Adrian Hirn and Winnifried Wollner

7 An optimal control problem for equations with *p*-structure and its finite element discretization

Abstract: We analyze a finite element approximation of an optimal control problem that involves an elliptic equation with p-structure (e. g., the p-Laplace) as a constraint. As the nonlinear operator related to the p-Laplace equation mapping the space $W_0^{1,p}(\Omega)$ to its dual $(W_0^{1,p}(\Omega))^*$ is not Gâteaux differentiable, first-order optimality conditions cannot be formulated in a standard way. Without using adjoint information, we derive novel a priori error estimates for the convergence of the cost functional for both variational discretization and piecewise constant controls.

Keywords: p-Laplacian, optimization, finite element method, optimality conditions, a priori error estimates

MSC 2010: 49K20, 49M25, 65N30, 65N12

7.1 Introduction

In this paper, for given $\alpha > 0$ and $u_d \in L^2(\Omega)$, we study the finite element discretization of the following elliptic optimal control problem:

$$\text{Minimize} \quad J(q,u) := \frac{1}{2}\|u - u_d\|_2^2 + \frac{\alpha}{2}\|q\|_2^2 \tag{7.1a}$$

subject to the PDE-constraints

$$-\operatorname{div} \boldsymbol{S}(\nabla u) = q \quad \text{in } \Omega, \tag{7.1b}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{7.1c}$$

Acknowledgement: The authors would like to thank the anonymous referees for their valuable comments helping to shorten and improve the manuscript.
W. Wollner is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Projektnummer 314067056.

Adrian Hirn, Hochschule Esslingen, Robert-Bosch-Straße 1, 73037 Göppingen, Germany, e-mail: adrian.hirn@hs-esslingen.de
Winnifried Wollner, Technische Universität Darmstadt, Fachbereich Mathematik, Dolivostr. 15, 64293 Darmstadt, Germany, e-mail: wollner@mathematik.tu-darmstadt.de

https://doi.org/10.1515/9783110695984-007

and for $q_a, q_b \in \mathbb{R}$, $q_a < q_b$, and, without loss of generality, $0 \in [q_a, q_b]$, the box-constraints

$$q_a \leq q(x) \leq q_b \quad \text{for a. a. } x \in \Omega, \tag{7.1d}$$

where for given $p > 1$ and $\varepsilon \geq 0$, the nonlinear vector field $\boldsymbol{S} : \mathbb{R}^d \to \mathbb{R}^d$ is supposed to have p-structure or, more precisely, (p, ε)-structure; see Assumption 7.2.1. Prototypical examples falling into this class are

$$\boldsymbol{S}(\nabla u) = |\nabla u|^{p-2} \nabla u \quad \text{or} \quad \boldsymbol{S}(\nabla u) = (\varepsilon^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u. \tag{7.2}$$

Throughout the paper, $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, is either a convex polyhedral domain or a bounded convex domain with smooth boundary $\partial\Omega \in C^2$. We include the case of curved boundary in our analysis as we need to assume certain conditions on the regularity of u that are so far only available for domains with smooth boundaries. For simplicity of exposition, we restrict the analysis to $d = 2$ in case of a curved boundary.

Equations with p-structure arise in various physical applications, such as the theory of plasticity, bimaterial problems in elastic-plastic mechanics, non-Newtonian fluid mechanics, blood rheology, and glaciology; see, e. g., [26, 32, 34] and the references therein. The first operator in (7.2) corresponds to the p-Laplace equation. For $\varepsilon > 0$, the second operator in (7.2) regularizes the degeneracy of the p-Laplacian as the modulus of the gradient tends to zero. For $\varepsilon = 0$, $(7.2)_2$ reduces to the p-Laplace operator $(7.2)_1$. Finite element (FE) approximations of the p-Laplace equation and related equations have been widely investigated; see [1, 22, 24, 30]. Attention to optimal control of quasi-linear PDEs is given, e. g., in [9, 11]. The extension to the parabolic case can be found in [6, 8]. The approximation of optimal control problems in the coefficient for the p-Laplacian by its ε-regularization is studied in [10]. For problem (7.1), some DWR-type a posteriori error estimates have been utilized in [25]. To the best knowledge of the authors, no a priori discretization error results are available for this problem class; by our work we want to fill this gap. We point out that the case $\varepsilon = 0$ is included in our analysis.

A standard procedure for the finite element analysis of an optimal control problem consists in deriving first-order optimality conditions and exploiting the properties of the adjoint state. Unfortunately, this analysis requires the existence of a suitable adjoint state. To see that this cannot be guaranteed by standard theory, note that the nonlinear operator related to the p-Laplace equation maps the space $W_0^{1,p}(\Omega)$ to its dual $(W_0^{1,p}(\Omega))^*$. Hence its formal derivative is a linear operator mapping $W_0^{1,p}(\Omega)$ into its dual. As it can be seen in the calculations yielding (7.36), the corresponding linear operator is positive and thus injective; further, it is clearly self-adjoint; see (7.31). Hence, unless $p = 2$, the linear operator cannot be surjective; see, e. g., the discussion in [27]. Hence standard KKT-theory is not applicable in the natural setting.

Despite this lack of standard theory, for $\varepsilon > 0$, we are able to show the existence of a suitable discrete adjoint state allowing a discrete optimality system suitable for a variational discretization in the spirit of [28]. Due to lack of first-order optimality conditions on the continuous level, we cannot attain additional regularity of the adjoint variable. Without additional regularity of these variables, we cannot expect more than qualitative convergence for them. Hence, to establish a priori error estimates, we follow techniques established for elliptic optimization problems with state [18, 35] or gradient-state [36] constraints, where also no convergence rates of the adjoint variable are available. Although in our analysis, we can adopt ideas from [36], we have to cope with several challenges due to the nonlinear degenerate PDE-constraint. For discretization of (7.1), we consider two possible approaches: (a) variational discretization with piecewise linear states and (b) piecewise linear states and piecewise constant controls. In case of (a) the control space is discretized implicitly by the discrete adjoint equation. We show that the sequence of discrete global minimizers $(\overline{q}_h, \overline{u}_h)$ for mesh size $h \in (0, 1]$ has a strong accumulation point $(\overline{q}, \overline{u})$ that is a global optimal solution to (7.1). Under a certain realistic regularity assumption for solutions of the state equation (Assumption 7.2.4), we prove a quantitative convergence estimate for the cost functional value for both variational discretization and piecewise constant controls in Theorems 7.7.2 and 7.7.3.

For the proof of these estimates, we combine methods from [36] with quasi-norm techniques from [22] to handle the degeneracy of the nonlinear operator. Our method does not require additional regularity of the control variable. The required regularity in Assumption 7.2.4 is verified for the p-Laplace equation on bounded convex domains with C^2-boundary in [13, 16].

The plan of the paper is as follows. In Section 7.2, we fix our notation and clarify the structure of the nonlinear vector field $\boldsymbol{S}$. Further, we state our assumption on the regularity of solutions to (7.1b)–(7.1c) (Assumption 7.2.4). Section 7.3 is concerned with the precise formulation of the optimal control problem (7.1) and its solvability. In Section 7.4, we describe its finite element discretization, followed in Section 7.5 by an analysis of the first-order optimality conditions. In Section 7.6, we collect and extend several results on the finite element approximation of the p-Laplace equation to apply them in Section 7.7 to the convergence analysis of the optimal control problem. There we verify without any regularity assumption that the sequence of discrete minimizers $(\overline{q}_h, \overline{u}_h)$ has a strong accumulation point $(\overline{q}, \overline{u})$, which is an optimal solution to (7.1). Under the regularity Assumption 7.2.4, we then prove a priori error estimates quantifying the order of convergence in the cost functional.

7.2 Preliminaries

To begin with, we clarify our notation and state important properties of the nonlinear operator in (7.1b). Further, we pose our assumption on the regularity of solutions to the state equation (7.1b)–(7.1c), which will be crucial for our analysis.

7.2.1 Notation

The set of all positive real numbers is denoted by $\mathbb{R}^+$. Let $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$. The Euclidean scalar product of two vectors $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d$ is denoted by $\boldsymbol{\xi} \cdot \boldsymbol{\eta}$. We set $|\boldsymbol{\eta}| := (\boldsymbol{\eta} \cdot \boldsymbol{\eta})^{1/2}$. We often use c as a generic constant whose value may change from line to line but does not depend on important variables. We write $a \sim b$ if there exist constants $c, C > 0$, independent of all relevant quantities, such that $cb \leq a \leq Cb$. Similarly, the notation $a \lesssim b$ stands for $a \leq Cb$.

Let $\omega \subset \Omega$ be a measurable nonempty set. The d-dimensional Lebesgue measure of ω is denoted by $|\omega|$. The mean value of a Lebesgue-integrable function f over ω is denoted by

$$\langle f \rangle_\omega := \fint_\omega f(x)\, \mathrm{d}x := \frac{1}{|\omega|} \int_\omega f(x)\, \mathrm{d}x.$$

For $\nu \in [1, \infty]$, $L^\nu(\Omega)$ stands for the Lebesgue space, and $W^{m,\nu}(\Omega)$ for the Sobolev space of order m. For $\nu > 1$, we denote by $W_0^{1,\nu}(\Omega)$ the Sobolev space with vanishing traces on $\partial\Omega$. The $L^\nu(\omega)$-norm is denoted by $\|\cdot\|_{\nu;\omega}$, and the $W^{m,\nu}(\omega)$-norm is denoted by $\|\cdot\|_{m,\nu;\omega}$. For $\nu \in (1, \infty)$ and $\frac{1}{\nu} + \frac{1}{\nu'} = 1$, i. e., $\nu' = \frac{\nu}{\nu-1}$, the dual space of $W_0^{1,\nu}(\omega)$ is denoted by $W^{-1,\nu'}(\omega) = (W_0^{1,\nu}(\omega))^*$, and for its dual norm, we write $\|\cdot\|_{-1,\nu';\omega}$. For the $L^2(\omega)$ inner product, we use the notation $(\cdot,\cdot)_\omega$. This notation of norms and inner products is also used for vector-valued functions. In case of $\omega = \Omega$, we usually omit the index Ω, e. g., $\|\cdot\|_\nu = \|\cdot\|_{\nu;\Omega}$.

We recall the important Poincaré inequality: For $\nu \in (1, \infty)$, we have

$$\|u\|_{\nu;\omega} \leq c_P \|\nabla u\|_{\nu;\omega} \quad \forall u \in W_0^{1,\nu}(\omega). \tag{7.3}$$

There exist diverse generalizations of Poincaré's inequality. We will make use of the following version, which goes back to [5]: Let ω be a bounded convex open subset of $\mathbb{R}^d$, $d \geq 1$, and let $\varphi : [0, \infty) \to [0, \infty)$ be continuous and convex with $\varphi(0) = 0$. Let $u : \omega \to \mathbb{R}^N$, $N \geq 1$, be in $W^{1,1}(\omega)$ such that $\varphi(|\nabla u|) \in L^1(\omega)$. Then

$$\int_\omega \varphi\left(\frac{|u(x) - \langle u \rangle_\omega|}{\delta}\right) \mathrm{d}x \leq \left(\frac{V_d \delta^d}{|\omega|}\right)^{1-\frac{1}{d}} \int_\omega \varphi\big(|\nabla u(x)|\big)\, \mathrm{d}x, \tag{7.4}$$

where δ is the diameter of ω, and V_d is the volume of the unit ball in $\mathbb{R}^d$.

7.2.2 Properties of the nonlinear operator

In this section, we state our assumptions on the nonlinear operator $\boldsymbol{S}$. Further, we discuss important properties of the nonlinear operator, and we indicate how it relates to so-called N-functions.

