Week 1 — History and Linear algebra

→ History (content from Q1)

"Theme" 1: Matter — continuous or discrete?
(Think atomists, Molecular theory, Avogadro, periodic table of elements, composition of an atom, etc.)

"Theme" 2: Light — wave or particle?
(early 1900s: light "packets"?)

"Theme" 3: Energy — Continuous or Discrete?
(Spectra, blackbody radiation, Bohr's model, $E_n = -\alpha^2 \frac{M_e c^2}{2} \frac{1}{n^2}$
Franck + Hertz: e^- collision experiment where they show energy absorption by mercury atoms is quantized)

"Theme" 4: Matter — Particle or wave?
(Matter waves! Schrödinger! e^- are waves: Davisson–Germer experiments to find wavelengths)

→ Groups

- Groups are <u>sets</u> w/ internal operator "+", $+: E \times E \to E$
$(a,b) \mapsto a+b$ (for group E).
- "+" has the properties:
  - Associative: $(a+b)+c = a+(b+c)$
  - Identity element: $e_+ + a = a + e_+ = a$ (e_+ exists)
  - Symmetric element: $b+a = a+b = e_+$ (b " for all a)
- An abelian group (or commutative group) <u>additionally</u> has the property that: $a+b = b+a$

Ex.

$(\mathbb{Z}, +)$ is an abelian group wrt $+$. $e_+ = 0$. Also $(\mathbb{R}, +)$ & $(\mathbb{C}, \cdot)$

The set of planar symmetries of an equilateral triangle is the smallest <u>non</u>-abelian group:

- Reflection: hold one corner, flip other two

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  A         A           C           B
B   C  →  C   B   or  B   A   or  A   C
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- Identity
- Rotation of 120°

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  A         C               A         B
B   C  →  A   B    and    B   C  →  C   A
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Say we call these transformations $ref_1, ref_2, ref_3, I, rot_1, rot_2$

Is this a group? Yes!

- $ref_1 + (rot_2 + ref_1) = (ref_1 + rot_2) + ref_1$ (assoc.)
- $rot_1(rot_2) = I$, $ref_1(ref_1) = I$, etc. (symm/ident.)

Week 2 - More linear algebra

→ Fields

- Fields are abelian groups with an extra $\cdot$ (mult.) operator which has these properties:
  - Commutative: $ab = ba$
  - Associative: $(ab)c = a(bc)$
  - Identity / Symmetry: $e_+ = 0$ and $e_\cdot = 1$
  - Distributive over sum: $a(b+c) = ab + ac$
- Note: Fields are abelian groups wrt $+$, but not necessarily $\cdot$ (though it can be, e.g. $(\mathbb{C}, \cdot)$). $F \setminus \{0\}$, or Field F with 0 removed, is abelian wrt $\cdot$, so Identity and symmetry hold.
- Also note that e^x is a homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}_0^+, \cdot)$ which makes $(\mathbb{R}_0^+, \cdot)$ abelian, since $(\mathbb{R}, +)$ is abelian
- And $e^{\alpha x}$ is the eigenfcn of d/dx ...

Ex.

$\mathbb{C}$, $\mathbb{R}$, and $\mathbb{Q}$ are fields, but are not all complete. ($\mathbb{C}$ is)

$\mathbb{Z}$ is not a field because only ± 1 have symmetric elts.

- $\mathbb{Q}$ is incomplete in two ways:
  1. Not all Cauchy sequences converge in $\mathbb{Q}$.
     Cauchy seq. $\{a_n\}_{n=1}^{\infty}$ has property that $\forall \varepsilon > 0$ in the given field, $\exists N \in \mathbb{Z}^+$ such that $\forall n, m > N$, $|a_n - a_m| < \varepsilon$
     ⇝ essentially, the series converges on an elt in field (?)
     e.g. F_i = i^{th} element in Fibonacci, and sequence where $x_n = F_n / F_{n-1}$ converges on irrational value ϕ.
  2. $\mathbb{Q}$ is not algebraically complete/closed. This means not all polynomial has roots in $\mathbb{Q}$. Formally:

$$P_m(x) = \sum_{i=0}^{m} c_i x^m \quad x, c_i \in F \quad (\text{polynomial } P_m)$$

if $\forall P_m$ you can find $\{\lambda_i\}_{i=1 \dots m}$ $\lambda_i \in F$ where

$$P_m(x) = c_m \prod_{i=1}^{m} (x - \lambda_i)$$

F is closed.

Since $x^2 - 2 = 0$ has no sln in $\mathbb{Q}$, $\mathbb{Q}$ is not algebraically closed.

