A free boundary problem for ∞–Laplace equation

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We consider a free boundary problem for the p-Laplacian

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u),$$

describing nonlinear potential flow past convex profile K with prescribed pressure gradient $|\nabla u(x)| = a(x)$ on the free stream line. The main purpose of this paper is to study the limit as $p \to \infty$ of the classical solutions of the problem above, existing under certain convexity assumptions on $a(x)$. We show, as one can expect, that the limit solves the corresponding problem for the ∞-Laplacian

$$\Delta_\infty u = \nabla^2 u \nabla u \cdot \nabla u,$$

in a certain weak sense, strong however, to guarantee the uniqueness. We show also that in the special case $a(x) \equiv a_0 > 0$ the limit coincides with an explicit solution, given by a distance function.

Key Words: free boundary problems; classical solutions; weak solutions; ∞–Laplacian; p–Laplacian.

1. INTRODUCTION

In the past few years there has been a renewed interest in geometric configurations in potential flow in fluid mechanics. This time, however, the focus is on nonlinear flows with power law generalization (see e.g. [AM], [HS1–3].) The latter refers to the *p–Laplace operator*

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u), \qquad 1 < p < \infty.$$

The peculiar nonlinearity and degeneracy of the operator make the problems of potential flow both more realistic and also much harder to analyze.

Our concern, in this paper, will be the limit case $p \to \infty$ of the work of A. Henrot and H. Shahgholian [HS1–3]. Let us first formulate the problem.

Let K be a compact convex subset in $\mathbf{R}^n$, and $a(x)$ a positive continuous function in $K^c = \mathbf{R}^n \setminus K$. For $p \in (1, \infty)$ consider the following Bernoulli-type free boundary problem: find a pair (u, Ω) where $\Omega \supset K$ is a domain in $\mathbf{R}^n$ and u is a nonnegative continuous function in $\overline{\Omega}$ such that

$$(FB_p) \qquad \begin{cases} \Delta_p u = 0 & \text{in } \Omega \setminus K \\ u = 0 \quad \text{and} \quad |\nabla u| = a & \text{on } \partial\Omega \\ u = 1 & \text{on } K. \end{cases}$$

Problem (FB_p), as mentioned earlier, describes nonlinear potential flow past the profile K and $\partial\Omega$ describes the stream line with prescribed pressure $|\nabla u| = a$. This problem for $p = 2$ is well studied and there is an extensive list of literature. We refer to the paper of A. Acker and R. Meyer [AM] for background and further references. We also mention the pioneering work of H. W. Alt and L. A. Caffarelli [AC] in this connection.

For $p \in (1, \infty)$, under suitable convexity assumptions on $a(x)$ and regularity assumptions on K to be specified later, problem (FB_p) admits a unique classical solution, which we denote by (u_p, Ω_p); see [HS3]. The main purpose of this paper is to study the behavior of the pair (u_p, Ω_p) as $p \to \infty$. For reader's convenience we outline some steps of the proof for $1 < p < \infty$ in the appendix.

It it known that the limits of p-harmonic functions (weak solutions of $\Delta_p u = 0$) as $p \to \infty$ are viscosity solutions of the equation $\Delta_\infty u = 0$ (in other words, ∞–harmonic function), where

$$\Delta_\infty u = \nabla^2 u \nabla u \cdot \nabla u,$$

see [BDM] (see also [Ar].) The operator Δ_∞ is known as the ∞–*Laplace operator*. Since the free boundary condition is the same for all problems (FB_p), we expect that the limit

$$(u_\infty, \Omega_\infty) = \lim_{p \to \infty} (u_p, \Omega_p),$$

if it exists, is a solution of the limiting problem

$$(FB_\infty) \qquad \begin{cases} \Delta_\infty u = 0 & \text{in } \Omega \setminus K \\ u = 0 \quad \text{and} \quad |\nabla u| = a & \text{on } \partial\Omega \\ u = 1 & \text{on } K \end{cases}$$

in a certain sense.

Another point of view is to treat problem (FB_∞) independently and interpret the limit process above as one of the ways of solving the problem. This approach gives rise to new kinds of questions such as uniqueness and regularity of solutions.

It is worth noting that in certain special cases one can write down a solution of (FB_∞) explicitly. For instance consider the case $a(x) \equiv a_0$. Then one can easily see that a pair (u, Ω) with

$$u(x) = 1 - a_0 \operatorname{dist}(x, K) \quad \text{and} \quad \Omega = \{\operatorname{dist}(x, K) < 1/a_0\} \tag{1.1}$$

solves problem (FB_∞) in the classical sense (see Section 3). Now one can ask whether the explicit solution (1.1) is a limit of solutions of (FB_p). We show that this is indeed the case and moreover (1.1) is the unique classical solution of (FB_∞) with $a(x) \equiv a_0$ (see Theorem 3.3.)

The complications in the case of more general $a(x)$ are connected with the fact that generally it is not known whether ∞–harmonic functions are C^1 regular or not, and the problem is how to interpret the free boundary condition. We overcome this difficulty by introducing the notion of the weak solution of (FB_∞). Then we show that the only weak solution of (FB_∞) is nothing but the limit $(u_\infty, \Omega_\infty)$ (see Theorem 4.4.)

We conclude the paper with the brief analysis of the limit $p \to 1+$. The following phenomenon takes place: the domains Ω_p shrink to K (in dimension $n \geq 2$.)

2. PRELIMINARIES

Throughout the paper we assume that K is a convex compact in $\mathbf{R}^n$, $n \geq 2$, which satisfies the *uniform interior ball condition*, i.e. there exists $\delta > 0$ such that for every $x \in \partial K$ there is a ball B of radius δ with the property $B \subset K$ and $\partial B \cap \partial K = \{x\}$.

Next the function $a(x)$ will be positive and continuous in $K^c = \mathbf{R}^n \setminus K$ with

$$\inf_{x \in K^c} a(x) = a_0 > 0. \tag{2.1}$$

We will assume also that $1/a(x)$ is locally concave in K^c. (Sometimes we will call such functions *harmonic concave*.) Moreover, assuming w.l.o.g. that the origin is an interior point of K, we will require

$$\lambda a(\lambda x) > a(x) \quad \text{for every } x \in K^c \text{ and } \lambda > 1. \tag{2.2}$$

We start with the definition.

DEFINITION 2.1. A pair (u, Ω), where Ω is a domain in $\mathbf{R}^n$ that contains K and u is a nonnegative continuous function on $\overline{\Omega}$, is called a *classical supersolution* of (FB_p) for $1 < p < \infty$ if

(i) $u \in C^1(\Omega \setminus K)$ and $\Delta_p u \leq 0$ in $\Omega \setminus K$ in the sense of distributions;
(ii) $\limsup_{y \to x} |\nabla u(y)| \leq a(x)$ for every $x \in \partial\Omega$;

(iii) $u \geq 1$ on K and $u = 0$ on $\partial\Omega$.

Analogously, we define *classical subsolutions* of (FB_p), by reversing the inequality signs in (i), (ii) and (iii) and replacing lim sup by lim inf in (ii).

A *classical solution* of (FB_p) is a pair (u, Ω) which is a classical sub- and supersolution at the same time.

As a straightforward consequence of condition (2.2), we have the following comparison principle.

LEMMA 2.2 (Comparison principle). *For problem (FB_p), every classical subsolution is smaller than every classical supersolution.*

More precisely, the lemma should be read: *If (u', Ω') is a classical sub- and (u, Ω) is a classical supersolution of (FB_p), then $\Omega' \subset \Omega$ and $u' \leq u$ in Ω'.*

Proof. We will consider first the case when u' and u are C^1 up to $\partial\Omega'$ and $\partial\Omega$ respectively. Suppose that $\Omega' \not\subset \Omega$. Then we have

$$\lambda_0 = \min\{\lambda \geq 1 \colon \lambda^{-1}\Omega' \subset \Omega\} > 1.$$

Here $\lambda^{-1}\Omega' = \{\lambda^{-1}x \colon x \in \Omega\} =: \Omega'_\lambda$. Then $\Omega'_{\lambda_0} \subset \Omega$ and there is a common point $x_0 \in \partial(\Omega'_{\lambda_0}) \cap \partial\Omega$. Consider the rescaled function

$$u'_{\lambda_0}(x) = u'(\lambda_0 x) \quad \text{in } \Omega'_{\lambda_0}.$$

Then usual comparison principle for p–harmonic functions says that $u'_{\lambda_0} \leq u$ in Ω'_{λ_0}. But $u'_{\lambda_0}(x_0) = u(x_0) = 0$, hence there must be

$$\lambda_0 a(\lambda_0 x_0) \leq |\nabla u_{\lambda_0}(x_0)| \leq |\nabla u(x_0)| \leq a(x_0),$$

which contradicts to (2.2). Therefore $\Omega' \subset \Omega$ and moreover $u' \leq u$ in Ω' by the comparison principle for p-harmonic functions.

The reasonings above are known as the *Lavrent'ev principle* (cf. [La]).