Assumption 7.2.1 (Nonlinear operator). We assume that the nonlinear operator $\boldsymbol{S} : \mathbb{R}^d \to \mathbb{R}^d$ belongs to $C^0(\mathbb{R}^d, \mathbb{R}^d) \cap C^1(\mathbb{R}^d \setminus \{\mathbf{0}\}, \mathbb{R}^d)$ and satisfies $\boldsymbol{S}(\mathbf{0}) = \mathbf{0}$. Furthermore, we assume that the operator $\boldsymbol{S}$ possesses (p, ε)-structure, i. e., there exist $p \in (1, \infty)$, $\varepsilon \in [0, \infty)$, and constants $C_0, C_1 > 0$ such that

$$\sum_{i,j=1}^{d} \partial_i S_j(\boldsymbol{\xi}) \eta_i \eta_j \geq C_0 (\varepsilon + |\boldsymbol{\xi}|)^{p-2} |\boldsymbol{\eta}|^2, \tag{7.5a}$$

$$\left|\partial_i S_j(\boldsymbol{\xi})\right| \leq C_1 (\varepsilon + |\boldsymbol{\xi}|)^{p-2} \tag{7.5b}$$

for all $\boldsymbol{\xi}$, $\boldsymbol{\eta} \in \mathbb{R}^d$ with $\boldsymbol{\xi} \neq \mathbf{0}$ and all $i, j \in \{1, \ldots, d\}$.

Important examples of nonlinear operators $\boldsymbol{S}$ satisfying Assumption 7.2.1 are those derived from a potential with (p, ε)-structure, i. e., there exists a convex function $\Phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ belonging to $C^1(\mathbb{R}_0^+) \cap C^2(\mathbb{R}^+)$ and satisfying $\Phi(0) = 0$ and $\Phi'(0) = 0$ such that for all $\boldsymbol{\xi} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ and $i = 1, \ldots, d$,

$$S_i(\boldsymbol{\xi}) = \partial_i(\Phi(|\boldsymbol{\xi}|)) = \Phi'(|\boldsymbol{\xi}|) \frac{\xi_i}{|\boldsymbol{\xi}|}. \tag{7.6}$$

If, in addition, Φ possesses (p, ε)-structure, i. e., if there exist $p \in (1, \infty)$, $\varepsilon \in [0, \infty)$, and constants $C_2, C_3 > 0$ such that for all $t > 0$,

$$C_2(\varepsilon + t)^{p-2} \leq \Phi''(t) \leq C_3(\varepsilon + t)^{p-2}, \tag{7.7}$$

then we can show (see [4]) that $\boldsymbol{S}$ satisfies Assumption 7.2.1. Note that (7.2) falls into this class. We will briefly discuss how the operator $\boldsymbol{S}$ with (p, ε)-structure relates to N-functions that are standard in the theory of Orlicz spaces; see [20]. We define the convex function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ by

$$\varphi(t) := \int_0^t (\varepsilon + s)^{p-2} s \, \mathrm{d}s. \tag{7.8}$$

The function φ belongs to $C^1(\mathbb{R}_0^+) \cap C^2(\mathbb{R}^+)$ and satisfies, uniformly in $t > 0$,

$$\min\{1, p-1\}(\varepsilon + t)^{p-2} \leq \varphi''(t) \leq \max\{1, p-1\}(\varepsilon + t)^{p-2}. \tag{7.9}$$

Therefore inequalities (7.5a) and (7.5b) defining the (p, ε)-structure of $\boldsymbol{S}$ can be expressed equivalently in terms of the convex function φ:

$$\sum_{i,j=1}^{d} \partial_i S_j(\boldsymbol{\xi}) \eta_i \eta_j \geq \widetilde{C}_0 \varphi''(|\boldsymbol{\xi}|) |\boldsymbol{\eta}|^2, \quad \left|\partial_i S_j(\boldsymbol{\xi})\right| \leq \widetilde{C}_1 \varphi''(|\boldsymbol{\xi}|) \quad \forall i, j = 1, \ldots, d.$$

The function φ is an example of an N-function satisfying the Δ_2-condition; see, e. g., [20, 23]. In view of (7.9), the function φ satisfies uniformly in t the equivalence

$$\varphi''(t)t \sim \varphi'(t). \tag{7.10}$$

Several studies on the finite element analysis of the p-Laplace equation indicate that a p-structure-adapted quasi-norm is crucial for error estimation. To this end, for given $\psi \in C^1([0,\infty))$, we introduce the family of shifted functions $\{\psi_a\}_{a\geq 0}$ by

$$\psi_a(t) := \int_0^t \psi_a'(s)\,\mathrm{d}s \quad \text{with} \quad \psi_a'(t) := \frac{\psi'(a+t)}{a+t}t. \tag{7.11}$$

For $\psi = \varphi$ given by (7.8), we have $\varphi_a(t) \sim (\varepsilon + a + t)^{p-2}t^2$ uniformly in $t \geq 0$. In [20] the following Young-type inequality is provided: For all $\delta > 0$, there exists $c(\delta) > 0$ such that for all s, t, $a \geq 0$,

$$s\varphi_a'(t) + \varphi_a'(s)t \leq \delta\varphi_a(s) + c(\delta)\varphi_a(t), \tag{7.12}$$

where the constant $c(\delta)$ only depends on p and δ (it is independent of ε and a).

We define the function $\boldsymbol{F} : \mathbb{R}^d \to \mathbb{R}^d$ associated with the nonlinear operator $\boldsymbol{S}$ with (p,ε)-structure by

$$\boldsymbol{F}(\boldsymbol{\xi}) := (\varepsilon + |\boldsymbol{\xi}|)^{\frac{p-2}{2}}\boldsymbol{\xi}, \tag{7.13}$$

where p and ε are the same as in Assumption 7.2.1. The vector fields $\boldsymbol{S}$ and $\boldsymbol{F}$ are closely related to each other as depicted by the following lemma provided by [19, 20].

Lemma 7.2.2. *For $p \in (1,\infty)$ and $\varepsilon \in [0,\infty)$, let $\boldsymbol{S}$ satisfy Assumption 7.2.1, and let $\boldsymbol{F}$, φ, and $\varphi_{|\boldsymbol{\xi}|}$ be defined by (7.13), (7.8), and (7.11), respectively. Then for all $\boldsymbol{\xi}$, $\boldsymbol{\eta} \in \mathbb{R}^d$, we have*

$$\begin{aligned}(\boldsymbol{S}(\boldsymbol{\xi}) - \boldsymbol{S}(\boldsymbol{\eta}))\cdot(\boldsymbol{\xi}-\boldsymbol{\eta}) &\sim (\varepsilon + |\boldsymbol{\xi}| + |\boldsymbol{\eta}|)^{p-2}|\boldsymbol{\xi}-\boldsymbol{\eta}|^2\\ &\sim \varphi_{|\boldsymbol{\xi}|}(|\boldsymbol{\xi}-\boldsymbol{\eta}|) \sim |\boldsymbol{F}(\boldsymbol{\xi}) - \boldsymbol{F}(\boldsymbol{\eta})|^2,\\ |\boldsymbol{S}(\boldsymbol{\xi}) - \boldsymbol{S}(\boldsymbol{\eta})| \sim \varphi_{|\boldsymbol{\xi}|}'(|\boldsymbol{\xi}-\boldsymbol{\eta}|) &\sim (\varepsilon + |\boldsymbol{\xi}| + |\boldsymbol{\eta}|)^{p-2}|\boldsymbol{\xi}-\boldsymbol{\eta}|,\end{aligned}$$

where the constants only depend on p; in particular, they are independent of $\varepsilon \geq 0$.

Due to Lemma 7.2.2, for all $u, v \in W^{1,p}(\Omega)$, we have the equivalence

$$(\boldsymbol{S}(\nabla u) - \boldsymbol{S}(\nabla v), \nabla u - \nabla v)_\Omega \sim \|\boldsymbol{F}(\nabla u) - \boldsymbol{F}(\nabla v)\|_2^2 \sim \int_\Omega \varphi_{|\nabla u|}(|\nabla u - \nabla v|)\,\mathrm{d}x \tag{7.14}$$

with constants only depending on p. We refer to each quantity in (7.14) as the *quasi-norm* or *natural distance* following, e. g., [1–3, 21]. It has been used very successfully in the finite element analysis of equations with p-structure.

The following lemma from [29] shows the connection between the natural distance and the Sobolev norms.

Lemma 7.2.3. *For $p \in (1,\infty)$ and $\varepsilon \in [0,\infty)$, let the operator $\boldsymbol{S}$ satisfy Assumption* 7.2.1, *and let $\boldsymbol{F}$ be defined by* (7.13). *Then for all $u, v \in W^{1,p}(\Omega)$, we have:*

(i) *in the case $p \in (1,2]$, with constants only depending on p,*

$$\begin{aligned} \|\nabla(u-v)\|_p^2 &\lesssim \|\boldsymbol{F}(\nabla u)-\boldsymbol{F}(\nabla v)\|_2^2 \, \|\varepsilon + |\nabla u| + |\nabla v|\|_p^{2-p}, \\ \|\boldsymbol{F}(\nabla u)-\boldsymbol{F}(\nabla v)\|_2^2 &\lesssim \|\nabla(u-v)\|_p^p; \end{aligned}$$

(ii) *in the case $p \in [2,\infty)$, with constants only depending on p,*

$$\|\nabla(u-v)\|_p^p \lesssim \|\boldsymbol{F}(\nabla u)-\boldsymbol{F}(\nabla v)\|_2^2 \lesssim \|\varepsilon + |\nabla u| + |\nabla v|\|_p^{p-2} \, \|\nabla(u-v)\|_p^2.$$

In particular, all constants appearing in (i) *and* (ii) *are independent of $\varepsilon \geq 0$.*

7.2.3 Regularity assumption

We impose our assumption on the regularity of solutions to the state equation, which will later enable us to derive a priori error estimates for the finite element approximation of (7.1).

Assumption 7.2.4. Let $q \in L^{\max\{2,p'\}}(\Omega)$, $\frac{1}{p}+\frac{1}{p'}=1$. Then the weak solution u to equation (7.1b)–(7.1c) with p-structure satisfies the regularity

$$\boldsymbol{S}(\nabla u) \in W^{1,2}(\Omega) \quad \text{and} \quad u \in W^{2,2}(\Omega),$$

and there exist positive constants c_1, c_2, γ such that

$$\|\boldsymbol{S}(\nabla u)\|_{1,2} \leq c_1 \|q\|_2 \quad \text{and} \quad \|u\|_{2,2} \leq c_2 \|q\|_{\max\{2,p'\}}^{\gamma}. \tag{7.15}$$

The regularity Assumption 7.2.4 is satisfied for certain data:

– In [13], it is shown that if $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded convex open set, $q \in L^2(\Omega)$, and $p \in (1,\infty)$, then the weak solution u to equation (7.1b)–(7.1c) with p-structure satisfies the regularity $\boldsymbol{S}(\nabla u) \in W^{1,2}(\Omega)$ with

$$C_1\|q\|_2 \leq \|\boldsymbol{S}(\nabla u)\|_{1,2} \leq C_2\|q\|_2,$$

where the constants C_1, C_2 only depend on p and d. In particular, the analysis carried out in [13] covers the p-Laplacian with $\boldsymbol{S}(\nabla u) = |\nabla u|^{p-2}\nabla u$.

