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ASYMPTOTIC BEHAVIOR OF THE BEST SOBOLEV TRACE CONSTANT IN EXPANDING AND CONTRACTING DOMAINS

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Abstract. We study the asymptotic behavior for the best constant and extremals of the Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ on expanding and contracting domains. We find that the behavior strongly depends on p and q. For contracting domains we prove that the behavior of the best Sobolev trace constant depends on the sign of $qN - pN + p$ while for expanding domains it depends on the sign of $q - p$. We also give some results regarding the behavior of the extremals, for contracting domains we prove that they converge to a constant when rescaled in a suitable way and for expanding domains we observe when a concentration phenomena takes place.

1. **Introduction.** Let Ω be a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$. Of importance in the study of boundary value problems for differential operators in Ω are the Sobolev trace inequalities. For any $1 < p < N$, and $1 < q \leq p^* = p(N-1)/(N-p)$ we have that $W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega)$ and hence the following inequality holds:

$$S_q \|u\|^p_{L^q(\partial\Omega)} \leq \|u\|^p_{W^{1,p}(\Omega)},$$

for all $u \in W^{1,p}(\Omega)$. This is known as the Sobolev trace embedding Theorem. The best constant for this embedding is the largest S_q such that the above inequality holds, that is,

$$S_q(\Omega) = \inf_{u \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p + |u|^p \, dx}{\left(\displaystyle\int_{\partial\Omega} |u|^q \, d\sigma\right)^{p/q}}. \tag{1}$$

Moreover, if $1 < q < p^*$ the embedding is compact and as a consequence we have the existence of extremals, i.e. functions where the infimum is attained, see

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[8]. These extremals are weak solutions of the following problem

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{q-2}u & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian and $\frac{\partial}{\partial\nu}$ is the outer unit normal derivative.

Standard regularity theory and the strong maximum principle, [16], show that any extremal u belongs to the class $C^{1,\alpha}_{\mathrm{loc}}(\Omega) \cap C^{\alpha}(\overline{\Omega})$ and that is strictly one signed in Ω, so we can assume that $u > 0$ in Ω. Let us fix p, q with $1 < q < p^*$ and Ω a bounded smooth domain in $\mathbb{R}^N$, C^1 is enough for our calculations. For $\mu > 0$ we consider the family of domains

$$\Omega_\mu = \mu\Omega = \{\mu x \ ; \ x \in \Omega\}.$$

The purpose of this work is to describe the asymptotic behavior of the best Sobolev trace constants $S_q(\Omega_\mu)$ as $\mu \to 0+$ and $\mu \to +\infty$.

As a precedent, see [4] for a detailed analysis of the behavior of extremals and best Sobolev constants in expanding domains for $p = 2$ and $q > 2$. In that paper it is proved that the extremals develop a peak near the point where the curvature of the boundary attains a maximum. In [5] and [13] a related problem in the half-space $\mathbb{R}^N_+$ for the critical exponent is studied. See also [6], [7] for other geometric problems that leads to nonlinear boundary conditions.

Let us call u_μ an extremal corresponding to Ω_μ. Making a change of variables, we go back to the original domain Ω. If we define $v_\mu(x) = u_\mu(\mu x)$, we have that $v_\mu \in W^{1,p}(\Omega)$ and

$$S_q(\Omega_\mu) = \mu^{(Nq-Np+p)/q} \frac{\displaystyle\int_\Omega \mu^{-p}|\nabla v_\mu|^p + |v_\mu|^p \, dx}{\left(\displaystyle\int_{\partial\Omega} |v_\mu|^q \, d\sigma\right)^{p/q}}. \tag{3}$$

We can assume, and we do so, that the functions u_μ are chosen so that

$$\int_{\partial\Omega} |v_\mu|^q \, d\sigma = 1.$$

We remark that the quantity (1) is not homogeneous under dilations or contractions of the domain. This is a remarkable difference with the study of the Sobolev embedding $W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)$. First, we deal with the case $\mu \to 0+$. As we will see the behavior of the Sobolev constant and extremals is very different when the domain is contracted than when it is expanded. Our first result is the following:

THEOREM 1.1. *Let $1 < q < p^*$, then*

$$\lim_{\mu \to 0+} \frac{S_q(\Omega_\mu)}{\mu^{(Nq-Np+p)/q}} = \frac{|\Omega|}{|\partial\Omega|^{p/q}} \tag{4}$$

and if we scale the extremals u_μ to the original domain Ω as $v_\mu(x) = u_\mu(\mu x)$, $x \in \Omega$, with $\|v_\mu\|_{L^q(\partial\Omega)} = 1$, then v_μ is nearly constant in the sense that $v_\mu \to |\partial\Omega|^{-1/q}$ in $W^{1,p}(\Omega)$.

Observe that the behavior of the Sobolev trace constant, strongly depends on p and q. If we call $\beta_{pq} = (Nq - Np + p)/q$ then we have that, as $\mu \to 0+$,

$$\begin{array}{ll} S_q \to 0 & \text{if } \beta_{pq} > 0, \\ S_q \to +\infty & \text{if } \beta_{pq} < 0, \\ S_q \to C \neq 0 & \text{if } \beta_{pq} = 0. \end{array}$$

Let us remark that the influence of the geometry of the domain appears in (4).

In the special case $p = q$, problem (2) becomes a nonlinear eigenvalue problem. For $p = 2$, this eigenvalue problem is known as the *Steklov* problem, [2]. In [8] it is proved, applying the Ljusternik-Schnirelman critical point Theory on C^1 manifolds, that there exists a sequence of variational eigenvalues $\lambda_k \nearrow +\infty$ and it is easy to see that the first eigenvalue $\lambda_1(\Omega)$ verifies $\lambda_1(\Omega) = S_p(\Omega)$. So Theorem 1.1 shows a difference in the behavior of the first eigenvalue of (2) with respect to the domain with the behavior of the first eigenvalue of the following Dirichlet problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where it is a well known fact that λ_1 increases as the domain decreases, see [1], [10].

The variational eigenvalues λ_k of (2) are characterized by

$$\frac{1}{\lambda_k} = \sup_{C \in C_k} \min_{u \in C} \frac{\|u\|^p_{L^p(\partial\Omega)}}{\|u\|^p_{W^{1,p}(\Omega)}}, \tag{5}$$

where $C_k = \{C \subset W^{1,p}(\Omega);\ C \text{ is compact, symmetric and } \gamma(C) \geq k\}$ and γ is the Krasnoselski genus (see [11]). It is shown in [9] that there exists a second eigenvalue for (2) and that it coincides with the second variational eigenvalue λ_2. Moreover, the following characterization of the second eigenvalue λ_2 holds

$$\lambda_2 = \inf_{u \in A} \left\{ \int_\Omega |\nabla u|^p + |u|^p \, dx \right\}, \tag{6}$$

where $A = \{u \in W^{1,p}(\Omega);\ \|u\|_{L^p(\partial\Omega)} = 1 \text{ and } |\partial\Omega^{\pm}| \geq c\}$, $\partial\Omega^+ = \{x \in \partial\Omega;\ u(x) > 0\}$ and $\partial\Omega^-$ is defined analogously. Concerning the eigenvalue problem, we have the following result.

Theorem 1.2. *There exists a constant* $\widetilde{\lambda}_2$ *such that*

$$\lim_{\mu \to 0+} \mu^{p-1} \lambda_2(\Omega_\mu) = \widetilde{\lambda}_2.$$

This constant $\widetilde{\lambda}_2$ *is the first nonzero eigenvalue of the following problem*

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega, \\ \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \tilde{\lambda} |u|^{p-2} u & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Moreover, if we take an eigenfunction $u_{2,\mu}$ *associated to* $\lambda_2(\Omega_\mu)$ *and scale it to* Ω *as in Theorem 1.1, we obtain that* $v_{2,\mu} \to \widetilde{v}_2$ *in* $W^{1,p}(\Omega)$, *where* $\widetilde{v}_2$ *is an eigenfunction of* (7) *associated to* $\widetilde{\lambda}_2$. *Also, every eigenvalue* $\lambda_2(\Omega_\mu) \leq \lambda(\Omega_\mu) \leq \lambda_k(\Omega_\mu)$ *of* (2) *(variational or not) behaves as* $\lambda(\Omega_\mu) \sim \mu^{1-p}$ *as* $\mu \to 0+$. *Finally, if* $\mu_j \to 0$ *and* $\lambda_j = \lambda(\Omega_{\mu_j})$ *is a sequence of eigenvalues such that there exists* λ *with*

$$\lim_{j \to \infty} \mu_j^{p-1} \lambda_j = \lambda,$$

let (v_j) be the sequence of associated eigenfunctions rescaled as in Theorem 1.1, then (v_j) has a convergent subsequence (v_{j_k}) and a limit v, that is an eigenfunction of (7) with eigenvalue λ.

