Discontinuous Gradient Constraints and the Infinity Laplacian

Article *in* International Mathematics Research Notices · January 2012

DOI: 10.1093/imrn/rnv214 · Source: arXiv

Petri Juutinen
University of Jyväskylä

Mikko Parviainen
University of Jyväskylä

Julio D. Rossi
Universidad de Buenos Aires

DISCONTINUOUS GRADIENT CONSTRAINTS AND THE INFINITY LAPLACIAN

PETRI JUUTINEN, MIKKO PARVIAINEN AND JULIO D. ROSSI

Abstract. Motivated by tug-of-war games and asymptotic analysis of certain variational problems, we consider the following gradient constraint problem: given a bounded domain $\Omega \subset \mathbb{R}^n$, a continuous function $f\colon \partial\Omega \to \mathbb{R}$ and a non-empty subset $D \subset \Omega$, find a solution to

$$\begin{cases} \min\{\Delta_\infty u, |Du| - \chi_D\} = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega, \end{cases}$$

where Δ_∞ is the infinity Laplace operator. We prove that this problem always has a solution that is unique if $\overline{D} = \overline{\operatorname{int} D}$. If this regularity condition on D fails, then solutions obtained from game theory and L^p-approximation need not coincide.

1. Introduction

The infinity Laplacian, introduced by Aronsson [1] in 1960's, is a second order quasilinear partial differential operator formally defined as

$$\Delta_\infty u(x) = D^2u(x)Du(x) \cdot Du(x) = \sum_{i,j=1}^{n} u_{ij}(x)u_i(x)u_j(x).$$

It is the "Laplacian of L^∞-variational problems": the equation $\Delta_\infty u(x) = 0$ is the Euler-Lagrange equation for the variational problem of finding absolute minimizers for the prototypical L^∞-functional

$$I(u) = ||Du||_{L^\infty(\Omega)}$$

with given boundary values, see e.g. [19]. The infinity Laplacian also arises from certain random turn games [27], [3] and mass transportation problems [14], and it appears in several applications, such as image reconstruction and enhancement [7], and the study of shape metamorphism [8].

In this paper, we are interested in the following gradient constraint problem involving the infinity Laplacian: given a bounded domain $\Omega \subset \mathbb{R}^n$, a continuous function $f\colon \partial\Omega \to \mathbb{R}$ and a non-empty subset $D \subset \Omega$, find a

Date: January 26, 2012.
2000 *Mathematics Subject Classification.* 35J92, 35D40, 35Q91, 91A15.
Key words and phrases. infinity Laplacian, tug of war games, gradient constraint.

viscosity solution to

$$
\begin{cases} \min\{\Delta_\infty u(x), |Du(x)| - \chi_D(x)\} = 0 & \text{in } \Omega \\ u(x) = f(x) & \text{on } \partial\Omega. \end{cases} \tag{1.1}
$$

Here $\chi_D \colon \Omega \to \mathbb{R}$ denotes the characteristic function of the set D, that is,

$$
\chi_D(x) = \begin{cases} 1, & \text{if } x \in D, \\ 0, & \text{if } x \in \Omega \setminus D. \end{cases}
$$

The study of gradient constraint problems of the type

$$
\min\{\Delta_\infty u(x), |Du(x)| - g(x)\} = 0, \tag{1.2}
$$

where $g \geq 0$, was initiated by Jensen in his celebrated paper [19]. He used the solutions of the equation $\min\{\Delta_\infty u, |Du| - \varepsilon\} = 0$ and its pair $\max\{\Delta_\infty u, \varepsilon - |Du|\} = 0$ to approximate the solutions of the infinity Laplace equation $\Delta_\infty u = 0$. In this way, he proved uniqueness for the infinity Laplace equation by first showing that it holds for the approximating equations. The same approach was used in the anisotropic case by Lindqvist and Lukkari in [22], and a variant of (1.2) appears in the so called ∞-eigenvalue problem, see e.g [21].

In general, the uniqueness of solutions for (1.2) is fairly easy to show if g is continuous and everywhere positive, and is known to hold, owing to Jensen's work, if $g \equiv 0$. However, the case $g \geq 0$ seems to have been entirely open before this paper. The situation resembles the one with the infinity Poisson equation $\Delta_\infty u = g$: the uniqueness is known to hold if $g > 0$ or $g \equiv 0$, and the case $g \geq 0$ is an outstanding open problem, see [27]. It is one of our main results in this paper that the uniqueness for (1.2) holds in the special case $g = \chi_D$ under the fairly mild regularity condition $\overline{D} = \overline{\operatorname{int} D}$ on the set D, see Theorem 4.1 below. Moreover, the uniqueness in general fails if this condition is not fulfilled.

Our interest in (1.1) arises only partially from the desire to generalize Jensen's results. Another reason for considering this problem is its connection to the boundary value problems

$$
\begin{cases} \Delta_p u = g & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1.3}
$$

where $\Delta_p u = \operatorname{div}(|Du|^{p-2} Du)$ is the p-Laplace operator, $1 < p < \infty$, and $g \geq 0$. It is not difficult to show that, up to selecting a subsequence, solutions u_p to (1.3) converge as $p \to \infty$ to a limit function that must satisfy (1.1) with $D = \{x \in \Omega \colon g(x) > 0\}$. However, different subsequences may, a priori, yield different limit functions. This possibility has been previously excluded in the cases $g \equiv 0$ and $g > 0$, the latter under the additional assumption that $f = 0$, see e.g. [5], [17]. Our results imply that the limit function is unique for *any* continuous functions $g \geq 0$ and f, see Theorem 7.1. In particular, the limit function depends on g only via the set $D = \{x \in \Omega \colon g(x) > 0\}$.

Needless to say, for $g \leq 0$ our techniques can be applied as well, and then we encounter the equation

$$\max\{\Delta_\infty u, \chi_D - |Du|\} = 0.$$

Since the results are identical, we omit this case.

Further motivation for considering (1.1) comes from its connection to game theory. Recently, Peres, Schramm, Sheffield and Wilson [27] introduced a two player random turn game called "tug-of-war", and showed that, as the step size converges to zero, the value functions of this game converge to the unique viscosity solution of the infinity Laplace equation $\Delta_\infty u = 0$. We define and study a variant of the tug-of-war game in which one of the players has the option to sell his/hers turn to the other player with a fixed price (that depends on the step size) when the game token is in the set D. It is then shown that the value functions of this new game converge to a solution of (1.1). Thus, besides its own interest, the game provides an alternative way to prove the existence of a solution to (1.1).

The boundary value problem (1.1) may have multiple solutions if the set D is irregular, that is, $\overline{D} \neq \overline{\operatorname{int} D}$. However, the limit of the value functions of our game is always the smallest solution and hence unique. We give several examples in which the game solution and the solution constructed by taking the limit as p goes to infinity in the p-Laplace problems $\Delta_p u = \chi_D$ are not the same. The possibility of having different solutions is also motivated by stability considerations. Somewhat similar results but on a different problem were recently obtained by Yu in [28].

Our main uniqueness result, that (1.1) has exactly one solution if $\overline{D} = \overline{\operatorname{int} D}$, is proved in a slightly unusual manner. Indeed, instead of proving directly a comparison principle for sub- and supersolutions of (1.1), we identify the solution in a way that guarantees its uniqueness, see Theorem 4.2 below for details. The intuition for this identification comes partially from the game theoretic interpretation of our problem. On the other hand, the uniqueness proof for (1.1) gave us a hint on how to prove similar result for the value functions of the game, and so these two sides complement each other nicely.

Due to the fact that the solutions need not be smooth and that the infinity Laplacian is not in divergence form, we use viscosity solutions when dealing with (1.1). However, since $\chi_D(x)$ can be viewed either as a function, defined at every point of Ω, or as an element of $L^\infty(\Omega)$, defined only almost everywhere, one can use either the standard notion or the L^∞-viscosity solutions. The first one fits well with the game theoretic approach, whereas L^∞-viscosity solutions are quite natural from the point of view of p-Laplace approximation. We have chosen to use mostly the standard notion of (continuous) viscosity solutions, mainly because this makes it easier to compare our results with what has been proved earlier. For completeness, we have included a short section explaining L^∞-viscosity solutions of (1.1).

Contents

2. Preliminaries

2.1. **Viscosity solutions for the gradient constraint problem.** To be on the safe side, we begin by recalling what is meant by viscosity solutions of the boundary value problem

$$\begin{cases} \min\{\Delta_\infty u, |Du| - \chi_D\} = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where

$$\Delta_\infty u(x) = \sum_{i,j=1}^{n} u_{ij}(x) u_i(x) u_j(x)$$

is the infinity Laplace operator and for $D \subset \Omega$,

$$\chi_D(x) = \begin{cases} 1, & x \in D, \\ 0, & x \in \Omega \setminus D. \end{cases}$$

First, the boundary condition "$u = f$ on $\partial\Omega$" is understood in the classical sense, that is, $\lim_{x\to z} u(x) = f(z)$ for all $z \in \partial\Omega$.

Second, to define viscosity solutions for the equation

$$\min\{\Delta_\infty u, |Du| - \chi_D\} = 0, \tag{2.5}$$

one needs to use the semicontinuous envelopes of χ_D. To this end, we denote by $\operatorname{int} D$ and $\overline{D}$, respectively, the (topological) interior and closure of the set D.

Definition 2.1. *An upper semicontinuous function $u\colon \Omega \to \mathbb{R}$ is a* viscosity subsolution *to* (2.5) *in Ω if, whenever $x \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $u - \varphi$ has a strict local maximum at x, then*

$$\min\{\Delta_\infty \varphi(x), |D\varphi(x)| - \chi_{\operatorname{int} D}(x)\} \geq 0. \tag{2.6}$$

A lower semicontinuous function $v\colon \Omega \to \mathbb{R}$ is a viscosity supersolution *to* (2.5) *in Ω if, whenever $x \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $v - \phi$ has a strict local minimum at x, then*

$$\min\{\Delta_\infty \phi(x), |D\phi(x)| - \chi_{\overline{D}}(x)\} \leq 0. \tag{2.7}$$

Finally, a continuous function $h\colon \Omega \to \mathbb{R}$ is a viscosity solution *to* (2.5) *in Ω if it is both a viscosity subsolution and a viscosity supersolution.*

Sometimes it is convenient to replace the condition "$u-\varphi$ has a strict local maximum at x" by the requirement that "φ touches u at x from above", and to write (2.6) in the form

$$\Delta_\infty \varphi(x) \geq 0 \quad \text{and} \quad |D\varphi(x)| - \chi_{\operatorname{int} D}(x) \geq 0.$$

Similarly, we sometimes replace "$v - \phi$ has a strict local minimum at x" by "ϕ touches v at x from below", and write (2.7) in the form

$$\Delta_\infty \phi(x) \leq 0 \quad \text{or} \quad |D\phi(x)| - \chi_{\overline{D}}(x) \leq 0.$$

The reader should notice that if $\operatorname{int} D$ is empty, then a solution u to the infinity Laplace equation $\Delta_\infty u = 0$ also satisfies $\min\{\Delta_\infty u, |Du| - \chi_D\} = 0$.

2.2. **Patching and Jensen's equation.** The main difficulty in proving the uniqueness of solutions for the infinity Laplace equation $\Delta_\infty u(x) = 0$ is the very severe degeneracy of the equation at the points where the gradient Du vanishes. To overcome this, several approximation methods have been introduced. The first one, due to Jensen [19], was to use the equation

$$\min\{\Delta_\infty u(x), |Du(x)| - \varepsilon\} = 0, \tag{2.8}$$

whose solutions are subsolutions of $\Delta_\infty u = 0$ and have (at least formally) a non-vanishing gradient. Another device, called "patching", appears in the papers by Barron and Jensen [4] and by Crandall, Gunnarsson and Wang [10], and it is based on the use of the eikonal equation. We show below that these two methods actually coincide. This fact was mentioned in [10], but no proof for it was given. The result will be crucial in the proof of our main uniqueness result, Theorem 4.1 below.

To proceed, we need some notation. We denote by

$$\operatorname{Lip}(u, B_r(x)) = \inf\{L \in \mathbb{R} : |u(z) - u(y)| \leq L\,|z - y| \ \text{ for } z, y \in B_r(x)\}$$

the least Lipschitz constant for u on the ball $B_r(x)$. Let $h\colon \Omega \to \mathbb{R}$ be the unique viscosity solution to the infinity Laplace equation $\Delta_\infty h = 0$ in Ω with boundary values $h = f$ on $\partial\Omega$. Then h is everywhere differentiable, see [12], and $|Dh(x)|$ equals to the pointwise Lipschitz constant of h,

$$L(h,x) := \lim_{r\to+0} \operatorname{Lip}(h, B_r(x))$$

for every $x \in \Omega$. Since the map $x \mapsto L(h,x)$ is upper semicontinuous, see e.g. [9], this implies that the set

$$V_\varepsilon := \{x \in \Omega\colon\ |Dh(x)| < \varepsilon\}$$

is an open subset of Ω. Now, define the "patched solution" $h_\varepsilon\colon \overline{\Omega} \to \mathbb{R}$ by first setting

$$h_\varepsilon = h \quad \text{in } \overline{\Omega} \setminus V_\varepsilon,$$

and then, for each connected component U of V_ε and $x \in U$, defining

$$h_\varepsilon(x) = \sup_{y\in\partial U} \left(h(y) - \varepsilon d_U(x,y)\right),$$

where $d_U(x,y)$ stands for the (interior) distance between x and y in U. The results collected in the following "patching lemma" are taken from [10].

Lemma 2.2. *It holds that*

(1) $\Delta_\infty h_\varepsilon \geq 0$ *in* Ω *in the viscosity sense,*
(2) $h_\varepsilon = h$ *on* $\overline{\Omega} \setminus V_\varepsilon$ *and* $h_\varepsilon \leq h$ *on* $\overline{\Omega}$,
(3) $L(h_\varepsilon, x) \geq \varepsilon$ *for* $x \in \Omega$,
(4) h_ε *is a viscosity solution to* $|Dh_\varepsilon| - \varepsilon = 0$ *in* V_ε.

Now, let $z_\varepsilon\colon \Omega \to \mathbb{R}$ be the unique viscosity solution to Jensen's equation (2.8) in Ω with $z_\varepsilon = f$ on $\partial\Omega$. Then it holds that the patched solution coincides with the solution to (2.8).

Theorem 2.3. *Let* $z_\varepsilon \in C(\overline{\Omega})$ *be the solution to* (2.8) *and* h_ε *be defined as above. Then* $z_\varepsilon = h_\varepsilon$ *in* Ω.

Proof. Without loss of generality, we may assume that $\varepsilon = 1$. Let us first show that h_1 is a supersolution to (2.8). Let $\phi \in C^2(\Omega)$ be such that $h_1 - \phi$ has a local minimum at $x \in \Omega$. If $x \in \Omega \setminus V_1$, then as $h(x) = h_1(x)$ and $h \geq h_1$ everywhere, we see that also $h - \phi$ has a local minimum at $x \in \Omega$. Since h satisfies $\Delta_\infty h = 0$, this implies

$$\min\{\Delta_\infty \phi(x), |D\phi(x)| - 1\} \leq \Delta_\infty \phi(x) \leq 0,$$

as desired. On the other hand, if $x \in V_1$, then $|D\phi(x)| - 1 \leq 0$ by Lemma 2.2, and again we have

$$\min\{\Delta_\infty \phi(x), |D\phi(x)| - 1\} \leq 0.$$

Thus h_1 is a supersolution to (2.8).

