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Nonlinear PDE's

T. Bhattacharya - E. DiBenedetto – J. Manfredi

LIMITS AS $p \to \infty$ OF $\Delta_p u_p = f$ AND RELATED EXTREMAL PROBLEMS

INTRODUCTION

1. The physical problem

A prismatic material rod subject to a torsional moment, for an extended period of time and at sufficiently high temperature, exhibits a permanent plastic deformation called *creep*. Modelling of such a phenomenon involves the differential operator $\Delta_p \equiv \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, for large p. Let the cross section Ω of the rod, be a bounded simply connected domain in $\mathbf{R}^2$, with smooth boundary $\partial\Omega$. The function $u : \Omega \to \mathbf{R}$ is the *stress potential* and typically satisfies an equation of the type

$$(1.1)\qquad \begin{cases} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = -\operatorname{div}(\omega x), & \text{in } \Omega, \quad \text{where } x \equiv (x_1, x_2) \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where ω is the *angular twist rate*. This equations is derived under the assumption that the components of strain and stress are linked by a power law referred to as the *creep-law* (see Kachanov [**Ka1**] Chapters IV and VIII; see also [**Ka2**],[**FLO**] and [**Ra**]). The knowledge of u permits to recover the distributions of stresses and strains in the rod. Let u_p be the unique solution of

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(1.1). If the creep is generated by a moment M_p, then

$$M_p = 2 \int_\Omega u_p dx. \tag{1.2}$$

The quantity

$$E_p = \int_\Omega |\nabla u_p|^p dx \tag{1.3}$$

is the *p–torsional stiffness*, i.e. the energy of the system. If $x \to \omega(x)$ is constant we have from (1.1), by multiplying by u_p and integrating over Ω

$$E_p = \omega M_p. \tag{1.4}$$

Given the moment M_p the problem consists in finding both the potential u_p and the angular twist rate $\omega(\cdot)$. In the quoted references ω is a–priori taken to be constant. In Part IV we give a detailed discussion of the model, prove (1.1)-(1.4) and present a way of calculating stress and strain distributions from u_p. We supply a rigorous proof of the physically intuitive fact that $x \to \omega(x)$ is constant and compute an explicit solution of (1.1) in the case of a rod with circular cross section.

According to Von Mises, Tresca, Saint–Venant yield criterion, the stress tensor in a ideally plastic homogeneous isotropic material, must have constant modulus. Physical evidence as well as explicit solutions (see [**Ka1, Ka2**] and Part IV of this paper) suggest that, as $p \to \infty$, the distribution of stresses computed from (1.1) tends to the distribution of stresses in the ideally plastic case. Even though the ideal plastic state is not associated with a potential, it is natural to ask whether if, and in what sense $\{u_p\}$ will converge, as $p \to \infty$, to an ideally plastic "potential" u_∞, whose gradient ∇u_∞ generates a distribution of stresses of constant modulus, i.e. $|\nabla u_\infty|$ =const.

2. The mathematical problem

Motivated by the remarks of the previous section we will study the limiting behavior of the family of problems (1.1) as $p \to \infty$, where $x \to \omega(x)$ is any given non negative function and Ω is a domain in $\mathbf{R}^N$. We will also study the family of problems

$$\text{(2.1)} \qquad \begin{cases} \operatorname{div}\left(|\nabla u_p|^{p-2}\nabla u_p\right) = -f\,, & \text{in } \Omega, \\ u_p \in W_o^{1,p}(\Omega)\,, & \text{on } \partial\Omega, \end{cases}$$

were f is a given non-negative function in Ω. The problem here is to identify the limit (or limits) u_∞ of the family $\{u_p\}$ in some suitable topology as well as the limiting p.d.e. satisfied by u_∞. It turns out that both the limit(s) of $\{u_p\}$ and the limiting partial differential equation depend upon ω and f only trough their support, i.e. they are determined only by the geometrical structure of the support of ω and f and are independent of other properties such as "size" or "mass-distribution" (see Part II §'s 3–5).

The partial differential problems in (2.1) are the Euler equations associated with the variational problem

$$\text{(2.2)} \qquad \varphi \to \min \int_\Omega \left\{ \frac{1}{p}|\nabla\varphi|^p - f\varphi \right\} dx\,, \qquad \varphi \in W_o^{1,p}(\Omega).$$

A similar variational formulation holds for (1.1). If we let *formally* $p \to \infty$ in (2.2) we obtain the extremal problem

$$\text{(2.3)} \qquad \varphi \to \min \int_\Omega -f\varphi dx\,, \qquad \varphi \in W^{1,\infty}(\Omega) \cap C_o(\bar{\Omega})$$

Here $C_o(\bar{\Omega})$ denotes the space of continuous functions in $\bar{\Omega}$ vanishing on $\partial\Omega$. However a less formal analysis reveals that as $p \to \infty$ the problem imposes an extra constraint to (2.3), namely $\|\nabla\varphi\|_{\infty,\Omega} = 1$ (see Part II, §2). This condition guarantees existence of solutions to (2.3). Uniqueness of such solutions again depends on the geometry of the support of f or ω.

The limiting p.d.e. is not of variational nature and it is given by (see Part II, §5)

$$\text{(2.4)} \qquad |\nabla u_\infty| = 1\,, \quad \text{in the set } [f > 0] \quad \text{in the viscosity sense,}$$

$$\text{(2.5)} \qquad \Delta_\infty u_\infty = 0\,, \quad \text{in the set } \overline{[f > 0]}^c \quad \text{in the viscosity sense,}$$

where for $\varphi \in C^2_{loc}(\Omega)$

$$\Delta_\infty \varphi \equiv \frac{1}{2}\nabla\varphi \cdot \nabla|\nabla\varphi|^2 = \sum_{i,j=1}^{N} \varphi_{x_i}\varphi_{x_j}\varphi_{x_i x_j}. \tag{2.6}$$

Even though these are not of divergence form they can be viewed as some sort of Euler equation of (2.3).

To summarize, the sequence of unconstrained variational problems (2.1) posed in uniformly convex Banach spaces yields a constrained extremal problem in a non reflexive Banach space. The corresponding sequence of second order elliptic partial differential equations yields (loosely speaking) a first order Hamilton-Jacobi equation in certain portions of Ω and a second order parabolic equation in the remaining part.

We conclude this section by recalling the definition of viscosity solution of (2.4) and (2.5) following Crandall–Lions [**CL**].

Viscosity solutions of (2.4). Let D be a domain in $\mathbf{R}^N$.

(2.6′) $u \in C_{loc}(D)$ is a local viscosity subsolution of $|\nabla u| = 1$ in D if
$\forall x_o \in D$, $\forall v \in C^1_{loc}(D)$ such that $x \to (u-v)(x)$ has a local maximum at x_o,
$|\nabla v(x_o)| \leq 1$.

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$|\nabla v(x_o)| \geq 1$.

A function $u \in C(D)$ is a viscosity solution of $|\nabla u| = 1$ in D if it is both a viscosity sub and super solution of that same equation.

Viscosity solutions of (2.5). Let D be a domain in $\mathbf{R}^N$.

(2.7) $u \in C_{loc}(D)$ is a local viscosity subsolution of $u = 0$ in D if
$\forall x_o \in D$, $\forall v \in C^2_{loc}(D)$ such that $x \to (u-v)(x)$ has a local maximum at x_o,
$v(x_o) \geq 0$.

Viscosity supersolutions and viscosity solutions are defined analogously.

3. Summary of results

Consider the family (1.1) with ω constant, say $\omega = 1$. Payne and Philippin [PP] have shown that

$$\lim_{p\to\infty} M_p = 2\int_\Omega d(x,\partial\Omega)dx$$

where $x \to d(x,\partial\Omega)$ is the distance function from $x \in \Omega$ to $\partial\Omega$. They also obtained several a–priori bounds for the p–torsional stiffness E_p, when Ω is a *convex* domain with boundary $\partial\Omega$ of class C^2. As a consequence of our results (see Proposition 2.1 and Corollary 2.1 of Part II) it will follow that

$$\text{(3.1)} \qquad \lim_{p\to\infty} u_p(\cdot) = d(\cdot,\partial\Omega) \quad \text{strongly in} \quad W_o^{1,m}(\Omega) \quad \forall m > 1.$$

In particular

$$\text{(3.2)} \qquad \lim_{p\to\infty} u_p(\cdot) = d(\cdot,\partial\Omega) \quad \text{in } C^\alpha(\bar{\Omega}) \quad \forall\alpha \in (0,1).$$

No convexity of Ω nor regularity of $\partial\Omega$ is needed here. Furthermore, from [CL], the function $u_\infty(\cdot) = d(\cdot,\partial\Omega)$ is the unique viscosity solution of

$$\text{(3.3)} \qquad |\nabla u| = 1 \qquad \text{in} \quad \Omega.$$

Payne and Philippin based their arguments on a maximum principle up to the boundary satisfied by $|\nabla u_p|$. This approach requires some smoothness of $\partial\Omega$ and on the fact that the right hand side of the first of (1.1) is constant. We bypass these restrictions by proving that, regardless of the structure of the right hand side of (1.1), the function of p

$$p \to \left(\frac{1}{|\Omega|}\int_\Omega |\nabla u_p|^p dx\right)^{\frac{p-1}{p}}$$

is monotone decreasing in p. This permits to establish that $\lim_{p\to\infty} u_p$ exists and identify it as $d(\cdot,\partial\Omega)$. The technique permits more general non–homogeneous terms in (1.1). In particular if $\mathrm{div}(\omega x) > 0$ in Ω we still have (3.1)-(3.3) regardless of the local behavior of $\mathrm{div}(\omega x) > 0$. If $x \to \omega(x)$ is compactly supported in Ω the possibility of identifying u_∞ depends on the geometry of $\mathrm{supp}[\omega]$. We refer to §3 of Part II for the precise statements and proofs. Here we examine more closely the case (2.1) i.e the case when the right hand side of (1.1) is replaced by any non-negative function f. If $f > 0$ in Ω then (3.1)-(3.3) continue to hold regardless of the structure or regularity of f. If f vanishes in a substantial portion of Ω then (3.1)-(3.3) are still valid within the interior of the set $[f > 0]$ and it is natural to ask whether they hold troughout Ω. This question can be answered in terms of the mutual structure of supp$[f]$ and the *ridge* $\mathcal{R}$ of Ω. The ridge $\mathcal{R}$ of Ω is the set of points of Ω where $\nabla d(\cdot, \partial\Omega)$ is discontinuous. Roughly speaking if supp$[f]$ covers the ridge then (3.1)-(3.3) continue to hold in the whole Ω. We refer to §4 of Part II for the precise statement. In general one always has

$$0 \leq u_\infty(x) \leq d(x, \partial\Omega), \qquad \forall x \in \Omega.$$

However simple examples (see §4 Part II) show that u_∞ may be uniquely determined and not equal to the distance function outside the support of f.

The limit function u_∞ is always a viscosity subsolution of $|\nabla u| = 1$. In fact this is true for any f even of variable sign (Proposition 5.1 of Part II). This fact is proved by establishing local a-priori gradient estimates on the solutions u_p of (2.1) independent of p, of the type

$$||\nabla u_p||_{\infty, B_R(x_o)} \leq \gamma^{\frac{1}{p}}, \qquad B_R(x_o) \equiv \{|x - x_o| < R\},$$

where γ is a constant dependent upon $|\Omega|, R$ and $||f||_{\infty,\Omega}$, but it is independent of p. Such estimate as well as some global ones are proved in Part III and may be of independent interest.

It is also true that the limit function u_∞ is always a viscosity solution of $\Delta_\infty u = 0$ in the open set $\Omega - \text{supp}[f]$. The equation $\Delta_\infty u = 0$ is obtained by formally expanding the divergence in $\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = 0$, dividing by $(p-2)\,|\nabla u|^{p-4}$ and letting $p \to \infty$. Properties of C^2 solutions of $\Delta_\infty u = 0$ have been studied by Aronsson in two nice papers [**Ar1,Ar2**]. In particular, Aronsson shows that the Dirichlet problem

$$\text{(3.4)} \qquad \begin{cases} \Delta_\infty u = 0 & \text{in } \Omega, \\ u - F = 0 & \text{in } \partial\Omega, \qquad F \in C^2(\bar{\Omega}), \end{cases}$$

does not have a C^2 solution even if Ω is a disk in $\mathbf{R}^2$. To remedy the situation Aronsson [**Ar1**] proposed as an alternative to (3.4) the following minimization problem

$$\text{(3.5)} \qquad \min\left\{||\nabla u||_{\infty,\Omega} : u - F \in W^{1,\infty}(\Omega) \cap C_o(\bar{\Omega})\right\}.$$

Solutions of this minimum problem are called ***absolute minimals.***

In Part I we briefly present Aronsson's ideas somewhat updated and propose a new class of solutions to (3.4) called ***variational solutions.*** These are obtained by solving

$$\begin{cases} \operatorname{div}\left(|\nabla u_p|^{p-2}\nabla u_p\right) = 0 & \text{in} \quad \Omega, \\ u - F = 0 & \text{in} \quad \partial\Omega, \end{cases}$$

and by taking the limit of u_p in $C^\alpha(\bar{\Omega})$ for $\alpha \in (0,1)$. Variational solutions are Lipschitz continuous in $\bar{\Omega}$ and it is always possible to solve (3.4) in the variational sense. Furthermore in §2 of Part I we prove that variational solutions are absolute minimals of (3.5) and are also viscosity solutions of (3.4). A major open issue here is their uniqueness.*

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*R. Jensen (Loyola University at Chicago) has recently announced a proof of uniqueness of viscosity solutions of $\Delta_\infty u = 0$.