Observe that the first eigenvalue of (7) is zero with associated eigenfunction a constant. Hence Theorem 1.1 says that the first eigenvalue and the first eigenfunction of our problem (2) converges to the ones of (7). Theorem 1.2 says that $\lambda(\Omega_\mu) \to +\infty$ as $\mu \to 0+$ for the remaining eigenvalues and that problem (7) is a limit problem for (2) when $\mu \to 0+$. We believe that Theorem 1.2 is our main result.

Now, we deal with the case $\mu \to +\infty$. In this case we find, as before, that the behavior strongly depends on p and q. We prove,

THEOREM 1.3. *Let $\beta_{pq} = (qN - pN + p)/q$. It holds*

1. *If* $1 < q < p$, $0 < c_1 \mu^{\beta_{pq}-1} \le S_q(\Omega_\mu) \le c_2 \mu^{\beta_{pq}-1}$.
2. *If* $p \le q < p^*$, $0 < c_1 \le S_q(\Omega_\mu) \le c_2 < \infty$.

For the lower bound in (2) in the case $p < q < p^*$ we have to assume that the corresponding extremals v_μ rescaled such that $\max_{\overline{\Omega}} v_\mu = 1$ verify $|\nabla v_\mu| \le C\mu$. Moreover, for all cases, we have that the corresponding extremals u_μ rescaled as in Theorem 1.1 concentrates at the boundary, in the sense that

$$
\begin{aligned}
&\int_\Omega |v_\mu|^p \, dx \le C\mu^{-\beta_{pq}} \to 0 \quad \text{as } \mu \to +\infty, && \text{if } q \ge p, \\
&\int_\Omega |v_\mu|^p \, dx \le C\mu^{-1} \to 0 \quad \text{as } \mu \to +\infty, && \text{if } q < p,
\end{aligned}
$$

with

$$
\int_{\partial\Omega} |v_\mu|^q \, d\sigma = 1.
$$

As before the behavior of the Sobolev trace constant depends on p and q. We have that, as $\mu \to +\infty$,

$$
\begin{aligned}
&S_q \to 0 && \text{if } \beta_{pq} - 1 < 0, \text{ i.e. } q < p, \\
&0 < c_1 \le S_q \le c_2 < \infty && \text{if } \beta_{pq} - 1 \ge 0, \text{ i.e. } q \ge p.
\end{aligned}
$$

The hypothesis $|\nabla v_\mu| \le C\mu$ is a regularity assumption, see [15] for $C^{1,\alpha}_{\mathrm{loc}}$ regularity results. As a consequence of our arguments we have that the extremals do not develop a peak if $1 < q < p$ as in this case we have that

$$
c_1 \le \int_{\partial\Omega} |v_\mu|^p \, d\sigma \le c_2,
$$

and

$$
\int_{\partial\Omega} |v_\mu|^q \, d\sigma = 1.
$$

For $p = q$ it is proved in [12] that the first eigenvalue $\lambda_1(\Omega_\mu) = S_p(\Omega_\mu)$ is isolated and simple. As a consequence of this if Ω is a ball the extremal v_μ is radial and hence it does not develop a peak. Finally, for $q > p$ the extremals develop peaking concentration phenomena in the sense that, for every $a > 0$,

$$
a^p |\partial\Omega \cap \{v_\mu > a\}| \to 0, \qquad \text{as } \mu \to +\infty,
$$

with $\max_{\overline{\Omega}} v_\mu = 1$. This is in concordance with the results of [4] where for $p = 2$, $q > 2$ they find that the extremals concentrates, with the formation of a peak, near a point of the boundary where the curvature maximizes. We believe that for

$q > p$, extremals develop a single peak as in the case $p = 2$. Nevertheless that kind of analysis needs some fine knowledge of the limit problem in $\mathbb{R}^N_+$ that is not yet available for the $p-$Laplacian.

Let us give an idea of the proof of the lower bounds. In the case $p = q$ we can obtain the lower bound by an approximation procedure. We replace $W^{1,p}(\Omega)$ by an increasing sequence of subspaces in the minimization problem. Then we prove a convergence result and find a uniform bound from below for the approximating problems. We believe that this idea can be used in other contexts. For the case $q > p$ we use our assumption $|\nabla v_\mu| \le C\mu$ to prove a reverse Hölder inequality for the extremals on the boundary that allows us to reduce to the case $p = q$.

Finally, for large μ, in the case $p = q$ we can prove that every eigenvalue is bounded.

THEOREM 1.4. *Let $\lambda_1(\Omega_\mu) \le \lambda(\Omega_\mu) \le \lambda_k(\Omega_\mu)$ be an eigenvalue of (2) in Ω_μ (variational or not). Then there exists two constants, $C_1, C_2 > 0$, independent of μ such that $0 < C_1 \le \lambda(\Omega_\mu) \le C_2 < +\infty$, for every μ large.*

The rest of the paper is organized as follows. In Section 2, we deal with the case $\mu \to 0$ and in Section 3, we study the case $\mu \to +\infty$. Throughout the paper, by C we mean a constant that may vary from line to line but remains independent of the relevant quantities.

2. **Behavior as $\mu \to 0+$.** In this section we focus on the case $\mu \to 0+$. First we prove Theorem 1.1 and then study the case where $q = p$ (the eigenvalue problem).

Let us begin with the following Lemma.

LEMMA 2.1. *Under the assumptions of Theorem 1.1, it follows that*

$$S_q(\Omega_\mu) \le \mu^{(Nq-Np+p)/q} \frac{|\Omega|}{|\partial\Omega|^{p/q}}.$$

Proof. Let us recall that

$$S_q(\Omega_\mu) = \inf_{u \in W^{1,p}(\Omega_\mu)\setminus\{0\}} \frac{\displaystyle\int_{\Omega_\mu} |\nabla u|^p + |u|^p \, dx}{\left(\displaystyle\int_{\partial\Omega_\mu} |u|^q \, d\sigma\right)^{p/q}}.$$

Then, taking $u \equiv 1$ it follows that

$$S_q(\Omega_\mu) \le \mu^{(Nq-Np+p)/q} \frac{|\Omega|}{|\partial\Omega|^{p/q}},$$

as we wanted to see. □

This Lemma shows that the ratio $S_q(\Omega_\mu)/\mu^{(Nq-Np+p)/q}$ is bounded. So a natural question will be to determine if it converges to some value. This is answered in Theorem 1.1 that we prove next.