- In [14], on certain domains, Lipschitz-continuous solutions are obtained whenever $q \in L^r(\Omega)$ with $r > d$ has mean-value zero and $\varepsilon > 0$. [13, Remark 2.7] claims that this implies the $W^{2,2}$-regularity.
- In [16], it is shown that, if $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, is a bounded domain with C^2-boundary, $p \in (1,2]$, and $q \in L^{p'}(\Omega)$, then the weak solution u of the p-Laplace equation (7.1b)–(7.1c) with $\boldsymbol{S}(\nabla u) = |\nabla u|^{p-2}\nabla u$ fulfills

$$u \in W^{2,2}(\Omega) \quad \text{with} \quad \|\nabla^2 u\|_2 \leq C\|q\|_{p'}^{\frac{1}{p-1}}.$$

- In [33], it is shown that if $\Omega \subset \mathbb{R}^2$ is either convex or has C^2-boundary, then for $p \in (1,2)$, the weak solution u of the p-Laplace equation (7.1b)–(7.1c) with $\boldsymbol{S}(\nabla u) = |\nabla u|^{p-2}\nabla u$ satisfies

$$q \in L^r \text{ with } r > 2 \quad \Longrightarrow \quad u \in W^{2,2}(\Omega).$$

As a consequence, Assumption 7.2.4 is satisfied for the p-Laplace equation in the case $p \in (1,2]$ if, e. g., $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded convex domain with C^2-boundary and $q \in L^{p'}(\Omega)$. Since later we take $q \in \mathcal{Q}_{\mathrm{ad}} \subset L^\infty(\Omega)$, we can weaken Assumption 7.2.4: It is sufficient to assume that $q \in L^\infty(\Omega)$ implies $\boldsymbol{S}(\nabla u) \in W^{1,2}(\Omega)$ and $u \in W^{2,2}(\Omega)$ with $\|\boldsymbol{S}(\nabla u)\|_{1,2} \lesssim \|q\|_\infty$ and $\|u\|_{2,2} \lesssim \|q\|_\infty^\gamma$ (replacing (7.15)).

7.3 Optimal control problem

In this section, we give a precise definition of the optimal control problem (7.1). For $\frac{1}{p} + \frac{1}{p'} = 1$, i. e., $p' = \frac{p}{p-1}$, the natural spaces for the states and controls are

$$\mathcal{V} := W_0^{1,p}(\Omega), \quad Q := L^{\max\{2,p'\}}(\Omega), \quad \mathcal{Q}_{\mathrm{ad}} := \{q \in Q \mid q_a \leq q \leq q_b \text{ a. e. in } \Omega\}.$$

The weak formulation of the state equation (7.1b)–(7.1c) reads as follows:

For a given control $q \in \mathcal{Q}_{\mathrm{ad}}$, find the state $u = u(q) \in \mathcal{V}$ such that

$$\big(\boldsymbol{S}(\nabla u), \nabla \varphi\big)_\Omega = (q, \varphi)_\Omega \quad \forall \varphi \in \mathcal{V}. \tag{7.16}$$

We now investigate stability and continuity properties of the solution $u \in \mathcal{V}$ with respect to the control q. It will be suitable for this to consider variations of q in $W^{-1,p'}(\Omega)$.

Lemma 7.3.1. *For all $p \in (1,\infty)$ and $q \in \mathcal{Q}_{\mathrm{ad}}$, there exists a unique solution $u = u(q) \in \mathcal{V}$ to* (7.16)*. This solution satisfies the a priori estimate*

$$\|\nabla u\|_p \leq c_1\big(\|q\|_{-1,p'}^{\frac{1}{p-1}} + c_2\varepsilon\big), \tag{7.17}$$

where $c_1 > 0$ only depends on Ω and p, and $c_2 = 1$ if $p < 2$ and $c_2 = 0$ otherwise.

Proof. Lemma 7.2.2 implies that the operator $-\operatorname{div}\boldsymbol{S}(\nabla\cdot) : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is strictly monotone. Using the theory of monotone operators (see [38, 41]), we can thus easily conclude that for each $q \in Q$, there exists a unique solution $u = u(q) \in \mathcal{V}$ to (7.16). The proof of (7.17) is standard and can be found, e. g., in [29]. $\square$

The next lemma states that the solution operator $q \mapsto u = u(q)$ is locally Hölder-continuous.

Lemma 7.3.2. *For $p \in (1,\infty)$, let $u_1 = u(q_1) \in \mathcal{V}$ and $u_2 = u(q_2) \in \mathcal{V}$ be the solutions to the state equation* (7.16) *for the right-hand side $q_1 \in \mathcal{Q}_{\mathrm{ad}}$ and $q_2 \in \mathcal{Q}_{\mathrm{ad}}$. Then there exist constants, only depending on p, Ω such that*

$$\|\boldsymbol{F}(\nabla u_1) - \boldsymbol{F}(\nabla u_2)\|_2 \lesssim \begin{cases} \|\varepsilon + |\nabla u_1| + |\nabla u_2|\|_p^{\frac{2-p}{2}} \|q_1 - q_2\|_{-1,p'} & \text{for } p \le 2, \\ \|q_1 - q_2\|_{-1,p'}^{\frac{p'}{2}} & \text{for } p \ge 2, \end{cases}$$

$$\|\nabla u_1 - \nabla u_2\|_p \lesssim \begin{cases} \|\varepsilon + |\nabla u_1| + |\nabla u_2|\|_p^{2-p} \|q_1 - q_2\|_{-1,p'} & \text{for } p \le 2, \\ \|q_1 - q_2\|_{-1,p'}^{\frac{1}{p-1}} & \text{for } p \ge 2. \end{cases}$$

Proof. As $u_1 \in \mathcal{V}$ and $u_2 \in \mathcal{V}$ solve (7.16) with right-hand sides q_1 and q_2, we have

$$(\boldsymbol{S}(\nabla u_1) - \boldsymbol{S}(\nabla u_2), \nabla\varphi)_\Omega = \langle q_1 - q_2, \varphi\rangle \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

Testing this equation with $\varphi = u_1 - u_2$ and employing Lemma 7.2.2, we get

$$\|\boldsymbol{F}(\nabla u_1) - \boldsymbol{F}(\nabla u_2)\|_2^2 \sim \langle q_1 - q_2, u_1 - u_2\rangle \le \|q_1 - q_2\|_{-1,p'} \|u_1 - u_2\|_{1,p} \tag{7.18}$$

for $p \in (1,\infty)$. Poincaré's inequality and Lemma 7.2.3 imply

$$\begin{aligned} &\|\boldsymbol{F}(\nabla u_1) - \boldsymbol{F}(\nabla u_2)\|_2^2 \\ &\quad \lesssim \|q_1 - q_2\|_{-1,p'} \|\nabla u_1 - \nabla u_2\|_p \\ &\quad \lesssim \begin{cases} \|q_1 - q_2\|_{-1,p'} \|\varepsilon + |\nabla u_1| + |\nabla u_2|\|_p^{\frac{2-p}{2}} \|\boldsymbol{F}(\nabla u_1) - \boldsymbol{F}(\nabla u_2)\|_2 & \text{for } p \le 2, \\ \|q_1 - q_2\|_{-1,p'} \|\boldsymbol{F}(\nabla u_1) - \boldsymbol{F}(\nabla u_2)\|_2^{\frac{2}{p}} & \text{for } p \ge 2. \end{cases} \end{aligned}$$

This yields the desired estimate in the natural distance. Using similar arguments, we have

$$\begin{aligned} \|q_1 - q_2\|_{-1,p'} \|\nabla u_1 - \nabla u_2\|_p &\overset{(7.18)}{\gtrsim} \|\boldsymbol{F}(\nabla u_1) - \boldsymbol{F}(\nabla u_2)\|_2^2 \\ &\gtrsim \begin{cases} \|\varepsilon + |\nabla u_1| + |\nabla u_2|\|_p^{p-2} \|\nabla u_1 - \nabla u_2\|_p^2 & \text{for } p \le 2, \\ \|\nabla u_1 - \nabla u_2\|_p^p & \text{for } p \ge 2. \end{cases} \end{aligned}$$

From this we obtain the desired estimate in the $W_0^{1,p}$-norm. $\square$

For given $\alpha > 0$ and $u_d \in L^2(\Omega)$, we define the cost functional $J : Q \times \mathcal{V} \to \mathbb{R}$ as

$$J(q,u) := \frac{1}{2}\|u - u_d\|_2^2 + \frac{\alpha}{2}\|q\|_2^2.$$

We aim to solve the following optimal control problem:

$$\text{Minimize } J(q,u) \quad \text{subject to (7.16) and } (q,u) \in \mathcal{Q}_{\mathrm{ad}} \times \mathcal{V}. \tag{P}$$

We tacitly let $J(q,u) = \infty$ whenever $u \notin L^2(\Omega)$. For the finite element analysis of (P), we will later utilize the following relation, which holds for all $(q_1,u_1),(q_2,u_2) \in Q \times \mathcal{V}$ due to the parallelogram law:

$$\begin{aligned} &\frac{1}{2}\left\|\frac{u_1 - u_2}{2}\right\|_2^2 + \frac{\alpha}{2}\left\|\frac{q_1 - q_2}{2}\right\|_2^2 + J\left(\frac{1}{2}(q_1 + q_2), \frac{1}{2}(u_1 + u_2)\right) \\ &\quad \le \frac{1}{2}J(q_1,u_1) + \frac{1}{2}J(q_2,u_2). \end{aligned} \tag{7.19}$$

Further, we often make use of the continuous embedding $W^{1,p}(\Omega) \subset L^2(\Omega)$ for $p \ge \frac{2d}{d+2}$. As a start, we deal with the existence of solutions to (P).

Theorem 7.3.3. *For $p \in (1,\infty)$ and $\varepsilon \ge 0$, the optimal control problem (P) has at least one globally optimal control $\overline{q} \in \mathcal{Q}_{\mathrm{ad}}$ with corresponding optimal state $\overline{u} = u(\overline{q}) \in \mathcal{V}$.*

Proof. The proof follows standard arguments; see [17, 31]. According to Lemma 7.3.1, for each control $q \in \mathcal{Q}_{\mathrm{ad}}$, the state equation (7.16) has a unique solution $u = u(q) \in \mathcal{V}$. The functional J is bounded from below. Thus there exists

$$j := \inf_{q \in \mathcal{Q}_{\mathrm{ad}}} J(q, u(q)).$$

Let $\{(q_n,u_n)\}_{n=1}^{\infty}$ be a minimizing sequence, i. e.,

$$q_n \in \mathcal{Q}_{\mathrm{ad}}, \quad u_n := u(q_n), \quad J(q_n,u_n) \to j \quad \text{as } n \to \infty.$$

As $\mathcal{Q}_{\mathrm{ad}}$ is nonempty, convex, closed, and bounded in $L^{\max\{p',2\}}(\Omega)$, it is weakly sequentially compact. Hence there exists a subsequence, denoted again by $\{q_n\}_{n=1}^{\infty}$, that weakly converges in $L^{\max\{p',2\}}(\Omega)$ to a function $\overline{q} \in \mathcal{Q}_{\mathrm{ad}}$,

$$q_n \rightharpoonup \overline{q} \quad \text{weakly in } L^{\max\{p',2\}}(\Omega)$$

and thus strongly in $W^{-1,p'}(\Omega)$. Then Poincaré's inequality and Lemma 7.3.2 imply

$$\|u_n - u(\overline{q})\|_{1,p} \lesssim \|\nabla u_n - \nabla u(\overline{q})\|_p \to 0 \quad (n \to \infty), \tag{7.20}$$

where in the case $p < 2$, (7.17) was also used. If $p \geq \frac{2d}{d+2}$, then from (7.20) we directly obtain that $u_n \to u(\overline{q})$ strongly in $L^2(\Omega)$. If $p < \frac{2d}{d+2}$, then we proceed differently. Since

$$\|u_n\|_2^2 \lesssim \|u_n - u_d\|_2^2 + \|u_d\|_2^2 \leq 2J(q_n, u_n) + \|u_d\|_2^2 \leq C,$$

we have in addition for some function $\overline{u} \in L^2(\Omega)$ that

$$u_n \rightharpoonup \overline{u} \quad \text{weakly in } L^2(\Omega),$$

and due to $p < 2$, $u_n \rightharpoonup \overline{u}$ weakly in $L^p(\Omega)$. In view of (7.20), we have $u_n \to u(\overline{q})$ strongly in $L^p(\Omega)$. As the weak limit is unique, this yields $\overline{u} = u(\overline{q})$, and hence

$$(q_n, u_n) \rightharpoonup (\overline{q}, \overline{u}) = (\overline{q}, u(\overline{q})) \quad \text{weakly in } L^2(\Omega) \times L^2(\Omega).$$

