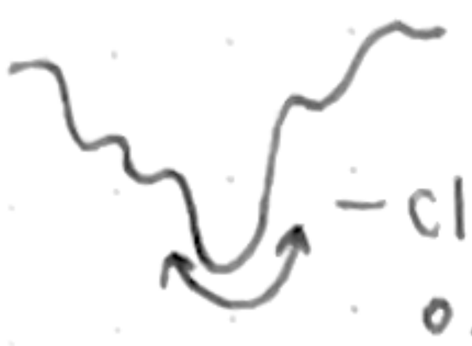
Video lecture – HARMONIC OSCILLATOR (in 1D)

Consider the harmonic oscillator where $F(x) = -k(x-x_0)$
This has quadratic potential $V(x) = \frac{1}{2}k(x-x_0)^2$

Any potential with a minimum that is sufficiently regular there can be approximated by the quadratic potential.

(some arbitrary potential...)



– close to the equilibrium position, system behaves like harmonic oscillator

Thus the harmonic oscillator is used to approximate/describe:
- vibration in molecules
- Phonons in solid state
- EMF (as a collection of harmonic oscillators)

CLASSICAL APPROACH: NEWTONIAN

$F = ma \rightarrow -kx = m\ddot{x}$ (say $x_0 = 0$)

$$\ddot{x} + \frac{k}{m}x = 0 \rightarrow \ddot{x} + \omega^2 x = 0 \text{ for } \omega \equiv \sqrt{\frac{k}{m}}$$

$\frac{1}{[T]^2} \downarrow$ $\quad \sqrt{\frac{1}{[T]^2}} \downarrow$

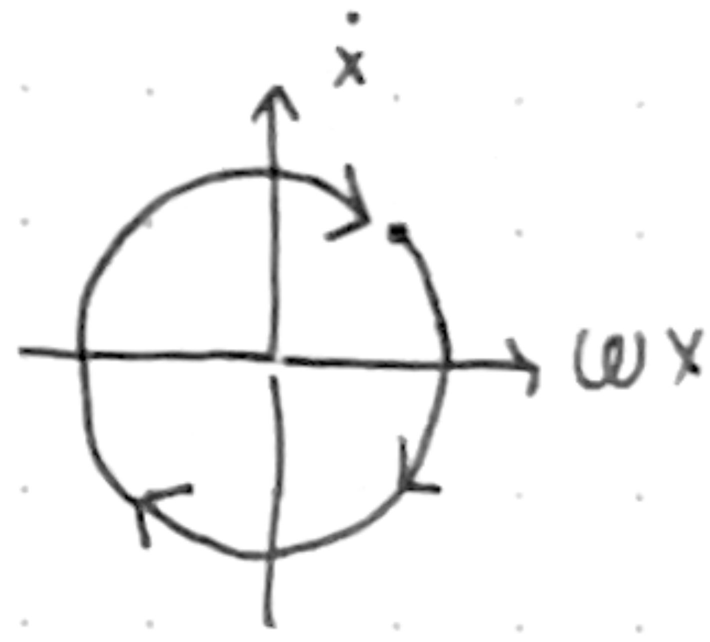
2nd order differential equation, so:

$x(t) = A\cos(\omega t + \phi)$, or $x(t) = B_1 \sin(\omega t) + B_2 \cos(\omega t)$

and $\qquad\qquad = C_1 e^{i\omega t} + C^* e^{-i\omega t}$ (result is real)

$\dot{x} = -\omega A \sin(\omega t + \phi)$

If we plot $(\omega x, \dot{x})$ where $\omega x = \omega A\cos(\omega t + \phi)$:



CLASSICAL APPROACH: CANONICAL

$H = \frac{p^2}{2m} + \frac{1}{2}kx^2$ (again, $x_0 = 0$)

$$\begin{cases} \dot{x} = \partial_p H = p/m \\ \dot{p} = -\partial_x H = -kx \end{cases}$$

OR use ω again. $k = m\omega^2$, so:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{\omega}{2}\left(\frac{p^2}{\omega m} + m\omega x^2\right) = \frac{\omega}{2}\left(\left(\frac{p}{\sqrt{m\omega}}\right)^2 + \left(\sqrt{m\omega}\,x\right)^2\right)$$

Call $P = \frac{p}{\sqrt{\omega m}}$ and $Q = \sqrt{m\omega}\,x$. We can actually show $\{Q, P\}_{PB}$ is the same as $\{x, p\}_{PB}$, thus $x, p \rightarrow Q, P$ is a canonical transformation. $\rightarrow$

$$\{Q,P\}_{PB} = \{\sqrt{m\omega}\,x, \frac{P}{\sqrt{m\omega}}\}_{PB} = \frac{\sqrt{m\omega}}{\sqrt{m\omega}}\{x,p\}_{PB} = 1$$
(since we know the Poisson bracket is bilinear)