The case when u' and u are not C^1 up to the boundary can be handled in various ways, reducing it to the case above. One of the ways is to consider $u' - \varepsilon$ and $u - \varepsilon$ on the level sets $\Omega'_\varepsilon = \{u' \geq \varepsilon\}$ and $\Omega_\varepsilon = \{u \geq \varepsilon\}$ respectively and deduce that the constant $\lambda_0 = \lambda_0(\varepsilon)$ as above but for Ω'_ε and Ω_ε tends to 1 as ε goes to $0+$. ∎

The comparison lemma above gives us the uniqueness of the classical solution for (FB_p) if it exists. The following existence theorem is due to A. Henrot and H. Shahgholian. For reader's convenience we give a brief sketch of the proof of this theorem at the end of this paper.

THEOREM 2.3 ([HS3]). *There exists a unique classical solution* (u_p, Ω_p) *of the free boundary problem* (FB_p). *Moreover* Ω_p *is convex and* $\partial\Omega_p$ *is* C^1 *regular.* ∎

Next we will use the favorable geometric situation to compare solutions of (FB_p) for different p. This lemma has appeared already in [Ja].

LEMMA 2.4. *Let* $\Omega \supset K$ *be a convex domain and* u *be the* p*–capacitary potential* ($1 < p < \infty$) *of the convex ring* $\Omega \setminus K$, *i.e. a continuous function on* $\overline{\Omega \setminus K}$ *such that*

$$\Delta_p u = 0 \text{ in } \Omega \setminus K; \quad u = 0 \text{ on } \partial\Omega; \quad u = 1 \text{ on } \partial K.$$

Then

$$\begin{aligned} \Delta_q u &\leq 0, && \text{if } 1 < q \leq p \text{ and} \\ \Delta_q u &\geq 0, && \text{if } p \leq q \leq \infty. \end{aligned}$$

Proof. By a result of J. Lewis [Le], $|\nabla u| > 0$ in $\Omega \setminus K$ and therefore u is a real analytic in $\Omega \setminus K$. Then one can write $\Delta_p u$ in a nondivergence form

$$\Delta_p u = |\nabla u|^{p-2}(\Delta u + |\nabla u|^{-2}(p-2)\Delta_\infty u). \tag{2.3}$$

Let us introduce operators

$$L_p u := |\nabla u|^{2-p}\Delta_p u = \Delta u + |\nabla u|^{-2}(p-2)\Delta_\infty u. \tag{2.4}$$

Then $L_q u$ has always the same sign as $\Delta_q u$ for every $1 < q < \infty$. Using now assumption that $\Delta_p u = 0$ in $\Omega \setminus K$ and hence $L_p u = 0$, we deduce that

$$L_q u = |\nabla u|^{-2}(q-p)\Delta_\infty u \quad \text{in } \Omega \setminus K. \tag{2.5}$$

Hence the lemma will follow as soon as we prove

$$\Delta_\infty u \geq 0. \tag{2.6}$$

To this end, take a point $x \in \Omega \setminus K$ and choose a coordinate system centered at x such that the n–axis is directed as ∇u. Let $u(x) = s$. By [Le] the level set $\mathcal{L}_s(u) = \{y : u(y) > s\}$ is convex and therefore its boundary $\ell_s(u) = \partial\mathcal{L}_s(u)$ can be given near the point x as a graph $\{y : y_n = f(y')\}$ of a convex C^∞ function f in

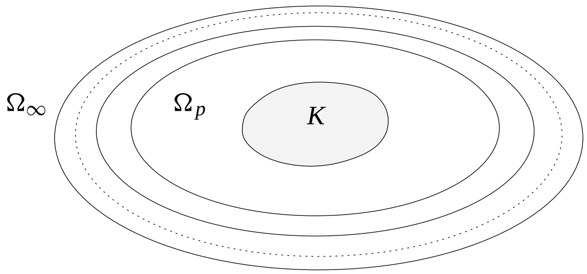

FIG. 1. The variational solution: $\Omega_\infty = \lim_{p\to\infty} \Omega_p$

the coordinate system chosen above (here $y' = (y_1, \dots, y_{n-1})$.) Differentiating twice the identity $u(y', f(y')) = s$ and using that $f_i(0) = 0$ we obtain

$$u_{ii} + u_n f_{ii} = 0 \quad \text{at } 0 \text{ for } i = 1, \dots, n-1. \tag{2.7}$$

The indices mean differentiation with respect to the corresponding y-coordinates. In particular $u_n = |\nabla u|$ at 0. Observe now that in these new coordinates the condition $\Delta_p u = 0$ at x means precisely

$$\sum_{i=1}^{n-1} u_{ii} + (p-1)u_{nn} = 0 \quad \text{at } 0. \tag{2.8}$$

From (2.7) and (2.8) we obtain

$$\Delta_\infty u(x) = u_{nn}u_n^2 = \frac{n-1}{p-1}\kappa u_n^3 \geq 0, \tag{2.9}$$

where $\kappa = 1/(n-1)\sum_{i=1}^{n-1} f_{ii}$ is the mean curvature of $\ell_s(u)$ at the point x, which is nonnegative. Since x was an arbitrary point in $\Omega \setminus K$, the proof of the lemma is complete. ▮

COROLLARY 2.5 (Variational solution). *The classical solutions (u_p, Ω_p) of (FB_p) are nondecreasing in $p \in (1, \infty)$ and are uniformly bounded. More precisely, $\Omega_{p'} \subset \Omega_{p''}$ and $u_{p'} \leq u_{p''}$ in $\Omega_{p'}$ for every $1 < p' \leq p'' < \infty$ and there exists a bounded domain Ω_0 and a function u_0 on Ω_0 such that $\Omega_p \subset \Omega_0$ and $u_p \leq u_0$ for all $1 < p < \infty$. Hence there exists*

$$(u_\infty, \Omega_\infty) = \lim_{p\to\infty}(u_p, \Omega_p) \tag{2.10}$$

which we will call a variational solution *of (FB_∞).*

Proof. The monotonicity part is an immediate corollary of Theorem 2.3, Lemma 2.4 and Lemma 2.2. As the uniform bound (u_0, Ω_0) we can use

$$u_0(x) = 1 - a_0 \operatorname{dist}(x, K), \qquad \Omega_0 = \{u_0 > 0\} = \{x\colon \operatorname{dist}(x, K) < 1/a_0\},$$

where, recall, $a_0 = \inf_{x\in K^c} a(x) > 0$. Indeed, the pair (u_0, Ω_0) is a classical supersolution of (FB_p) for all $p \in (1, \infty)$ and hence the statement follows. (See also *Step 1* in the proof of Theorem 3.3.) ▮

3. CLASSICAL SOLUTIONS FOR $a(x) \equiv a_0$

Precisely as for (FB_p), we define the classical (sub- and super-) solutions of (FB_∞).

DEFINITION 3.1. A pair (u, Ω), where Ω is a domain in $\mathbf{R}^n$ that contains K and u is a nonnegative continuous function on $\overline{\Omega}$, is called a *classical supersolution* of (FB_∞) if

(i) $u \in C^1(\Omega \setminus K)$ and $\Delta_\infty u \le 0$ in $\Omega \setminus K$ in the viscosity sense;
(ii) $\limsup_{y\to x} |\nabla u(y)| \le a(x)$ for every $x \in \partial\Omega$;
(iii) $u \ge 1$ on K and $u = 0$ on $\partial\Omega$.

Classical subsolutions and *classical solutions* of (FB_∞) are defined by the respective modifications of (i)–(iii) as in Definition 2.1.

Remark 3.2. Lemma 2.2 remains valid for $p = \infty$ with the same proof. The only properties of the operator needed is the comparison principle and the possibility to rescale sub- and supersolutions. Observe, however, that the comparison principle for Δ_∞ is nontrivial and was proved by R. Jensen in [Je].

As we have mentioned in the introduction, problem (FB_∞) has a simple explicit solution (u, Ω) given by (1.1) when $a(x) \equiv a_0 > 0$. From the other hand we know from Corollary 2.5 that there is always a variational solution. The next theorem says that these two are the same.

THEOREM 3.3 (Classical solution). *In the case $a(x) \equiv a_0 > 0$ the variational solution $(u_\infty, \Omega_\infty)$ of (FB_∞) as defined in Corollary 2.5 is given by*

$$u_\infty(x) = 1 - a_0 \operatorname{dist}(x, K) \quad \textit{and} \quad \Omega_\infty = \{\operatorname{dist}(x, K) < 1/a_0\}. \tag{3.1}$$

Moreover, $(u_\infty, \Omega_\infty)$ is the unique classical solution of (FB_∞).