Part I. About ∞-harmonic functions

1. Aronsson extension problem and absolute minimals

In a beautiful paper [**Ar1**], Aronsson has given a characterization of C^2–solutions of

$$\Delta_\infty u = \frac{1}{2}\nabla u \cdot \nabla\left(|\nabla u|^2\right) = 0. \tag{1.1}$$

We shall recall some facts from [**Ar1**] to be used below. If E is a subset of $\mathbf{R}^N$ and f is defined on E let

$$\mu(f,E) \equiv \sup_{\substack{x,y\in E\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|}. \tag{1.2}$$

Note that if Ω is a domain and $f \in W^{1,\infty}(\Omega)$ then

$$\|\nabla f\|_{\infty,\Omega} \le \mu(f,\Omega). \tag{1.3}$$

In (1.3) equality holds if Ω is convex and strict inequality is possible for non convex Ω. This can be seen from the following example

$$\Omega \equiv \left\{x_1^2 + x_2^2 \ge 1\right\} - \{x_1 \le 0\}\ , \qquad f(x_1,x_2) = \arctan\frac{x_1}{x_2}\ ,$$
$$\|\nabla f\|_{\infty,\Omega} = 1 \qquad \text{and} \qquad \mu(f,\Omega) = +\infty.$$

As indicated in [**Ar1**], the definition of $\mu(f,\Omega)$ must be modified so that equality holds in (1.3) for every domain Ω. Define for $x, y \in \Omega$

$$d_\Omega(x,y) \equiv \inf_\Gamma[\text{ length of }\Gamma] \tag{1.4}$$

where Γ is a Lipschitz curve in Ω joining x and y, and set

$$\mu_\Omega(f) \equiv \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|f(x)-f(y)|}{d_\Omega(x,y)}. \tag{1.5}$$

We now have

$$\mu_\Omega(f) = \|\nabla f\|_{\infty,\Omega} \tag{1.6}$$

in the sense that a function $f \in C(\Omega)$ is in $W^{1,\infty}(\Omega)$ if and only if $\mu_\Omega(f) < \infty$. Moreover, equality (1.6) holds and for convex Ω we have $\mu_\Omega(f) = \mu(f, \Omega)$. It is obvious that $\|\nabla f\|_{\infty,\Omega} \leq \mu_\Omega(f)$. To prove the reverse inequality let $x, y \in \Omega$ and a Lipschitz curve Γ joining them be fixed. For every polygonal of vertices $x \equiv x_1, x_2, \ldots, x_{m-1}, x_m \equiv y$ and $x_i \in \Gamma$, $\forall i = 1, 2, \ldots, m$ we have

$$\begin{aligned} |f(x) - f(y)| \leq \sum_{i=1}^{m-1} |f(x_i) - f(x_{i+1})| &\leq \sum_{i=1}^{m-1} |x_i - x_{i+1}| \, \|\nabla f\|_{\infty,\Omega} \\ &\leq \text{ length of } \Gamma \|\nabla f\|_{\infty,\Omega}. \end{aligned}$$

Let now $f \in L^\infty(\Omega)$ satisfying

$$\begin{aligned} &\exists M > 0 \quad \text{such that } \forall F \subset \Omega : \bar{F} \subset \Omega \\ &\forall h \in \mathbf{R}^N \ : \ |h| < \operatorname{dist}(F, \partial\Omega) \, , \ \forall x \in F \\ &\|f(x+h) - f(x)\|_{L^\infty(F)} \leq |h| M. \end{aligned}$$

Then $f \in W^{1,\infty}(\Omega)$ and $\|\nabla f\|_{\infty,\Omega} \leq M$.

The problem of extending such an f to a continuous function up to the closure of Ω is in general ill posed as shown by the previous example. However if $\partial\Omega$ is Lipschitz, any $f \in W^{1,\infty}(\Omega)$ has a continuous extension to $\bar{\Omega}$. Moreover

$$\mu_\Omega(f) = \mu_{\bar{\Omega}}(f) = \|\nabla f\|_{\infty,\Omega}.$$

Conversely if $f \in W^{1,\infty}(\Omega)$ has a continuous extension to $\bar{\Omega}$, then $\mu_\Omega(f) = \mu_{\bar{\Omega}}(f)$. Here the definition (1.4) of $d_\Omega(\cdot,\cdot)$ is extended to $\bar{\Omega}$ by the limiting process

$$d_{\bar{\Omega}}(x, y) = \liminf_{n\to\infty} d_\Omega(x_n, y_n) \, , \quad x_n \, , \, y_n \in \Omega \, , \quad x_n \to x \, , \, y_n \to y.$$

Note that $d_{\bar{\Omega}}$ may not be a metric in $\bar{\Omega}$.

Absolute minimals.

We say that $f \in W^{1,\infty}(\Omega)$ is an absolute minimal if for any subdomain D of Ω such that $\bar{D} \subset \Omega$ we have

$$\mu_{\bar{D}}(f) = \mu_{\partial D}(f) = \sup_{\substack{x,y \in \partial D \\ x \neq y}} \frac{|f(x) - f(y)|}{d_D(x, y)}. \tag{1.7}$$

The extension problem (Ω, g).

Let a function $g : \partial\Omega \to \mathbf{R}$ be given such that

$$0 < \mu_{\partial\Omega}(g) < \infty.$$

A function $f \in W^{1,\infty}(\Omega)$ is a solution of the extension problem (Ω, g) if

$$f \in C(\bar{\Omega}), \quad f = g \text{ on } \partial\Omega \quad \text{and} \quad \mu_{\bar{\Omega}}(f) = \mu_{\partial\Omega}(g).$$

Aronsson **[Ar1]** proves that this problem always has a solution and that a function f is the *unique* solution of the extension problem (Ω, g) *if and only if*

$$f \in C^1(\Omega) \qquad \text{and} \qquad |\nabla f| \text{ is constant.}$$

Thus, in general the extension problem (Ω, g) has more than one solution. All solutions however must agree on a set E, called the *uniqueness set*, which is determined as follows. A point x is in E if and only if there are points $y_1, y_2 \in \partial\Omega$ such that the following two conditions are satisfied

$$|g(y_1) - g(y_2)| = \mu_{\partial\Omega}(g) d_\Omega(y_1, y_2) \quad \text{and} \tag{1.8}$$

$$d_\Omega(x, y_1) + d_\Omega(x, y_2) = d_\Omega(y_1, y_2). \tag{1.9}$$

For example if Ω is convex, E is the union of segments joining boundary points satisfying (1.8).

Suppose now $f \in W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$ is a solution of the extension problem (Ω, g), and let h be any other function in $W^{1,\infty}(\Omega) \cap C(\bar{\Omega})$ such that $h = g$ on $\partial\Omega$. Then we must have

$$\begin{aligned} \|\nabla f\|_{\infty,\Omega} &= \mu_\Omega(f) = \mu_{\bar{\Omega}}(f) \\ &= \mu_{\partial\Omega}(g) = \mu_{\partial\Omega}(h) \\ &\le \mu_{\bar{\Omega}}(h) = \mu_\Omega(h) = \|\nabla h\|_{\infty,\Omega}. \end{aligned} \tag{1.10}$$

Therefore solutions of the extension problem (Ω, g) are minima of the L^∞-norm of the gradient. Their non uniqueness stems from the fact that $L^\infty(\Omega)$ is not a strictly convex Banach space.

It follows from these remarks that an absolute minimal f in Ω is a solution of the extension problem $(D, f|_{\partial D})$ for any subdomain D of Ω such that $\bar{D} \subset \Omega$ and it minimizes the L^∞-norm of the gradient on any such subdomain of Ω.

It is shown in **[Ar1]** that absolute minimals satisfy a comparison principle with respect to affine functions. Aronsson used this to prove the remarkable fact that among all the solutions of any extension problem (Ω, g) there is always one that is an absolute minimal. He also proved the following

THEOREM 1.1. *If $f \in C^2(\Omega)$, then f is an absolute minimal in Ω if and only if it satisfies*

$$\Delta_\infty f = 0 \quad .$$

2. Variational solutions of $\Delta_\infty u = 0$

Let $F \in W^{1,\infty}(\Omega)$ and for each $p \in (1,\infty)$ let u_p denote the unique solution of the problem

$$\text{(2.1)} \qquad \begin{cases} \Delta_p u_p = 0 & \text{in} \quad \Omega \\ u_p - F \in W_o^{1,p}(\Omega). \end{cases}$$

Multiply the first of (2.1) by $(u_p - F)$ and integrate over Ω. Standard calculations give

$$\int_\Omega |\nabla u_p|^p \, dx \le \int_\Omega |\nabla F|^p \, dx.$$

If $m_o > 1$ is fixed, by Hölder inequality, for $p > m_o$ we have

$$\begin{aligned} \int_\Omega |\nabla u_p|^{m_o} \, dx &\le \left(\int_\Omega |\nabla u_p|^p \, dx \right)^{\frac{m_o}{p}} |\Omega|^{1-\frac{m_o}{p}} \\ &\le \|\nabla F\|_{\infty,\Omega}^{m_o} |\Omega| . \end{aligned}$$

That is,

$$\text{(2.2)} \qquad \left(\frac{1}{|\Omega|} \int_\Omega |\nabla u_p|^{m_o} \, dx \right)^{\frac{1}{m_o}} \le \|\nabla F\|_{\infty,\Omega}.$$

Therefore the net $\{u_p\}_{p \ge m_o}$ is uniformly bounded in $W^{1,m_o}(\Omega)$, and we may select a subsequence indexed with p_{m_o} such that

$$u_{p_{m_o}} \to u_\infty \qquad \text{weakly in} \quad W^{1,m_o}(\Omega).$$

Letting $p \to \infty$ in (2.2) along the subsequence $\{u_{p_{m_o}}\}$ we obtain, by the weak lower semicontinuity of the $L^{m_o}(\Omega)$–norm

$$\left(\frac{1}{|\Omega|} \int_\Omega |\nabla u_\infty|^{m_o} \, dx \right)^{\frac{1}{m_o}} \leq \|\nabla F\|_{\infty,\Omega}. \tag{2.3}$$

This inequality holds true for any sequence $\{m_i\}_{i\in \mathbf{N}} \nearrow \infty$. Indeed, such a sequence being fixed, we may select by diagonalization a subsequence $\{u_{p'}\}$ out of $\{u_{p_{m_o}}\}$ such that

$$u_{p'} \rightharpoonup u_\infty \qquad \text{weakly in} \quad W^{1,m_i}(\Omega) \qquad \forall i \in \mathbf{N}.$$

Writing (2.3) for all m_i and letting $m_i \to \infty$ gives

$$\|\nabla u_\infty\|_{\infty,\Omega} \leq \|\nabla F\|_{\infty,\Omega}. \tag{2.4}$$

Moreover by the Sobolev embedding theorem [**Ad**]

$$u_{p'} \longrightarrow u_\infty \qquad \text{in } C^\alpha(\mathcal{K}) \qquad \forall \alpha \in (0,1)$$

for every compact subset $\mathcal{K}$ of Ω. Moreover if either $\partial\Omega$ has the cone property or $F \in C^1(\bar\Omega)$ (see [**Ad**]) then

$$u_{p'} \longrightarrow u_\infty \qquad \text{in } C^\alpha(\bar\Omega) \qquad \forall \alpha \in (0,1).$$

The limits so obtained we call *variational solutions* of $\Delta_\infty u = 0$ in Ω.

PROPOSITION 2.1. *Variational solutions are absolute minimals.*

PROOF: Let u_∞ be a variational solution of $\Delta_\infty u = 0$ corresponding to some subsequence of $\{u_p\}$, say $\{u_{p'}\}$. Fix any subdomain D of Ω such that $\bar D \subset \Omega$, and for $p \in (1,\infty)$ let v_p denote the unique solution of

$$\begin{cases} \Delta_p v_p = 0 & \text{in} \quad D \\ v_p - u_\infty \in W_o^{1,p}(D). \end{cases} \tag{2.5}$$

By the limiting process indicated above we may select subsequences $\{v_{p'}\}$ converging to some v_∞ weakly in $W^{1,m}(D)$ for all $m > 1$ and in $C^\alpha(\mathcal{K})$ for all $\alpha \in (0,1)$ for every compact subset $\mathcal{K}$ of D. We claim that for the subsequence

$\{v_{p'}\}$ corresponding to $\{u_{p'}\}$ we have $v_\infty = u_\infty$. Indeed $u_{p'}$ and $v_{p'}$ are both p'-harmonic in D and by the comparison principle

$$\|u_{p'} - v_{p'}\|_{\infty,D} \le \|u_{p'} - v_{p'}\|_{\infty,\partial D} = \|u_{p'} - u_\infty\|_{\infty,\partial D} \to 0 \text{ as } p' \to \infty.$$

Suppose now that $G \in W^{1,\infty}(D) \cap C(\bar{D})$ is a solution of the (D, u_∞) extension problem. Arguing as in the derivation of (2.4) we have

$$\|\nabla v_\infty\|_{\infty,D} \le \|\nabla G_\infty\|_{\infty,D}.$$

Then

$$\begin{aligned} \mu_{\bar{D}}(u_\infty) = \mu_{\bar{D}}(v_\infty) = \mu_D(v_\infty) &= \|\nabla v_\infty\|_{\infty,D} \\ &\le \|\nabla G_\infty\|_{\infty,D} = \mu_D(G) = \mu_{\bar{D}}(G) \\ &= \mu_{\partial D}(G) = \mu_{\partial D}(v_\infty) = \mu_{\partial D}(u_\infty). \end{aligned}$$

PROPOSITION 2.2. *Variational solutions are viscosity solutions*

PROOF: Let u_∞ be a variational solution corresponding to a subsequence $\{u_q\}_{q\in\mathbf{N}}$ out of the net $\{u_p\}$. Fix $x_o \in \Omega$ and let $v \in C^2_{loc}(\Omega)$ be such that $(u_\infty - v)$ has a local maximum at x_o, so that for some $R > 0$

$$(u_\infty - v)(x_o) \ge (u_\infty - v)(x), \quad \forall x \in B_R(x_o) \equiv \{|x - x_o| < R\} \text{ for some } R \in (0,1).$$

We will show that

$$\Delta_\infty v(x_o) \equiv \left(\nabla v \cdot \nabla |\nabla v|^2\right)(x_o) \ge 0.$$

Let

$$C = \sup_{q\in\mathbf{N}} \max\left\{\|u_q\|_{\infty,B_R(x_o)}\,;\ \|v\|_{\infty,B_R(x_o)}\right\}$$

and consider the sequence of functions

$$x \to u_q(x) - v(x) - k|x - x_o|^a, \qquad k = \frac{4C}{R^a}, \quad a > 2 \text{ fixed}.$$

On the boundary of $B_R(x_o)$ we have

$$u_q(x) - v(x) - k|x - x_o|^a \le C - kR^a = -3C$$

and for $x = x_o$, $u_q(x_o) - v(x_o) \geq -2C$. Therefore the function $x \to u_q(x) - v(x) - k|x - x_o|^a$ attains its maximum in $\bar{B}_R(x_o)$ at some point x_q in the interior of $B_R(x_o)$.

We claim that the sequence $\{x_q\}$ has x_o as its only accumulation point. Indeed if $x^* \neq x_o$ is an accumulation point of $\{x_q\}$ and if $\{x_{q'}\}$ is a subsequence converging to x^* we must have for all q'

$$u_{q'}(x_{q'}) - v(x_{q'}) - k\,|x_{q'} - x_o|^a \geq u_{q'}(x) - v(x) - k\,|x - x_o|^a \ , \quad \forall x \in B_R(x_o).$$

Setting $x = x_o$ in this inequality and letting $q' \to \infty$ we obtain

$$u_\infty(x^*) - v(x^*) - k\,|x^* - x_o| \geq u_\infty(x_o) - v(x_o)$$

which contradicts the fact that $(u_\infty - v)$ takes its maximum in $\bar{B}_R(x_o)$ at x_o.

