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Julio D. Rossi
Universidad de Buenos Aires

Alfredo Miranda
Universidad de Buenos Aires

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Monotone iterations of two obstacle problems with different operators

Irene Gonzálvez[1] · **Alfredo Miranda**[2] · **Julio D. Rossi**[2]

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Abstract
In this paper we analyze iterations of the obstacle problem for two different operators. We solve iteratively the obstacle problem from above or below for two different differential operators with obstacles given by the previous functions in the iterative process. When we start the iterations with a super or a subsolution of one of the operators this procedure generates two monotone sequences of functions that we show that converge to a solution to the two membranes problem for the two different operators. We perform our analysis in both the variational and the viscosity settings.

Mathematics Subject Classification 35J20 · 35J92 · 35D40

1 Introduction

The main goal in this paper is to find solutions to the two membranes problem as limits of sequences obtained by iterating the obstacle problem. Let us first describe the obstacle problem (from above or below) and the then two membranes problem (as described in [8]).

✉ Irene Gonzálvez
irene.gonzalvez@uam.es

Alfredo Miranda
amiranda@dm.uba.ar

Julio D. Rossi
jrossi@dm.uba.ar

1 Departamento de Matemáticas, Universidad Autónoma de Madrid, Campus de Cantoblanco, 28049 Madrid, Spain

2 Departamento de Matemáticas, FCEyN, Universidad de Buenos Aires, Pabellón I, Ciudad Universitaria, 1428 Buenos Aires, Argentina

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1.1 The obstacle problem

The obstacle problem is one of the main problems in the mathematical study of variational inequalities and free boundary problems. The problem is to find the equilibrium position of an elastic membrane whose boundary is fixed, and which is constrained to lie above a given obstacle. In mathematical terms, given an operator L (notice that here we can consider fully nonlinear problems of the form $Lu = F(D^2u, Du, u, x)$) that describes the elastic configuration of the membrane, a bounded Lipschitz domain Ω and a boundary datum f, the obstacle problem from below (here solutions are assumed to be above the obstacle) reads as

$$\begin{cases} u \geq \phi & \text{in } \Omega, \\ Lu \geq 0 & \text{in } \Omega, \\ Lu = 0 & \text{in } \{u > \phi\}, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

or equivalently

$$\begin{cases} \min\{Lu, u - \phi\} = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

We will denote by

$$u = \underline{\mathrm{O}}(L, \phi, f)$$

the solution to (1.1).

When the operator L is in divergence form and is associated to an energy functional $E(u)$ the problem becomes a variational problem and a solution can be obtained minimizing E in the set of functions in the appropriate Sobolev space that are above the obstacle and take the boundary datum. We refer to Sect. 2 for details.

The obstacle problem can be also stated as follows: we look for the smallest supersolution of L (with boundary datum f) that is above the obstacle. This formulation is quite convenient when dealing with fully nonlinear problems using viscosity solutions. We will assume here that the problem (1.1) has a unique continuous viscosity solution (for general theory of viscosity solutions we refer to [9] and [12]). This is guaranteed if L has a comparison principle and one can construct barriers close to the boundary so that the boundary datum f is taken continuously. See Sect. 3.

We can also consider the obstacle problem from above (here solutions are assumed to be below the obstacle)

$$\begin{cases} v \leq \varphi & \text{in } \Omega, \\ Lv \leq 0 & \text{in } \Omega, \\ Lv = 0 & \text{in } \{u < \varphi\}, \\ v = g & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

or equivalently

$$\begin{cases} \max\{Lv, v-\varphi\} = 0 & \text{in } \Omega, \\ v = g & \text{on } \partial\Omega. \end{cases}$$

In this case, the obstacle problem can be viewed as follows: we look for the largest subsolution of L (with boundary datum g) that is below the obstacle. We will denote by

$$v = \overline{\mathrm{O}}(L, \varphi, g)$$

the solution to (1.2).

For general references on the obstacle problem (including regularity of solutions, that in some cases are proved to be optimally $C^{1,1}$) we just mention [5], [6], [13], [24], the survey [21] and references therein.

1.2 The two membranes problem

Closely related to the obstacle problem, one of the systems that attracted the attention of the PDE community is the two membranes problem. This problem models the behaviour of two elastic membranes that are clamped at the boundary of a prescribed domain, they are assumed to be ordered, one membrane is above the other, and they are subject to different external forces (the membrane that is on top is pushed down and the one that is below is pushed up). The main assumption here is that the two membranes do not penetrate each other (they are assumed to be ordered in the whole domain). This situation can be modeled by two obstacle problems; the lower membrane acts as an obstacle from below for the free elastic equation that describes the location of the upper membrane, while, conversely, the upper membrane is an obstacle from above for the equation for the lower membrane. When the equations that obey the two membranes have a variational structure this problem can be tackled using calculus of variations (one aims to minimize the sum of the two energies subject to the constraint that the functions that describe the position of the membranes are always ordered inside the domain, one is bigger or equal than the other), see [27]. On the other hand, when the involved equations are not variational the analysis relies on monotonicity arguments (using the maximum principle). Once existence of a solution (in an appropriate sense) is obtained a lot of interesting questions arise, like uniqueness, regularity of the involved functions, a description of the contact set, the regularity of the contact set, etc. See [7, 8, 26], the dissertation [28] and references therein. More concretely, given two differential operators $L_1(D^2u, Du, u, x)$ and $L_2(D^2v, Dv, v, x)$ the mathematical formulation the two membranes problem (with Dirichlet boundary conditions) is the following:

Definition 1.1 A pair (u, v) is called a solution to the two membranes problem if it solves

$$\begin{cases} \min\left\{L_1(u)(x), (u-v)(x)\right\} = 0, \ x \in \Omega, \\ \max\left\{L_2(v)(x), (v-u)(x)\right\} = 0, \ x \in \Omega, \\ u(x) = f(x), \qquad v(x) = g(x), \qquad x \in \partial\Omega. \end{cases}$$

With our previous notation, this system can be written as

$$u = \underline{\mathrm{O}}(L_1, v, f) \qquad \text{and} \qquad v = \overline{\mathrm{O}}(L_2, u, g).$$

Remark that, in general, the two membranes problem as stated in Definition 1.1 does not have uniqueness. To see this, just take a solution to $L_1(D^2u) = 0$ with $u|_{\partial\Omega} = f$ and v the solution to the obstacle problem for $L_2(D^2v)$ with u as obstacle from above, that is, $v = \overline{\mathrm{O}}(L_2, u, g)$. One can easily check that the pair (u, v) is a solution to the general formulation of the two membranes problem stated Definition 1.1. Analogously, one can take a solution to $L_2(D^2v) = 0$ with $v|_{\partial\Omega} = g$ and u the solution to the corresponding obstacle problem, $u = \underline{\mathrm{O}}(L_1, v, f)$, to obtain another solution to the two membranes problem. In general, these two pairs do not coincide. For example, consider Ω as the interval $(0, 1)$, $f \equiv 1$, $g \equiv 0$ on $\partial\Omega$ and the operators $L_1(u'') = u'' - 10$ and $L_2(v'') = v'' + 2$. The solution to $L_1(u'') = 0$ with $u(0) = u(1) = 1$ is $\widehat{u}(x) = -5x(1-x) + 1$ and the solution to $L_2(v'') = 0$ with $v(0) = v(1) = 0$ is $\tilde{v}(x) = x(1-x)$ for $x \in (0, 1)$. Let $\widehat{v} = \overline{\mathrm{O}}(L_2, \widehat{u}, g)$ and $\tilde{u} = \underline{\mathrm{O}}(L_1, \tilde{v}, f)$. Both pairs $(\widehat{u}, \widehat{v})$ and $(\tilde{u}, \tilde{v})$ are solutions to the two membranes problem but they are different. Since $\widehat{u}$ is a solution and $\tilde{u}$ is a supersolution to L_1 with the same boundary datum, by the comparison principle, we get $\widehat{u} \leq \tilde{u}$. On the other hand, since $\tilde{u}$ is the solution to the obstacle problem for $L_1(u'')$ with $\tilde{v}$ as obstacle from below, $\tilde{u} \geq \tilde{v}$. Since $\widehat{u}$ and $\tilde{v}$ are not ordered, we obtain that $\widehat{u} \not\equiv \tilde{u}$.

The two membranes problem for the Laplacian with a right hand side, that is, for $L_1(D^2u) = -\Delta u + h_1$ and $L_2(D^2v) = -\Delta v - h_2$, was considered in [27] using variational arguments. Latter, in [7] the authors solve the two membranes problem for two different fractional Laplacians of different order (two linear non-local operators defined by two different kernels). Notice that in this case the problem is still variational. In these cases an extra condition appears, namely, the sum of the two operators vanishes, $L_1(D^2u) + L_2(D^2v) = 0$, inside Ω. Moreover, this extra condition together with the variational structure is used to prove a $C^{1,\gamma}$ regularity result for the solution.

The two membranes problem for a fully nonlinear operator was studied in [7, 8, 26]. In particular, in [8] the authors consider a version of the two membranes problem for two different fully nonlinear operators, $L_1(D^2u)$ and $L_2(D^2v)$. Assuming that L_1 is convex and that $L_2(X) = -L_1(-X)$, they prove that solutions are $C^{1,1}$ smooth.

We also mention that a more general version of the two membranes problem involving more than two membranes was considered by several authors (see for example [1, 10, 11]).

1.3 Description of the main results

As we mentioned at the beginning of the introduction, our main goal is to obtain solutions to the two membranes problems as limits of iterations of the obstacle problem. For iterations of the obstacle problem in a different context we refer to [4].

Let us consider two different operators L_1 and L_2 (with boundary data f and g respectively, we assume that $f > g$) and generate two sequences iterating the obstacle problems from above and below. Given an initial function v_0, take u_0 as the solution to the obstacle problem from below for L_1 with boundary datum f and obstacle v_0, that is,

$$u_0 = \underline{\mathrm{O}}(L_1, v_0, f).$$

Now, we use this u_0 as the obstacle from above for L_2 with datum g and obtain

$$v_1 = \overline{\mathrm{O}}(L_2, u_0, g).$$

We can iterate this procedure (solving the obstacle problem for L_1 or L_2 with boundary data f or g and obstacle the previous u or v) to obtain two sequences $\{u_n\}_n$, $\{v_n\}_n$, given by

$$u_n = \underline{\mathrm{O}}(L_1, v_n, f), \qquad v_n = \overline{\mathrm{O}}(L_2, u_{n-1}, g).$$

Our main goal here is to show that, when the initial function v_0 is a subsolution for L_2 with boundary datum g, then both sequences of functions $\{u_n\}_n$, $\{v_n\}_n$ are nondecreasing sequences that converge to a limit pair that gives a solution of the two membranes problem, i.e., there exists a pair of functions (u_∞, v_∞) such that

$$u_n \to u_\infty \qquad \text{and} \qquad v_n \to v_\infty$$

and the limit functions satisfy

$$u_\infty = \underline{\mathrm{O}}(L_1, v_\infty, f) \qquad \text{and} \qquad v_\infty = \overline{\mathrm{O}}(L_2, u_\infty, g).$$

An analogous result can be obtained when we start the iteration with u_0 a supersolution to L_1 with boundary datum f and consider $\{u_n\}_n$, $\{v_n\}_n$, given by

$$u_n = \underline{\mathrm{O}}(L_1, v_{n-1}, f), \qquad v_n = \overline{\mathrm{O}}(L_2, u_n, g).$$

In this case the sequences are non-increasing sequences and also converge to a solution to the two membranes problem.

We will study this iterative scheme using two very different frameworks for the solutions. First, we deal with variational methods and understand solutions in the weak sense (this imposes that L_1 and L_2 must be in divergence form) and next we deal with viscosity solutions (allowing L_1 and L_2 to be general elliptic operators).