Thus the Hamiltonian is preserved: $H = \frac{\omega}{2}(Q^2+P^2)$
which is much more symmetric:

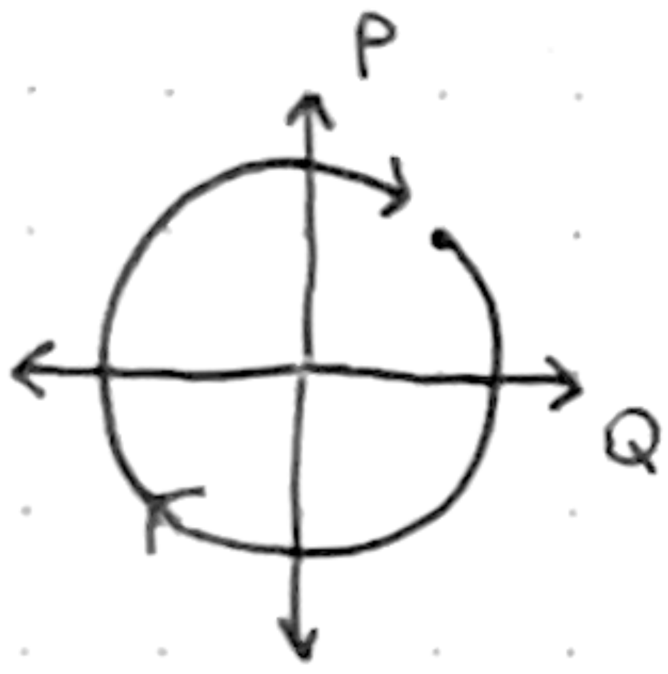
$$\begin{cases} \dot{Q} = \partial_P H = \omega P \\ \dot{P} = -\partial_Q H = -\omega Q \end{cases}$$

And, we can define $\eta = Q + iP$ (Q & P are real, so η stores/preserves the info of both)

$$\dot{\eta} = \dot{Q} + i\dot{P} = \omega P - i\omega Q = -i\omega(Q+iP) = \underline{-i\omega\eta}$$
which yields closed form solution:

$$\eta(t) = \eta_0 e^{-i\omega t}$$ (very similar to QM solutions)

Clearly η is in the complex plane with real part Q, imaginary part P, so its phase space corresponds to complex plane and so we once again have



QM APPROACH: OPERATOR ALGEBRA

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2 \quad \text{and } [\hat{x},\hat{p}] = i\hbar,\ k = m\omega^2 \text{ (knowns)}$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \frac{\hbar\omega}{2}\left(\frac{\hat{p}^2}{m\hbar\omega} + \frac{m\omega}{\hbar}\hat{x}^2\right)$$

$[E]$

$$= \underbrace{\frac{\hbar\omega}{2}}_{[E]=[\text{Action}][\frac{1}{T}]}\underbrace{\left(\left(\frac{\hat{p}}{\sqrt{m\hbar\omega}}\right)^2 + \left(\sqrt{\frac{m\omega}{\hbar}}\hat{x}\right)^2\right)}_{\text{dimensionless!}}$$

Call $\hat{P} = \frac{\hat{p}}{\sqrt{m\hbar\omega}}$ and $\hat{Q} = \sqrt{\frac{m\omega}{\hbar}}\hat{x}$, similar to previous approach

$$[\hat{Q},\hat{P}] = \left[\sqrt{\frac{m\omega}{\hbar}}\hat{x}, \frac{1}{\sqrt{m\hbar\omega}}\hat{p}\right] = \frac{1}{\hbar}(i\hbar) = i$$

Thus we can define $\hat{H} = \frac{\hbar\omega}{2}(\hat{Q}^2+\hat{P}^2)$. Why are we doing this? Because solving for the eigenfunctions/values is hard otherwise...

Define $\hat{a} = (\hat{Q}+i\hat{P})/\sqrt{2}$ and $\hat{a}^\dagger = (\hat{Q}-i\hat{P})/\sqrt{2}$. These are <u>not</u> self-adjoint or normal ($\hat{N} = \hat{A} + i\hat{B}$ where $[\hat{A},\hat{B}] = 0$, "close" to being self-adjoint). However, we can define $\hat{n} = \hat{a}^\dagger \hat{a}$, which evidently is self-adjoint.