Next at x_q,

$$\nabla u_q(x_q) - \nabla(x_q) - ak\,(x_q - x_o)\,|x_q - x_o|^{a-2} = 0.$$

Without loss of generality we may assume $|\nabla v(x_o)| \neq 0$. Then for q large enough we must have $|\nabla u_q(x_q)| \neq 0$. Therefore $|\nabla u_q(x)|$ will not vanish in a small neighborhood of x_q depending on q. In particular u_q is C^∞ in such a neighborhood. Thus

$$|\nabla u_q(x_q)|^2\,\Delta u_q(x_q) + (q-2)\langle D^2 u_q(x_q) \nabla u_q(x_q)\,,\ \nabla u_q(x_q)\rangle = 0$$

where $D^2 u_q(x_q)$ is the hessian matrix of u_q evaluated at x_q. Also in the sense of matrices

$$D^2 u_q(x_q) - D^2 v(x_q) - a|x_q - x_o|^{a-2}\left[\mathbf{I} + (a-2)\frac{(x_q - x_o)_i (x_q - x_o)_j}{|x_q - x_o|^2}\right] \leq 0.$$

In particular

$$\Delta u_q(x_q) - \Delta v(x_q) - a|x_q - x_o|^{a-2}\,[N + (a-2)] \leq 0.$$

It follows that

$$\begin{aligned}
&\left|\nabla v(x_q) + ak(x_q - x_o)|x_q - x_o|^{a-2}\right|^2 \left(\Delta v(x_q) + a|x_q - x_o|^{a-2}[N + a - 2]\right) + \\
&(q-2)\left\langle\left(D^2 v(x_q) + a|x_q - x_o|^{a-2}\left[\mathbf{I} + (a-2)\frac{(x_q - x_o)_i (x_q - x_o)_j}{|x_q - x_o|^2}\right]\right)\right. \\
&\left(\nabla v(x_q) + ak(x_q - x_o)|x_q - x_o|^{a-2}\right)\,,\left(\nabla v(x_q) + ak(x_q - x_o)|x_q - x_o|^{a-2}\right)\Big\rangle \geq 0.
\end{aligned}$$

Divide by $(q-2)$ and let $q \to \infty$. Since $a > 2$ this gives

$$\langle (D^2 v(x_o))\, \nabla v(x_o)\, ,\, \nabla v(x_o)\rangle \geq 0$$

i.e.,

$$\Delta_\infty v(x_o) \geq 0.$$

Analogously if $v \in C^2_{loc}(\Omega)$ is such that $(u_\infty - v)$ has a local minimum at x_o, a symmetric argument shows that $\Delta_\infty v(x_o) \leq 0$.

3. About $\Delta_\infty u = 0$ in Ω

There are at least four ways of interpreting the equation $\Delta_\infty u = 0$. Namely

I. **C²-solutions.** This class is too small to solve the Dirichlet problem (see the example on page 425 of [**Ar2**]).

II. Absolute minimals. The Dirichlet problem can always be solved within this class.

III. Variational solutions in the sense of §2.

IV. Viscosity solutions in the sense of Crandall–Lions [**CL**].

From the results of Aronsson [**Ar1, Ar2**] and the remarks of the previous section we have the following relationships between them:

(3.1)
$$(C^2 - \text{sol.}) \subset (\text{Abs.minimals})$$
$$(C^2 - \text{sol.}) \subset (\text{Viscosity Sol.})$$

(3.2)
$$(\text{Variational Sol.}) \subset (\text{Abs.minimals})$$
$$(\text{Variational Sol.}) \subset (\text{Viscosity Sol.})$$

The function $u(x_1, x_2) = |x_1|^{\frac{4}{3}} - |x_2|^{\frac{4}{3}}$ is a variational solution in $\{|x| < 1\}$ and not a C^2–solution. See Lemma 4 in [**Ar4**]. Thus, the class of C^2-solutions is strictly contained in the class

Part II. Limits of torsional problems

1. The variational problem

Let Ω be a bounded domain in $\mathbf{R}^N$, $N \geq 2$ and for $p > 1$ let $u_p \in W_o^{1,p}(\Omega)$ be the unique weak solution of

$$\text{(1.1)} \qquad \begin{cases} \operatorname{div}\left(|\nabla u_p|^{p-2}\nabla u_p\right) + \div (x\omega) = 0\,, & \text{in } \Omega \\ u_p = 0 & \text{in } \partial\Omega \end{cases}$$

where $x \to \omega(x)$ is a given non-negative bounded function. When $N = 2$ the problem is related to the creeping phenomenon of a long prismatic elasto–plastic rod under the action of a moment parallel to its axis. The function u_p is a potential whose gradient permits to recover the distribution of stresses in the rod. The function $\omega(\cdot)$ is called the *angular twist rate* and in most models is taken to be constant. The quantity

$$\text{(1.2)} \qquad E_p \equiv E_p(\omega, \Omega) = \int_\Omega |\nabla u_p|^p dx$$

is the (p,ω)–*torsional stiffness* of the rod. We refer to Part IV for a description of the physical problem and a derivation of the model. There we prove that in general ω is a radial function, and discuss how the limiting case $p \to \infty$ is related to the pure plastic state of the material.

The weak formulation of (1.1) is

$$\text{(1.3)} \qquad \int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla\varphi dx = -\int_\Omega \omega\nabla\varphi \cdot x dx\,, \qquad \forall\varphi \in W_o^{1,p}(\Omega)$$

This and (1.2) imply the Poincaré type inequality

$$\text{(1.4)} \qquad \left| -\int_\Omega \omega\nabla\varphi \cdot x dx \right| \leq E_p^{\frac{p-1}{p}} \left(\int_\Omega |\nabla\varphi|^p dx \right)^{\frac{1}{p}}\,, \qquad \forall\varphi \in W_o^{1,p}(\Omega).$$

Therefore

$$(1.5) \qquad \max_{\substack{\varphi\in W_o^{1,p}(\Omega)\\ \varphi\neq 0}} \left(\frac{-\int\limits_\Omega \omega\nabla\varphi\cdot x dx}{\left(\int\limits_\Omega |\nabla\varphi|^p dx\right)^{\frac{1}{p}}} \right)^{\frac{p}{p-1}} \leq E_p$$

where, by possibly changing φ into $-\varphi$ we have assumed that

$$-\int\limits_\Omega \omega\nabla\varphi\cdot x dx \geq 0.$$

By taking $\varphi = u_p$ in (1.3) we have

$$E_p = \int\limits_\Omega |\nabla u_p|^p dx = -\int_\Omega \omega\nabla u_p\cdot x dx.$$

From this

$$E_p = \frac{\left(\int\limits_\Omega |\nabla u_p|^p dx\right)^{\frac{p}{p-1}}}{\left(\int\limits_\Omega |\nabla u_p|^p dx\right)^{\frac{1}{p-1}}} = \left(\frac{-\int\limits_\Omega \omega\nabla u_p\cdot x dx}{\left(\int\limits_\Omega |\nabla u_p|^p dx\right)^{\frac{1}{p}}} \right)^{\frac{p}{p-1}}$$

$$\leq \max_{\substack{\varphi\in W_o^{1,p}(\Omega)\\ \varphi\neq 0}} \left(\frac{-\int\limits_\Omega \omega\nabla\varphi\cdot x dx}{\left(\int\limits_\Omega |\nabla\varphi|^p dx\right)^{\frac{1}{p}}} \right)^{\frac{p}{p-1}}.$$

Combining these inequality yields the

Thompson principle for the torsional stiffness.

$$(1.6) \qquad E_p = \max_{\substack{\varphi\in W_o^{1,p}(\Omega)\\ \varphi\neq 0}} \left(\frac{-\int\limits_\Omega \omega\nabla\varphi\cdot x dx}{\left(\int\limits_\Omega |\nabla\varphi|^p dx\right)^{\frac{1}{p}}} \right)^{\frac{p}{p-1}}.$$

Remark 1.1. If $\Omega\subset\Omega'$ and if $x\to\omega(x)$ is defined in Ω', then $E_p(\Omega)\leq E_p(\Omega')$, since $u_p\in W_o^{1,p}(\Omega)$ extended to be zero outside Ω, is in contention to compute $E_p(\Omega')$.

Remark 1.2. Suppose $\omega \equiv 1$. Then $E_p(\Omega) \leq E_p(\hat{\Omega})$ where $\hat{\Omega}$ is the Steiner symmetrization of Ω (see[**PS**]), since symmetrization decreases $\|\nabla\varphi\|_{p,\Omega}$.

Remark 1.3. If φ maximizes the functional in the Thompson principle then $\varphi = \lambda u_p$ for some $\lambda > 0$. This follows by choosing λ so that

$$\int_\Omega \omega\nabla\varphi \cdot x dx = \lambda \int_\Omega \omega\nabla u_p \cdot x dx$$

and using the fact that solutions of (1.1) are unique.

From the weak formulation (1.3) with $\varphi = u_p$ it follows that

$$E_p = -\int_\Omega \nabla u_p \cdot \mathbf{V} dx \tag{1.7}$$

where $\mathbf{V} \in L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N)$ is any vector valued function whose weak divergence equals $\operatorname{div}(\omega x)$. By Hölder inequality

$$E_p \leq E_p^{\frac{1}{p}} \left(\int_\Omega |\mathbf{V}|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \tag{1.8}$$

with equality if $\mathbf{V} = |\nabla u_p|^{p-2}\nabla u_p$. We formulate these remarks as

Dirichlet principle.

$$E_p = \min\left\{ \int_\Omega |\mathbf{V}|^{\frac{p}{p-1}} dx \; : \; \mathbf{V} \in L^{\frac{p}{p-1}}(\Omega; \mathbf{R}^N)\,,\ \operatorname{div}\mathbf{V} = \operatorname{div}(\omega x) \text{ in } \mathcal{D}'(\Omega) \right\}. \tag{1.9}$$

A function $u_p \in W^{1,p}(\Omega)$ is a weak solution of (1.1) if and only if it minimizes the functional

$$J_p(\varphi) = \int_\Omega \left\{ \frac{1}{p}|\nabla\varphi|^p + \omega\nabla\varphi \cdot x \right\} dx\,, \qquad \varphi \in W_o^{1,p}(\Omega). \tag{1.10}$$

Letting $p \to \infty$ in (1.10) gives *formally*

$$J_\infty(\varphi) = \int_\Omega \omega\nabla\varphi \cdot x dx\,, \qquad \varphi \in W^{1,\infty}(\Omega) \cap C_o(\bar{\Omega}).$$

The minimum of $J_\infty(\varphi)$ among all $\varphi \in W^{1,\infty}(\Omega) \cap C_o(\bar\Omega)$ is obviously $-\infty$. The minimization of $J_\infty(\cdot)$ is meaningful if we impose an extra constraint on φ. We will show that the problem of minimizing $J_p(\varphi)$, $\varphi \in W_o^{1,p}(\Omega)$ tends to the problem of minimizing

$$(1.11) \qquad J_\infty(\varphi) \quad \text{among all } \varphi \in W^{1,\infty}(\Omega) \cap C_o(\bar\Omega) \quad \text{such that } \|\nabla\varphi\|_{\infty,\Omega} = 1.$$

This functional gives rise to two basic questions. The first is whether there exists a solution and if so if it is unique. The second is to find the Euler equation (i.e.; the p.d.e. corresponding to (1.1) as $p \to \infty$) if any, and if so in what sense the p.d.e. holds.

THEOREM 1.1. *There exists at least one solution $\varphi_\infty \in W^{1,\infty}(\Omega) \cap C_o(\bar\Omega)$ of the minimum problem*

$$(1.12) \qquad \varphi \longrightarrow \min J_\infty(\varphi), \qquad \|\nabla\varphi\|_{\infty,\Omega} = 1.$$

Moreover

$$(1.13) \qquad |\varphi_\infty(x)| \le d(x, \partial\Omega), \ \forall x \in \bar\Omega$$

where $x \to d(x,\partial\Omega)$ denotes the distance function from x to $\partial\Omega$.

Questions of uniqueness and identification of the Euler equations are discussed in §3,4.

2. The limit of u_p as $p \to \infty$

For $p > N$ one has $W_o^{1,p}(\Omega) = W^{1,p}(\Omega) \cap C_o(\bar\Omega)$ for any domain Ω. See [Ma] for a complete proof. In particular $d(\cdot,\partial\Omega) \in W_o^{1,p}(\Omega)$ for all $p > N$. Therefore $d(\cdot,\partial\Omega)$ is a test function for the Thompson principle.

From the Dirichlet principle, (1.8) and Hölder inequality it follows that

$$(2.1) \quad \forall 1 < q < p,\ \forall \mathbf{V} \in L^{\frac{q}{q-1}}(\Omega) \quad \text{such that } \operatorname{div}\mathbf{V} = \operatorname{div}(\omega x) \ \text{in } \mathcal{D}'(\Omega),$$

$$\left(|\Omega|^{-1}E_p\right)^{\frac{p-1}{p}} \le \left(|\Omega|^{-1}\int_\Omega |\mathbf{V}|^{\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}} \le \left(|\Omega|^{-1}\int_\Omega |\mathbf{V}|^{\frac{q}{q-1}}dx\right)^{\frac{q-1}{q}}.$$

Therefore

$$\left(|\Omega|^{-1}E_p\right)^{\frac{p-1}{p}} \leq \inf_{\substack{\mathbf{V}\in L^{\frac{q}{q-1}}(\Omega)\\ +\mathbf{v}=\omega x \text{ in } \mathcal{D}'(\Omega)}} \left(|\Omega|^{-1}\int_{\Omega} |\mathbf{V}|^{\frac{q}{q-1}}dx\right)^{\frac{q-1}{q}} = \left(|\Omega|^{-1}E_q\right)^{\frac{q-1}{q}}.$$

We summarize

LEMMA 2.1. *The function $p \to \left(|\Omega|^{-1}E_p\right)^{\frac{p-1}{p}}$ is monotonically decreasing as $p \to \infty$.*

It follows that $\{E_p\}$ has a limit as $p \to \infty$ and we set

(2.2) $$E_\infty = \lim_{p\to\infty} E_p.$$

Fix $m > 1$. Then by Hölder inequality for all $p > m$

$$\int_{\Omega} |\nabla u_p|^m dx \leq \left(\int_{\Omega} |\nabla u_p|^p dx\right)^{\frac{m}{p}} |\Omega|^{1-\frac{m}{p}} \leq E_p^{\frac{m}{p}} |\Omega|^{1-\frac{m}{p}}.$$

If $m_o < m_1 < m_2 < \cdots$ is an increasing sequence or real numbers, by an argument similar to that leading to (2.4) in §2 of part I, a subsequence $\{u_{p'}\}$ can be selected out of the net $\{u_p\}$ such that

$$u_{p'} \to u_\infty \in W^{1,\infty}(\Omega)\cap C_o(\bar{\Omega}), \qquad \text{weakly in } W^{1,m_i}(\Omega), \quad \forall i \in \mathbf{N}.$$

Moreover

(2.3) $$\|\nabla u_\infty\|_{\infty,\Omega} \leq \lim_{p\to\infty}\left(\frac{E_p}{|\Omega|}\right)^{\frac{1}{p}} = 1.$$

Different subsequences out of $\{u_p\}$ may generate different limits. However all the limiting functions obtained by this procedure must satisfy (2.3).

LEMMA 2.2. *Let u_∞ be one such limit. Then*

(2.4) $$|u_\infty(x)| \leq d(x,\partial\Omega)$$

where $x \to d(x,\partial\Omega)$ denotes the distance function from x to the boundary of Ω.

PROOF: From (2.3), for all $x, y \in \bar{\Omega}$

$$|u_\infty(x) - u_\infty(y)| \leq |x-y|$$

and (2.4) follows for $y \in \partial\Omega$ since $u_\infty \in W^{1,\infty}(\Omega) \cap C_o(\bar{\Omega})$.

