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FINITE ELEMENT APPROXIMATION OF THE p -LAPLACIAN

JOHN W. BARRETT AND W. B. LIU

ABSTRACT. In this paper we consider the continuous piecewise linear finite element approximation of the following problem: Given $p \in (1, \infty)$, f , and g , find u such that

$$-\nabla \cdot (|\nabla u|^{p-2} \nabla u) = f \quad \text{in } \Omega \subset \mathbb{R}^2, \quad u = g \quad \text{on } \partial\Omega.$$

The finite element approximation is defined over Ω^h , a union of regular triangles, yielding a polygonal approximation to Ω . For sufficiently regular solutions u , achievable for a subclass of data f , g , and Ω , we prove optimal error bounds for this approximation in the norm $W^{1,q}(\Omega^h)$, $q = p$ for $p < 2$ and $q \in [1, 2]$ for $p > 2$, under the additional assumption that $\Omega^h \subseteq \Omega$. Numerical results demonstrating these bounds are also presented.

1. INTRODUCTION

Let Ω be a bounded open set in \mathbb{R}^2 with a Lipschitz boundary $\partial\Omega$. Given $p \in (1, \infty)$, $f \in L^2(\Omega)$, and $g \in W^{1-1/p,p}(\partial\Omega)$, we consider the following problem:

(\mathcal{P}) Find $u \in W_g^{1,p}(\Omega) \equiv \{v \in W^{1,p}(\Omega) : v = g \text{ on } \partial\Omega\}$ such that

$$\int_{\Omega} |\nabla u|^{p-2} (\nabla u, \nabla v)_{\mathbb{R}^2} d\Omega = \int_{\Omega} f v d\Omega \quad \forall v \in W_0^{1,p}(\Omega),$$

where $|v|^2 = (v, v)_{\mathbb{R}^2}$. Throughout we adopt the standard notation $W^{m,q}(D)$ for Sobolev spaces on D with norm $\|\cdot\|_{W^{m,q}(D)}$ and seminorm $|\cdot|_{W^{m,q}(D)}$. We note that the seminorm $|\cdot|_{W^{1,q}(D)}$ and the norm $\|\cdot\|_{W^{1,q}(D)}$ are equivalent on $W_0^{1,q}(D)$.

Problem (\mathcal{P}) above is the weak formulation of the Dirichlet problem for the p -Laplacian

$$(1.1) \quad -\nabla \cdot (|\nabla u|^{p-2} \nabla u) = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega.$$

The well-posedness of (\mathcal{P}) is well established, and one can refer to, for example, Glowinski and Marrocco [5] or the account in Ciarlet [4]. Of course, one can study more general boundary conditions and the presence of lower-order terms in the differential operator. However, for ease of exposition, we just consider (\mathcal{P}), although most of our results can be adapted to more general

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problems. From Glowinski and Marrocco [5], or Ciarlet [4], (\mathcal{P}) is equivalent to the following minimization problem:

(\mathcal{Q}) Find $u \in W_g^{1,p}(\Omega)$ such that

$$(1.2a) \quad J_\Omega(u) \leq J_\Omega(v) \quad \forall v \in W_g^{1,p}(\Omega),$$

where

$$(1.2b) \quad J_\Omega(v) \equiv \frac{1}{p} \int_\Omega |\nabla v|^p d\Omega - \int_\Omega f v d\Omega.$$

It is easily established that $J_\Omega(\cdot)$ is strictly convex and continuous on $W_g^{1,p}(\Omega)$. Further, $J_\Omega(\cdot)$ is Gateaux differentiable with

$$(1.3) \quad J'_\Omega(u)(v) \equiv \int_\Omega |\nabla u|^{p-2} (\nabla u, \nabla v)_{\mathbb{R}^2} d\Omega - \int_\Omega f v d\Omega \quad \forall v \in W_0^{1,p}(\Omega).$$

Hence, there exists a unique solution to (\mathcal{Q}), and (\mathcal{Q}) is equivalent to (\mathcal{P}), its Euler equation. In addition, we have that

$$(1.4) \quad \|u\|_{W^{1,p}(\Omega)} \leq C[\|f\|_{L^2(\Omega)}^{1/(p-1)} + \|g\|_{W^{1-1/p,p}(\partial\Omega)}].$$

The problem (\mathcal{P}) occurs in many mathematical models of physical processes: nonlinear diffusion and filtration, see Philip [8]; power-law materials, see Atkinson and Champion [1]; and quasi-Newtonian flows, see Atkinson and Jones [2], for example.

It is the purpose of this paper to analyze the finite element approximation of (\mathcal{P}). Let Ω^h be a polygonal approximation to Ω defined by $\overline{\Omega^h} \equiv \bigcup_{\tau \in T^h} \overline{\tau}$, where T^h is a partitioning of Ω^h into a finite number of disjoint open regular triangles τ , each of maximum diameter bounded above by h . In addition, for any two distinct triangles, their closures are either disjoint, or have a common vertex, or a common side. Let $\{P_j\}_{j=1}^J$ be the vertices associated with the triangulation T^h , where P_j has coordinates (x_j, y_j) . Throughout we assume that $P_j \in \partial\Omega^h$ implies $P_j \in \partial\Omega$, and that $\Omega^h \subseteq \Omega$. We note that, owing to the elliptic degeneracy of the p -Laplacian and the limited regularity of the solution u , see below, it is not a simple matter to extend the results in this paper to the case $\Omega^h \not\subseteq \Omega$. Associated with T^h is the finite-dimensional space

$$(1.5) \quad S^h \equiv \{\chi \in C(\overline{\Omega^h}) : \chi|_\tau \text{ is linear } \forall \tau \in T^h\} \subset W^{1,p}(\Omega^h).$$

Let $\pi_h : C(\overline{\Omega^h}) \rightarrow S^h$ denote the interpolation operator such that for any $v \in C(\overline{\Omega^h})$, the interpolant $\pi_h v \in S^h$ satisfies $\pi_h v(P_j) = v(P_j)$, $j = 1, \dots, J$. We recall the following standard approximation results. For $m = 0$ or 1 , and for all $\tau \in T^h$, we have (a) for $q \in [1, \infty]$, $s \in [1, \infty]$, provided $W^{2,s}(\tau) \hookrightarrow W^{m,q}(\tau)$,

$$(1.6a) \quad |v - \pi_h v|_{W^{m,q}(\tau)} \leq Ch^{2(1/q-1/s)} h^{2-m} |v|_{W^{2,s}(\tau)} \quad \forall v \in W^{2,s}(\tau);$$

and (b) for $q > 2$,

$$(1.6b) \quad |v - \pi_h v|_{W^{m,q}(\tau)} \leq Ch^{1-m} |v|_{W^{1,q}(\tau)} \quad \forall v \in W^{1,q}(\tau).$$

In (1.6a) we have noted the imbedding $W^{2,1}(\tau) \hookrightarrow C(\overline{\tau})$; see, for example, p. 300 in Kufner et al. [6].