Proof. Set

$$u(x) = 1 - a_0 \operatorname{dist}(x, K) \quad \text{and} \quad \Omega = \{\operatorname{dist}(x, K) < 1/a_0\}.$$

Step 1. Show that for every $p \in (1, \infty)$ there holds

$$\Omega_p \subset \Omega \quad \text{and} \quad u_p \leq u \text{ in } \Omega_p. \tag{3.2}$$

By Lemma 2.2 it is sufficient to show that (u, Ω) is a classical supersolution of (FB_p). Since the gradient and boundary value conditions (ii) and (iii) in Definition 2.1 are readily satisfied the only condition to be verified is $\Delta_p u \leq 0$. The latter holds due to concavity of u. Hence (3.2) is proved. Passing to the limit in (3.2) as $p \to \infty$ we obtain

$$\Omega_\infty \subset \Omega \quad \text{and} \quad u_\infty \leq u \text{ in } \Omega_\infty. \tag{3.3}$$

Step 2. In order to reverse inequalities in (3.3), we construct explicit subsolutions of (FB_p) from (u, Ω) as follows. Assume for the moment that K has C^2 regular boundary; then the distance function $\operatorname{dist}(x, K)$ is C^2 in K^c as well. For $\varepsilon > 0$ let

$$\phi_\varepsilon(s) = \frac{e^{\varepsilon s} - 1}{\varepsilon} \tag{3.4}$$

and $\delta_\varepsilon \in (0, 1)$ be such that

$$\phi_\varepsilon(1 - \delta_\varepsilon) = 1. \tag{3.5}$$

(More explicitly $\delta_\varepsilon = 1 - (1/\varepsilon)\log(1 + \varepsilon)$.) Define

$$u^\varepsilon(x) = \phi_\varepsilon(u - \delta_\varepsilon) \tag{3.6}$$

in

$$\Omega^\varepsilon = \{x \colon u(x) > \delta_\varepsilon\} = \{x \colon \operatorname{dist}(x, K) < 1/a_0 - \delta_\varepsilon\}.$$

We claim that

$$(u^\varepsilon, \Omega^\varepsilon) \text{ is a classical subsolution of } (FB_p) \text{ for } p > p(\varepsilon). \tag{3.7}$$

By construction, the gradient and boundary value conditions are satisfied and we have to verify only that $\Delta_p u^\varepsilon \geq 0$ in $\Omega^\varepsilon \setminus K$ for p large. For this purpose, take a point $x \in \Omega^\varepsilon \setminus K$ and choose a coordinate system $(y_1, \dots, y_n)$ centered at x with n axis directed as $\nabla u^\varepsilon(x)$ (or, equivalently, as $\nabla u(x)$.) These are the same coordinates used in the proof of Lemma 2.4. Observe that the level sets of u^ε are those of u, but parameterized in another way. Assume $u(x) = s$. Then the level line $\ell_s(u) = \ell_{\phi_\varepsilon(s-\delta_\varepsilon)}(u^\varepsilon)$ can be represented near x as a graph $\{y\colon y_n = f(y')\}$ of a convex C^2 function. In new coordinates we can calculate that at x

$$u^\varepsilon_{jj} = \phi''_\varepsilon(u-\delta_\varepsilon)u_j^2 + \phi'_\varepsilon(u-\delta_\varepsilon)u_{jj} \tag{3.8}$$

for $j = 1, \dots, n$. Using that $u_i = 0$ for $i = 1, \dots, n-1$, $u_n = a_0$ and $u_{nn} = 0$ at x as well as a formula (2.7) (which is still valid) we obtain

$$u^\varepsilon_{ii} = -\phi'_\varepsilon(u-\delta_\varepsilon)a_0 f_{ii}(x) \quad \text{for } i = 1, \dots, n-1 \tag{3.9}$$

and

$$u^\varepsilon_{nn} = \phi''_\varepsilon(u-\delta_\varepsilon)a_0^2 \tag{3.10}$$

at point x. Observe now that ϕ_ε satisfies the differential equation

$$\phi''_\varepsilon = \varepsilon\phi'_\varepsilon.$$

Employing this fact we find that

$$|\nabla u^\varepsilon|^{2-p}\Delta_p u^\varepsilon = \sum_{i=1}^{n-1} u^\varepsilon_{ii} + (p-1)u^\varepsilon_{nn} = \phi'_\varepsilon(u-\delta_\varepsilon)a_0((p-1)a_0\varepsilon - (n-1)\kappa), \tag{3.11}$$

where $\kappa = (1/(n-1))\sum_{i=1}^{n-1} f_{ii}$ is the mean curvature of the level line $\ell_s(u) = \ell_{(1-s)/a_0}(\operatorname{dist}(\cdot, K))$ at x. Recall now that we assume K has a uniform interior ball property hence the mean curvature of ∂K at every point is bounded by a finite constant, say κ_0. But then the mean curvature of every level line of the distance function $\operatorname{dist}(x, K)$ at every point is not greater than κ_0. Hence we have $\kappa \le \kappa_0$ for the κ in (3.11). If now p is so large that

$$(p-1)a_0\varepsilon - (n-1)\kappa_0 \ge 0,$$

then from (3.11) we will have immediately $\Delta_p u^\varepsilon \ge 0$ at x. This means that (3.7) is true with

$$p(\varepsilon) = 1 + \frac{n-1}{a_0\varepsilon}\kappa_0.$$

Letting $\varepsilon \to 0$ we conclude that

$$\Omega \subset \Omega_\infty \quad \text{and} \quad u \le u_\infty \text{ in } \Omega \tag{3.12}$$

in the case when K has C^2 regular boundary. If not, let $K_\eta \subset K$ be a strictly convex subset with C^2 boundary approximating K as $\eta \to 0$. Then proceeding as above we prove first that $\{x\colon \operatorname{dist}(x, K_\eta) < 1/a_0\} \subset \Omega_\infty$ and $1 - a_0 \operatorname{dist}(x, K_\eta) \leq u_\infty(x)$. Letting $\eta \to 0$ we prove (3.12) in the general case.

Step 3. The uniqueness of $(u_\infty, \Omega_\infty)$ follows readily from Lemma 2.2, see Remark 3.2.

The proof of the theorem is complete. ∎

4. WEAK SOLUTIONS FOR GENERAL $a(x)$

Now we return to the general case, when $a(x)$ is a positive harmonic concave function for $x \in K^c$, satisfying (2.1) and (2.2). In this case we don't have an explicit solution as for the case $a(x) \equiv a_0$, but we can still hope that the variational solution is a classical solution of (FB_∞). The first difficulty we face is that u_∞ may fail to be C^1. However, as we will see later, the family $\{u_p\}$ of classical solutions of (FB_p) is uniformly bounded in the Lipschitz norm, and hence u_∞ is at least Lipschitz.

The main purpose of this section is to introduce a notion of a weak solution of (FB_∞) that doesn't use the differentiability properties of u.

DEFINITION 4.1. A pair (u, Ω), where Ω is a domain in $\mathbf{R}^n$ that contains K and u is a nonnegative continuous function on $\overline{\Omega}$, is called a *weak supersolution* of (FB_∞) if

(i) $\Delta_\infty u \leq 0$ in $\Omega \setminus K$ in the viscosity sense;
(ii) $\limsup_{y\to x} u(y)/\operatorname{dist}(y, \partial\Omega) \leq a(x)$ for every $x \in \partial\Omega$;
(iii) $u \geq 1$ on K and $u = 0$ on $\partial\Omega$.

Analogously, we define *weak subsolutions* of (FB_∞), by reversing the inequality signs in (i), (ii) and (iii) and replacing lim sup by lim inf in (ii).

A *weak solution* of (FB_∞) is a pair (u, Ω) which is a weak sub- and supersolution at the same time.

Remark 4.2. The comparison principle, Lemma 2.2, still works for the weak solutions defined above, with a minor modification in the proof. This implies in particular the uniqueness of the weak solution.

The following lemma provides compatibility of the above definition with the definition of classical solutions.

LEMMA 4.3. *A classical (super-, sub-) solution of (FB_∞) is a weak (super-, sub-) solution of (FB_∞).*

Proof. 1) *Supersolutions.* This case is simple. For $y \in \Omega \setminus K$ we have

$$u(y) = u(y) - u(z) = (y - z) \cdot \nabla u(\xi) \leq \operatorname{dist}(y, \partial\Omega)|\nabla u(\xi)|,$$

where $z = z(y) \in \partial\Omega$ is chosen such that $\operatorname{dist}(y, \partial\Omega) = |y - z|$ and ξ is a certain point on the line segment, joining y and z. Since $y \to x \in \partial\Omega$ implies $\xi \to x$, we obtain

$$\limsup \frac{u(y)}{\operatorname{dist}(y, \partial\Omega)} \leq \limsup |\nabla u(y)| \leq a(x)$$

as $\Omega \setminus K \ni y \to x \in \partial\Omega$.

2) *Subsolutions.* This case needs more attention. Let (u, Ω) be a classical subsolution of (FB_∞), and consider the function $v(x) = u(x)/a(x)$. Assume first $a(x)$ is C^1 regular. Then v is C^1 as well and its gradient is given by

$$\nabla v = \frac{\nabla u}{a} - u\frac{\nabla a}{a^2}.$$

In particular

$$\liminf_{\Omega\setminus K \ni y \to x} |\nabla v(y)| \geq 1$$

for every $x \in \partial\Omega$. Then we can proceed as in the proof of Claim 1 in [Vo]. Let $y \in \Omega \setminus K$ and $v(y) = s$. Suppose $0 < s < s_\varepsilon$, where s_ε (for small $\varepsilon > 0$) is chosen such that for $z \in \Omega \setminus K$

$$v(z) < s_\varepsilon \quad \text{implies} \quad 1 - \varepsilon < |\nabla v(z)|. \tag{4.1}$$

Find now a curve $\gamma_y(t)$ which solves

$$\frac{d}{dt}\gamma_y(t) = \frac{\nabla v(\gamma_y(t))}{|\nabla v(\gamma_y(t))|^2} \quad \text{and} \quad \gamma_y(s) = y. \tag{4.2}$$

There exist at least one solution $\gamma_y(t)$ of (4.2) defined on the whole interval $(0, s)$ with a continuous extension to $[0, s]$ and which satisfies

$$v(\gamma_y(t)) = t \quad \text{for } t \in [0, s]\,.$$

Let ℓ_y denote the length of the curve $\gamma_y\,[0, s]$. Then from (4.1)

$$v(y) \geq (1 - \varepsilon)\ell_y \geq (1 - \varepsilon)\operatorname{dist}(y, \partial\Omega).$$

Consequently

$$\liminf \frac{u(y)}{\operatorname{dist}(y, \partial\Omega)} = a(x) \liminf \frac{v(y)}{\operatorname{dist}(y, \partial\Omega)} \geq a(x)$$

as $\Omega \setminus K \ni y \to x \in \partial\Omega$ and the lemma follows in the case $a(x)$ is C^1 regular.