Proof of Theorem 1.1. Let $u_\mu \in W^{1,p}(\Omega_\mu)$ be a extremal for $S_q(\Omega_\mu)$ and define $v_\mu(x) = u_\mu(\mu x)$, we have that $v_\mu \in W^{1,p}(\Omega)$. We can assume that the functions u_μ are chosen so that

$$\int_{\partial\Omega} |v_\mu|^q \, d\sigma = 1.$$

Equation (3) and Lemma 2.1 give, for $\mu < 1$,

$$\|v_\mu\|^p_{W^{1,p}(\Omega)} \le \int_\Omega \mu^{-p}|\nabla v_\mu|^p + |v_\mu|^p\,dx \le \frac{|\Omega|}{|\partial\Omega|^{p/q}},$$

so there exists a function $v \in W^{1,p}(\Omega)$ and a sequence $\mu_j \to 0+$ such that

$$\begin{aligned} v_{\mu_j} &\rightharpoonup v \quad \text{weakly in } W^{1,p}(\Omega),\\ v_{\mu_j} &\to v \quad \text{in } L^p(\Omega),\\ v_{\mu_j} &\to v \quad \text{in } L^q(\partial\Omega). \end{aligned}$$

Moreover,

$$\int_\Omega |\nabla v_\mu|^p\,dx \le \frac{|\Omega|}{|\partial\Omega|^{p/q}}\mu^p.$$

Hence $\nabla v_\mu \to 0$ in $L^p(\Omega)$. It follows that the limit v is a constant and must verify $\int_{\partial\Omega}|v|^q = 1$, hence $v = \mathit{constant} = |\partial\Omega|^{-1/q}$ and so the full sequence v_μ converges weakly in $W^{1,p}(\Omega)$ to v. From our previous bounds we have

$$v_\mu \to \frac{1}{|\partial\Omega|^{1/q}} \text{ in } L^p(\Omega) \qquad \text{and} \qquad \int_\Omega |\nabla v_\mu|^p\,dx \to 0.$$

Therefore, we have strong convergence, $v_\mu \to |\partial\Omega|^{-1/q}$in $W^{1,p}(\Omega)$. The proof is finished. $\square$

Now we turn our attention to the case $p = q$ which is a nonlinear eigenvalue problem. We recall that Theorem 1.1 says that $\lambda_1(\Omega_\mu) = S_p(\Omega_\mu) \sim \mu \to 0$. First we focus on the behavior of the second eigenvalue λ_2. For the proof of Theorem 1.2 we need the following Lemmas. We believe that these results have independent interest.

Lemma 2.2. *Let $h \in L^{p'}(\partial\Omega)$. Then, problem*

$$\begin{cases} \Delta_p w = 0 & \text{in } \Omega,\\ |\nabla w|^{p-2}\frac{\partial w}{\partial \nu} = h(x) & \text{on } \partial\Omega, \end{cases} \tag{8}$$

has a weak solution if and only if $\int_{\partial\Omega} h(x)\,d\sigma = 0$. Moreover, the solution is unique up to an additive constant.

Proof. It is straightforward to check that if there exists a weak solution to (8) then $\int_{\partial\Omega} h(x)\,d\sigma = 0$.

Now, let $X = \{w \in W^{1,p}(\Omega);\ \int_\Omega w\,dx = 0\}$. By a standard compactness argument, one can verify that the following Poincare inequality holds,

$$\|w\|_{L^p(\Omega)} \le C\|\nabla w\|_{L^p(\Omega)}, \tag{9}$$

for every $w \in X$ and some constant C. Let us now define

$$\Phi(w) = \int_\Omega |\nabla w|^p\,dx - \int_{\partial\Omega} h(x)w\,d\sigma. \tag{10}$$

Critical points of Φ in $W^{1,p}(\Omega)$ are weak solutions of (8). By (9), Φ is a strictly convex, bounded below functional on X, and so there exists a unique function $w \in X$ such that $\Phi'(w)(v) = 0$ for every $v \in X$. Now, using the fact that $\int_{\partial\Omega} h(x)\,d\sigma = 0$, it is easy to see that $\Phi'(w)(v) = 0$ for every $v \in W^{1,p}(\Omega)$ and the proof is now complete. $\square$

Now we find a variational characterization of the first non-zero eigenvalue of the limit problem (7).

LEMMA 2.3. *Let $\tilde{\lambda}_2$ be defined by*

$$\tilde{\lambda}_2 = \inf_{u \in Y - \{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{\displaystyle\int_{\partial\Omega} |u|^p \, d\sigma}, \tag{11}$$

where $Y = \{u \in W^{1,p}(\Omega);\ \int_{\partial\Omega} |u|^{p-2} u \, d\sigma = 0\}$. Then the infimum is attained.

Proof. Let u_n be a minimizing sequence with $\|u_n\|_{L^p(\partial\Omega)} = 1$. By a compactness argument we can extract a subsequence, that we still call u_n, such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega), \\ u_n &\to u \quad \text{in } L^p(\Omega), \\ u_n &\to u \quad \text{in } L^p(\partial\Omega). \end{aligned}$$

Hence $u \in Y - \{0\}$, $\|u\|_{L^p(\partial\Omega)} = 1$. Moreover, we have that

$$\int_\Omega |\nabla u|^p \, dx \le \liminf \int_\Omega |\nabla u_n|^p \, dx = \tilde{\lambda}_2.$$

Therefore u is a minimizer. □

Now we are ready to deal with the proof of Theorem 1.2 which is the main result of the paper.

Proof of Theorem 1.2. We can assume that $0 \in \Omega$ and then we can take $u(x) = x_1$ in the characterization of λ_2 given by (6) to obtain

$$\lambda_2(\Omega_\mu) \le \frac{|\Omega_\mu| + \displaystyle\int_{\Omega_\mu} |x_1|^p \, dx}{\displaystyle\int_{\partial\Omega_\mu} |x_1|^p \, d\sigma} = \mu^{1-p} \frac{|\Omega| + \mu^p \displaystyle\int_\Omega |y_1|^p \, dy}{\displaystyle\int_{\partial\Omega} |y_1|^p \, d\sigma} \le C\mu^{1-p}.$$

Hence if we consider $v_{2,\mu}$ any eigenfunction associated to $\lambda_2(\Omega_\mu)$ normalized with $\|v_{2,\mu}\|_{L^p(\partial\Omega)} = 1$ we get

$$C\mu^{1-p} \ge \lambda_2(\Omega_\mu) = \mu^{1-p} \left(\int_\Omega |\nabla v_{2,\mu}|^p \, dx + \mu^p \int_\Omega |v_{2,\mu}|^p \, dx \right).$$

Therefore $\|\nabla v_{2,\mu}\|_{L^p(\Omega)} \le C$. As we have that $\|v_{2,\mu}\|_{L^p(\partial\Omega)} = 1$, it follows that $\|v_{2,\mu}\|_{W^{1,p}(\Omega)} \le C$, hence we can extract a subsequence $\mu_j \to 0+$ such that

$$\begin{aligned} v_{2,\mu_j} &\rightharpoonup \tilde{v}_2 \quad \text{weakly in } W^{1,p}(\Omega), \\ v_{2,\mu_j} &\to \tilde{v}_2 \quad \text{in } L^p(\Omega), \\ v_{2,\mu_j} &\to \tilde{v}_2 \quad \text{in } L^p(\partial\Omega). \end{aligned}$$

Therefore we have that

$$\int_{\partial\Omega} |\tilde{v}_2|^p \, d\sigma = 1.$$

As it is proved in [9], $|\{v_{2,\mu_j} > 0\} \cap \partial\Omega|, |\{v_{2,\mu_j} < 0\} \cap \partial\Omega| > c$ independent of μ_j, then $\tilde{v}_2$ changes sign. Hence, we get

$$\int_\Omega |\nabla \tilde{v}_2|^p \, dx \ne 0.$$

Taking a subsequence, if necessary, we can assume that

$$\frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} \to \bar{\lambda} \qquad \text{as } \mu \to 0+$$

and, as

$$\frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} = \int_\Omega |\nabla v_{2,\mu}|^p \, dx + \mu^p \int_\Omega |v_{2,\mu}|^p \, dx,$$

passing to the limit

$$0 \neq \int_\Omega |\nabla \tilde{v}_2|^p \, dx \leq \liminf \int_\Omega |\nabla v_{2,\mu}|^p \, dx = \bar{\lambda},$$

hence we obtain that $\bar{\lambda} \neq 0$.