To prove that h_1 is a subsolution to (2.8), let $\varphi \in C^2(\Omega)$ be such that $h_1 - \varphi$ has a local maximum at $x \in \Omega$. We need to prove that

$$\Delta_\infty \varphi(x) \geq 0 \text{ and } |D\varphi(x)| - 1 \geq 0.$$

By Lemma 2.2, the first inequality holds no matter where x lies, and the second is true if $x \in V_1$. Thus we only have to show that

$$|D\varphi(x)| - 1 \geq 0$$

if $x \in \Omega \setminus V_1$. But this is true because

$$\max_{\overline{B}_r(x)} \varphi - \varphi(x) \geq \max_{\overline{B}_r(x)} h_1 - h_1(x) \geq rL(h_1, x) \geq r;$$

see e.g. [9] for the second inequality. The proof of $z_1 = h_1$ is completed upon recalling the uniqueness result for (2.8) in [19]. □

3. Existence of solutions. A variational approach

In this section, we prove the existence of a viscosity solution to the gradient constraint problem (2.4) by showing that solutions u_p to

$$\begin{cases} \Delta_p u = \chi_D & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

converge uniformly, as $p \to \infty$, to a function that satisfies (2.4). In fact, we prove a slightly more general result and show that the convergence holds true even if we replace χ_D in (3.9) by a non-negative function $g \in L^\infty(\Omega)$ for which $D = \{x \in \Omega\colon g(x) > 0\}$ and a certain non-degeneracy assumption is valid, see (3.10) below.

We begin by recalling the definitions of a weak and viscosity solution for the equation $\Delta_p u = g$, where $g \geq 0$ is bounded but not necessarily continuous. Since we are mainly interested in what happens when $p \to \infty$, we assume throughout this section that $p > \max\{2, n\}$.

Definition 3.1. *A function $u \in W^{1,p}(\Omega) \cap C(\Omega)$ is a* weak solution *of $\Delta_p u = g$ in Ω if it satisfies*

$$-\int_\Omega |Du|^{p-2} Du \cdot D\varphi \, dx = \int_\Omega g\varphi \, dx,$$

for every $\varphi \in C_0^\infty(\Omega)$.

By a weak solution to the boundary value problem (3.9) we mean a function $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ that is a weak solution to $\Delta_p u = g$ in Ω and satisfies $u = f$ on $\partial\Omega$. We also suppose that f is Lipschitz continuous to begin with, and fix a Lipschitz function $F\colon \overline{\Omega} \to \mathbb{R}$ such that $F = f$ on $\partial\Omega$, $\mathrm{Lip}(F, \Omega) = \mathrm{Lip}(f, \partial\Omega)$, and $||F||_{L^\infty(\Omega)} = ||f||_{L^\infty(\partial\Omega)}$. Since $p > n$ and f is Lipschitz, the conditions $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ and $u = f$ on $\partial\Omega$ are equivalent to the statement that $u - F \in W_0^{1,p}(\Omega)$, see [26].

Lemma 3.2. *There exists a unique weak solution $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ to $\Delta_p u = g$ with fixed Lipschitz continuous boundary values f, and it is characterized as being a minimizer for the functional*

$$F_p(u) = \int_\Omega \frac{|Du|^p}{p}\,dx + \int_\Omega gu\,dx$$

in the set of functions $\{u \in W^{1,p}(\Omega) \ : \ u = f \text{ on } \partial\Omega\}$.

Proof. The functional F_p is coercive and weakly semicontinuous, hence the minimum is attained. Moreover, this minimum is a weak solution to $\Delta_p u = g$ in the sense of Definition 3.1. Uniqueness follows from the strict convexity of the functional. For details, we refer to Giusti's monograph [16]. $\square$

Due to the possible discontinuity of the right hand side $g(x)$, we use semicontinuous envelopes when defining viscosity solutions. Denote

$$g_*(x) = \liminf_{y \to x} g(y),$$

and

$$g^*(x) = \limsup_{y \to x} g(y).$$

We recall that g_* is lower semicontinuous, g^* upper semicontinuous, and $g_* \le g \le g^*$. We assume that g is non-degenerate in the sense that

$$g_*(x) > 0 \qquad \text{for all } x \in \operatorname{int} D, \tag{3.10}$$

with $D = \{x \in \Omega\colon g(x) > 0\}$. This condition is used in Theorem 3.6.

Notice that if $g(x) = \chi_D(x)$, then

$$g_*(x) = \chi_{\operatorname{int} D}(x) \quad \text{ and } \quad g^*(x) = \chi_{\overline{D}}(x).$$

Thus (3.10) holds in this case. Observe also that since we assume that $p \ge 2$, the equation $\Delta_p u = g$ is not singular at the points where the gradient vanishes, and thus $x \mapsto \Delta_p \phi(x) = (p-2)|D\phi|^{p-4}\Delta_\infty \phi(x) + |D\phi|^{p-2}\Delta\phi(x)$ is well defined and continuous for any $\phi \in C^2(\Omega)$.

Definition 3.3. *An upper semicontinuous function $u\colon \Omega \to \mathbb{R}$ is a* viscosity subsolution *to $\Delta_p u = g$ in Ω if, whenever $x \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $u - \varphi$ has a strict local maximum at x, then*

$$\Delta_p \varphi(x) \ge g_*(x).$$

A lower semicontinuous function $v\colon \Omega \to \mathbb{R}$ is a viscosity supersolution *to $\Delta_p u = g$ in Ω if, whenever $x \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $v - \phi$ has a strict local minimum at x, then*

$$\Delta_p \phi(x) \le g^*(x).$$

Finally, a continuous function $h\colon \Omega \to \mathbb{R}$ is a viscosity solution *to $\Delta_p u = g$ in Ω if it is both a viscosity subsolution and a viscosity supersolution.*

Proposition 3.4. *A continuous weak solution of $\Delta_p u = g$ is a viscosity solution.*

Proof. Let $x \in \Omega$ and choose a test function ϕ touching u at x from below, that is, $u(x) = \phi(x)$ and $u - \phi$ has a strict minimum at x. We want to show that

$$(p-2)|D\phi|^{p-4}\Delta_\infty \phi(x) + |D\phi|^{p-2}\Delta\phi(x) \le g^*(x).$$

If this is not the case, then there exists a radius $r > 0$ such that

$$(p-2)|D\phi|^{p-4}\Delta_\infty \phi(y) + |D\phi|^{p-2}\Delta\phi(y) > g^*(y),$$

for every $y \in B_r(x)$. Set $m = \inf_{|y-x|=r}(u-\phi)(y)$ and let $\psi(y) = \phi(y)+m/2$. This function ψ verifies $\psi(x) > u(x)$ and

$$\operatorname{div}(|D\psi|^{p-2}D\psi) > g^*(y) \ge g(y),$$

which, upon integration by parts, implies that ψ is a weak subsolution to $\Delta_p u = g$ in $B_r(x)$. Thus we have that $\psi \le u$ on $\partial B_r(x)$, u is a weak solution and ψ a weak subsolution to $\Delta_p v = g$, which by the comparison principle (for weak solutions) implies that $u \ge \psi$ in $B_r(x)$. But this contradicts the inequality $\psi(x) > u(x)$.

This proves that u is a viscosity supersolution. The proof of the fact that u is a viscosity subsolution runs along similar lines. □

Next we prove that there is a subsequence of weak solutions to

$$\begin{cases} \Delta_p u = g & \text{in } \Omega \\ u_p = f & \text{on } \partial\Omega \end{cases} \tag{3.11}$$

that converges uniformly as $p \to \infty$.

Lemma 3.5. *There exists a function $u_\infty \in W^{1,\infty}(\Omega)$ and a subsequence of solutions to the above problem,* (3.11)*, such that*

$$\lim_{p\to\infty} u_p(x) = u_\infty(x)$$

uniformly in $\overline{\Omega}$ as $p \to \infty$.

Proof. Recall that we assumed that f is Lipschitz continuous. Let h_p be the unique p-harmonic function with boundary values f, that is, $h_p \in W^{1,p}(\Omega)$ satisfies

$$\begin{cases} \Delta_p h_p = 0 & \text{in } \Omega \\ h_p = f & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

Now we can use $u_p - h_p \in W_0^{1,p}(\Omega)$ (note that u_p and h_p agree on $\partial\Omega$) as a test-function in the weak formulations of $\Delta_p u = g$ and (3.12), and by subtracting the resulting equations obtain

$$\int_\Omega (|Du_p|^{p-2}\,Du_p - |Dh_p|^{p-2}\,Dh_p)\cdot(Du_p - Dh_p)\,dx = \int_\Omega g(u_p - h_p)\,dx.$$

Using the well-known vector inequality

$$2^{2-p}\,|a-b|^p \le (|a|^{p-2}\,a - |b|^{p-2}\,b)\cdot(a-b),$$

and Hölder's and Sobolev's inequalities (see [15, p.164] for the constants), this yields

$$\begin{aligned}&\fint_\Omega |Du_p - Dh_p|^p \, dx \\ &\qquad \le 2^{p-2} \, ||g||_{L^\infty(\Omega)} \left(\frac{|\Omega|}{\omega_n}\right)^{1/n} \left(\fint_\Omega |Du_p - Dh_p|^p \, dx\right)^{1/p},\end{aligned}$$

where ω_n is the measure of unit ball in $\mathbb{R}^n$, $|\Omega|$ the measure of Ω, and $\fint$ denotes the averaged integral $\fint_\Omega = \frac{1}{|\Omega|}\int_\Omega$. Thus

$$\left(\fint_\Omega |Du_p - Dh_p|^p \, dx\right)^{1/p} \le 2^{\frac{p-2}{p-1}} \, ||g||_{L^\infty(\Omega)}^{1/(p-1)} \left(\frac{|\Omega|}{\omega_n}\right)^{1/n(p-1)},$$

and if $n < m < p$, Hölder's inequality gives

$$\left(\fint_\Omega |Du_p - Dh_p|^m \, dx\right)^{1/m} \le 2 \max\left\{1, ||g||_{L^\infty(\Omega)} \left(\frac{|\Omega|}{\omega_n}\right)^{1/n}\right\}. \tag{3.13}$$

We infer from Morrey's inequality that the sequence $\{u_p - h_p\}_{p\ge m}$ is bounded in $C^{0,1-n/m}(\overline{\Omega})$, the space of $(1-n/m)$-Hölder continuous functions. Thus, in view of Arzelà-Ascoli's theorem, there is $v \in C(\overline{\Omega})$ such that (up to selecting a subsequence) $u_p - h_p \to v$ as $p \to \infty$. Since the constant on the right-hand side of (3.13) is independent of m, a diagonal argument shows that $v \in W^{1,\infty}(\Omega)$. Moreover, as $h_p \to h$ uniformly in $\overline{\Omega}$, where $h \in W^{1,\infty}(\Omega)$ is the unique solution to $\Delta_\infty h = 0$ that agrees with f on $\partial\Omega$ (see [19]), we have that $u_p \to u_\infty := v + h \in W^{1,\infty}(\Omega)$ uniformly in $\overline{\Omega}$. □

Theorem 3.6. *A uniform limit u_∞ of a subsequence u_p as $p \to \infty$ is a viscosity solution to* (2.4).

Proof. From the uniform convergence it is clear that u_∞ is continuous and satisfies $u_\infty = f$ on $\partial\Omega$.

Next, to show that u_∞ is a supersolution, assume that $u_\infty - \phi$ has a strict minimum at $x \in \Omega$. We have to check that

$$\Delta_\infty \phi(x) \le 0 \quad \text{or} \quad |D\phi(x)| - \chi_{\overline{D}}(x) \le 0. \tag{3.14}$$

By the uniform convergence of u_p to u_∞ there are points x_p such that $u_p - \phi$ has a minimum at x_p and $x_p \to x$ as $p \to \infty$. At those points we have

$$(p-2)|D\phi|^{p-4}\Delta_\infty\phi(x_p) + |D\phi|^{p-2}\Delta\phi(x_p) \le g^*(x_p).$$

Let us suppose that $|D\phi(x)| > \chi_{\overline{D}}(x)$, since otherwise (3.14) clearly holds. Then $|D\phi(x)| > 0$, and hence $|D\phi(x_p)| > 0$ for p large enough by continuity. Thus we may divide by $(p-2)|D\phi(x_p)|^{p-4}$ in the inequality above and obtain

$$\Delta_\infty\phi(x_p) \le \frac{1}{p-2}|D\phi|^2\Delta\phi(x_p) + \frac{g^*(x_p)}{(p-2)|D\phi(x_p)|^{p-4}}. \tag{3.15}$$

Notice that if $x \notin \overline{D} = \operatorname{supp} g$, then (3.15) implies $\Delta_\infty \phi(x) \leq 0$. On the other hand, if $x \in \overline{D}$, then $|D\phi(x)| > \chi_{\overline{D}}(x) = 1$, which implies $|D\phi(x_p)|^{p-4} \to \infty$ as $p \to \infty$. In view of (3.15), this gives again $\Delta_\infty \phi(x) \leq 0$. Thus (3.14) is valid.

To show that u_∞ is also a subsolution to (2.4), we fix $\varphi \in C^2(\Omega)$ such that $u_\infty - \varphi$ has a strict maximum at $x \in \Omega$. We have to check that

$$\Delta_\infty \varphi(x) \geq 0 \quad \text{and} \quad |D\varphi(x)| - \chi_{\operatorname{int} D}(x) \geq 0. \tag{3.16}$$

By the uniform convergence of u_p to u_∞, there are points x_p such that $u_p - \varphi$ has a maximum at x_p and $x_p \to x$ as $p \to \infty$. At those points we have

$$(p-2)|D\varphi|^{p-4}\Delta_\infty \varphi(x_p) + |D\varphi|^{p-2}\Delta\varphi(x_p) \geq g_*(x_p). \tag{3.17}$$

If $x \notin \operatorname{int} D$ and $|D\varphi(x)| = 0$, then (3.16) clearly holds. On the other hand, if $x \in \operatorname{int} D$, then by (3.10), $g_*(x_p) > 0$ for p large. In view of (3.17), this implies that we must have $D\varphi(x_p) \neq 0$. Thus, we can divide in (3.17) by $(p-2)|D\varphi(x_p)|^{p-4}$ to obtain

$$\Delta_\infty \varphi(x_p) \geq -\frac{1}{p-2}|D\psi|^2 \Delta\varphi(x_p) + \frac{g_*(x_p)}{(p-2)|D\varphi(x_p)|^{p-4}}. \tag{3.18}$$

Since g_* is non-negative, letting $p \to \infty$ in (3.18) yields $\Delta_\infty \varphi(x) \geq 0$. Moreover, if $x \in \operatorname{int} D$ and $|D\varphi(x)| < \chi_{\operatorname{int} D}(x) = 1$, then as $g_*(x) > 0$, the right side of (3.18) tends to infinity, whereas the left side remains bounded, a contradiction. Therefore we must also have $|D\varphi(x)| - \chi_{\operatorname{int} D}(x) \geq 0$, which concludes the proof. $\square$

Theorem 3.6 says that the boundary value problem (2.4) has a solution if f is assumed to be Lipschitz. Owing to Jensen's uniqueness results, this restriction can be removed.

Theorem 3.7. *The gradient constraint problem* (2.4) *has at least one solution for any* $f \in C(\partial\Omega)$ *and* $D \subset \Omega$.

Proof. Let f_j be a sequence of Lipschitz functions converging to f uniformly on $\partial\Omega$ and let u_j be a solution to (2.5) such that $u_j = f_j$ on $\partial\Omega$, provided by Theorem 3.6. Since u_j is a subsolution to the infinity Laplace equation, for every $\Omega' \subset\subset \Omega$ there is $C > 0$ such that

$$||Du_j||_{L^\infty(\Omega')} \leq C,$$

see e.g. [2]. Thus, up to selecting a subsequence, there is a locally Lipschitz continuous u such that $u_j \to u$ locally uniformly. Moreover, it follows from the standard stability results for viscosity solutions, see [13], that u is a solution to (2.5).