For continuous $a(x)$, take a C^1 regularization $a_\varepsilon(x) \le a(x)$, converging pointwise to $a(x)$ as $\varepsilon \to 0$, and use it in the reasonings above to obtain

$$\liminf u(y)/\operatorname{dist}(y, \partial\Omega) \ge a_\varepsilon(x).$$

Then we let $\varepsilon \to 0$ to complete the proof. ▮

One of the main purposes of this paper is to prove the following theorem.

THEOREM 4.4 (Weak solution). *The variational solution $(u_\infty, \Omega_\infty)$ of (FB_∞) as defined in Corollary 2.5 is the only weak solution of (FB_∞).*

5. $(u_\infty, \Omega_\infty)$ IS A WEAK SUBSOLUTION

The statement in the title of this section constitutes the relatively easy part in the proof of Theorem 4.4.

LEMMA 5.1. *The variational solution $(u_\infty, \Omega_\infty)$ is a weak subsolution of (FB_∞).*

Proof. We use similar reasonings as those in the proof of Lemma 4.3.

Let (u_p, Ω_p) be the classical solution of (FB_p) for $p > 1$, $y \in \Omega_p \setminus K$ and $u_p(y) = s$. Consider the curve $\gamma_y(t)$ which solves

$$\frac{d}{dt}\gamma_y(t) = \frac{\nabla u_p(\gamma_y(t))}{|\nabla u_p(\gamma_y(t))|^2} \quad \text{and} \quad \gamma_y(s) = y. \tag{5.1}$$

On this curve the following relation is satisfied

$$u_p(\gamma_y(t)) = t$$

and one can infer from this that $\gamma_y(t)$ can be defined on the interval $(0, s)$ with a continuous extension to $[0, s]$. The curve γ_y has the following nice property.

$$|\nabla u_p(\gamma_y(t))| \nearrow \ \text{as } t \nearrow. \tag{5.2}$$

To prove this statement, fix $z_0 = \gamma_y(t_0)$ for some $t_0 \in (0, s)$. Choose a new coordinate system in $\mathbf{R}^n$ with the origin at z_0 and with the n–axis directed as $\nabla u_p(z_0)$. Then $\gamma_y'(t_0)$ is directed as the n–axis as well. Therefore

$$\frac{d}{dt}|\nabla u_p(\gamma_y(t))|\Big|_{t=t_0} = |\gamma'(t_0)|(|\nabla u_p|)_n(z_0) = |\gamma'(t_0)|(u_p)_{nn}(z_0), \tag{5.3}$$

where the subscript n means the differentiation along the n–axis. On the other hand we have

$$(u_p)_{nn} = \frac{n-1}{p-1}\kappa(u_p)_n \geq 0 \tag{5.4}$$

at z_0, where $\kappa \geq 0$ is the mean curvature of the level line $\{u_p = t_0\}$ at z_0, as it follows from the computations in the proof of Lemma 2.4, cf. (2.9). Combining (5.3) and (5.4), we prove (5.2).

Let us now approach the free boundary $\partial\Omega_p$ along the curve $\gamma_y(t)$ as $t \to 0+$. Then in the limit

$$|\nabla u_p(\gamma_y(0)| = a(\gamma_y(0)).$$

Together with (5.2) this implies

$$|\nabla u_p(y)| \geq a(\gamma_y(0)) \quad \text{for } y \in \Omega_p \setminus K. \tag{5.5}$$

More generally, we have

$$|\nabla u_p(\gamma_y(t))| \geq a(\gamma_y(0)) \quad \text{for } t \in [0, s]\,. \tag{5.6}$$

To proceed, we suppose $p \geq p_0 > 1$ and $0 < s \leq \sigma < s_0 < 1$. Set

$$c_0 = \inf a(y) \quad \text{and} \quad L_0 = \sup \frac{|a(y) - a(z)|}{|y - z|}$$

for $y, z \in \overline{\Omega}_\infty \setminus \{u_{p_0} > s_0\}$. Then $c_0 > 0$ and $L_0 < \infty$. Using (5.6), we can infer very rough estimate

$$|\nabla u_p(\gamma_y(t))| \geq c_0 \quad \text{for } t \in [0, s]\,.$$

If we denote by ℓ_y the length of the curve $\gamma_y\,[0, s]$, then the estimate above will imply

$$u_p(y) \geq c_0 \ell_y$$

and consequently

$$|y - \gamma_y(0)| \leq \ell_y \leq \frac{u_p(y)}{c_0}.$$

Then by the definition of L_0 and the above estimate

$$a(\gamma_y(0)) \geq a(y) - L_0|y - \gamma_y(0)| \geq a(y) - \frac{L_0}{c_0}u_p(y) \geq a(y) - \frac{L_0}{c_0}\sigma.$$

Going back again to (5.6), integrating along the curve $\gamma_y[0, s]$ and applying the previous estimate, we obtain

$$u_p(y) \geq a(\gamma_y(0))\ell_y \geq \left(a(y) - \frac{L_0}{c_0}\sigma\right) \operatorname{dist}(y, \partial\Omega_p)$$

for every $p \geq p_0$ and $y \in \Omega_p \setminus K$ with $u_p(y) \leq \sigma < s_0$.

Letting $p \to \infty$, we obtain

$$u_\infty(y) \geq \left(a(y) - \frac{L_0}{c_0}\sigma\right) \operatorname{dist}(y, \partial\Omega_\infty)$$

for every $y \in \Omega_\infty \setminus K$ with $u_\infty(y) \leq \sigma < s_0$. It is clear now that the lemma follows. ▌

6. UNIFORM GRADIENT BOUND FOR u_p AS $p \to \infty$

In this section we make the first step in proving that $(u_\infty, \Omega_\infty)$ is a weak supersolution of (FB_∞).

We remind that we assume K to satisfy the uniform interior ball condition. That is, there exists $\delta > 0$ such that for every $x \in \partial K$ one can find a ball B of radius δ such that $B \subset K$ and $\partial B \cap \partial K = \{x\}$. As a consequence we have the following lemma.

LEMMA 6.1. *There exists a constant $M > 0$ such that for the classical solution (u_p, Ω_p) of (FB_p) we have*

$$|\nabla u_p| \leq M \quad \text{in } \Omega_p \setminus K$$

for all $p \geq p_0 > 1$. The constant M here depends only on δ in the uniform interior ball condition for K, on p_0, on $d_0 = \inf_{y \in \partial\Omega_{p_0}} \operatorname{dist}(y, K)$, and on the space dimension n.

In particular, u_∞ is Lipschitz continuous.

Proof. We mostly follow the proof of Lemma 2.3 in [HS1]. Observe first that $|\nabla u_p|$ satisfies the maximum principle in $\Omega_p \setminus K$, namely

$$\max_{\Omega_p \setminus K} |\nabla u_p| \leq \max_{\partial K \cup \partial\Omega_p} |\nabla u_p|.$$

This follows from Payne-Philippin's inequality

$$L_{u_p}(|\nabla u_p|^p) \geq 0,$$

with the elliptic operator

$$L_{u_p}(v) := |\nabla u_p|^{p-2}\Delta v + (p-2)|\nabla u_p|^{p-4}\nabla^2 v \nabla u_p \cdot \nabla u_p.$$

For the proof apply Lemma 1 in [PP] with the particular choices $\alpha = -1$, $\beta = 0$, γ and $v = 0$ to obtain

$$L_{u_p}(|\nabla u_p|^p) \geq \frac{p(p-1)^2}{4}|\nabla u_p|^{p-6}(\nabla u_p \cdot \nabla(|\nabla u_p|^2))^2.$$

Now, having the gradient maximum principle it suffices to prove that ∇u_p is uniformly bounded on $\partial\Omega_p \cup \partial K$. Since $|\nabla u_p| = 1$ on $\partial\Omega_p$ we need only a uniform gradient bound on ∂K.