Taking $\varphi \equiv 1$ in the weak form of the equation satisfied by $v_{2,\mu}$ we get that

$$\mu^p \int_\Omega |v_{2,\mu}|^{p-2} v_{2,\mu} \, dx = \frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} \int_{\partial\Omega} |v_{2,\mu}|^{p-2} v_{2,\mu} \, d\sigma.$$

Passing again to the limit we have that

$$\tilde{v}_2 \in Y = \left\{ u \in W^{1,p}(\Omega);\ \int_{\partial\Omega} |u|^{p-2} u \, d\sigma = 0 \right\}.$$

Let w be a function where the infimum (11) is attained with $\|w\|_{L^p(\partial\Omega)} = 1$. As $w \in A$ (see (6)), we have

$$\int_\Omega |\nabla w|^p + \mu^p |w|^p \, dx \geq \frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} = \int_\Omega |\nabla v_{2,\mu}|^p + \mu^p |v_{2,\mu}|^p \, dx.$$

Taking the limit as $\mu \to 0+$ we get

$$\tilde{\lambda}_2 = \int_\Omega |\nabla w|^p \, dx \geq \lim_{\mu \to 0} \frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} \geq \int_\Omega |\nabla \tilde{v}_2|^p \, dx \geq \inf_{\|z\|_{L^p(\partial\Omega)}=1, z \in Y} \int_\Omega |\nabla z|^p = \tilde{\lambda}_2.$$

Therefore

$$\lim_{\mu \to 0} \frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} = \tilde{\lambda}_2$$

and

$$\int_\Omega |\nabla v_{2,\mu}|^p \, dx \to \int_\Omega |\nabla \tilde{v}_2|^p \, dx,$$

from where it follows that $v_{2,\mu} \to \tilde{v}_2$ strongly in $W^{1,p}(\Omega)$. Once again, we pass to the limit as $\mu \to 0+$ in the weak formulation satisfied by $v_{2,\mu}$ to get that $\tilde{v}_2$ is an eigenfunction associated to $\tilde{\lambda}_2$. By the characterization of $\tilde{\lambda}_2$ given in Lemma 11 we get that this is the first non-zero eigenvalue for problem (7).

Now we find the behavior of the remaining eigenvalues. Let $\lambda(\Omega_\mu)$ be an eigenvalue (variational or not). Then, as the variational eigenvalues $\lambda_k(\Omega_\mu)$ form an unbounded sequence, there exists k such that $\lambda_2(\Omega_\mu) \leq \lambda(\Omega_\mu) \leq \lambda_k(\Omega_\mu)$. Now, let $x_1, \ldots, x_k \in \partial\Omega$ and $r = r(k)$ be such that $dist(x_i, x_j) > 2r$. Let $\phi \in C^\infty(\Omega)$ be a nonnegative function with support $B(0, r)$ and let $\phi_j(x) = \phi(x - x_j)$.

Now, let us define $S_k = span\{\phi_1, \ldots, \phi_k\} \cap \{v \in W^{1,p}(\Omega);\ \|v\|_{W^{1,p}(\Omega)} = 1\}$ and $S_{k,\mu} = \{v(x/\mu);\ v \in S_k\}$, then $\gamma(S_k) = \gamma(S_{k,\mu}) = k$. Hence

$$\frac{1}{\lambda_k(\Omega_\mu)} = \sup_{\gamma(S) \geq k} \inf_{u \in S} \frac{\displaystyle\int_{\partial\Omega_\mu} |u|^p \, d\sigma}{\displaystyle\int_{\Omega_\mu} |\nabla u|^p + |u|^p \, dx} \geq \inf_{u \in S_{k,\mu}} \frac{\displaystyle\int_{\partial\Omega_\mu} |u|^p \, d\sigma}{\displaystyle\int_{\Omega_\mu} |\nabla u|^p + |u|^p \, dx}.$$

Changing variables we get,

$$\frac{1}{\lambda_k(\Omega_\mu)} \geq \mu^{p-1} \inf_{v\in S_k} \frac{\displaystyle\int_{\partial\Omega} |v|^p \, d\sigma}{\displaystyle\int_\Omega |\nabla v|^p + \mu^p |v|^p \, dx}. \tag{12}$$

As ϕ_i have disjoint support,

$$\|v\|^p_{L^p(\Omega)} = \left\| \sum_{i=1}^k a_i \phi_i \right\|^p_{L^p(\Omega)} = \sum_{i=1}^k |a_i|^p \|\phi_i\|^p_{L^p(\Omega)} \leq \sum_{i=1}^k |a_i|^p \|\phi\|^p_{L^p(B(0,r))}$$

and

$$\|\nabla v\|^p_{L^p(\Omega)} = \left\| \sum_{i=1}^k a_i \nabla\phi_i \right\|^p_{L^p(\Omega)} = \sum_{i=1}^k |a_i|^p \|\nabla\phi_i\|^p_{L^p(\Omega)} \leq \sum_{i=1}^k |a_i|^p \|\nabla\phi\|^p_{L^p(B(0,r))}.$$

As the boundary of Ω is regular we have that there exists a constant C_k such that

$$\|v\|^p_{L^p(\partial\Omega)} = \left\| \sum_{i=1}^k a_i \phi_i \right\|^p_{L^p(\partial\Omega)} = \sum_{i=1}^k |a_i|^p \|\phi_i\|^p_{L^p(\partial\Omega)} \geq C_k \sum_{i=1}^k |a_i|^p.$$

Using these estimates in (12) we obtain

$$0 < c \leq \frac{\lambda_2(\Omega_\mu)}{\mu^{1-p}} \leq \frac{\lambda(\Omega_\mu)}{\mu^{1-p}} \leq \frac{\lambda_k(\Omega_\mu)}{\mu^{1-p}} \leq C_k < +\infty$$

and the result follows.

Finally we study the convergence of the eigenvalues and eigenfunctions corresponding to the rest of the spectrum. By our hypotheses we have that

$$\lim_{j\to\infty} \frac{\lambda_j}{\mu_j^{1-p}} = \lambda.$$

As v_j is bounded in $W^{1,p}(\Omega)$ we can extract a subsequence (that we still call v_j) such that

$$\begin{aligned} v_j &\rightharpoonup v \quad \text{weakly in } W^{1,p}(\Omega), \\ v_j &\to v \quad \text{in } L^p(\Omega), \\ v_j &\to v \quad \text{in } L^p(\partial\Omega). \end{aligned}$$

Using that v_j are solutions of (2), we obtain

$$\int_\Omega |\nabla v_j|^{p-2} \nabla v_j \nabla\phi + \mu_j^p |v_j|^{p-2} v_j \phi \, dx = \frac{\lambda_j}{\mu_j^{1-p}} \int_{\partial\Omega} |v_j|^{p-2} v_j \phi \, d\sigma. \tag{13}$$

Taking $\phi \equiv 1$ we get

$$\int_\Omega \mu_j^p |v_j|^{p-2} v_j \, dx = \frac{\lambda_j}{\mu_j^{1-p}} \int_{\partial\Omega} |v_j|^{p-2} v_j \, d\sigma.$$

The limit as $j \to \infty$ gives us

$$0 = \lambda \int_{\partial\Omega} |v|^{p-2} v \, d\sigma$$

and, as $\lambda \neq 0$, we obtain that

$$0 = \int_{\partial\Omega} |v|^{p-2} v \, d\sigma. \tag{14}$$

By Lemma 2.2 and (14), there exists a unique $w \in W^{1,p}(\Omega)$ with

$$\int_{\partial\Omega} |w|^{p-2} w \, d\sigma = 0$$

that satisfies

$$\begin{cases} \Delta_p w = 0 & \text{in } \Omega, \\ |\nabla w|^{p-2} \frac{\partial w}{\partial \nu} = \lambda |v|^{p-2} v & \text{on } \partial\Omega. \end{cases} \tag{15}$$