→ Vector Spaces

- Vector spaces are spaces over a field. They are not fields but are abelian groups wrt $+$, where $\vec{0}$ (null vector) is the neutral element.
- They have "multiply by scalar" operation with properties: →

- Distributive over sum in V: $c(u+v) = cu + cv$, $u, v \in V$
- " " " " F: $(a+b)v = av + bv$ $v \in V$
- Associative for multiplication in F: $(ab)V = a(bv)$ $v \in V$
- Neutral wrt "1" in F: $1 \cdot V = V$, $v \in V$

Ex. $(\mathbb{R}^3, \mathbb{R})$

→ Linear dependence

- Vector set, $\{v_i\}_{i \in I_n}$ for $v_i \in V$, is linearly dependent if $\exists \{c_i\}$ where not all c_i's are 0, where $\sum_i c_i v_i = \vec{0}$

→ Generators

- A set created by forming linear combinations with $\{v_i\}$ is "generated" by $\{v_i\}$
- if $\{v_i\}$ is a vector space, generated set is also a vector space
- A set of all linear combos of a set S from vector space V is a vector space, called span (S). vectors in S generate span(S).

→ Basis

- A linearly independent set of generators of vector space V is V's basis.
  - These are orthogonal and their cardinality = $\dim(V)$

Ex. Vector spaces

- $(\mathbb{R}^n, \mathbb{R})$ and $(\mathbb{C}^n, \mathbb{C})$ are vector spaces
- The set of vectors along some line which passes through the origin $\vec{0}$ is a vector space, and the line itself is a subspace of the space it's in (e.g. $\mathbb{R}^3$).

→ Vector subspace

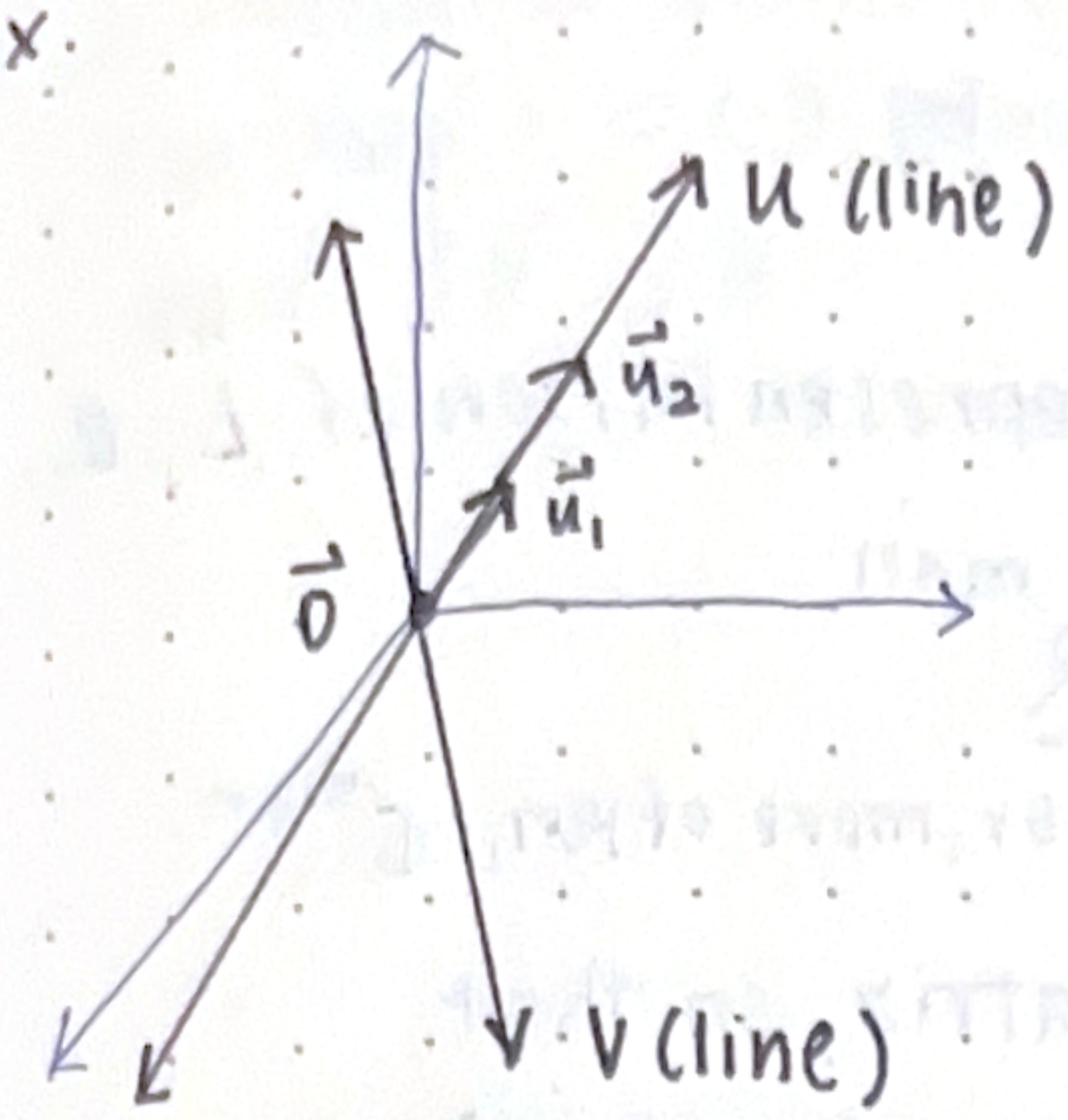
- $W \subset (V, F)$ is a subspace if:
  - $\vec{0} \in W$
  - $u + v \in W$ $\forall u, v \in W$
  - $cv \in W$ $\forall c \in F$, $\forall v \in W$