$\hat{a}$ is sometimes called the "annihilation" operator and $\hat{a}^\dagger$ the "creation" operator, and $\hat{n}$ the number operator. $\hat{a}$ and $\hat{a}^\dagger$ are also called the ladder operators.

We can write $\hat{Q}$ as $\dfrac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}$ and $\hat{P}$ as $\dfrac{\hat{a}-\hat{a}^\dagger}{i\sqrt{2}}$, which means we can re-write $\hat{H}$:

$$\hat{Q}^2 = \frac{1}{2}(\hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})$$

$$\hat{P}^2 = -\frac{1}{2}(\hat{a}^2 + \hat{a}^{\dagger 2} - \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a})$$

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{Q}^2 + \hat{P}^2) = \frac{\hbar\omega}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})$$

$$\to [\hat{a},\hat{a}^\dagger] = \left[\frac{1}{\sqrt{2}}(\hat{Q}+i\hat{P}), \frac{1}{\sqrt{2}}(\hat{Q}-i\hat{P})\right]$$

$$= \frac{1}{2}([\hat{Q},\hat{Q}] + [\hat{Q},-i\hat{P}] + [i\hat{P},\hat{Q}] + [i\hat{P},-i\hat{P}])$$

$$= \frac{1}{2}([\hat{Q},-i\hat{P}] + [i\hat{P},\hat{Q}])$$

$$= \frac{1}{2}(-i[\hat{Q},\hat{P}] + i[\hat{P},\hat{Q}])$$

$$= \frac{1}{2}(-i[\hat{Q},\hat{P}] - i[\hat{Q},\hat{P}]) = -i[\hat{Q},\hat{P}] = 1$$

$$\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1$$

$\hat{H} = \dfrac{\hbar\omega}{2}(2\hat{a}^\dagger\hat{a} + 1) = \hbar\omega(\hat{n} + \frac{1}{2})$ which means we can find the eigenvalues/vectors of $\hat{H}$ if we find them for $\hat{n}$.

<u>EIGENVALUES OF $\hat{n}$ (AND EIGENVECTORS)</u>

$\hat{n}|n,\alpha\rangle = n|n,\alpha\rangle$ $n \in \mathbb{R}$, for eigenvalue n and index α to resolve degeneracy. Assume $\langle n,\alpha|n,\alpha\rangle = 1$.

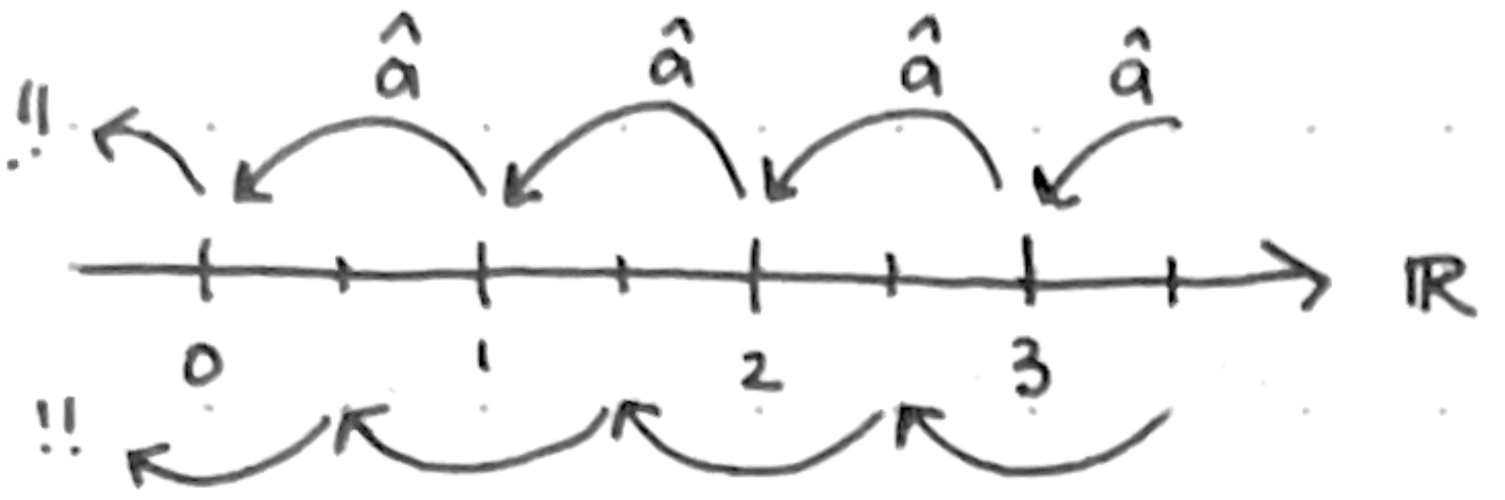
$$\|\hat{a}|n,\alpha\rangle\|^2 = \langle n,\alpha|\hat{a}^\dagger\hat{a}|n,\alpha\rangle \geq 0 \text{ (by definition... }^2)$$
$$= \langle n,\alpha|\hat{n}|n,\alpha\rangle = n\langle n,\alpha|n,\alpha\rangle = n$$