Combining (13), the variational formulation of (15) with $\phi = v_j - w$ and the fact that we are dealing with a strongly monotone operator (see [3]), we get

$$\begin{aligned} \alpha \quad & \|\nabla v_j - \nabla w\|^p_{L^p(\Omega)} \leq \int_\Omega (|\nabla v_j|^{p-2}\nabla v_j - |\nabla w|^{p-2}\nabla w)(\nabla v_j - \nabla w)\, dx \\ = \quad & -\mu_j^p \int_\Omega |v_j|^{p-2} v_j (v_j - w)\, dx + \frac{\lambda_j}{\mu_j^{1-p}} \int_{\partial\Omega} |v_j|^{p-2} v_j (v_j - w)\, d\sigma \\ & -\lambda \int_{\partial\Omega} |v|^{p-2} v (v_j - w)\, d\sigma \\ \leq \quad & C\mu_j^p + \left(\frac{\lambda_j}{\mu_j^{1-p}} - \lambda \right) \int_{\partial\Omega} |v_j|^{p-2} v_j (v_j - w)\, d\sigma \\ & +\lambda \int_{\partial\Omega} (|v_j|^{p-2} v_j - |v|^{p-2} v)(v_j - w)\, d\sigma. \end{aligned}$$

The first two terms go to zero as $j \to \infty$. Concerning the last one, we have that it is bounded by

$$\begin{aligned} & (\|v_j\|_{L^p(\partial\Omega)} + \|v\|_{L^p(\partial\Omega)})^{p-2} \|v_j - v\|_{L^p(\partial\Omega)} \|v_j - w\|_{L^p(\partial\Omega)} && \text{if } p \geq 2, \\ & M \|v_j - v\|^{p-1}_{L^p(\partial\Omega)} \|v_j - w\|_{L^p(\partial\Omega)} && \text{if } p < 2. \end{aligned}$$

Therefore, taking the limit $j \to \infty$, we get $\nabla v_j \to \nabla w$ in $L^p(\Omega)$ and as $\nabla v_j \rightharpoonup \nabla v$ weakly in $L^p(\Omega)$ we conclude that $\nabla v = \nabla w$ and so $v = w$ and $v_j \to v$ strongly in $W^{1,p}(\Omega)$. Finally, taking limits in (13) we obtain that v is a weak solution of (7) as we wanted to prove. □

3. **Behavior as $\mu \to +\infty$.** In this section we study the behavior of the Sobolev constant in expanding domains, that is when $\mu \to +\infty$. To clarify the exposition we divide the proof of Theorem 1.3 in several Lemmas. Let us begin by the upper bounds.

LEMMA 3.1. *Let $p = q$, then there exists a constant $C > 0$ such that $S_p(\Omega_\mu) = \lambda_1(\Omega_\mu) \leq C$, for every μ large.*

Proof. We have $p = q$ and look for a bound on the first eigenvalue $\lambda_1(\Omega_\mu)$. Changing variables as before we have that

$$\lambda_1(\Omega_\mu) = \inf_{v \in W^{1,p}(\Omega)} \frac{\mu \left(\displaystyle\int_\Omega \mu^{-p} |\nabla v|^p + |v|^p \, dx \right)}{\displaystyle\int_{\partial\Omega} |v|^p d\sigma}.$$

We choose $v(x)$ such that $v = a = \mathit{constant}$ on $\partial\Omega$ and $v = 0$ in $\Omega_r = \{x \in \Omega\ ;\ \mathit{dist}(x, \partial\Omega) \geq r\}$ with $|\nabla v| \leq C/r$. We fix a such that

$$\int_{\partial\Omega} |v|^p \, d\sigma = 1,$$

that is $a = |\partial\Omega|^{-1/p}$. As for r small we have that $|\Omega \setminus \Omega_r| \sim r|\partial\Omega|$ we get

$$\int_\Omega |v|^p \, d\sigma \leq Cr.$$

Using that $|\nabla v| \leq C/r$ we obtain

$$\int_\Omega |\nabla v|^p \, d\sigma \leq \frac{C}{r^{p-1}},$$

therefore

$$\lambda_1(\Omega_\mu) \leq C\mu \left(C\frac{\mu^{-p}}{r^{p-1}} + Cr \right).$$

Finally, choose $r = \mu^{-1}$ to obtain the desired result. $\square$

LEMMA 3.2. *Let $p < q < p^*$, then there exists a constant $C > 0$ such that $S_q(\Omega_\mu) \leq C$, for every μ large.*

Proof. As we mentioned in the introduction, we have that

$$S_q(\Omega_\mu) = \mu^{(Nq-Np+p)/q} \inf_{v \in W^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega \mu^{-p} |\nabla v|^p + |v|^p \, dx}{\left(\displaystyle\int_{\partial\Omega} |v|^q \, d\sigma \right)^{p/q}}. \tag{16}$$

Now, let us choose a point $x_0 \in \partial\Omega$ and let $\phi \in C^\infty(\Omega)$ with support $B(x_0, \mu^{-1})$, and $\|\phi\|^q_{L^q(\partial\Omega)} = 1$.

Arguing as in Section 2, we have that

$$\mu^{(Nq-Np+p)/q} \int_\Omega |\phi|^p \, dx \leq C,$$

and

$$\mu^{(Nq-Np+p)/q} \mu^{-p} \int_\Omega |\nabla \phi|^p \, dx \leq C.$$

Therefore, taking $\phi = v$ in (16), we get $S_q(\Omega_\mu) \leq C$, and this ends the proof. $\square$

LEMMA 3.3. *Let $1 < q < p$, then we have $S_q(\Omega_\mu) \leq C\mu^{(N-1)(q-p)/q}$, for some constant $C > 0$. Remark that this says that $\lim_{\mu\to\infty} S_q(\Omega_\mu) = 0$.*

Proof. We observe that the same calculations of Lemma 3.2 show that S_q is bounded independently of μ for $1 < q < p$. Now, as in the case $p = q$ (Lemma 3.1), let us take $v(x)$ such that $v = a = constant$ on $\partial\Omega$ and $v = 0$ in $\Omega_r = \{x \in \Omega\ ;\ dist(x, \partial\Omega) \geq r\}$. We fix a such that

$$\int_{\partial\Omega} |v|^q \, d\sigma = 1.$$

Using the same arguments as in Lemma 3.1 we get

$$S_q(\Omega_\mu) \leq C\mu^{(Nq-Np+p)/q} \left(C\frac{\mu^{-p}}{r^{p-1}} + Cr \right)$$

and choosing $r = \mu^{-1}$ we obtain $S_q(\Omega_\mu) \leq C\mu^{(Nq-Np+p-q)/q}$. □

Now let us prove that the extremals concentrates at the boundary.

LEMMA 3.4. *Let $1 < q < p^*$. The extremals concentrate at the boundary in the sense that*

$$\int_\Omega |v_\mu|^p \, dx \to 0 \qquad as\ \mu \to +\infty,$$

while

$$\int_{\partial\Omega} |v_\mu|^q \, d\sigma = 1.$$

Proof. Let v_μ be an extremal such that $\|v_\mu\|_{L^q(\partial\Omega)} = 1$. From our previous bound we get, for $p = q$,

$$\mu^{1-p} \int_\Omega |\nabla v_\mu|^p \, dx + \mu \int_\Omega |v_\mu|^p \, dx \leq C$$

Hence

$$\int_\Omega |v_\mu|^p \, dx \leq \frac{C}{\mu} \to 0 \qquad \text{as } \mu \to +\infty.$$

Now we turn back to the case $1 < q < p$. We have, from our previous calculations,

$$S_q(\Omega_\mu) \leq C\mu^{(Nq-Np+p-q)/q}.$$

Hence

$$\int_\Omega |v_\mu|^p \, dx \leq C\mu^{(N-1)(q-p)/q} \to 0 \qquad \mu \to +\infty.$$

Finally, for $p < q < p^*$ we get that

$$\mu^{(Nq-Np+p)/q} \int_\Omega |v_\mu|^p \, dx \leq C$$

and therefore, as we are in the case $q > p$ and so $Nq > p(N-1)$, we get

$$\int_\Omega |v_\mu|^p \, dx \leq \frac{C}{\mu^{(Nq-Np+p)/q}} \to 0 \qquad \text{as } \mu \to +\infty.$$

The proof is now complete. □

To get the bound from below for λ_1 in the case $p = q$ we use the following idea, first we replace the minimization problem in $W^{1,p}(\Omega)$ with a minimization problem in a sequence of increasing subspaces and next we find that for an adequate choice of the subspaces we get a uniform lower bound for the approximate problems. This idea combined with a convergence result for the approximations gives the desired result. So, let us first state and prove the convergence result. Since this procedure works for every $1 < q < p^*$ we prove it in full generality.