Thus we only need to prove that $u = f$ on $\partial\Omega$. Let h_j and z_j be the unique solutions to the infinity Laplace equation and Jensen's equation (2.8) (with $\varepsilon = 1$), respectively, such that $h_j = z_j = f_j$ on $\partial\Omega$. By comparison, $z_j \leq u_j \leq h_j$ on $\overline{\Omega}$, and hence $z \leq u \leq h$ on $\overline{\Omega}$, where h and z are the unique solutions to the infinity Laplace equation and (2.8), respectively, with $h = z = f$ on $\partial\Omega$ (cf. [19, Corollary 3.14]). In particular, $u = f$ on $\partial\Omega$, as desired. □

We close this section by proving a sharp a priori bound for the solutions of (2.4) that are obtained using the p-Laplace approximation. To this end, we will again assume that f is Lipschitz and recall that F denotes a Lipschitz extension of f that satisfies $||F||_{L^\infty(\Omega)} = ||f||_{L^\infty(\partial\Omega)}$.

Lemma 3.8. *A uniform limit u_∞ of a subsequence u_p as $p \to \infty$ satisfies*

$$||Du_\infty||_{L^\infty(\Omega)} \le \max\{1, \mathrm{Lip}(f)\}.$$

Proof. Recall that we have denoted by u_p the minimizer of

$$F_p(u) = \frac{1}{p}\int_\Omega |Du|^p\,dx + \int_\Omega gu\,dx$$

on the set $K = \{u \in W^{1,p}(\Omega) : u = f \text{ on } \partial\Omega\}$. Then $F \in K$ and we have

$$\begin{aligned} \frac{1}{p}\int_\Omega |Du_p|^p\,dx + \int_\Omega gu_p\,dx &\le \frac{1}{p}\int_\Omega |DF|^p\,dx + \int_\Omega gF\,dx \\ &\le \frac{\mathrm{Lip}(f)^p|\Omega|}{p} + ||g||_{L^\infty(\Omega)}\,||f||_{L^\infty(\partial\Omega)}\,|\Omega|\,. \end{aligned}$$

This together with Hölder's inequality implies

$$\begin{aligned} \int_\Omega |Du_p|^p\,dx &\le \mathrm{Lip}(f)^p|\Omega| + pC - p\int_\Omega gu_p\,dx \\ &\le \mathrm{Lip}(f)^p|\Omega| + pC + p\,||u_p||_{L^p(\Omega)}\,||g||_{L^{p'}(\Omega)}\,. \end{aligned} \tag{3.19}$$

By Sobolev's inequality, we have

$$\begin{aligned} ||u_p||_{L^p(\Omega)} &\le ||u_p - F||_{L^p(\Omega)} + ||F||_{L^p(\Omega)} \\ &\le C(\Omega, n)\,||Du_p - DF||_{L^p(\Omega)} + ||f||_{L^\infty(\partial\Omega)}\,|\Omega|^{1/p} \\ &\le C(\Omega, n)\,||Du_p||_{L^p(\Omega)} + C(n, \Omega)(\mathrm{Lip}(f) + ||f||_{L^\infty(\partial\Omega)}). \end{aligned} \tag{3.20}$$

By combining (3.19) and (3.20), we obtain

$$\int_\Omega |Du_p|^p\,dx \le \mathrm{Lip}(f)^p|\Omega| + pC + Cp\|Du_p\|_{L^p(\Omega)},$$

that is,

$$\|Du_p\|_{L^p(\Omega)} \le \left(C\mathrm{Lip}(f)^p + Cp + Cp\|Du_p\|_{L^p(\Omega)}\right)^{1/p}, \tag{3.21}$$

where the positive constant C depends on n, Ω, f and g, but is independent of p for $p > n$. Observe that

$$(Ca^p + p\,b + p\,c)^{1/p} \to \max\{a, 1\} \quad \text{as } p \to \infty$$

and recall the preliminary bound

$$\begin{aligned}\|Du_p\|_{L^p(\Omega)} &\le \|Dh_p\|_{L^p(\Omega)} + 2\,|\Omega|^{1/p}\max\left\{1, ||g||_{L^\infty(\Omega)}\left(\frac{|\Omega|}{\omega_n}\right)^{1/n}\right\}\\ &\le C(n,\Omega)(\mathrm{Lip}(f) + ||g||_{L^\infty(\Omega)} + 1)\end{aligned}$$

that was obtained in course of the proof of Lemma 3.5. Combining these facts with (3.21) we get

$$||Du_\infty||_{L^\infty(\Omega)} \le \max\{1, \mathrm{Lip}(f)\}.$$

This ends the proof. □

Remark 3.9. *For future reference, we note that all the results in this section, except for Theorem 3.6, hold for any bounded g without any sign restrictions.*

4. Uniqueness and comparison results

In this section, we show that under a suitable topological assumption on D, the problem (2.4), that is, $\min\{\Delta_\infty u, |Du|-\chi_D\} = 0$ with fixed boundary values, $u = f$ on $\partial\Omega$, has a unique solution. In addition, if the condition is not satisfied, the uniqueness is lost.

Theorem 4.1. *Suppose that $\overline{\operatorname{int} D} = \overline{D}$. Then the problem* (2.4) *has a unique solution.*

To prove Theorem 4.1, we show that a solution u of (2.4) can be characterized in the following way. Let $h \in C(\overline{\Omega})$ be the unique solution to $\Delta_\infty h = 0$ satisfying $h = f$ on $\partial\Omega$, and denote

$$\mathcal{A} = \{x \in \Omega \colon\ |Dh(x)| < 1\}, \qquad \mathcal{B} = \mathcal{A} \cap D;$$

recall that h is everywhere differentiable, as proved in [12]. Let further $z \in C(\overline{\Omega})$ be the unique solution to the Jensen's equation

$$\min\{\Delta_\infty z, |Dz| - 1\} = 0, \tag{4.22}$$

also satisfying $z = f$ on $\partial\Omega$. Then, if $\overline{\operatorname{int} D} = \overline{D}$, we have $u(x) = z(x)$ for all $x \in \mathcal{B}$ and $\Delta_\infty u(x) = 0$ in $\Omega \setminus \overline{\mathcal{B}}$. But the solution to $\Delta_\infty v = 0$ in $\Omega \setminus \overline{\mathcal{B}}$ with the boundary values z on $\partial\mathcal{B}$ and f on $\partial\Omega$ is unique, and thus u is unique.

Theorem 4.2. *Suppose that $\overline{\operatorname{int} D} = \overline{D}$. Let $u \in C(\overline{\Omega})$ be a solution to* (2.4)*. Then $u(x) = z(x)$ for all $x \in \mathcal{B}$ and $\Delta_\infty u(x) = 0$ in $\Omega \setminus \overline{\mathcal{B}}$.*

Proof. Observe first that since u is a supersolution of the Jensen's equation (4.22) and $u = z$ on $\partial\Omega$, we have $u \ge z$ in Ω. On the other hand, since $\Delta_\infty u \ge 0$ and $u = h$ on $\partial\Omega$, we have $u \le h$ in Ω. Thus

$$z(x) \le u(x) \le h(x) \quad \text{for all } x \in \Omega.$$

Next we recall Theorem 2.3, which implies that $z(x) = h(x)$ in $\Omega \setminus \mathcal{A}$. This implies that

$$u(x) = h(x) = z(x) \qquad \text{in } \Omega \setminus \mathcal{A},$$

and, in particular, that $\Delta_\infty u(x) = 0$ in $\Omega \setminus \overline{\mathcal{A}}$. Moreover, as $\Delta_\infty u = 0$ in $\Omega \setminus \overline{D}$ by the fact that it satisfies (2.4), we have

$$\Delta_\infty u(x) = 0 \qquad \text{in } \Omega \setminus \overline{\mathcal{B}}.$$

To prove that $u = z$ in $\mathcal{B}$ we argue by contradiction and suppose that there is $\hat{x} \in \operatorname{int} \mathcal{B}$ such that $u(\hat{x}) - z(\hat{x}) > 0$. If u were smooth, we would have $|Du(\hat{x})| \geq 1$ by the second part of the equation, and from $\Delta_\infty u \geq 0$ it would follow that $t \mapsto |Du(\gamma(t))|$ is non-decreasing along the curve γ for which $\gamma(0) = \hat{x}$ and $\dot{\gamma}(t) = Du(\gamma(t))$. Using this information and the fact that $|z(x) - z(y)| \leq |x - y|$ in $\mathcal{A}$, we could then follow γ to $\partial\mathcal{A}$ to find a point y where $u(y) > z(y)$; but this is a contradiction since u and z coincide on $\partial\mathcal{A}$.

To overcome the fact that u need not be smooth and to make the formal steps outlined above rigorous, let $\delta > 0$ and

$$u_\delta(x) = \sup_{y \in \Omega} \left\{ u(y) - \frac{1}{2\delta} |x - y|^2 \right\}$$

be the standard sup-convolution of u. Observe that since u is bounded in Ω, we in fact have

$$u_\delta(x) = \sup_{y \in B_{R(\delta)}(x)} \left\{ u(y) - \frac{1}{2\delta} |x - y|^2 \right\}$$

with $R(\delta) = 2\sqrt{\delta \,||u||_{L^\infty(\Omega)}}$. We assume that $\delta > 0$ is so small that

(1) $\hat{x} \in (\operatorname{int} \mathcal{B})_\delta := \{x \in \operatorname{int} \mathcal{B} \colon \operatorname{dist}(x, \partial\mathcal{B}) > R(\delta)\}$; and
(2) for $\mathcal{A}_\delta := \{x \in \mathcal{A} \colon \operatorname{dist}(x, \partial\mathcal{A}) > R(\delta)\}$, it holds

$$\sup_{x \in (\operatorname{int} \mathcal{B})_\delta} (u_\delta - z) > \sup_{x \in \partial\mathcal{A}_\delta} (u_\delta - z).$$

Regarding the second condition, recall that $u_\delta \to u$ locally uniformly when $\delta \to 0$, and that $u = z$ on $\partial\mathcal{A}$.

Next we observe that since u is a solution to (2.4), it follows that $\Delta_\infty u_\delta \geq 0$ and $|Du_\delta| - \chi_{(\operatorname{int} D)_\delta} \geq 0$ in Ω_δ; see e.g. [20]. In particular, since u_δ is semiconvex and thus twice differentiable a.e., there exists $x_0 \in (\operatorname{int} \mathcal{B})_\delta$ such that

$$u_\delta(x_0) - z(x_0) > \sup_{x \in \partial\mathcal{A}_\delta} (u_\delta - z),$$

u_δ is (twice) differentiable at x_0, and

$$|Du_\delta(x_0)| = L(u_\delta, x_0) \geq 1.$$

Now let $r_0 = \frac{1}{2} \operatorname{dist}(x_0, \partial\mathcal{A}_\delta)$ and let $x_1 \in \partial B_{r_0}(x_0)$ be a point such that

$$\max_{y \in \overline{B}_{r_0}(x_0)} u_\delta(y) = u_\delta(x_1).$$

Since $\Delta_\infty u_\delta \geq 0$, the increasing slope estimate, see [9], implies

$$1 \leq L(u_\delta, x_0) \leq L(u_\delta, x_1) \quad \text{and} \quad u_\delta(x_1) \geq u_\delta(x_0) + |x_0 - x_1| \,.$$

By defining $r_1 = \frac{1}{2}\operatorname{dist}(x_1, \partial\mathcal{A}_\delta)$, choosing $x_2 \in \partial B_{r_1}(x_1)$ so that

$$\max_{y \in \overline{B}_{r_1}(x_1)} u_\delta(y) = u_\delta(x_2),$$

and using the increasing slope estimate again yields

$$1 \leq L(u_\delta, x_0) \leq L(u_\delta, x_1) \leq L(u_\delta, x_2)$$

and

$$u_\delta(x_2) \geq u_\delta(x_1) + |x_1 - x_2| \geq u_\delta(x_0) + |x_0 - x_1| + |x_1 - x_2| \,.$$

Repeating this construction gives a sequence (x_k) such that $x_k \to a \in \partial\mathcal{A}_\delta$ as $k \to \infty$ and

$$u_\delta(x_k) \geq u_\delta(x_0) + \sum_{j=0}^{k-1} |x_j - x_{j+1}| \quad \text{for } k = 1, 2, \ldots$$

On the other hand, since $|z(x) - z(y)| \leq |x - y|$ whenever the line segment $[x, y]$ is contained in $\mathcal{A}$ (see [10]), we have

$$z(x_k) \leq z(x_0) + \sum_{j=0}^{k-1} |x_j - x_{j+1}| \,.$$

Thus, by continuity,

$$u_\delta(a) - z(a) = \lim_{k\to\infty} u_\delta(x_k) - z(x_k) \geq u_\delta(x_0) - z(x_0) > \sup_{x \in \partial\mathcal{A}_\delta} (u_\delta - z).$$

But this is impossible because $a \in \partial\mathcal{A}_\delta$. Hence the theorem is proved. □

Remark 4.3. *The proof of Theorem 4.2 shows that the uniqueness for* (2.4) *in fact holds under the weaker (but less explicit) assumption*

$$\overline{\operatorname{int}\mathcal{B}} = \overline{\mathcal{B}}.$$

Remark 4.4. *Under the assumption $\overline{\operatorname{int} D} = \overline{D}$, we have that the unique solution $u \in C(\overline{\Omega})$ to* (2.4) *satisfies $u(x) = z(x)$ for all $x \in \overline{D}$. This follows from the fact that for $x \in \Omega \setminus \mathcal{A}$, $z(x) = h(x)$ by Theorem 2.3.*

In addition to uniqueness, we also have a comparison principle for the equation $\min\{\Delta_\infty u, |Du| - \chi_D\} = 0$.

Theorem 4.5. *Suppose that $\overline{\operatorname{int} D} = \overline{D}$. Then if v_1 is a subsolution and v_2 is a supersolution to* (2.4)*, we have $v_1 \leq v_2$.*

Proof. Let u be the unique solution to (2.4). We will show that $v_1 \leq u$ and $u \leq v_2$ in Ω.

Since v_2 is a supersolution to Jensen's equation (4.22), we have by Jensen's comparison theorem [19] that $v_2 \geq z$ in Ω. In particular, owing to Remark 4.4, $v_2 \geq u = z$ in $\overline{D}$. On the other hand, in $\Omega \setminus \overline{D}$ we have $\Delta_\infty u = 0$ by Theorem 4.2 and $\Delta_\infty v_2 \leq 0$, and so $v_2 \geq u$ in $\Omega \setminus \overline{D}$ as well.

As for the other inequality $u \geq v_1$, we notice first that it suffices to prove that $u = z \geq v_1$ in $\mathcal{B}$. Indeed, since $\Delta_\infty u = 0$ in $\Omega \setminus \mathcal{B}$ again by Theorem 4.2 and $\Delta_\infty v_1 \geq 0$ in Ω, it follows that $u \geq v_1$ in $\mathcal{B}$ implies $u \geq v_1$ in Ω.

To prove $u = z \geq v_1$ in $\mathcal{B}$, we simply observe that we can repeat the argument used in the second part of the proof of Theorem 4.2, which proved that $u \leq z$ in $\mathcal{B}$, as this argument only used the fact that u was a subsolution to (2.4). □

4.1. **Non-uniqueness.** In this section, we discuss various situations where there are more than one solution to (2.4). For convenience of exposition, we assume that $\mathrm{Lip}(f) \leq 1$, which implies that the infinity harmonic extension h of f satisfies $||Dh||_{L^\infty(\Omega)} \leq 1$. This simplifies the notation as we have

$$D = D \cap \{|Dh| \leq 1\}.$$

Let us first establish the non-uniqueness in the easy case $\operatorname{int} D = \emptyset$.