Hence $n \geq 0$ and $\hat{a}|n,\alpha\rangle \neq 0$ if $n > 0$. Why do we care about $\hat{a}|n,\alpha\rangle$? Because it's an eigenvector of $\hat{n}$!

$$\hat{n}\hat{a}|n,\alpha\rangle = \hat{a}^{+}\hat{a}\,\hat{a}|n,\alpha\rangle = (\hat{a}\hat{a}^{+} - 1)\hat{a}|n,\alpha\rangle = \hat{a}(\hat{a}^{+}\hat{a} - 1)|n,\alpha\rangle$$
$$= \hat{a}(\hat{n}-1)|n,\alpha\rangle = (n-1)\hat{a}|n,\alpha\rangle$$

(if $\|\hat{a}|n,\alpha\rangle\| \neq 0$, of course. so, if $n \neq 0$...)
Acting on an eigenvector corresponding to eigenvalue n with $\hat{a}$ produces an eigenvector with eigenvalue $n-1$.

Thus we can continue to apply $\hat{a}$ until we get $n=0$, at which point $\hat{a}|0,\alpha\rangle$ is the null vector and not an eigenvector, so we can't continue.



Note we know $n \geq 0$, so we can't have eigenvalues < 0. This means we can't "start" at a non-integer eigenvalue and continually apply $\hat{a}$. We would avoid the $n=0$ case, but arrive at a negative eigenvalue.

$\therefore$ If we find pos. integer eigenvalue n, $n-1, n-2, \ldots, 0$ are also eigenvalues.

$\hat{a}^{+}|n,\alpha\rangle$ is also an eigenvector of $\hat{n}$:

$$\hat{n}\hat{a}^{+}|n,\alpha\rangle = \hat{a}^{+}\underline{\hat{a}\hat{a}^{+}}|n,\alpha\rangle = \hat{a}^{+}(\hat{a}^{+}\hat{a}+1)|n,\alpha\rangle$$
$$= \hat{a}^{+}(\hat{n}+1)|n,\alpha\rangle = (n+1)\hat{a}^{+}|n,\alpha\rangle$$

So applying $\hat{a}^{+}$ on an eigenvector of $\hat{n}$ yields an eigenvector of $\hat{n}$ with an incremented eigenvalue.

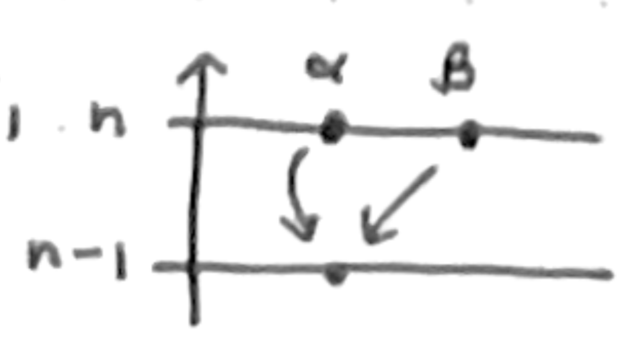
We know $\|\hat{a}|n,\alpha\rangle\| \neq 0$ if $n>0$, let's also verify $\hat{a}^{+}|n,\alpha\rangle$ is not the null vector:

$$\|\hat{a}^{+}|n,\alpha\rangle\|^2 = \langle n,\alpha|\hat{a}\hat{a}^{+}|n,\alpha\rangle = \langle n,\alpha|\hat{a}^{+}\hat{a}+1|n,\alpha\rangle$$
$$= \langle n,\alpha|\hat{n}+1|n,\alpha\rangle = n+1 \neq 0. \checkmark$$

$$\therefore \sigma(\hat{n}) = \mathbb{N}_0 = \{0,1,2,\ldots\}$$

DEGENERACY

Say we have two eigenvectors of $\hat{n}$, $|n,\alpha\rangle$ and $|n,\beta\rangle$ so that $\langle n,\alpha|n,\beta\rangle = 0$. Are $\hat{a}|n,\alpha\rangle$ and $\hat{a}|n,\beta\rangle$ also orthogonal?
If they are not,