Now we want to describe a general approximation procedure for S_q. These results are essentially contained in [14] but we reproduce the main arguments here in order to make the paper self-contained.

The Sobolev trace constant S_q can be characterized as

$$S_q = \inf_{v\in W^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla v|^p + |v|^p \, dx; \qquad \int_{\partial\Omega} |v|^q \, d\sigma = 1. \right\}. \tag{17}$$

As we have already mentioned, the idea is to replace the space $W^{1,p}(\Omega)$ with a subspace V_h in the minimization problem (17). To this end, let V_h be an increasing sequence of closed subspaces of $W^{1,p}(\Omega)$, such that

$$\begin{aligned} &\left\{ u_h \in V_h; \ \int_{\partial\Omega} |u_h|^q \, d\sigma = 1 \right\} \neq \emptyset \\ \text{and} \quad & \\ &\lim_{h\to 0} \inf_{u_h\in V_h} \|v - u_h\|_{W^{1,p}(\Omega)} = 0, \quad \forall \|v\|_{W^{1,p}(\Omega)} = 1. \end{aligned} \tag{18}$$

We observe that the only requirement on the subspaces V_h is (18). This allows us to choose V_h as the usual finite elements spaces, for example.

With this sequence of subspaces V_h we define our approximation of S_q by

$$S_{q,h} = \inf_{u_h\in V_h} \left\{ \int_\Omega |\nabla u_h|^p + |u_h|^p \, dx; \qquad \int_{\partial\Omega} |u_h|^q \, d\sigma = 1 \right\}. \tag{19}$$

We have that, under hypothesis (18), $S_{q,h}$ approximates S_q when $h \to 0$.

Theorem 3.1. *Let v be an extremal for* (17). *Then, there exists a constant C independent of h such that,*

$$|S_q - S_{q,h}| \leq C \inf_{u_h\in V_h} \|u_h - v\|_{W^{1,p}(\Omega)},$$

for every h small enough.

Proof. As $V_h \subset W^{1,p}(\Omega)$ we have that

$$S_q \leq S_{q,h}. \tag{20}$$

Let us choose $w \in V_h$ such that $\|w - v\|_{W^{1,p}(\Omega)} \leq \inf_{V_h} \|v - u_h\|_{W^{1,p}(\Omega)} + \varepsilon$. We have

$$\begin{aligned} S_{q,h}^{1/p} &= \|u_h\|_{W^{1,p}(\Omega)} \leq \frac{\|w\|_{W^{1,p}(\Omega)}}{\|w\|_{L^q(\partial\Omega)}} \\ &\leq \frac{\|w - v\|_{W^{1,p}(\Omega)} + \|v\|_{W^{1,p}(\Omega)}}{\|w\|_{L^q(\partial\Omega)}} \\ &= \left(\frac{\|w - v\|_{W^{1,p}(\Omega)} + S_q^{1/p}}{\|w\|_{L^q(\partial\Omega)}} \right). \end{aligned}$$

Now we use that

$$|\|w\|_{L^q(\partial\Omega)} - 1| \leq |\|w\|_{L^q(\partial\Omega)} - \|v\|_{L^q(\partial\Omega)}| \leq \|w - v\|_{L^q(\partial\Omega)} \leq C\|w - v\|_{W^{1,p}(\Omega)}$$

and hypothesis (18) to obtain that for every h small enough,

$$S_{q,h} \leq \left(\frac{\|w - v\|_{W^{1,p}(\Omega)} + S_q^{1/p}}{1 - C\|w - v\|_{W^{1,p}(\Omega)}} \right)^p \leq S_q + C\|w - v\|_{W^{1,p}(\Omega)}. \tag{21}$$

The result follows from (20) and (21). $\square$

Now we prove a result regarding the convergence of the approximate extremals. We will not use it but it completes the analysis of the approximations.

THEOREM 3.2. *Let u_h be a function in V_h where the infimum* (19) *is archived. Then from any sequence $h \to 0$ we can extract a subsequence $h_j \to 0$ such that u_{h_j} converges strongly to an extremal in $W^{1,p}(\Omega)$. That is, there exists an extremal of* (17), *v, with*

$$\lim_{h_j \to 0} \|u_{h_j} - v\|_{W^{1,p}(\Omega)} = 0.$$

Proof. Theorem 3.1 and hypothesis (18) gives that

$$\lim_{h \to 0} \|u_h\|^p_{W^{1,p}(\Omega)} = \lim_{h \to 0} S_{q,h} = S_q.$$

Hence there exists a constant C such that for every h small enough, $\|u_h\|_{W^{1,p}(\Omega)} \le C$. Therefore we can extract a subsequence, that we denote by u_{h_j}, such that

$$\begin{aligned} u_{h_j} &\rightharpoonup w && \text{weakly in } W^{1,p}(\Omega), \\ u_{h_j} &\to w && \text{strongly in } L^p(\Omega), \\ u_{h_j} &\to w && \text{strongly in } L^q(\partial\Omega). \end{aligned} \tag{22}$$

Hence, from the $L^q(\partial\Omega)$ convergence we have,

$$1 = \lim_{h_j \to 0} \int_{\partial\Omega} |u_{h_j}|^q \, d\sigma = \int_{\partial\Omega} |w|^q \, d\sigma.$$

Therefore w is an admissible function in the minimization problem (17). Now we observe that, if v is an extremal,

$$\begin{aligned} \|v\|^p_{W^{1,p}(\Omega)} &\le \|w\|^p_{W^{1,p}(\Omega)} \le \liminf_{h_j \to 0} \|u_{h_j}\|^p_{W^{1,p}(\Omega)} \\ &\le \lim_{h_j \to 0} \|u_{h_j}\|^p_{W^{1,p}(\Omega)} = \lim_{h_j \to 0} S_{q,h} = S_q = \|v\|^p_{W^{1,p}(\Omega)}, \end{aligned}$$

and therefore,

$$\lim_{h_j \to 0} \|u_{h_j}\|_{W^{1,p}(\Omega)} = \|w\|_{W^{1,p}(\Omega)} = S_q^{1/p}. \tag{23}$$

The space $W^{1,p}(\Omega)$ being uniformly convex, the weak convergence, (22), and the convergence of the norms, (23), imply the convergence in norm. Therefore $u_{h_j} \to w$ in $W^{1,p}(\Omega)$. This limit w verifies $\|w\|^p_{W^{1,p}(\Omega)} = S_q$ and $\|w\|_{L^q(\partial\Omega)} = 1$. Hence it is an extremal and we have that $\lim_{h_j \to 0} \|u_{h_j} - w\|_{W^{1,p}(\Omega)} = 0$. $\square$

With these convergence results we can prove the lower bound in the case $p = q$.

LEMMA 3.5. *Let $p = q$, then $S_p(\Omega_\mu) = \lambda_1(\Omega_\mu) \ge C$, for every μ large.*

Proof. Let us choose a particular subspace V_h of $W^{1,p}(\Omega)$. As the boundary of Ω is smooth, we can define new coordinates near the boundary as follows. As before we denote by $\Omega_r = \{x \in \Omega; dist(x, \partial\Omega) \ge r\}$ and by $\partial\Omega_r = \{x \in \Omega; dist(x, \partial\Omega) = r\}$ and we use the following construction. We define $\Phi(\xi, r) = \xi - r\nu(\xi)$, where $\nu(\xi)$ is the exterior normal vector at $\xi \in \partial\Omega$. $\Phi : \partial\Omega \times (0, R) \mapsto \Omega \setminus \overline{\Omega}_R$. We recall that Φ is a difeomorphism if R is small enough. With this application Φ we can define a triangulation as follows. First, choose a uniform regular triangulation of size h of the set $\partial\Omega \times (0, R)$. Now, by the application Φ we can get a triangulation of the strip $\Omega \setminus \overline{\Omega}_R$. In fact, we can select as nodes x_{ij} the points $\Phi(\xi_i, r_j)$, where

(ξ_i, r_j) is a node of the uniform mesh of $\partial\Omega \times (0, R)$. Our space V_h is defined by all the continuous functions in $W^{1,p}(\Omega)$ that are linear over each triangle of the strip $\Omega \setminus \overline{\Omega}_R$. This space is the usual space of linear finite elements in special triangulations defined using the mapping Φ, see [3] for detailed information on the finite elements method.