Variational operators. To simplify the notation and the arguments, when we deal with the problem using variational methods, we will concentrate in the particular choice of L_1 and L_2 as two different $p-$Laplacians; that is, we let

$$\mathcal{L}_p(w) = -\Delta_p w + h_p \qquad \text{and} \qquad \mathcal{L}_q(w) = -\Delta_q w + h_q,$$

where $\Delta_p w = \operatorname{div}(|\nabla w|^{p-2}\nabla w)$, $\Delta_q w = \operatorname{div}(|\nabla w|^{q-2}\nabla w)$ are two different $p-$Laplacians and h_p, h_q are two given functions. These operators are associated to the energies

$$E_p(w) = \frac{1}{p}\int_\Omega |\nabla w|^p + \int_\Omega h_p w \qquad \text{and} \qquad E_q(w) = \frac{1}{q}\int_\Omega |\nabla w|^q + \int_\Omega h_q w.$$

These energies are naturally well-posed in the Sobolev spaces $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$, respectively.

In this variational framework the obstacle problem can be solved minimizing the energy in the appropriate set of functions. In fact, the solution to the obstacle problem from below, $u = \underline{\mathrm{O}}(\mathcal{L}_p, \varphi, f)$, can be obtained as the minimizer of the energy E_p among functions are constrained to be above the obstacle, that is,

$$E_p(u) = \min_{w \in \underline{\Lambda}^p_{f,\varphi}} \Big\{ E_p(w) \Big\}, \tag{1.3}$$

with

$$\underline{\Lambda}^p_{f,\varphi} = \Big\{ w \in W^{1,p}(\Omega) : \quad w - f \in W^{1,p}_0(\Omega), \quad w \geq \varphi \ \text{ a.e } \ \Omega \Big\}.$$

The minimizer in (1.3) exists and is unique provided the set $\underline{\Lambda}^p_{f,\varphi}$ is not empty. We refer to Sect. 2 for details and precise definitions.

Due to the fact that the problem satisfied by the limit of the sequences constructed iterating the obstacle problem from above and below does not have uniqueness (see the comments after Definition 1.1) we need to rely on monotonicity of the sequences to obtain convergence (rather than use compactness arguments that provide only convergence along subsequences).

We notice at this point that, in this variational setting, one can solve the two membranes problem just minimizing the total energy $E(u, v) = E_p(u) + E_q(v)$ in the subset of $W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ given by $\{(u, v) : u - f \in W^{1,p}_0(\Omega), v - g \in W^{1,q}_0(\Omega), u \geq v \text{ a.e } \Omega\}$. The fact that there is a unique pair that minimizes $E(u, v)$ follows from the strict convexity of the functional using the direct method of calculus of variations. One can check that the minimizing pair is in fact a solution to the two membranes problem given in Definition 1.1. However, in general, there are other solutions to the two membranes problem in the sense of Definition 1.1 that are not minimizers of $E(u, v)$. Since solutions according to Definition 1.1 are, in general, not unique, we observe that the limit that we prove to exist for the sequences that we construct iterating the obstacle problem from above and below does not necessarily converge to the unique minimizer to $E(u, v)$.

Fully nonlinear operators. On the other hand, when we deal with viscosity solutions to the obstacle problems we use the general framework described in [12] and we understand sub and supersolutions applying the operator to a smooth test function that touches the graph of the sub/supersolution from above or below at some point in the domain. Remark that here we can consider fully nonlinear elliptic operators that need not be in divergence form.

When we develop the viscosity theory of the iterations in this general viscosity setting it is more difficult to obtain estimates in order to pass to the limit in the sequences $\{u_n\}_n$, $\{v_n\}_n$ and hence we have to rely again on monotonicity (that is obtained using comparison arguments).

In the viscosity framework there is no gain in considering two particular operators. Therefore, we will state and prove the results for two general second order elliptic operators that satisfy the comparison principle. However, if one looks for a model problem one can think on two normalized $p-$Laplacians, that is, let

$$L_1(w) = -\beta_1 \Delta w - \alpha_1 \Delta_\infty w + h_p \quad \text{and} \quad L_2(w) = -\beta_2 \Delta w - \alpha_2 \Delta_\infty w + h_q.$$

Here $\Delta w = trace(D^2 w)$ (the usual Laplacian), $\Delta_\infty w = \langle D^2 w \frac{\nabla w}{|\nabla w|}, \frac{\nabla w}{|\nabla w|} \rangle$ (the normalized infinity Laplacian), α_i, β_i are nonnegative coefficients and h_p, h_q are continuous functions. This operator, the normalized $p-$Laplacian, appears naturally when one considers game theory to solve nonlinear PDEs, we refer to [19, 22, 23, 27] and the recent books [3, 18].

Let us finish the introduction with a short paragraph concerning the equivalence between weak and viscosity solutions for divergence form operators. For the Dirichlet problem $\Delta u = 0$ the notions of weak and viscosity solutions coincide (and in fact the Dirichlet problem has a unique classical solution), see [16] and [25]. Moreover, the equivalence between weak and viscosity solutions include quasi-linear equations, [17, 20], and some non-local equations, [2, 14].

The paper is organized in two sections; in Sect. 2 we deal with that variational setting of the problem and in Sect. 3 we analyze the problem using viscosity theory.

2 Variational solutions

In this section we will obtain two sequences of solutions to variational obstacle problems whose limits constitute a pair of functions that is a variational solution to the two membranes problem.

Let us start by introducing some notations and definitions. We follow [15]. For short, we write

$$\begin{aligned} \mathcal{L}_p(w) &= -\Delta_p w + h_p, & E_p(w) &= \frac{1}{p}\int_\Omega |\nabla w|^p + \int_\Omega h_p w, \\ \mathcal{L}_q(w) &= -\Delta_q w + h_q, & E_q(w) &= \frac{1}{q}\int_\Omega |\nabla w|^q + \int_\Omega h_q w, \end{aligned}$$

for the operators and their associated energies. In this variational context, we now introduce the definition of weak super and subsolutions.

Definition 2.1 Given $p \in (1, \infty)$ and $h_p \in W^{-1,p}(\Omega)$ we say that $u \in W^{1,p}(\Omega)$ is a weak supersolution (resp. subsolution) for $\mathcal{L}_p$ if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w + \int_\Omega h_p w \geq 0 \ \text{ (resp. } \leq 0)$$

for all non-negative $w \in W_0^{1,p}(\Omega)$ and in this case we written $\mathcal{L}_p u \geq 0$ (resp. $\mathcal{L}_p u \leq 0$) weakly in Ω. We say that $u \in W^{1,p}(\Omega)$ is a weak solution for $\mathcal{L}_p$ if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla w + \int_\Omega h_p w = 0$$

for all $w \in W_0^{1,p}(\Omega)$ and we write $\mathcal{L}_p u = 0$ weakly in Ω.

Definition 2.2 Given $p,\ q \in (1, \infty)$, let $f \in W^{1,p}(\Omega)$, $\varphi \in L^1(\Omega)$, $h_p \in W^{-1,p}$ and

$$\underline{\Lambda}^p_{f,\varphi} = \Big\{ w \in W^{1,p}(\Omega) : \quad w - f \in W_0^{1,p}(\Omega), \quad w \geq \varphi \ \text{ a.e } \ \Omega \Big\}.$$

When $\underline{\Lambda}^p_{f,\varphi} \neq \emptyset$, we say $u \in \underline{\Lambda}^p_{f,\varphi}$ is the solution of the p -Laplacian upper obstacle problem in a variational sense with obstacle φ and boundary datum f if

$$E_p(u) = \min_{w \in \underline{\Lambda}^p_{f,\varphi}} \Big\{ E_p(w) \Big\},$$

i.e, u minimizes the energy associated with the operator $\mathcal{L}_p := -\Delta_p + h_p$ in the set of functions that are above the obstacle and take the boundary datum, $\underline{\Lambda}^p_{f,\varphi}$. In this case, we denote $u = \underline{\mathrm{O}}(\mathcal{L}_p, \varphi, f)$. In Proposition 1 we show that this solution exists and is unique provided the set $\underline{\Lambda}^p_{f,\varphi}$ is not empty.

Analogously, for $g \in W^{1,q}(\Omega)$, $\varphi \in L^1(\Omega)$, $h_q \in W^{-1,q}$, we define

$$\overline{\Lambda}^q_{g,\varphi} = \Big\{ w \in W^{1,q}(\Omega) : \quad w - g \in W_0^{1,q}(\Omega), \quad w \leq \varphi \ \text{ a.e } \ \Omega \Big\}.$$

When $\overline{\Lambda}^q_{g,\varphi} \neq \emptyset$, we say that $v \in \overline{\Lambda}^q_{g,\varphi}$ is the solution of the q -Laplacian lower obstacle problem in a varational sense with obstacle φ and boundary g data if

$$E_q(v) = \min_{w \in \overline{\Lambda}^q_{g,\varphi}} \Big\{ E_q(w) \Big\}.$$

For short, we denote v as $v = \overline{\mathrm{O}}(\mathcal{L}_q, \varphi, g)$.

Proposition 1 *Let be $h_p \in W^{-1,p}(\Omega)$ and $f \in W^{1,p}(\Omega)$. If $\underline{\Lambda}^p_{f,\varphi} \neq \emptyset$ there exists a unique $u \in \underline{\Lambda}^p_{f,\varphi}$ such that $E_p(u) \leq E_p(w)$ for all $w \in \underline{\Lambda}^p_{f,\varphi}$, i.e, there exists a unique $u \in \underline{\Lambda}^p_{f,\varphi}$ that minimizes the energy E_p in $\underline{\Lambda}^p_{f,\varphi}$. That is, $u = \underline{O}(\mathcal{L}_p, \varphi, f)$. This function satisfies*

$$\begin{cases} u \geq \varphi & \text{a.e. in } \ \Omega, \\ \mathcal{L}_p u \geq 0 & \text{weakly in } \ \Omega, \\ \mathcal{L}_p u = 0 & \text{weakly in } \ \{u > \varphi\} \text{ (for } \varphi \text{ continuous)}, \\ u = f & \text{a.e. in } \ \partial\Omega. \end{cases}$$

Analogously, let $h_q \in W^{-1,q}(\Omega)$ and $g \in W^{1,q}(\Omega)$, if $\overline{\Lambda}^q_{g,\varphi} \neq \emptyset$, there exists a unique $v \in \overline{\Lambda}^q_{g,\varphi}$ such that $E_q(v) \leq E_q(w)$ for all $w \in \overline{\Lambda}^q_{g,\varphi}$. That is, $v = \overline{O}(\mathcal{L}_q, \varphi, g)$. This function satisfies

$$\begin{cases} v \leq \varphi & \text{a.e. in } \ \Omega, \\ \mathcal{L}_q v \leq 0 & \text{weakly in } \ \Omega, \\ \mathcal{L}_q v = 0 & \text{weakly in } \ \{v < \varphi\} \text{ (for } \varphi \text{ continuous)}, \\ v = g & \text{a.e. in } \ \partial\Omega. \end{cases}$$

Proof Let us prove existence and uniqueness of the solution to the obstacle problem from below. The proof is contained in Theorem 3.21 in [15] but we reproduce some details here for completeness. Suppose that $\underline{\Lambda}^p_{f,\varphi} \neq \emptyset$. Since E_p is a coercive and strictly convex functional in $W^{1,p}(\Omega)$, we obtain that all minimizing sequences $u_j \in \underline{\Lambda}^p_{f,\varphi}$ converge weakly to the same limit $u \in W^{1,p}(\Omega)$ (that we want to show that is the unique minimizer of E_p in $\underline{\Lambda}^p_{f,\varphi}$). Let us show that $u \in \underline{\Lambda}^p_{f,\varphi}$. Rellich–Kondrachov's Theorem implies that, taking a subsequence, $u_j \to u$ strongly in $L^p(\Omega)$. Then, taking a subsequence, we have that $u_j \to u$ a.e. in Ω. Thus, since $u_j \geq \varphi$ a.e. Ω and $u_j - f \in W^{1,p}_0(\Omega)$ for all $j \geq 1$, we get $u \geq \varphi$ a.e. Ω and $u - f \in W^{1,p}_0(\Omega)$.