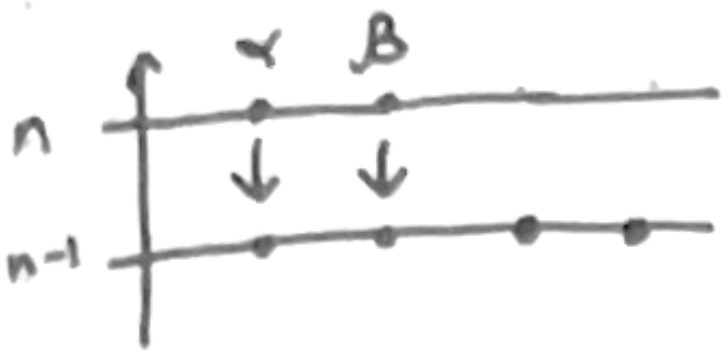
then the degeneracy of n is not the same as the degeneracy of $n-1$.

Let's see:

$\langle n,\alpha|\hat{a}^\dagger\hat{a}|n,\beta\rangle = n\langle n,\alpha|n,\beta\rangle = 0$.

So yes, $\hat{a}$ maps a set of orthogonal eigenvectors to an equally sized set of orthogonal eigenvectors (at eigenvalue $n-1$).

It is possible, however, that $n-1$ has a <u>higher</u> degeneracy than n:



so let's do the same analysis for $\hat{a}^\dagger$: are eigenvectors $\hat{a}^\dagger|n,\alpha\rangle$ and $\hat{a}^\dagger|n,\beta\rangle$ orthogonal?

$\langle n,\alpha|\hat{a}\hat{a}^\dagger|n,\beta\rangle = \langle n,\alpha|\hat{a}^\dagger\hat{a}+1|n,\beta\rangle = (n+1)(0)$
$= 0$.

So: $\deg(n-1) \geq \deg(n)$, and $\deg(n) \geq \deg(n-1)$ which tells us <u>all</u> energy levels have the <u>same</u> degeneracy.

If eigenvalue 0 has no degeneracy, neither does 1, 2, 3 ... <u>no</u> eigenvalue is degenerate:

$\hat{a}|n=0,\alpha\rangle = 0$ and so is not an eigenvector. Recall that we define $\hat{a}$ earlier; $\hat{a} = (\hat{Q}+i\hat{P})/\sqrt{2}$. We also defined $\hat{Q}$ and $\hat{P}$ in terms of $\hat{x}$ and $\hat{p}$. So,

$$\hat{a} = (\hat{Q}+i\hat{P})/\sqrt{2} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} + i\frac{\hat{p}}{\sqrt{m\hbar\omega}}\right) \text{ and}$$

$$\hat{a}|0,\alpha\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} + i\frac{\hat{p}}{\sqrt{m\hbar\omega}}\right)|0,\alpha\rangle = 0$$ (can mult. both sides by $\sqrt{2}$ to remove...)