The functional J is convex and continuous on $L^2(\Omega) \times L^2(\Omega)$, so it is weakly lower-semicontinuous. Thus, from this weak convergence we conclude

$$j = \lim_{n\to\infty} J(q_n, u_n) \geq \liminf_{n\to\infty} J(q_n, u_n) \geq J(\overline{q}, \overline{u}) \geq j,$$

i. e., $\overline{q}$ is an optimal control. The proof is completed. □

7.4 Finite element discretization

In this section, we introduce the discretization of the optimal control problem (7.1). We assume that Ω is either a convex polygonal/polyhedral domain or a bounded convex domain with C^2-boundary, where for the latter case, we assume that $d = 2$. Let $\mathbb{T}_h$ be a shape regular decomposition of Ω into d-dimensional simplices such that if Ω is polyhedral, then $\overline{\Omega} = \bigcup_{K\in\mathbb{T}_h} \overline{K}$, or if Ω has a curved boundary, then the corner points of

$$\overline{\Omega_h} := \bigcup_{K\in\mathbb{T}_h} \overline{K} \subset \overline{\Omega}$$

belong to $\partial\Omega$. By h_K we denote the diameter of a cell $K \in \mathbb{T}_h$, and by ρ_K the supremum of diameters of inscribed balls. The mesh parameter h represents the maximum diameter of the cells, i. e., $h := \max\{h_K;\ K \in \mathbb{T}_h\}$. We assume that $\mathbb{T}_h$ is nondegenerate (see [7]), i. e.,

$$\max_{K\in\mathbb{T}_h} \frac{h_K}{\rho_K} \leq \kappa_0 \quad \forall h. \tag{7.21}$$

For $K \in \mathbb{T}_h$, we define the set of neighbors N_K and the neighborhood S_K by

$$
\begin{aligned}
N_K &:= \{K' \in \mathbb{T}_h : \overline{K'} \cap \overline{K} \neq \emptyset\},\\
S_K &:= \operatorname{int} \bigcup_{K' \in N_K} \overline{K'}.
\end{aligned}
\tag{7.22}
$$

The sets S_K are open, bounded, and connected. The nondegeneracy (7.21) of $\mathbb{T}_h$ implies the following properties of $\mathbb{T}_h$, in which all constants are independent of h:

$$
|S_K| \sim |K| \quad \text{for all } K \in \mathbb{T}_h \quad \text{and} \quad \# N_K \leq m_0 \quad \text{for some } m_0 \in \mathbb{N}. \tag{7.23}
$$

If $\partial\Omega$ is curved, then we need to be able to estimate the integral over the part of the domain not covered by Ω_h. As the treatment of curved boundaries is not the purpose of the paper, in this case, we restrict ourselves to space dimension $d = 2$ for ease of presentation.

To this end, we need the following:

Lemma 7.4.1. *Let Ω be a convex domain with C^2-boundary, and let $d = 2$. Then for $u \in W^{1,2}(\Omega)$, the integral of $|u|^2$ on the stripe $\Sigma_h := \overline{\Omega} \setminus \overline{\Omega_h}$ is bounded by*

$$
\int_{\Sigma_h} |u(x)|^2 \, \mathrm{d}x \lesssim h^2 \|u\|_{1,2;\Omega}^2. \tag{7.24}
$$

Its proof can be found, e. g., in the proof of [37, Satz 3.3].

Let $\mathcal{P}_m(K)$ be the set of polynomials on K of degree less than or equal to m. For the discretization of the state equation, we employ the space $\mathcal{V}_h \subset W^{1,\infty}(\Omega)$ of linear finite elements on the triangulation $\mathbb{T}_h$,

$$
\hat{\mathcal{V}}_h = \{u_h \in C(\overline{\Omega_h}) \mid u_h|_K \in \mathcal{P}_1(K) \text{ for all } K \in \mathbb{T}_h \text{ and } u_h|_{\partial\Omega_h} = 0\}.
$$

If Ω is polyhedral, then $\Omega_h = \Omega$, and we set $\mathcal{V}_h = \hat{\mathcal{V}}_h$. Otherwise, we define $\mathcal{V}_h$ as the space of functions $u_h \in \hat{\mathcal{V}}_h$ extended to $\overline{\Omega}$ by setting $u_h = 0$ on the stripe $\Sigma_h := \overline{\Omega} \setminus \overline{\Omega_h}$. For the discretization of the controls, we study two approaches:

(a) Variational discretization: The control variable is not explicitly discretized.

(b) Piecewise constant controls on the family of triangulations $\{\mathbb{T}_h\}$ introduced for the discretization of the state variables:

$$
\hat{Q}_h^0 = \{q_h : \overline{\Omega_h} \to \mathbb{R} \mid q_h|_K \in \mathcal{P}_0(K) \text{ for all } K \in \mathbb{T}_h\}.
$$

If Ω is polyhedral, then we define $Q_h^0 = \hat{Q}_h^0$; otherwise,

$$
Q_h^0 = \{q_h \in Q \mid q_h|_{\Omega_h} \in \hat{Q}_h^0\}.
$$

The discrete admissible set is $\mathcal{Q}_{h,\mathrm{ad}}^0 := Q_h^0 \cap \mathcal{Q}_{\mathrm{ad}}$.

Note that for the case of curved boundary, the functions in Q_h^0 are not restricted to a finite-dimensional set on the stripe $\Sigma_h = \overline{\Omega} \setminus \overline{\Omega_h}$. Hence this space is formally infinite-dimensional. However, due to the control cost $\frac{\alpha}{2}\|q\|_2^2$ and the fact that $q_h|_{\Sigma_h}$ has no influence on the discrete solution of (7.28), the optimal discrete control will satisfy $q_h|_{\Sigma_h} = 0$ due to our choice $0 \in [q_a, q_b]$. This means that an implementation can work on the finite-dimensional set $\hat{Q}_h^0$. In the subsequent analysis, $\mathcal{Q}_{h,\mathrm{ad}}$ equals either $\mathcal{Q}_{\mathrm{ad}}$ in case of variational discretization or $\mathcal{Q}_{h,\mathrm{ad}}^0$ in case of cellwise constant discretization.

Moreover, we introduce $\hat{\Pi}_h : L^1(\Omega_h) \to \hat{Q}_h^0$ as the natural extension of the L^2-projection, i. e., for $q \in L^1(\Omega_h)$, we define $\hat{\Pi}_h q \in \hat{Q}_h^0$ by

$$(\hat{\Pi}_h q, \varphi_h)_{\Omega_h} = (q, \varphi_h)_{\Omega_h} \quad \forall \varphi_h \in \hat{Q}_h^0. \tag{7.25}$$

If Ω is polyhedral, then we set $\Pi_h = \hat{\Pi}_h$. Otherwise, we define $\Pi_h : Q \to Q_h^0$ by $(\Pi_h q)|_{\Omega_h} = \hat{\Pi}_h q$ and $(\Pi_h q)|_{\Sigma_h} = q|_{\Sigma_h}$ for all $q \in Q$. It is well known that the operator Π_h satisfies for any $\nu \in [1, \infty]$ the stability estimate

$$\|\Pi_h q\|_{\nu;\Omega_h} \le \|q\|_{\nu;\Omega_h} \quad \forall q \in L^\nu(\Omega_h). \tag{7.26}$$

From the stability property (7.26) we can derive an interpolation estimate for Π_h.

Lemma 7.4.2. *There exists a constant $c > 0$ independent of h such that for all $q \in W^{m,\nu}(\Omega)$ with $m \in \{0, 1\}$ and $\nu \in (1, \infty)$, we have*

$$\|q - \Pi_h q\|_{-1,\nu} + h\|q - \Pi_h q\|_\nu \le c h^{m+1} \|q\|_{m,\nu}. \tag{7.27}$$

We omit the proof, as it is a standard consequence of orthogonality, the definition of the norms, and standard error estimates for quasi-interpolation operators noting that by the definition of Π_h the boundary stripe Σ_h induces no error.

The Galerkin approximation of (7.16) consists in replacing the Banach space $\mathcal{V}$ by the finite element space $\mathcal{V}_h$:

For a given control $q \in \mathcal{Q}_{\mathrm{ad}}$, find the discrete state $u_h = u_h(q) \in \mathcal{V}_h$ with

$$\big(\boldsymbol{S}(\nabla u_h), \nabla \varphi_h\big)_\Omega = (q, \varphi_h)_\Omega \quad \forall \varphi_h \in \mathcal{V}_h. \tag{7.28}$$

The existence of a unique solution u_h to (7.28) and an a priori estimate for u_h in $W^{1,p}(\Omega)$ follow by using similar arguments as in the continuous case.

Lemma 7.4.3. *For each $p \in (1, \infty)$, there exists a unique solution $u_h \in \mathcal{V}_h$ to (7.28). This discrete solution satisfies the a priori estimate*

$$\|\nabla u_h\|_p \le c_1 \big(\|q\|_{-1,p'}^{\frac{1}{p-1}} + c_2 \varepsilon\big), \tag{7.29}$$

where $c_1 > 0$ only depends on Ω and p, and $c_2 = 1$ if $p < 2$ and $c_2 = 0$ otherwise.

The following lemma is a discrete version of Lemma 7.3.2.

Lemma 7.4.4. *For $p \in (1,\infty)$, let $u_{h1} = u_h(q_1) \in \mathcal{V}_h$ and $u_{h2} = u_h(q_2) \in \mathcal{V}_h$ be the solutions to the discrete equation (7.28) for the right-hand sides $q_1 \in \mathcal{Q}_{\mathrm{ad}}$ and $q_2 \in \mathcal{Q}_{\mathrm{ad}}$. Then there exist constants, only depending on p and Ω, such that*

$$\|\boldsymbol{F}(\nabla u_{h1}) - \boldsymbol{F}(\nabla u_{h2})\|_2 \lesssim \begin{cases} \|\varepsilon + |\nabla u_{h1}| + |\nabla u_{h2}|\|_p^{\frac{2-p}{2}} \|q_1 - q_2\|_{-1,p'} & \text{for } p \le 2, \\ \|q_1 - q_2\|_{-1,p'}^{\frac{p'}{2}} & \text{for } p \ge 2, \end{cases}$$

$$\|\nabla u_{h1} - \nabla u_{h2}\|_p \lesssim \begin{cases} \|\varepsilon + |\nabla u_{h1}| + |\nabla u_{h2}|\|_p^{2-p} \|q_1 - q_2\|_{-1,p'} & \text{for } p \le 2, \\ \|q_1 - q_2\|_{-1,p'}^{\frac{1}{p-1}} & \text{for } p \ge 2. \end{cases}$$

Proof. The proof follows along the same lines as the proof of Lemma 7.3.2 if the space $\mathcal{V}$ is replaced by $\mathcal{V}_h$. □

Now let us consider the discrete optimal control problem. The discrete analog to (P) reads:

$$\text{Minimize } J(q_h, u_h) \quad \text{subject to (7.28) and } (q_h, u_h) \in \mathcal{Q}_{h,\mathrm{ad}} \times \mathcal{V}_h. \tag{P_h}$$

Following the same arguments used for the proof of Theorem 7.3.3, we can conclude the existence of a solution to (P_h).

Lemma 7.4.5. *For each $h > 0$, there exists an optimal control $\overline{q}_h$ with corresponding optimal state $\overline{u}_h$ of the minimization problem (P_h).*

7.5 Discrete optimality system

In this section, we are concerned with an optimality system for (P_h), which can be utilized for practical computation of the discrete optimal solution. We will close this section with a discussion on the continuous optimality system. For ease of exposition, we restrict ourselves to the particular nonlinear operator $(7.2)_2$, i. e., for

$$a(u)(\varphi) := (\boldsymbol{S}(\nabla u), \nabla\varphi)_\Omega, \quad \boldsymbol{S}(\nabla u) = (\varepsilon^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u,$$

we consider the discrete variational formulation of the state equation (7.28),

$$a(u_h)(\varphi_h) = (q, \varphi_h)_\Omega \quad \forall \varphi_h \in \mathcal{V}_h. \tag{7.30}$$

On the discrete level, it can be shown that the semilinear form a is Gâteaux differentiable for each $\varepsilon > 0$ with Gâteaux derivative

$$\begin{aligned} a'(v_h)(w_h, \varphi_h) = &\int_\Omega (\varepsilon^2 + |\nabla v_h|^2)^{\frac{p-2}{2}} \nabla w_h \cdot \nabla \varphi_h \, \mathrm{d}x \\ &+ (p-2) \int_\Omega (\varepsilon^2 + |\nabla v_h|^2)^{\frac{p-4}{2}} (\nabla v_h \cdot \nabla w_h)(\nabla v_h \cdot \nabla \varphi_h) \, \mathrm{d}x. \end{aligned} \tag{7.31}$$

Since this will be crucial for this section, here we limit ourselves to $\varepsilon > 0$.