The finite element approximation of (\mathcal{P}) that we wish to consider is:

(\mathcal{P}^h) Find $u^h \in S_g^h$ such that

$$(1.7a) \quad \int_{\Omega^h} |\nabla u^h|^{p-2} (\nabla u^h, \nabla v^h)_{\mathbb{R}^2} d\Omega^h = \int_{\Omega^h} f v^h d\Omega^h \quad \forall v^h \in S_0^h,$$

where

$$(1.7b) \quad S_0^h \equiv \{\chi \in S^h : \chi = 0 \text{ on } \partial\Omega^h\}$$

and

$$(1.7c) \quad S_g^h \equiv \{\chi \in S^h : \chi = g^h \text{ on } \partial\Omega^h\},$$

where $g^h \in S^h$ is chosen to approximate the Dirichlet boundary data. If $p > 2$, then $u \in W^{1,p}(\Omega)$ implies $u \in C(\overline{\Omega})$, and so we set $g^h \equiv \pi_h u$. For the explicit error bounds derived in §3 for $p < 2$, we assume that $u \in C(\overline{\Omega})$, and so once again set $g^h \equiv \pi_h u$. However, for the abstract analysis of this and the next section, g^h can be arbitrary. The corresponding minimization problem is:

(\mathcal{Q}^h) Find $u^h \in S_g^h$ such that

$$(1.8) \quad J_{\Omega^h}(u^h) \leq J_{\Omega^h}(v^h) \quad \forall v^h \in S_g^h.$$

The well-posedness of (\mathcal{P}^h) \equiv (\mathcal{Q}^h) follows in an analogous way to that of (\mathcal{P}) and (\mathcal{Q}), see Glowinski and Marrocco [5] or Ciarlet [4], and

$$(1.9) \quad \|u^h\|_{W^{1,p}(\Omega^h)} \leq C[\|f\|_{L^2(\Omega^h)}^{1/(p-1)} + \|g^h\|_{W^{1,p}(\Omega^h)}].$$

We note that for $p = 2$, problem (\mathcal{P}) reduces to the weak formulation of the linear Laplacian, and hence the regularity of u and the finite element error analysis are well established in this case. For $p \neq 2$, the regularity of u is less well established, as (1.1) is then a degenerate quasi-linear elliptic problem. It is well known, see Example 3.1 in §3, that u has limited regularity for infinitely smooth data f , g , and Ω . Therefore, there is no benefit in considering higher-order finite element approximations, and hence our restriction to continuous piecewise linears from the outset. Lieberman [7] has proved that if $\partial\Omega \in C^{1,\beta}$, then g is the trace of a function $\in C^{1,\gamma}(\overline{\Omega})$ for $\beta, \gamma \in (0, 1)$, and if $f \in L^\infty(\Omega)$, then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. However, for explicit finite element error bounds one requires global regularity results on the second, or maybe higher, derivatives of u . Unfortunately, such results are not available at present in the literature, but it is an active area of research worldwide.

The following error bounds were proved in Glowinski and Marrocco [5] for the case $\Omega^h \equiv \Omega$ and $g \equiv 0$:

If $u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$, then

$$(1.10) \quad \|u - u^h\|_{W^{1,p}(\Omega)} \leq \begin{cases} Ch^{1/(3-p)} & \text{if } p \leq 2, \\ Ch^{1/(p-1)} & \text{if } p \geq 2, \end{cases}$$

where throughout this paper C denotes a generic positive constant independent of h . Chow [3], employing an approach of Tyukhtin [9], improved these error bounds. He proved that

$$(1.11a) \quad \|u - u^h\|_{W^{1,p}(\Omega)} \leq C \|u - v^h\|_{W^{1,p}(\Omega)}^{p/2} \quad \forall v^h \in S_0^h \quad \text{if } p \leq 2,$$

and

$$(1.11b) \quad \|u - u^h\|_{W^{1,p}(\Omega)} \leq C(\|\pi_h u\|_{W^{1,p}(\Omega)}) \|u - \pi_h u\|_{W^{1,p}(\Omega)}^{2/p} \quad \text{if } p > 2,$$

and hence, if $u \in W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$, it follows from (1.6a) that

$$(1.12) \quad \|u - u^h\|_{W^{1,p}(\Omega)} \leq \begin{cases} Ch^{p/2} & \text{if } p \leq 2, \\ Ch^{2/p} & \text{if } p \geq 2. \end{cases}$$

It is the purpose of this paper to prove optimal error bounds. The layout is as follows. In the next section we prove an abstract error bound for the approximation (\mathcal{P}^h) of (\mathcal{P}) . In §3 we study the case $p \in (1, 2)$ and prove an optimal $W^{1,p}$ error bound, that is, $O(h)$, provided that $u \in W^{3,1}(\Omega) \cap C^{2,(2-p)/p}(\overline{\Omega})$. Thus, this optimal error bound requires a stronger regularity assumption on u than that for the bound (1.12) in the case $p < 2$. In §4 we study the case $p > 2$ and first show that the bound (1.12) for $p > 2$ can be achieved under the weaker regularity requirement $u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$. Second, under the additional assumption $|f| \geq \rho > 0$ a.e. in Ω , we prove an optimal $W^{1,4/3}$ error bound. We note that the above regularity requirements on u for these optimal error bounds are achievable for a subclass of data f , g , and Ω . In §5 we show that the error bounds derived in the previous sections hold for the fully practical scheme of employing numerical integration on the right-hand side of (1.7) if f is sufficiently smooth. Finally, we report on some numerical examples, which confirm these optimal error bounds.

2. AN ABSTRACT ERROR BOUND

We first prove a lemma, which is a generalization of Lemmas 5.1, 5.2, 5.3 and 5.4 in Glowinski and Marrocco [5].

Lemma 2.1. *For all $p > 1$ and $\delta \geq 0$ there exist positive constants C_1 and C_2 such that for all $\xi, \eta \in \mathbb{R}^2$, $\xi \neq \eta$,*

$$(2.1a) \quad ||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \leq C_1 |\xi - \eta|^{1-\delta} (|\xi| + |\eta|)^{p-2+\delta}$$

and

$$(2.1b) \quad (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta)_{\mathbb{R}^2} \geq C_2 |\xi - \eta|^{2+\delta} (|\xi| + |\eta|)^{p-2-\delta}.$$

Proof. The approach is similar to that in Glowinski and Marrocco [5]. For all $\xi, \eta \in \mathbb{R}^2$, $\xi \neq \eta$, let

$$(2.2) \quad G_1(\xi, \eta) \equiv ||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| / [|\xi - \eta|^{1-\delta} (|\xi| + |\eta|)^{p-2+\delta}].$$

We wish to prove that G_1 is bounded above. For any $\varepsilon > 0$, G_1 is continuous on

$$D_\varepsilon \equiv \{(\xi, \eta) : |\xi - \eta| \geq \varepsilon \text{ and } (|\xi| + |\eta|) \leq 1/\varepsilon\}.$$

In addition, we note that for all $\xi, \eta \in \mathbb{R}^2$, $\xi \neq \eta$,

$$(2.3) \quad \begin{aligned} G_1(\xi, \eta) &\equiv G_1(\eta, \xi), & G_1(\lambda\xi, \lambda\eta) &\equiv G_1(\xi, \eta) \quad \text{for all } \lambda \in \mathbb{R}^+, \\ G_1(0, \eta) &= 1 \quad \text{and} \quad G_1(A\xi, A\eta) &\equiv G_1(\xi, \eta) \quad \text{if } A^T A = I, \end{aligned}$$

i.e., A is a rotation matrix. Therefore, without loss of generality we can take $\xi = e_1 \equiv (1, 0)$. Since

$$(2.4) \quad G_1(e_1, \eta) \rightarrow 1 \quad \text{as } |\eta| \rightarrow \infty,$$

it remains to show that $\limsup G_1(e_1, \eta) < \infty$ as $|e_1 - \eta| \rightarrow 0$.