$\langle x|$

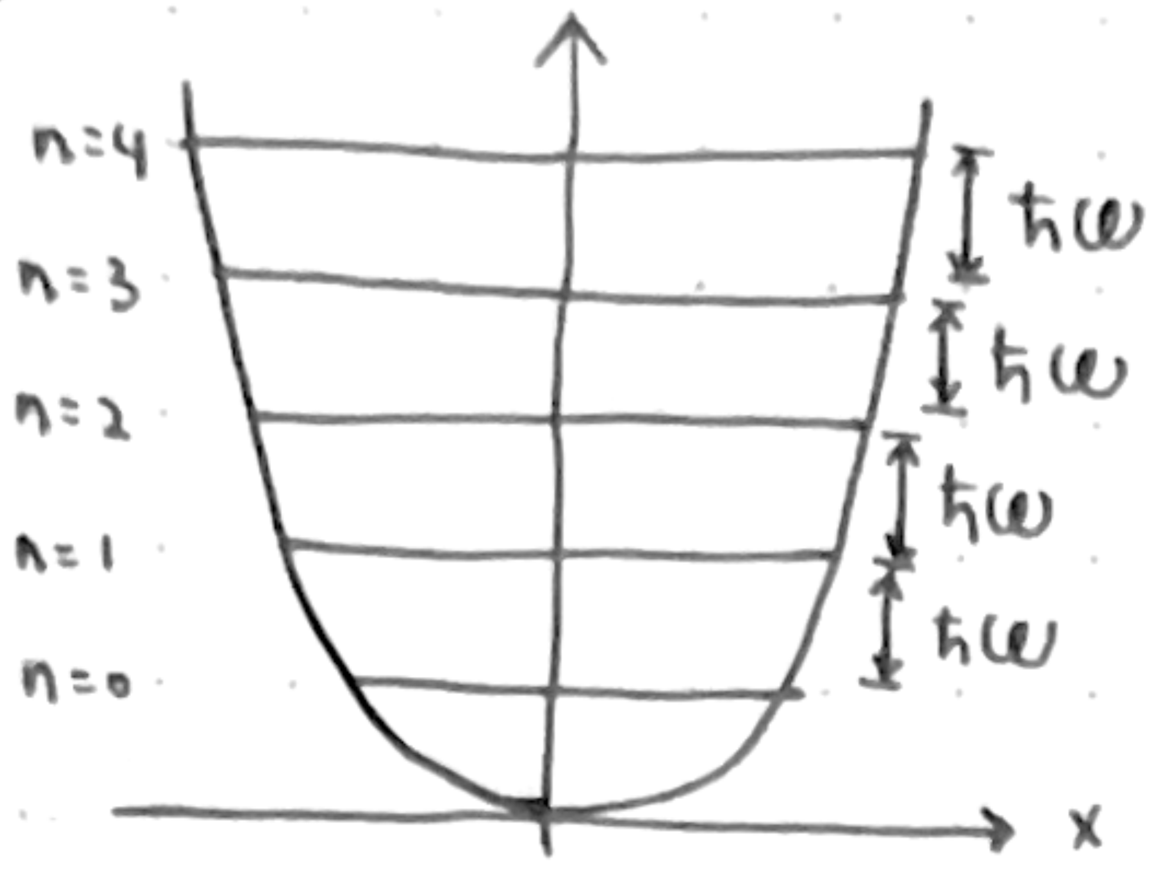
$$\langle x|\sqrt{\frac{m\omega}{\hbar}}\,\hat{x} + i\frac{\hat{p}}{\sqrt{m\hbar\omega}}|0,\alpha\rangle = \left(\sqrt{\frac{m\omega}{\hbar}}\,x + i\frac{-i\hbar}{\sqrt{m\hbar\omega}}\frac{d}{dx}\right)\langle x|0,\alpha\rangle = 0$$

(multiply both sides by $\sqrt{\frac{m\omega}{\hbar}}$)

$\left(\frac{m\omega}{\hbar}x + \frac{d}{dx}\right)\psi_{0,\alpha}(x) = 0$ which is a linear, 1^{st} order, homogenous differential equation. Thus it has <u>one</u> linearly independent solution for $\langle x|0,\alpha\rangle = \psi_{0,\alpha}(x)$, meaning that eigenvalue 0 is <u>non degenerate</u>. So all eigenvalues of $\hat{n}$ are non degenerate!

<u>HAMILTONIAN</u>: $\hat{H} = \hbar\omega(\hat{n}+1/2)$, so eigenvalues look like $E_n = \hbar\omega(n+1/2)$ for eigenvalues of $\hat{n}$, $n \in \mathbb{N}_0$.

$E_0 = \frac{1}{2}\hbar\omega$ is above the minimum possible energy; it's called the "zero point energy". Subsequent E_n's are evenly spaced by $\hbar\omega$



This equidistant property allows us to do the following: Say we have some linear combination of eigenvalues of $\hat{n}$, $|\psi_0\rangle$ so that

$$|\psi_0\rangle = \sum_n |n\rangle c_n$$

then, after some time t has passed, we would have $|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi_0\rangle$, and given that we know $\hat{H} = \hbar\omega(n+1/2)$, $|\psi(t)\rangle = \sum_n |n\rangle c_n \, e^{-i\hbar\omega(n+1/2)t/\hbar}$

$$= e^{-i\omega t/2} \sum_n |n\rangle c_n e^{-i\omega n t}$$

so: If $\omega t = 2\pi m \rightarrow n\omega t = 2\pi n m$ (for some integer m), then we have $|\psi(t)\rangle = e^{-i\omega t/2} \sum_n |n\rangle c_n e^{-i2\pi n m} = e^{-i\omega t/2} \sum_n |n\rangle c_n$.

$e^{-i\omega t/2}$ is just a global phase factor, so for sometime, $|\psi(t)\rangle = |\psi_0\rangle$. Hence there is an intrinsic period $T = \frac{2\pi}{\omega}$ that guarantees the motion is rigorously periodic with period T, no matter the initial conditions.