Now we can define the adjoint problem associated with (7.28): Find $z_h \in \mathcal{V}_h$ such that

$$a'(u_h)(\varphi_h, z_h) = (u_h - u_d, \varphi_h)_\Omega \quad \forall \varphi_h \in \mathcal{V}_h. \tag{7.32}$$

The next lemma concerns the unique solvability of the discrete adjoint problem.

Lemma 7.5.1. *Let $p \in (1, \infty)$, $\varepsilon > 0$, and $b \in W^{-1,\max\{p',2\}}(\Omega)$ with $p' = \frac{p}{p-1}$. For each $h > 0$, there exists a unique solution $z_h \in \mathcal{V}_h$ to*

$$a'(u_h)(\varphi_h, z_h) = \langle b, \varphi_h \rangle \quad \forall \varphi_h \in \mathcal{V}_h, \tag{7.33}$$

where u_h solves (7.28). *The adjoint solution z_h satisfies the a priori estimate*

$$\|\nabla z_h\|_{\min\{p,2\}} \le c \|b\|_{-1,\max\{p',2\}}, \tag{7.34}$$

where the constant c only depends on p, ε, Ω, q_a, q_b.

Proof. First, we prove that if there exists a solution z_h to problem (7.33), then z_h is uniquely determined. To this end, we assume that z_h^1 and z_h^2 are two functions satisfying (7.33). Setting $\xi_h := z_h^1 - z_h^2$, we observe that

$$a'(u_h)(\varphi_h, \xi_h) = 0 \quad \forall \varphi_h \in \mathcal{V}_h. \tag{7.35}$$

We recall that u_h is uniformly bounded in $W^{1,p}(\Omega)$ by the data; see (7.29). In the case $p \le 2$, we may estimate the quantity $a'(u_h)(\xi_h, \xi_h)$ as follows:

$$\begin{aligned} a'(u_h)(\xi_h, \xi_h) \overset{(7.31)}{\ge} &\int_\Omega (\varepsilon^2 + |\nabla u_h|^2)^{\frac{p-2}{2}} |\nabla \xi_h|^2 \, \mathrm{d}x \\ &+ (p-2) \int_\Omega (\varepsilon^2 + |\nabla u_h|^2)^{\frac{p-4}{2}} |\nabla u_h|^2 |\nabla \xi_h|^2 \, \mathrm{d}x \end{aligned}$$

$$\begin{aligned} &\geq (p-1)\int_\Omega (\varepsilon^2+|\nabla u_h|^2)^{\frac{p-2}{2}}|\nabla \xi_h|^2\,\mathrm{d}x \\ &\geq (p-1)\int_\Omega (\varepsilon+|\nabla u_h|)^{p-2}|\nabla \xi_h|^2\,\mathrm{d}x. \end{aligned}$$

Using Hölder's inequality, $q(x) \in [q_a, q_b]$ a. e., and (7.29) for $p \leq 2$ and $\varepsilon > 0$, we arrive at

$$a'(u_h)(\xi_h,\xi_h) \geq (p-1)\|\varepsilon+|\nabla u_h|\|_p^{p-2}\|\nabla\xi_h\|_p^2 \geq c\|\nabla\xi_h\|_p^2.$$

In the case $p > 2$, we can bound the quantity $a'(u_h)(\xi_h,\xi_h)$ from below as follows:

$$\begin{aligned} a'(u_h)(\xi_h,\xi_h) &\overset{(7.31)}{=} \int_\Omega (\varepsilon^2+|\nabla u_h|^2)^{\frac{p-2}{2}}\nabla\xi_h\cdot\nabla\xi_h\,\mathrm{d}x \\ &\quad + (p-2)\int_\Omega (\varepsilon^2+|\nabla u_h|^2)^{\frac{p-4}{2}}(\nabla u_h\cdot\nabla\xi_h)(\nabla u_h\cdot\nabla\xi_h)\,\mathrm{d}x \\ &\geq \int_\Omega (\varepsilon^2+|\nabla u_h|^2)^{\frac{p-2}{2}}|\nabla\xi_h|^2\,\mathrm{d}x \geq \int_\Omega \varepsilon^{p-2}|\nabla\xi_h|^2\,\mathrm{d}x = \varepsilon^{p-2}\|\nabla\xi_h\|_2^2. \end{aligned}$$

To sum up, we may deduce that there exists a constant $c = c(p,\varepsilon,\Omega,q_a,q_b)$ with

$$a'(u_h)(\xi_h,\xi_h) \geq c\|\nabla\xi_h\|_{\min\{p,2\}}^2. \tag{7.36}$$

From (7.35), (7.36), and Poincaré's inequality we infer $\xi_h \equiv 0$, and hence $z_h^1 = z_h^2$. Since system (7.33) is linear and the space $\mathcal{V}_h$ is finite-dimensional, we can conclude from the uniqueness that there exists a solution z_h. For the proof of (7.34), we test (7.33) with $\varphi_h := z_h$. Then we can apply the same arguments that led to (7.36) to obtain

$$\|b\|_{-1,\max\{p',2\}}\|z_h\|_{1,\min\{p,2\}} \geq \langle b, z_h\rangle = a'(u_h)(z_h,z_h) \gtrsim \|\nabla z_h\|_{\min\{p,2\}}^2.$$

Together with Poincaré's inequality, this yields the statement. □

With the help of the discrete adjoint state, we can now formulate an optimality system for (P_h):

Lemma 7.5.2. *Let $\varepsilon > 0$. If a control $\overline{q}_h \in \mathcal{Q}_{h,\mathrm{ad}}$ with state $\overline{u}_h = u_h(\overline{q}_h) \in \mathcal{V}_h$ is an optimal solution to problem (P_h), then there exists an adjoint state $\overline{z}_h \in \mathcal{V}_h$, so that*

$$a(\overline{u}_h)(\varphi_h) = (\overline{q}_h,\varphi_h)_\Omega \quad \forall\varphi_h \in \mathcal{V}_h, \tag{7.37a}$$

$$a'(\overline{u}_h)(\overline{z}_h,\varphi_h) = (\overline{u}_h - u_d,\varphi_h)_\Omega \quad \forall\varphi_h \in \mathcal{V}_h, \tag{7.37b}$$

$$(\alpha\overline{q}_h + \overline{z}_h, \delta q_h - \overline{q}_h)_\Omega \geq 0 \quad \forall\delta q_h \in \mathcal{Q}_{h,\mathrm{ad}}. \tag{7.37c}$$

Remark 7.5.3. It is well known that the variational inequality (7.37c) has a pointwise almost everywhere representation; see, e. g., [40]. Indeed, (7.37c) can be rewritten using the projection $P_{[q_a,q_b]}$ onto the interval $[q_a, q_b]$ defined by

$$P_{[q_a,q_b]}(f(x)) = \min(q_b, \max(q_a, f(x))). \tag{7.38}$$

In the case $\mathcal{Q}_{h,\mathrm{ad}} = \mathcal{Q}_{\mathrm{ad}}$, a control $\bar{q}_h \in \mathcal{Q}_{h,\mathrm{ad}}$ solving (P_h) necessarily satisfies (7.37), and thus the control $\bar{q}_h$ and the solution $\bar{z}_h$ of (7.37b) satisfy the projection formula

$$\bar{q}_h = P_{[q_a,q_b]}\left(-\frac{1}{\alpha}\bar{z}_h\right).$$

In the case $\mathcal{Q}_{h,\mathrm{ad}} = \mathcal{Q}^0_{h,\mathrm{ad}}$, we have

$$\bar{q}_h = P_{[q_a,q_b]}\left(-\frac{1}{\alpha}\Pi_h\bar{z}_h\right),$$

where Π_h denotes the L^2-projection on $\mathcal{Q}^0_{h,\mathrm{ad}}$.

The question arises whether an analogous optimality system for (P) can be formulated. A closer look at (7.31) however reveals that this is not an easy task: The natural regularity for the state $u \in W^{1,p}(\Omega)$ is not sufficient to formulate a well-defined, invertible, Gâteaux derivative, as already discussed in the introduction.

Still, we can pass to the limit in the discrete adjoint. According to the a priori estimate (7.34) and Lemma 7.4.3, the discrete adjoint solution $\bar{z}_h$ is uniformly bounded in $W^{1,\min\{p,2\}}(\Omega)$ for $p \geq \frac{2d}{d+2}$ as it is shown by the following calculation:

$$\begin{aligned} \|\nabla \bar{z}_h\|_{\min\{p,2\}} \leq c\|u_h - u_d\|_{-1,\max\{p',2\}} &= \sup_{\varphi\in W_0^{1,\min\{p,2\}}(\Omega)} \frac{(u_h - u_d, \varphi)_\Omega}{\|\varphi\|_{1,\min\{p,2\}}} \\ &\leq \sup_{\varphi\in W_0^{1,\min\{p,2\}}(\Omega)} \frac{\|u_h - u_d\|_2\|\varphi\|_2}{\|\varphi\|_{1,\min\{p,2\}}} \lesssim \|u_h\|_{1,p} + \|u_d\|_2 \lesssim C. \end{aligned}$$

Hence there exists a function $\bar{z} \in W_0^{1,\min\{p,2\}}(\Omega)$ such that, up to a subsequence,

$$\bar{z}_h \rightharpoonup \bar{z} \quad \text{weakly in } W_0^{1,\min\{p,2\}}(\Omega) \quad (h \to 0). \tag{7.39}$$

Due to the compact embedding $W^{1,\min\{p,2\}}(\Omega) \subset L^2(\Omega)$ for $p > \frac{2d}{d+2}$, we get

$$\bar{z}_h \to \bar{z} \quad \text{strongly in } L^2(\Omega) \quad (h \to 0). \tag{7.40}$$

Consequently, the projection formula yields strong convergence of the controls

$$\bar{q}_h = P_{[q_a,q_b]}\left(-\frac{1}{\alpha}\bar{z}_h\right) \to \bar{q} = P_{[q_a,q_b]}\left(-\frac{1}{\alpha}\bar{z}\right) \quad \text{in } L^2(\Omega) \quad (h \to 0)$$

for $p > \frac{2d}{d+2}$ in the case of variational discretization; see Remark 7.5.3. This also shows the additional regularity $q \in W^{1,\min\{p,2\}}(\Omega)$ for any such limit point.

7.6 FE approximation of the p-Laplace equation

Before analyzing the convergence of the discretized optimal control problem, we collect and extend several results regarding the FE approximation of the p-Laplace equation. The first lemma states that the Galerkin approximation is a quasi-best-approximation with respect to the natural distance.

Lemma 7.6.1 (Best-approximation in quasinorms). *For $\varepsilon \geq 0$ and $p \in (1, \infty)$, let $u \in \mathcal{V}$ be the unique solution of (7.16), and let $u_h \in \mathcal{V}_h$ its finite element approximation, i. e., $u_h \in \mathcal{V}_h$ is the unique solution of (7.28). Then, for $\Sigma_h := \overline{\Omega} \setminus \overline{\Omega_h}$,*

$$\|\boldsymbol{F}(\nabla u) - \boldsymbol{F}(\nabla u_h)\|_{2;\Omega} \lesssim \inf_{\varphi_h \in \mathcal{V}_h} \|\boldsymbol{F}(\nabla u) - \boldsymbol{F}(\nabla \varphi_h)\|_{2;\Omega}, \tag{7.41a}$$

$$\|\boldsymbol{F}(\nabla u) - \boldsymbol{F}(\nabla u_h)\|_{2;\Omega} \lesssim \inf_{\varphi_h \in \mathcal{V}_h} \|\boldsymbol{F}(\nabla u) - \boldsymbol{F}(\nabla \varphi_h)\|_{2;\Omega_h} + \|\boldsymbol{F}(\nabla u)\|_{2;\Sigma_h}, \tag{7.41b}$$

where the constants only depend on p (they are independent of h and ε).