Let us prove that u verifies the properties stated in the proposition. Let us start with the fact that $\mathcal{L}_p u \geq 0$ weakly in Ω. Since $\underline{\Lambda}^p_{f,\varphi}$ is a convex set, $u + t(v - u) \in \underline{\Lambda}^p_{f,\varphi}$ for all $v \in \underline{\Lambda}^p_{f,\varphi}$ and $t \in [0, 1]$. Thanks to the fact that E_p reaches its minimum in the set $\underline{\Lambda}^p_{f,\varphi}$ at u, we have that

$$i(t) := E_p[u + t(v - u)]$$

satisfies $i(t) \geq i(0)$ and therefore $i'(0) \geq 0$. Let us compute

$$i'(t) = \int_\Omega [|\nabla u + t\nabla(v - u)|^{p-2}(\nabla u + t\nabla(v - u)) \cdot \nabla(v - u) - h_p(v - u)]dx.$$

Taking $t = 0$, we get

$$i'(0) = \int_{\Omega} [|\nabla u|^{p-2}\nabla u \cdot \nabla(v-u) - h_p(v-u)]dx \geq 0. \tag{2.4}$$

If we consider $w \in W_0^{1,p}(\Omega)$ $w \geq 0$ a.e. Ω, and $v = u + tw \in \underline{\Lambda}_{f,\varphi}^p$ for $t \geq 0$ small enough, if we come back to (2.4), we obtain

$$t\int_{\Omega} [|\nabla u|^{p-2}\nabla u \cdot \nabla w - h_p w]dx \geq 0$$

with $t \geq 0$. This implies that

$$\mathcal{L}_p u \geq 0$$

weakly in Ω.

Finally, we show that

$$\mathcal{L}_p u = 0$$

weakly in $\{u(x) > \varphi(x)\}$ when φ is continuous. From Section 3.26 in [15] we have that the solution to the obstacle problem with continuous obstacle becomes also continuous (after a redefinition in a set of measure zero). Therefore, for a continuous obstacle the set $\{x : u(x) > \varphi(x)\}$ is an open set. Take a ball

$$\overline{B_r} \subset \{x : u(x) > \varphi(x)\},$$

and consider a nonnegative $w \in W_0^{1,p}(B_r) \cap C_0(\overline{B_r})$. Using that $\varphi < u$ in ∂B_r, we get $v = u + tw \in \underline{\Lambda}_{f,\varphi}^p$ for $|t|$ small enough. Then, let us come back to (2.4) and obtain

$$t\int_{B_r} [|\nabla u|^{p-2}\nabla u \cdot \nabla w - h_p w]dx \geq 0.$$

In this case t can be positive or negative. Thus, we get

$$\int_{B_r} [|\nabla u|^{p-2}\nabla u \nabla w - h_p w]dx = 0,$$

which implies

$$\mathcal{L}_p u = 0$$

weakly in B_r and therefore in $\{u > \varphi\}$.

The other case (an obstacle from above) is analogous. □

This minimizer of the energy is in fact the infimum of weak supersolutions are above/below the obstacle.

Proposition 2 *Let $u = \underline{O}(\mathcal{L}_p, \varphi, f)$. This function satisfies*

$$u = \inf \left\{ w \in \underline{\Lambda}^p_{f,\varphi} : \ \mathcal{L}_p w \geq 0 \ \textit{weakly in} \ \Omega \right\}, \tag{2.5}$$

Analogously, let $v = \overline{O}(\mathcal{L}_q, \varphi, g)$, then

$$v = \sup \left\{ w \in \overline{\Lambda}^q_{g,\varphi} : \ \mathcal{L}_q w \leq 0 \ \textit{weakly in} \ \Omega \right\}. \tag{2.6}$$

Proof Let us consider

$$\overline{u} = \inf \left\{ w \in \underline{\Lambda}^p_{f,\varphi} : \ \mathcal{L}_p w \geq 0 \ \text{weakly in} \ \Omega \right\}.$$

We will prove that $\overline{u} = u$ in Ω.

Let us start proving that $u \geq \overline{u}$. By Proposition 1, we have

$$u \in \left\{ w \in \underline{\Lambda}^p_{f,\varphi} : \ \mathcal{L}_p w \geq 0 \ \text{weakly in} \ \Omega \right\},$$

and then we get

$$u \geq \overline{u} = \inf \left\{ w \in \underline{\Lambda}^p_{f,\varphi} : \ \mathcal{L}_p w \geq 0 \ \text{weakly in} \ \Omega \right\}.$$

Now, let us prove that $u \leq \overline{u}$. Given $w \in \underline{\Lambda}^p_{f,\varphi}$ such that $\mathcal{L}_p w \geq 0$ weakly in Ω, we have

$$\int_\Omega [|\nabla w|^{p-2} \nabla w \cdot \nabla v - h_p v] dx \geq 0 \quad \text{for all } v \in W_0^{1,p}(\Omega)$$

and then, using (2.4), we get

$$\int_\Omega [|\nabla u|^{p-2} \nabla u \cdot \nabla (v-u) - h_p (v-u)] dx \geq 0 \quad \text{for all } v \in \underline{\Lambda}^p_{f,\varphi}(\Omega). \tag{2.7}$$

Let us consider $z = \min\{w, u\} \in \underline{\Lambda}^p_{f,\varphi}$, then $u - z \in W_0^{1,p}(\Omega)$, $u - z \geq 0$. Then

$$0 \leq \int_\Omega [|\nabla w|^{p-2} \nabla w \cdot \nabla (u-z) - h_p (u-z)] dx$$

and, using (2.7), we obtain

$$0 \geq \int_\Omega [|\nabla u|^{p-2} \nabla u \cdot \nabla (u-z) - h_p (u-z)] dx.$$

Substracting these inequalities we conclude that

$$0 \leq \int_{\Omega} [(|\nabla w|^{p-2}\nabla w - |\nabla u|^{p-2}\nabla u) \cdot \nabla(u - z)]dx,$$

which implies

$$0 \leq \int_{\{u>z\}} [(|\nabla w|^{p-2}\nabla w - |\nabla u|^{p-2}\nabla u) \cdot \nabla(u - w)]dx$$

using that $(|b|^{p-2}b - |a|^{p-2}a)(a - b) \leq 0$ (with a strict inequality for $a \neq b$) we obtain $|\{u > z\}| = 0$. Then, $u = z \leq w$. Thus, we have obtained that $u \leq \overline{u}$.

The other case is analogous. □

Remark 1 Notice that Proposition 2 implies that the solution to the obstacle problem is monotone with respect to the obstacle in the sense that, for $\varphi_1 \geq \varphi_2$, we have $u_1 = \underline{\mathrm{O}}(\mathcal{L}_p, \varphi_1, f) \geq u_2 = \underline{\mathrm{O}}(\mathcal{L}_p, \varphi_2, f)$.

Now, we are ready to introduce the definition of a weak solution to the two membranes problem.

Definition 2.3 Let $f \in W^{1,p}(\Omega)$ and $g \in W^{1,q}(\Omega)$ be two functions such that $f \geq g$ in $\partial\Omega$ in the sense of traces. Take $h_p \in W^{-1,p}(\Omega)$ and $h_q \in W^{-1,q}(\Omega)$. We say that the pair (u, v) with $u \in \underline{\Lambda}^p_{f,v}$, $v \in \overline{\Lambda}^q_{g,u}$ is a solution of the two membranes problem if

$$(u, v) \text{ satisfies } \begin{cases} E_p(u) = \min\limits_{w \in \underline{\Lambda}^p_{f,v}} \Big\{E_p(w)\Big\}, \\ E_q(v) = \min\limits_{w \in \overline{\Lambda}^q_{g,u}} \Big\{E_q(w)\Big\}, \end{cases}$$

i.e.,

$$u = \underline{\mathrm{O}}(\mathcal{L}_p, v, f) \text{ and } v = \overline{\mathrm{O}}(\mathcal{L}_q, u, g),$$

Remark 2 In general, given a pair (f, g), the solution to the two membranes problem is not unique.

2.1 Iterative method

The main result of this section reads as follows.

Theorem 2.4 *For $p \geq q$ let $f \in W^{1,p}(\Omega)$ and $g \in W^{1,q}(\Omega)$ be two functions such that $f \geq g$ in $\partial\Omega$ in the sense of traces and take $h_p \in W^{-1,p}(\Omega)$ and $h_q \in W^{-1,q}(\Omega)$. Let us consider v_0 a weak subsolution of $\mathcal{L}_q v = 0$ in Ω such that $v_0 - g \in W^{1,q}_0(\Omega)$ and then define inductively the sequences*

$$u_n = \underline{O}(\mathcal{L}_p, v_n, f), \qquad v_n = \overline{O}(\mathcal{L}_q, u_{n-1}, g).$$

Both sequences of functions $\{u_n\}_{n=0}^{\infty} \subset W^{1,p}(\Omega)$, $\{v_n\}_{n=0}^{\infty} \subset W^{1,q}(\Omega)$ *converge strongly in* $W^{1,p}(\Omega)$ *and* $W^{1,q}(\Omega)$*, respectively. Moreover, the limits of the sequences are a solution of the two membranes problem. That is, there exists a pair of functions* $u_\infty \in \underline{\Lambda}^p_{f,v_\infty}$ *and* $v_\infty \in \overline{\Lambda}^q_{g,u_\infty}$ *such that*

$$u_n \to u_\infty \;\; \textit{strongly in} \;\; W^{1,p}(\Omega) \;\; \textit{and} \;\; v_n \to v_\infty \;\; \textit{strongly in} \;\; W^{1,q}(\Omega),$$

and, in addition, the limit pair (u_∞, v_∞) *satisfies*

$$u_\infty = \underline{O}(\mathcal{L}_p, v_\infty, f) \;\; \textit{and} \;\; v_\infty = \overline{O}(\mathcal{L}_q, u_\infty, g).$$

Proof Notice that from the definition of the sequences as solutions to the obstacle problem we have

$$v_n \le u_n$$

a.e. Ω for each $n \in \mathbb{N}$. Since we assumed that the boundary data are ordered, $f \ge g$, and functions in $W^{1,p}(\Omega)$ and in $W^{1,q}(\Omega)$ have a trace on the boundary, we can also say that $u_n \ge v_n$ a.e. on the boundary (with respect to the $(N-1)$-dimensional measure). From the construction of the sequences we obtain that the sets $\underline{\Lambda}^p_{f,v_n}$ and $\overline{\Lambda}^q_{g,u_n}$ are not empty, in fact we have that $u_{n-1} \in \underline{\Lambda}^p_{f,v_n}$ (since v_n solves an obstacle problem from above with obstacle u_{n-1}, we have $v_n \le u_{n-1}$) and $v_n \in \overline{\Lambda}^q_{g,u_n}$ (since u_n solves an obstacle problem from below with obstacle v_n, we have $v_n \le u_n$). Hence the sequences $\{u_n\}$ and $\{v_n\}$ are well defined.

We will divide our arguments in several steps.

First step. First, we prove a monotonicity result. The two sequences are non-decreasing.

Let us see that $v_0 \le v_1$ a.e. Ω. This is due to the fact that v_0 is a weak subsolution of $\mathcal{L}_q v = 0$ and $v_0 \in \overline{\Lambda}^q_{g,u_0}$ and by (2.6) we have

$$v_1 = \sup \Big\{ w \in \overline{\Lambda}^q_{g,u_0} : \;\; \mathcal{L}_q w \le 0 \;\; \text{weakly in} \;\; \Omega \Big\}.$$

Then, we conclude that $v_0 \le v_1$.