Next from (1.5) it follows that for $\varphi \in W^{1,\infty}(\Omega) \cap C_o(\bar{\Omega})$, $\varphi \neq 0$ arbitrary but fixed

$$\frac{-\int_\Omega \omega \nabla\varphi \cdot x dx}{\left(\int_\Omega |\nabla\varphi|^p dx\right)^{\frac{1}{p}}} \leq (E_p)^{\frac{p-1}{p}}.$$

Letting $p \to \infty$ along p'

$$\text{(2.5)} \qquad \frac{-\int_\Omega \omega \nabla\varphi \cdot x dx}{\|\nabla\varphi\|_{\infty,\Omega}} \leq E_\infty, \qquad \forall \varphi \in W^{1,\infty}(\Omega) \cap C_o(\bar{\Omega}), \ \varphi \neq 0.$$

In the Thompson principle (1.6) equality holds if $\varphi = u_p$. Then letting $p \to \infty$ along p'

$$\text{(2.6)} \qquad E_\infty = -\int_\Omega \omega \nabla u_\infty \cdot x dx.$$

Therefore since $\|\nabla u_\infty\|_{\infty,\Omega} \leq 1$

$$\text{(2.7)} \qquad E_\infty \leq \frac{-\int_\Omega \omega \nabla u_\infty \cdot x dx}{\|\nabla u_\infty\|_{\infty,\Omega}} \leq \sup_{\substack{\varphi \in W^{1,\infty}(\Omega) \cap C_o(\bar{\Omega}), \varphi \neq 0 \\ \|\nabla\varphi\|_{\infty,\Omega} \leq 1}} \left(\frac{-\int_\Omega \omega \nabla\varphi \cdot x dx}{\|\nabla\varphi\|_{\infty,\Omega}} \right).$$

Combining (2.7) and (2.5) implies

$$\inf_{\substack{\varphi \in W^{1,\infty}(\Omega) \cap C_o(\bar{\Omega}), \varphi \neq 0 \\ \|\nabla\varphi\|_{\infty,\Omega} \leq 1}} \left\{ \|\nabla\varphi\|_{\infty,\Omega} + \frac{1}{E_\infty} \int_\Omega \omega \nabla\varphi \cdot x dx \right\} = 0.$$

Therefore if $\varphi = u_\infty$

$$\|\nabla u_\infty\|_{\infty,\Omega} + \frac{1}{E_\infty} \int_\Omega \omega \nabla u_\infty \cdot x dx \geq 0$$

and in view of (2.6) $\|\nabla u_\infty\|_{\infty,\Omega} \geq 1$. Thus

$$\text{(2.8)} \qquad \|\nabla u_\infty\|_{\infty,\Omega} = 1.$$

With this information at hand, it follows from (2.6) that

$$\text{(2.9)} \qquad E_\infty \leq \sup_{\substack{\varphi \in W^{1,\infty}(\Omega) \cap C_o(\bar{\Omega}), \varphi \neq 0 \\ \|\nabla\varphi\|_{\infty,\Omega} = 1}} \left(-\int_\Omega \omega \nabla\varphi \cdot x dx \right).$$

This and (2.5) proves Theorem 1.1.

The upper bound (2.4) suggests that in some cases $u_\infty(x) = d(x, \partial\Omega)$ for all $x \in \Omega$. If so, the solution of the minimum problem (1.12), whose existence is claimed by theorem 1.1, is unique. We will study this question in the next sections. Here we give a preliminary result

PROPOSITION 2.1. *Assume* $\operatorname{div}(\omega x) > 0$ *in* Ω. *Then, the whole net* $\{u_p\} \to u_\infty$ *strongly in* $W_o^{1,m}(\Omega)$, *for all* $m > 1$ *and*

$$u_\infty(x) = d(x, \partial\Omega)\,, \qquad \forall x \in \Omega.$$

PROOF: Let u_∞ be any solution of the minimum problem (1.12). Then

$$\int_\Omega \omega \nabla u_\infty \cdot x dx \leq \int_\Omega \omega \nabla d(x, \partial\Omega) \cdot x dx.$$

That is,

$$\int_\Omega [u_\infty(x) - d(x, \partial\Omega] \operatorname{div}(\omega x) dx \geq 0$$

which together with (2.4) gives $u_\infty(x) = d(x, \partial\Omega)$.

Since every subsequence of $\{u_p\}$ has itself a subsequence converging to $d(\cdot, \partial\Omega)$ weakly in $W_o^{1,m}(\Omega)$, it follows that the full net $\{u_p\}_{p\geq m}$ converges to $d(\cdot, \partial\Omega)$ in $C^{o,\alpha}(\bar\Omega)$, where $\alpha = 1 - N/m$, and $\{\nabla u_p\}_{p\geq m}$ converges weakly in $L^p(\Omega)$ to $\nabla d(\cdot, \partial\Omega)$.

It remains to be shown that $u_p \to d(\cdot, \partial\Omega)$ strongly in $W_o^{1,m}(\Omega)$. From Clarkson's inequalities we have for $p, q > m$

$$\begin{aligned}\int_\Omega \frac{|\nabla u_p + \nabla u_q|^m}{2^m}\, dx + \int_\Omega \frac{|\nabla u_p - \nabla u_q|^m}{2^m}\, dx &\leq \frac{1}{2}\int_\Omega |\nabla u_p|^m\, dx + \frac{1}{2}\int_\Omega |\nabla u_q|^m\, dx \\ &\leq \frac{1}{2}\left[E_p^{\frac{m}{p}}\, |\Omega|^{1-\frac{m}{p}} + E_q^{\frac{m}{q}}\, |\Omega|^{1-\frac{m}{q}}\right]\end{aligned}$$

Therefore

$$\limsup_{p,q\to\infty} \int_\Omega \frac{|\nabla u_p + \nabla u_q|^m}{2^m}\, dx \leq |\Omega|.$$

On the other hand, from the weak lower semicontinuity of the L^m–norm,

$$|\Omega| = \int_\Omega |\nabla d(x, \partial\Omega)|^m\, dx \leq \liminf_{p,q\to\infty} \int_\Omega \frac{|\nabla u_p + \nabla u_q|^m}{2^m}\, dx.$$

Thus, we conclude

$$\limsup_{p,q\to\infty} \int_\Omega \frac{|\nabla u_p - \nabla u_q|^m}{2^m}\, dx = 0.$$

Corollary 2.1. *If $x \longrightarrow \omega(x)$ is a given positive constant, then*

$$\{u_p\} \longrightarrow d(\cdot, \partial\Omega) \ \textit{in}\ C^\alpha(\bar{\Omega}),\ \forall \alpha \in (0,1). \tag{2.10}$$

The convergence in integral mean

$$\int_\Omega u_p dx \longrightarrow \int_\Omega d(x, \partial\Omega) dx$$

for convex Ω whose boundary $\partial\Omega$ is class C^2, is due to Payne and Philippin (see[**PP**]). The convergence in (2.10) is in $C^\alpha(\bar{\Omega})$ for *every* domain Ω.

Remark 2.1. The proof of Proposition 2.1 shows that only the positivity of $\mathrm{div}(\omega x)$ is relevant to the result. In particular the conclusion still holds if $\mathrm{div}(\omega x)$ is replaced by any bounded, positive function f (see §4).

Remark 2.2. We will show in Part IV that, in the case of torsional creep (N=2) in a cylindrical rod of cross section Ω, the function $\omega(\cdot)$ must be constant.

It is in general false that $u_\infty(x) = d(x, \partial\Omega)$ as shown by the following

Counterexample. Let $\Omega \subset \mathbf{R}^2$ be the square

$$\|x\|_\infty \equiv \max\{|x_1|, |x_2|\}$$

and let $x \to \omega(x)$ be given by

$$\omega = \begin{cases} 1, & \text{if} \quad |x| = \|x\|_2 \equiv \sqrt{x_1^2 + x_2^2} < 1, \\ 0, & \text{in} \quad \Omega - \{|x| < 1\}. \end{cases}$$

If $d(x, \partial\Omega) = 1 - \|x\|_\infty$ were an extremal for the functional (1.11),

$$(2.11) \qquad -\int_\Omega \omega \|x\|_\infty dx \le \int_\Omega \omega \nabla\varphi \cdot x dx \,, \quad \forall \varphi \in W^{1,\infty}(\Omega) \cap C_o(\bar{\Omega}) \,,\ \|\nabla\varphi\|_{\infty,\Omega} = 1.$$

Select

$$\varphi = \begin{cases} 1 - |x| & \text{if} \quad |x| < 1, \\ 0 & \text{in} \quad \Omega - \{|x| < 1\}. \end{cases}$$

Then it would follow from (2.11) that

$$\int_{\{|x|<1\}} (\|x\|_\infty - \|x\|_2)\, dx \ge 0$$

which is a contradiction.

Remark 2.3. The same conclusion holds if ω is *any* non-negative bounded function supported in the ball $\{|x| < 1\}$.

These remarks suggest that the identification of $u_\infty(x)$ as $d(x, \partial\Omega)$ is only dependent upon the signum of $\text{div}(\omega x)$ and the support of ω. We will study separately these two cases in the next sections.

3. Identifying $\mathbf{u}_\infty$ within the support of ω

We will assume that supp$[\omega]$ is contained in a ball B_R of radius R centered at the origin and itself contained in Ω. We let u_∞ be one of the minimizers of (1.11) obtained as limit of some subsequence $\{u_q\}$, $q \in \mathbf{N}$ out of the net $\{u_p\}$.

PROPOSITION 3.1. *Assume $\omega \geq 0$ and that there exists $R > 0$ such that*

$$supp[\omega] \subset B_R \subset \Omega.$$

Then

$$u_\infty(x) \leq R - |x|, \qquad \forall x \in supp[\omega]. \tag{3.1}$$

If $supp[\omega] \cap \partial\Omega$ is non-empty, then

$$u_\infty(x) = R - |x|, \qquad \forall x \in supp[\omega]. \tag{3.2}$$

If $supp[\omega]$ is a ball $B_\rho \subset \Omega$, then

$$u_\infty(x) = (\rho - |x|)_+, \qquad \forall x \in \Omega. \tag{3.3}$$

PROOF: Since u_∞ is a minimizer

$$\int_{\{supp[\omega]\}} \omega\,[\nabla u_\infty - \nabla(R - |x|)_+] \cdot x dx \leq 0.$$

However $[\nabla u_\infty - \nabla(R - |x|)_+] \cdot x \geq 0$, for all $x \in$ supp$[\omega]$. Therefore $\nabla u_\infty(x) = -x/|x|$, for all $x \in$ supp$[\omega]$ and u_∞ must have the form

$$u_\infty(x) = \lambda - |x|, \qquad \forall x \in \text{supp}[\omega] \quad \text{for some } \lambda \in \mathbf{R}. \tag{3.4}$$

PROOF OF (3.2)

Let $x_o \in$ supp$[\omega] \cap \partial\Omega$. By Lemma 2.2,

$$|\lambda - |x|| \leq d(x, \partial\Omega), \qquad \forall x \in \text{supp}[\omega],$$

i.e.

$$|x| - d(x, \partial\Omega) \leq \lambda \leq |x| + d(x, \partial\Omega).$$

For $x = x_o$ this implies $\lambda = R$.

PROOF OF (3.1)

If for some $x_o \in \text{supp}[\omega] \cap \partial\Omega$ then the conclusion follows from (3.2). Let us assume that $\text{supp}[\omega] \subset \Omega$ and there exists $\eta \in (0,1)$ such that

$$\text{supp}[\omega] \subset B_R \subset B_{R+\eta} \subset \Omega.$$

Let $x \to \zeta(x)$ be a non-negative piecewise smooth cutoff function in $B_{R+\eta}$ such that $\zeta \equiv 1$ in B_R and

$$\nabla\zeta = \begin{cases} -\frac{1}{\eta}\frac{x}{|x|}, & \text{in } B_{R+\eta} - B_R, \\ 0, & \text{in } B_R. \end{cases}$$

The minimizer u_∞ is generated by some subsequence $\{u_q\}$, $q \in \mathbf{N}$ out of the net $\{u_p\}$. In the weak formulation (1.3) with $p = q$ take ζ as testing function to obtain

$$-\frac{1}{\eta}\int_{B_{R+\eta}-B_R} |\nabla u_q|^{q-2}\nabla u_q \cdot \frac{x}{|x|}dx = 0$$

and letting $\eta \to 0$

$$\int_{\partial B_R} |\nabla u_q|^{q-2}\frac{\partial u_q}{\partial\nu}d\sigma = 0 \tag{3.5}$$

where ν is the exterior unit normal to ∂B_R and $d\sigma$ is the $(N-1)$–dimensional surface measure on ∂B_R. It follows that for any domain S containing $\text{supp}[\omega]$,

$$\int_{\partial S} |\nabla u_q|^{q-2}\frac{\partial u_q}{\partial\nu}d\sigma = 0$$

where we have assumed that ∂S has outer unit normal ν a.e. with respect to the surface measure $d\sigma$.

Within $\text{supp}[\omega]$ the minimizer has the structure (3.4). If $\lambda > R$, then $u_\infty > 0$ on $\text{supp}[\omega]$. Therefore, since $u_q \to u_\infty$ in $C^\alpha(\bar\Omega)$ for all $\alpha \in (0,1)$, for q large enough

$$u_q > 0, \qquad \text{on} \quad \text{supp}[\omega].$$

Since $u_q = 0$ on $\partial\Omega$ and

$$-\text{div}|\nabla u_q|^{q-2}\nabla u_q = 0, \qquad \text{in} \quad \Omega - \text{supp}[\omega], \tag{3.6}$$

we must have $u_q \geq 0$ in $\Omega - \text{supp}[\omega]$. Therefore

$$\frac{\partial u_q}{\partial \nu} \leq 0\,, \qquad \text{on} \quad \partial\Omega. \tag{3.7}$$

Integrating (3.6) over $\Omega - B_R$ and using (3.5) we get

$$\int_{\partial\Omega} |\nabla u_q|^{q-2} \frac{\partial u_q}{\partial \nu} d\sigma = 0.$$

This boundary integral is well defined since by the results of [**ChDi**], $x \to \nabla u_q(x)$ is Hölder continuous up to the boundary $\partial\Omega$. In view of (3.7) we must have $\frac{\partial u_q}{\partial \nu} = 0$ on $\partial\Omega$. This is a contradiction unless $u_q \equiv 0$ in $\Omega - B_R$. Indeed since

$$u_q = 0 \qquad \text{and} \qquad \frac{\partial u_q}{\partial \nu} = 0 \qquad \text{on } \partial\Omega,$$

we may extend u_q with zero outside Ω to obtain a non-negative q-harmonic function in $\mathbf{R}^N - B_R$. This however would contradict the Harnack inequality.

Proof of (3.3)

If $\lambda < \rho$, then $u_\infty < 0$ in $\partial\text{supp}[\omega] \equiv \partial B_\rho$ and an argument almost identical to the one above gives $\lambda = \rho$. Thus $u_\infty(x) = \rho - |x|$ in B_ρ. In particular $u_\infty = 0$ on ∂B_ρ. Therefore $u_q \to 0$ in $\Omega - \text{supp}[\omega]$.

Remark 3.1. The meaning of parts (3.2) and (3.3) of the Proposition is that of finding an analog of the uniqueness set in the Aronsson extension problem (see §1 in Part I).