Let $\eta = (1 + \rho \cos \theta, \rho \sin \theta)$. Then a simple calculation yields that

$$(2.5) \quad \lim_{\rho \rightarrow 0} G_1(e_1, \eta) = \begin{cases} 0 & \text{if } \delta > 0, \\ 2^{2-p} \{1 + p(p-2) \cos^2 \theta\}^{1/2} & \text{if } \delta = 0. \end{cases}$$

Hence the desired result (2.1a).

Similarly, we prove (2.1b). Let

$$(2.6) \quad G_2(\xi, \eta) \equiv |\xi - \eta|^{2+\delta} (|\xi| + |\eta|)^{p-2-\delta} / (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta)_{\mathbb{R}^2}.$$

From Glowinski and Marrocco [5] we have that

$$(2.7) \quad (|\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta)_{\mathbb{R}^2} > 0 \quad \text{if } \xi \neq \eta.$$

Therefore, we only need to prove that G_2 is bounded above. In addition, the results (2.3) and (2.4) hold for G_2 .

Setting $\eta = (1 + \rho \cos \theta, \rho \sin \theta)$, a simple calculation yields that

$$(2.8) \quad \lim_{\rho \rightarrow 0} G_2(e_1, \eta) = \begin{cases} 0 & \text{if } \delta > 0, \\ 2^{p-2} \{1 + (p-2) \cos^2 \theta\}^{-1} & \text{if } \delta = 0. \end{cases}$$

Hence the desired result (2.1b). \square

The inequality (2.1a) was proved in Glowinski and Marrocco [5] for $p \in (1, 2]$ with $\delta = 2 - p$, and for $p \geq 2$ with $\delta = 0$; similarly, (2.1b) was proved for $p \in (1, 2]$ with $\delta = 0$, and for $p \geq 2$ with $\delta = p - 2$.

For $p \in (1, \infty)$ and $\sigma \geq 0$ we define for any $v \in W^{1,p}(\Omega^h)$

$$(2.9) \quad |v|_{(p,\sigma)} \equiv \int_{\Omega^h} (|\nabla u| + |\nabla v|)^{p-\sigma} |\nabla v|^\sigma d\Omega^h,$$

where u is the solution of (\mathcal{P}) . We prove the following results for later use.

Lemma 2.2. For $p \in (1, \sigma]$ we have

$$(2.10a) \quad |v|_{(p,\sigma)}^{\sigma/p} \leq |v|_{W^{1,p}(\Omega^h)}^\sigma \leq C[|u|_{W^{1,p}(\Omega^h)} + |v|_{W^{1,p}(\Omega^h)}]^{\sigma-p} |v|_{(p,\sigma)},$$

and for $p \in [\sigma, \infty)$,

$$(2.10b) \quad |v|_{W^{1,p}(\Omega^h)}^p \leq |v|_{(p,\sigma)} \leq C[|u|_{W^{1,p}(\Omega^h)} + |v|_{W^{1,p}(\Omega^h)}]^{p-\sigma} |v|_{W^{1,p}(\Omega^h)}^\sigma.$$

Hence, (2.9) is well defined for $v \in W^{1,p}(\Omega^h)$.

Proof. Setting $w \equiv (|\nabla u| + |\nabla v|)^{p-\sigma}$, we first consider the case $p \in (1, \sigma]$. The left inequality in (2.10a) follows immediately from noting that $w \leq |\nabla v|^{p-\sigma}$. Applying Hölder's inequality, we have

$$\begin{aligned} |v|_{W^{1,p}(\Omega^h)}^{\sigma/p} &\equiv \left\{ \int_{\Omega^h} w^{-p/\sigma} [w^{1/\sigma} |\nabla v|]^p d\Omega^h \right\}^{\sigma/p} \\ &\leq \left\{ \int_{\Omega^h} w^{-p/(\sigma-p)} d\Omega^h \right\}^{(\sigma-p)/p} |v|_{(p,\sigma)} \\ &= \left\{ \int_{\Omega^h} (|\nabla u| + |\nabla v|)^p d\Omega^h \right\}^{(\sigma-p)/p} |v|_{(p,\sigma)}. \end{aligned}$$

The right inequality in (2.10a) follows by noting that for all $\gamma \in [0, \infty)$ there exists $C_\gamma > 0$ such that $|a + b|^\gamma \leq C_\gamma(|a|^\gamma + |b|^\gamma)$ for all $a, b \in \mathbb{R}$.

The inequalities (2.10b) can be proved in a similar manner. \square

The next theorem is the natural generalization of the result in §7 of Chow [3]. We use the minimization property of u^h and Lemma 2.1, whereas Chow uses the Glowinski and Marrocco version of Lemma 2.1.

Theorem 2.1. *Let u and u^h be the unique solutions of $(\mathcal{P}) \equiv (\mathcal{Q})$ and $(\mathcal{P}^h) \equiv (\mathcal{Q}^h)$, respectively. Then for any $\delta_1 \in [0, 2)$ and $\delta_2 \geq 0$, and any $v^h \in S_g^h$, it follows that*

$$(2.11) \quad |u - u^h|_{(p, 2+\delta_2)} \leq C|u - v^h|_{(p, 2-\delta_1)}.$$

Proof. We have for any $v^h \in S_g^h$ that

$$\begin{aligned} J_{\Omega^h}(v^h) - J_{\Omega^h}(u) &= \int_0^1 J'_{\Omega^h}(u + s(v^h - u))(v^h - u) ds \\ &= \int_0^1 [J'_{\Omega^h}(u + s(v^h - u))([u + s(v^h - u)] - u) \\ &\quad - J'_{\Omega^h}(u)([u + s(v^h - u)] - u)] \frac{ds}{s} \\ &\quad + J'_{\Omega^h}(u)(v^h - u) \\ &\equiv A(v^h) + J'_{\Omega^h}(u)(v^h - u), \end{aligned} \quad (2.12a)$$

where from (1.3)

$$\begin{aligned} A(v^h) &\equiv \int_0^1 \left[\int_{\Omega^h} \{[|\nabla(u + s(v^h - u))|]^{p-2} \right. \\ &\quad \times \nabla(u + s(v^h - u)) - |\nabla u|^{p-2} \nabla u \nabla(v^h - u)\} d\Omega^h \Big] ds. \end{aligned} \quad (2.12b)$$