Also we can see that $u_0 \le u_1$ a.e. Ω. In fact, from (2.5), we have

$$u_0 = \inf \Big\{ w \in \underline{\Lambda}^p_{f,v_0} : \;\; \mathcal{L}_p w \ge 0 \;\; \text{weakly in} \;\; \Omega \Big\}$$

and then, using that $v_1 \ge v_0$ a.e. Ω, we obtain that $u_1 \in \underline{\Lambda}^q_{f,v_0}$ and u_1 is a weak supersolution for $\mathcal{L}_p u = 0$. Then, $u_0 \le u_1$ a.e. in Ω.

Now, to obtain the general case, we just use an inductive argument. Suppose $v_{n-1} \le v_n$ and $u_{n-1} \le u_n$ a.e. Ω. Since $v_n \le u_n$, we have $v_n \in \overline{\Lambda}^q_{g,u_n}$. From (2.6) we have

$$v_{n+1} = \sup \Big\{ w \in \overline{\Lambda}^q_{g,u_n} : \;\; \mathcal{L}_q w \le 0 \;\; \text{weakly in} \;\; \Omega \Big\} \;\; \text{and} \;\; \mathcal{L}_q v_n \le 0 \;\; \text{weakly in} \;\; \Omega$$

which implies $v_n \le v_{n+1}$ a.e. Ω. On the other hand, the last inequality and the fact that $v_{n+1} \le u_{n+1}$ give us $u_{n+1} \in \underline{\Lambda}^p_{f,v_n}$. Hence, (2.5) implies that

$$u_n = \inf \left\{ w \in \underline{\Lambda}^p_{f,v_n} : \ \mathcal{L}_p w \ge 0 \ \text{ weakly in } \ \Omega \right\} \ \text{ and } \ \mathcal{L}_p u_{n+1} \ge 0 \ \text{ weakly in } \ \Omega,$$

and we conclude that $u_n \le u_{n+1}$ a.e. Ω.

Second step. Our next step is to show that the sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are bounded in $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$, respectively. To prove this fact let us take $w \in W^{1,q}(\Omega)$ the solution to $\mathcal{L}_q w = 0$ with boundary datum g. That is, w satisfies

$$\mathcal{L}_q w = 0 \quad \text{weakly in } \ \Omega \ \text{ and } \ w = g \ \text{ in } \ \partial\Omega \ \text{ in the sense of traces.}$$

By (2.6), we have that $\mathcal{L}_q v_n \le 0$ weakly in Ω, and thanks to $v_n - w \in W_0^{1,q}(\Omega)$ and the comparison principle for $\mathcal{L}_q$, we obtain that

$$v_n \le w$$

a.e. Ω. Let us consider $z = \underline{\mathrm{O}}(\mathcal{L}_p, w, f)$. Since $v_n \le w$ we have $z \in \underline{\Lambda}^p_{f,v_n}$ for all $n \in \mathbb{N}$. Now, since u_n is the solution to the obstacle problem, u_n minimizes the energy among functions in the set $\underline{\Lambda}^p_{f,v_n}$. Then, as z is in the former set, we get

$$E_p(u_n) \le E_p(z)$$

for all $n \in \mathbb{N}$, that is, we have

$$\int_\Omega \frac{|\nabla u_n|^p}{p} + \int_\Omega h_p u_n \le \int_\Omega \frac{|\nabla z|^p}{p} + \int_\Omega h_p z.$$

We can rewrite this inequality to obtain

$$\int_\Omega |\nabla u_n|^p \le C(z, p, h_p) + C(p, h_p)\|u_n\|_{L^p(\Omega)}. \tag{2.8}$$

Using that $u_n - f \in W_0^{1,p}(\Omega)$, from Poincare's inequality, we obtain

$$\|u_n - f\|_{L^p(\Omega)} \le C\|\nabla u_n - \nabla f\|_{L^p(\Omega)} \ \text{ for all } \ n \in \mathbb{N}.$$

Then, by the triangle inequality we get

$$\|u_n\|_{L^p(\Omega)} \le C(f) + C\|\nabla u_n\|_{L^p(\Omega)} \ \text{ for all } \ n \in \mathbb{N}.$$

If we come back to (2.8), we obtain

$$\int_\Omega |\nabla u_n|^p \le C(z, p, h_p, f) + C(p, h_p, f)\|\nabla u_n\|_{L^p(\Omega)}.$$

Now, if we use Young's inequality $ab \le \varepsilon a^p + C(\varepsilon)b^{p'}$ with $\varepsilon = \frac{1}{2}$ in the last term, we conclude

$$\int_\Omega |\nabla u_n|^p \le C(z, p, h_p, f) \ \text{ for all } \ n \in \mathbb{N}.$$

Thus, we get the desired bound for u_n, there exists a constant C independent of n such that

$$\|u_n\|_{W^{1,p}(\Omega)} \le C \ \text{ for all } \ n \in \mathbb{N}.$$

Now, we prove that $\{v_n\}$ is bounded in $W^{1,q}(\Omega)$. Since $v_0 \le v_n \le u_{n-1}$ we have that $v_0 \in \underline{\Lambda}^q_{g,u_{n-1}}$ for all $n \in \mathbb{N}$. This implies

$$E_q(v_n) \le E_q(v_0).$$

That is,

$$\int_\Omega \frac{|\nabla v_n|^q}{q} + \int_\Omega h_q v_n \le \int_\Omega \frac{|\nabla v_0|^q}{q} + \int_\Omega h_q v_0.$$

Now, we just proceed as before to obtain

$$\|v_n\|_{W^{1,q}(\Omega)} \le C \ \text{ for all } \ n \in \mathbb{N}.$$

Third step. We show strong convergence to a solution of the two membranes problem.

Since $\{u_n\}$ and $\{v_n\}$ are bounded in $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$ respectively, there exist subsequences $\{u_{n_j}\} \subset \{u_n\}$ and $\{v_{n_j}\} \subset \{v_n\}$ such that

$$u_{n_j} \rightharpoonup u_\infty \ \text{ weakly in } \ W^{1,p}(\Omega) \quad \text{and} \quad v_{n_j} \rightharpoonup v_\infty \ \text{ weakly in } \ W^{1,q}(\Omega),$$

as $n_j \to \infty$.

By the Rellich–Kondrachov theorem, there exists a subsequence $\{u_{n_j}\}$ that converges strongly in $L^p(\Omega)$ and a subsequence $\{v_{n_j}\}$ that converges strongly in $L^q(\Omega)$. Extracting again a subsequence if needed, we get

$$u_{n_j} \longrightarrow u_\infty \quad \text{and} \quad v_{n_j} \longrightarrow v_\infty \ \text{ a.e. } \Omega,$$

as $n_j \to \infty$. Using that $\{u_n\}$ and $\{v_n\}$ are increasing we obtain that the entire sequences converge pointwise (and weakly) to the unique limits (that are given by the supremums of the sequences). This implies that $u_\infty \ge v_\infty$ a.e. Ω, $u_\infty = f$ and $v_\infty = g$ a.e. $\partial\Omega$ in the sense of traces. This gives us $u_\infty \in \underline{\Lambda}^p_{f,v_\infty}$ and $v_\infty \in \overline{\Lambda}^q_{g,u_\infty}$.

Let us define $u = \underline{\mathrm{O}}(\mathcal{L}_p, v_\infty, f)$ and $v = \overline{\mathrm{O}}(\mathcal{L}_q, u_\infty, g)$. Our next step is to prove that $u = u_\infty$ and $v = v_\infty$ a.e. Ω and to get that the convergence is strong in the corresponding Sobolev spaces.

First, we begin with the case of the upper membranes u_n. Since $u = \underline{\mathrm{O}}(\mathcal{L}_p, v_\infty, f)$ and $u_\infty \in \underline{\Lambda}^p_{f,v_\infty}$ we have that $E_p(u) \le E_p(u_\infty)$. On the other hand, since $\{v_n\}$ is an increasing sequence we have $u_\infty \ge v_\infty \ge v_n$ for each $n \in \mathbb{N}$ wich implies $u \in \underline{\Lambda}^p_{f,v_n}$. Then, we have that $E_p(u_n) \le E_p(u)$.

On the other hand, by the semicontinuity of the norm, $u_n = u_\infty$ in $\partial\Omega$ and by weak convergence, we obtain

$$\|\nabla u_\infty\|_{L^p(\Omega)} \le \liminf_{n\to\infty} \|\nabla u_n\|_{L^p(\Omega)}, \quad \int_\Omega h_p u_n \longrightarrow \int_\Omega h_p u_\infty \;\text{ as }\; n \to \infty.$$

Then, we have

$$E_p(u_\infty) \le \liminf_{n\to\infty} E_p(u_n) \le \limsup_{n\to\infty} E_p(u_n) \le E_p(u) \le E_p(u_\infty). \tag{2.9}$$

Thus, $E_p(u_\infty) = E_p(u)$. And since the energy minimizer is unique, $u_\infty = u$ a.e. Ω. Moreover, by (2.9) and the weak convergence, we have obtained that $\{u_n\}$ is a minimizing sequence of the energy and the gradient of u_n converges strongly to the gradient of u_∞ as n goes to infinity. Then, we conclude that

$$u_n \longrightarrow u_\infty \;\text{ strongly in }\; W^{1,p}(\Omega) \;\text{ as }\; n \to \infty.$$

Once we have seen that $u_\infty = \underline{\mathrm{O}}(\mathcal{L}_p, v_\infty, f)$, let us prove that $v_\infty = \overline{\mathrm{O}}(\mathcal{L}_q, u_\infty, g)$. Again, since $v_\infty \in \overline{\Lambda}^q_{g,u_\infty}$, we have that $E_p(v) \le E_p(v_\infty)$.

Note that we cannot repeat the previous argument step by step because we do not know if v is in $\overline{\Lambda}^q_{g,u_{n-1}}$. Though, it is enough to change a little our strategy (we use at this point that $p \ge q$). We construct a subsequence $\{\tilde{v}_n\}$ such that $\tilde{v}_n \in \overline{\Lambda}^q_{g,u_{n-1}}$ and $E_p(\tilde{v}_n)$ converges to $E_p(v)$ as n goes to infinity. We define $\{\tilde{v}_n\}$ as

$$\tilde{v}_n = v - u_\infty + u_n.$$

Since $p \ge q$ we have that $\tilde{v}_n \in W^{1,q}(\Omega)$, $\tilde{v}_n - g \in W^{1,q}_0(\Omega)$ and $E_q(\tilde{v}_n)$ goes to $E_q(v)$ as $n \to \infty$ due to the Rellick–Kondrachov Compactness Theorem and the fact that u_n converges to u_∞ in $W^{1,p}(\Omega)$. Also we have the inequality $\tilde{v}_n \le u_n$ because $v \le u_\infty$. Then, $\tilde{v}_n \in \overline{\Lambda}^q_{g,u_n}$ which implies $E_q(v_{n+1}) \le E_q(\tilde{v}_n)$. Besides, by the semicontinuity of the norm, the fact that $v_n = v_\infty$ on $\partial\Omega$ and the weak convergence, we obtain

$$\|\nabla v_\infty\|_{L^q(\Omega)} \le \liminf_{n\to\infty} \|\nabla v_{n+1}\|_{L^q(\Omega)}, \quad \int_\Omega h_q v_{n+1} \longrightarrow \int_\Omega h_q v_\infty \;\text{ as }\; n \to \infty.$$

As a consequence,

$$\begin{aligned} E_q(v_\infty) &\le \liminf_{n\to\infty} E_q(v_{n+1}) \le \limsup_{n\to\infty} E_q(v_{n+1}) \\ &\le \limsup_{n\to\infty} E_q(\tilde{v}_n) = E_q(v) \le E_p(v_\infty). \end{aligned}$$

Then, $E_q(v_\infty) = E_q(v)$. By the uniqueness of the minimizer, $v_\infty = v$ a.e. Ω. Furthermore, from the same reasons as in the case of the upper membranes, we have that the entire sequence converges strongly,

$$v_n \longrightarrow v_\infty \text{ strongly in } W^{1,q}(\Omega) \text{ as } n \to \infty.$$

This ends the proof. □

If $p < q$ we can also construct a pair of sequences that converges in $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$ to a solution of the two membranes problem. In this case we just have to start the iterations of the obstacle problems with u_0 a weak supersolution to $\mathcal{L}_p u = 0$.

