak{S}_\varepsilon)\Big|_{\varepsilon=0} = \int_{\mathcal{S}} f(\omega)\big(b_{10}\varsigma^* + b_{20}g^* + \sigma\upsilon^* + l^* b_{30} - \partial_n \mathbf{u} \cdot \partial_n \mathbf{h}^*\big)\, dS.$$

It remains to note that by virtue of identity (1.33),

$$(g^*, \varsigma^*, \upsilon^*, l^*) = (g, \varsigma, \upsilon, l), \qquad \partial_n \mathbf{h}^* = \partial_n \mathbf{h} \quad \text{on} \quad \mathcal{S},$$

and the proof of theorem is completed. ◻

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