From (2.12b) and (2.1a) we have that

$$\begin{aligned} |A(v^h)| &\leq C_1 \int_0^1 s^{1-\delta_1} \int_{\Omega^h} (|\nabla[u + s(v^h - u)]| + |\nabla u|)^{p-2+\delta_1} \\ &\quad \times |\nabla(v^h - u)|^{2-\delta_1} d\Omega^h ds \\ &\leq C|u - v^h|_{(p, 2-\delta_1)}, \end{aligned} \quad (2.13)$$

where we have noted that for all v_1, v_2 , and $s \in [0, 1]$

$$(2.14) \quad \frac{1}{2}s(|\nabla v_1| + |\nabla v_2|) \leq |\nabla[v_1 + sv_2]| + |\nabla v_1| \leq 2(|\nabla v_1| + |\nabla v_2|).$$

From (2.12b), (2.1b), and (2.14) we have that

$$\begin{aligned} |A(v^h)| &\geq C_2 \int_0^1 s^{1+\delta_2} \int_{\Omega^h} (|\nabla[u + s(v^h - u)]| + |\nabla u|)^{p-2+\delta_2} \\ &\quad \times |\nabla(v^h - u)|^{2+\delta_2} d\Omega^h ds \\ &\geq C|u - v^h|_{(p, 2+\delta_2)}. \end{aligned} \quad (2.15)$$

From (1.8) and (2.12) we have that for all $v^h \in S_g^h$

$$\begin{aligned} A(u^h) + J'_{\Omega^h}(u)(u^h - u) &\equiv J_{\Omega^h}(u^h) - J_{\Omega^h}(u) \leq J_{\Omega^h}(v^h) - J_{\Omega^h}(u) \\ &\equiv A(v^h) + J'_{\Omega^h}(u)(v^h - u). \end{aligned} \quad (2.16)$$

Therefore, it follows from (2.16), (2.13), and (2.15) that

$$(2.17) \quad |u - u^h|_{(p, 2+\delta_2)} \leq C|u - v^h|_{(p, 2-\delta_1)} + J'_{\Omega^h}(u)(v^h - u^h).$$

As Ω^h is Lipschitz, $\Omega^h \subseteq \Omega$ and $\chi \equiv v^h - u^h \in S_0^h$, we can extend χ to be zero on $\Omega \setminus \Omega^h$. Denoting this extension by $\hat{\chi}$, we have that $\hat{\chi} \in W_0^{1,p}(\Omega)$ and hence from (\mathcal{P}) that $J'_{\Omega^h}(u)(\chi) \equiv J'_\Omega(u)(\hat{\chi}) = 0$. Therefore, the desired result (2.11) follows from (2.17). \square

3. ERROR BOUNDS FOR $p \in (1, 2)$

Assuming that $u \in W^{2,1}(\Omega)$, which implies that $u \in C(\overline{\Omega})$, we can set $g^h \equiv \pi_h u$ in (1.7c). Choosing $\delta_2 = 0$ in (2.11) and noting (2.10a), (1.4), (1.9), and (1.6a), we have for all $\delta_1 \in [0, 2)$ and for all $v^h \in S_g^h$

$$(3.1) \quad |u - u^h|_{W^{1,p}(\Omega^h)}^2 \leq C|u - u^h|_{(p,2)} \leq C|u - v^h|_{(p,2-\delta_1)}.$$

Choosing $\delta_1 = 2 - p$ and noting (2.10a) yield that for all $v^h \in S_g^h$

$$(3.2) \quad |u - u^h|_{W^{1,p}(\Omega^h)}^2 \leq C|u - v^h|_{(p,p)} \leq C|u - v^h|_{W^{1,p}(\Omega^h)}^p.$$

From a Poincaré inequality we have for all $q \in [1, \infty)$, for all $v \in W^{1,q}(\Omega)$, and for all $v^h, w^h \in S_g^h$ that

$$(3.3) \quad \|v - w^h\|_{W^{1,q}(\Omega^h)} \leq C\|v - v^h\|_{W^{1,q}(\Omega^h)} + C|v - w^h|_{W^{1,q}(\Omega^h)}.$$

Hence, from (3.2) with $v^h \equiv \pi_h u$, (3.3), and (1.6a) we have that

$$(3.4a) \quad \|u - u^h\|_{W^{1,p}(\Omega^h)} \leq C\|u - \pi_h u\|_{W^{1,p}(\Omega^h)} + C|u - \pi_h u|_{W^{1,p}(\Omega^h)}^{p/2}$$

$$(3.4b) \quad \leq Ch^{p/2} \quad \text{if } u \in W^{2,p}(\Omega),$$

the generalization of the results (1.11a) and (1.12), for $p < 2$, of Chow [3] to the case of nonhomogeneous boundary data g and $\Omega^h \subseteq \Omega$. Below we prove an optimal $W^{1,p}$ error bound for sufficiently regular u , based on choosing $\delta_1 = 0$ in (3.1).

Lemma 3.1. *Let $\alpha \in (-1, 0)$. If $v \in W^{2,1}(\Omega)$, then*

$$(3.5) \quad \int_{\Omega} |v|^\alpha |\nabla v|^2 d\Omega < \infty.$$

Proof. We have that

$$\begin{aligned} \int_{\Omega} |v|^\alpha |v_x|^2 dx dy &\equiv \frac{1}{\alpha + 1} \int_{\Omega} (\text{sign}(v)|v|^{\alpha+1})_x v_x dx dy \\ &\equiv \frac{1}{\alpha + 1} \left\{ \int_{\partial\Omega} \text{sign}(v)|v|^{\alpha+1} v_x dy - \int_{\Omega} \text{sign}(v)|v|^{\alpha+1} v_{xx} dx dy \right\} \end{aligned}$$

and a similar identity with v_x replaced by v_y . The desired result (3.5) then follows from the imbedding $W^{2,1}(\Omega) \hookrightarrow C(\overline{\Omega})$ and the trace inequality $\|\cdot\|_{L^1(\partial\Omega)} \leq C\|\cdot\|_{W^{1,1}(\Omega)}$. \square

Theorem 3.1. *If $u \in W^{3,1}(\Omega) \cap C^{2,\alpha}(\overline{\Omega})$, with $\alpha > 0$, then it follows that*

$$(3.6a) \quad \|u - u^h\|_{W^{1,p}(\Omega^h)}^2 \leq C[h^2 + h^{p(1+\alpha)}],$$

and hence, if $u \in W^{3,1}(\Omega) \cap C^{2,(2-p)/p}(\overline{\Omega})$, then

$$(3.6b) \quad \|u - u^h\|_{W^{1,p}(\Omega^h)} \leq Ch.$$

Proof. As $u \in C^{2,\alpha}(\overline{\Omega})$, we have from (1.6a) that for all $\tau \in T^h$ and for all $(x, y) \in \overline{\tau}$

$$(3.7) \quad |\nabla(u - \pi_h u)(x, y)| \leq Ch|H[u]|_{L^\infty(\tau)} \leq ChH[u](x, y) + Ch^{1+\alpha},$$

where $H[u] \equiv |u_{xx}| + |u_{xy}| + |u_{yy}|$.