Theorem 2.5 *For $p < q$ let $f \in W^{1,p}(\Omega)$ and $g \in W^{1,q}(\Omega)$ be two functions such that $f \geq g$ in $\partial\Omega$ in the sense of traces and take $h_p \in W^{-1,p}(\Omega)$ and $h_q \in W^{-1,q}(\Omega)$. Take u_0 a weak supersolution to $\mathcal{L}_p u = 0$ in Ω such that $u_0 - f \in W_0^{1,p}(\Omega)$ and then let*

$$u_n = \underline{O}(\mathcal{L}_p, v_{n-1}, f), \qquad v_n = \overline{O}(\mathcal{L}_q, u_n, g).$$

Both sequences $\{u_n\}_{n=0}^{\infty} \subset W^{1,p}(\Omega)$ and $\{v_n\}_{n=0}^{\infty} \subset W^{1,q}(\Omega)$ converge strongly in $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$, respectively. Moreover, the limits of the sequences are a solution of the two membranes problem. That is, there exists a pair of functions $u_\infty \in \underline{\Lambda}^p_{f,v_\infty}$ and $v_\infty \in \overline{\Lambda}^q_{g,u_\infty}$ such that

$$u_n \to u_\infty \text{ strongly in } W^{1,p}(\Omega) \text{ and } v_n \to v_\infty \text{ strongly in } W^{1,q}(\Omega),$$

and, in addition, the limit pair (u_∞, v_∞) satisfies

$$u_\infty = \underline{O}(\mathcal{L}_p, v_\infty, f) \text{ and } v_\infty = \overline{O}(\mathcal{L}_q, u_\infty, g).$$

Proof The difference between these sequences and the sequences defined in Theorem 2.4 is that, here, $\{u_n\}$ and $\{v_n\}$ are decreasing sequences.

Due to the monotonicity of the sequences, we can prove that u_n converges weakly to some u_∞ in $W^{1,p}(\Omega)$ and v_n converges to some v_∞ weakly in $W^{1,q}(\Omega)$ as n goes to infinity.

The strong convergence of v_n to v_∞ is given in the same way that we got strong convergence for u_n to u_∞ in Proposition 2.4 thanks to $\{v_n\}$ is a decreasing sequence. On the other hand, for $\{u_n\}$ we can reproduce the proof for $\{v_n\}$ in proposition (2.4) because $W^{1,q}(\Omega) \hookrightarrow W^{1,p}(\Omega)$ continuously. □

Remark 3 An alternative idea to deal with the case $p < q$ runs as follows: It can be easily proved that

$$u = \underline{\mathrm{O}}(-\Delta_p + h_p, \varphi, f) \text{ if and only if } u = \overline{\mathrm{O}}(-\Delta_p - h_p, -\varphi, -f).$$

Then, if $p < q$ we consider

$$\begin{aligned} p' &= q, \quad h_{p'} = -h_q, \quad f' = -g, \\ q' &= p, \quad h_{q'} = -h_p, \quad g' = -f. \end{aligned}$$

For the problem with the new set of parameters, p', q', $h_{p'}$, $h_{q'}$, f' and g', we can apply the iterative method described in Theorem 2.4 and get a pair (u'_∞, v'_∞) such that $u_\infty \in \underline{\Lambda}^{p'}_{f',v'_\infty}$ and $v'_\infty \in \overline{\Lambda}^{q'}_{g',u'_\infty}$ such that $u'_\infty = \underline{\mathrm{O}}(-\Delta_{p'} + h_{p'}, v'_\infty, f')$ and $v'_\infty = \overline{\mathrm{O}}(-\Delta_{q'} + h_{q'}, u'_\infty, g')$. Thus, $u_\infty = -v'_\infty$ and $v_\infty = -u'_\infty$ are in $\underline{\Lambda}^{p}_{f,v_\infty}$ and $\overline{\Lambda}^{q}_{g,u_\infty}$ respectively and

$$u_\infty = \underline{\mathrm{O}}(\mathcal{L}_p, v_\infty, f), \quad \text{and} \quad v_\infty = \overline{\mathrm{O}}(\mathcal{L}_q, u_\infty, g),$$

i.e, (u_∞, v_∞) is a solution for the original two membranes problem.

Remark 4 We remark that the limit depends strongly on the initial function from where we start the iterations. We would like to highlight that $\{u_n\}$ and $\{v_n\}$ converge to u_∞ and v_∞ strongly in $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$ respectively due to the fact that both sequences are monotone. The monotonicity, in turn, depends on the initial function for the iteration. Specifically, when $p \geq q$, the monotonicity arises because the initial datum v_0 is a weak subsolution of $\mathcal{L}_q v = 0$. On the other hand, when $p \leq q$ it stems from the fact that u_0 is a weak supersolution of $\mathcal{L}_p u = 0$.

If one considers the pair (u, v) obtained as follows: let u be the solution to $\mathcal{L}_p u = 0$ in Ω with $u = f$ on $\partial\Omega$ and v the solution to the corresponding obstacle problem $v = \overline{\mathrm{O}}(\mathcal{L}_q, u, g)$, then, as we have mentioned in the introduction this pair (u, v) is a solution to the two membranes problem. Now, if one starts the iteration procedure with $u_0 = u$ we obtain that the sequences converge after only one step. In fact we get $v_1 = \overline{\mathrm{O}}(\mathcal{L}_q, u, g)$ and next the iteration gives again, that is, we have $u = \underline{\mathrm{O}}(\mathcal{L}_p, v_1, f)$. This is due to the fact that u is a solution to $\mathcal{L}_p u = 0$ in the whole Ω with $u \geq v_1$ and therefore it is the solution to the obstacle problem (with v_1 as obstacle from below). Analogously, if one starts with v the solution to $\mathcal{L}_q v = 0$ in Ω with $v = g$ on $\partial\Omega$ we obtain u as the solution to the corresponding obstacle problem $u = \underline{\mathrm{O}}(\mathcal{L}_p, v, f)$. This pair (u, v) is also a solution to the two membranes problem, but, in general the two pairs are different.

2.2 Extension to the two membranes problem for nonlocal operators

If we consider

$$\mathcal{L}_p := (-\Delta_p)^s + h_p \quad \text{and} \quad \mathcal{L}_q := (-\Delta_q)^t + h_q$$

with

$$(-\Delta_p)^s(u)(x) = \int_{\mathbb{R}^N} \frac{|u(y) - u(x)|^{p-2}(u(y) - u(x))}{|x - y|^{N+sp}} dy,$$

the fractional $p-$Laplacian. Notice that $\mathcal{L}_p$ and $\mathcal{L}_q$ are associated with the energies

$$E_p^s(w) = \frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)-u(x)|^p}{|x-y|^{N+sp}} dy\, dx + \int_\Omega h_p w$$

and

$$E_q^t(w) = \frac{1}{q}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)-u(x)|^q}{|x-y|^{N+tq}} dy\, dx + \int_\Omega h_q w.$$

Our iterative method of the Theorem 2.4 gives us a pair (u_∞, v_∞) that is solution of the "new" two membranes problem whenever

$$W^{s,p}(\Omega) \hookrightarrow W^{t,q}(\Omega) \ \text{ continuously.}$$

To be able to define a variational solution of the obstacle problem for these new $\mathcal{L}_p$ and $\mathcal{L}_q$ is needed that $\underline{\Lambda}^p_{f,\varphi}$ and $\overline{\Lambda}^q_{g,\varphi}$ are closed sets under the weak topology. This is given by Mazur's Theorem if $W^{s,p}(\Omega)$ and $W^{t,q}(\Omega)$ are reflexive Banach space and $\underline{\Lambda}^p_{f,\varphi}$ and $\overline{\Lambda}^q_{g,\varphi}$ are closed convex sets (with the norm) in $W^{s,p}(\Omega)$ and $W^{t,q}(\Omega)$ respectively.

3 Viscosity solutions

In this section we aim to prove similar convergence results for iterations of the obstacle problem to a pair that solves the two membranes problems when the involved operators are not variational. Here we will consider two nonlinear elliptic operators and we understand solutions in the viscosity sense. To define the sequences as solutions to the corresponding obstacle problems we need to assume that the involved operators verify the following set of conditions:

Hypothesis The operators $\mathcal{L}_1$ and $\mathcal{L}_2$ verify

- $\mathcal{L}w = F(D^2w, \nabla w) - h(x)$ with F continuous in both coordinates, such that the comparison principle holds.
- If f is continuous, there exists a unique $w \in C(\overline{\Omega})$ solution to $\mathcal{L}w = 0$ in Ω, $w = f$ in $\partial\Omega$.
- If φ, ψ and f, g are continuous, then the corresponding solutions to the obstacle problems, $u = \underline{\mathcal{O}}(\mathcal{L}_1, \varphi, f)$ and $v = \overline{\mathcal{O}}(\mathcal{L}_2, \psi, g)$ are continuous. See Definition 3.4 below.

To be precise, working in the viscosity sense, we need to introduce the definition of semicontinuous functions.

Definition 3.1 $f : \Omega \longrightarrow \mathbb{R}$ is a lower semicontinuous function, l.s.c, at $x \in \Omega$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$f(x) \le f(y) + \varepsilon \ \text{ for all } \ y \in B_\delta(x).$$

If a function f is l.s.c. at every $x \in \Omega$, we say that f is l.s.c. in Ω. The lower semicontinuous envelope of f is

$$f_* = \sup\{h : \Omega \longrightarrow \mathbb{R} \ \text{ l.s.c.} : \quad h \leq f.\}$$

On the other hand, $g : \Omega \longrightarrow \mathbb{R}$ is a upper semicontinuous function, u.s.c, at $x \in \Omega$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$g(y) \leq g(x) + \varepsilon \ \text{ for all } \ y \in B_\delta(x).$$

If a functions g is u.s.c. at every $x \in \Omega$, we say that it is u.s.c. in Ω. The upper semicontinuous envelope of g is

$$g^* = \inf\{h : \Omega \longrightarrow \mathbb{R} \ \text{ u.s.c.} : \quad h \geq f.\}$$

Now, we give the precise definition of sub and supersolutions in the viscosity sense.

Definition 3.2 The function $u : \Omega \longrightarrow \mathbb{R}$ is called a $\mathcal{L}$-viscosity supersolution if its lower semicontinuous envelope function, u_*, satisfies the following: for every $\phi \in C^2(\overline{\Omega})$ such that ϕ touches u_* at $x \in \Omega$ strictly from below, that is, $u_* - \phi$ has a strict minimum at x with $u_*(x) = \phi(x)$, we have

$$\mathcal{L}\phi(x) \geq 0.$$

In this case, we write $\mathcal{L}u \geq 0$ in the viscosity sense.

Conversely, u is called a $\mathcal{L}$-viscosity subsolution if its upper semicontinuous envelope function, u^*, satisfies that for every $\phi \in C^2(\overline{\Omega})$ such that ϕ touches u^* at $x \in \Omega$ strictly from above, that is, $\phi - u^*$ has a strict minimum at x with $u^*(x) = \phi(x)$, we have

$$\mathcal{L}\phi(x) \leq 0$$

and we write $\mathcal{L}u \leq 0$ in the viscosity sense.