Fix now $p_0 > 1$ and consider the function u_{p_0} near ∂K. For $x \in \partial K$ let $B_\delta(z_x)$ be a touching ball at x from inside of K. By the definition of d_0 we have also $B_{\delta+d_0}(z_x) \subset \Omega_{p_0}$. If one denote by v_{p_0} the p_0–capacitary potential of the ring $B_{\delta+d_0}(z_x) \setminus B_\delta(z_x)$ then by the comparison principle we will have $v_{p_0} \leq u_{p_0}$. Since also $v_{p_0}(x) = u_{p_0}(x)$ we may conclude

$$|\nabla u_{p_0}(x)| \leq |\nabla v_{p_0}(x)| = M(\delta, d_0, p_0, n).$$

Take now $p > p_0$. From Corollary 2.5 we know that $u_{p_0} < u_p < 1$. Also $u_p = u_{p_0} = 1$ on ∂K and consequently

$$|\nabla u_p(x)| \leq |\nabla u_{p_0}(x)| \leq M \quad \text{for } x \in \partial K.$$

The proof is complete. ▮

7. ON ∞-HARMONIC FUNCTIONS NEAR SINGULAR CONVEX BOUNDARY POINTS

In this section we study the behavior of ∞–harmonic functions near a singular convex boundary point. The results are more or less independent from the rest of the paper, and have the interest of their own. More specifically we prove the following result.

THEOREM 7.1. *Let D be a convex open set with x_0 a singular boundary point and u a viscosity solution of $\Delta_\infty u = 0$ in $D \cap B_R(x_0)$ for some $R > 0$. Suppose moreover u vanishes on $\partial D \cap B_R(x_0)$. Then $u(x) = O(|x - x_0|^{1+\varepsilon})$ as $x \to x_0$ for some $\varepsilon > 0$.*

Such a result for p–harmonic functions is very well known, see e.g. in [Kr], [Do].

Proof. The convexity of D helps us to reduce the problem to the 2-dimensional case. Namely, that the point x_0 is a singular boundary point of D means precisely that there are at least 2 supporting hyperplanes to D at x_0. Therefore after a necessary translation and rotation we may assume $x_0 = 0$ and that there is a cone Γ in $\mathbf{R}^2$ of aperture less than π and with vertex at the origin, such that $D \subset \Gamma \times \mathbf{R}^{n-2}$.

Suppose at the moment that we know the existence of a barrier v in the 2-dimensional cone Γ that has the following properties:

(i) $v \in C(\overline{\Gamma})$ and $\Delta_\infty v \leq 0$ in the viscosity sense;
(ii) $v > 0$ in $\overline{\Gamma} \setminus \{0\}$ and $v(0) = 0$;
(iii) $v(x) = O(|x|^{1+\varepsilon})$ for some $\varepsilon > 0$.

Then using standard arguments together with Jensen's maximum principle we obtain $u(x) = O(|x|^{1+\varepsilon})$, as needed.

Let us point out that a barrier as above, symmetric with respect to the cone's axis, cannot be C^2 regular, the reason being that ∞–superharmonicity implies concavity of the barrier on the symmetry axis. The barriers that we will construct next are only $C^{1,1/3}$ regular.

LEMMA 7.2. *For every $\theta \in (0, \pi/2)$ there is $\varepsilon > 0$ and a $C^{1,1/3}$ regular function $F(\alpha)$ for $\alpha \in (-\theta, \theta)$ such that (using complex notations) v, given by*

$$v(re^{i\alpha}) = r^{1+\varepsilon} F(\alpha), \tag{7.1}$$

satisfies (i)–(iii) *above in* $\Gamma = \{re^{i\alpha} : \alpha \in (-\theta, \theta)\}$.

Proof. Let us compute formally the ∞–Laplacian for v as in (7.1). We will obtain

$$\Delta_\infty v(re^{i\alpha}) = r^{-(1-3\varepsilon)/2}(\varepsilon(1+\varepsilon)^3 F(\alpha)^3 + F'(\alpha)^2((1+\varepsilon)(1+2\varepsilon)F(\alpha) + F''(\alpha))).$$

This can be rewritten as

$$\Delta_\infty v(re^{i\alpha}) = r^{-(1-3\varepsilon)/2} L_\varepsilon(F)(\alpha),$$

where

$$L_\varepsilon(F) = \varepsilon(1+\varepsilon)^3 F^3 + (F')^2((1+\varepsilon)(1+2\varepsilon)F + F'').$$

Substitute $F = e^f$. This will give

$$L_\varepsilon(e^f) = e^{3f}(\varepsilon(1+\varepsilon)^3 + (f')^4 + (f')^2((1+\varepsilon)(1+2\varepsilon) + f'')),$$

or

$$L_\varepsilon(e^f) = e^{3f}(f')^2 \ell_\varepsilon(f'),$$

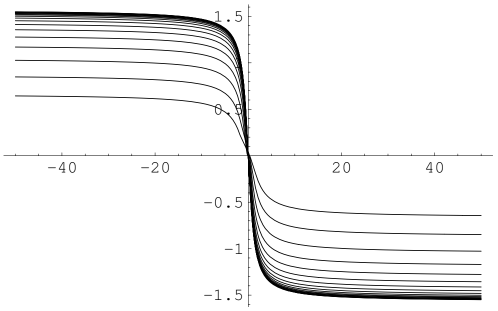

FIG. 2. Graphs of $\phi_\varepsilon(y)$ as $\varepsilon \to 0$.

where

$$\ell_\varepsilon(y) = (1+\varepsilon)(1+2\varepsilon) + \frac{\varepsilon(1+\varepsilon)^3}{y^2} + y^2 + y'$$

a first order differential operator. Let us now consider the ODE

$$\ell_\varepsilon(y) = 0.$$

After integration we will get

$$-\arctan\left(\frac{y(\alpha)}{1+\varepsilon}\right) + \sqrt{\frac{\varepsilon}{1+\varepsilon}}\arctan\left(\frac{y(\alpha)}{\sqrt{\varepsilon(1+\varepsilon)}}\right) = \alpha + C,$$

where C is a constant. Choose $C = 0$ and consider the function

$$\phi_\varepsilon(y) = -\arctan\left(\frac{y}{1+\varepsilon}\right) + \sqrt{\frac{\varepsilon}{1+\varepsilon}}\arctan\left(\frac{y}{\sqrt{\varepsilon(1+\varepsilon)}}\right)$$

on the whole real axis. Then ϕ_ε is a smooth strictly decreasing function with $\phi(0) = 0$. Using that the image of arctan is the interval $(-\pi/2, \pi/2)$ we easily obtain that the image of ϕ_ε is the interval $(-\theta_\varepsilon, \theta_\varepsilon)$ with

$$\theta_\varepsilon = \frac{\pi}{2}\left(1 - \sqrt{\frac{\varepsilon}{1+\varepsilon}}\right).$$

Then the inverse function $\psi_\varepsilon = \phi_\varepsilon^{-1}$ is well defined on the interval $(-\theta_\varepsilon, \theta_\varepsilon)$. Moreover, since the derivative

$$\phi_\varepsilon'(y) = -\frac{y^2}{(y^2+\varepsilon+\varepsilon^2)(y^2+(1+\varepsilon)^2)}$$

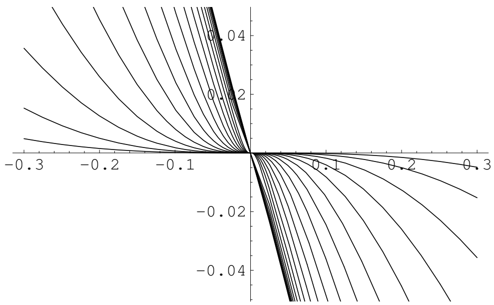

FIG. 3. $\phi_\varepsilon(y)$ near 0 as $\varepsilon \to 0$.

is strictly negative for $y \neq 0$, $\psi_\varepsilon(\alpha)$ is C^∞ everywhere on $(-\theta_\varepsilon, \theta_\varepsilon)$ except $\alpha = 0$. To examine the behavior of $\psi_\varepsilon(\alpha)$ near $\alpha = 0$ we look at the asymptotic development of $\phi_\varepsilon(y)$ near $y = 0$:

$$\phi_\varepsilon(y) = -\frac{y^3}{3\varepsilon(1+\varepsilon)^3} + O(y^5).$$

This implies that ψ_ε is only $C^{1/3}$ regular at the origin.

Let us choose now $\varepsilon > 0$ so that $\theta < \theta_\varepsilon$ and go backwards in the constructions above. Set $f(\alpha) = \int_0^\alpha \psi_\varepsilon(t)dt$ and $F(\alpha) = e^f(\alpha)$. Then F is C^∞ everywhere in the cone $\Gamma_\varepsilon = \{re^{i\alpha} \colon \alpha \in (-\theta_\varepsilon, \theta_\varepsilon)\} \supset \Gamma$ except the symmetry axis $\alpha = 0$ where it is only $C^{1,1/3}$ regular.

By construction, $v(re^{i\alpha}) = r^{1+\varepsilon}F(\alpha)$ will satisfy $\Delta_\infty v = 0$ in the classical sense off the symmetry axis $\alpha = 0$. We claim that in the viscosity sense $\Delta_\infty v = 0$ in the whole Γ_ε. First observe that there are *no* paraboloids touching v by below at the points on the symmetry axis $\alpha = 0$, otherwise F would be $C^{1,1}$ regular. This implies immediately that $\Delta_\infty v \leq 0$ in Γ_ε in the viscosity sense. Let now P be a paraboloid touching v by above at a point on the axis $\alpha = 0$. Since v is C^1 regular, at the touching point $\nabla P = \nabla v$, which is directed along the axis. Observe now that on the axis $\alpha = 0$ v is a convex function $r^{1+\varepsilon}F(0)$ and hence we must have $\nabla^2 P \nabla P \cdot \nabla P \geq 0$. This implies that $\Delta_\infty v \geq 0$ in the viscosity sense in Γ_ε. Hence the claim follows.

