Tusk Sinusoidal model, V2 and estimation algorithm

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Description of the sinusoidal model

The observations along the tusk are denoted Y_i for $i = 1, \dots, n$, with the corresponding position along the tusk denoted x_i. The model is the following

$$Y_i = f(x_i, \varphi) + \varepsilon_i$$

with ε_i a random noise assumed to be normally distributed with mean 0 and variance ω^2. The regression function is a periodic sinusoidal function

$$f(x, \varphi) = A \sin(g(x) + b) + B \sin(2g(x) + 2b + \pi/2)$$

with function g defined as

$$g(x) = \int_0^x \xi_u du$$

and finally ξ_u is assumed to be a random square root process

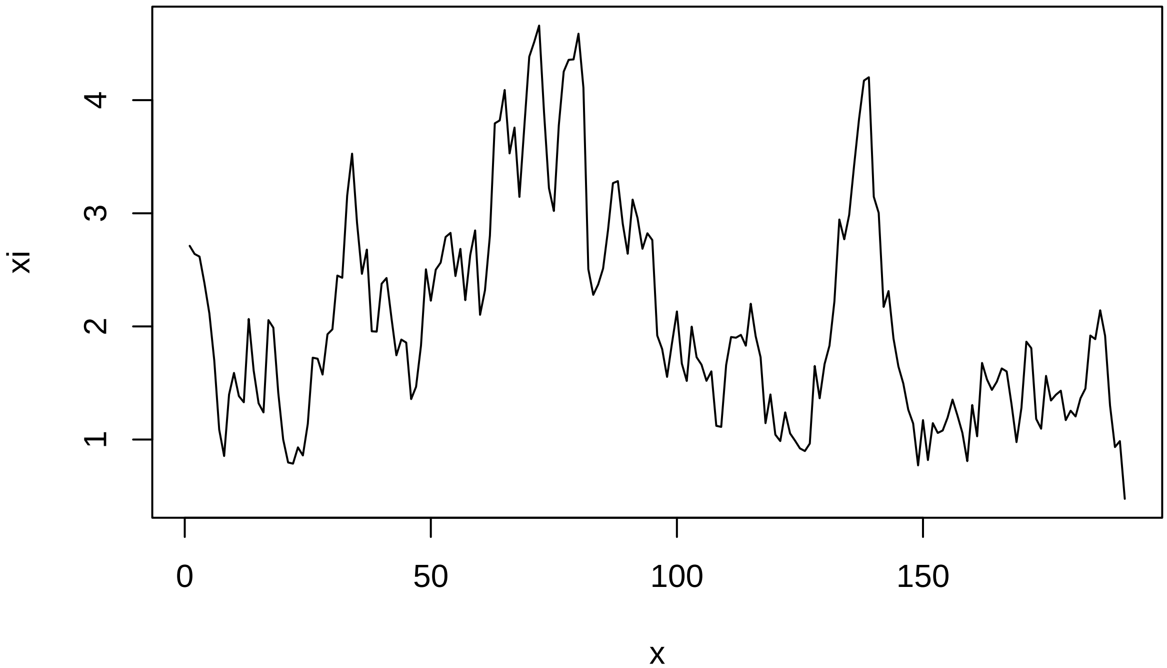
$$d\xi_u = -\beta(\xi_u - a)du + \sigma\sqrt{\xi_i}dW_u$$