→ Sum/direct sum of Subspaces

- If $U, W \subset (V, F)$ then set $U + W = \{\vec{u} + \vec{w} \mid \vec{u} \in U, \vec{w} \in W\} \subset (V, F)$ also.
- If $\forall \vec{v} \in V$ $\exists!$ $\vec{u} \in U, \vec{w} \in W \mid \vec{u} + \vec{w} = \vec{v}$, then $U + W$ is a direct sum.
  $\exists!$ — only one
  direct sum — $U \oplus W$
- $U \oplus W$ iff U has basis $\{\vec{u}_1 \ldots \vec{u}_n\}$ and W has basis $\{\vec{w}_1 \ldots \vec{w}_m\}$ and $\{\vec{u}_1 \ldots \vec{u}_n, \vec{w}_1 \ldots \vec{w}_m\}$ is linearly independent.
  - (another way to say this is: $U \oplus W$ iff $U \cap W = \{\vec{0}\}$)
  - $\dim(U \oplus W) = \dim(U) + \dim(W)$
  - Can also say: If $U + W \neq U \oplus W$, $\exists \vec{v} \in U + W \mid \exists (\vec{u}_1, \vec{w}_1) \neq (\vec{u}_2, \vec{w}_2)$

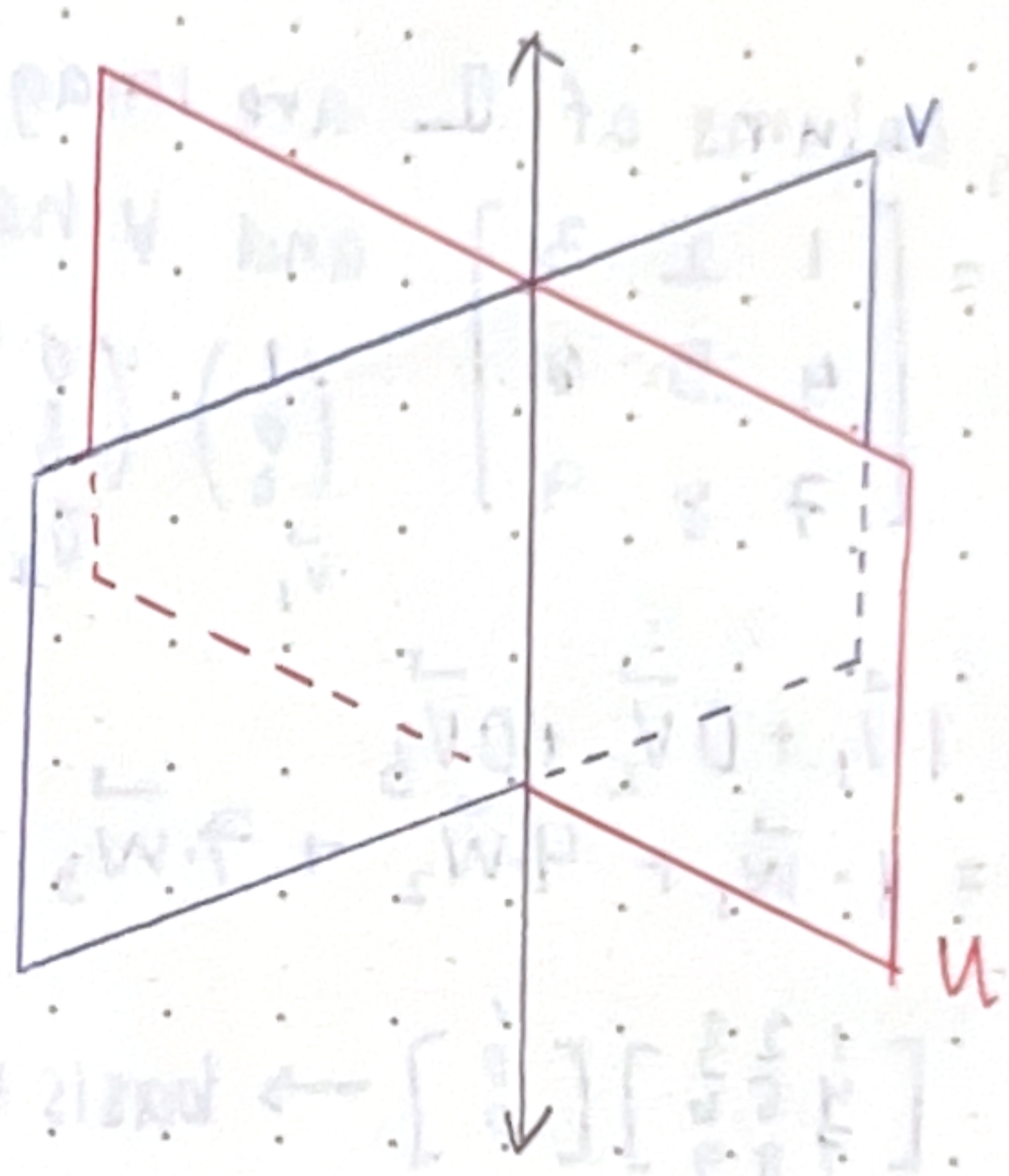
where $\vec{v} = \vec{u}_1 + \vec{w}_1 = \vec{u}_2 + \vec{w}_2 \rightarrow \vec{u}_1 - \vec{u}_2 = \vec{w}_2 - \vec{w}_1$. Since $\vec{u}_1 - \vec{u}_2$ must be in U, $\vec{w}_2 - \vec{w}_1$ must be in W, $\vec{u}_1 - \vec{u}_2 = \vec{w}_2 - \vec{w}_1 \in U \cap W$.
= There is a non-trivial intersection $\neq \{\vec{0}\}$.

EX.



- $\vec{u}_1$ and $\vec{u}_2$ are along U, $\{\vec{u}_1, \vec{u}_2\}$ is a vector space, too! $\{\vec{u}_1\}$ is a basis/gen. for this set.
- $U \cap V = \{\vec{0}\}$, so $U \oplus V$
- U and V each have basis of 1 vector. Together, 2 vectors define a plane! (if they are linearly independent.)