The rest of properties (i)-(iii) are easily verified for v and the proof of the lemma is complete. ▮

Theorem 7.1 is proved. ▮

Remark 7.3. Observe that in the case $\theta < \pi/4$ one can take as a barrier the function

$$v(x, y) = x^{4/3} - |y|^{4/3} = r^{4/3}(\cos(\alpha)^{4/3} - |\sin(\alpha)|^{4/3})$$

in the lemma above. One can even check that it is precisely the function that appears in the proof for $\varepsilon = 1/3$.

8. C^1 REGULARITY OF $\partial\Omega_\infty$

The results of this section are straightforward consequences of Theorem 7.1.

LEMMA 8.1 (C^1 regularity). *The level sets $\{u_\infty > s\}$ are convex sets with C^1 regular boundary $\{u_\infty = s\}$ for every $s \in [0, 1)$.*
In particular, $\partial\Omega_\infty$ is C^1 regular.

Proof. Let $p_0 > 1$ and $s_0 \in (0, 1)$. Then from the proof of Lemma 5.1 we know that for $c_0 = \min\{a(x): x \in \overline{\Omega}_\infty \setminus \{u_{p_0} > s_0\}\} > 0$ we have

$$|\nabla u_p(y)| \geq c_0 \quad \text{whenever } p \geq p_0 \text{ and } 0 < u_p(y) < s_0.$$

This implies in particular, that

$$u_p(y) - s \geq c_0 \operatorname{dist}(y, \{u_p = s\}) \quad \text{whenever } p \geq p_0 \text{ and } s < u_p(y) < s_0.$$

Letting $p \to \infty$ we obtain

$$u_\infty(y) - s \geq c_0 \operatorname{dist}(y, \{u_\infty = s\}) \quad \text{whenever } s < u_\infty(y) < s_0. \tag{8.1}$$

Now, Theorem 7.1 says that (8.1) would be impossible if $\{u_\infty = s\}$ were not C^1 regular. Indeed, suppose there is a singular point y_0 on $\{u_\infty = s\}$. Since $\{u_\infty > s\}$ is a convex open set, we can find a unit vector e and a small $\delta > 0$ such that the truncated cone $\mathcal{C} = \mathcal{C}(y_0, e, \delta) = \{y \in \mathbf{R}^n: |y - y_0| < \delta,\ \operatorname{angle}(y - y_0, e) < \delta\}$ is contained in $\{u_\infty > s\}$. Then along the symmetry axis of $\mathcal{C}$, for $y = y_0 + \eta e$ with $0 < \eta < \delta$ we will have

$$\operatorname{dist}(y, \{u_\infty = s\}) \geq \tan(\delta)|y - y_0|.$$

This implies by (8.1)

$$u_\infty(y) - s \geq c_0 \tan(\delta)|y - y_0|,$$

contradicting Theorem 7.1. The lemma is proved. ▮

The following corollary is a kind of compensation for the missing C^1 regularity of u_∞.

COROLLARY 8.2. *There is a unit-vector field* $\nu \in C(\overline{\Omega}_\infty \setminus K)$ *such that*

$$\nabla u_\infty(y) = \nu(y)|\nabla u_\infty(y)| \quad \textit{a.e. in } \Omega_\infty \setminus K. \tag{8.2}$$

Proof. For every $y \in \overline{\Omega}_\infty \setminus K$ there is a unique supporting hyperplane to $\{u_\infty < u_\infty(y)\}$ at y. Denote by $\nu(y)$ the unit normal vector of this hyperplane, pointing into $\{u_\infty < u_\infty(y)\}$. Then Lemma 8.1 implies the continuity of ν up to $\partial\Omega_\infty$.

To see the identity (8.2), recall that u_∞ is Lipschitz continuous and therefore differentiable a.e. in $\Omega_\infty \setminus K$. At the differentiability points y where $|\nabla u_\infty| \neq 0$, evidently $\nabla u_\infty/|\nabla u_\infty|$ is normal to the level set and hence coincides with $\nu(y)$. The proof is complete. ▮

9. STABLE SOLUTIONS OF THE BOUNDARY VALUE PROBLEM

The disadvantage of the classical and weak solutions defined in the previous sections is that we don't know generally their stability under the limit. In this section we intend to give a notion of a stable solution, in this sense. As the source of the definitions and notations we use *User's guide to viscosity solutions* [CIL] of M. G. Crandall, H. Ishii and P. L. Lions.

Suppose we have a family $F_p\colon \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \times \mathcal{S}(n) \to \mathbf{R}$, of upper semicontinuous *proper* operators, where $1 < p < \infty$ and $\mathcal{S}(n)$ is the set of symmetric $n \times n$ matrices. The operator F is proper if

$$F(x, r, \eta, X) \le F(x, s, \eta, Y) \quad \text{whenever } r \le s \text{ and } Y \le X,$$

where $r, s \in \mathbf{R}$, $x, \eta \in \mathbf{R}^n$, $X, Y \in \mathcal{S}(n)$ and $\mathcal{S}(n)$ is equipped with its usual order. Observe, the operator $-\Delta_p$ and not Δ_p is proper!

Suppose next, we have a family of locally compact subsets $\mathcal{O}_p \subset \mathbf{R}^n$ and functions $u_p \in \mathrm{LSC}(\mathcal{O}_p)$ (lower semicontinuous functions on $\mathcal{O}_p$), which solve

$$F_p(x, u_p, \nabla u_p, \nabla^2 u_p) \ge 0 \quad \text{on } \mathcal{O}_p$$

in the viscosity sense. Denote

$$\mathcal{O}_\infty = \limsup_{p\to\infty} \mathcal{O}_p$$

$$\begin{aligned} &= \{x\colon \exists x_{p_k} \in \mathcal{O}_{p_k} \text{ such that } x_{p_k} \to x\} \quad \text{and} \\ u_\infty(x) &= \liminf_{p\to\infty}{}_* u_p(x) \quad \text{on } \mathcal{O}_\infty \\ &= \inf\{\liminf_{k\to\infty} u_{p_k}(x_{p_k})\colon x_{p_k} \in \mathcal{O}_{p_k}, x_{p_k} \to x\}. \end{aligned}$$

Then the following theorem holds.

THEOREM 9.1. *If the operators F_p converge locally uniformly to an operator F_∞, then*

$$F_\infty(x, u_\infty, \nabla u_\infty, \nabla^2 u_\infty) \geq 0 \quad \textit{on } \mathcal{O}_\infty$$

in the viscosity sense.

The theorem is an immediate corollary of the following lemma, which is a variation of the Proposition 4.3 in [CIL].

LEMMA 9.2. *For every $\hat{x} \in \mathcal{O}_\infty$ and $(\eta, X) \in J^{2,-}_{\mathcal{O}_\infty} u_\infty(\hat{x})$ there exists a sequence $\hat{x}_{p_k} \in \Omega_{p_k}$ and $(\eta_k, X_k) \in J^{2,-}_{\mathcal{O}_{p_k}} u_{p_k}(\hat{x}_{p_k})$ with $p_k \to \infty$ such that*

$$(\hat{x}_{p_k}, u_{p_k}(\hat{x}_{p_k}), \eta_k, X_k) \to (\hat{x}, u_\infty(\hat{x}), \eta, X).$$

Recall that given a function u on a subset $\mathcal{O}$ of $\mathbf{R}^n$ and a point $\hat{x} \in \mathcal{O}$, the second-order *subjet* $J^{2,-}_{\mathcal{O}} u(\hat{x})$ is the set of pairs $(\eta, X) \in \mathbf{R}^n \times \mathcal{S}(n)$ such that

$$u(x) \geq u(\hat{x}) + \eta(x - \hat{x}) + \frac{1}{2} X(x - \hat{x}) \cdot (x - \hat{x}) + o(|x - \hat{x}|^2) \tag{9.1}$$

as $\mathcal{O} \ni x \to \hat{x}$. In addition to subjets we will also need their closures

$$\begin{aligned} \overline{J}^{2,-}_{\mathcal{O}} u(\hat{x}) = \{(\eta, X) \in \mathbf{R}^n \times \mathcal{S}(n)\colon \exists x_k \in \mathcal{O} \text{ and } (\eta_k, X_k) \in J^{2,-}_{\mathcal{O}} u(x_k) \\ \text{such that } (x_k, u(x_k), \eta_k, X_k) \to (\hat{x}, u(\hat{x}), \eta, X)\}. \end{aligned}$$

Proof of Lemma 9.2. We refer to the proof of Proposition 4.3 in [CIL]. The only essential difference is that the locally compact sets $\mathcal{O}_p$ do not vary with p there, but this does not affect on the proof. ∎

Having Theorem 9.1 in mind, we give the following definition.