Lemma 4.6. *Suppose that* $\operatorname{int} D = \emptyset$*, and* f *is Lipschitz continuous with constant* $L \leq 1$*. Let* $v_\alpha \in C(\overline{\Omega})$ *be the unique function satisfying*

$$\begin{cases} \Delta_\infty v_\alpha = 0 & \text{in } \Omega \setminus \overline{D} \\ v_\alpha = f & \text{on } \partial\Omega \\ v_\alpha(x) = \sup\limits_{y \in \partial\Omega} \Big(f(y) - \alpha\,|x - y| \Big) & \text{for } x \in \overline{D}. \end{cases} \tag{4.23}$$

Then v_α *is a solution to* (2.4) *for every* $\alpha \in [L, 1]$.

Proof. We show first that v_α is a subsolution to $\min\{\Delta_\infty u, |Du| - \chi_D\} = 0$. To this end, as $\operatorname{int} D = \emptyset$, we only have to verify that

$$\Delta_\infty v_\alpha \geq 0 \qquad \text{in } \Omega.$$

This is clearly true in $\Omega \setminus \overline{D}$, so let us suppose that $\hat{x} \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $v_\alpha - \varphi$ has a strict local maximum at $\hat{x} \in \overline{D}$. Since for each $y \in \partial\Omega$,

$$x \mapsto f(y) - \alpha\,|x - y|$$

is a viscosity subsolution of the infinity Laplace equation, it follows that

$$g(x) = \sup_{y \in \partial\Omega} \Big(f(y) - \alpha\,|x - y| \Big)$$

is a subsolution as well. By the comparison principle, this implies that $v_\alpha(x) \geq g(x)$ for all $x \in \Omega$. In particular, since $v_\alpha(\hat{x}) = g(\hat{x})$, this means that also $g - \varphi$ has a strict local maximum at $\hat{x}$. As g is a subsolution, this implies $\Delta_\infty \varphi(\hat{x}) \geq 0$, as desired.

To show that v_α is a supersolution it suffices to prove that $|Dv_\alpha| - 1 \le 0$ in D. But this is evident from the fact that $\operatorname{Lip}(v_\alpha, \overline{\Omega}) = \alpha \le 1$, which holds because v_α is the absolutely minimizing Lipschitz extension of its boundary values to $\Omega \setminus \overline{D}$, see e.g. [9], and $L \le \alpha$. □

Remark 4.7. *In the special case of zero boundary values $f \equiv 0$, Lemma 4.6 says that v_α, the unique function that satisfies*

$$\begin{cases} \Delta_\infty v_\alpha = 0 & \text{in } \Omega \setminus \overline{D} \\ v_\alpha(x) = 0 & \text{on } \partial\Omega \\ v_\alpha(x) = -\alpha \operatorname{dist}(x, \partial\Omega) & \text{for } x \in \overline{D}, \end{cases} \tag{4.24}$$

is a solution to (2.4) *for every $\alpha \in [0,1]$.*

Note also that when $\operatorname{int}(D) = \emptyset$ *and $\overline{D}$ contains more than a single point, there are more solutions to* (2.4) *than just the ones described in Lemma 4.6. In fact, let us take any subset $A \subset \overline{D}$ and let z be the solution to*

$$\begin{cases} \Delta_\infty z = 0 & \text{in } \Omega \setminus A \\ z = 0 & \text{on } \partial\Omega \\ z = -\operatorname{dist}(x, \partial\Omega) & \text{for } x \in A. \end{cases} \tag{4.25}$$

Then αz is also a solution to (2.4)*, with $f = 0$, for every $\alpha \in [0,1]$.*

Next we will show that the uniqueness question in the general case reduces to the uniqueness for (4.27) (see below), where $D \setminus \overline{\operatorname{int} D}$ has empty interior.

If $\operatorname{int} D \ne \emptyset$, then $\overline{\operatorname{int} D}$ satisfies the condition of Theorem 4.1 for uniqueness. That is, $\overline{\operatorname{int}(\overline{\operatorname{int} D})} = \overline{\overline{\operatorname{int} D}}$, and thus there exists a unique solution u_0 to

$$\begin{cases} \min\{\Delta_\infty u, |Du| - \chi_{\overline{\operatorname{int} D}}\} = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{4.26}$$

Define $f_0 \in C(\partial(\Omega \setminus \overline{\operatorname{int} D}))$ by setting $f_0(x) = f(x)$, if $x \in \partial\Omega$, and $f_0(x) = u_0(x)$, if $x \in \partial(\overline{\operatorname{int} D})$.

Lemma 4.8. *Suppose that $\overline{\operatorname{int} D} \ne \overline{D}$ and $\operatorname{int} D \ne \emptyset$. Let v be any solution to*

$$\begin{cases} \min\{\Delta_\infty v, |Dv| - \chi_{D \setminus \overline{\operatorname{int} D}}\} = 0 & \text{in } \Omega \setminus \overline{\operatorname{int} D} \\ v = f_0 & \text{on } \partial(\Omega \setminus \overline{\operatorname{int} D}), \end{cases} \tag{4.27}$$

and define $w \colon \Omega \to \mathbb{R}$ by

$$w(x) = \begin{cases} v(x), & \text{if } x \in \Omega \setminus \overline{\operatorname{int} D} \\ u_0(x), & \text{if } x \in \overline{\operatorname{int} D}. \end{cases}$$

Then w is a solution to

$$\begin{cases} \min\{\Delta_\infty w, |Dw| - \chi_D\} = 0 & \text{in } \Omega \\ w = f & \text{on } \partial\Omega. \end{cases}$$

Proof. Let us first show that w is a supersolution. Let $\phi \in C^2(\Omega)$ be a function touching w from below at $x \in \Omega$. If $x \in \Omega \setminus \overline{\operatorname{int} D}$, then, as $w = v$ in $\Omega \setminus \overline{\operatorname{int} D}$, we have

$$\min\{\Delta_\infty \phi(x), |D\phi(x)| - \chi_{\overline{D\setminus \overline{\operatorname{int} D}}}(x)\} \le 0.$$

But then also

$$\min\{\Delta_\infty \phi(x), |D\phi(x)| - \chi_{\overline{D}}(x)\} \le 0,$$

as desired. On the other hand, if $x \notin \Omega \setminus \overline{\operatorname{int} D}$, then $x \in \overline{\operatorname{int} D}$, and we have $w(x) = u_0(x)$. Moreover, as u_0 is a solution and v a subsolution to $\Delta_\infty h = 0$ in $\Omega \setminus \overline{\operatorname{int} D}$, we have $v \le u_0$ in $\Omega \setminus \overline{\operatorname{int} D}$ by comparison principle. Thus $w \le u_0$ in Ω and ϕ touches u_0 from below at x, and thus

$$\min\{\Delta_\infty \phi(x), |D\phi(x)| - \chi_{\overline{\operatorname{int} D}}(x)\} \le 0.$$

This clearly implies

$$\min\{\Delta_\infty \phi(x), |D\phi(x)| - \chi_{\overline{D}}(x)\} \le 0,$$

and we have shown that w is a supersolution.

To check that w is also a subsolution, we fix a function $\varphi \in C^2(\Omega)$ touching w from above at $x \in \Omega$. We want to show that

$$\Delta_\infty \varphi(x) \ge 0 \quad \text{and} \quad |D\varphi(x)| \ge \chi_{\operatorname{int} D}(x).$$

First, if $x \in \operatorname{int} D = \operatorname{int}(\overline{\operatorname{int} D})$, then $w = u_0$ in a neighborhood of x, and thus

$$\min\{\Delta_\infty \varphi(x), |D\varphi(x)| - \chi_{\operatorname{int}(\overline{\operatorname{int} D})}(x)\} \ge 0,$$

by (4.26). On the other hand, if $x \notin \operatorname{int} D$, then it suffices to show that $\Delta_\infty \varphi(x) \ge 0$. This is clearly true if $x \in \Omega \setminus \overline{\operatorname{int} D}$, so we may assume that $x \in \partial(\overline{\operatorname{int} D})$. But by Theorem 4.2, $u_0(x) = z(x)$, where z is the unique solution to Jensen's equation (4.22), and $u_0 \ge z$ in Ω, so that φ touches also z from above at x. Hence $\Delta_\infty \varphi(x) \ge 0$, and we are done. $\square$

Lemma 4.8 above shows that the uniqueness question in the general case reduces to the uniqueness for (4.27). This type of situation was already dealt with in Lemma 4.6. However, the reader should notice that in (4.27), $\operatorname{Lip}(f_0) = 1$, and thus Lemma 4.6 cannot be used to deduce that there are more than one solution.

Next we present examples showing that under conditions $\overline{\operatorname{int} D} \ne \overline{D}$ and $\operatorname{int} D \ne \emptyset$ (and $f \equiv 0$), problem (2.4) may have either a unique solution or multiple solutions, depending on the geometry.

Example 4.9. Suppose that $\Omega = B_2(0)$, $f \equiv 0$, and $D = B_1(0) \cup D_1$, where $D_1 \subset B_2 \setminus \overline{B}_1$ is any non-empty set with empty interior. Clearly $\overline{\operatorname{int} D} \ne \overline{D}$ and $\operatorname{int} D \ne \emptyset$. We claim that the only solution to $\min\{\Delta_\infty z, |Dz| - \chi_D\} = 0$ is $u(x) = |x| - 2$. First, as $|x| - 2$ is a solution to

$$\min\{\Delta_\infty u, |Du| - 1\} = 0,$$

it follows that any solution v to

$$\min\{\Delta_\infty v, |Dv| - \chi_D\} = 0$$

satisfies $v \geq u$, since v is a supersolution to the first equation as well.

On the other hand, it follows from Theorem 4.2 that u is the unique solution to

$$\min\{\Delta_\infty u, |Du| - \chi_{B_1(0)}\} = 0$$

satisfying the boundary condition $u = 0$ on $\partial\Omega$. Thus, owing to Theorem 4.5, we also have $v \leq u$, since v is a subsolution to the previous equation as well.

For the second example, we need the following lemma:

Lemma 4.10. *Let u_0 be the unique solution to*

$$\begin{cases} \min\{\Delta_\infty u, |Du| - \chi_{\overline{\operatorname{int} D}}\} = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{4.28}$$

Then u_0 is the largest solution to (2.4).

Proof. We only need to show that u_0 is a solution to (2.4) as its maximality then follows directly from the comparison principle, Theorem 4.5. To this end, we first show that u_0 is a supersolution. Let $\phi \in C^2(\Omega)$ be a function touching u_0 from below at $x \in \Omega$. Then, by (4.28),

$$0 \geq \min\{\Delta_\infty \phi, |D\phi| - \chi_{\overline{\operatorname{int} D}}\} \geq \min\{\Delta_\infty \phi, |D\phi| - \chi_{\overline{D}}\}.$$

Thus u_0 is a supersolution to (2.4).

The subsolution case is quite similar. Let $\varphi \in C^2(\Omega)$ be a function touching u_0 from above at $x \in \Omega$. By $\operatorname{int}(\overline{\operatorname{int} D}) = \operatorname{int} D$ and (4.28), it follows that

$$0 \leq \min\{\Delta_\infty \varphi, |D\varphi| - \chi_{\operatorname{int}(\overline{\operatorname{int} D})}\} = \min\{\Delta_\infty \varphi, |D\varphi| - \chi_{\operatorname{int} D}\}.$$

Thus u_0 is also a subsolution to (2.4). $\square$

Example 4.11. Let $n = 2$, $\Omega = \,]-1,1[^2$, $f \equiv 0$, and $D = D_0 \cup \{x_0\}$, where $D_0 = B_{1/2}(0)$ and $x_0 \in \Omega \setminus \overline{B}_{1/2}(0)$ is to be chosen.

Let u be the unique solution to

$$\begin{cases} \min\{\Delta_\infty u, |Du| - \chi_{D_0}\} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, by Lemma 4.10, u is a solution to (2.4). Moreover, by Theorem 4.2,

$$u(x) = -\operatorname{dist}(x, \partial\Omega)$$

for all $x \in \overline{D}_0$. By Lemma 4.8, equation (4.27), which in this case reads

$$\begin{cases} \min\{\Delta_\infty v, |Dv| - \chi_{\{x_0\}}\} = 0 & \text{in } \Omega \setminus \overline{\operatorname{int} D} = \Omega \setminus \overline{D}_0 \\ v = u & \text{on } \partial(\Omega \setminus \overline{D}_0), \end{cases} \tag{4.29}$$

determines the uniqueness of a solution to (2.4). We show that there exist several solutions for this problem.

Choose $x_0 \in \Omega \setminus \overline{D}_0$ so that $-\operatorname{dist}(x, \partial\Omega)$ is not differentiable at x_0, and let u_1 be the unique function such that

$$\begin{cases} u_1(x) = -\operatorname{dist}(x, \partial\Omega), & x \in \overline{D}_0 \cup \{x_0\} \\ u_1 = 0, & \text{on } \partial\Omega \\ \Delta_\infty u_1 = 0, & \text{in } \Omega \setminus (\overline{D}_0 \cup \{x_0\}). \end{cases}$$

We can now check that u_1 is a solution to (4.29). Indeed, that u_1 is subsolution follows easily, because $\operatorname{int}\{x_0\} = \emptyset$, $\Delta_\infty u_1 = 0$ in $\Omega \setminus (\overline{D}_0 \cup \{x_0\})$, and there are no C^2 functions touching u_1 from above at x_0.

To show that u_1 is also a supersolution, let $\phi \in C^2(\Omega \setminus \overline{D}_0)$ be a function touching u_1 from below at $x \in \Omega \setminus \overline{D}_0$. The case $x \in \Omega \setminus (\overline{D}_0 \cup \{x_0\})$ is again clear, and we may assume that $x = x_0$. Since $||Du_1||_{L^\infty(\Omega)} \leq 1$ by the definition of u_1 and properties of infinity harmonic functions, it follows that $|D\phi(x_0)| \leq 1 = \chi_{\overline{D}}(x_0)$. Hence we have shown that u_1 is also a supersolution to (2.4).

Finally, let us observe that $u(x_0) \neq u_1(x_0)$. The reason is that u is, as a solution to the infinity Laplace equation $\Delta_\infty h = 0$, differentiable at x_0 (see [12]), but u_1 is not, because it touches $-\operatorname{dist}(x, \partial\Omega)$ from above at x_0.

4.2. **Minimal and maximal solutions.** It turns out that the boundary value problem (2.4) has always a maximal and a minimal solution. The maximal solution $\bar{u}$ has been characterized in Lemma 4.10, and the minimal solution $\underline{u}$ can be constructed as follows. Given any $D \subset \Omega$, define

$$D_i = (D + B(0, \tfrac{1}{i})) \cap \Omega = \{x \in \Omega\colon\ \operatorname{dist}(x, D) < \tfrac{1}{i}\}.$$

Then D_i is open and hence $\overline{D_i} = \overline{\operatorname{int} D_i}$. By Theorem 4.1, there is a unique solution u_i to

$$\begin{cases} \min\{\Delta_\infty u, |Du| - \chi_{D_i}\} = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases}$$

By the comparison principle, Theorem 4.5, the sequence (u_i) is monotone, and, as each u_i is a subsolution to the infinity Laplace equation, locally equicontinuous. Moreover, $u_1 \leq u_i \leq h$ in $\overline{\Omega}$, where $\Delta_\infty h = 0$ in Ω and $h = f$ on $\partial\Omega$. Hence $u_i \to u$ locally uniformly in Ω for some u satisfying $u = f$ on $\partial\Omega$. It follows easily from the standard stability results for viscosity solutions that u is a solution to (2.4). Moreover, Theorem 4.5 implies that any solution v to (2.4) satisfies $v \geq u_i$ for all i's, and hence u must be the minimal solution. We will give an alternative characterization for the minimal solution in Section 5 below.