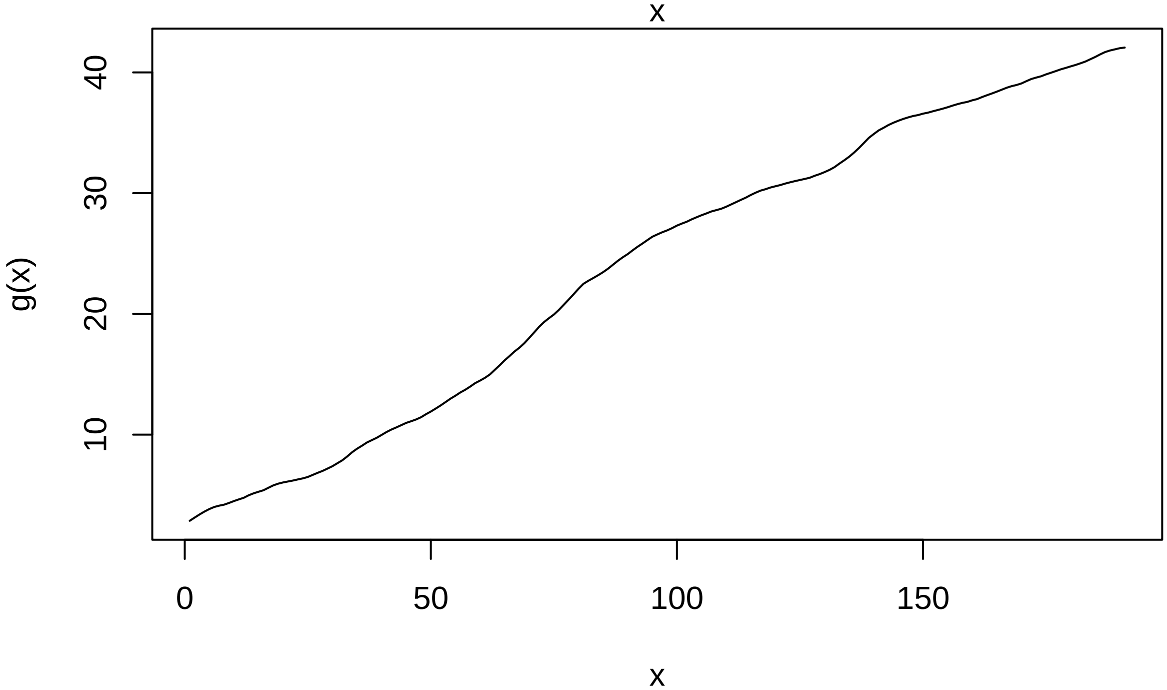
We can calculate the moments of ξ_u:

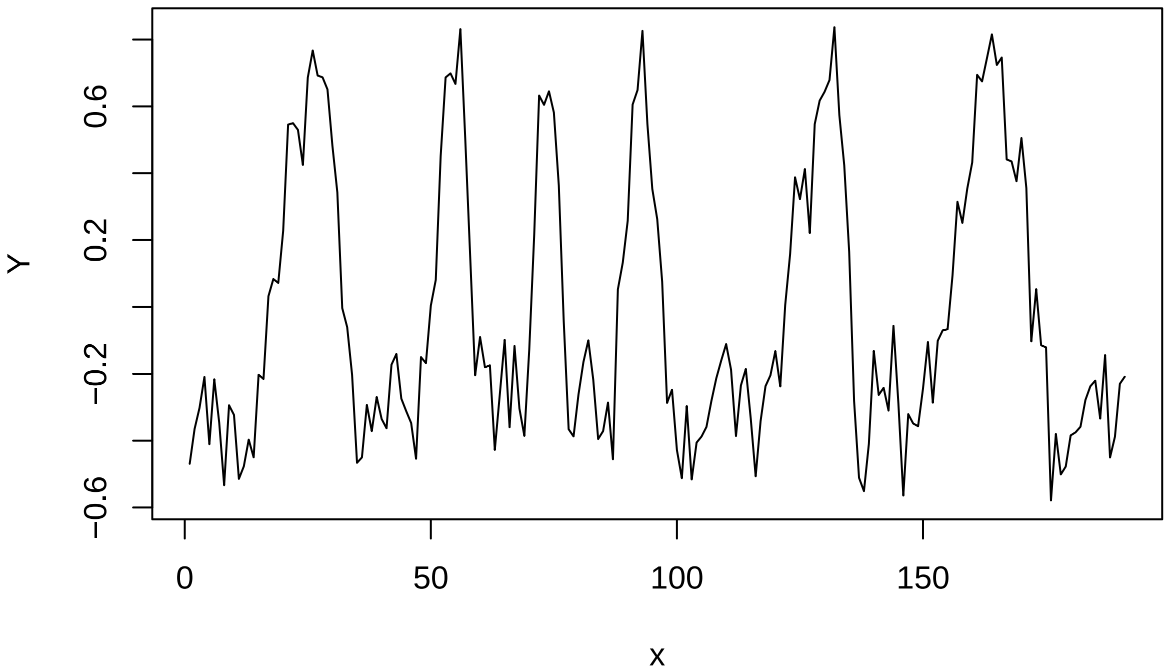
$$\begin{aligned}
\mathbb{E}(\xi_{k\Delta}|\xi_{(k-1)\Delta}) &= \xi_{(k-1)\Delta}e^{-\beta\Delta} + a(1 - e^{-\beta\Delta}) && (1)\\
Var(\xi_{k\Delta}|\xi_{(k-1)\Delta}) &= \xi_{(k-1)\Delta}\frac{\sigma^2}{\beta}\psi(1-\psi) + \frac{a\sigma^2}{2\beta}(1-\psi)^2 && (2)\\
& && (3)
\end{aligned}$$

The objective is to estimate the parameters $\varphi = (A, B, a, b)$; $\psi = e^{-\beta\Delta}$ where Δ is the step size between two observations and $\gamma^2 = \frac{\sigma^2}{\beta}(1-\psi)$.