It is easy to check that the function $q(t) \equiv (a + t)^{p-2}t^2$ with $a \geq 0$ is increasing on \mathbb{R}^+ and hence that $q(|t_1| + |t_2|) \leq 2[q(|t_1|) + q(|t_2|)]$ for all $t_1, t_2 \in \mathbb{R}$. Therefore, we have from (3.1) with $\delta_1 = 0$ and $v^h \equiv \pi_h u$, (3.7), and the above that

$$(3.8) \quad \begin{aligned} |u - u^h|_{W^{1,p}(\Omega^h)}^2 &\leq C \int_{\Omega^h} (|\nabla u| + |\nabla(u - \pi_h u)|)^{p-2} |\nabla(u - \pi_h u)|^2 d\Omega^h \\ &\leq Ch^2 \int_{\Omega^h} (|\nabla u| + ChH[u])^{p-2} (H[u])^2 d\Omega^h \\ &\quad + Ch^{2(1+\alpha)} \int_{\Omega^h} (|\nabla u| + Ch^{1+\alpha})^{p-2} d\Omega^h \\ &\leq Ch^{p(1+\alpha)} + Ch^2 \int_{\Omega^h} (|\nabla u|)^{p-2} (H[u])^2 d\Omega^h. \end{aligned}$$

Setting $v_1 \equiv u_x$ and $v_2 \equiv u_y$, we have from (3.5), as $v_1, v_2 \in W^{2,1}(\Omega)$, that

$$(3.9) \quad \begin{aligned} \int_{\Omega^h} (|\nabla u|)^{p-2} (H[u])^2 d\Omega^h &\leq C \int_{\Omega^h} (v_1^2 + v_2^2)^{(p-2)/2} (|\nabla v_1|^2 + |\nabla v_2|^2) d\Omega^h \\ &\leq C \int_{\Omega^h} [|v_1|^{p-2} |\nabla v_1|^2 + |v_2|^{p-2} |\nabla v_2|^2] d\Omega^h < \infty. \end{aligned}$$

Combining (3.8) and (3.9) yields the result (3.6a) and hence (3.6b) with $\|\cdot\|_{W^{1,p}(\Omega^h)}$ replaced by $|\cdot|_{W^{1,p}(\Omega^h)}$. The results (3.6) then follow by noting (3.3), (1.6a), and that $u \in W^{3,1}(\Omega)$ implies $u \in W^{2,p}(\Omega)$. \square

We note that one can prove (3.6b) under alternative regularity requirements on u , e.g., $u \in W^{3,p}(\Omega)$. However, we will not exploit this here. We now show that the regularity requirements on u in Theorem 3.1 hold for a model problem.

Example 3.1. We consider a radially symmetric version of problem (\mathcal{P}) . Let $\Omega \equiv \{r: r < 1\}$, $f(x, y) \equiv F(r)$, $f \in L^q(\Omega)$ for $q > 2$, and g be constant, where $r \equiv (x^2 + y^2)^{1/2}$. Then

$$(3.10a) \quad u(x, y) \equiv U(r) \equiv - \int_r^1 \text{sign}(Z(t)) |Z(t)|^{1/(p-1)} dt + g,$$

$$(3.10b) \quad U'' \equiv (|Z|^{(2-p)/(p-1)} Z') / (p-1)$$

and

$$(3.10c) \quad U''' \equiv C_1 \text{sign}(Z) |Z|^{(3-2p)/(p-1)} (Z')^2 + C_2 |Z|^{(2-p)/(p-1)} Z''$$

for some constants C_i , where

$$(3.10d) \quad z(x, y) \equiv Z(r) \equiv (|U'|^{p-2} U')(r) = -\frac{1}{r} \int_0^r t F(t) dt.$$

It is a simple matter to deduce from (3.10d) that

$$(3.11a) \quad f \in C^{0,q}(\overline{\Omega}) \Rightarrow z \in C^{1,q}(\overline{\Omega}) \quad \forall q \in [0, 1]$$

and

$$(3.11b) \quad f \in W^{1,q}(\Omega) \text{ for } q > 1 \Rightarrow z \in W^{2,1}(\Omega) \Rightarrow z \in C(\overline{\Omega}).$$

It follows from (3.10b) and (3.11a) that

$$(3.12a) \quad p \in (1, \frac{3}{2}] \text{ and } f \in C^{0,\beta}(\overline{\Omega}) \text{ for } \beta \in [0, 1] \Rightarrow u \in C^{2,\beta}(\overline{\Omega}),$$

$$(3.12b) \quad p \in [\frac{3}{2}, 2) \text{ and } f \in C^{0,\beta}(\overline{\Omega}) \text{ for } \beta \in [0, (2-p)/(p-1)] \Rightarrow u \in C^{2,\beta}(\overline{\Omega}),$$

and from (3.10c), (3.11b), and Lemma 3.1 that

$$(3.12c) \quad p \in (1, 2) \text{ and } f \in W^{1,q}(\Omega) \text{ for } q > 1 \Rightarrow u \in W^{3,1}(\Omega).$$

4. ERROR BOUNDS FOR $p > 2$

Let $g^h \equiv \pi_h u$ in (1.7c). From (2.11) with $\delta_1 = 0$ and $\delta_2 = p - 2$, (2.10b), (1.4), (1.9), and (1.6b) it follows that

$$(4.1) \quad |u - u^h|_{W^{1,p}(\Omega^h)}^p \leq |u - u^h|_{(p,p)} \leq C|u - \pi_h u|_{(p,2)} \leq C|u - \pi_h u|_{W^{1,p}(\Omega^h)}^2,$$

and hence it follows from (3.3) and (1.6a) that

$$(4.2a) \quad \|u - u^h\|_{W^{1,p}(\Omega^h)} \leq C\|u - \pi_h u\|_{W^{1,p}(\Omega^h)} + C|u - \pi_h u|_{W^{1,p}(\Omega^h)}^{2/p}$$

$$(4.2b) \quad \leq Ch^{2/p} \quad \text{if } u \in W^{2,p}(\Omega),$$

the generalization of the results (1.11b) and (1.12), for $p > 2$, of Chow [3] to the case of nonhomogeneous boundary data g and $\Omega^h \subseteq \Omega$. Alternatively, assuming $u \in W^{1,\infty}(\Omega)$, we have from (2.11) with $\delta_1 = 2 - s$, $s \in [1, 2]$, and $\delta_2 = p - 2$ and (1.6b) that

$$(4.3) \quad |u - u^h|_{W^{1,p}(\Omega^h)}^p \leq |u - u^h|_{(p,p)} \leq C|u - \pi_h u|_{(p,s)} \leq C|u - \pi_h u|_{W^{1,s}(\Omega^h)}^s.$$

In addition, we note from (1.6b) that for $u \in W^{1,\infty}(\Omega)$

$$(4.4) \quad \|u - \pi_h u\|_{W^{1,q}(\Omega^h)} \leq C\|u - \pi_h u\|_{W^{1,s}(\Omega^h)}^{s/q} \quad \text{if } q > s.$$

Hence, it follows from (4.3), (3.3), (4.4), and (1.6a) that if $u \in W^{1,\infty}(\Omega) \cap W^{2,s}(\Omega)$, $s \in [1, 2]$, then

$$(4.5a) \quad \|u - u^h\|_{W^{1,p}(\Omega^h)} \leq C\|u - \pi_h u\|_{W^{1,p}(\Omega^h)} + C|u - \pi_h u|_{W^{1,s}(\Omega^h)}^{s/p}$$

$$(4.5b) \quad \leq C\|u - \pi_h u\|_{W^{1,s}(\Omega^h)}^{s/p} \leq Ch^{s/p}.$$

Choosing $f \equiv 1$ and $g \equiv 0$ in Example 3.1 yields that $u(x, y) \equiv C(1 - r^{p/(p-1)})$, and so $u \in W^{2,s}(\Omega)$ only if $s < 2(p-1)/(p-2)$. Therefore, in general u rarely belongs to $W^{2,p}(\Omega)$ in order for (4.2b) to guarantee that the error converges at least at the rate of $h^{2/p}$ in $W^{1,p}$. However, from (4.5b) we see that this rate is ensured under the far weaker regularity requirement of $u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$, and this is satisfied by the example above.