4. Identifying $\mathbf{u}_\infty$ for a non negative source f

We will examine how $\text{div}(\omega x) \geq 0$ permits to identify the minimizers u_∞ and yield uniqueness. Since only the signum of $\text{div}(\omega x)$ is relevant, we replace this term by a non-negative function $f : \Omega \to \mathbf{R}^+$. The arguments of the previous sections can be repeated to conclude

THEOREM 4.1. *There exists at least one solution* $u_\infty \in W^{1,\infty}(\Omega) \cap C_o(\bar{\Omega})$ *of the extremal problem*

$$(4.1) \qquad \varphi \longrightarrow \max \int_\Omega \varphi f dx\,, \qquad \forall \varphi \in W^{1,\infty}(\Omega) \cap C_o(\bar{\Omega})\,,\ \|\nabla\varphi\|_{\infty,\Omega} = 1.$$

Moreover

$$0 \le u_\infty(x) \le d(x, \partial\Omega)\,, \qquad \forall x \in \bar{\Omega}.$$

The solution is constructed as limit of a subsequence of solutions of

$$(4.2) \qquad \begin{cases} \operatorname{div}|\nabla u_p|^{p-2}\nabla u_p = -f & \text{in } \Omega \\ u_p = 0 & \text{on } \partial\Omega. \end{cases}$$

From Remark 2.1 it follows that if $f > 0$ in Ω then

$$(4.3) \qquad u_\infty = d(x, \partial\Omega)\,, \qquad \forall x \in \bar{\Omega}.$$

The proof shows that formula (4.3) holds for all $x \in \operatorname{supp}[f]$. Here we study cases when $u_\infty(x) = d(x,\partial\Omega)$, for $x \in \bar{\Omega}$ even if $\operatorname{supp}[f]$ is a proper subset of Ω satisfying meas $(\Omega - \operatorname{supp}[f]) > 0$.

Let $\mathcal{R}$ denote the ridge of Ω, i.e. the set of discontinuities of the gradient of the distance function. The set $\mathcal{R}$ can be alternatively defined as all the points in Ω equidistant from at least two points in $\partial\Omega$. See [**EH**].

THEOREM 4.2. *The function* $u_\infty = d(\cdot, \partial\Omega)$ *is the only extremal of (4.1) if and only if* $\mathcal{R}$ *is contained in* supp$[f]$.

PROOF (SUFFICIENCY): Let u_∞ be any extremal of (4.1). Within $\operatorname{supp}[f]$ we have $u_\infty(x) = d(x,\partial\Omega)$ and on the set $\Omega_o \equiv \Omega - \operatorname{supp}[f]$ it satisfies

$$\sup_{(x,y)\in\bar{\Omega}_o} \frac{|u_\infty(x) - u_\infty(y)|}{|x-y|} = \sup_{(x,y)\in\partial\Omega_o} \frac{|u_\infty(x) - u_\infty(y)|}{|x-y|}.$$

That is, u_∞ is a solution of the extension problem $(\Omega_o\,,\ d(x,\partial\Omega))$ (see §1, Part I).

Now $x \to d(x,\partial\Omega)$ is a solution of the extension problem $(\Omega_o\,,\ d(x,\partial\Omega))$ and it is $C^1(\Omega_o)$ since Ω_o does not contain points of discontinuities of $x \to \nabla d(x,\partial\Omega)$. Therefore it is the only solution. Thus $u_\infty(x) = d(x,\partial\Omega)$ for all $x \in \bar{\Omega}$.

PROOF (NECESSITY)

In Ω_o, the function u_∞ a the solution of the extension problem $(\Omega_o\,,\, d(x,\partial\Omega))$. However $x \to d(x,\partial\Omega)$ is one such solution and if there are points in $\mathcal{R}$ not covered by supp$[f]$, it is not of class $C^1(\Omega_o)$. Therefore by Aronsson theorem it cannot be the only one.

As an example consider the case when Ω is the ball B_R centered about the origin. If $0 \in$ supp$[f]$, then $x \to d(x,\partial\Omega)$ is the only extremal for (4.1). If for example

$$f \equiv \begin{cases} 1 \quad , & \frac{R}{2} < \;\; |x| < R \\ 0 \quad , & \qquad |x| < \frac{R}{2}, \end{cases}$$

then each member of the family

$$\varphi_\theta = \begin{cases} R - |x| \quad , & \theta < |x| < R, \quad \theta \in [0, \frac{R}{2}), \\ R - \theta \quad , & 0 < |x| \le \theta \end{cases}$$

is an extremal for (4.1). In particular for $\theta = 0$ we recover the distance function. However, the only extremal constructed as a limit of u_p's is the one corresponding to $\theta = R/2$.

Remark 4.1. The conclusion of the Proposition is independent of the structure of f. Only the geometrical structure of supp$[f]$ is relevant.

Remark 4.2. In the torsion problem of a cylindrical rod (see part IV) $f =$ div(ωx) and ω, the angular twist rate is a positive constant. Since $f(x) = 2\omega$ for all $x \in \Omega$, the extremal problem (4.1) has $u_\infty(x) = d(x,\partial\Omega)$ as the only solution.

It is shown in §3 of part IV that if a moment M parallel to the axis of the rod is applied

$$\int_\Omega u_p dx = \frac{1}{2} M \,.$$

From (4.1) with $f = 2\omega$ it follows that as $p \to \infty$

$$4\omega \int_\Omega d(x,\partial\Omega) dx = M$$

which gives a way of computing the twist rate for the case of a pure plastic rod if a given moment M is applied.

5. The limiting p.d.e.

Consider the family of problems (4.2) where we assume that f is continuous and non-negative in Ω. We let u_∞ be any extremal of (4.1) obtained as limit of some subsequence $\{u_q\}$, $q \in \mathbb{N}$ out of the net $\{u_p\}$. Such an extremal satisfies a partial differential equation which, roughly speaking can be viewed as the Euler equation associated with the functional in (4.1).

PROPOSITION 5.1. *Let $f \in L^\infty(\Omega)$. Then any u_∞ satisfies*

$$|\nabla u_\infty| \leq 1 \qquad \text{in the viscosity sense}. \tag{5.1}$$

Remark 5.1. The proposition continues to hold for rather general f even depending upon ∇u_p (see Lemma 1.1 of Part III).

PROPOSITION 5.2. *Let $f \in L^\infty(\Omega) \cap C(\Omega)$, and $f \geq 0$. Then any u_∞ satisfies*

$$|\nabla u_\infty| = 1 \quad \text{on } [f > 0] \quad \text{in the viscosity sense} \tag{5.2}$$

$$\Delta_\infty u_\infty = 0 \quad \text{on } \overline{[f > 0]}^c \quad \text{in the viscosity sense.} \tag{5.3}$$

Remark 5.2. On the set $\overline{[f > 0]}^c$, u_∞ is also a variational solution of

$$\begin{cases} \Delta_\infty u_\infty = 0 & \text{in } \overline{[f > 0]}^c, \\ u_\infty(x) = d(x, \partial\Omega) & \text{on } \partial\overline{[f > 0]}^c. \end{cases}$$

This follows from the arguments of §2 of Part I.

Remark 5.3. Once we have $|\nabla u_\infty| \leq 1$ a.e., it follows that u_∞ satisfies $|\nabla u_\infty| \leq 1$ in the viscosity sense. However, we prefer to present a different proof emphasizing the approximation of the ∞-Laplacian by the p-Laplacian as $p \to \infty$. The estimates needed to make this idea precise are in Part III, where we prove gradient estimates uniform in p.

PROOF OF PROPOSITION 5.1: Fix $x_o \in \Omega$ and let $v \in C^2_{loc}(\Omega)$ be such that $(u_\infty - v)$ has a local maximum at x_o, so that

$$(u_\infty - v)(x_o) \geq (u_\infty - v)(x) \quad \forall x \in B_R(x_o) \equiv \{|x - x_o| < R\} \quad \text{for some } R \in (0,1).$$

We have to show that

$$|\nabla v(x_o)| \leq 1. \tag{5.4}$$

Let

$$C = \sup_{q \in \mathbf{N}} \max \left\{ \|u_q\|_{\infty, B_R(x_o)} \; ; \; \|v\|_{\infty, B_R(x_o)} \right\}$$

and consider the sequence of functions

$$x \to u_q(x) - v(x) - k|x - x_o|^a, \qquad k = \frac{4C}{R}, \; a > 2.$$

Proceeding as in the proof of Proposition 2.2 of Part I, for $q \in \mathbf{N}$, the functions

$$x \to u_q(x) - v(x) - k|x - x_o|^a$$

attain their maximum at some x_q in the interior of $B_R(x_o)$, and the sequence $\{x_q\}$ has x_o as its only accumulation point. Therefore

$$\nabla u_q(x_q) = \nabla v(x_q) + ka(x_q - x_o)|x_q - x_o|^{a-2}.$$

Since $x_q \to x_o$, there exist some q^* such that $x_q \in B_{\frac{R}{2}}(x_o)$, for all $q \geq q^*$. If (5.4) does not hold, there exists some $\delta \in (0,1)$ such that $|\nabla v(x_o)| \geq 1 + \delta$. By choosing q^* sufficiently large we have

$$|\nabla u_q(x_q)| > 1 + \frac{\delta}{2} - ka|x_q - x_o|^{a-1} \geq 1 + \frac{\delta}{4}, \qquad \forall q \geq q^*. \tag{5.5}$$

By Lemma 1.1 of Part III, for all $x \in B_{\frac{R}{2}}(x_o)$

$$|\nabla u_q(x)| \leq \left(\frac{\gamma}{R^N}\right)^{\frac{1}{q}} \left(\int_{B_R(x_o)} (1 + |\nabla u_q|)^q \, dx \right)^{\frac{1}{q}} \tag{5.6}$$

where γ is a constant independent of q. Next, in the weak formulation of (4.2) take u_p as a testing function to obtain by standard calculations

$$\int_\Omega (1 + |\nabla u_p|)^p \, dx \leq \gamma \|f\|_{\infty, \Omega}$$

where γ depends upon $|\Omega|$ and it is independent of p. This in (5.6) implies that there exists a constant γ depending upon $\|f\|_{\infty\Omega}$, $|\Omega|$ and R, such that

$$|\nabla u_q(x)| \leq \gamma^{\frac{1}{q}}, \qquad \forall x \in B_{\frac{R}{2}}(x_o).$$

For q sufficiently large this contradicts (5.5) and proves (5.4).

PROOF OF PROPOSITION 5.2: It will suffice to show that on the set $[f>0]$

$$|\nabla u_\infty| \geq 1 \qquad \text{in the viscosity sense .}$$

Fix $x_o \in [f>0]$ and let $v \in C^2_{loc}(\Omega)$ be such that $(u_\infty - v)$ has a local *minimum* at x_o, so that

$$(u_\infty - v)(x_o) \leq (u_\infty - v)(x) \quad \forall x \in B_R(x_o) \equiv \{|x-x_o|<R\} \quad \text{for some } R \in (0,1).$$

The radius R can be taken to be so small that $B_R(x_o) \subset [f>0]$. We have to prove that

$$|\nabla v(x_o)| \geq 1 \quad .$$

Introduce the sequence of functions

$$x \longrightarrow u_q(x) - v(x) + k|x-x_o|^a\,, \qquad k = \frac{4C}{R}\,, \quad a > 2$$

where C is defined as before. Then, for all $q \in \mathbf{N}$, the functions

$$x \longrightarrow u_q(x) - v(x) - k|x-x_o|^a$$

attain their minimum at some x_q in the interior of $B_R(x_o)$, and the sequence $\{x_q\}$ converges to x_o. Assume first $u_q \in C^2(B_R(x_o))$. Then (4.2) holds pointwise in $B_R(x_o)$, and we have

$$\begin{aligned} &|\nabla u_q(x_q)|^{q-2}\,\Delta u_q(x_q) + \\ &(q-2)\,|\nabla u_q(x_q)|^{q-4}\,\langle D^2 u_q(x_q)\nabla u_q(x_q)\,,\,\nabla u_q(x_q)\rangle = -f(x_q) \end{aligned} \tag{5.7}$$

where $D^2 u_q(x_q)$ is the hessian matrix of u_q evaluated at x_q. From the minimum property of x_q we have

$$\nabla u_q(x_q) = \nabla v(x_q) - ka(x_q - x_o)|x_q - x_o|^{a-2}. \tag{5.8}$$

Also, in the sense of matrices

$$D^2 u_q(x_q) - D^2 v(x_q) + ka|x_q - x_o|^{a-2}\left[\mathbf{I} + (a-2)\frac{(x_q-x_o)_i(x_q-x_o)_j}{|x_q-x_o|^2}\right] \geq 0. \tag{5.9}$$

In particular

$$\Delta u_q(x_q) - \Delta v(x_q) + ka|x_q - x_o|^{a-2}[N + (a-2)] \geq 0. \tag{5.10}$$

In (5.7) substitute $\nabla u_q(x_q)$ from (5.8) and minorate the terms involving the second derivatives by using (5.9) and (5.10) in a way similar to the proof of Proposition 2.2 of Part I. In the resulting inequality we divide by

$$(q-2)|\nabla u_q(x_q)|^{q-4} = (q-2)|\nabla v(x_q) - ka(x_q - x_o)|x_q - x_o|^{a-2}|^{q-4}$$

to obtain

$$\begin{gathered}
\frac{1}{q-2}\left|\nabla v(x_q) + ak(x_q - x_o)|x_q - x_o|^{a-2}\right|^2 \times \\
(\Delta v(x_q) + a|x_q - x_o|^{a-2}[N + a - 2]) + \\
\langle\left(D^2 v(x_q) - ka|x_q - x_o|^{a-2}\left[\mathbf{I} + (a-2)\frac{(x_q - x_o)_i (x_q - x_o)_j}{|x_q - x_o|^2}\right]\right) \\
(\nabla v(x_q) - ak(x_q - x_o)|x_q - x_o|^{a-2})\ , (\nabla v(x_q) - ak(x_q - x_o)|x_q - x_o|^{a-2})\rangle \leq \\
-\frac{f(x_q)}{(q-2)|\nabla v(x_q) - ka(x_q - x_o)|x_q - x_o|^{a-2}|^{q-4}}.
\end{gathered} \tag{5.11}$$

Letting $q \to \infty$

$$\Delta_\infty v(x_o) \leq -\frac{f(x_o)}{\lim_{q\to\infty}(q-2)|\nabla v(x_q)|^{q-4}}.$$

Therefore if $|\nabla v(x_o)| \leq 1 - \delta$ for some $\delta \in (0,1)$ the right hand side tends to infinity contradicting the fact that $v \in C^2(B_R(x_o))$.

If u_q is not $C^2(B_R(x_o))$ we approximate it with the solutions of

$$\begin{cases} \operatorname{div}\left((|\nabla u_{q,\varepsilon}|^2 + \varepsilon)^{\frac{q-2}{2}}\ \nabla u_{q,\varepsilon}\right) = -f & \text{in } B_{2R}(x_o), \\ u_{q,\varepsilon} = u_q & \text{on } \partial B_{2R}(x_o). \end{cases}$$

For $q \in \mathbf{N}$ fixed we work with $u_{q,\varepsilon}$ since $u_{q,\varepsilon} \in C^2(B_R(x_o))$ and $u_{q,\varepsilon} \to u_q$ is in $C^{1,\alpha}(\bar{B}_R(x_o))$, for some $\alpha \in (0,1)$ depending upon q. We arrive to an expression similar to (5.11) where

$$|\nabla v(x_q) - ka(x_q - x_o)|x_q - x_o|^{a-2}|^2$$

is replaced by

$$|\nabla v(x_{q_\varepsilon}) - ka(x_{q_\varepsilon} - x_o)|x_{q_\varepsilon} - x_o|^{a-2}|^2 + \varepsilon$$

for a sequence of points $\{x_{q_\varepsilon}\} \to x_q$. We then let $\varepsilon \to 0$ and then let $q \to \infty$ to obtain the result.