DEFINITION 9.3. Let D be an open subset of $\mathbf{R}^n$ and Γ a relatively open portion of ∂D. Then a lower semicontinuous function u on $D \cup \Gamma$ is a *viscosity* *solution* of

$$(BV) \qquad \begin{cases} F(x,u,\nabla u,\nabla^2 u)\geq 0 & \text{in } D\\ B(x,u,\nabla u,\nabla^2 u)\geq 0 & \text{on } \Gamma\end{cases}$$

with F and B proper operators, if u is a viscosity solution of

$$G_+(x,u,\nabla u,\nabla^2 u)\geq 0 \quad \text{on } D\cup\Gamma,$$

where

$$G_+(x,r,\eta,X)=\begin{cases} F(x,r,\eta,X) & \text{if } x\in D\\ \max(F(x,r,\eta,X),B(x,r,\eta,X)) & \text{if } x\in\Gamma.\end{cases}$$

Explicitly, this means

$$\begin{cases} F(x,u(x),\eta,X)\geq 0 & \text{for } (\eta,X)\in J^{2,-}_D u(x),\ x\in D\\ \max(F(x,u(x),\eta,X),B(x,u(x),\eta,X))\geq 0 & \text{for } (\eta,X)\in \overline{J}^{2,-}_{D\cup\Gamma} u(x),\ x\in\Gamma.\end{cases}$$

Suppose now we have a sequence of continuous functions u_p that solve

$$F_p\geq 0 \text{ in } D_p \quad \text{and} \quad B_p\geq 0 \text{ on } \Gamma_p$$

in the viscosity sense, for continuous proper operators F_p and B_p, open subsets D_p of $\mathbf{R}^n$ and relatively open $\Gamma_p\subset\partial D_p$. Suppose also that D_p and Γ_p converges in the Hausdorff metric to D_∞ and $\Gamma_\infty\subset\partial D_\infty$ and that u_p converges locally uniformly to a continuous function u_∞. Then the following assertion holds.

THEOREM 9.4. *If the operators F_p and B_p converges locally uniformly to operators F_∞ and B_∞ respectively as $p\to\infty$ and u_p are as above, then the limit u_∞ solves*

$$F_\infty\geq 0 \textit{ in } D_\infty \quad \textit{and} \quad B_\infty\geq 0 \textit{ on } \Gamma_\infty$$

in the viscosity sense. ▮

10. (u_∞,Ω_∞) IS A WEAK SUPERSOLUTION

In this section we prove the second counterpart of Theorem 4.4.

LEMMA 10.1. *The variational solution (u_∞,Ω_∞) is a weak supersolution of (FB_∞).*

Proof. Take a sequence of points $y_k \in \Omega_\infty \setminus K$, converging to $x_0 \in \partial\Omega_\infty$. Let $d_k = \operatorname{dist}(y_k, \partial\Omega_\infty)$ and $x_k \in \partial\Omega_\infty$ be such that $|y_k - x_k| = d_k$. Then without loss of generality we may assume $e_k = (y_k - x_k)/d_k$ converge to $e_0 = (1, 0, \dots, 0)$. We want to prove that

$$\limsup_{k\to\infty} \frac{u_\infty(y_k)}{d_k} \le a(x_0).$$

Define u_∞ to be identically zero off Ω_∞ and consider the functions

$$v_k(x) = \frac{u_\infty(x_k + d_k x)}{d_k} \quad x \in \mathbf{R}^n.$$

Since u_∞ is Lipschitz continuous, the family $\{v_k\}$ is uniformly Lipschitz. Hence we may extract a subsequence converging uniformly on compacts to a Lipschitz function v_0. Assume that this is the whole sequence. Then

$$\frac{u_\infty(y_k)}{d_k} = v_k(e_k) \to v_0(e_0)$$

and we will be done if we show

$$v_0(e_0) \le a(x_0).$$

In fact, the following identity holds:

$$v_0(x) = a(x_0)(x \cdot e_0)^+ \quad x \in \mathbf{R}^n. \tag{10.1}$$

We prove this important fact in several steps.

Step 1. Show that $\{v_0 > 0\} = \Pi_0 = \{x\colon x \cdot e_0 > 0\}$.

Let $D_k = \{v_k > 0\}$. Then if we fix $s_0 \in (0, 1)$, we can find a constant c_0 such that

$$v_k(x) \ge \min\{c_0 \operatorname{dist}(x, D_k^c), s_0/d_k\}; \tag{10.2}$$

see the proof of Lemmas 5.1 and 8.1.

Observe next, that $D_k \subset \Pi_k = \{x\colon x \cdot e_k > 0\}$ by definition and that $0 \in \partial D_k \cap \partial \Pi_k$. Since Ω_∞ has C^1 regular boundary and D_k are dilations of Ω_∞ with a coefficient $1/d_k \to \infty$, we will have that D_k converge to Π_0 and (10.2) will give

$$v_0(x) \ge c_0 \operatorname{dist}(x, \Pi_0^c).$$

Hence the positivity set of v_0 contains Π_0. Since v_k are 0 off Π_k, v_0 is 0 off Π_0. This proves the statement.

Step 2. There is $\alpha \geq a(x_0)$ such that $v_0(x) = \alpha(x \cdot e_0)^+$.

Let ν be the vector field in Corollary 8.2 and set

$$\nu_k(x) = \nu(x_k + d_k x).$$

Then

$$\nabla v_k(x) = |\nabla v_k(x)| \nu_k(x) \quad \text{a.e. in } D_k.$$

Observe now, that $\nu_k(x) \to \nu(x_0) = e_0$ uniformly on compacts subsets of Π_0. Suppose $\phi \in L^2_{\mathrm{loc}}(\Pi_0, \mathbf{R}^n)$ is a weak limit of a subsequence of ∇v_k (note, ∇v_k are uniformly bounded in L^∞ norm). Then $e \cdot e_0 = 0$ implies $\phi \cdot e_0 = 0$ a.e. in Π_0. This means $\phi = |\phi| e_0$ a.e. in Π_0. Besides, we will have $\nabla v_0 = \phi$ in the distributional sense and therefore $\partial_e v_0 = 0$ for any direction e orthogonal to e_0. Therefore $v_0(x) = f(x \cdot e_0)$ in Π_0 for a certain function f, positive and continuous. Besides, f will be nondecreasing.

Next, observe that v_0 satisfies $\Delta_\infty v_0 = 0$ in the viscosity sense, since v_k satisfy $\Delta_\infty v_k = 0$ in $D_k \setminus \{v_k \geq 1/d_k\}$. Then we leave to the reader as a good exercise to show that for v_0 of the special form $f(x \cdot e_0)$ as above this means $f''(t)(f'(t))^2 = 0$ in the viscosity sense. This, in turn, implies that f is a linear function, i.e. $f(t) = \alpha t + \beta$. Since $f(0) = 0$ we obtain $f(t) = \alpha t$ and consequently $v_0(x) = \alpha(x \cdot e_0)^+$ everywhere in $\mathbf{R}^n$.

That $\alpha \geq a(x_0)$ follows simply from the observation that $\alpha = v_0(e_0)$ and

$$v_0(e_0) = \lim v_k(e_k) = \lim(u_\infty(y_k)/d_k) \geq a(x_0),$$

by Lemma 5.1.

Step 3. v_0 is a viscosity solution (see Section 9) of a boundary value problem

$$\begin{cases} -\Delta_\infty v_0 \geq 0 & \text{in } \Pi_0 \\ -\nabla v_0 \cdot e_0 + a(x_0) \geq 0 & \text{on } \partial \Pi_0. \end{cases} \tag{10.3}$$

To prove this statement, we are going to apply Theorem 9.4. We first refine our constructions.

Let $d_{k,p} = \operatorname{dist}(y_k, \partial\Omega_p)$ and $x_{k,p} \in \partial\Omega_p$ be such that $d_{k,p} = |y_k - x_{k,p}|$. Let also $e_{k,p} = (y_k - x_{k,p})/d_{k,p}$. Define next

$$v_{k,p}(x) = \frac{u_p(x_{k,p} + d_{k,p} x)}{d_{k,p}} \quad \text{for } x \in \mathbf{R}^n,$$

with the convention that $u_p = 0$ off Ω_p.

Now, choose p_k so large that $\tilde{e}_k := e_{k,p_k} \to e_0$ and moreover $\tilde{v}_k := v_{k,p_k} \to v_0$ uniformly on compacts. If now $\widetilde{D}_k = \{\tilde{v}_k > 0\}$ and $R > 0$ is fixed, then $\widetilde{D}_k \cap B_R(0)$ and $\partial \widetilde{D}_k \cap B_R(0)$ converge in the Hausdorff distance to $\Pi_0 \cap B_R(0)$ and $\partial \Pi_0 \cap B_R(0)$ respectively. Moreover, if

$$\tilde{\nu}_k(x) = \nabla \tilde{v}_k(x)/|\nabla \tilde{v}_k(x)| \quad \text{and} \quad \tilde{a}_k(x) = a(x_{k,p_k} + d_{k,p_k} x)$$

then $\tilde{\nu}_k(x)$ and $\tilde{a}_k(x)$ converge uniformly in $B_R(0)$ to e_0 and $a(x_0)$.