Proof. For polyhedral Ω, the lemma is proven in [22]. Using Lemma 7.2.2 and the Galerkin orthogonality ($\mathcal{V}_h \subset \mathcal{V}$), we can deduce that for arbitrary $\varphi_h \in \mathcal{V}_h$,

$$\begin{aligned}
\int_\Omega \varphi_{|\nabla u|}(|\nabla u - \nabla u_h|)\, \mathrm{d}x &\sim \int_\Omega (\boldsymbol{S}(\nabla u) - \boldsymbol{S}(\nabla u_h)) \cdot (\nabla u - \nabla u_h)\, \mathrm{d}x \\
&\sim \int_\Omega (\boldsymbol{S}(\nabla u) - \boldsymbol{S}(\nabla u_h)) \cdot (\nabla u - \nabla \varphi_h)\, \mathrm{d}x \\
&\lesssim \int_\Omega \varphi'_{|\nabla u|}(|\nabla u - \nabla u_h|)|\nabla u - \nabla \varphi_h|\, \mathrm{d}x.
\end{aligned}$$

Applying Young's inequality (7.12) to the shifted function $\varphi_{|\nabla u|}$, we obtain

$$\int_\Omega \varphi_{|\nabla u|}(|\nabla u - \nabla u_h|)\, \mathrm{d}x \lesssim \int_\Omega \varphi_{|\nabla u|}(|\nabla u - \nabla \varphi_h|)\, \mathrm{d}x.$$

Using Lemma 7.2.2 and taking the infimum over all $\varphi_h \in \mathcal{V}_h$, we arrive at statement (7.41a). We have $\varphi_h|_{\Sigma_h} = 0$ for all $\varphi_h \in \mathcal{V}_h$, and thus (7.41b) easily follows from inequality (7.41a). □

The Scott–Zhang interpolation operator $j_h : W_0^{1,1}(\Omega) \to \mathcal{V}_h$ (see [39]) is defined so that it fulfills $j_h v = v$ for all $v \in \mathcal{V}_h$ and preserves homogeneous boundary conditions.

It is also suitable for interpolation in quasi-norms as it satisfies the following property (see [22]): For all $v \in W^{1,p}(\Omega)$ and $K \in \mathbb{T}_h$, we have

$$\fint_K \left|\boldsymbol{F}(\nabla v) - \boldsymbol{F}(\nabla j_h v)\right|^2 \,\mathrm{d}x \lesssim \inf_{\boldsymbol{\eta}\in\mathbb{R}^d} \fint_{S_K} \left|\boldsymbol{F}(\nabla v) - \boldsymbol{F}(\boldsymbol{\eta})\right|^2 \,\mathrm{d}x, \tag{7.42}$$

where the constant only depends on p. In particular, it is independent of h and ε. On the basis of (7.42), it is a simple matter to derive an interpolation estimate in quasi-norms: As the function $\boldsymbol{F}$ is surjective, (7.42) implies

$$\fint_K \left|\boldsymbol{F}(\nabla v) - \boldsymbol{F}(\nabla j_h v)\right|^2 \,\mathrm{d}x \lesssim \inf_{\boldsymbol{\xi}\in\mathbb{R}^d} \fint_{S_K} \left|\boldsymbol{F}(\nabla v) - \boldsymbol{\xi}\right|^2 \,\mathrm{d}x. \tag{7.43}$$

Now let us assume that $v \in W^{1,p}(\Omega)$ satisfies the regularity $\boldsymbol{F}(\nabla v) \in W^{1,2}(\Omega)^d$. If we choose $\boldsymbol{\xi} = \langle \boldsymbol{F}(\nabla v)\rangle_{S_K}$ in (7.43), then we can apply Poincaré's inequality (7.4):

$$\fint_K \left|\boldsymbol{F}(\nabla v) - \boldsymbol{F}(\nabla j_h v)\right|^2 \,\mathrm{d}x \lesssim \fint_{S_K} h_K^2 \left|\nabla \boldsymbol{F}(\nabla v)\right|^2 \,\mathrm{d}x. \tag{7.44}$$

To obtain a global version of (7.44), we sum inequality (7.44) over all elements $K \in \mathbb{T}_h$ and use the mesh properties (7.23):

$$\left\|\boldsymbol{F}(\nabla v) - \boldsymbol{F}(\nabla j_h v)\right\|_{2;\Omega_h} \le ch\left\|\nabla \boldsymbol{F}(\nabla v)\right\|_{2;\Omega_h}. \tag{7.45}$$

Combining Lemma 7.6.1 and (7.45), we obtain an error estimate in quasi-norms.

Lemma 7.6.2. *For $\varepsilon \ge 0$ and $p \in (1,\infty)$, let $u \in \mathcal{V}$ be the unique solution of* (7.16), *and let $u_h \in \mathcal{V}_h$ be its finite element approximation, i. e., $u_h \in \mathcal{V}_h$ is the unique solution of* (7.28). *In case of curved $\partial\Omega$, we require that $d = 2$.*

(i) *If u satisfies the regularity assumption $\boldsymbol{F}(\nabla u) \in W^{1,2}(\Omega)^d$, then*

$$\left\|\boldsymbol{F}(\nabla u) - \boldsymbol{F}(\nabla u_h)\right\|_2 \le ch\left\|\boldsymbol{F}(\nabla u)\right\|_{1,2}.$$

(ii) *If u satisfies $\boldsymbol{S}(\nabla u) \in W^{1,2}(\Omega)^d$ and $u \in W^{2,2}(\Omega)$, then*

$$\left\|\boldsymbol{F}(\nabla u) - \boldsymbol{F}(\nabla u_h)\right\|_2 \le ch\left\|\boldsymbol{S}(\nabla u)\right\|_{1,2}^{\frac{1}{2}}\|u\|_{2,2}^{\frac{1}{2}}.$$

All constants c only depend on p (they are independent of h and ε).

Proof. (i) For polyhedral Ω, the statement directly follows by combining (7.41a) and (7.45). If $\partial\Omega$ is curved, then we use (7.41b) and (7.45) and take into account

Lemma 7.4.1:

$$\|\boldsymbol{F}(\nabla u)-\boldsymbol{F}(\nabla u_h)\|_{2;\Omega} \overset{(7.41b)}{\lesssim} \inf_{\varphi_h\in\mathcal{V}_h}\|\boldsymbol{F}(\nabla u)-\boldsymbol{F}(\nabla\varphi_h)\|_{2;\Omega_h}+\|\boldsymbol{F}(\nabla u)\|_{2;\Sigma_h}$$
$$\lesssim \|\boldsymbol{F}(\nabla u)-\boldsymbol{F}(\nabla j_h u)\|_{2;\Omega_h}+\|\boldsymbol{F}(\nabla u)\|_{2;\Sigma_h} \overset{(7.45),\,(7.24)}{\lesssim} ch\|\boldsymbol{F}(\nabla u)\|_{1,2;\Omega}.$$

(ii) From (7.45), the pointwise estimate with constants independent of ε (cf. [4])

$$|\nabla\boldsymbol{F}(\nabla v)|^2 \sim |\nabla\boldsymbol{S}(\nabla v)||\nabla^2 v|,$$

and the Hölder inequality, we get that for all $v\in W^{2,2}(\Omega)$ with $\boldsymbol{S}(\nabla v)\in W^{1,2}(\Omega)^d$,

$$\|\boldsymbol{F}(\nabla v)-\boldsymbol{F}(\nabla j_h v)\|_{2;\Omega_h} \lesssim h\|\nabla\boldsymbol{S}(\nabla v)\|_{2;\Omega_h}^{\frac{1}{2}}\|\nabla^2 v\|_{2;\Omega_h}^{\frac{1}{2}}. \tag{7.46}$$

If Ω is polyhedral, then the statement directly follows by combining (7.41a) and (7.46). If $\partial\Omega$ is curved, then we use (7.41b), (7.46), Lemma 7.2.2, and Lemma 7.4.1:

$$\begin{aligned}\|\boldsymbol{F}(\nabla u)-\boldsymbol{F}(\nabla u_h)\|_{2;\Omega}^2 &\lesssim \inf_{\varphi_h\in\mathcal{V}_h}\|\boldsymbol{F}(\nabla u)-\boldsymbol{F}(\nabla\varphi_h)\|_{2;\Omega_h}^2+\|\boldsymbol{F}(\nabla u)\|_{2;\Sigma_h}^2\\ &\lesssim \|\boldsymbol{F}(\nabla u)-\boldsymbol{F}(\nabla j_h u)\|_{2;\Omega_h}^2+(\boldsymbol{S}(\nabla u),\nabla u)_{\Sigma_h}\\ &\lesssim h^2\|\nabla\boldsymbol{S}(\nabla u)\|_{2;\Omega_h}\|\nabla^2 u\|_{2;\Omega_h}+\|\boldsymbol{S}(\nabla u)\|_{2;\Sigma_h}\|\nabla u\|_{2;\Sigma_h}\\ &\lesssim h^2\|\nabla\boldsymbol{S}(\nabla u)\|_{2;\Omega_h}\|\nabla^2 u\|_{2;\Omega_h}+h\|\boldsymbol{S}(\nabla u)\|_{1,2;\Omega}\cdot h\|u\|_{2,2;\Omega}.\end{aligned}$$

Taking the square root, we obtain the stated a priori estimate. □

If the solution u of the state equation satisfies the regularity assumption (7.15), according to Lemma 7.6.2(ii), the error measured in the natural distance can be bounded in terms of the control q. From this we get error bounds in the $W^{1,p}$-norm.

Corollary 7.6.3. *For $p\in(1,\infty)$, $\varepsilon\geq 0$, and any $q\in\mathcal{Q}_{\text{ad}}$, let $u=u(q)\in\mathcal{V}$ be the solution of* (7.16)*, and let $u_h=u_h(q)\in\mathcal{V}_h$ be its discrete approximation, i. e., the solution of* (7.28)*. In case of curved $\partial\Omega$, we require that $d=2$. If u satisfies Assumption* 7.2.4*, there exist constants only depending on p with*

$$\|\boldsymbol{F}(\nabla u)-\boldsymbol{F}(\nabla u_h)\|_2 \leq ch, \quad \|\nabla u-\nabla u_h\|_p \leq \begin{cases} ch & \text{for } p\leq 2,\\ ch^{\frac{2}{p}} & \text{for } p\geq 2.\end{cases} \tag{7.47}$$

In particular, the constants do not depend on the mesh size h and ε.

Proof. The error estimate in the natural distance directly follows from Lemma 7.6.2(ii), Assumption 7.2.4, and the inequality $\|q\|_{\max\{2,p'\}}\lesssim\max\{|q_a|,|q_b|\}$:

$$\|\boldsymbol{F}(\nabla u)-\boldsymbol{F}(\nabla u_h)\|_2 \leq Ch\|q\|_2^{\frac{1}{2}}\|q\|_{\max\{2,p'\}}^{\frac{\gamma}{2}} \leq Ch.$$

To derive the error estimates in the $W^{1,p}$-norm, we apply Lemma 7.2.3:

$$\|\boldsymbol{F}(\nabla u) - \boldsymbol{F}(\nabla u_h)\|_2^2 \gtrsim \begin{cases} \|\varepsilon + |\nabla u| + |\nabla u_h|\|_p^{p-2} \|\nabla u - \nabla u_h\|_p^2 & \text{for } p \le 2, \\ \|\nabla u - \nabla u_h\|_p^p & \text{for } p \ge 2, \end{cases}$$

and, if $p \le 2$, then use the stability Lemmas 7.3.1 and 7.4.3 and the inequality $\|q\|_{p'} \lesssim \max\{|q_a|, |q_b|\}$. □

If higher regularity is not available for the solution of state equation (7.16), then we still have the strong convergence of its finite element approximation in $W^{1,p}(\Omega)$.