A LITTLE MORE ABOUT EIGENVECTORS $|n\rangle$

So far, all we know about $|n\rangle$ is that $\| \hat{a}|n\rangle \|^2 = n$ and $\| \hat{a}^\dagger |n\rangle \|^2 = n+1$ which tells us that $\hat{a}|n\rangle$ itself has a length of $\sqrt{n}$ (and eigenvalue $n-1$) and $\hat{a}^\dagger|n\rangle$ has length $\sqrt{n+1}$ and eigenvalue $n+1$:

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle e^{i\theta} \qquad \hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle e^{i\theta}$$

for some global phase factor $e^{i\theta}$. We want to choose phases so $e^{i\theta}$ do not apply at all. In fact, we choose the phase for all $|n\rangle$ by setting the phase for state $|0\rangle$:

From $\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$, we can rewrite indices to be one less: $\hat{a}^\dagger|n-1\rangle = \sqrt{n}\,|n\rangle \longrightarrow |n\rangle = \frac{\hat{a}^\dagger|n-1\rangle}{\sqrt{n}}$. We can, by the same relation, say $|n-1\rangle = \frac{\hat{a}^\dagger|n-2\rangle}{\sqrt{n-1}}$ which means $|n\rangle = \frac{(\hat{a}^\dagger)^2|n-2\rangle}{\sqrt{n}\sqrt{n-1}}$ and so on, until we get to $|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle$

OPERATORS IN THE NUMBER ($\hat{n}$) BASIS

$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$\hat{H} = \hbar\omega(\hat{n} + 1/2)$

$\hat{x} = \sqrt{\frac{\hbar}{m\omega}}\frac{1}{\sqrt{2}}(\hat{a}+\hat{a}^\dagger)$ and $\hat{p} = \sqrt{\frac{m\hbar\omega}{2}}\frac{1}{i}(\hat{a}-\hat{a}^\dagger)$

Clearly, $\hat{n}$ is diagonal in the number basis and so is $\hat{H}$.
$\langle n|\hat{n}|m\rangle = n\,\delta_{nm}$ and $\langle n|\hat{H}|m\rangle = \hbar\omega(n+\frac{1}{2})\,\delta_{nm}$ which look like:

$$\mathbb{n} = \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & & & 0 & \\ & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ 0 & & & & \ddots \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad \mathbb{H} = \hbar\omega \begin{bmatrix} 1/2 & & & 0 & \\ & 3/2 & & & \\ & & 5/2 & & \\ & & & 7/2 & \\ 0 & & & & \ddots \end{bmatrix}$$

$\langle n|\hat{a}|m\rangle = \sqrt{m}\,\langle n|m-1\rangle = \sqrt{m}\,\delta_{n,m-1}$ so $\hat{a}$ and $\hat{a}^\dagger$ have matrices:

$$\mathbb{a} = \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & & 0 & \\ & 0 & \sqrt{2} & & \\ & & 0 & \sqrt{3} & \\ & & & 0 & \sqrt{4} \\ 0 & & & & \ddots \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad \mathbb{a}^\dagger = \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & & & 0 & \\ 1 & 0 & & & \\ & \sqrt{2} & 0 & & \\ & & \sqrt{3} & 0 & \\ 0 & & & \sqrt{4} & \ddots \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

(upper diagonal is non-zero) index of row < index of col

(lower diagonal is non-zero) index of col < index of row

Finally, because $\hat{x}$ and $\hat{p}$ are combinations of $\hat{a}$ and $\hat{a}^\dagger$:

$$\mathbb{x} = \begin{bmatrix} 0 & 1 & & 0 & \\ 1 & 0 & \sqrt{2} & & \\ & \sqrt{2} & 0 & \sqrt{3} & \\ & & \sqrt{3} & 0 & \sqrt{4} \\ 0 & & & \sqrt{4} & \ddots \end{bmatrix} \sqrt{\frac{\hbar}{2m\omega}} \qquad \mathbb{p} = \sqrt{\frac{m\hbar\omega}{2}}\,\frac{1}{i} \begin{bmatrix} 0 & 1 & & 0 & \\ 1 & 0 & \sqrt{2} & & \\ & \sqrt{2} & 0 & \sqrt{3} & \\ & & \sqrt{3} & 0 & \sqrt{4} \\ 0 & & & \sqrt{4} & \ddots \end{bmatrix}$$

Taking the expectation value of $\hat{x}$ and $\hat{p}$ for states $|n\rangle$ is fairly simple. Call the coefficient on $\hat{x}$ A and the coefficient on $\hat{p}$ B:

$\langle n|\hat{x}|n\rangle = A\langle n|\,\hat{a}+\hat{a}^\dagger|n\rangle = A\langle n|\,(\sqrt{n}\,|n-1\rangle + \sqrt{n+1}\,|n+1\rangle)$

and $|n\rangle$ is orthogonal to $|n\pm1\rangle$, so $\langle n|\hat{x}|n\rangle = 0$.