Set next for $p > 1$

$$F_p(x, u, \nabla u, \nabla^2 u) = -\frac{|\nabla u|^{4-p}}{p-2} \Delta_p u = -\frac{|\nabla u|^2}{p-2} \Delta u - \Delta_\infty u.$$

and observe that F_p converges locally uniformly on compacts to $F_\infty = -\Delta_\infty$ as $p \to \infty$.

Fix now large $R > 0$. Then $\tilde{v}_k$ is a classical and therefore viscosity solution of the following boundary value problem

$$\begin{cases} F_p(x, u, \nabla u, \nabla^2 u) \geq 0 & \text{in } \widetilde{D}_k \cap B_R(0) \\ -\nabla u \cdot \tilde{\nu}_k(x) + \tilde{a}_k(x) \geq 0 & \text{on } \partial \widetilde{D}_k \cap B_R(0) \end{cases} \tag{10.4}$$

Applying now Theorem 9.4, we conclude immediately that v_0 is a viscosity solution of

$$\begin{cases} -\Delta_\infty v \geq 0 & \text{in } \Pi_0 \cap B_R(0) \\ -\nabla v \cdot e_0 + a(x_0) \geq 0 & \text{on } \partial \Pi_0 \cap B_R(0) \end{cases} \tag{10.5}$$

for every R. Since R is arbitrary, (10.3) follows.

Step 4. In the representation $v_0(x) = \alpha (x \cdot e_0)^+$, we have $\alpha \leq a(x_0)$.

This follows from (10.3). Indeed, for $\beta < \alpha$, we have

$$v_0(x) = \alpha(x \cdot e_0) \geq \beta(x \cdot e_0) + (x \cdot e_0)^2 \quad \text{for } x \in \Pi_0,$$

as $x \to 0$, which means that $(\beta e_0, e_0^t e_0) \in J^{2,-}_{\overline{\Pi}_0} v_0(0)$. Hence we must have by the definition of the viscosity supersolution that

$$\max\{-\Delta_\infty(\beta(x \cdot e_0) + (x \cdot e_0)^2), -\nabla(\beta(x \cdot e_0) + (x \cdot e_0)^2) + a(x_0)\} \geq 0$$

at $x = 0$, or

$$\max\{-2\beta^2, -\beta + a(x_0)\} \geq 0.$$

This implies

$$-\beta + a(x_0) \geq 0$$

and since $\beta < \alpha$ was arbitrary,

$$\alpha \le a(x_0).$$

The identity (10.1), Lemma 10.1 and consequently Theorem 4.4 are proved. ∎

11. THE LIMIT AS $p \to 1$

We would like to conclude with a brief discussion of what happens with the classical solutions (u_p, Ω_p) of (FB_p) as $p \to 1+$. The answer is extremely simple.

THEOREM 11.1. *As $p \to 1+$, the domains Ω_p collapse to K, provided $n \ge 2$.*

Remark 11.2. We stress here that the space dimension $n \ge 2$. For $n = 1$ the statement of Theorem 11.1 fails, since all the problems (FB_p) are the same for all $p \in (1, \infty)$.

Proof of Theorem 11.1. We follow a scheme, similar to that in the proof of Theorem 3.3. For large E denote

$$\phi_E(s) = \frac{e^{Es} - 1}{E}$$

and let $\delta_E > 0$ be such that $\phi_E(\delta_E) = 1$, or explicitly

$$\delta_E = \frac{\log(1+E)}{E}.$$

Note that $\delta_E \to 0$ as $E \to \infty$. Let

$$u^E(x) = \phi_E(\delta_E - a_0 \operatorname{dist}(x, K))$$

in

$$\Omega^E = \{x \colon \operatorname{dist}(x, K) < \delta_E / a_0\}.$$

Assume for a moment that K has C^2 regular strictly convex boundary, such that the mean curvature $\kappa(x) \ge \kappa_1 > 0$ on ∂K. Then we claim that

(11.1) (u^E, Ω^E) is a classical supersolution of (FB_p) for $1 < p < p(E)$.

The free boundary conditions are easily verified, hence we need to prove only that $\Delta_p u^E \le 0$. Computing as in the proof of Theorem 3.3, we obtain

$$|\nabla u^E|^{2-p}\Delta_p u^E = \phi'_E a_0((p-1)a_0 E - (n-1)\kappa),$$

where κ is the mean curvature at a point on the level line $\{\operatorname{dist}(x, K) = s\}$ with $0 < s < \delta_E/a_0$. Then $\kappa \ge \kappa_1/(1+s\kappa_1)$ and consequently, if E is so large that $\delta_E/a_0 \le 1$,

$$\kappa \ge \kappa_2 = \kappa_1/(1+\kappa_1).$$

Hence $\Delta_p u^E \le 0$ as soon as

$$(p-1)a_0 E - (n-1)\kappa_2 \le 0$$

or equivalently

$$p < 1 + \frac{(n-1)\kappa_2}{a_0 E}.$$

Thus (11.1) and consequently the theorem follow in the case when K has a strictly convex C^2 regular boundary. For general K we can find $K_\eta \supset K$ that converges to K as $\eta \to 0$, with the properties as above, so that we can conclude $\lim_{p\to 1+} \Omega_p \subset K_\eta$. Then letting $\eta \to 0$ we complete the proof of the theorem. ▮

APPENDIX: CLASSICAL SOLUTIONS OF (FB_p)

The main objective of this appendix is to outline the main steps in the proof of Theorem 2.3 due to A. Henrot and H. Shahgholian [HS3]. We will make the same assumptions on the compact K and function $a(x)$ in $K^c = \mathbf{R}^n \setminus K$ as in the beginning of Section 2.

Let $\mathcal{E}^* = \mathcal{E}^*(K, a(x))$ be the subclass of all classical supersolutions (u, Ω) of (FB_p) (see Definition 2.1) with convex support Ω.

Consider the intersection

$$\Omega^* = \bigcap_{(u,\Omega)\in\mathcal{E}^*} \Omega$$

and set u^* to be the p-capacitary potential of the convex ring $\Omega^* \setminus K$

$$\Delta_p u^* = 0 \quad \text{in } \Omega^* \setminus K, \qquad u^* = 0 \quad \text{on } \partial\Omega, \qquad u^* = 1 \quad \text{on } K.$$

In order for this definition to have a meaning one needs to have

$$(*) \qquad \mathcal{E}^* \ne \emptyset \quad \text{and} \quad \Omega^* \supset\supset K.$$

Using the assumptions $a(x) \geq a_0 > 0$ in K^c and the uniform interior ball condition for K, we can construct explicit (radially symmetric) supersolutions and subsolutions to guarantee $(*)$.

Then the following assertion holds.

PROPOSITION A.1. *(u^*, Ω^*) is the classical solution of (FB_p).*

Here is the sketch of the proof.

Step 1. Prove that (u^*, Ω^*) is a classical supersolution of (FB_P). This is relatively easy to show, using that $v = |\nabla u|^p$ is a subsolution of the linearized p-Laplacian

$$L_u(v) = |\nabla u|^{p-2}\Delta v + (p-2)|\nabla u|^{p-4}\nabla^2 v \nabla u \cdot \nabla u \geq 0,$$

provided u is p-harmonic.

Step 2. Prove that $\limsup |\nabla u^*(y)| = a(x)$ as $y \to x$, $y \in \Omega^*$, for every *extreme point* $x \in \partial\Omega^*$. The latter means that x is not a convex combination of points on $\partial\Omega^*$ other than x. The points $x \in \partial\Omega^*$ with a hyperplane Π touching $\partial\Omega^*$ at x only are dense among all extreme points, so it is enough to prove the statement only for them. If in an arbitrary small neighborhood of such x we have $|\nabla u^*| < a$, then we can cut from Ω^* a small cap by a parallel translation of the supporting hyperplane Π, thus constructing a new supersolution; see [HS2, Lemma 3.4]. This, however, will contradict the minimality of Ω^*.

Step 3. Observe that *Step 2* implies C^1 regularity of $\partial\Omega^*$. Indeed, let $x \in \partial\Omega^*$ with two supporting hyperplanes. Then by barrier arguments, we will have $|\nabla u^*(y)| \to 0$ as $y \to x$, a contradiction.

Step 4. Having C^1 regularity of $\partial\Omega^*$, one can show that in fact $\liminf |\nabla u^*(y)| = a(x)$ as $y \to x$, $y \in \Omega^*$, for every extreme point $x \in \partial\Omega^*$. This can be shown, e.g. by the technique used in Section 5 of this paper.

Step 5. Now we have that the free boundary condition is satisfied for all points on $\partial\Omega^*$ except those contained in line segments $\ell \subset \partial\Omega^*$. To fill these points we use the assumption that $1/a(x)$ is concave along with the following nice observation due to P. Laurence and E. Stredulinsky [LS].

LEMMA A.2. *Let D be a convex domain with C^1 regular boundary ∂D and $\ell \subset \partial D$ a line segment. Let also u be a nonnegative C^1 function with convex level sets, defined in D and continuously vanishing on ∂D. Then $1/|\nabla u|$ is convex on ℓ.*

This completes the proof of the proposition and hence Theorem 2.3 follows.

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