Lemma 7.6.4. *For $p \in (1, \infty)$ and $\varepsilon \ge 0$, let $u \in \mathcal{V}$ be the unique solution of the state equation* (7.16)*, and let $u_h \in \mathcal{V}_h$ be the unique solution of its discrete approximation* (7.28)*, each for the right-hand side $q \in W^{-1,p'}(\Omega)$. Then u_h converges strongly in $\mathcal{V}$ to u as $h \to 0$, i. e.,*

$$\lim_{h\to 0} \|u - u_h\|_{1,p} = 0. \tag{7.48}$$

Proof. The lemma is proven in [12] for the case of polyhedral Ω.. Let Ω be a bounded convex domain with $\partial\Omega \in C^2$. Since $C_0^\infty(\Omega)$ is dense in $\mathcal{V}$, there exists a sequence $(\Phi_n) \subset C_0^\infty(\Omega)$ such that

$$\|u - \Phi_n\|_{1,p} \to 0 \quad (n \to \infty). \tag{7.49}$$

Let $i_h : C(\overline{\Omega}) \to \mathcal{V}_h$ denote the Lagrange interpolation operator; see [15]. On the stripe $\Sigma_h = \overline{\Omega} \setminus \overline{\Omega_h}$, we can set $(i_h\Phi)|_{\Sigma_h} = 0$ for $\Phi \in C(\overline{\Omega})$. Applied to Φ_n, for all $K \in \mathbb{T}_h$, it satisfies

$$\|\Phi_n - i_h\Phi_n\|_{1,p;K} \le c|K|^{1/p} h_K \|\Phi_n\|_{2,\infty;K}. \tag{7.50}$$

Because of Lemma 7.6.1, the finite element solution u_h fulfills

$$\begin{aligned} \|\boldsymbol{F}(\nabla u) - \boldsymbol{F}(\nabla u_h)\|_2 &\lesssim \inf_{\varphi_h \in \mathcal{V}_h} \|\boldsymbol{F}(\nabla u) - \boldsymbol{F}(\nabla \varphi_h)\|_2 \lesssim \|\boldsymbol{F}(\nabla u) - \boldsymbol{F}(\nabla i_h\Phi_n)\|_2 \\ &\lesssim \|\boldsymbol{F}(\nabla u) - \boldsymbol{F}(\nabla\Phi_n)\|_2 + \|\boldsymbol{F}(\nabla\Phi_n) - \boldsymbol{F}(\nabla i_h\Phi_n)\|_2. \end{aligned} \tag{7.51}$$

As the support of Φ_n, $\operatorname{supp}(\Phi_n)$, is compact and $\operatorname{supp}(\Phi_n) \subset \Omega$, there exists $h_0 = h_0(n) > 0$ such that

$$h < h_0(n) \quad \Rightarrow \quad \operatorname{supp}(\Phi_n) \subset \overline{\Omega_h} \quad \text{with} \quad \overline{\Omega_h} = \bigcup_{K \in \mathbb{T}_h} \overline{K}.$$

We can then infer from (7.50) that for $h < h_0(n)$, we have the estimate

$$\|\Phi_n - i_h\Phi_n\|_{1,p;\Omega} \le c|\Omega|^{1/p} h \|\Phi_n\|_{2,\infty;\Omega}. \tag{7.52}$$

As the natural distance relates to the $W_0^{1,p}$-norm (see Lemma 7.2.3), we can combine (7.51) and (7.52) to obtain, for each $n \in \mathbb{N}$,

$$\lim_{h\to 0}\|\boldsymbol{F}(\nabla u) - \boldsymbol{F}(\nabla u_h)\|_2 \lesssim \|\boldsymbol{F}(\nabla u) - \boldsymbol{F}(\nabla \Phi_n)\|_2.$$

Employing Lemma 7.2.3 and recalling (7.49), from this we infer the statement. □

7.7 Convergence of the approximation of the optimal control problem

This section contains the main results of the paper: Without assuming any regularity, for the case of piecewise constant controls, we show that the sequence of discrete global optimal solutions $(\overline{q}_h, \overline{u}_h)$ has a strong accumulation point $(\overline{q}, \overline{u}) \in \mathcal{Q}_{\mathrm{ad}} \times \mathcal{V}$ that is a global optimal solution of the original optimal control problem. Under the regularity Assumption 7.2.4, we then prove a priori error estimates quantifying the order of convergence for both variational discretization and piecewise constant controls.

Theorem 7.7.1 (Convergence of global minimizers). *For $\varepsilon \in [0,\infty)$ and $p \in [\frac{2d}{d+2},\infty)$, let $\boldsymbol{S}$ satisfy Assumption 7.2.1. For each $h > 0$, let $\overline{q}_h \in \mathcal{Q}_{h,\mathrm{ad}}$ be a discrete global optimal control, and let $\overline{u}_h = u_h(\overline{q}_h) \in \mathcal{V}_h$ be the corresponding discrete optimal state, i. e., $(\overline{q}_h, \overline{u}_h)$ solves (P_h). Then the sequence $(\overline{q}_h, \overline{u}_h)$ has a weak accumulation point $(\overline{q}, \overline{u}) \in \mathcal{Q}_{\mathrm{ad}} \times \mathcal{V}$. Further, any weak accumulation point is also a strong accumulation point, i. e., up to a subsequence,*

$$\overline{q}_h \to \overline{q} \quad \text{in } L^2(\Omega), \quad \overline{u}_h \to \overline{u} \quad \text{in } W^{1,p}(\Omega) \quad (h \to 0).$$

Moreover, any such point $(\overline{q}, \overline{u})$ is a global optimal solution of (P).

Proof. For each $h > 0$, let $(\overline{q}_h, \overline{u}_h)$ be a global solution of (P_h). Weak accumulation points of $(\overline{q}_h, \overline{u}_h)$ exist in $L^{\max\{p',2\}}(\Omega) \times W_0^{1,p}(\Omega)$ due to the uniform a priori bounds

$$\|\overline{q}_h\|_{\max\{p',2\}} \lesssim \max\{|q_a|, |q_b|\}, \quad \|\overline{u}_h\|_{1,p} \overset{(7.29)}{\lesssim} C \quad \text{uniformly in } h. \tag{7.53}$$

Now let $(\bar{q}, \bar{u})$ be a global minimizer of (P), whose existence is ensured by Theorem 7.3.3.

For piecewise constant controls, i. e., $\mathcal{Q}_{h,\mathrm{ad}} = \mathcal{Q}^0_{h,\mathrm{ad}}$, we define Π_h as the L^2-projection given in Section 7.4. For variational controls, i. e., $\mathcal{Q}_{h,\mathrm{ad}} = \mathcal{Q}_{\mathrm{ad}}$, we set $\Pi_h = \mathrm{Id}$. Then the sequence of minimizers $(\overline{q}_h, \overline{u}_h)$ satisfies

$$J(\overline{q}_h, \overline{u}_h) \le J(\Pi_h\overline{q}, u_h(\Pi_h\overline{q})). \tag{7.54}$$

Using $W^{1,p}(\Omega) \subset L^2(\Omega)$ for $p \geq \frac{2d}{d+2}$, we can then infer

$$\|u_h(\Pi_h\overline{q}) - u(\overline{q})\|_2 \leq \underbrace{\|u_h(\Pi_h\overline{q}) - u_h(\overline{q})\|_2}_{\to 0 \text{ due to Lemma 7.4.4, (7.29), (7.27)}} + \underbrace{\|u_h(\overline{q}) - u(\overline{q})\|_2}_{\to 0 \text{ due to Lemma 7.6.4}} \overset{h\to 0}{\longrightarrow} 0.$$

Therefore inequality (7.54) implies

$$\limsup_{h\to 0} J(\overline{q}_h, \overline{u}_h) \leq \limsup_{h\to 0} J(\Pi_h\overline{q}, u_h(\Pi_h\overline{q})) = J(\overline{q}, u(\overline{q})).$$

Hence any weak limit (q, u) of $(\overline{q}_h, \overline{u}_h)$ in $L^{\max\{p',2\}}(\Omega) \times W^{1,p}(\Omega)$ satisfies

$$J(q, u) \leq \liminf_{h\to 0} J(\overline{q}_h, \overline{u}_h) \leq \limsup_{h\to 0} J(\overline{q}_h, \overline{u}_h) \leq J(\bar{q}, \bar{u}),$$

since J is weakly lower semicontinuous. Analogously to the proof of Theorem 7.3.3, we can show that (q, u) solves (7.16). Hence (q, u) is a global minimizer of (P), and

$$J(\overline{q}_h, \overline{u}_h) \to J(q, u) \quad (h \to 0). \tag{7.55}$$

Further, $\overline{q}_h \rightharpoonup q$ weakly in $L^{\max\{p',2\}}(\Omega)$ implies that $\overline{q}_h \to q$ strongly in $W^{-1,p'}(\Omega)$. We apply Lemmas 7.6.4 and 7.4.4 (together with the bound (7.29) in the case $p \leq 2$) to see that

$$\|\overline{u}_h - u\|_{1,p} \leq \underbrace{\|u_h(\overline{q}_h) - u_h(q)\|_{1,p}}_{\to 0 \text{ due to Lemma 7.4.4, (7.29)}} + \underbrace{\|u_h(q) - u(q)\|_{1,p}}_{\to 0 \text{ due to Lemma 7.6.4}} \overset{h\to 0}{\longrightarrow} 0$$

and thus to obtain the strong convergence $\overline{u}_h \to u$ in $W_0^{1,p}(\Omega)$. By the parallelogram law (7.19), for $\hat{q} = \frac{1}{2}(\overline{q}_h + q)$ and $\hat{u} = \frac{1}{2}(\overline{u}_h + u)$, we get

$$\frac{1}{8}\|\overline{u}_h - u\|^2 + \frac{\alpha}{8}\|\overline{q}_h - q\|_2^2 \leq \frac{1}{2}J(q, u) + \frac{1}{2}J(\overline{q}_h, \overline{u}_h) - J(\hat{q}, \hat{u}).$$

We set $\tilde{u} = u(\hat{q})$. Then $J(q, u) \leq J(\hat{q}, \tilde{u})$, and hence

$$\frac{1}{8}\|\overline{u}_h - u\|^2 + \frac{\alpha}{8}\|\overline{q}_h - q\|_2^2 \leq \frac{1}{2}J(\overline{q}_h, \overline{u}_h) - \frac{1}{2}J(q, u) + \big(J(\hat{q}, \tilde{u}) - J(\hat{q}, \hat{u})\big). \tag{7.56}$$

In view of (7.55), the first difference on the right-hand side of (7.56) goes to zero as $h \to 0$. For the last sum, in parenthesis, we notice that

$$2J(\hat{q}, \tilde{u}) - 2J(\hat{q}, \hat{u}) = \|\tilde{u} - u_d\|_2^2 - \|\hat{u} - u_d\|_2^2. \tag{7.57}$$

As already shown, $\overline{u}_h \to u$ in $W^{1,p}(\Omega)$, and hence $\hat{u} \to u$ in $W^{1,p}(\Omega)$. Moreover, as $\overline{q}_h \to q$ in $W^{-1,p'}(\Omega)$, we have $\hat{q} \to q$ in $W^{-1,p'}(\Omega)$, and thus by Lemma 7.3.2

$$\tilde{u} = u(\hat{q}) \to u = u(q) \quad \text{strongly in } W^{1,p}(\Omega).$$

By our assumption on p, $p \geq \frac{2d}{d+2}$, we therefore obtain

$$\tilde{u} \to u \quad \text{strongly in } L^2(\Omega), \quad \hat{u} \to u \quad \text{strongly in } L^2(\Omega).$$

Combining this, (7.57), (7.55), and (7.56), we conclude that $\overline{q}_h \to q$ strongly in $L^2(\Omega)$. □