In Theorem 3.6, we proved the existence of a solution to (2.4) using p-Laplace approximation. Next we show that this "variational" solution is, in general, neither the minimal nor the maximal solution.

Example 4.12. Let $\Omega = B_1(0)$, $D = \{0\}$, and $f \equiv 0$. Then, by Lemma 4.6, the maximal solution is $\bar{u}(x) \equiv 0$ and the minimal solution $\underline{u}(x) = |x| - 1$. Since $|D| = 0$, it follows that the unique solution u_p to

$$\begin{cases} \Delta_p v = \chi_D & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

is $u_p \equiv 0$. Hence, in this case, the (unique) variational solution is the maximal solution $\bar{u}(x) \equiv 0$.

Now if Ω and f are as above, and $D = \Omega \cap (\mathbb{R} \setminus \mathbb{Q})^n$ (that is, D consists of points in Ω with irrational coordinates), then, again by Lemma 4.6, the maximal solution is $\bar{u}(x) \equiv 0$ and the minimal solution $\underline{u}(x) = |x| - 1$. However, since $\chi_D(x) = 1$ a.e., it follows that u_p is a solution also to $\Delta_p v = 1$ in Ω. Then it is well-known, see e.g. [5] and the references therein, that $u_p \to -\operatorname{dist}(x, \partial\Omega) = \underline{u}(x)$ as $p \to \infty$. Hence, in this case, the (unique) variational solution is the minimal solution $\underline{u}(x)$.

Example 4.13. Let $\Omega = B_4(0)$, $f \equiv 0$, and $D = \{0\} \cup \hat{D}$, where $\hat{D} = (B_3 \setminus B_2) \cap (\mathbb{R} \setminus \mathbb{Q})^n$ (that is, $\hat{D}$ consists of the points in the annulus $B_3 \setminus B_2$ having irrational coordinates). Then, as $\operatorname{int} D = \emptyset$, $\bar{u} \equiv 0$ and $\underline{u}(x) = 4 - |x|$. However, the results in [5] imply that

$$u_\infty(x) = \lim_{p \to \infty} u_p(x) = -\operatorname{dist}(x, \partial\Omega)$$

in $B_3 \setminus B_2$ and that u_∞ is a solution to the infinity Laplace equation in B_2; thus in this case the (unique) variational solution u_∞ is neither the minimal nor the maximal solution.

5. Games

In this section, we consider a variant of the tug-of-war game introduced by Peres, Schramm, Sheffield and Wilson in [27], and show that the value functions of this game converge, as the step size tends to zero, to the minimal solution of (2.4).

As before, let Ω be a bounded open set and $D \subset \Omega$. For a fixed $\varepsilon > 0$, consider the following two-player zero-sum-game. If $x_0 \in \Omega \setminus D$, then the players play a tug-of-war game as described in [27], that is, a fair coin is tossed and the winner of the toss is allowed to move the game token to any $x_1 \in \overline{B}_\varepsilon(x_0)$. On the other hand, if $x_0 \in D \cap \Omega$, then Player II, the player seeking to minimize the final payoff, can either sell the turn to Player I with the price $-\varepsilon$ or decide that they toss a fair coin and play tug-of-war. If Player II sells the turn, then Player I can move the game token to any $x_1 \in \overline{B}_\varepsilon(x_0)$. After the first round, the game continues from x_1 according to the same rules.

This procedure yields a possibly infinite sequence of game states $x_0, x_1, \ldots$ where every x_k is a random variable. The game ends when the game token hits Γ_ε, the boundary strip of width ε given by

$$\Gamma_\varepsilon = \{x \in \mathbb{R}^n \setminus \Omega \,:\, \operatorname{dist}(x, \partial\Omega) < \varepsilon\}.$$

We denote by $x_\tau \in \Gamma_\varepsilon$ the first point in the sequence of game states that lies in Γ_ε, so that τ refers to the first time we hit Γ_ε.

At this time the game ends with the terminal payoff given by $F(x_\tau)$, where $F : \Gamma_\varepsilon \to \mathbb{R}$ is a given Borel measurable continuous *payoff function*. Player I earns $F(x_\tau)$ while Player II earns $-F(x_\tau)$.

A strategy S_{I} for Player I is a function defined on the partial histories that gives the next game position $S_{\mathrm{I}}(x_0, x_1, \dots, x_k) = x_{k+1} \in \overline{B}_\varepsilon(x_k)$ if Player I wins the toss. Similarly Player II plays according to a strategy S_{II}. In addition, we define a decision variable, which tells when Player II decides to sell a turn

$$\theta(x_0, \dots, x_k) = \begin{cases} 1, & x_k \in D \text{ and Player II sells a turn,} \\ 0, & \text{otherwise.} \end{cases}$$

The one step transition probabilities will be

$$\begin{aligned} &\pi_{S_{\mathrm{I}}, S_{\mathrm{II}}, \theta}(x_0, \dots, x_k, A) \\ &\qquad = \big(1 - \theta(x_0, \dots, x_k)\big) \frac{1}{2} \Big(\delta_{S_{\mathrm{I}}(x_0,\dots,x_k)}(A) + \delta_{S_{\mathrm{II}}(x_0,\dots,x_k)}(A) \Big) \\ &\qquad\quad + \theta(x_0, \dots, x_k) \delta_{S_{\mathrm{I}}(x_0,\dots,x_k)}(A). \end{aligned}$$

By using the Kolmogorov's extension theorem and the one step transition probabilities, we can build a probability measure $\mathbb{P}^{x_0}_{S_{\mathrm{I}}, S_{\mathrm{II}}, \theta}$ on the game sequences. The expected payoff, when starting from x_0 and using the strategies $S_{\mathrm{I}}, S_{\mathrm{II}}$, is

$$\begin{aligned} &\mathbb{E}^{x_0}_{S_{\mathrm{I}}, S_{\mathrm{II}}, \theta} \left[F(x_\tau) - \varepsilon \sum_{i=0}^{\tau-1} \theta(x_0, \dots, x_i) \right] \\ &\qquad = \int_{H^\infty} \Big(F(x_\tau) - \varepsilon \sum_{i=0}^{\tau-1} \theta(x_0, \dots, x_i) \Big) \, d\mathbb{P}^{x_0}_{S_{\mathrm{I}}, S_{\mathrm{II}}, \theta}, \end{aligned} \tag{5.30}$$

where $F \colon \Gamma_\varepsilon \to \mathbb{R}$ is the given continuous function prescribing the terminal payoff.

The *value of the game for Player I* is given by

$$u^\varepsilon_{\mathrm{I}}(x_0) = \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}}, \theta} \mathbb{E}^{x_0}_{S_{\mathrm{I}}, S_{\mathrm{II}}, \theta} \left[F(x_\tau) - \varepsilon \sum_{i=0}^{\tau-1} \theta(x_0, \dots, x_i) \right]$$

while the *value of the game for Player II* is given by

$$u^\varepsilon_{\mathrm{II}}(x_0) = \inf_{S_{\mathrm{II}}, \theta} \sup_{S_{\mathrm{I}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}}, S_{\mathrm{II}}, \theta} \left[F(x_\tau) - \varepsilon \sum_{i=0}^{\tau-1} \theta(x_0, \dots, x_i) \right].$$

Intuitively, the values $u^\varepsilon_{\mathrm{I}}(x_0)$ and $u^\varepsilon_{\mathrm{II}}(x_0)$ are the best expected outcomes each player can guarantee when the game starts at x_0. Observe that if the game does not end almost surely, then the expectation (5.30) is undefined. In this case, we define $\mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}},\theta}$ to take value $-\infty$ when evaluating $u^{\varepsilon}_{\mathrm{I}}(x_0)$ and $+\infty$ when evaluating $u^{\varepsilon}_{\mathrm{II}}(x_0)$.

We start the analysis of our game with the statement of the *Dynamic Programming Principle* (DPP).

Lemma 5.1 (DPP)**.** *The value function for Player I satisfies for $x \in \Omega$*

$$u^{\varepsilon}_{I}(x) = \min\left\{\frac{1}{2}\sup_{y\in\overline{B}_{\varepsilon}(x)} u^{\varepsilon}_{I}(y) + \frac{1}{2}\inf_{y\in\overline{B}_{\varepsilon}(x)} u^{\varepsilon}_{I}(y);\ \sup_{y\in\overline{B}_{\varepsilon}(x)} u^{\varepsilon}_{I}(y) - \varepsilon\chi_D(x)\right\}$$

and $u^{\varepsilon}_{I}(x) = F(x)$ in Γ_{ε}. The value function for Player II, u^{ε}_{II}, satisfies the same equation.

If $u^{\varepsilon}_{\mathrm{I}} = u^{\varepsilon}_{\mathrm{II}}$, we say that the game has a value. Our game has a value, but we postpone the proof of this fact until Theorem 5.7. First we prove that the value u_{ε} of the game converges to the minimal solution of (2.4).

Theorem 5.2. *Let u_{ε} be the family of game values for a Lipschitz continuous boundary data F, and let u be the minimal solution to* (2.4) *with $F = f$ on $\partial\Omega$. Then*

$$u_{\varepsilon} \to u \quad \textit{uniformly in } \overline{\Omega}$$

as $\varepsilon \to 0$.

As a first step, we prove that, up to selecting a subsequence, $u_{\varepsilon} \to u$ as $\varepsilon \to 0$ for some Lipschitz function u.

Theorem 5.3. *Let u_{ε} be a family of game values for a Lipschitz continuous boundary data F. Then there exists a Lipschitz continuous function u such that, up to selecting a subsequence,*

$$u_{\varepsilon} \to u \quad \textit{uniformly in } \overline{\Omega}$$

as $\varepsilon \to 0$.

Proof. Since Ω is bounded, it suffices to prove asymptotic Lipschitz continuity for the family u_{ε} and then use the asymptotic version of Arzelà-Ascoli lemma from [24]. We prove the required oscillation estimate by arguing by contradiction: If there exists a point where the oscillation

$$A(x) := \sup_{y\in\overline{B}_{\varepsilon}(x)} u_{\varepsilon}(y) - \inf_{y\in\overline{B}_{\varepsilon}(x)} u_{\varepsilon}(y)$$

is large compared to the oscillation of the boundary data, then the DPP takes the same form as for the standard tug-of-war game. Intuitively, the tug-of-war never reduces the oscillation when playing to sup or inf directions. Thus we can iterate this idea up to the boundary to show that the oscillation of the boundary data must be larger than it actually is, which is the desired contradiction.

To be more precise, we claim that

$$A(x) \leq 4 \max\{\mathrm{Lip}(F); 1\}\varepsilon,$$

for all $x \in \Omega$. Aiming for a contradiction, suppose that there exists $x_0 \in \Omega$ such that

$$A(x_0) > 4 \max\{\mathrm{Lip}(F); 1\}\varepsilon.$$

In this case, we have that

$$\begin{aligned} u_\varepsilon(x_0) &= \min\Big\{\frac{1}{2}\sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon(y) + \frac{1}{2}\inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon(y); \sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon(y) - \varepsilon\chi_D\Big\} \\ &= \frac{1}{2}\sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon(y) + \frac{1}{2}\inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon(y). \end{aligned} \tag{5.31}$$

The reason is that the alternative

$$\frac{1}{2}\sup_{y\in\overline{B}_\varepsilon(x_0)} u_\varepsilon(y) + \frac{1}{2}\inf_{y\in\overline{B}_\varepsilon(x_0)} u^\varepsilon(y) > \sup_{y\in\overline{B}_\varepsilon(x_0)} u_\varepsilon(y) - \varepsilon\chi_D$$

would imply

$$A(x_0) = \sup_{y\in\overline{B}_\varepsilon(x_0)} u_\varepsilon(y) - \inf_{y\in\overline{B}_\varepsilon(x_0)} u_\varepsilon(y) < 2\varepsilon\chi_D \leq 2\varepsilon, \tag{5.32}$$

which is a contradiction with $A(x_0) > 4 \max\{\mathrm{Lip}(F); 1\}\varepsilon$. It follows from (5.31) that

$$\sup_{y\in\overline{B}_\varepsilon(x_0)} u_\varepsilon(y) - u_\varepsilon(x_0) = u_\varepsilon(x_0) - \inf_{y\in\overline{B}_\varepsilon(x_0)} u_\varepsilon(y) = \frac{1}{2}A(x_0).$$

Let $\eta > 0$ and take $x_1 \in \overline{B}_\varepsilon(x_0)$ such that

$$u_\varepsilon(x_1) \geq \sup_{y\in\overline{B}_\varepsilon(x_0)} u_\varepsilon(y) - \frac{\eta}{2}.$$

We obtain

$$u_\varepsilon(x_1) - u_\varepsilon(x_0) \geq \frac{1}{2}A(x_0) - \frac{\eta}{2} \geq 2\max\{\mathrm{Lip}(F); 1\}\varepsilon - \frac{\eta}{2},$$

and, since $x_0 \in \overline{B}_\varepsilon(x_1)$, also

$$\sup_{y\in\overline{B}_\varepsilon(x_1)} u_\varepsilon(y) - \inf_{y\in\overline{B}_\varepsilon(x_1)} u_\varepsilon(y) \geq 2\max\{\mathrm{Lip}(F); 1\}\varepsilon - \frac{\eta}{2}.$$

Arguing as before, (5.31) also holds at x_1, since otherwise the above inequality would lead to a contradiction similarly as (5.32) for small enough η. Thus

$$\sup_{y\in\overline{B}_\varepsilon(x_1)} u_\varepsilon(y) - u_\varepsilon(x_1) = u_\varepsilon(x_1) - \inf_{y\in\overline{B}_\varepsilon(x_1)} u_\varepsilon(y) \geq 2\max\{\mathrm{Lip}(F); 1\}\varepsilon - \frac{\eta}{2},$$

so that

$$\begin{aligned} A(x_1) &= \sup_{y\in\overline{B}_\varepsilon(x_1)} u_\varepsilon(y) - u_\varepsilon(x_1) + u_\varepsilon(x_1) - \inf_{y\in\overline{B}_\varepsilon(x_1)} u_\varepsilon(y) \\ &\geq 4\max\{\mathrm{Lip}(F); 1\}\varepsilon - \eta. \end{aligned}$$

Iterating this procedure, we obtain $x_i \in \overline{B}_\varepsilon(x_{i-1})$ such that

$$u_\varepsilon(x_i) - u_\varepsilon(x_{i-1}) \geq 2\max\{\mathrm{Lip}(F);1\}\varepsilon - \frac{\eta}{2^i} \tag{5.33}$$

and

$$A(x_i) \geq 4\max\{\mathrm{Lip}(F);1\}\varepsilon - (\sum_{j=0}^{i-1}\frac{\eta}{2^j}). \tag{5.34}$$

We can proceed with an analogous argument considering points where the infimum is nearly attained to obtain x_{-1}, x_{-2},... such that $x_{-i} \in \overline{B}_\varepsilon(x_{-(i-1)})$, and (5.33) and (5.34) hold. Since u_ε is bounded, there must exist k and l such that $x_k, x_{-l} \in \Gamma_\varepsilon$, and we have

$$\begin{aligned}\frac{|F(x_k) - F(x_{-l})|}{|x_k - x_{-l}|} &\geq \frac{\displaystyle\sum_{j=-l+1}^{k} u_\varepsilon(x_j) - u_\varepsilon(x_{j-1})}{\varepsilon(k+l)}\\ &\geq 2\max\{\mathrm{Lip}(F);1\} - \frac{2\eta}{\varepsilon},\end{aligned}$$

a contradiction. Therefore

$$A(x) \leq 4\max\{\mathrm{Lip}(F);1\}\varepsilon,$$

for every $x \in \Omega$. □

In order to prove Theorem 5.2, we define a modified game: the difference to the previous game is that Player II can sell turns in the whole Ω and not just when the token is in D. We refer to our original game as D-game and to the modified game as Ω-game.