- Here, $U + V = \mathbb{R}^3$ but U and V intersect in a line $\neq \vec{0}$, so they don't make a direct sum
- $U + W$ is still a subspace of $\mathbb{R}^3$, though (so also a vector space.
- U and V each need 2 vectors in their basis. Then there's a total of 4 vectors — in $\mathbb{R}^3$, 4 vectors must include at least 2 linearly dep. vectors.



→ *Linear Applications*

- a linear app. is a fcn $\hat{L}: V \rightarrow W$, $\vec{v} \mapsto \hat{L}\vec{v}$ for vector spaces V, W with properties:
  - $\forall v, u \in V$, $\hat{L}(\vec{u} + \vec{v}) = \hat{L}\vec{u} + \hat{L}\vec{v}$
  - $\forall v \in V, \forall c \in F$, $\hat{L}(c\vec{v}) = c\hat{L}\vec{v}$
- Say V has $\{\vec{v}_i\}_{i=1..n}$ basis, $\dim(V) = n$ and W has $\{\vec{w}_i\}_{i=1...m}$ basis, $\dim(W) = m$.

$\forall \vec{v} \in V$, $\vec{v} = \sum c_i \vec{v}_i$ (linear combo of bases)

$\hat{L}\vec{v}$ then can be written as:

$$\sum_i c_i \underline{\underline{\hat{L}\vec{v}_i}} \quad \text{where } \hat{L}\vec{v}_i \in \underline{\underline{W}} \text{ (def of } \hat{L}\text{)}$$

and, $\exists$ value $L_{ji} \in F \mid \underline{\underline{\hat{L}\vec{v}_i}} = \sum_{j=1}^{m} w_j L_{ji} \longrightarrow$

$\hat{L}\vec{v}$ can now be written as: $\sum_{i=1}^{n}\sum_{j=1}^{m} c_i \vec{w}_j L_{ji}$

$= \sum_{j=1}^{m} \vec{w}_j \sum_{i=1}^{n} L_{ji} c_i$ (since w_j only depends on j)

- Say $b_j = \sum_{i=1}^{n} L_{ji} c_i$, then $\hat{L}\vec{v} = \sum_{j=1}^{m} \vec{w}_j \cdot b_j$
- This L_{ji} is an element of the Matrix Representation of $\hat{L}$, $\mathbb{L}$.