Finally, u is a $\mathcal{L}$-viscosity solution if it is both a $\mathcal{L}$-viscosity supersolution and a $\mathcal{L}$-viscosity subsolution and we denote $\mathcal{L}u = 0$ in the viscosity sense.

Now, let us introduce three results concerning the limit of non-decreasing sequences of viscosity sub or supersolutions. Although these results are not difficult to prove, they will be crucial for the proof of our main Theorem 3.9 in this section.

Proposition 3 *Let be $\{v_n\}_{n=0}^{\infty} \subset C(\overline{\Omega})$ a non-decreasing sequence of functions, continuous up to the boundary such that v_n is a $\mathcal{L}$-viscosity subsolution for each $n \in \mathbb{N}$ and $v_n \longrightarrow v_\infty$ pointwise. Then, v_∞ is a $\mathcal{L}$-viscosity subsolution.*

Proof Let be $\phi \in C^2(\overline{\Omega})$ such that $\phi - v_\infty^*$ has a strict minimum at $x_0 \in \Omega$ and $\phi(x_0) = v_\infty^*(x_0)$.

Since v_∞^* is upper semicontinuous, $\phi - v_\infty^*$ is lower semicontinuous. Then, fixed $r > 0$ small enough, $\phi - v_\infty^*$ reaches a minimum in $D_r := \overline{\Omega} \setminus B_r(x_0)$. Say in $z_\infty \in D_r$. Thanks to $-v_\infty^* = (-v_\infty)_*$ and that the sequence $\{v_n\}$ is non-decreasing, we have

$$0 < (\phi - v_\infty^*)(z_\infty) \leq (\phi - v_\infty^*)(z) \leq (\phi - v_\infty)(z) \leq (\phi - v_n)(z) \tag{3.10}$$

for each $z \in D_r$ and for each $n \in \mathbb{N}$.

On the other hand, by the definition of pointwise value of the lower semicontinuous envelope,

$$\begin{aligned} 0 = (\phi - v_\infty^*)(x_0) &= (\phi + (-v_\infty)_*)(x_0) \\ &= \inf_{\left\{ \{x_k\}_{k=0}^\infty \subset \Omega:\ x_k \to_{k\to\infty} x_0 \right\}} \lim_{k\to\infty} (\phi - v_\infty)(x_k). \end{aligned}$$

Then, given $\varepsilon > 0$ there exists a sequence $\{y_k\}$ within $B_{r/2}(x_0)$ such that

$$\lim_{k\to\infty} |(\phi - v_\infty)(y_k)| \leq \varepsilon/3.$$

Thus, there is $k_0 \in \mathbb{N}$ such that $|(\phi - v_\infty)(y_{k_0})| \leq 2\varepsilon/3$. Since v_n converges to v_∞ pointwise, there exists $n_0 \in \mathbb{N}$ such that $|(\phi - v_\infty)(y_{k_0}) - (\phi - v_n)(y_{k_0})| \leq \varepsilon/3$ for all $n \geq n_0$.

In short, for each $\varepsilon > 0$ there exists a $y \in B_{r/2}(x_0)$ and $n_0 \in \mathbb{N}$ such that

$$(\phi - v_n)(y) \leq \varepsilon \quad \text{for all} \quad n \geq n_0.$$

Since, $\phi - v_n$ is a continuous function, given $\varepsilon < (\phi - v_\infty^*)(z_\infty)$ one can see that the minimum of $\phi - v_n$ in $\overline{\Omega}$ is reached at some $x_n^r \in B_{r/2}(x_0)$ for all $n \geq n_0$ thanks to the above inequality and (3.10). Note that n_0 depends on r, so we write $n_0 = n_0(r)$. We have $n_0(r) \to \infty$ and $x_n^r \to x_0$ as $r \to 0^+$. Due to the fact that v_n is a $\mathcal{L}$- viscosity subsolution, we have

$$\mathcal{L}\phi(x_n^r) \leq 0.$$

Since $\mathcal{L}$ is a continuous operator we have

$$0 \geq \lim_{r\to\infty} (\mathcal{L}\phi)(x_n^r) = (\mathcal{L}\phi)\big(\lim_{r\to\infty} x_n^r\big) = (\mathcal{L}\phi)(x_0).$$

Therefore v_∞ is a $\mathcal{L}$-viscosity subsolution. □

Lemma 3.3 *Let $\{u_n\} \subset C(\overline{\Omega})$ a non-decreasing sequence of continuous up to the boundary functions. If u_n converge pointwise to some function u_∞, then u_∞ is a lower semicontinuous function in $\overline{\Omega}$.*

Proof Let be $x_0 \in \Omega$. Since $u_n \longrightarrow u_\infty$ pointwise, we have that for each $\varepsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that $u_\infty(x_0) \leq u_{n_0}(x_0) + \varepsilon$. On the other hand, due to the fact that

u_{n_0} is continuous there exist a δ such that $u_{n_0}(x_0) \leq u_{n_0}(y) + \varepsilon$ for all $y \in B_\delta(x_0)$. Moreover, thanks to the fact that the sequence $\{u_n\}$ is non-decreasing we get that, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$u_\infty(x_0) \leq u_\infty(y) + \varepsilon \quad \text{for all} \quad y \in B_\delta(x_0).$$

The proof is finished. □

Proposition 4 *Let $\{u_n\}_{n=0}^\infty \subset C(\overline{\Omega})$ be a non-decreasing sequence of continuous up to the boundary functions such that u_n is a $\mathcal{L}$-viscosity supersolution for each $n \in \mathbb{N}$ and $u_n \longrightarrow u_\infty$ pointwise. Then, u_∞ is a $\mathcal{L}$-viscosity supersolution.*

Proof By Lemma 3.3 we know that u_∞ is a lower semicontinuous function. To prove that u_∞ is a $\mathcal{L}$-viscosity supersolution let us consider $\phi \in C^2(\overline{\Omega})$ such that $u_\infty - \phi$ has a strict minimum at $x_0 \in \Omega$ such that $(u_\infty - \phi)(x_0) = 0$. We want to obtain that $(\mathcal{L}\phi)(x_0) \geq 0$.

Let $r > 0$ be a fixed small radius. Since $u_\infty - \phi$ is lower semicontinuous, these functions reach a positive minimum in $\overline{\Omega} \setminus B_r(x_0)$, say in $z_\infty \in \overline{\Omega} \setminus B_r(x_0)$, then, we have

$$0 < (u_\infty - \phi)(z_\infty) \leq (u_\infty - \phi)(x) \quad \text{for all} \quad x \in \overline{\Omega} \setminus B_r(x_0).$$

Also, since $u_n - \phi$ is continuous, there exists $z_n \in \overline{\Omega} \setminus B_r(x_0)$ where $u_n - \phi$ reaches a minimum in $\overline{\Omega} \setminus B_r(x_0)$. We claim that $(u_n - \phi)(z_n)$ converges to $(u_\infty - \phi)(z_\infty)$ as n to infinity. Let us continue the proof assuming this fact and prove it after finishing our argument.

The claim implies that there exists $n_0 = n_0(r)$ such that

$$0 < \frac{1}{2}(u_\infty - \phi)(z_\infty) \leq (u_n - \phi)(z_n) \leq (u_n - \phi)(x) \tag{3.11}$$

for all $x \in \overline{\Omega} \setminus B_r(x_0)$ and $n \geq n_0$.

On the other hand, since u_n converges to u_∞ pointwise and $(u_\infty - \phi)(x_0) = 0$, there exists $n_1 = n_1(r)$ such that

$$(u_n - \phi)(x_0) \leq \frac{1}{2}(u_\infty - \phi)(z_\infty) \quad \text{for all } n \geq n_1.$$

Thus, the above and (3.11) imply $u_n - \phi$ reaches its minimum in $\overline{\Omega}$ in the interior of the ball $B_r(x_0)$ for all $n \geq n_3 = \max\{n_0, n_1\}$. Call this point x_n^r. Then, since u_n is a $\mathcal{L}$-viscosity supersolution, we get

$$(\mathcal{L}\phi)(x_n^r) \geq 0 \quad \text{for all} \quad n \geq n_3(r) \quad \text{with} \quad x_n^r \in B_r(x_0).$$

Due to the fact that $x_n^r \to x_0$ as $r \to 0^+$, we have

$$(\mathcal{L}\phi)(x_0) = \lim_{r \to 0^+} (\mathcal{L}\phi)(x_n^r) \geq 0.$$

Therefore, u_∞ is a $\mathcal{L}$-viscosity supersolution in Ω.

Now, we will prove the claim. Let us define $w_\infty = u_\infty - \phi$, $w_n = u_n - \phi$ and $a_n = w_n(z_n)$. Thanks to $\{u_n\}$ is a pointwise increasing sequence, $\{a_n\}$ is also an increasing sequence,

$$a_n = w_n(z_n) \leq w_n(z_{n+1}) \leq w_{n+1}(z_{n+1}) = a_{n+1}.$$

Moreover, the sequence is bounded above by $w_\infty(z_\infty)$ due to the fact that u_n converges pointwise to u_∞. Hence,

$$a_n \leq w_n(z_n) \leq w_n(z_\infty) \leq w_\infty(z_\infty).$$

Then, there exists a number $a_\infty \leq w_\infty(z_\infty)$ such that a_n converges to a_∞ as n tends to infinity.

Suppose that $a_\infty < w_\infty(z_\infty)$. Then, because $\{z_n\}$ is a collection of points in a compact set, there exists a subsequence $\{z_{n_j}\} \subset \{z_n\}$ and a point $\widetilde{z} \in \overline{\Omega} \setminus B_r(x_0)$ such that

$$z_{n_j} \longrightarrow \widetilde{z} \ \text{ as } \ n_j \to \infty.$$

Since w_n is a lower semicontinuous function, for each $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, \widetilde{z})$ such that

$$w_\infty(\widetilde{z}) \leq w_\infty(y) + \varepsilon \ \text{ for all } \ y \in B_\delta(\widetilde{z}).$$

Due to the fact that z_{n_j} converges to $\widetilde{z}$, we can suppose $z_{n_j} \in B_\delta(\widetilde{z})$ for all n_j. Then

$$w_\infty(\widetilde{z}) \leq w_\infty(z_{n_j}) + \varepsilon \ \text{ for all } \ n_j.$$

For each n_j there exists a $m_j \in \mathbb{N}$ such that

$$w_\infty(z_{n_j}) \leq w_k(z_{n_j}) + \varepsilon \ \text{ for all } \ k \geq m_j.$$

Observe that we can construct a sequence $\{(n_j, k_j)\}$ where $k_j \geq m_j$ for all $j \in \mathbb{N}$ and $(n_j, k_j) \longrightarrow (\infty, \infty)$ as $j \to \infty$ such that

$$w_\infty(z_{n_j}) \leq w_{k_j}(z_{n_j}) + \varepsilon \ \text{ for all } \ j \in \mathbb{N}.$$

Since $a_\infty < w_\infty(z_\infty)$ there exists $\varepsilon_0 > 0$ such that

$$a_\infty + 2\varepsilon_0 < w_\infty(z_\infty).$$

For that ε_0 the above argument implies that there exists a sequence $\{(n_j, k_j)\}$ that goes to (∞, ∞) as $j \to \infty$ such that

$$a_\infty + 2\varepsilon_0 < w_\infty(z_\infty) \leq w_{k_j}(z_{n_j}) + 2\varepsilon_0 \ \text{ for all } \ j \in \mathbb{N}$$

and therefore,

$$a_\infty < w_{k_j}(z_{n_j}) \text{ for all } j \in \mathbb{N}.$$

On the other hand, since $\{a_n\}$ is a increasing sequence and $a_{k_j} = w_{k_j}(z_{k_j})$ we have that

$$w_{k_j}(z_{k_j}) \leq a_\infty < w_{k_j}(z_{n_j}) \text{ for all } j \in \mathbb{N}.$$

Thanks to the fact that for each $j \in \mathbb{N}$ the functions w_{k_j} are continuous, $\overline{\Omega} \setminus B_r(x_0)$ is connected and z_{n_j} and z_{k_j} are in $\overline{\Omega} \setminus B_r(x_0)$, there exists $y_j \in \overline{\Omega} \setminus B_r(x_0)$ such that

$$w_{k_j}(y_j) = a_\infty.$$

Thus, there is a subsequence $\{y_{j_i}\} \subset \{y_j\}$ and a point $y \in \overline{\Omega} \setminus B_r(x_0)$ such that $y_{j_i} \to y$ as $i \to \infty$. Due to the fact that w_∞ is a lower semicontinuous function we have

$$w_\infty(y) \leq \lim_{y_{j_i} \to y} w_{k_{j_i}}(y_{j_i}) = a_\infty.$$

However, this is a contradiction because $y \in \overline{\Omega} \setminus B_r(x_0)$ and that implies

$$a_\infty < w_\infty(z_\infty) \leq w_\infty(y).$$

Therefore, we have the desired equality $a_\infty = w_\infty(z_\infty)$. □

Now, we are ready to introduce the definition of a solution to the obstacle problem in the viscosity sense.