Lemma 5.4 (DPP, Ω-game)**.** *The value function for Player I satisfies for* $x \in \Omega$

$$u_I^\varepsilon(x) = \min\Big\{\frac{1}{2}\sup_{y\in\overline{B}_\varepsilon(x)} u_I^\varepsilon(y) + \frac{1}{2}\inf_{y\in\overline{B}_\varepsilon(x)} u_I^\varepsilon(y);\ \sup_{y\in\overline{B}_\varepsilon(x)} u_I^\varepsilon(y) - \varepsilon\Big\}$$

and $u_I^\varepsilon(x) = F(x)$ *in* Γ_ε*. The value function for Player II,* u_{II}^ε*, satisfies the same equation.*

It will be shown in Theorem 5.6 that this game has a value $u_\varepsilon = u_{\mathrm{I}}^\varepsilon = u_{\mathrm{II}}^\varepsilon$. We start by showing that the value of the game converges to the unique solution of Jensen's equation $\min\{\Delta_\infty u, |Du| - 1\} = 0$.

Theorem 5.5. *Let* u_ε *be the family of values of the* Ω*-game for a Lipschitz continuous boundary data* F*, and let* u *be the unique solution to* (4.22) *with* $u = F$ *on* $\partial\Omega$*. Then*

$$u_\varepsilon \to u \quad \textit{uniformly in } \overline{\Omega}.$$

Proof. The convergence of a subsequence to a Lipschitz continuous function u follows by the same argument as in Theorem 5.3.

By pulling towards a boundary point, we see that $u = F$ on $\partial\Omega$, and we can focus our attention on showing that u satisfies Jensen's equation in the viscosity sense. To establish this, we consider an asymptotic expansion related to our operator.

Fix a point $x \in \Omega$ and $\phi \in C^2(\Omega)$. Let x_1^ε and x_2^ε be a minimum point and a maximum point, respectively, for ϕ in $\overline{B}_\varepsilon(x)$. As in [23], we obtain the asymptotic expansion

(5.35)

$$\begin{aligned}
&\min\left\{\frac{1}{2}\max_{y\in\overline{B}_\varepsilon(x)}\phi(y)+\frac{1}{2}\min_{y\in\overline{B}_\varepsilon(x)}\phi(y);\ \max_{y\in\overline{B}_\varepsilon(x)}\phi(y)-\varepsilon\right\}-\phi(x)\\
&\geq \min\left\{\frac{\varepsilon^2}{2}D^2\phi(x)\left(\tfrac{x_1^\varepsilon-x}{\varepsilon}\right)\cdot\left(\tfrac{x_1^\varepsilon-x}{\varepsilon}\right)+o(\varepsilon^2);\right.\\
&\qquad\qquad \left.\left(D\phi(x)\cdot\tfrac{x_2^\varepsilon-x}{\varepsilon}-1\right)\varepsilon+\frac{\varepsilon^2}{2}D^2\phi(x)\left(\tfrac{x_2^\varepsilon-x}{\varepsilon}\right)\cdot\left(\tfrac{x_2^\varepsilon-x}{\varepsilon}\right)+o(\varepsilon^2)\right\}.
\end{aligned}$$

Suppose that $u-\phi$ has a strict local minimum at x and that $D\phi(x)\neq 0$. By the uniform convergence, for any $\eta_\varepsilon > 0$ there exists a sequence (x_ε) converging to x such that

$$u_\varepsilon(x)-\phi(x)\geq u_\varepsilon(x_\varepsilon)-\phi(x_\varepsilon)-\eta_\varepsilon,$$

that is, $u_\varepsilon-\phi$ has an approximate minimum at x_ε. Moreover, considering $\tilde{\phi}=\phi-u_\varepsilon(x_\varepsilon)-\phi(x_\varepsilon)$, we may assume that $\phi(x_\varepsilon)=u_\varepsilon(x_\varepsilon)$. Thus, by recalling the fact that u_ε satisfies the DPP, we obtain

$$\eta_\varepsilon\geq -\phi(x_\varepsilon)+\min\left\{\frac{1}{2}\max_{y\in\overline{B}_\varepsilon(x)}\phi(y)+\frac{1}{2}\min_{y\in\overline{B}_\varepsilon(x)}\phi(y);\ \max_{y\in\overline{B}_\varepsilon(x)}\phi(y)-\varepsilon\right\}.$$

By choosing $\eta_\varepsilon=o(\varepsilon^2)$, using (5.35) and dividing by ε^2, we have

$$\begin{aligned}
0\geq \min&\left\{\frac{1}{2}D^2\phi(x)\left(\tfrac{x_1^\varepsilon-x}{\varepsilon}\right)\cdot\left(\tfrac{x_1^\varepsilon-x}{\varepsilon}\right)+\frac{o(\varepsilon^2)}{\varepsilon^2};\right.\\
&\left.\left(D\phi(x)\cdot\tfrac{x_2^\varepsilon-x}{\varepsilon}-1\right)\frac{1}{\varepsilon}+\frac{1}{2}D^2\phi(x)\left(\tfrac{x_2^\varepsilon-x}{\varepsilon}\right)\cdot\left(\tfrac{x_2^\varepsilon-x}{\varepsilon}\right)+\frac{o(\varepsilon^2)}{\varepsilon^2}\right\}.
\end{aligned}$$

Since $D\phi(x)\neq 0$, by letting $\varepsilon\to 0$, we conclude that

$$\Delta_\infty\phi(x)\leq 0 \quad\text{or}\quad |D\phi(x)|-1\leq 0.$$

This shows that u is a viscosity supersolution to Jensen's equation (4.22), provided that $D\phi(x)\neq 0$. On the other hand, if $D\phi=0$, then $\Delta_\infty\phi(x)=0$ and the same conclusion follows.

To prove that u is a viscosity subsolution, we consider a function φ that touches u from above at $x \in \Omega$ and observe that a reverse inequality to (5.35) holds at the point of touching. Arguing as above, one can deduce that

$$\Delta_\infty \varphi(x) \geq 0 \quad \text{and} \quad |D\varphi(x)| - 1 \geq 0.$$

To finish the proof, we observe that since the viscosity solution of (4.22) is unique, all the subsequential limits of (u_ε) are equal. □

Next we prove that the value u_ε of the D-game converges to the minimal solution of $\min\{\Delta_\infty u, |Du| - \chi_D\} = 0$.

Proof of Theorem 5.2. Let h_ε, u_ε and z_ε denote the values of the standard tug-of-war, the D-game, and the Ω-game, respectively. Since Player II has more options in D-game and again more in Ω-game, we have

$$z_\varepsilon(x) \leq u_\varepsilon(x) \leq h_\varepsilon(x) \quad \text{for all } x \in \Omega. \tag{5.36}$$

As in the proof of Theorem 4.2, we denote by h the unique solution to the infinity Laplace equation,

$$\mathcal{A} = \{x \in \Omega \colon\ |Dh(x)| < 1\}, \qquad \mathcal{B} = \mathcal{A} \cap D,$$

and by z the unique solution to the Jensen's equation (4.22).

We claim that

$$u_\varepsilon \to z \quad \text{in} \quad \mathcal{B}.$$

Striving for a contradiction, suppose that there is $x_0 \in \mathcal{B}$ such that

$$u_\varepsilon(x_0) - z(x_0) > C \tag{5.37}$$

for all $\varepsilon > 0$. We recall from Theorem 5.5 and [27] that

$$z_\varepsilon \to z \quad \text{and} \quad h_\varepsilon \to h \quad \text{uniformly in } \overline{\Omega}.$$

Moreover, Theorem 2.3 implies that $z(x) = h(x)$ in $\Omega \setminus \mathcal{A}$ and this together with (5.36) yields

$$h = z \leq u_\varepsilon + o(1) \leq z + o(1) = h + o(1) \tag{5.38}$$

in $\Omega \setminus \mathcal{A}$ with a uniform error term.

We will next show that

$$\delta(x_0) := \sup_{y \in \overline{B}(x_0)} u_\varepsilon(y) - u_\varepsilon(x_0) \geq \varepsilon. \tag{5.39}$$

Indeed, looking at the DPP in Lemma 5.1, we have two alternatives. The first alternative is

$$u_\varepsilon(x_0) = \frac{1}{2}\Big\{ \inf_{y \in \overline{B}(x_0)} u_\varepsilon(y) + \sup_{y \in \overline{B}(x_0)} u_\varepsilon(y) \Big\} < \sup_{y \in \overline{B}(x_0)} u_\varepsilon(y) - \varepsilon \chi_D(x_0).$$

Since $x_0 \in D$, this implies that

$$2\varepsilon < \sup_{y\in\overline{B}(x_0)} u_\varepsilon(y) - \inf_{y\in\overline{B}(x_0)} u_\varepsilon(y)$$

and

$$\sup_{y\in\overline{B}(x_0)} u_\varepsilon(y) - u_\varepsilon(x_0) = u_\varepsilon(x_0) - \inf_{y\in\overline{B}(x_0)} u_\varepsilon(y), \tag{5.40}$$

from which we deduce $\delta(x_0) > \varepsilon$. The second alternative is

$$u_\varepsilon(x_0) = \sup_{y\in\overline{B}(x_0)} u_\varepsilon(y) - \varepsilon \tag{5.41}$$

which implies $\delta(x_0) = \varepsilon$, and the claim (5.39) follows.

Let $\eta > 0$ and choose a point x_1 so that

$$u_\varepsilon(x_1) \geq \sup_{y\in\overline{B}(x_0)} u_\varepsilon(y) - \eta 2^{-1}.$$

It follows from (5.39) that $u_\varepsilon(x_1) - \inf_{y\in\overline{B}(x_1)} u_\varepsilon(y) \geq \varepsilon - \eta 2^{-1}$. Moreover, in the case of the first alternative at x_1 it holds that $\delta(x_1) \geq \varepsilon - \eta 2^{-1}$ by the equation similar to (5.40), and in the case of the second alternative, the equation similar to (5.41) implies that $\delta(x_1) = \varepsilon$.

We iterate the argument and obtain a sequence of points (x_k) such that

$$u_\varepsilon(x_k) \geq u_\varepsilon(x_0) + k\varepsilon - \eta \sum_{i=1}^{\infty} 2^{-i}. \tag{5.42}$$

The sequence exits $\mathcal{A}$ in a finite number of steps, i.e., there exists a first point x_{k_0} in the sequence such that $x_{k_0} \in \Omega \setminus \mathcal{A}$. This follows from (5.42) and the boundedness of u_ε. On the other hand, since $|z(x) - z(y)| \leq |x - y|$ whenever the line segment $[x, y]$ is contained in $\mathcal{A}$ (see [10]), we have

$$z(x_{k_0}) \leq z(x_0) + k\varepsilon + C\varepsilon,$$

where the term $C\varepsilon$ is due to the last step being partly outside $\mathcal{A}$. By this estimate, (5.38) and (5.42), we obtain

$$o(1) \geq u_\varepsilon(x_{k_0}) - z(x_{k_0}) \geq u_\varepsilon(x_0) - z(x_0) - \eta - C\varepsilon.$$

This gives a contradiction with (5.37) provided we choose η and ε small enough.

We have

$$u_\varepsilon \to z \quad \text{in} \quad \mathcal{A} \cap D \quad \text{and} \quad u_\varepsilon \to h \quad \text{in} \quad \Omega \setminus \mathcal{A}.$$

But in $\mathcal{A} \setminus D$, the D-game is just a tug-of-war, and by [27], u_ε converges to the unique solution to

$$\begin{cases} \Delta_\infty u = 0, & \text{in } \mathcal{A} \setminus D \\ u = h, & \text{on } \partial A \setminus D \\ u = z, & \text{on } \partial D \cap \mathcal{A}. \end{cases}$$

This ends the proof. $\square$

Now let us show that there is a value for the Ω-game.

Theorem 5.6. *The Ω-game has a value, i.e. $u^\varepsilon_I = u^\varepsilon_{II}$.*

Proof. By definition, $u^\varepsilon_{\mathrm{I}} \le u^\varepsilon_{\mathrm{II}}$, and thus it remains to prove the opposite inequality. Observe that by pulling towards a boundary point, Player II can end the game almost surely and thus $u^\varepsilon_{\mathrm{II}} < \infty$. Let

$$\delta(x) := \sup_{y\in \overline{B}_\varepsilon(x)} u^\varepsilon_{\mathrm{II}}(y) - u^\varepsilon_{\mathrm{II}}(x).$$

Suppose that Player I uses a strategy S^0_{I}, in which she always chooses to step to a point that almost maximizes $u^\varepsilon_{\mathrm{II}}$, that is, to a point x_k such that

$$u^\varepsilon_{\mathrm{II}}(x_k) \ge \sup_{y\in \overline{B}_\varepsilon(x_{k-1})} u^\varepsilon_{\mathrm{II}}(y) - \eta 2^{-k},$$

for a fixed $\eta > 0$. We claim that $m_k = u^\varepsilon_{\mathrm{II}}(x_k) - \eta 2^{-k}$ is a submartingale. Indeed, it follows by the DPP that

$$u^\varepsilon_{\mathrm{II}}(x_k) - \inf_{\overline{B}_\varepsilon(x_k)} u^\varepsilon_{\mathrm{II}}(y) \le \sup_{\overline{B}_\varepsilon(x_k)} u^\varepsilon_{\mathrm{II}}(y) - u^\varepsilon_{\mathrm{II}}(x_k) = \delta(x_k), \tag{5.43}$$

and thus

$$\mathbb{E}^{x_0}_{S^0_{\mathrm{I}},S_{\mathrm{II}},\theta}[u^\varepsilon_{\mathrm{II}}(x_k) - \eta 2^{-k} \,|\, x_0,\dots,x_{k-1}] \ge u^\varepsilon_{\mathrm{II}}(x_{k-1}) - \eta 2^{-(k-1)}.$$

From the submartingale property it follows that the limit $\lim_{k\to\infty} m_{\tau\wedge k}$ exists by the martingale convergence theorem. Furthermore, at every point $x \in \Omega$ either

$$u^\varepsilon_{\mathrm{II}}(x) = \frac{1}{2}\left\{ \inf_{y\in \overline{B}_\varepsilon(x)} u^\varepsilon_{\mathrm{II}}(y) + \sup_{y\in \overline{B}_\varepsilon(x)} u^\varepsilon_{\mathrm{II}}(y)\right\} < \sup_{y\in \overline{B}_\varepsilon(x)} u^\varepsilon_{\mathrm{II}}(y) - \varepsilon$$

implying

$$\varepsilon < \sup_{y\in \overline{B}_\varepsilon(x)} u^\varepsilon_{\mathrm{II}}(y) - u^\varepsilon_{\mathrm{II}}(x), \tag{5.44}$$

or

$$u^\varepsilon_{\mathrm{II}}(x) = \sup_{y\in \overline{B}_\varepsilon(x)} u^\varepsilon_{\mathrm{II}}(y) - \varepsilon.$$

Hence

$$\sup_{y\in \overline{B}_\varepsilon(x)} u^\varepsilon_{\mathrm{II}}(y) - u^\varepsilon_{\mathrm{II}}(x) = \varepsilon. \tag{5.45}$$

Thus $\delta(x) \ge \varepsilon$ always. On the other hand, there are arbitrary long sequences of moves made by Player I. Indeed, if Player II sells a turn, then Player I gets to move, and otherwise this is a consequence of the zero-one law. Since m_k is a bounded submartingale, these two facts imply that the game must end almost surely.