$\hat{L} : \underset{\text{dim } n}{V} \to \underset{\text{dim } m}{W}$ which means $\mathbb{L} \in \mathbb{R}^{m\times n}$

└ or, more often, $\mathbb{C}^{m\times n}$

$$\mathbb{L} = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & & \vdots \\ \vdots & & \ddots & \\ L_{m1} & & \cdots & L_{mn} \end{bmatrix}$$

($m\times n$ matrix, so that $(m\times n)(n\times 1) = (m\times 1)$ or w/e...)

Here, colums of $\mathbb{L}$ are images of values in V's basis, e.g.

if $\mathbb{L} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and V has basis: $\underset{\vec{v}_1}{\begin{pmatrix}1\\0\\0\end{pmatrix}}$ $\underset{\vec{v}_2}{\begin{pmatrix}0\\1\\0\end{pmatrix}}$ and $\underset{\vec{v}_3}{\begin{pmatrix}0\\0\\1\end{pmatrix}}$ then

$\vec{v}_1 = 1\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3$

$\hat{L}\vec{v}_1 = 1\cdot\vec{w}_1 + 4\cdot\vec{w}_2 + 7\cdot\vec{w}_3$ ($\begin{bmatrix}1\\0\\0\end{bmatrix}$ selects first elt in rows)

$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} \to$ basis vec $\begin{bmatrix}1\\0\\0\end{bmatrix}$ maps to $\begin{bmatrix}1\\4\\7\end{bmatrix}$

→ Scalar product / inner product

- $\langle\cdot|\cdot\rangle : V\times V \mapsto F$, V is vector space (V, F). F is often $\mathbb{C}$, $(\vec{a},\vec{b}) \mapsto \langle\vec{a}|\vec{b}\rangle$ has properties:
  - $\langle\vec{a}|\vec{a}\rangle \in \mathbb{R}$, $\langle\vec{a}|\vec{a}\rangle \geq 0$, and $\langle\vec{a}|\vec{a}\rangle = 0$ iff $\vec{a} = \vec{0}$ (pos. def.)
  - $\forall \alpha, \beta \in F$, $\forall \vec{a}, \vec{b}, \vec{c} \in V$, then $\langle\vec{c}|\alpha\vec{a} + \beta\vec{b}\rangle = \alpha\langle\vec{c}|\vec{a}\rangle + \beta\langle\vec{c}|\vec{b}\rangle$ (linear in second entry)
  - $\forall \vec{a}, \vec{b} \in V$, $\langle\vec{a}|\vec{b}\rangle = \langle\vec{b}|\vec{a}\rangle^*$
- This means $\langle\cdot|\cdot\rangle$ is "anti-linear" in 1st entry, meaning that:

  $\langle\alpha\vec{a} + \beta\vec{b}|\vec{c}\rangle = \alpha^*\langle\vec{a}|\vec{c}\rangle + \beta^*\langle b|c\rangle$

  (since this is $\langle\vec{c}|\alpha\vec{a} + \beta\vec{b}\rangle^*$ according to the third property,

  and $= (\alpha\langle\vec{c}|\vec{a}\rangle + \beta\langle\vec{c}|\vec{b}\rangle)^*$

  $= \alpha^*\langle\vec{c}|\vec{a}\rangle^* + \beta^*\langle\vec{c}|\vec{b}\rangle^*$

  $= \alpha^*\langle\vec{a}|\vec{c}\rangle + \beta^*\langle\vec{b}|\vec{c}\rangle$ ✓.)
- Also call $\langle\cdot|\cdot\rangle$ "Sequilinear"

- if $\vec{a}, \vec{b} \in \mathbb{C}^n$, $\langle \vec{a} | \vec{b} \rangle = \sum_{i=1}^{n} a_i^* b_i = 0$, then $\vec{a}$ and $\vec{b}$ are orthogonal (perpendicular)
- This is also called "Pre-Hilbert" Space (not complete, so...)

$\perp$ indicates geometry!

Further: $\langle \cdot | \cdot \rangle$ allows norms & distance: $\|\vec{v}\| = \sqrt{\langle \vec{v} | \vec{v} \rangle}$ and $\|\vec{x} - \vec{y}\|$