The proof of (5.3) is identical to that of Proposition 2.2 of Part I.

Remark 5.3. The limiting equations (5.2)-(5.3) are independent of f. They only depend on supp$[f]$.

The identification of $u_\infty(\cdot)$ as $d(\cdot, \partial\Omega)$ holds only on the set $[f > 0]$. We conclude this section by observing that in general the two equations (5.2) and (5.3) cannot be replaced by (5.3) alone, as shown by the following

Counterexample.

Let Ω be the disc $\{|x| < 1\}$. Then $u_\infty(x) = 1 - |x|$ is *not* a viscosity solution of $\Delta_\infty u = 0$ in Ω.

Let $x \to v(x) = 1 + \mathbf{a} \cdot x + b|x|^2$ where $\mathbf{a} \in \mathbf{R}^N$ and $b \in \mathbf{R}$. We have $u_\infty(0) = v(0) = 1$, and

$$u_\infty(x) \leq 1 + \mathbf{a} \cdot x + b|x|^2\,, \qquad \forall |x| < \frac{1}{2}$$

if $\mathbf{a}$ and b are chosen to satisfy $|\mathbf{a}| < \frac{1}{2}$, $|b| < \frac{1}{2}$. Therefore $(u_\infty - v)$ has a local maximum at $x = 0$, within the ball $B_{\frac{1}{2}}(0)$. By direct calculation

$$v_{x_i} = a_i + 2bx_i\,, \qquad v_{x_i x_j} = 2b\delta_{ij}$$

and

$$\begin{aligned}\Delta_\infty v = v_{x_i} v_{x_j} v_{x_i x_j} &= (a_i + 2bx_i)(a_j + 2bx_j)2b\delta_{ij} \\ &= 2b|\mathbf{a} + bx|^2.\end{aligned}$$

For $x = 0$, $\Delta_\infty v = 2b|\mathbf{a}|^2$ and the assertion follows from the fact that the signum of b is arbitrary provided $|b| < \frac{1}{2}$.

Part III. Gradient estimates uniform in p

1. Interior estimates

Let $u_p \in W^{1,p}_{loc}(\Omega)$ be a local weak solution of

$$-\mathrm{div}|\nabla u_p|^{p-2}\nabla u_p = f(x,u,\nabla u_p), \qquad p > 2, \tag{1.1}$$

where $f : \Omega \times \mathbf{R}^{N+1} \to \mathbf{R}$ satisfies

$$|f(x,u_p,\nabla u_p)| \leq \gamma\left(1 + |\nabla u_p|^{p-1}\right), \quad \text{a.e. } x \in \Omega, \tag{1.2}$$

for a given positive constant γ independent of p. We will follow the standard convention of denoting by γ a generic positive constant, independent of p, whose value may change from formula to formula.

We will establish interior gradient estimates for the family $\{u_p\}$,$p > 2$ uniform with respect to the parameter p.

PROPOSITION 1.1. *There exists a constant γ depending only upon N and independent of p, such that for every compact subset $\mathcal{K}$ of Ω*

$$\|\nabla u_p\|_{\infty,\mathcal{K}} \leq \left(\frac{\gamma}{L^N}\right)^{\frac{1}{p}} \left(\int_\Omega (1 + |\nabla u_p|)^p\, dx\right)^{\frac{1}{p}}, \qquad \textit{where} \tag{1.3}$$

$$L = \min\{diam(\mathcal{K})\,;\, dist(\partial\Omega, \mathcal{K})\}.$$

For $R > 0$ let $B_R(x_o) \equiv \{|x - x_o| < R\}$ denote the ball of radius R about x_o. If $x_o \equiv 0$ we let $B_R(0) \equiv B_R$. Without loss of generality we may take $R \leq 1$.

The proof of Proposition 1.1 rests upon the following

LEMMA 1.1. *Let $u \in W^{1,p}_{loc}(\Omega)$ be a local weak solution of (1.1) and assume (1.2) holds. There exists a constant γ depending only upon N and independent of p such that*

$$\forall x_o \in \Omega\,,\ \forall \rho \leq \frac{R}{2} \quad \textit{such that} \quad B_R(x_o) \subset \Omega \tag{1.6}$$

$$\|\nabla u\|_{\infty,B_\rho(x_o)} \leq \gamma^{\frac{1}{p}} R^{-\frac{N}{p}} \left(\int_{B_R(x_o)} (1 + |\nabla u|)^p\, dx\right)^{\frac{1}{p}}.$$

Remark 1.1. It is known that $x \to \nabla u$ is locally Hölder continuous in Ω (see [**Mn,Di**]). The point of the Lemma is to prove that the constant γ in (1.6) is independent of p.

Remark 1.2. In view of the results of §2, Part II it is impossible to obtain $C^{1,\alpha}_{loc}(\Omega)$ estimates uniform in p.

Remark 1.3. The proof shows that (1.3) is also true for systems, i.e. when u and f take values in $\mathbf{R}^m$, $m \in \mathbf{N}$.

2. Estimates up to the boundary

Gradient estimates up to the boundary $\partial\Omega$ will be derived under the assumption that Ω is convex and that the function f on the right hand side of (1.1) is bounded. Let $u_p \in W^{1,p}_o(\Omega)$ be a weak solution of

$$-\mathrm{div}|\nabla u_p|^{p-2}\nabla u_p = f \in L^\infty(\Omega) \tag{2.1}$$

and assume Ω is convex.

PROPOSITION 2.1. *There exists a constant γ depending only upon N, $\|f\|_{\infty,\Omega}$, and independent of p, such that*

$$\|\nabla u_p\|_{\infty,\Omega} \le \gamma \left(diam[\Omega]^{-N} \int_\Omega (1 + |\nabla u_p|)^p \, dx \right)^{\frac{1}{p}}. \tag{2.2}$$

Remark 2.1. A gradient estimate of up to the boundary was derived by Payne and Philippin [**PP**], for convex domains Ω with the additional assumption that the boundary $\partial\Omega$ is of class C^2.

The proof of (2.2) is based on the following

LEMMA 2.1. *There exists constant γ depending only upon N and $\|f\|_{\infty,\Omega}$ such that for all $P \in \Omega$*

$$|u_p(P)| \le \gamma dist[P\,;\,\partial\Omega] \left(diam[\Omega]^{-N} \int_\Omega (1 + |\nabla u_p|)^p \, dx \right)^{\frac{1}{p}}. \tag{2.3}$$

Proof of Proposition 2.1: Fix $P \in \Omega$, let P_o be its projection on $\partial\Omega$ and construct the sphere $B_r(P)$ of center P and radius $r = \frac{1}{2}|P - P_o|$. By Lemma 1.1

$$(2.4) \qquad |\nabla u_p|(P) \leq \gamma \left(r^{-N} \int_{B_{\frac{r}{2}}(P)} (1 + |\nabla u_p|)^p \, dx \right)^{\frac{1}{p}} .$$

Next in the weak formulation of (2.1) take the testing function $u_p \zeta^p$ where $x \to \zeta(x)$ is a piecewise smooth cutoff function in $B_r(P)$ that equals one on $B_{r/2}(P)$ and such that $|\nabla\zeta| < \frac{2}{r}$. Standard calculations give

$$(2.5) \qquad \int_{B_{r/2}(P)} |\nabla u_p|^p dx \leq \frac{\gamma}{r^p} \int_{B_r(P)} |u_p|^p + \gamma \|f\|_{\infty,\Omega} \int_{B_r(P)} |u_p| dx.$$

By Lemma 2.1 for all $x \in B_r(P)$

$$|u_p(x)| \leq \gamma r \left(\text{diam}[\Omega]^{-N} \int_\Omega (1 + |\nabla u_p|)^p \, dx \right)^{\frac{1}{p}} .$$

Combining this with (2.5) and (2.4) proves the Proposition.

3. Proof of Lemma 1.1

Fix $x_o \in \Omega$ and $R > 0$ such that $B_R(x_o) \subset \Omega$. Without loss of generality we may assume $x_o \equiv 0$. For $\sigma \in (0,1)$ consider the concentric balls $B_{(1-\sigma)R}$ and let $x \to \zeta(x)$ be a piecewise smooth cutoff function in B_R such that

$$\begin{cases} \zeta(x) = 0, & x \in \partial B_R \\ \zeta(x) = 1 & x \in B_{(1-\sigma)R} \\ |\nabla\zeta| \leq (\sigma R)^{-1} . \end{cases}$$

The calculations below are somewhat formal and can be justified by a difference-quotient argument as indicated in [Uhl]. From (1.1) by differentiation (3.1)

$$-\text{div}\left\{ |\nabla u|^{p-2} \nabla u_{x_j} + (p-2)|\nabla u|^{p-4} u_{x_k} u_{x_k x_j} \nabla u \right\} = \frac{\partial}{\partial x_j} f , \quad j = 1, 2, \ldots, N.$$

In the weak formulation of (3.1) take the testing functions

$$\varphi = u_{x_j} v\zeta^2 , \qquad v = \left(|\nabla u|^2 - h\right)_+ \equiv \max\left\{\left(|\nabla u|^2 - h\right)_+ ; 0\right\},$$

where $h \geq 1$ has to be chosen. Integrating over B_R and adding over $j = 1, 2, \ldots, N$ we obtain by standard calculations

$$\begin{aligned} &\int_{B_R} \left\{ \frac{1}{2} |\nabla u|^{p-2} |\nabla v|^2 + \sum_{j=1}^{N} |\nabla u|^{p-2} v \left|\nabla u_{x_j}\right|^2 \right\} \zeta^2 dx + \\ &\frac{p-2}{2} \int_{B_R} \left\{ \frac{1}{2} |\nabla u|^{p-4} |\nabla v|^2 v + |\nabla u|^{p-4} |\nabla v \cdot \nabla u|^2 \right\} \zeta^2 dx = \\ &- \int_{B_R} \left\{ |\nabla u|^{p-2} \nabla v v + (p-2) |\nabla u|^{p-4} (\nabla v \cdot \nabla u) \nabla u v \right\} \zeta \nabla \zeta dx \\ &- \int_{B_R} f(x, u, \nabla u) \frac{\partial}{\partial x_j} \left(u_{x_j} v\right) dx. \end{aligned} \tag{3.2}$$

We observe that $|\nabla u|^{p-2} v \leq |\nabla u|^{p+2}$ and set

$$A_h \equiv [v > h] \cap B_R ; \qquad z \equiv |\nabla u|^{\frac{p+2}{2}} . \tag{3.3}$$

On the set A_h , since $h \geq 1$

$$|f(x, u, \nabla u)| \leq \gamma |\nabla u|^{p-1} \qquad \text{a.e. } A_h .$$

Then the first integral on the right hand side of (3.2) is majorized by

$$\frac{1}{8} \int_{B_R} |\nabla u|^{p-2} |\nabla v|^2 \zeta^2 dx + \frac{p-2}{8} \int_{B_R} |\nabla u|^{p-4} |\nabla v \cdot \nabla u|^2 \zeta^2 dx + \gamma p \frac{1}{(\sigma R)^2} \int_{A_h} z^2 dx.$$

The integral involving f_o is estimated by

$$\frac{1}{2} \int_{B_R} \sum_{j=1}^{N} |\nabla u|^{p-2} \left|\nabla u_{x_j}\right|^2 v\zeta^2 dx + \gamma p \frac{1}{(\sigma R)^2} \int_{A_h} z^2 dx.$$

These estimates in (3.2) give

$$\int_{B_R} |\nabla u|^{p-2} |\nabla v|^2 \zeta^2 dx \leq \frac{\gamma p}{(\sigma R)^2} \int_{A_h} z^2 dx \tag{3.4}$$

for a constant γ independent of p and h.

Since $|\nabla u|^{p-2}|\nabla v|^2 = \left(\frac{2}{p+2}\right)^2 |\nabla z|^2$ and $h \geq 1$ is arbitrary, we rewrite (3.4) as

$$\int_{A_k} |\nabla [(z-k)_+\zeta]|^2 \, dx \leq \frac{\gamma p^3}{(\sigma R)^2} \int_{A_k} z^2 dx \tag{3.5}$$

where $k \geq 1$ is arbitrary and

$$A_k \equiv [z > k] \cap B_R .$$

Inequality (3.5) holds true for every pair of balls B_R , $B_{(1-\sigma)R}$ for all $\sigma \in (0,1)$ and for all $k \geq 1$.

Let $\tau \in (0,1)$ be fixed and consider the sequence of radii

$$\rho_n = (1-\tau)R + \frac{\tau}{2^n}R, \quad \tilde{\rho}_n = \frac{\rho_n + \rho_{n+1}}{2} = (1-\tau)R + \frac{3\tau}{2^{n+2}}R, \quad n = 0, 1, \ldots$$

and the corresponding sequences of nested balls

$$B_n = B_{\rho_n} , \quad \tilde{B}_n = B_{\tilde{\rho}_n} , \qquad B_{n+1} \subset \tilde{B}_n \subset B_n .$$

Consider also the sequence of increasing levels

$$k_n = \left(1 - 2^{-(n+1)}\right) k_o , \qquad n = 0, 1, \ldots ,$$

where $k_o \geq 2$ has to be chosen, and set

$$A_n \equiv [z > k_n] \cap B_n ; \qquad \tilde{A}_n \equiv [z > k_n] \cap \tilde{B}_n .$$

Write (3.5) for $\tilde{k}_n = \frac{1}{2}(k_n + k_{n+1})$ and for the pair of balls B_n , $\tilde{B}_n$, for which $\sigma = 2^{n+2}/\tau$. In such a case ζ is a cutoff function in $\tilde{B}_n$ that equals one on B_{n+1} and such that we have the bound $|\nabla\zeta| \leq 2^{n+1}/\tau R$.