Below we prove error bounds in weaker norms, $\|\cdot\|_{W^{1,q}(\Omega^h)}$ with $q \in [1, p)$.

Lemma 4.1. For all $t \in [2, p]$ and $q \in [1, t]$ for which

$$(4.6a) \quad \int_{\Omega} |\nabla u|^{-(p-t)q/(t-q)} d\Omega < \infty \quad \text{if } q \in [1, t)$$

and

$$(4.6b) \quad |\nabla u|^{-(p-t)} \in L^\infty(\Omega) \quad \text{if } q = t,$$

we have for $u \in W^{1,\infty}(\Omega) \cap W^{2,s}(\Omega)$ with $s \in [1, 2]$ that

$$(4.7) \quad \|u - u^h\|_{W^{1,q}(\Omega^h)} \leq Ch^{s/t}.$$

Proof. Choosing $\delta_1 = 2 - s$ and $\delta_2 = t - 2$ in (2.11), noting (4.3) and (4.6), and applying a Hölder inequality, we obtain that

$$(4.8) \quad |u - u^h|_{W^{1,q}(\Omega^h)}^t \leq C|u - u^h|_{(p,t)} \leq C|u - \pi_h u|_{(p,s)} \leq C|u - \pi_h u|_{W^{1,s}(\Omega^h)}^s.$$

The desired result (4.7) then follows from (4.8), (1.6a), (3.3), and (4.4). \square

To improve on the $h^{s/p}$ convergence rate for the error in (4.5b), we wish to take $t \in [2, p)$, which gives rise to the restrictions (4.6) on u ; that is, we require $\{(x, y) \in \Omega: |\nabla u(x, y)| = 0\}$ to have zero measure and a growth condition on $|\nabla u|^{-1}$. From inspection we see that the weakest growth restriction on $|\nabla u|^{-1}$ for a fixed t is needed when $q = 1$. We now look for sufficient conditions on u and the data f in order for these restrictions to hold.

Lemma 4.2. If $u \in W^{1,\infty}(\Omega) \cap W^{2,s}(\Omega)$, $s \in [1, \infty]$, then there exists an $M \in L^s(\Omega)$ such that

$$(4.9) \quad |f| \leq M|\nabla u|^{p-2} \quad \text{a.e. in } \Omega.$$

Proof. Let $\nabla u \equiv (v_1, v_2) \in [W^{1,s}(\Omega)]^2$ and $v \equiv (v_1^2 + v_2^2)^{1/2} \equiv |\nabla u| \in L^\infty(\Omega)$. As $|v_1/v| + |v_2/v|$ is bounded and $\nabla v \equiv (v_1 \nabla v_1 + v_2 \nabla v_2)/v$, it follows that $v \in W^{1,s}(\Omega)$. In addition, we have that

$$(4.10) \quad \begin{aligned} f &= -\operatorname{div}(v^{p-2}v_1, v^{p-2}v_2) \\ &= -v^{p-2}\{[(v_1)_x + (v_2)_y] + (p-2)[v_1v_x + v_2v_y]/v\}. \end{aligned}$$

Hence the desired result (4.9). \square

Under the assumption that $\{(x, y) \in \Omega: f(x, y) = 0\}$ has zero measure, the inequality (4.9), for example, yields for $t \geq 2$ and $1 \leq q < t < p$ that

$$(4.11) \quad \int_{\Omega} |\nabla u|^{-(p-t)q/(t-q)} d\Omega \leq \int_{\Omega} [M/|f|]^{(p-t)q/[(p-2)(t-q)]} d\Omega.$$

Therefore, with $M \in L^s(\Omega)$, for a given $s \in [1, \infty]$, and imposing a growth condition on $|f|^{-1}$, one can choose appropriate t and q so that (4.6a) and hence (4.7) hold. Below we give an example of such a result.

Theorem 4.1. Let $u \in W^{1,\infty}(\Omega) \cap W^{2,s}(\Omega)$, $s \in [1, \infty]$. If $|f|^{-\gamma} \in L^1(\Omega)$ for some $\gamma \in (0, \infty)$, or if $|f|^{-1} \in L^\infty(\Omega)$ we set $\gamma = \infty$, then we have for $q \in [1, p)$ that

$$(4.12a) \quad \|u - u^h\|_{W^{1,q}(\Omega^h)} \leq \begin{cases} Ch^{2/t} & \text{if } s \geq 2, \\ Ch^{s/t} & \text{if } s \in [1, 2), \end{cases}$$

where

$$(4.12b) \quad t = \max\{2, q[(s + \gamma)p + (p - 2)\gamma s]/[(s + \gamma)q + (p - 2)\gamma s]\}.$$

Proof. First a simple calculation yields that t satisfying (4.12b) is such that $t \in [2, p)$ and $t > q$. Setting

$$\eta \equiv q(p - t)/[(p - 2)(t - q)],$$

we conclude that $\eta \leq \gamma s/(s + \gamma)$ and hence $s\eta \leq \gamma(s - \eta)$, and if γ is finite then $\eta < s$. Therefore, from (4.11), the assumptions on f and Hölder's inequality we have

$$(4.13) \quad \begin{aligned} \int_{\Omega} |\nabla u|^{-(p-t)q/(t-q)} d\Omega &\leq \int_{\Omega} (M|f|^{-1})^{\eta} d\Omega \\ &\leq \left(\int_{\Omega} M^s d\Omega \right)^{\eta/s} \left(\int_{\Omega} |f|^{-\eta s/(s-\eta)} d\Omega \right)^{(s-\eta)/s} \leq C. \end{aligned}$$

Similarly, (4.13) holds if γ is infinite, as $\eta \leq s$. The desired result (4.12a) then follows from (4.6a) and (4.7). \square

We note that for fixed q , γ , and s the right-hand side of (4.12b) tends to $\max\{(2, q[(s + \gamma) + \gamma s]/\gamma s)\}$ as $p \rightarrow \infty$. Therefore, the error bound (4.12a) does not degenerate as $p \rightarrow \infty$, unlike (4.2b) and (4.5b).