Simulation of the model






Y
0.6
0.2
−0.2
−0.6
0
50
100
150
x

Estimation with the EM algorithm

The EM algorithm is based on the complete log-likelihood of the model. The solution of the hidden process ξ_x is

$$\xi_{x+\Delta} = \xi_x \psi + a * (1-\psi) + \int_x^{x+\Delta} \sqrt{\xi_s} \sigma e^{\beta(s-x)} dW_s$$

such that the transition density is approximated

$$p(\xi_{x+\delta}|\xi_x) \approx \mathcal{N}\left(\xi_x\psi + a(1-\psi), \gamma^2\left(\psi\xi_x + \frac{a(1-\psi)}{2}\right)\right)$$

The approximate complete log-likelihood is thus

$$\begin{aligned}
\log L(Y, \xi, \theta) &= \sum_{i=1}^n \log p(Y_i|\xi_i) + \sum_{i=1}^n \log p(\xi_i|\xi_{i-1}) + \log p(\xi_1) \\
&\approx -\sum_{i=1}^n \frac{(Y_i - f(x_i, \varphi))^2}{2\omega^2} - \frac{n}{2}\log(\omega^2) \\
&\quad -\sum_{i=1}^n \frac{(\xi_i - \xi_{i-1}\psi - a(1-\psi))^2}{\gamma^2\left(\psi\xi_x + \frac{a(1-\psi)}{2}\right)} - \frac{1}{2}\sum_{i=1}^n \log(\gamma^2\left(\psi\xi_{i-1} + \frac{a(1-\psi)}{2}\right))
\end{aligned}$$

The EM algorithm proceeds at iteration k with the two following steps, given the current value of the parameters θ^k

- E step: calculation of $Q(\theta, \theta^k)$
- M step: update of the parameters $\theta^{k+1} = \arg\max_\theta Q(\theta, \theta^k)$

E step

The condition distribution $p(\xi|Y; \theta^k)$ is not explicit because the regression function is not linear. We should proceed with a MCMC algorithm to sample from this distribution. This will lead to a stochastic version of the EM algorithm, namely the SAEM algorithm.

M step

We need the sufficient statistics to update the algorithm.

The statistics are

$$\begin{aligned}
S_1(\xi_i) &= \frac{1}{n}\sum_{i=1}^n (Y_i - f(x_i(\xi_i), \varphi))^2 \\
S_2(\xi_i) &= \sum_{i=1}^n \xi_{i-1}\xi_i \\
S_3(\xi_i) &= \sum_{i=1}^n \xi_{i-1}^2 \\
S_4(\xi_i) &= \sum_{i=1}^n \xi_i^2
\end{aligned}$$

The update of the parameters are based on these statistics.

SAEM algorithm

The steps of the SAEM algorithm are

- E step: simulation of a new trajectory ξ^k with a MCMC algorithm targeting $p(\xi|Y;\theta^k)$ as stationary distribution
- SA step: stochastic approximation of the sufficient statistics

$$\begin{aligned}
s_1^k &= s_1^{k-1} + (1-\alpha_k)(S_1(\xi^k) - s_1^{k-1}) \\
s_2^k &= s_2^{k-1} + (1-\alpha_k)(S_2(\xi^k) - s_2^{k-1}) \\
s_3^k &= s_3^{k-1} + (1-\alpha_k)(S_3(\xi^k) - s_3^{k-1}) \\
s_4^k &= s_4^{k-1} + (1-\alpha_k)(S_4(\xi^k) - s_4^{k-1})
\end{aligned}$$

- M step: update of θ^k using the sufficient statistics s^k

$$\begin{aligned}
\widehat{\varphi}^k &= \arg\min_{\varphi} \sum_{i=1}^{n} \left(y_i - f(x_i(\xi_i^k), \varphi)\right)^2 \\
\widehat{\omega^2}^k &= s_1^k \\
\widehat{\gamma^2}^k &= \frac{1}{n} \sum_{i=1}^{n} \frac{\left(\xi_i - \xi_{i-1}\widehat{\psi} - \widehat{a}(1-\widehat{\psi})\right)^2}{\widehat{\psi}\xi_{i-1} + \widehat{a}/2(1-\widehat{\psi})}
\end{aligned}$$