To prove the rates of convergence, we follow the approach presented in [36]. First, let us deal with the variational discretization, i. e., only the state space is discretized. For brevity of presentation, we name this problem

$$\text{Minimize } J(q_h, u_h) \quad \text{subject to (7.28) and } (q_h, u_h) \in \mathcal{Q}_{\text{ad}} \times \mathcal{V}_h. \tag{P_s}$$

Theorem 7.7.2 (Convergence rates for variational discretization). *For $\varepsilon \in [0, \infty)$ and $p \in [\frac{2d}{d+2}, \infty)$, let Assumptions 7.2.1 and 7.2.4 be satisfied. For each $h > 0$, let $(\overline{q}_h, \overline{u}_h) \in \mathcal{Q}_{\text{ad}} \times \mathcal{V}_h$ be a global solution of the semidiscretized optimization problem (P_s), and let $(\overline{q}, \overline{u}) \in \mathcal{Q}_{\text{ad}} \times \mathcal{V}$ be a global solution of (P). Then there exists a constant $c > 0$ independent of h such that*

$$\left|J(\overline{q}, \overline{u}) - J(\overline{q}_h, \overline{u}_h)\right| \leq ch^{\min\{1, \frac{2}{p}\}}. \tag{7.58}$$

Proof. We define $u_h \in \mathcal{V}_h$ as the solution to (7.28) for the control $\overline{q}$. For $p \in (1, \infty)$, Corollary 7.6.3 provides us with the estimate

$$\|\nabla\overline{u} - \nabla u_h\|_p \leq ch^{\min\{1, \frac{2}{p}\}}.$$

The continuous embedding $W^{1,p}(\Omega) \subset L^2(\Omega)$ for $p \geq \frac{2d}{d+2}$ implies

$$\|\overline{u} - u_h\|_2 \leq c\|\overline{u} - u_h\|_{1,p} \leq ch^{\min\{1, \frac{2}{p}\}}. \tag{7.59}$$

Note the following elementary inequality for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\eta} \in \mathbb{R}^d$:

$$\left||\boldsymbol{\xi}_1 - \boldsymbol{\eta}|^2 - |\boldsymbol{\xi}_2 - \boldsymbol{\eta}|^2\right| = \left|(\boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 - 2\boldsymbol{\eta}) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)\right| \leq 2(|\boldsymbol{\xi}_1| + |\boldsymbol{\xi}_2| + |\boldsymbol{\eta}|)|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|.$$

From this and (7.59) we infer the estimate

$$\begin{aligned}\left|J(\overline{q}, \overline{u}) - J(\overline{q}, u_h)\right| &= \left|\frac{1}{2}\int_\Omega |\overline{u} - u_d|^2 \,\mathrm{d}x - \frac{1}{2}\int_\Omega |u_h - u_d|^2 \,\mathrm{d}x\right| \\ &\leq \int_\Omega (|\overline{u}| + |u_h| + |u_d|)|\overline{u} - u_h| \,\mathrm{d}x \\ &\leq (\|\overline{u}\|_2 + \|u_h\|_2 + \|u_d\|_2)\|\overline{u} - u_h\|_2 \\ &\leq ch^{\min\{1, \frac{2}{p}\}},\end{aligned} \tag{7.60}$$

where we have also used (7.17) and (7.29) for the last inequality. As the pair $(\overline{q}, u_h)$ is admissible for (P_s), the inequality

$$J(\overline{q}_h, \overline{u}_h) \le J(\overline{q}, u_h) \tag{7.61}$$

is fulfilled, and, consequently,

$$J(\overline{q}_h, \overline{u}_h) - J(\overline{q}, \overline{u}) \le J(\overline{q}, u_h) - J(\overline{q}, \overline{u}) \overset{(7.60)}{\le} ch^{\min\{1, \frac{2}{p}\}}. \tag{7.62}$$

Note that $\|\overline{q}_h\|_{\max\{p',2\}} \le C$ uniformly in $h \in (0,1]$. To obtain the reverse inequality of (7.62), starting from $(\overline{q}_h, \overline{u}_h)$, we construct $(\overline{q}_h, \hat{u})$ by defining $\hat{u} \in \mathcal{V}$ as the solution to (7.16). Note that $(\overline{q}_h, \hat{u})$ are feasible for the exact optimal control problem (P), although both $\overline{q}_h$ and $\hat{u}$ depend on h. As a result, we have

$$J(\overline{q}, \overline{u}) \le J(\overline{q}_h, \hat{u}). \tag{7.63}$$

We can precisely use the same arguments as for (7.60) to obtain

$$\left|J(\overline{q}_h, \overline{u}_h) - J(\overline{q}_h, \hat{u})\right| \le ch^{\min\{1, \frac{2}{p}\}}. \tag{7.64}$$

Combining inequalities (7.60), (7.61), (7.63), and (7.64), we finally arrive at

$$\begin{aligned} -ch^{\min\{1, \frac{2}{p}\}} &\overset{(7.60)}{\le} J(\overline{q}, \overline{u}) - J(\overline{q}, u_h) \overset{(7.61)}{\le} J(\overline{q}, \overline{u}) - J(\overline{q}_h, \overline{u}_h) \\ &\overset{(7.63)}{\le} J(\overline{q}_h, \hat{u}) - J(\overline{q}_h, \overline{u}_h) \overset{(7.64)}{\le} ch^{\min\{1, \frac{2}{p}\}}. \end{aligned}$$

This establishes the statement. □

Now let us deal with the case where the control space is discretized. To this end, we adapt the theory presented in [36] to our situation. To quantify the order of convergence, some regularity of the optimal control $\overline{q}$ is usually required. In the linear setting, additional regularity of $\overline{q}$ can be proven by deriving additional regularity of the adjoint state z. As we have seen in our discussion in Section 7.5, such additional regularity can only be shown in the case of the variational discretization.

Theorem 7.7.3 (Convergence rates for piecewise constant controls). *For $\varepsilon \in [0, \infty)$ and $p \in [\frac{2d}{d+2}, \infty)$, let Assumptions 7.2.1 and 7.2.4 be satisfied. For each $h > 0$ and $\mathcal{Q}_{h,\mathrm{ad}} = \mathcal{Q}^0_{h,\mathrm{ad}}$, let $\overline{q}_h \in \mathcal{Q}_{h,\mathrm{ad}}$ be a discrete optimal control, and let $\overline{u}_h = u_h(\overline{q}_h) \in \mathcal{V}_h$ be the corresponding discrete optimal state, i. e., $(\overline{q}_h, \overline{u}_h)$ are global solutions of (P_h). Further, let $(\overline{q}, \overline{u}) \in \mathcal{Q}_{\mathrm{ad}} \times \mathcal{V}$ be a global solution of (P). Then there exists a constant $c > 0$ independent of h such that*

$$\left|J(\overline{q}, \overline{u}) - J(\overline{q}_h, \overline{u}_h)\right| \le ch^{\min\{1, \frac{1}{p-1}\}}. \tag{7.65}$$

Proof. We have already proven the existence of an accumulation point $(\overline{q}, \overline{u})$ in Theorem 7.7.1 and assume from now on that $(\overline{q}_h, \overline{u}_h)$ converges to this limit.

Let $(\hat{q}_h, \hat{u}_h) \in \mathcal{Q}_{h,\mathrm{ad}} \times \mathcal{V}$ be a global solution of the following auxiliary problem in which only the control variable is discretized:

$$\text{Minimize } J(q_h, u_h) \quad \text{subject to (7.16) and } (q_h, u_h) \in \mathcal{Q}_{h,\mathrm{ad}} \times \mathcal{V}. \tag{7.66}$$

To derive the stated error estimate, we split the error as follows:

$$\left|J(\overline{q}, \overline{u}) - J(\overline{q}_h, \overline{u}_h)\right| \le \left|J(\overline{q}, \overline{u}) - J(\hat{q}_h, \hat{u}_h)\right| + \left|J(\hat{q}_h, \hat{u}_h) - J(\overline{q}_h, \overline{u}_h)\right|. \tag{7.67}$$

By repeating the proof of Theorem 7.7.2 we can estimate the second term on the right-hand side of (7.67) as

$$\left|J(\hat{q}_h, \hat{u}_h) - J(\overline{q}_h, \overline{u}_h)\right| \le ch^{\min\{1, \frac{2}{p}\}}. \tag{7.68}$$

This is possible as all constants appearing in this proof are only dependent of $\|\hat{q}_h\|_{\max\{p',2\}}$ (and the regularity of $\mathbb{T}_h$ and characteristics of $\boldsymbol{S}$). Note that $\|\hat{q}_h\|_{\max\{p',2\}}$ is uniformly bounded in $h \in (0,1]$.

Thus it is sufficient to estimate the first term on the right-hand side of (7.67). To this end, we use again similar arguments as in the proof of Theorem 7.7.2. Let us set $q_h := \Pi_h \overline{q}$, where Π_h stands for the L^2-projection onto Q_h^0. It is clear that $\Pi_h \colon \mathcal{Q}_{\mathrm{ad}} \to \mathcal{Q}_{h,\mathrm{ad}}^0$.

Let $u_h \in \mathcal{V}$ be the solution to the state equation (7.16) for control q_h. From Lemma 7.3.2 we deduce the estimate

$$\|\nabla \overline{u} - \nabla u_h\|_p \lesssim \begin{cases} \|\varepsilon + |\nabla \overline{u}| + |\nabla u_h|\|_p^{2-p} \|\overline{q} - \Pi_h \overline{q}\|_{-1,p'} & \text{for } p \le 2, \\ \|\overline{q} - \Pi_h \overline{q}\|_{-1,p'}^{\frac{1}{p-1}} & \text{for } p \ge 2. \end{cases} \tag{7.69}$$

Due to the uniform a priori bounds (7.17) and (7.29) and the stability of Π_h, in the case $p \le 2$, there exists a constant $C > 0$ independent of h such that

$$\left\|\varepsilon + |\nabla \overline{u}| + |\nabla u_h|\right\|_p^{2-p} \le C.$$

Employing Lemma 7.4.2, we can bound the right-hand side of (7.69) by

$$\|\nabla \overline{u} - \nabla u_h\|_p \le ch^{\min\{1, \frac{1}{p-1}\}}. \tag{7.70}$$

From the convexity of J we conclude

$$\begin{aligned} J(\overline{q}, \overline{u}) &\ge J(q_h, u_h) + \left\langle J'(q_h, u_h), (\overline{q} - q_h, \overline{u} - u_h)\right\rangle \\ &= J(q_h, u_h) + \alpha(q_h, \overline{q} - q_h)_\Omega + (u_h - u_d, \overline{u} - u_h)_\Omega. \end{aligned}$$

Since (q_h, u_h) is feasible for (7.66), the inequality

$$\begin{aligned} 0 &\le J(\hat{q}_h, \hat{u}_h) - J(\overline{q}, \overline{u}) \le J(q_h, u_h) - J(\overline{q}, \overline{u}) \\ &\le \alpha(q_h, q_h - \overline{q})_\Omega + (u_h - u_d, u_h - \overline{u})_\Omega \end{aligned}$$

follows. The last term on the right-hand side is bounded by

$$(u_h - u_d, u_h - \overline{u})_\Omega \le \|u_h - u_d\|_2 \|u_h - \overline{u}\|_2 \overset{(7.70)}{\le} c h^{\min\{1, \frac{1}{p-1}\}}$$

for $p \ge \frac{2d}{d+2}$. Moreover,

$$(q_h, q_h - \overline{q})_\Omega = (q_h, \Pi_h \overline{q} - \overline{q})_\Omega = 0$$

due to the definition of the L^2-projection. To sum up, we get

$$0 \le J(\hat{q}_h, \hat{u}_h) - J(\overline{q}, \overline{u}) \le c h^{\min\{1, \frac{1}{p-1}\}}. \tag{7.71}$$

Combining (7.67), (7.68), and (7.71), we conclude the statement noting that $h^{\min\{1, \frac{1}{p-1}\}} \ge h^{\min\{1, \frac{2}{p}\}}$. □

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