Corollary 4.1. *Let $u \in W^{1,\infty}(\Omega) \cap W^{2,s}(\Omega)$, $s \in [1, \infty]$. Suppose that there exists a constant $\rho > 0$ such that $|f| \geq \rho$ a.e. in Ω ; then for $q \in [1, p)$ we have that*

$$(4.14a) \quad \|u - u^h\|_{W^{1,q}(\Omega^h)} \leq \begin{cases} Ch^{2/t} & \text{if } s \geq 2, \\ Ch^{s/t} & \text{if } s \in [1, 2), \end{cases}$$

where

$$(4.14b) \quad t = \max\{2, q[p + (p - 2)s]/[q + (p - 2)s]\}.$$

Hence, we have that for $q = 2s/(1 + s)$

$$(4.14c) \quad \|u - u^h\|_{W^{1,q}(\Omega^h)} \leq \begin{cases} Ch & \text{if } s \geq 2, \\ Ch^{s/2} & \text{if } s \in [1, 2). \end{cases}$$

Proof. The result (4.14a,b) follows directly from setting $\gamma = \infty$ in (4.12). The result (4.14c) then follows from (4.14a,b) by noting that $t = 2$ if $q = 2s/(1 + s)$. \square

5. NUMERICAL EXAMPLES

The standard Galerkin method analyzed in the previous sections requires the term $\int_{\Omega^h} f v^h d\Omega^h$ for all $v^h \in S_0^h$ to be integrated exactly. This is difficult in practice, and it is computationally more convenient to consider a scheme where numerical integration is applied to this term. With $\overline{\Omega^h} \equiv \bigcup_{\tau \in T^h} \bar{\tau}$ and $\{a_i\}_{i=1}^3$ being the vertices of a triangle τ , we define the quadrature rule

$$(5.1) \quad Q_{\tau}(v) \equiv \frac{1}{3} \text{meas}(\tau) \sum_{i=1}^3 v(a_i) \equiv \int_{\tau} \pi_h v d\tau$$

approximating $\int_{\tau} v \, d\tau$ for $v \in C(\bar{\tau})$. Then, for $v, w \in C(\overline{\Omega^h})$, we set

$$(5.2) \quad (v, w)^h \equiv \sum_{\tau \in T^h} Q_{\tau}(vw) \equiv \int_{\Omega^h} \pi_h(vw) \, d\Omega^h$$

as an approximation to $\int_{\Omega^h} vw \, d\Omega^h$.

The fully practical finite element approximation of (\mathcal{P}) that we wish to consider is:

$(\hat{\mathcal{P}}^h)$ Find $\hat{u}^h \in S_g^h$ such that

$$(5.3) \quad \int_{\Omega^h} |\nabla \hat{u}^h|^{p-2} (\nabla \hat{u}^h, \nabla v^h)_{\mathbb{R}^2} \, d\Omega^h = (f, v^h)^h \quad \forall v^h \in S_0^h.$$

The corresponding minimization problem is:

$(\hat{\mathcal{E}}^h)$ Find $\hat{u}^h \in S_g^h$ such that

$$(5.4a) \quad \hat{J}_{\Omega^h}(\hat{u}^h) \leq \hat{J}_{\Omega^h}(v^h) \quad \forall v^h \in S_g^h,$$

where

$$(5.4b) \quad \hat{J}_{\Omega^h}(v^h) \equiv \frac{1}{p} \int_{\Omega^h} |\nabla v^h|^p \, d\Omega^h - (f, v^h)^h.$$

The well-posedness of $(\hat{\mathcal{P}}^h) \equiv (\hat{\mathcal{E}}^h)$ follows in an analogous way to that of (\mathcal{P}) and (\mathcal{E}) , and

$$(5.5) \quad \|\hat{u}^h\|_{W^{1,p}(\Omega^h)} \leq C[\|f\|_{L^\infty(\Omega^h)}^{1/(p-1)} + \|g^h\|_{W^{1,p}(\Omega^h)}].$$

We now bound the error $\hat{u}^h - u$. First we have the analogue of Theorem 2.1.

Theorem 5.1. *Let u and \hat{u}^h be the unique solutions of $(\mathcal{P}) \equiv (\mathcal{E})$ and $(\hat{\mathcal{P}}^h) \equiv (\hat{\mathcal{E}}^h)$, respectively. Let $f \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$. Then for any $\delta_1 \in [0, 2)$ and $\delta_2 \geq 0$, and any $v^h \in S_g^h$, it follows that*

$$(5.6) \quad |u - \hat{u}^h|_{(p, 2+\delta_2)} \leq C|u - v^h|_{(p, 2-\delta_1)} + Ch^2 \|v^h - \hat{u}^h\|_{W^{1,1}(\Omega^h)}.$$

Proof. The proof follows exactly that of Theorem 2.1 with \hat{u}^h and \hat{J}_{Ω^h} instead of u^h and J_{Ω^h} in (2.16). However, whereas $J'_{\Omega^h}(u)(v^h - u^h) = 0$, we now have for all $v^h \in S_g^h$

$$(5.7a) \quad \begin{aligned} \hat{J}'_{\Omega^h}(u)(v^h - \hat{u}^h) &\equiv J'_{\Omega^h}(u)(v^h - \hat{u}^h) \\ &\quad - \int_{\Omega^h} f(v^h - \hat{u}^h) \, d\Omega^h + (f, v^h - \hat{u}^h)^h \\ &\equiv - \int_{\Omega^h} f(v^h - \hat{u}^h) \, d\Omega^h + (f, v^h - \hat{u}^h)^h \end{aligned}$$

and

$$(5.7b) \quad \begin{aligned} \left| \int_{\Omega^h} f(v^h - \hat{u}^h) \, d\Omega^h - (f, v^h - \hat{u}^h)^h \right| &\leq C\|(I - \pi_h)[f(v^h - \hat{u}^h)]\|_{L^1(\Omega^h)} \\ &\leq Ch^2 |f(v^h - \hat{u}^h)|_{W^{2,1}(\Omega^h)} \leq Ch^2 \|v^h - \hat{u}^h\|_{W^{1,1}(\Omega^h)}, \end{aligned}$$

provided $f \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$. Hence, we obtain the desired result (5.6). \square

In particular, assuming $u \in W^{2,1}(\Omega)$ if $p < 2$, we have for $v^h \equiv g^h \equiv \pi_h u$ that for any $\delta_1 \in [0, 2)$ and $\delta_2 \geq 0$

$$(5.8) \quad |u - \hat{u}^h|_{(p, 2+\delta_2)} \leq C|u - \pi_h u|_{(p, 2-\delta_1)} + Ch^2.$$

Hence, it is a simple matter to check that the results of the previous sections hold for \hat{u}^h as well as u^h if $f \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$. We note that this constraint on f can be weakened and is imposed here for ease of exposition only.

We now report on some numerical results with the fully practical approximation (5.3). For computational ease we took Ω to be the square $[0, 1] \times [0, 1]$. This was partitioned into uniform right-angled triangles by dividing it first into equal squares of sides of length $1/N$ and then into triangles by inserting the SW-NE diagonals. We imposed homogeneous Neumann data on the sides $x = 0$ and $y = 0$ and Dirichlet data on the sides $x = 1$ and $y = 1$. Therefore, the problem can be viewed as a Dirichlet problem over $[-1, 1] \times [-1, 1]$, and so our error analysis applies directly.