By a similar argument utilizing the fact $\delta(x) \geq \varepsilon$, we see that

$$u^\varepsilon_{\mathrm{II}}(x_k) - \varepsilon \sum_{i=0}^{k-1} \theta(x_0, \dots, x_i) - \eta 2^{-k}$$

is a submartingale as well. It then follows from Fatou's lemma and the optional stopping theorem that

$$\begin{aligned}
u^\varepsilon_{\mathrm{I}}(x_0) &= \sup_{S_{\mathrm{I}}} \inf_{S_{\mathrm{II}},\theta} \mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}},\theta}[F(x_\tau) - \varepsilon \sum_{i=0}^{\tau-1} \theta(x_0, \dots, x_i)] \\
&\geq \inf_{S_{\mathrm{II}},\theta} \mathbb{E}^{x_0}_{S^0_{\mathrm{I}},S_{\mathrm{II}},\theta}[F(x_\tau) - \varepsilon \sum_{i=0}^{\tau-1} \theta(x_0, \dots, x_i) - \eta 2^{-\tau}] \\
&\geq \inf_{S_{\mathrm{II}},\theta} \limsup_{k\to\infty} \mathbb{E}^{x_0}_{S^0_{\mathrm{I}},S_{\mathrm{II}},\theta}[u^\varepsilon_{\mathrm{II}}(x_{\tau\wedge k}) - \varepsilon \sum_{i=0}^{\tau\wedge k-1} \theta(x_0, \dots, x_i) - \eta 2^{-(\tau\wedge k)}] \\
&\geq \inf_{S_{\mathrm{II}},\theta} \mathbb{E}_{S^0_{\mathrm{I}},S_{\mathrm{II}},\theta}[u^\varepsilon_{\mathrm{II}}(x_0) - \eta] = u^\varepsilon_{\mathrm{II}}(x_0) - \eta.
\end{aligned}$$

This implies that $u^\varepsilon_{\mathrm{I}} \geq u^\varepsilon_{\mathrm{II}}$. □

Now, let us prove the analogous statement for the D-game.

Theorem 5.7. *The D-game has a value, i.e. $u^\varepsilon_I = u^\varepsilon_{II}$.*

Proof. The proof is quite similar to the proof of Theorem 5.6, but when tug-of-war is played outside D we have to make sure that

$$\delta(x) = \sup_{y\in \overline{B}_\varepsilon(x)} u^\varepsilon_{\mathrm{II}}(y) - u^\varepsilon_{\mathrm{II}}(x)$$

is large enough. This is done by using the backtracking strategy, cf. Theorem 2.2 of [27].

Fix $\eta > 0$ and a starting point $x_0 \in \Omega$, and set $\delta_0 = \min\{\delta(x_0), \varepsilon\}/2$. We suppose for now that $\delta_0 > 0$, and define

$$X_0 = \Big\{ x \in \Omega \,:\, \delta(x) > \delta_0 \Big\}.$$

Observe that $D \subset X_0$ by estimates similar to (5.44) and (5.45).

We consider a strategy S^0_I for Player I that distinguishes between the cases $x_k \in X_0$ and $x_k \notin X_0$. First, if $x_k \in X_0$, then she always chooses to step to a point x_{k+1} satisfying

$$u^\varepsilon_{\mathrm{II}}(x_{k+1}) \geq \sup_{y\in \overline{B}_\varepsilon(x_k)} u^\varepsilon_{\mathrm{II}}(y) - \eta_{k+1} 2^{-(k+1)},$$

where $\eta_{k+1} \in (0, \eta]$ is small enough to guarantee that $x_{k+1} \in X_0$. Thus if $x_k \in X_0$ and Player I gets to choose the next position (by winning the coin toss or through the selling of the turn by the other player), for

$$m_k = u^\varepsilon_{\mathrm{II}}(x_k) - \eta 2^{-k}$$

it holds that

$$\begin{aligned} m_{k+1} &\geq u^{\varepsilon}_{\mathrm{II}}(x_k) + \delta(x_k) - \eta_{k+1}2^{-(k+1)} - \eta 2^{-(k+1)} \\ &\geq u^{\varepsilon}_{\mathrm{II}}(x_k) + \delta(x_k) - \eta 2^{-k} \\ &= m_k + \delta(x_k). \end{aligned}$$

On the other hand, if Player II wins the toss and moves from $x_k \in X_0$ to $x_{k+1} \in X_0$, it holds, in view of (5.43), that

$$m_{k+1} \geq u^{\varepsilon}_{\mathrm{II}}(x_k) - \delta(x_k) - \eta 2^{-(k+1)} > m_k - \delta(x_k).$$

In the case $x_k \notin X_0$, we set

$$m_k = u^{\varepsilon}_{\mathrm{II}}(y_k) - \delta_0 d_k - \eta 2^{-k},$$

where y_k denotes the last game position in X_0 up to time k, and d_k is the distance, measured in number of steps, from x_k to y_k along the graph spanned by the previous points $y_k = x_{k-j}, x_{k-j+1}, \ldots, x_k$ that were used to get from y_k to x_k. The strategy for Player I in this case is to backtrack to y_k, that is, if she wins the coin toss, she moves the token to one of the points $x_{k-j}, x_{k-j+1}, \ldots, x_{k-1}$ closer to y_k so that $d_{k+1} = d_k - 1$. Thus if Player I wins and $x_k \notin X_0$ (whether $x_{k+1} \in X_0$ or not),

$$m_{k+1} \geq \delta_0 + m_k.$$

To prove the desired submartingale property for m_k, there are three more cases to be checked. If Player II wins the toss and he moves to a point $x_{k+1} \notin X_0$ (whether $x_k \in X_0$ or not), it holds that

$$\begin{aligned} m_{k+1} &= u^{\varepsilon}_{\mathrm{II}}(y_k) - d_{k+1}\delta_0 - \eta 2^{-(k+1)} \\ &\geq u^{\varepsilon}_{\mathrm{II}}(y_k) - d_k\delta_0 - \delta_0 - \eta 2^{-k} \\ &= m_k - \delta_0. \end{aligned}$$

If Player II wins the coin toss and moves from $x_k \notin X_0$ to $x_{k+1} \in X_0$, then

$$m_{k+1} = u^{\varepsilon}_{\mathrm{II}}(x_{k+1}) - \eta 2^{-(k+1)} \geq -\delta(x_k) + u^{\varepsilon}_{\mathrm{II}}(x_k) - \eta 2^{-k} \geq -\delta_0 + m_k$$

where the first inequality is due to (5.43), and the second follows from the fact $m_k = u^{\varepsilon}_{\mathrm{II}}(y_k) - d_k\delta_0 - \eta 2^{-k} \leq u^{\varepsilon}_{\mathrm{II}}(x_k) - \eta 2^{-k}$.

Taking into account all the different cases, we see that m_k is a bounded (from above) submartingale, and since Player I can assure that $m_{k+1} \geq m_k + \delta_0$ if she wins a coin toss, the game must again terminate almost surely. We can now conclude the proof similarly as in the case of Theorem 5.6; recall that $\delta(x_k) \geq \varepsilon$ whenever x_k in D.

Finally, let us remove the assumption that $\delta(x_0) > 0$. If $\delta(x_0) = 0$ for $x_0 \in X$, then Player I adopts a strategy of pulling towards a boundary point until the game token reaches a point x_0' such that $\delta(x_0') > 0$ or x_0' is outside Ω. It holds that $u^{\varepsilon}_{\mathrm{II}}(x_0) = u^{\varepsilon}_{\mathrm{II}}(x_0')$, because by (5.43) it cannot happen that $\delta(x) = \sup_{y\in\overline{B}_\varepsilon(x)} u^\varepsilon_{\mathrm{II}}(y) - u^\varepsilon_{\mathrm{II}}(x) = 0$ and $u^\varepsilon_{\mathrm{II}}(x) - \inf_{y\in\overline{B}_\varepsilon(x)} u^\varepsilon_{\mathrm{II}}(y) > 0$ simultaneously. Thus we can repeat the proof also in this case. □

6. L^∞-viscosity solutions

In this section, we outline another approach to the problem (2.4)

$$\begin{cases} \min\{\Delta_\infty u, |Du| - \chi_D\} = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases}$$

The idea is to regard χ_D as a bounded, measurable function, defined only up to a set of measure zero, and to accommodate the set of test-functions to this interpretation. This point of view fits well with the approximation of (2.4) by the equations $\Delta_p u_p = \chi_D$, but it turns out to be incompatible with the game approach at least in some cases.

6.1. **The approximating p-Laplace equations.** We begin by recalling the definition of L^∞-viscosity solutions for the approximating p-Laplace equations. For simplicity, we consider only the equation

$$\Delta_p u = \chi_D, \tag{6.46}$$

and leave the more general version $\Delta_p u = g$, with g non-negative and bounded, to the reader. As before, the boundary conditions are understood in the classical sense. For more on L^∞-viscosity solutions, see e.g. [6].

Definition 6.1. *A continuous function $u\colon \Omega \to \mathbb{R}$ is an L^∞-*viscosity sub-solution *of* (6.46) *if, whenever $x \in \Omega$ and $\varphi \in W^{2,\infty}_{loc}(\Omega)$ are such that $u - \varphi$ has a strict local maximum at x, then*

$$\operatorname*{ess\,limsup}_{y\to x} \left(\Delta_p \varphi(y) - \chi_D(y)\right) \geq 0.$$

*A continuous function $v\colon \Omega \to \mathbb{R}$ is an L^∞-*viscosity supersolution *of* (6.46) *if, whenever $x \in \Omega$ and $\phi \in W^{2,\infty}_{loc}(\Omega)$ are such that $v - \phi$ has a strict local minimum at x, then*

$$\operatorname*{ess\,liminf}_{y\to x} \left(\Delta_p \phi(y) - \chi_D(y)\right) \leq 0.$$

*Finally, a continuous function $h\colon \Omega \to \mathbb{R}$ is an L^∞-*viscosity solution *of* (6.46) *if it is both a viscosity subsolution and a viscosity supersolution.*

Proposition 6.2. *A continuous weak solution of* (6.46) *is an L^∞-viscosity solution.*

Proof. The proof is almost the same as that of Proposition 3.4. The difference is that the counter proposition holds almost everywhere. Nonetheless, the proof utilizes weak solutions and thus a set of measure zero makes no difference. We leave the details to the reader. □

6.2. **The gradient constraint problem.**

Definition 6.3. *A continuous function* $u\colon \Omega \to \mathbb{R}$ *is an* L^∞-viscosity subsolution *of* (2.5) *if, whenever* $x \in \Omega$ *and* $\varphi \in W^{2,\infty}_{loc}(\Omega)$ *are such that* $u - \varphi$ *has a strict local maximum at* x*, then*

$$\operatorname*{ess\,limsup}_{y\to x}\Big(\min\{\Delta_\infty \varphi(y), |D\varphi(y)| - \chi_D(y)\}\Big) \geq 0.$$

A continuous function $v\colon \Omega \to \mathbb{R}$ *is an* L^∞-viscosity supersolution *of* (2.5) *if, whenever* $x \in \Omega$ *and* $\phi \in W^{2,\infty}_{loc}(\Omega)$ *are such that* $v - \phi$ *has a strict local minimum at* x*, then*

$$\operatorname*{ess\,liminf}_{y\to x}\Big(\min\{\Delta_\infty \phi(y), |D\phi(y)| - \chi_D(y)\}\Big) \leq 0.$$

Finally, a continuous function $h\colon \Omega \to \mathbb{R}$ *is an* L^∞-viscosity solution *of* (2.5) *if it is both a viscosity subsolution and a viscosity supersolution.*

Our next result says that L^∞-viscosity solutions are viscosity solutions (in the sense of Definition 2.1). This holds for subsolutions and supersolutions as well. However, the converse is not true, as explained after Corollary 6.5.

Lemma 6.4. *An* L^∞*-viscosity subsolution* u *to* (2.4) *is a viscosity subsolution. Similarly, an* L^∞*-viscosity supersolution* v *to* (2.4) *is a viscosity supersolution.*

Proof. Let us first prove the claim about subsolutions. Let $\varphi \in C^2(\Omega)$ and $x \in \Omega$ be such that $u - \varphi$ has a strict local maximum at x. We want to show that

$$\Delta_\infty \varphi(x) \geq 0 \quad \text{and} \quad |D\varphi(x)| - \chi_{\operatorname{int} D}(x) \geq 0. \tag{6.47}$$

Since u is an L^∞-viscosity subsolution to (2.4), we have

$$\operatorname*{ess\,limsup}_{y\to x}\Big(\min\{\Delta_\infty \varphi(y), |D\varphi(y)| - \chi_D(y)\}\Big) \geq 0. \tag{6.48}$$

Observe that since the map $y \mapsto \Delta_\infty \varphi(y)$ is continuous, we immediately have $\Delta_\infty\varphi(x) \geq 0$. If $x \in \operatorname{int} D$, then also $y \mapsto \chi_D(y)$ is continuous in a neighborhood of x, and (6.48) implies (6.47) as desired. On the other hand, if $x \notin \operatorname{int} D$, then $|D\varphi(x)| - \chi_{\operatorname{int} D}(x) \geq 0$ holds trivially, and we are done.

To prove the supersolution case, let $\phi \in C^2(\Omega)$ and $x \in \Omega$ be such that $u - \phi$ has a strict local minimum at x. We want to show that

$$\Delta_\infty \phi(x) \leq 0 \quad \text{or} \quad |D\phi(x)| - \chi_{\overline{D}}(x) \leq 0. \tag{6.49}$$

Since u is an L^∞-viscosity supersolution to (2.4), we have

$$\operatorname*{ess\,liminf}_{y\to x}\Big(\min\{\Delta_\infty \phi(y), |D\phi(y)| - \chi_D(y)\}\Big) \leq 0. \tag{6.50}$$

Let us suppose that $\Delta_\infty \phi(x) > 0$. Then $|D\phi(x)| > 0$, and we must have $x \in \overline{D}$, for otherwise we would contradict (6.50). In fact, (6.50) implies that

$$|D\phi(x)| \leq \operatorname{ess} \limsup_{y \to x} \chi_D(y) \leq 1 = \chi_{\overline{D}}(x),$$

which completes the proof. □

Lemma 6.4 implies that if (2.4) has a unique viscosity solution, then it also has a unique L^∞-viscosity solution. In particular, we have

Corollary 6.5. *Suppose that $\overline{D} = \overline{\operatorname{int} D}$. Then* (2.4) *has a unique L^∞-viscosity solution.*

Proof. By Theorem 4.1, (2.4) has a unique viscosity solution. But by Lemma 6.4, any L^∞-viscosity solution is a viscosity solution, and thus there can be at most one. The existence of an L^∞-viscosity solution follows from L^p approximation, see Lemma 6.7 below. □

It is quite obvious that the uniqueness for L^∞-viscosity solutions holds in certain cases where there are several viscosity solutions to the problem (2.4). For example, if $|D| = 0$, then u is an L^∞-viscosity solution to (2.4) if and only if it is a solution to the infinity Laplace equation. On the other hand, Lemma 4.6 shows that if $|D| = 0$ and $\operatorname{Lip}(f, \partial\Omega) < 1$, then there are multiple viscosity solutions to (2.4). This also shows that a viscosity solution need not be an L^∞-viscosity solution.