Since $\{k_n\}$ is increasing, the integral on the right hand side is majorized by

$$\int_{A_{\tilde{k}_n}} z^2 dx \leq \gamma \int_{A_{\tilde{k}_n}} \left[(z-\tilde{k})_+^2 + \tilde{k}_n^2\right] dx \leq \gamma \int_{A_{\tilde{k}_n}} (z-k_n)_+^2 dx + \gamma k_o^2 \left|A_{\tilde{k}_n}\right| .$$

Also

$$\int_{A_n} (z-k_n)_+^2 \, dx \geq \int_{A_{\tilde{k}_n}} (z-k_n)_+^2 \, dx \geq \left(\tilde{k}_n - k_n\right)^2 \left|A_{\tilde{k}_n}\right| \geq 2^{-2n} k_o^2 \left|A_{\tilde{k}_n}\right| . \tag{3.6}$$

Combining these remarks in (3.5) we deduce the recursive inequalities

$$\int_{\tilde{B}_n} \left|\nabla\left[(z-\tilde{k}_n)_+\zeta\right]\right|^2 dx \le \frac{4^n\gamma p^3}{R^2}\int_{A_n}(z-k_n)_+^2\,dx. \tag{3.7}$$

In what follows we assume $N>2$. If $N=2$ the arguments are similar modulo minor modifications. By the Sobolev embedding theorem

$$\begin{aligned}\int_{B_{n+1}}(z-k_{n+1})_+^2\,dx &\le \int_{\tilde{B}_n}\left[\left(z-\tilde{k}_n\right)_+\zeta\right]^2 dx \\ &\le \left(\int_{\tilde{B}_n}\left|\nabla\left[\left(z-\tilde{k}_n\right)_+\zeta\right]\right|^2\right)\left|[z>\tilde{k}_n]\cap B_n\right|^{\frac{2}{N}}.\end{aligned} \tag{3.8}$$

Arguing as in (3.6)

$$\left|[z>\tilde{k}_n]\cap B_n\right|^{\frac{2}{N}} \le \gamma 2^{\frac{4n}{N}}k_o^{-\frac{4}{N}}\left[\int_{B_n}(z-k_n)_+^2\right]^{\frac{2}{N}}.$$

Combining this with (3.7) and (3.8) we deduce that there exist constants γ, $b>1$ such that

$$\int_{B_{n+1}}(z-k_{n+1})_+^2\,dx \le \frac{\gamma p b^n}{(\tau T)^2}k_o^{-\frac{4}{N}}\left(\int_{B_n}(z-k_n)_+^2\,dx\right)^{1+\frac{2}{N}}. \tag{3.9}$$

It follows from Lemma 5.6 of **[LSU]** p.95, that there exists a constant γ independent of p such that if k_o is chosen to satisfy

$$k_o = \gamma\max\left\{\left(\frac{p}{\tau R}\right)^N\int_{B_o}z^2dx\,;\,1\right\}$$

then

$$\int_{B_{n+1}}(z-k_n)_+^2\,dx \longrightarrow 0 \text{ as } n\to\infty, \text{ i.e. } \|z\|_{\infty,B_\infty}\le k_o.$$

Recalling the definitions of B_o, B_∞, and z we have

$$\|\nabla u\|_{\infty,B_{(1-\tau)R}}^{\frac{p+2}{2}} \le \frac{\gamma p^N}{(\tau R)^{\frac{N}{2}}}\left(\int_{B_R}(1+|\nabla u|)^{p+2}\,dx\right)^{\frac{1}{2}},$$

and

(3.10) $\forall R>0$ such that $B_R \subset \Omega$, $\forall \tau \in (0,1)$

$$\|\nabla u\|_{\infty, B_{(1-\tau)R}} \le \frac{\gamma^{\frac{1}{p}}}{(\tau R)^{\frac{N}{p+2}}} \left(\int_{B_R} (1+|\nabla u|)^{p+2}\, dx \right)^{\frac{1}{p+2}},$$

for a constant γ independent of p. The proof is now concluded by means of an interpolation process as follows. Consider the increasing sequence of radii

$$R_s = \frac{1}{2} R \sum_{i=0}^{s} 2^{-i}, \qquad s = 0,1,2,\ldots$$

and set

$$F_s \equiv \|(1+|\nabla u|)\|_{\infty, B_{R_s}}; \qquad G_o \equiv \left(\int_{B_R} (1+|\nabla u|)^p\, dx \right)^{\frac{1}{p}}.$$

Since $B_{R_o} \equiv B_{R/2}$, to prove the Lemma we have to show that there exist a constant γ independent of p such that

(3.11)
$$F_o \le \gamma G_o.$$

Write (1.17) over the pair of balls $B_s \subset B_{s+1}$ for which $\tau = 2^{-(s+1)}$. We obtain

$$\forall s = 0,1,2,\ldots \text{ and } \gamma \text{ independent of } p$$
$$F_s \le \gamma^{\frac{1}{p}} 2^{\frac{Ns}{p+2}} F_{s+1}^{\frac{2}{p+2}} G_o^{\frac{p}{p+2}}.$$

Let $\delta \in (0,1)$ to be chosen. Then by Young inequality

$$F_s \le \delta F_{s+1} + \gamma^{\frac{1}{p}} \delta^{-\frac{2}{p}} G_0 2^{\frac{Ns}{p}}.$$

Iterate these inequalities for $s = 0,1,2,\ldots$ to obtain

$$F_o \le \delta^s F_s + \left(\frac{\gamma}{\delta^2}\right)^{\frac{1}{p}} G_o \sum_{i=1}^{s} (2^{\frac{N}{p}} \delta)^i.$$

Now (3.11) follows by chosing $\delta = 2^{-(N+1)}$ and by letting $s \to \infty$.

4. Proof of Lemma 2.1

Fix $P \in \Omega$ let P_o be its projection on $\partial\Omega$ and let $r = |P - P_o|$. Set also

$$d = \mathrm{diam}[\Omega].$$

After a translation and a rotation of coordinates we may assume $P_o \equiv 0$ and $P \equiv (0, 0, \ldots, r)$. Construct the ball B_d centered at 0 and radius d. Since Ω is convex, it is contained in

$$B_d^+ \equiv B_d \cap \{x_N > 0\}.$$

Let $v \in W_o^{1,p}(B_d^+)$ be the unique solution of

$$\begin{cases} -\mathrm{div}|\nabla v|^{p-2}\nabla v = f^+ & \text{in } B_d^+ \\ v = 0 & \text{on } \partial B_d^+, \end{cases} \tag{4.1}$$

where $f^+ \equiv \max\{f\,;\,0\}$. By the comparison principle $v \geq 0$. Also since $v = 0$ On $x_N = 0$, the reflected function

$$\tilde{v} = \begin{cases} v(\bar{x}, x_N) & x_N \geq o \\ -v(\bar{x}, -x_N) & x_N < o, \end{cases} \qquad \bar{x} \equiv (x_1, x_2, \ldots, x_{N-1})$$

solves

$$\begin{cases} -\mathrm{div}|\nabla \tilde{v}|^{p-2}\nabla \tilde{v} = \tilde{f}^+ & \text{in } B_d \\ \tilde{v} = 0 & \text{on } \partial B_d, \end{cases}$$

By Lemma 1.1

$$\|\nabla v\|_{\infty, B^+_{\frac{d}{2}}} \leq \gamma \left(d^{-N} \int_{B_d^+} (1 + |\nabla v|)^p \, dx \right)^{\frac{1}{p}} \tag{4.2}$$

for a constant γ independent of p. Since $u = 0$ on $\partial\Omega$, $u^+ \equiv \max\{u\,;\,0\}$ is a weak subsolution of

$$\begin{cases} -\mathrm{div}|\nabla u^+|^{p-2}\nabla u^+ \leq f^+ & \text{in } B_d^+ \\ u^+ = 0 & \text{on } \partial B_d^+. \end{cases}$$

This implies the inequalities

$$u^+ \leq v \qquad \text{in } B_d^+ \quad \text{and} \tag{4.3}$$

$$\int_{B_d^+} |\nabla v|^p dx \leq \gamma \left(1 + \int_\Omega |\nabla u|^p dx \right). \tag{4.4}$$

for a constant γ dependent upon $\|f\|_{\infty,\Omega}$ and d^N but independent of p.

Inequality (4.3) follows from the comparison principle. To prove (4.4) in the weak formulation of (4.1) take the testing function $(v-u) \in W_o^{1,p}(B_d^+)$.

From (4.3), (4.4) and (4.2) it follows that

$$
\begin{aligned}
u^+(P) &\le v(P) - v(P_o) \le |P - P_o| \|\nabla v\|_{\infty, B^+_{d/2}} \\
&\le \gamma |P - P_o| \left(d^{-N} \int_\Omega (1 + |\nabla u|)^p \, dx \right)^{\frac{1}{p}} .
\end{aligned}
$$

Repeating the same argument for u^- proves the Lemma.

Part IV. The physical problem

1. The creep law

Let D be a domain in $\mathbf{R}^3$ with smooth boundary ∂D and consider an elastoplastic incompressible body subject to a steady state deformation starting from the reference configuration D. If we let $\mathbf{v} : D \to \mathbf{R}^3$ denote the deformation velocity and assume the deformation is small, the (infinitesimal) strain-rate is the symmetric tensor

$$\eta = \frac{1}{2}\left(\nabla \mathbf{v} + (\nabla \mathbf{v})^t\right) \tag{1.1}$$

where $(\nabla \mathbf{v})^t$ denotes the transpose of the matrix $\nabla \mathbf{v}$. The incompressibility condition div $\mathbf{v} = 0$ yields

$$\text{trace } \eta = 0. \tag{1.2}$$

We let τ be the stress tensor, denote with τ_{ij} $i, j = 1, 2, 3$ its components and let

$$\sigma \equiv \sum_{i=1}^{3} \tau_{ii}$$

be its trace. Similarly we denote with η_{ij} the components of η.

If $\mathbf{b} \equiv (b_1, b_2, b_3) : D \to \mathbf{R}^3$ is the distribution of body forces then the stress tensor τ must satisfy the equilibrium conditions

$$\operatorname{div} \tau = \mathbf{b}. \tag{1.3}$$

We introduce a new tensor whose trace (mean pressure) is normalized to zero, i.e.

$$\tau_o = \tau - \frac{1}{3}\sigma \mathbf{I} \qquad \text{(the stress deviator)}$$

where $\mathbf{I}$ is the 3×3 identity matrix. The norm $T = \sqrt{\text{trace}(\tau_o \tau_o^t)}$ of τ_o is called the *tangential stress intensity* and in terms of τ is given by

$$T = \frac{1}{\sqrt{3}} \sqrt{(\tau_{11} - \tau_{22})^2 + (\tau_{22} - \tau_{33})^2 + (\tau_{33} - \tau_{11})^2 + 3\sum_{i \neq j} \tau_{ij}^2}.$$

A material exhibits a creep when subject to a sufficiently large load or torsion moment, for an extended period of time and at sufficiently high temperature, it deforms permanently (see Kachanov [**Ka1**]). The temperature is assumed to be constant and the process is so slow that the strain–rate η can be taken to be time–independent. A material creep is modelled by a relation between strain–rate and stress of the form

$$\eta_{ij} = f(T)(\tau_{ij} - \frac{1}{3}\sigma\delta_{ij}); \qquad i,j = 1,2,3 \tag{1.4}$$

where δ_{ij} is the Kronecker symbol. The form of $f(T)$ is determined experimentally and is given (up to a constant normalized to one) by

$$f(T) = T^{p-2}, \qquad p > 2. \tag{1.5}$$

We refer to [**Ka1**] for further comments on the power law (1.5).

It follows from (1.4) that

$$T = |\eta|^{\frac{1}{p-1}} \qquad \text{and} \quad \tau_{ij} = |\eta|^{-\frac{p-2}{p-1}}\eta_{ij} + \frac{1}{3}\sigma\delta_{ij}.$$

Therefore the equilibrium conditions (1.3) and the incompressibility (1.2) give the system of equations

$$\begin{cases} \operatorname{div}\left(|\eta|^{q-2}\eta\right) = \mathrm{b} - \frac{1}{3}\nabla\sigma, \\ \operatorname{trace}\, \eta = 0, \qquad q = \frac{p}{p-1}. \end{cases} \tag{1.6}$$

The system (1.6) has four equations and the unknowns are σ and the three components $v_i\,, i = 1,2,3$ of the velocity $\mathbf{v}$.

Remark 1.1. If we were to assume $\sigma = 0$ in (1.6) then the system would be overdetermined. We will comment on such occurrence later.

2. Torsional creeps

Suppose a prismatic elastoplastic rod of length ℓ and cross section Ω is under steady torsion. Here Ω is a simply connected domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$ and the reference configuration is $D = \Omega_\ell \equiv \Omega \times (-\ell, 0)$. The vertical axis, of unit vector $\mathbf{k}$, coincides with the axis about which the torsion takes place.

Points in Ω are denoted with $x \equiv (x_1, x_2)$ and points in D with (x, x_3). Assume that $x \in \Omega \times \{0\}$ at $x_3 = -\ell$, has a steady angular displacement $\alpha(x)\mathbf{k}$ along the x_3 axis. We let $\omega(x) = \alpha(x)/\ell$ be the angular twist rate per unit length, so that for all $x_3 \in (-\ell, 0)$ the angular displacement of (x, x_3) is $x_3\omega(x)\mathbf{k}$, and the deformation velocity at $(x, x_3) \in D$ has the first two components $v_1 = -\omega x_2 x_3$; $v_2 = \omega x_1 x_3$.

The cross sections of the rod rotate as a solid body but do not remain plane. The vertical component of the displacement rate is given by $v_3 = \omega\psi(x)$. The function ψ is called the warping function and it is unknown.

Thus the displacement rate function is given by

$$\mathbf{v}(x, x_3) = \Big(-x_2 x_3, x_1 x_3, \psi(x)\Big)\omega(x) \tag{2.1}$$

and the infinitesimal strain-rate takes the form

$$\eta = \begin{pmatrix} -\omega_{x_1} x_2 x_3 & \frac{1}{2}x_3\left(x_1\omega_{x_1} - x_2\omega_{x_2}\right) & \frac{1}{2}\left(-\omega x_2 + (\omega\psi)_{x_1}\right) \\ \frac{1}{2}x_3\left(x_1\omega_{x_1} - x_2\omega_{x_2}\right) & \omega_{x_1} x_2 x_3 & \frac{1}{2}\left(\omega x_1 + (\omega\psi)_{x_2}\right) \\ \frac{1}{2}\left(-\omega x_2 + (\omega\psi)_{x_1}\right) & \frac{1}{2}\left(\omega x_1 + (\omega\psi)_{x_2}\right) & 0 \end{pmatrix}. \tag{2.2}$$

Since trace$(\eta) = 0$ we have $-\omega_{x_1} x_2 + \omega_{x_2} x_1 = 0$,i.e.

(2.3) $\qquad \omega \qquad$ must be assigned to be a radial function.

It follows from (1.4) that

$$\begin{cases} f(T)\tau_{13} = \frac{1}{2}\big((\omega\psi)_{x_1} - \omega x_2\big), \\ f(T)\tau_{23} = \frac{1}{2}\big((\omega\psi)_{x_2} + \omega x_1\big). \end{cases} \tag{2.4}$$

Moreover

$$[f(T)T]^2 = \frac{1}{4}x_3^2(\rho\omega_\rho)^2 + (\eta_{13}^2 + \eta_{23}^2); \qquad \rho = |x|. \tag{2.5}$$

The basic assumptions we make are

(i)- The body forces are negligible, i.e. $\mathbf{b} = 0$

(ii)- The rod is so long that τ_{33} is independent of x_3 .

Remark 2.1. To our knowledge creep models assume apriori that ω is constant troughout Ω, (see [**Ka1, FLO, Ra**]). We will show that in fact this is a consequence of the physical assumptions (i)-(ii) above.

It follows from (2.2), (1.4) and (1.2) that $\tau_{33} = \frac{1}{3}\sigma$ and therefore in view of (ii), σ is independent of x_3. We then have

LEMMA 2.1. *Assume σ is independent of x_3 and let ω be a radial Lipschitz function in Ω. Then a necessary condition for (1.6) to have a solution of the form (2.2) is that ω and σ are constant.*

We assume the result of the Lemma for the moment and postpone the proof until §4. The third equilibrium equation of (1.3), $\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\tau_{i3} = 0$, yields

$$\frac{\partial}{\partial x_1}\tau_{13} + \frac{\partial}{\partial x_2}\tau_{23} = 0\,; \qquad \forall x_3 \in (-\ell, 0).$$

Let us introduce a potential $u_p : \Omega \to \mathbf{R}$ satisfying

$$\frac{\partial}{\partial x_1}u_p = -\tau_{23}; \qquad \frac{\partial}{\partial x_2}u_p = \tau_{13}. \tag{2.6}$$

Since the boundary $\partial\Omega$ is in equilibrium, if $\nu \equiv (\nu_1, \nu_2)$ is the outward unit normal to $\partial\Omega$, $\sum_{i=1}^{3} \tau_{i3}\nu_i = 0$. Therefore u_p is constant on $\partial\Omega$, for all $x_3 \in (-\ell, 0)$ and after renormalization we may assume $u_p = 0$ there.