We computed our approximation (5.3) by solving the equivalent minimization problem (5.4). We used a Polak-Ribière conjugate gradient method, which worked reasonably well for the values of p reported here. We did not experiment with the augmented Lagrangian approach advocated by Glowinski and Marrocco [5], but this conjugate gradient approach was far superior to the gradient method suggested by Wei [10].

For our test problems we consider solutions of the radially symmetric problem, Example 3.1, extended to the unit square. In the first three examples we took for various values of p and γ

$$(5.9) \quad \begin{aligned} f &\equiv F(r) \equiv r^\sigma \quad \text{and} \\ u &\equiv U(r) \equiv (p-1)[1/(\sigma+2)]^{1/(p-1)}[1 - r^{(\sigma+p)/(p-1)}]/(\sigma+p). \end{aligned}$$

In all the examples, f is sufficiently smooth, so that the error bounds for u^h in the previous sections hold for \hat{u}^h as well.

Example 5.1. This is (5.9) with $\sigma = 0$ and $p = 1.5$. It follows from (3.12) that $u \in W^{3,1}(\Omega) \cap C^{2,1}(\bar{\Omega})$, and so from Theorem 3.1 we expect $O(h)$ convergence in $W^{1,1.5}(\Omega)$. This is certainly achieved by inspecting Table 5.1, where we adopt the notation $0.8233(-3) \equiv 0.8233 \times 10^{-3}$. In fact, \hat{u}^h is converging to u at the rate $O(h^2)$ in $L^\infty(\Omega)$, and there is a superconvergence for $\pi_h u - \hat{u}^h$ in $W^{1,1}(\Omega)$ and $W^{1,p}(\Omega)$. \square

TABLE 5.1

| N | $\ \pi_h u - \hat{u}^h\ _{W^{1,1}(\Omega)}$ | $\ \pi_h u - \hat{u}^h\ _{W^{1,p}(\Omega)}$ | $\ \pi_h u - \hat{u}^h\ _{L^\infty(\Omega)}$ |
|-----|---|---|--|
| 10 | 0.8233(-3) | 0.4823(-3) | 0.8150(-3) |
| 20 | 0.2061(-3) | 0.1207(-3) | 0.2034(-3) |
| 40 | 0.5196(-4) | 0.3043(-4) | 0.5109(-4) |
| 80 | 0.1235(-4) | 0.723(-5) | 0.1263(-4) |

Example 5.2. This is (5.9) with $\sigma = 0$ and $p = 4$. It follows from Example 3.1 and §4 that $u \in W^{2,s}(\Omega)$, with $s < 3$, and from (4.14c) we expect $O(h)$ convergence in $W^{1,1}(\Omega)$. From Table 5.2 we see this is achieved. In fact, $\pi_h u - \hat{u}^h$ exhibits superconvergence in $W^{1,1}(\Omega)$. \square

TABLE 5.2

| N | $\ \pi_h u - \hat{u}^h\ _{W^{1,1}(\Omega)}$ | $\ \pi_h u - \hat{u}^h\ _{W^{1,p}(\Omega)}$ | $\ \pi_h u - \hat{u}^h\ _{L^\infty(\Omega)}$ |
|-----|---|---|--|
| 10 | 0.1789(−2) | 0.4486(−2) | 0.3790(−2) |
| 20 | 0.5049(−3) | 0.2519(−2) | 0.1585(−2) |
| 40 | 0.1376(−3) | 0.1414(−2) | 0.6493(−3) |
| 80 | 0.3659(−4) | 0.7936(−3) | 0.2625(−3) |

Example 5.3. Here we take (5.9) with $\sigma = 7$ and $p = 4$. It follows from Example 3.1 that $u \in W^{2,\infty}(\Omega)$. From (4.12), as $s = \infty$ and $\gamma < 2/7$, we have with $q = 1$ that $t > 32/11$. Therefore, for all $\varepsilon > 0$ we have that

$$(5.10) \quad \|u - \hat{u}^h\|_{W^{1,1}(\Omega)} \leq Ch^{(11-\varepsilon)/16}.$$

We note that a sharper bound, $h^{(14-\varepsilon)/19}$, can be obtained by noting that for this model problem $|U'(r)| \geq Cr^{8/3}$ and applying (4.6) and (4.7) directly. From Table 5.3 we see that the above bounds are pessimistic. In fact, we have $O(h^2)$ convergence in $L^\infty(\Omega)$ and $\pi_h u - \hat{u}^h$ exhibits superconvergence in $W^{1,1}(\Omega)$. \square

TABLE 5.3

| N | $\ \pi_h u - \hat{u}^h\ _{W^{1,1}(\Omega)}$ | $\ \pi_h u - \hat{u}^h\ _{W^{1,p}(\Omega)}$ | $\ \pi_h u - \hat{u}^h\ _{L^\infty(\Omega)}$ |
|-----|---|---|--|
| 10 | 0.8153(−3) | 0.5988(−2) | 0.5014(−2) |
| 20 | 0.2164(−2) | 0.1931(−2) | 0.1235(−2) |
| 40 | 0.5918(−3) | 0.8893(−3) | 0.2989(−3) |
| 80 | 0.1429(−3) | 0.1952(−3) | 0.6449(−4) |

Finally we consider an example for $p > 2$, where $\{(x, y) \in \Omega: f(x, y) = 0\}$ does not have zero measure.

Example 5.4. We take

$$(5.11a) \quad F(r) \equiv \begin{cases} 0 & \text{for } r \leq a, \\ 4^{p-1}(r-a)^{(3p-4)}[2 + (a/r) - 3p] & \text{for } r \geq a \end{cases}$$

and

$$(5.11b) \quad U(r) \equiv \begin{cases} 0 & \text{for } r \leq a, \\ (r-a)^4 & \text{for } r \geq a \end{cases}$$

with $a = 0.3$ and $p = 4$. At present the only global error estimate we have for this case is the result (4.5b). Clearly, this is pessimistic from inspecting Table 5.4, where once again we see $O(h^2)$ convergence in $L^\infty(\Omega)$ and $\pi_h u - \hat{u}^h$ is superconvergent in $W^{1,1}(\Omega)$. We note that the maximum error did not occur in the disc $\{r: r \leq 0.3\}$. \square

TABLE 5.4

| N | $\ \pi_h u - \hat{u}^h\ _{W^{1,1}(\Omega)}$ | $\ \pi_h u - \hat{u}^h\ _{W^{1,p}(\Omega)}$ | $\ \pi_h u - \hat{u}^h\ _{L^\infty(\Omega)}$ |
|-----|---|---|--|
| 10 | 0.5879(-1) | 0.4653(-1) | 0.3080(-1) |
| 20 | 0.1553(-1) | 0.1182(-1) | 0.7930(-2) |
| 40 | 0.4332(-2) | 0.3118(-2) | 0.1992(-2) |
| 80 | 0.1139(-2) | 0.1706(-2) | 0.4923(-3) |

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