More generally, by mimicking the proof of Theorem 4.2, one can prove the following

Theorem 6.6. *Let $D \subset \Omega$ be the set in* (2.4) *and suppose that there exists $D' \subset \Omega$ for which $\overline{D'} = \overline{\operatorname{int} D'}$ and the symmetric difference*

$$D \triangle D' = (D \setminus D') \cup (D' \setminus D)$$

has measure zero. Then the problem (2.4) *has a unique L^∞-viscosity solution.*

Finally, we address the question of existence of L^∞-viscosity solutions. Recall from Lemma 3.5 that if f is Lipschitz, there exists a subsequence of (u_p), where $\Delta_p u_p = \chi_D$ in Ω and $u = f$ on $\partial\Omega$, and a function $u_\infty \in W^{1,\infty}(\Omega)$ such that

$$\lim_{p \to \infty} u_p(x) = u_\infty(x)$$

uniformly in $\overline{\Omega}$. We already know that u_∞ is a viscosity solution to (2.4), and next we show that it is also an L^∞-viscosity solution to this equation.

Lemma 6.7. *A uniform limit u_∞ of a subsequence u_p as $p \to \infty$ is an L^∞-viscosity solution to* (2.4).

Proof. That $u = f$ on $\partial\Omega$ is immediate from the uniform convergence.

Now, let us first check that u_∞ is an L^∞-viscosity subsolution. To this end, let us fix $\varphi \in W^{2,\infty}_{loc}(\Omega)$ such that $u - \varphi$ has a strict local maximum at some $x \in \Omega$. By the uniform convergence of a subsequence u_p to u_∞ there are points x_p such that $u_p - \varphi$ has a minimum at x_p and $x_p \to x$ as $p \to \infty$. At those points we have

$$\operatorname*{ess\,limsup}_{y \to x_p} \Big(\Delta_p \varphi(y) - \chi_D(y) \Big) \geq 0.$$

We show first that

$$\operatorname*{ess\,limsup}_{y \to x} \Delta_\infty \varphi(y) \geq 0. \tag{6.51}$$

We argue by contradiction, and suppose that there is $r > 0$ and $\varepsilon > 0$ such that

$$\Delta_\infty \varphi \leq -\varepsilon < 0 \quad \text{a.e. in } B_r(x).$$

Observe that this implies $|D\varphi| > 0$ a.e. in $B_r(x)$. Denoting

$$M_1 = \|D\varphi\|_{L^\infty(B_{2r}(x))} \qquad \text{and} \qquad M_2 = \|D^2\varphi\|_{L^\infty(B_{2r}(x))},$$

we have

$$\Delta\varphi + (p-2)\,|D\varphi|^{-2}\,\Delta_\infty\varphi \leq nM_2 - (p-2)\frac{\varepsilon}{M_1^2}$$

a.e. in $B_r(x)$. In particular, for p large enough this expression is negative, and hence we have that

$$\begin{aligned}\Delta_p\varphi &= |D\varphi|^{p-2}\Big(\Delta\varphi + (p-2)\,|D\varphi|^{-2}\,\Delta_\infty\varphi\Big)\\ &\leq \left(\frac{\varepsilon}{M_2}\right)^{(p-2)/2}(nM_2 - (p-2)\frac{\varepsilon}{M_1^2}) < 0\end{aligned}$$

a.e. in $B_r(x)$. This contradicts the fact that

$$\operatorname*{ess\,limsup}_{y \to x_p} \Delta_p\varphi(y) \geq \operatorname*{ess\,limsup}_{y \to x_p} \Big(\Delta_p\varphi(y) - \chi_D(y)\Big) \geq 0,$$

and thus (6.51) must hold.

Next we show that

$$\operatorname*{ess\,limsup}_{y \to x} (|D\varphi(y)| - \chi_D(y)) \geq 0. \tag{6.52}$$

We again argue by contradiction, and suppose that there is $r > 0$ and $\varepsilon > 0$ such that

$$|D\varphi| - \chi_D \leq -\varepsilon < 0 \quad \text{a.e. in } B_r(x).$$

Thus $|B_r(x) \setminus D| = 0$ and $|D\varphi| \leq 1 - \varepsilon$ a.e. in $B_r(x)$. This implies that

$$\begin{aligned}\Delta_p\varphi - \chi_D &= |D\varphi|^{p-2}\Big(\Delta\varphi + (p-2)\,|D\varphi|^{-2}\,\Delta_\infty\varphi\Big) - 1\\ &\leq (1-\varepsilon)^{p-2}(n+p-2)M_2 - 1,\end{aligned}$$

a.e. in $B_r(x)$. The last expression on the right is negative if p is large enough, and we arrive to a contradiction by arguing as above. Hence (6.52) is valid, and together with (6.51) this implies that u_∞ is an L^∞-viscosity subsolution to (2.4).

To prove that u_∞ is also an L^∞-viscosity supersolution, we fix $\phi \in W^{2,\infty}_{loc}(\Omega)$ such that $u - \phi$ has a strict local minimum at some $x \in \Omega$. Again set $M_1 = \|D\phi\|_{L^\infty(B_{2r}(x))}$ and $M_2 = \|D^2\phi\|_{L^\infty(B_{2r}(x))}$. We have to show that

$$\operatorname*{ess\,liminf}_{y\to x} \Big(\min\{\Delta_\infty \phi(y),\, |D\phi(y)| - \chi_D(y)\}\Big) \le 0.$$

Suppose this is not the case. Then there are $r, \varepsilon > 0$ such that

$$\Delta_\infty \phi(y) \ge \varepsilon \quad \text{and} \quad |D\phi(y)| - \chi_D(y) \ge \varepsilon$$

a.e. in $B_r(x)$. Then

$$\Delta\phi + (p-2)\,|D\phi|^{-2}\,\Delta_\infty\phi \ge -nM_2 + (p-2)\frac{\varepsilon}{M_1^2} > 0 \quad \text{a.e. in } B_r(x)$$

for p large enough, and hence for such p's,

$$\begin{aligned}
&\Delta_p\phi - \chi_D = |D\phi|^{p-2}\Big(\Delta\phi + (p-2)\,|D\phi|^{-2}\,\Delta_\infty\phi\Big) - \chi_D \\
&\ge (\chi_D(y)+\varepsilon)^{p-2}\,((p-2)\frac{\varepsilon}{M_1^2} - nM_2) - \chi_D \\
&\ge \min\{\varepsilon^{p-2}((p-2)\frac{\varepsilon}{M_1^2} - nM_2), (1+\varepsilon)^{p-2}\,((p-2)\frac{\varepsilon}{M_1^2} - nM_2) - 1\} > 0
\end{aligned}$$

a.e. in $B_r(x)$. Recalling that by the uniform convergence of u_p to u_∞ there are points x_p such that $u_p - \phi$ has a minimum at x_p with $x_p \to x$ as $p \to \infty$, and that u_p's are L^∞-viscosity supersolutions to (3.9), we have a contradiction. □

7. An application: asymptotic behavior for p-Laplace problems

Given functions $g \in L^\infty(\Omega)$ and $f\colon \partial\Omega \to \mathbb{R}$ that is Lipschitz continuous, we consider, for every $p > 2$, the solution u_p to the elliptic problem

$$\begin{cases} \Delta_p u = g & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{7.53}$$

Our aim is to apply the results of the preceding sections to study the limit as $p \to \infty$ of the functions u_p. In particular, we want to see how this limit depends on the data f and g.

The case $f = 0$ was already considered in [17] (see also [5], [18]), where the authors prove that there is a uniform limit that depends on g. In particular, it is proved there that when g does not change sign and $f = 0$ then the limit is unique and depends only on the support of g.

7.1. **The case** $\mathrm{Lip}(f, \partial\Omega) \le 1$**.** The solution to (7.53) for a given p admits a variational characterization, namely, it is the unique minimizer of the functional

$$J_p(u) = \frac{1}{p}\int_\Omega |Du|^p\,dx + \int_\Omega gu\,dx$$

in the set $K_p = \{u \in W^{1,p}(\Omega) : u = f \text{ on } \partial\Omega\}$.

We proved in Lemma 3.8 that any subsequential limit u_∞ of u_p's satisfies

$$\|Du_\infty\|_{L^\infty(\Omega)} \le \max\{\mathrm{Lip}(f), 1\} = 1,$$

and from this it follows that u_∞ minimizes the functional

$$J_\infty(u) = \int_\Omega gu$$

in the set $K_\infty = \{u \in W^{1,\infty}(\Omega) : \|Du\|_{L^\infty(\Omega)} \le 1 \text{ and } u = f \text{ on } \partial\Omega\}$. Indeed, since for any $v \in K_\infty$,

$$\int_\Omega gu_p\,dx \le \frac{1}{p}\int_\Omega |Du_p|^p\,dx + \int_\Omega gu_p\,dx \le \frac{|\Omega|}{p} + \int_\Omega gv\,dx,$$

the claim follows from the uniform convergence $u_p \to u_\infty$.

In certain cases, the problem of minimizing J_∞ has clearly a unique solution. For example, if $g > 0$ in Ω, then the unique minimizer is given by

$$u(x) = \max_{y\in\partial\Omega}\Big\{f(y) - |x-y|\Big\}.$$

However, if $\mathrm{Lip}(f, \partial\Omega) > 1$, then it is not so easy to identify u_∞ as a minimizer of some variational problem, and we have to do something else.

7.2. **The general case.** Let g be continuous and non-negative. Then the non-degeneracy condition (3.10) clearly holds, and thus Lemma 3.6 implies that any subsequential limit u_∞ of u_p's is a viscosity solution to

$$\min\{\Delta_\infty u, |Du| - \chi_D\} = 0, \tag{7.54}$$

where $D = \{x \in \Omega\colon g(x) > 0\}$. Since, by the continuity of g, we have $\overline{\mathrm{int}\,D} = \overline{D}$, Theorem 4.1 says that (7.54) has a unique solution. Therefore, recalling Remark 4.3, we have proved the following result:

Theorem 7.1. *Let $g \ge 0$ be continuous. Then, the limit of the solutions u_p as $p \to \infty$ is characterized by being the unique solution to* (7.54) *with boundary datum f. In particular, the limit depends on g only through the set*

$$\mathrm{supp}(g) \cap \{x \in \Omega\colon |Dh(x)| < 1\},$$

where h stands for the unique solution to the infinity Laplace equation with $h = f$ on $\partial\Omega$.

References

[1] G. Aronsson. *Extensions of functions satisfying Lipschitz conditions*. Ark. Mat. 6 (1967), 551–561.

[2] G. Aronsson, M.G. Crandall and P. Juutinen, *A tour of the theory of absolutely minimizing functions*. Bull. Amer. Math. Soc., 41 (2004), 439–505.

[3] E.N. Barron, L.C. Evans and R. Jensen, *The infinity laplacian, Aronsson's equation and their generalizations*. Trans. Amer. Math. Soc. 360, (2008), 77–101.

[4] E.N. Barron and R. Jensen, *Minimizing the L^∞ norm of the gradient with an energy constraint*, Comm. Partial Differential Equations, 30, (2005), no. 10-12, 1741–1772.

[5] T. Bhattacharya, E. DiBenedetto and J.J. Manfredi. *Limits as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems*. Rend. Sem. Mat. Univ. Politec. Torino, (1991), 15–68.

[6] L. Caffarelli, M. G. Crandall, M. Kocan, and A. Swiech. *On viscosity solutions of fully nonlinear equations with measurable ingredients*, Comm. Pure Appl. Math. 49 (1996), no. 4, 365–397.

[7] V. Caselles, J.M. Morel and C. Sbert, *An axiomatic approach to image interpolation*, IEEE Trans. Image Process. 7 (1998), no. 3, 376–386.

[8] G. Cong, M. Esser, B. Parvin and G. Bebis, *Shape metamorphism using p-Laplacian equation*, Proc. of the 17th International Conference on Pattern Recognition ICPR 2004.

[9] M.G. Crandall, L. C. Evans and R. F. Gariepy. *Optimal Lipschitz extensions and the infinity Laplacian*, Calc. Var. Partial Differential Equations 13 (2001), no. 2, 123–139.

[10] M.G. Crandall, G. Gunnarsson and P. Y. Wang. *Uniqueness of infinity harmonic functions and the eikonal equation*, Comm. Partial Differential Equations, 32, 1587–1615 (2007).

[11] L.C. Evans. Partial Differential Equations. Grad. Stud. Math. 19, Amer. Math. Soc., 1998.

[12] L.C. Evans and C. K. Smart, *Everywhere differentiability of infinity harmonic functions*. Calc. Var. Partial Differential Equations 42 (2011), no. 1-2, 289–299.

[13] M.G. Crandall, H. Ishii and P.L. Lions. *User's guide to viscosity solutions of second order partial differential equations*. Bull. Amer. Math. Soc., 27 (1992), 1–67.

[14] J. García-Azorero, J.J. Manfredi, I. Peral and J.D. Rossi, *The Neumann problem for the ∞-Laplacian and the Monge-Kantorovich mass transfer problem*. Nonlinear Anal., 66, (2007), no. 2, 349–366.

[15] D. Gilbarg and N.S. Trudinger, Elliptic partial differential equations of second order, Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[16] E. Giusti. Direct methods in the calculus of variations. World Scientific, 2003.

[17] H. Ishii and P. Loreti. *Limits of solutions of p-Laplace equations as p goes to infinity and related variational problems*. SIAM J. Math. Anal., 37, (2005), no. 2, 411–437.

[18] U. Janfalk, *Behaviour in the limit, as $p \to \infty$, of minimizers of functionals involving p-Dirichlet integrals*, SIAM J. Math. Anal., 27, (1996), no. 2, 341–360.

[19] R. Jensen, *Uniqueness of Lipschitz extensions: minimizing the sup norm of the gradient*, Arch. Rational Mech. Anal., 123, (1993), 51–74.

[20] R. Jensen, P.-L. Lions, and P. E. Souganidis, *A uniqueness result for viscosity solutions of second order fully nonlinear partial differential equations*, Proc. Amer. Math. Soc., 102, (1988), no. 4, 975–978.

[21] P. Juutinen, P. Lindqvist and J.J. Manfredi, *The ∞-eigenvalue problem*, Arch. Ration. Mech. Anal., 148, (1999), no. 2, 89–105.

[22] P. Lindqvist and T. Lukkari, *A curious equation involving the ∞-Laplacian*, Adv. Calc. Var., 3, (2010), 409–421.

[23] J.J. Manfredi, M. Parviainen and J.D. Rossi, *An asymptotic mean value characterization of p-harmonic functions.* Proc. Amer. Math. Soc., 138, (2010), 881–889.

[24] J.J. Manfredi, M. Parviainen and J.D. Rossi, *On the definition and properties of p-harmonious functions.* To appear in Ann. Sc. Norm. Super. Pisa Cl. Sci.

[25] J.J. Manfredi, M. Parviainen and J.D. Rossi, *Dynamic programming principle for tug-of-war games with noise.* To appear in ESAIM. Control, Optimization and Calculus of Variations, COCV.

[26] O. Martio, *Counterexamples for unique continuation*. Manuscripta Math., 60, (1988), 21–47.

[27] Y. Peres, O. Schramm, S. Sheffield and D. Wilson, *Tug-of-war and the infinity Laplacian*, J. Amer. Math. Soc., 22, (2009), 167–210.

[28] Y. Yu, *Maximal and minimal solutions of an Aronsson equation: L^∞ variational problems versus the game theory*. Calc. Var. Partial Differential Equations, 37, (2010), no. 1-2, 63–74.

Petri Juutinen
Department of Mathematics and Statistics
University of Jyväskylä
P.O. Box 35, 40014 University of Jyväskylä,
Finland
petri.juutinen@jyu.fi

Mikko Parviainen
Department of Mathematics and Statistics
University of Jyväskylä
P.O. Box 35, 40014 University of Jyväskylä
Finland
mikko.j.parviainen@jyu.fi

Julio D. Rossi
Departamento de Análisis Matemático,
Universidad de Alicante,
Ap. correos 99, 03080, Alicante,
Spain.
On leave from Dpto. de Matemáticas, FCEyN
Universidad de Buenos Aires, 1428
Buenos Aires, Argentina.
jrossi@dm.uba.ar