Definition 3.4 Given $\varphi : \overline{\Omega} \to \mathbb{R}$ and $f \in C(\partial\Omega)$ such that $f \geq \varphi^*$ in $\partial\Omega$. We say that $u : \Omega \longrightarrow \mathbb{R}$ is a viscosity solution of the $\mathcal{L}$-lower obstacle problem with obstacle φ and boundary datum f and we denote $u = \underline{\mathcal{O}}(\mathcal{L}, \varphi, f)$ if u satisfies:

$$\begin{cases} u_* \geq \varphi^* & \text{in } \Omega, \\ u_* = f & \text{in } \partial\Omega, \\ \mathcal{L}u \geq 0 & \text{in } \Omega \text{ in the viscosity sense,} \\ \mathcal{L}u = 0 & \text{in } \{u_* > \varphi^*\} \text{ in the viscosity sense.} \end{cases}$$

Whereas, given ψ and g such that $f \leq \psi_*$ on $\partial\Omega$, we say that v is a viscosity solution of the $\mathcal{L}$-upper obstacle problem with obstacle ψ and boundary datum g, and we denote $v = \overline{\mathcal{O}}(\mathcal{L}, \psi, g)$, if v satisfies:

$$\begin{cases} v^* \leq \psi_* & \text{in } \Omega, \\ v^* = g & \text{in } \partial\Omega, \\ \mathcal{L}v \leq 0 & \text{in } \Omega \text{ in the viscosity sense,} \\ \mathcal{L}v = 0 & \text{in } \{v^* < \psi_*\} \text{ in the viscosity sense.} \end{cases}$$

Remark 5 Note that, when u is a viscosity solution of the $\mathcal{L}$-lower obstacle problem with obstacle φ and boundary datum f, then, its lower and upper semicontinuous envelopes, u_* and u^*, are also viscosity solutions for the same problem.

The same applies to the upper obstacle, if $v = \overline{\mathcal{O}}(\mathcal{L}, \psi, g)$, then, $v_* = \overline{\mathcal{O}}(\mathcal{L}, \psi, g)$ and $v^* = \overline{\mathcal{O}}(\mathcal{L}, \psi, g)$.

However, we have uniqueness of solutions to the obstacle problem up to semicontinuous envelopes.

Lemma 3.5 *Given φ and f defined as before*

(a) *if $f \geq \varphi^*$ in $\partial\Omega$, there exists at most one lower semicontinuous function u such that $u = \underline{\mathcal{O}}(\mathcal{L}, \varphi, f)$.*
(b) *if $f \leq \varphi^*$ in $\partial\Omega$, there exists at most one upper semicontinuous function u such that $u = \overline{\mathcal{O}}(\mathcal{L}, \varphi, f)$.*

Proof We will only prove item (a), the other item is analogous. (a) Suppose that there exists two lower semicontinuous functions solutions u_1 and u_2, let us prove that $u_2 \geq u_1$.

In the set $\{u_1 = \varphi^*\}$ we have $u_2 \geq \varphi^* = u_1$. In the open set $\{u_1 > \varphi^*\}$ we have that $\mathcal{L}u_2 \geq 0$ and $\mathcal{L}u_1 = 0$ and $u_2 \geq u_1$ in $\partial\{u_1 > \varphi^*\}$. Then, by the Comparison Principle we obtain that $u_2 \geq u_1$ in $\{u_1 > \varphi^*\}$. Thus, we conclude that

$$u_2 \geq u_1 \quad \text{in} \quad \Omega.$$

The reverse inequality follows interchanging the roles of u_1 and u_2. □

In view of the previous lemma, from now we will suppose that $u = \underline{\mathcal{O}}(\mathcal{L}, \varphi, f)$ is lower semicontinuous and $v = \overline{\mathcal{O}}(\mathcal{L}, \varphi, f)$ is upper semicontinuous.

Next, we show that when the obstacles and the boundary data are ordered then the solutions to the obstacle problems are also ordered.

Lemma 3.6 (a) *Given φ_1 and φ_2 functions such that $\varphi_1 \leq \varphi_2$ (a.e.) and $f_1, f_2 \in C(\partial\Omega)$ such that $f_1 \leq f_2$, if $f_1 \geq \varphi_1^*$ and $f_2 \geq \varphi_2^*$ on $\partial\Omega$, let us consider $u_1 = \underline{\mathcal{O}}(\mathcal{L}, \varphi_1, f)_1$ and $u_2 = \underline{\mathcal{O}}(\mathcal{L}, \varphi_2, f_2)$, then*

$$u_1 \leq u_2 \quad in \quad \Omega.$$

(b) *Given ψ_1 and ψ_2 functions such that $\psi_1 \leq \psi_2$ (a.e.) and $g_1, g_2 \in C(\partial\Omega)$ such that $g_1 \leq g_2$, if $g_1 \leq \psi_{1*}$ and $g_2 \leq \psi_{2*}$ on $\partial\Omega$, let us consider $v_1 = \overline{\mathcal{O}}(\mathcal{L}, \psi_1, g_1)$*

and $v_2 = \overline{\mathcal{O}}(\mathcal{L}, \psi_2, g_2)$*, then*

$$v_1 \le v_2 \ \ in \ \ \Omega.$$

Proof We will prove item (a), the other case is analogous.

(a) We will consider two cases:

In the set $\{u_1 = \varphi_1^*\}$ we have $u_2 \ge \varphi_2^* \ge \varphi_1^* = u_1$.

In the open set $\{u_1 > \varphi_1^*\}$ we have that $\mathcal{L}u_2 \ge 0$ and $\mathcal{L}u_1 = 0$ and $u_2 \ge u_1$ in $\partial\{u_1 > \varphi_1^*\}$. Then, by the Comparison Principle we obtain that $u_2 \ge u_1$ in $\{u_1 > \varphi^*\}$.

Thus $u_2 \ge u_1$ in Ω. □

Lemma 3.7 *Given* φ *and* f *as before, let us consider* $u = \underline{\mathcal{O}}(\mathcal{L}, \varphi, f)$*, then*

$$u = \min\Big\{w : \Omega \longrightarrow \mathbb{R} \ \ \textit{lower semicontinuous} : \\ \left.\begin{array}{l} w \ge \varphi^* \ \ in \ \ \Omega, \ \ w \ge f \ \ on \ \ \partial\Omega, \\ \mathcal{L}w \ge 0 \ \ in \ \ \Omega \quad in\ the\ viscosity\ sense \end{array}\right\}.$$

In the same way, if we take $v = \overline{\mathcal{O}}(\mathcal{L}, \varphi, f)$*, then*

$$v = \max\Big\{w : \Omega \longrightarrow \mathbb{R} \ \ \textit{upper semicontinuous} : \\ \left.\begin{array}{l} w \le \varphi_* \ \ in \ \ \Omega, \ \ w \le f \ \ on \ \ \partial\Omega, \\ \mathcal{L}w \le 0 \ \ in \ \ \Omega \quad in\ the\ viscosity\ sense \end{array}\right\}.$$

Proof Let us prove the first claim. Consider

$$\overline{u} = \min\Big\{w : \Omega \longrightarrow \mathbb{R} \ \ \text{lower semicontinuous} : \\ \left.\begin{array}{l} w \ge \varphi^* \ \ \text{in} \ \ \Omega, \ \ w \ge f \ \ \text{on} \ \ \partial\Omega, \\ \mathcal{L}w \ge 0 \ \ \text{in} \ \ \Omega \quad \text{in the viscosity sense} \end{array}\right\}.$$

Since $u \ge \varphi^*$ in Ω, $u = f$ on $\partial\Omega$ and $\mathcal{L}u \ge 0$ in Ω in viscosity sense, we get

$$\overline{u} \le u.$$

Now, let us consider

$$w \in \Big\{w : \Omega \longrightarrow \mathbb{R} \ \ \text{lower semicontinuous} : \\ \left.\begin{array}{l} w \ge \varphi^* \ \ \text{in} \ \ \Omega, \ \ w \ge f \ \ \text{on} \ \ \partial\Omega, \\ \mathcal{L}w \ge 0 \ \ \text{in} \ \ \Omega \quad \text{in the viscosity sense} \end{array}\right\}.$$

In the set $\{u = \varphi^*\}$ we have $w \ge \varphi^* = u$.

In the open set $\{u > \varphi^*\}$ we have that $\mathcal{L}w \geq 0$ and $\mathcal{L}u = 0$ in the viscosity sense and $w \geq u$ in $\partial\{u > \varphi^*\}$. Then, by the Comparison Principle we obtain that $w \geq u$ in $\{u_1 > \varphi^*\}$.

Then, $u \leq w$ in Ω. Taking minimum we get

$$u \leq \overline{u}.$$

The other case is analogous. $\square$

Now, we introduce the definition of a solution to the two membranes problem in the viscosity sense (this is just Definition 1.1 with solutions understood in the viscosity sense).

Definition 3.8 Given $f, g \in C(\partial\Omega)$ with $f \geq g$, let $\mathcal{L}_1$ and $\mathcal{L}_2$ by two operators that satisfy the hypothesis at the begining of the section. We say a pair of functions (u, v) is a solution of the two membranes problem with boundary data (f, g) if

$$u = \underline{\mathcal{O}}(\mathcal{L}_1, v, f) \quad \text{and} \quad v = \overline{\mathcal{O}}(\mathcal{L}_2, u, g).$$

Our main result is the following.

Theorem 3.9 *Given f, g and $\mathcal{L}_1, \mathcal{L}_2$, let us consider $v_0 \in C(\overline{\Omega})$ a $\mathcal{L}_2$-viscosity subsolution such that $v_0 \leq g$ in $\partial\Omega$ and then define inductively the sequences*

$$u_n = \underline{\mathcal{O}}(\mathcal{L}_1, v_n, f), \quad \textit{and} \quad v_n = \overline{\mathcal{O}}(\mathcal{L}_2, u_{n-1}, g).$$

Both sequences of functions $\{u_n\}_{n=0}^{\infty} \subset C(\overline{\Omega})$, $\{v_n\}_{n=0}^{\infty} \subset C(\overline{\Omega})$ converge to some limit functions u_∞ and v_∞, respectively. Moreover, theses limits are a solution of the two membranes problem with boundary data (f, g), that is,

$$u_\infty = \underline{\mathcal{O}}(\mathcal{L}_1, v_\infty, f) \quad \textit{and} \quad v_\infty = \overline{\mathcal{O}}(\mathcal{L}_2, u_\infty, g).$$

In addition, the functions u_∞ and v_∞ are lower semicontinuous in $\bar{\Omega}$ and continuous in the interior of the set where u_∞ touches v_∞, i.e, in the interior of $\{u_\infty = v_\infty\}$.