Let us call u_h the functions in V_h. We have indexed the nodes x_{ij} in a way such that $x_{i1} \in \partial\Omega$ and x_{ij} is at distance $j-1$ (in nodes) from the boundary, $\partial\Omega$. We denote by u_{ij} the value of u_h at the node x_{ij} and by a_{ij} the value of the gradient of u_h on the triangle T_{ij}. We assume that the index i runs from 1 to l and j from 1 to k_0. Remark that $k_0 \sim R/h$ and $l \sim |\partial\Omega|/h^{N-1}$.

We want to find a lower bound (independent of h and μ) on the approximation of the first eigenvalue,

$$\lambda_{1,h}(\Omega_\mu) = \inf_{u_h \in V_h} \left\{ \mu^{1-p} \int_\Omega |\nabla u_h|^p \, dx + \mu \int_\Omega |u_h|^p \, dx; \qquad \int_{\partial\Omega} |u_h|^p \, d\sigma = 1 \right\}.$$

To this end we consider a function $u_h \in V_h$ such that

$$\int_{\partial\Omega} |u_h|^p \, d\sigma = 1,$$

that is

$$\sum_{i=1}^{l} |u_{i1}|^p h^{N-1} \geq C_1$$

Let k be the first integer in $[1, k_0]$ such that

$$\sum_{i=1}^{l} |u_{ik}|^p h^{N-1} \leq \frac{C_1}{2}$$

First, let us observe that if $k = k_0$ (there are k_0 triangles between the two boundaries of $\Omega \setminus \Omega_r$), then we have

$$\begin{aligned} \mu \int_\Omega |u_h|^p \, dx \geq & \quad \mu \sum_{j=2}^{k_0} \sum_{i=1}^{l} \int_{T_{ij}} |u_h|^p \, dx \geq C\mu \sum_{j=2}^{k_0} \sum_{i=1}^{l} |u_{ij}|^p h^N \\ = & \quad Ch\mu \sum_{j=2}^{k_0} \sum_{i=1}^{l} |u_{ij}|^p h^{N-1} \geq Ch\mu k_0 \frac{C_1}{2}. \end{aligned}$$

As $k_0 \sim R/h$ we get that

$$\begin{aligned} \lambda_{1,h}(\Omega_\mu) &= \inf_{u_h \in V_h} \left\{ \mu^{1-p} \int_\Omega |\nabla u_h|^p \, dx + \mu \int_\Omega |u_h|^p \, dx; \qquad \int_{\partial\Omega} |u_h|^p \, d\sigma = 1 \right\} \\ &\geq \inf_{u_h \in V_h} \left\{ \mu \int_\Omega |u_h|^p \, dx; \qquad \int_{\partial\Omega} |u_h|^p \, d\sigma = 1 \right\} \geq C\mu > 1 \end{aligned}$$

and we are done. Hence let us assume that $k < k_0$. As before we can bound the term $\mu \int_\Omega |u_h|^p$ by

$$\mu \int_\Omega |u_h|^p \, dx \geq C\mu \sum_{j=2}^{k} \sum_{i=1}^{l} |u_{ij}|^p h^N = Ch\mu \sum_{j=2}^{k} \sum_{i=1}^{l} |u_{ij}|^p h^{N-1} \geq Ch\mu k \frac{C_1}{2}. \quad (24)$$

Now we observe that

$$u_{i1} - u_{ik} = \sum_{j=1}^{k} a_{ij} h.$$

Using this fact we get,

$$\begin{aligned} C &\le \left| \left(\frac{1}{l} \sum_{i=1}^{l} |u_{i1}|^p \right)^{1/p} - \left(\frac{1}{l} \sum_{i=1}^{l} |u_{ik}|^p \right)^{1/p} \right| \\ &\le \left(\frac{1}{l} \sum_{i=1}^{l} |u_{i1} - u_{ik}|^p \right)^{1/p} = \left(\frac{k^p}{l} \sum_{i=1}^{l} \left| \frac{1}{k} \sum_{j=1}^{k} a_{ij} h \right|^p \right)^{1/p}. \end{aligned}$$

Hence we get

$$\frac{Cl}{k^{p-1}h^p} \le \sum_{i=1}^{l} \frac{1}{k} \sum_{j=1}^{k} |a_{ij}|^p$$

and finally,

$$\mu^{1-p} \int_\Omega |\nabla u_h|^p \, dx \ge \frac{C\mu^{1-p} l h^{N-1}}{k^{p-1}h^{p-1}} \ge \frac{C\mu^{1-p}}{k^{p-1}h^{p-1}}. \tag{25}$$

Using (24) and (25) we obtain

$$\begin{aligned} \lambda_{1,h}(\Omega_\mu) &= \inf_{u_h \in V_h} \left\{ \mu^{1-p} \int_\Omega |\nabla u_h|^p \, dx + \mu \int_\Omega |u_h|^p \, dx; \quad \int_{\partial\Omega} |u_h|^p \, d\sigma = 1 \right\} \\ &\ge C(\mu h k) + \frac{C}{(\mu h k)^{p-1}}. \end{aligned}$$

Hence, if we call $\tau = \mu h k$ we get that

$$\lambda_{1,h}(\Omega_\mu) \ge F(\tau) \equiv C\tau + \frac{C}{\tau^{p-1}} \ge C.$$

Since the subspaces that we have chosen verify hypotheses (18), we can use the convergence result, Theorem 3.1, to get that $\lambda_1(\Omega_\mu) = \lim_{h\to 0} \lambda_{1,h}(\Omega_\mu) \ge C$. □

Let us look at the case $1 < q < p$ more carefully, and obtain a bound from below using the lower bound obtained for $\lambda_1(\Omega_\mu)$.

LEMMA 3.6. *Let $1 < q < p$. Then, for every μ large, $S_q(\Omega_\mu) \ge C\mu^{\beta_{pq}-1}$. Moreover this shows that, if v is an extremal,*

$$c_1 \left(\int_{\partial\Omega} |v|^q \, d\sigma \right)^{1/q} \ge \left(\int_{\partial\Omega} |v|^p \, d\sigma \right)^{1/p} \ge c_2 \left(\int_{\partial\Omega} |v|^q \, d\sigma \right)^{1/q}.$$

Hence there is no peaking formation in this case.

Proof. As we mentioned in the introduction, we have that

$$\begin{aligned}
S_q(\Omega_\mu) = \;& \mu^{(Nq-Np+p)/q} \inf_{v\in W^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega \mu^{-p}|\nabla v|^p + |v|^p\,dx}{\left(\displaystyle\int_{\partial\Omega} |v|^q\,d\sigma\right)^{p/q}} \\
= \;& \mu^{\beta_{pq}-1} \inf_{v\in W^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega \mu^{1-p}|\nabla v|^p + \mu|v|^p\,dx}{\left(\displaystyle\int_{\partial\Omega} |v|^q\,d\sigma\right)^{p/q}} \\
= \;& \mu^{\beta_{pq}-1} \inf_{v\in W^{1,p}(\Omega)} \frac{\displaystyle\int_\Omega \mu^{1-p}|\nabla v|^p + \mu|v|^p\,dx}{\displaystyle\int_{\partial\Omega} |v|^p\,d\sigma} \frac{\displaystyle\int_{\partial\Omega} |v|^p\,dx}{\left(\displaystyle\int_{\partial\Omega} |v|^q\,d\sigma\right)^{p/q}}.
\end{aligned}$$

Using that $1 < q < p$ we get that, by Holder's inequality

$$\frac{\displaystyle\int_{\partial\Omega} |v|^p\,dx}{\left(\displaystyle\int_{\partial\Omega} |v|^q\,d\sigma\right)^{p/q}} \geq C.$$