Computing $\frac{\partial}{\partial x_2}\Big(f(T)\tau_{13}\Big)$, $\frac{\partial}{\partial x_1}\Big(f(T)\tau_{23}\Big)$ from (2.4) and subtracting we arrive at

$$-\mathrm{div}(f(T)\nabla u_p) = \mathrm{div}(x\omega). \tag{2.7}$$

Computing the quantity $\eta_{13}^2 + \eta_{23}^2$ from (1.4) and (2.6) we find

$$\begin{aligned} \eta_{13}^2 + \eta_{23}^2 &= f(T)^2|\nabla u_p|^2 \\ &= \frac{1}{4}\{\omega^2\rho^2 + |\nabla(\omega\psi)|^2 + 2\omega x_1(\omega\psi)_{x_2} + 2\omega x_2(\omega\psi)_{x_1}\}. \end{aligned} \tag{2.8}$$

Moreover from (2.5)

$$|\nabla u_p| = T.$$

Therefore (2.7) yields the boundary value problem

$$(2.9) \qquad \begin{cases} -\text{div}(|\nabla u_p|^{p-2}\nabla u_p) = \text{div}(x\omega); \text{ in } \Omega \\ u_p = 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Finally the warping function ψ is recovered from the first two equations in (2.4). Namely

$$(2.10) \qquad \begin{cases} \frac{1}{2}\Delta(\omega\psi) = \frac{\partial}{\partial x_2}\Big(f(T)\tau_{13}\Big) + \frac{\partial}{\partial x_1}\Big(f(T)\tau_{23}\Big) \\ \nabla(\omega\psi)\cdot\nu = \omega(x_2, -x_1)\cdot\nu \qquad \text{on } \partial\Omega. \end{cases}$$

Next we will comment on the physical meaning of letting $p \to \infty$.

3. The ideal plastic case and the limit as $p \to \infty$

Experimental data as well as explicit solutions of (2.5) in cases of torsion in circular bars, bending, spheres etc.. (see Kachanov [**Ka1**] and §5 here) indicate that the pure ideal plastic situation is best described when $p \to \infty$.

As observed in §1, $T = |\eta|^{\frac{1}{p-1}}$. Thus proceeding formally $T \to 1$ as $p \to \infty$. Assuming $\sigma = 0$ this results in $|\nabla u| = 1$, where u is the limit of u_p. We will make this statement precise later.

The Von Mises, Tresca-Saint Venant yield-criteria for ideal plasticity of a homogeneous isotropic material ([**Ka 2**]) requires the *tangential stress intensity* T to be constant, i.e.

$$\sqrt{\tau_{13}^2 + \tau_{23}^2} = \gamma$$

where γ is the *minimum plastic yield*. The distribution of stresses is found by letting $|\nabla u| = T$ and solving (in some sense)

$$(3.1) \qquad \begin{cases} |\nabla u| = \text{ const}, \qquad \text{in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Therefore as $p \to \infty$ we expect u_p to converge (See part II) to a solution u_∞ of (3.1).

To further characterize, from a physical point of view such "expected" limiting function u, let us compute the torsional moment M of the distribution of forces (τ_{13}, τ_{23}) in Ω. We have

$$
\begin{aligned}
M &= \int_\Omega (x_1\tau_{23} - x_2\tau_{13})dx = -\int_\Omega \nabla u \cdot x dx \\
&= \int_\Omega (2u - div(xu))dx = 2\int_\Omega u dx
\end{aligned}
$$

since $u = 0$ on $\partial\Omega$.

Among the solutions of (3.1) we choose the one that maximizes M. Let $\gamma > 0$ be a given constant. It follows from a result of Aronsson [**Ar3**] that among all the functions $\varphi \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ vanishing on $\partial\Omega$ and such that $|\nabla\varphi| = \gamma$, *a.e.* Ω, there exist a unique one u_γ that maximizes the torsional moment $M_\gamma = 2\int_\Omega u_\gamma dx$. Normalizing $\gamma = 1$, Aronsson result implies that the stress potential $u(x)$ in ideally plastic torsion equals $d(x, \partial\Omega)$.

Suppose now that a torsional moment M is given. Set

$$
\int_\Omega u dx = \frac{M}{2} \tag{3.2}
$$

and minimize $\|\nabla u\|_{\infty,\Omega}$ among all the Lipschitz functions in Ω vanishing on $\partial\Omega$. According to Aronsson [**Ar1**] this problem has a unique solution u satisfying $|\nabla u(x)| = const$ a.e. Ω. The problem of finding u with constant $|\nabla u|$ and satisfying (3.2) can be thought of as that of finding the "minimum plastic yield" $\gamma = |\nabla u|$ *that would support the given torsional momentum* M.

From the results of Part II it follows that as $p \to \infty$, $\lim_{p\to\infty} u_p(x) = d(x, \partial\Omega)$ in any C^α-norm, for all $\alpha \in (0,1)$. This gives a mathematical justification of the physical evidence suggesting that the distribution of stresses of a steady state torsional creep approaches the distribution of stresses of an ideal plastic torsion as the exponent in the creep power law (1.5) tends to ∞. It also gives a constructive way of calculating the minimizers of (3.1) and its converse, whose existence is established in [**Ar1**].

4. Proof of Lemma 2.1

We assume we have a smooth solution of (1.6), satisfying in addition

$$|\eta| > \delta \quad \text{in } \Omega \quad \text{for some arbitrary } \delta \in (0,1)$$

so that the first three equations of the system (1.6) can be rewritten as

$$|\eta|^2 \mathrm{div}\eta - \frac{p-2}{2(p-1)}\eta \cdot \nabla|\eta|^2 = \frac{1}{3|\eta|^{p-4}}\nabla\sigma. \tag{4.1}$$

To simplify the calculations we write $\eta = x_3\eta_o + \eta_1$ where

$$\eta_o = \begin{pmatrix} -\omega_{x_1}x_2 & \frac{1}{2}(\omega_{x_1}x_1 - \omega_{x_2}x_2) & 0 \\ \frac{1}{2}(\omega_{x_1}x_1 - \omega_{x_2}x_2) & \omega_{x_1}x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.2}$$

and

$$\eta_1 = \begin{pmatrix} 0 & 0 & \frac{1}{2}(-\omega x_2 + (\omega\psi)_{x_1}) \\ 0 & 0 & \frac{1}{2}(\omega x_1 + (\omega\psi)_{x_2}) \\ \frac{1}{2}(-\omega x_2 + (\omega\psi)_{x_1}) & \frac{1}{2}(\omega x_1 + (\omega\psi)_{x_2}) & 0 \end{pmatrix}. \tag{4.3}$$

Then by a simple calculation

(i) $\quad |\eta|^2 = x_3^2|\eta_o|^2 + |\eta_1|^2$

(ii) $\quad \nabla|\eta|^2 = (x_3^2\nabla|\eta_o|^2 + \nabla|\eta_1|^2\,,\ 2x_3|\eta_o|^2)$

(iii) $\quad \mathrm{div}\eta = x_3\mathrm{div}\eta_o + \mathrm{div}\eta_1.$

The first term on the left hand side of the system (4.1) is

$$x_3^3|\eta_o|^2\mathrm{div}\eta_o + x_3^2|\eta_o|^2\mathrm{div}\eta_1 + x_3|\eta_1|^2\mathrm{div}\eta_o + |\eta_1|^2\mathrm{div}\eta_1$$

whereas the second term can be rewritten as

$$x_3^3\eta_o\nabla|\eta_o|^2 + x_3^2\eta_1\nabla|\eta_o|^2 + x_3\eta_o\nabla|\eta_1|^2 + \eta_1\nabla|\eta_1|^2$$

up to the factor $-(p-2)/2(p-1)$. If σ is independent of x_3, these imply the equations

(a) $\quad |\eta_o|^2\mathrm{div}\eta_o = \dfrac{p-2}{2(p-1)}\eta_o\nabla|\eta_o|^2$

(b) $\quad |\eta_o|^2\mathrm{div}\eta_1 = \dfrac{p-2}{2(p-1)}\eta_1\nabla|\eta_o|^2$

(c) $\quad |\eta_1|^2\mathrm{div}\eta_o = \dfrac{p-2}{p-1}\eta_o\nabla|\eta_1|^2$

(d) $\quad |\eta_1|^2\mathrm{div}\eta_1 = \dfrac{p-2}{2(p-1)}\eta_1\nabla|\eta_1|^2.$

The first implies

$$\operatorname{div}\left(|\eta_0|^{q-2}\eta_o\right) = 0\,, \qquad q = \frac{p}{p-1}. \tag{4.4}$$

We will show that (4.4) implies that if ω is Lipschitz continuous in Ω then $\omega = const$ and consequently $\eta_o = 0$. Therefore (b) and (c) above will be trivially satisfied. Finally (d) corresponds to having imposed the third of (1.3) and it is equivalent to (2.6). This in (4.1) will imply that σ is constant.

Returning to (4.4), it follows from (4.2) and the fact that ω is radial

$$\eta_o = \frac{\omega_\rho}{\rho}\begin{pmatrix} -x_1x_2 & \frac{1}{2}(x_1^2 - x_2^2) \\ \frac{1}{2}(x_1^2 - x_2^2) & x_1x_2 \end{pmatrix}, \qquad |\eta_o|^2 = \frac{1}{2}(\rho\omega_\rho)^2.$$

Therefore from (4.4)

$$\begin{cases} \operatorname{div}\varphi(\rho)\left(-x_1x_2 \;\; \frac{1}{2}(x_1^2 - x_2^2)\right) = 0 \\ \operatorname{div}\varphi(\rho)\left(\frac{1}{2}(x_1^2 - x_2^2) \;\; -x_1x_2\right) = 0 \end{cases} \tag{4.5}$$

where

$$\varphi(\rho) = (\rho\omega_\rho)^{q-2}\,\frac{\omega_\rho}{\rho}. \tag{4.6}$$

Expanding the differentiation in (4.5) we obtain

$$\begin{cases} x_2\varphi(\rho) - \frac{\varphi'(\rho)}{\rho}(x_1\,,x_2)\cdot\left(-x_1x_2\,,\frac{x_1^2-x_2^2}{2}\right) = 0 \\ x_1\varphi(\rho) + \frac{\varphi'(\rho)}{\rho}(x_1\,,x_2)\cdot\left(\frac{x_1^2-x_2^2}{2}\,,x_1x_2\right) = 0. \end{cases}$$

These in turn both imply the o.d.e.

$$4\varphi(\rho) + \rho\varphi'(\rho) = 0$$

whose solutions are of the form $\varphi(\rho) = C\rho^{-4}$, $C \in \mathbf{R}$. In terms of $\omega(\rho)$ we have

$$(\rho\omega_\rho)^{q-2}\,\frac{\omega_\rho}{\rho} = C\rho^{-4} \qquad \text{i.e.} \quad \omega(\rho) = -\frac{q-1}{q}|C|^{\frac{1}{q-1}}\rho^{-p} + h$$

for an arbitrary constant h. It follows that if ω is Lipschitz continuous in Ω then C must be zero.

5. Some exact solutions

Consider a cylinder with uniform circular cross section Ω in $\mathbf{R}^2$, of radius R, and subject to a torsional moment along the axis of symmetry, i.e. the x_3–axis. For each preassigned ω the problem in (2.9) has a unique solution. Let us assume ω is a given positive constant. Then using (2.9) we obtain

$$
\begin{aligned}
&-\frac{d}{d\rho}\left\{\rho|u_p'|^{p-2}u_p'\right\} = \omega\rho\,; \qquad \text{in } \Omega, \\
&u_p(R) = u_p'(0) = 0,
\end{aligned}
\tag{6.1}
$$

where $u_p = u_p(\rho)$, $\rho = |x|$ and $u_p' = \frac{d}{d\rho}u_p$. Integration yields

$$
u_p' = -(\omega\rho)^{\frac{1}{p-1}}\,, \qquad \text{and} \quad u_p = \frac{p-1}{p}(\omega)^{\frac{1}{p-1}}\left(R^{\frac{p}{p-1}} - \rho^{\frac{p}{p-1}}\right). \tag{6.2}
$$

An expression for the moment follows from (3.2), i.e.

$$
M = \left\{\frac{2\pi R^{2+\frac{p}{p-1}}}{2+\frac{p}{p-1}}\right\}\omega^{\frac{1}{p-1}}.
$$

Also

$$
u_p' = -\left\{\frac{(2+\frac{p}{p-1})M}{2\pi R^{2+\frac{p}{p-1}}}\right\}\rho^{\frac{1}{p-1}},
$$

and thus as $p \to \infty$

$$
|\nabla u_p| \longrightarrow \frac{3M}{2\pi R^3}.
$$

This limiting process has been done by keeping ω constant. This is referred to as limiting creep (see [**Ka1**]). Now we will keep M fixed and let $p \to \infty$. Then $\omega = \omega_p$ will depend upon p and we will deduce the asymptotic behavior of ω_p. This will also highlight the dependence of u_p on the quantity $\omega_p^{\frac{1}{p-1}}$.Set

$$
L = \frac{3M}{2\pi R^3} \qquad \text{and} \quad M_o = \frac{2\pi R^3}{3}.
$$

It is clear that if

1. $L < 1$, then $\omega_p \to 0$ as $p \to \infty$, i.e. in the limiting creep state, the rod resists all moments $M < M_o$, without any creep twist rate;
2. $L > 1$, then $\omega_p \to \infty$ as $p \to \infty$, i.e. the rod cannot resist moments $M > M_o$;

3. $L = 1$, then by a simple calculation $\omega_p \to e^{\frac{1}{3}}/R$ as $p \to \infty$. Furthermore, since $|\nabla u_\infty| = L$, limiting creep behavior occurs only when $|\nabla u_\infty| = 1$.

Let us compute the warping function ψ for this example. From (2.10) and the fact that u_p is radial it follows, after some elementary calculations that

$$\Delta(\omega\psi) = 0 \qquad \text{in } \Omega \qquad \nabla(\omega\psi)\cdot\nu = 0 \quad \text{on } \partial\Omega.$$

Therefore $(\omega\psi)$ is a constant for the entire circular shaft and ψ may be taken to be zero. Thus there is no warping of a circular rod under pure torsion.

ADDED In PROOF: We have recently learned that B. Kawohl (On a family of torsional creep problems, to appear in J. Reine Angew. Math.) has obtained a result similar. to Proposition 2.1 in Part II in the case $\operatorname{div}(wx) = 1$. His method, based in Egorov's theorem, is different than ours.

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T. BHATTACHARYA - E. DIBENEDETTO
Department of Mathematics – Northwestern University
Evanston – Illinois 60208 – U.S.A.

J. MANFREDI
Department of Mathematics And Statistics
University of Pittsburgh
Pittsburgh – Pennsylvania 15260 – U.S.A.