Proof As before, we divide the proof in several steps.

First step. Let us start proving that $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are non-decreasing.

Let us see that $v_0 \leq v_1$. Recall that $v_1 = \overline{\mathcal{O}}(\mathcal{L}_2, u_0, g)$ and $u_0 = \underline{\mathcal{O}}(\mathcal{L}_1, v_0, f)$, then $v_1 \leq u_0$ and $v_0 \leq u_0$.

In the set $\{v_1 = u_0\}$ we have $v_0 \leq u_0 = v_1$.

In the open set $\{v_1 < u_0\}$ we have that $\mathcal{L}_2 v_1 = 0$ and $\mathcal{L}_2 v_0 \leq 0$ and $v_1 \geq v_0$ in $\partial\{v_1 > u_0\}$. Then, by the Comparison Principle we obtain that $v_1 \geq v_0$ in $\{v_1 > u_0\}$.

Thus, $v_0 \leq v_1$ in Ω.

Now, using Lemma 3.6 we have $u_0 \leq u_1$. By induction we can continue and obtain $u_n \leq u_{n+1}$ and $v_n \leq v_{n+1}$ for all $n \geq 1$.

Second step. Let us prove that $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are bounded.

Let $w \in C(\overline{\Omega})$ be a $\mathcal{L}_2$-solution with $w = g$ in $\partial\Omega$. Then, since $\mathcal{L}_2 w = 0$, we get $\mathcal{L}_2 v_n \leq 0$. Moreover, we have $w \geq v_n$ in $\partial\Omega$. Hence, applying the Comparison Principle, we get $w \geq v_n$ in $\bar{\Omega}$ for all $n \geq 0$. This implies, together with the fact that $v_n \geq v_0$, that $\{v_n\}$ is bounded.

Now, if we consider $z = \underline{\mathcal{O}}(\mathcal{L}_1, w, f)$, using that $v_n \leq w$ by Lemma 3.6 $u_n \leq z$ and $z \in C(\overline{\Omega})$. This implies that $\{u_n\}$ is bounded.

Third step. Let us define

$$\lim_{n\to\infty} u_n(x) = u_\infty(x) \qquad \text{and} \qquad \lim_{n\to\infty} v_n(x) = v_\infty(x)$$

for all $x \in \overline{\Omega}$. Using that $\{u_n\}$ are continuous and the sequence is increasing we obtain that u_∞ is lower semicontinuous. We also have $u_\infty = f$ and $v_\infty = g$ in $\partial\Omega$. Let us consider

$$v = \overline{\mathcal{O}}(\mathcal{L}_2, u_\infty, g).$$

This function v is well defined and upper semicontinuous. Our goal is to prove that $v = v_\infty$.

Using that $u_n \leq u_\infty$, we get $v_n \leq v$ for all $n \geq 0$ by Lemma 3.6. Then, we have that $v_\infty \leq v$ and $v_\infty^* \leq v^* = v$. This implies that $v_\infty^* \leq u_\infty$. On the other hand, by Proposition 3, we have that $\mathcal{L}_2 v_\infty^* \leq 0$ in Ω. We also have $v_\infty^* \geq g$ in $\partial\Omega$. Now, we consider $w \in C(\Omega)$ a solution to

$$\begin{cases} \mathcal{L}_2 w = 0 & \text{in } \ \Omega \ \text{ in the viscosity sense,} \\ w = g & \text{in } \ \partial\Omega. \end{cases}$$

By the Comparison Principle we have that $v_n \leq w$ in $\overline{\Omega}$. Then $v_\infty \leq w$ in $\overline{\Omega}$ and $v_\infty^* \leq w^* = w$ in $\overline{\Omega}$. This implies $v_\infty^* \leq g$ on $\partial\Omega$ and, therefore, $v_\infty^* = g$ on $\partial\Omega$. Finally, we will prove that $\mathcal{L}_2 v_\infty^* = 0$ in the (open) set $\{v_\infty^* < u_\infty\}$. Let us consider $x_0 \in \{v_\infty^* < u_\infty\}$, and call $\delta = u_\infty(x_0) - v_\infty^*(x_0)$. Let us consider $n_0 \geq 0$ such that $u_\infty(x_0) - u_{n_0}(x_0) < \frac{\delta}{4}$. Using that u_{n_0} is continuous, there exists $\eta_1 = \eta_1(n_0) > 0$ such that $|u_{n_0}(x_0) - u_{n_0}(y)| < \frac{\delta}{4}$ for all $y \in B_{\eta_1}(x_0)$. On the other hand, there exists $\eta_2 > 0$ such that $v_\infty^*(y) < v_\infty^*(x_0) + \frac{\delta}{4}$ for all $y \in B_{\eta_2}(x_0)$. Gathering the previous estimates we obtain

$$\begin{aligned} u_{n_0}(y) - v_\infty^*(y) = & \underbrace{u_{n_0}(y) - u_{n_0}(x_0)}_{>-\frac{\delta}{4}} + \underbrace{u_{n_0}(x_0) - u_\infty(x_0)}_{>-\frac{\delta}{4}} \\ & + \underbrace{u_\infty(x_0) - v_\infty^*(x_0)}_{=\delta} + \underbrace{v_\infty^*(x_0) - v_\infty^*(y)}_{>-\frac{\delta}{4}} > \frac{\delta}{4} \end{aligned}$$

for all $y \in B_\eta(x_0)$ with $\eta = \min\{\eta_1, \eta_2\}$. Now, we have that $u_n(y) \geq u_{n_0}(y)$ and $v_n(y) \leq v_\infty^*(y)$ for all $n \geq n_0$. Thus

$$u_n(y) - v_n(y) \geq u_{n_0}(y) - v_\infty^*(y) > \frac{\delta}{4} > 0$$

for all $n \geq n_0$ and for all $y \in B_\eta(x_0)$. Then $B_\eta(x_0) \subseteq \{u_n > v_n\}$ for all $n \geq n_0$. Using that $\mathcal{L}_2 v_n = 0$ in $B_\eta(x_0)$ and taking the limit we obtain $\mathcal{L}_2 v_\infty = 0$ in $B_\eta(x_0)$ which is the same that $\mathcal{L}_2 v_\infty^* = 0$ in $B_\eta(x_0)$ because v_∞ is lower semicontinuous. As a consequence, $\mathcal{L}_2 v_n \geq 0$ in $B_\eta(x_0)$. In particular, $\mathcal{L}_2 v_n \geq 0$ in $B_\eta(x_0)$ for all $n \geq n_0$. Then, by Proposition 4, $\mathcal{L}_2 v_\infty \geq 0$ in $B_\eta(x_0)$ in the viscosity sense. Since previously we just proved that v_∞ is a $\mathcal{L}_2$-viscosity subsolution in Ω, we obtain that v_∞ is a $\mathcal{L}_2$-viscosity solution in $B_\eta(x_0)$. Therefore, v_∞ is a $\mathcal{L}_2$-viscosity solution in the set $\{v_\infty^* < u_\infty\}$.

Putting all together, since u_∞ is lower semicontinuous, we obtain

$$\begin{cases} v_\infty^* \leq u_\infty & \text{in } \ \Omega \\ v_\infty^* = g & \text{in } \ \partial\Omega, \\ \mathcal{L}_2 v_\infty \leq 0 & \text{in } \ \Omega \ \text{ in the viscosity sense,} \\ \mathcal{L}_2 v_\infty = 0 & \text{in } \ \{v_\infty^* < u_\infty\} \ \text{ in the viscosity sense.} \end{cases}$$

By uniqueness of the obstacle problem we get $v_\infty = v = \overline{\mathcal{O}}(\mathcal{L}_2, u_\infty, g)$.

Now, let us define $u = \underline{\mathcal{O}}(\mathcal{L}_1, v_\infty, f)$. Using $v_n \leq v_\infty^*$, we have $u_{n-1} \leq u$. Then, taking the limit, $u_\infty \leq u$. On the other hand we have that $\mathcal{L}_1 u_\infty \geq 0$ in Ω in viscosity sense due to Proposition 4 and $u_\infty = f$ in $\partial\Omega$. Moreover, since u_∞ is lower semicontinuous, taking lim sup in the inequality $u_n \geq v_n$, we obtain $u_\infty \geq v_\infty^*$ in $\overline{\Omega}$. Then, using the Lemma 3.7, we get $u \leq u_\infty$. Thus, we conclude that $u_\infty = u$.

Finally, we have that u_∞ and v_∞ are lower semicontinuous functions in $\bar{\Omega}$ by Lemma 3.3. Moreover, u_∞ is continuous in the interior of $\{u_\infty = v_\infty^*\}$ because u_∞ is a lower semicontinuous and, by definition, v_∞^* is an upper semicontinuous function. This also implies that v_∞^* is a continuous function in the interior of $\{u_\infty = v_\infty^*\}$. Then, $v_\infty = v_\infty^*$ there and therefore v_∞ is a continuous function in the interior of the set $\{u_\infty = v_\infty\}$. □

Remark 6 If u_∞ or v_∞ are continuous at $\partial\{u_\infty = v_\infty^*\}$, then, both functions u_∞ and v_∞ are continuous in $\bar{\Omega}$ and therefore the sequences defined in the previous theorem converge uniformly in the whole $\overline{\Omega}$.

In fact, without loss of generality, suppose that u_∞ is continuous in $\partial\{u_\infty = v_\infty^*\}$. We will prove the above using the Comparison Principle for $\mathcal{L}_1$. Since $\mathcal{L}_1 u_\infty = 0$ in $\{u_\infty > v_\infty^*\}$ in the viscosity sense, we have in particular that u_∞^* is a $\mathcal{L}_1$-subsolution and $u_{\infty *}$ a $\mathcal{L}_1$-supersolution in $\{u_\infty > v_\infty^*\}$ in the viscosity sense. Moreover, since $f > g$ in $\partial\Omega$, the boundary $\partial\{u_\infty > v_\infty^*\}$ is the disjoint union of $\partial\Omega$ and $\partial\{u_\infty = v_\infty^*\}$. In $\partial\Omega$, we have $u_\infty^* = u_{\infty *}$ by construction of u_n. And in $\partial\{u_\infty = v_\infty^*\}$ we have also that $u_\infty^* = u_{\infty *}$ because we have supposed u_∞ continuous across that boundary. Then, $u_\infty^* = u_{\infty *}$ in $\partial\{u_\infty > v_\infty^*\}$. Therefore, by the Comparison Principle of $\mathcal{L}_1$, $u_\infty^* \leq u_{\infty *}$ in $\{u_\infty > v_\infty^*\}$ and therefore u_∞ is continuous in $\overline{\{u_\infty > v_\infty^*\}}$. Moreover, as we have seen in the proof of the above theorem, u_∞ is also continuous in $\{u_\infty > v_\infty^*\}$. Then u_∞ is continuous in $\overline{\Omega}$. As a consequence, using the latest hypothesis concerning the operator $\mathcal{L}_2$, v_∞ is continuous in the whole $\overline{\Omega}$, because v_∞ is the solution of the upper obstacle problem with continuous boundary datum and continuous obstacle.

Finally, since the continuous functions u_∞ and v_∞ are the limit of the sequences of continuous functions $\{u_n\}$ and $\{v_n\}$ in a compact set $\bar{\Omega}$, the convergence is uniform.

Remark 7 As happens in the variational setting, we also have here that the limit depends strongly on the initial function from where we start the iterations. Moreover, the convergence of (u_n, v_n) to a solution of the two membranes problem relies on the monotonicity of the sequences $\{u_n\}$ and $\{v_n\}$. This property comes from the fact that the initial function v_0 is a $\mathcal{L}_2$-viscosity subsolution. For this discussion in the variational context, we refer to the arguments given in Remark 4.

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