Hence, using our previous lower bound for $\lambda_1(\Omega_\mu)$ we get that there exists a constant C such that $S_q(\Omega_\mu) \geq C\mu^{\beta_{pq}-1}$. The upper bound proved in Lemma 3.3, $S_q(\Omega_\mu) \leq C\mu^{\beta_{pq}-1}$, gives that

$$\begin{aligned}
C\mu^{\beta_{pq}-1} \geq S_q(\Omega_\mu) = \mu^{\beta_{pq}-1} & \frac{\displaystyle\int_\Omega \mu^{1-p}|\nabla v_\mu|^p + \mu|v_\mu|^p\,dx}{\displaystyle\int_{\partial\Omega} |v_\mu|^p\,d\sigma} \frac{\displaystyle\int_{\partial\Omega} |v_\mu|^p\,dx}{\left(\displaystyle\int_{\partial\Omega} |v_\mu|^q\,d\sigma\right)^{p/q}} \\
\geq C\mu^{\beta_{pq}-1} & \frac{\displaystyle\int_{\partial\Omega} |v_\mu|^p\,dx}{\left(\displaystyle\int_{\partial\Omega} |v_\mu|^q\,d\sigma\right)^{p/q}}.
\end{aligned}$$

Hence

$$\int_{\partial\Omega} |v_\mu|^p\,dx \leq C\left(\int_{\partial\Omega} |v_\mu|^q\,d\sigma\right)^{p/q}.$$

This ends the proof. $\square$

To finish the proof of Theorem 1.3 we need the following Lemma.

LEMMA 3.7. *Let $p < q < p^*$. Then, for large μ, $S_q(\Omega_\mu) \geq C$. Moreover, the extremals concentrates in the sense that $a^p|\partial\Omega \cap \{v_\mu > a\}| \to 0$, as $\mu \to +\infty$, with* $\max_{\overline{\Omega}} v_\mu = 1$.

Proof. First we prove that there exists a constant C such that $S_q(\Omega_\mu) \geq C$. Let v_μ be an extremal in Ω. By rescaling v_μ we can obtain an extremal $\tilde{v}_\mu$ such that

$\max_{\overline{\Omega}} \tilde{v}_\mu = 1$. That is, $0 < \tilde{v}_\mu \le 1$ and there exits a point $x_0 \in \partial\Omega$ with $\tilde{v}_\mu(x_0) = 1$. Arguing as in Lemma 3.6 we have

$$S_q(\Omega_\mu) = \mu^{\beta_{pq}-1} \frac{\displaystyle\int_\Omega \mu^{1-p}|\nabla \tilde{v}_\mu|^p + \mu|\tilde{v}_\mu|^p\, dx}{\displaystyle\int_{\partial\Omega} |\tilde{v}_\mu|^p\, d\sigma} \frac{\displaystyle\int_{\partial\Omega} |\tilde{v}_\mu|^p\, dx}{\left(\displaystyle\int_{\partial\Omega} |\tilde{v}_\mu|^q\, d\sigma\right)^{p/q}}. \tag{26}$$

As $\tilde{v}_\mu$ satisfies (2), by our hypothesis, we have that $|\nabla \tilde{v}_\mu| \le C\mu$. Hence

$$\{x \in \partial\Omega;\ \tilde{v}_\mu(x) \ge 1/2\} \supseteq B(x_0, c/\mu) \cap \partial\Omega.$$

As $q > p$ and $0 < \tilde{v}_\mu \le 1$ we have that

$$\int_{\partial\Omega} |\tilde{v}_\mu|^p\, d\sigma \ge \int_{\partial\Omega} |\tilde{v}_\mu|^q\, d\sigma.$$

Therefore

$$\begin{aligned}
\mu^{\beta_{pq}-1} \frac{\displaystyle\int_{\partial\Omega} |\tilde{v}_\mu|^p\, dx}{\left(\displaystyle\int_{\partial\Omega} |\tilde{v}_\mu|^q\, d\sigma\right)^{p/q}} &\ge \mu^{\beta_{pq}-1}\left(\int_{\partial\Omega} |\tilde{v}_\mu|^p\, dx\right)^{(q-p)/q} \\
&\ge C\mu^{\beta_{pq}-1}\left(\int_{\partial\Omega\cap B(x_0,c/\mu)} \frac{1}{2^p}\, dx\right)^{(q-p)/q} \ge C.
\end{aligned}$$

Using this bound and the lower bound for $S_p(\Omega_\mu)$ in (26) we get the desired lower bound. Next, we prove the concentration property for the extremals. Using the same arguments as before, we get

$$a^p|\partial\Omega \cap \{\tilde{v}_\mu > a\}| \le \int_{\partial\Omega} |\tilde{v}_\mu|^p\, d\sigma \le \frac{C}{\mu^{N-1}} \to 0, \qquad \text{as } \mu \to +\infty,$$

with $\max_{\overline{\Omega}} \tilde{v}_\mu = 1$. This proves the concentration phenomena. □

We end the article proving that every eigenvalue is bounded as $\mu \to +\infty$.

Proof of Theorem 1.4. The idea is similar as the one used in the proof of Theorem 1.2, see Section 2. Let $x_1, \dots, x_k \in \partial\Omega$ such that $dist(x_i, x_j) > 2\mu$ and let $\phi_j \in C^\infty(\Omega)$ with support $B(x_j, \mu)$ and $\max \phi_j = 1$. Now, let us define $S_k = span\{\phi_1, \dots, \phi_k\} \cap \{u \in W^{1,p}(\Omega);\ \|u\|_{W^{1,p}(\Omega)} = 1\}$ and $S_{k,\mu} = \{v(x/\mu);\ v \in S_k\}$. Then, $\gamma(S_k) = \gamma(S_{k,\mu}) = k$. Hence

$$\frac{1}{\lambda_k(\Omega_\mu)} = \sup_{\gamma(S)\ge k} \inf_{u\in S} \frac{\displaystyle\int_{\partial\Omega_\mu} |u|^p\, d\sigma}{\displaystyle\int_{\Omega_\mu} |\nabla u|^p + |u|^p\, dx} \ge \inf_{u\in S_{k,\mu}} \frac{\displaystyle\int_{\partial\Omega_\mu} |u|^p\, d\sigma}{\displaystyle\int_{\Omega_\mu} |\nabla u|^p + |u|^p\, dx}.$$

Changing variables we get,

$$\frac{1}{\lambda_k(\Omega_\mu)} \ge \mu^{p-1} \inf_{v\in S_k} \frac{\displaystyle\int_{\partial\Omega} |v|^p\, d\sigma}{\displaystyle\int_\Omega |\nabla v|^p + \mu^p|v|^p\, dx}. \tag{27}$$

As ϕ_i have disjoint support,

$$\|v\|^p_{L^p(\Omega)} = \left\|\sum_{i=1}^k a_i\phi_i\right\|^p_{L^p(\Omega)} = \sum_{i=1}^k |a_i|^p \|\phi_i\|^p_{L^p(\Omega)} \le C\sum_{i=1}^k |a_i|^p \mu^{-N}$$

and

$$\|\nabla v\|^p_{L^p(\Omega)} = \left\|\sum_{i=1}^k a_i\nabla\phi_i\right\|^p_{L^p(\Omega)} = \sum_{i=1}^k |a_i|^p \|\nabla\phi_i\|^p_{L^p(\Omega)} \le C\sum_{i=1}^k |a_i|^p \mu^{-N+p}.$$

As the boundary of Ω is regular we have that there exists a constant C such that

$$\|v\|^p_{L^p(\partial\Omega)} = \left\|\sum_{i=1}^k a_i\phi_i\right\|^p_{L^p(\partial\Omega)} = \sum_{i=1}^k |a_i|^p \|\phi_i\|^p_{L^p(\partial\Omega)} \ge C\sum_{i=1}^k |a_i|^p \mu^{1-N}.$$

Using these estimates we get $0 < c \le \lambda_1(\Omega_\mu) \le \lambda(\Omega_\mu) \le \lambda_k(\Omega_\mu) \le C_k < +\infty$. □

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