The same is true for $\langle n|\hat{p}|n\rangle$.

we can also find the standard deviation:

$$\langle n|\hat{x}^2|n\rangle = A^2 \langle n|\,\hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\,|n\rangle$$
$$= A^2 \langle n|\,\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\,|n\rangle = A^2\langle n|\,\hat{n} + \hat{n} + 1\,|n\rangle$$
$$= A^2(2n+1)$$

$$\langle n|\hat{p}^2|n\rangle = -|B|^2 \langle n|\,\hat{a}^2 + \hat{a}^{\dagger 2} - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger\,|n\rangle$$
$$= -|B|^2\langle n|\,-\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger\,|n\rangle$$
$$= |B|^2(2n+1)$$

$$\sigma_x\sigma_p = A|B|(2n+1) = \sqrt{\frac{\hbar}{m\omega}}\,\frac{1}{\sqrt{2}}\sqrt{\frac{m\hbar\omega}{2}}\,(2n+1)$$
$$= \frac{\hbar}{2}(2n+1) \geq \frac{\hbar}{2}$$

we see minimum uncertainty $\frac{\hbar}{2}$ at $n=0$ (later we'll see this implies the ground state of the Harmonic oscillator is a gaussian).

$|n\rangle$ IN SPATIAL REPRESENTATIONS

we want wavefunction $\langle Q|n\rangle = \psi_n(Q)$, and the related $\langle x|n\rangle = \phi_n(x)$.

$\hat{Q}|Q\rangle = Q|Q\rangle$, $\langle Q|Q'\rangle = \delta(Q-Q')$, $\hat{1} = \int_{-\infty}^{\infty} dQ\,|Q\rangle\langle Q|$ and

$\hat{x}|x\rangle = x|x\rangle$, $\langle x|x'\rangle = \delta(x-x')$, $\hat{1} = \int_{-\infty}^{\infty} dx\,|x\rangle\langle x|$

$\hat{x} = \sqrt{\frac{\hbar}{m\omega}}\,\hat{Q}$ so $|x\rangle = c\,|Q = \sqrt{\frac{m\omega}{\hbar}}\,x\rangle$ for some constant c, which we can find in two ways:

1. $\hat{1} = \int dx\,|x\rangle\langle x| = \int\sqrt{\frac{\hbar}{m\omega}}\,dQ\,|Q\rangle\langle Q|\,c^2$, $c^2 = \sqrt{\frac{m\omega}{\hbar}} \rightarrow c = \sqrt[4]{\frac{m\omega}{\hbar}}$

2. $\langle x|x'\rangle = \delta(x-x') = c^2\langle Q = \sqrt{\frac{m\omega}{\hbar}}\,x\,|\,Q' = \sqrt{\frac{m\omega}{\hbar}}\,x'\rangle$
$$= c^2\,\delta\left(\sqrt{\frac{m\omega}{\hbar}}\,(x-x')\right) = \sqrt{\frac{\hbar}{m\omega}}\,c^2\,\delta(x-x')$$

Then, we know:

$$\langle x|n\rangle = \phi_n(x) = \sqrt[4]{\frac{m\omega}{\hbar}}\;\psi_n\left(Q = \sqrt{\frac{m\omega}{\hbar}}\,x\right)$$

Now we just need $\psi_n(Q)$. To find this, we use the fact that $\hat{a}|0\rangle = 0$ and make use of the fact that $\hat{a} = (\hat{Q} + i\hat{P})/\sqrt{2}$.

$\hat{a}|0\rangle = \frac{1}{\sqrt{2}}(\hat{Q}+i\hat{P})|0\rangle = 0$

$\langle Q| \curvearrowleft$ $\langle Q|\frac{1}{\sqrt{2}}(\hat{Q}+i\hat{P})|0\rangle$ $\hat{Q}$ is self-adjoint, so $\langle Q|\hat{Q} = Q\langle Q|$, but

what about $\langle Q|\hat{P}$? Given we know $[\hat{Q}, \hat{P}] = i$, we can say it is

$-i\frac{d}{dQ}\langle Q|$... (somehow) or we can do $\langle Q|\hat{P} = \frac{1}{c}\langle x|\frac{\hat{p}}{\sqrt{m\hbar\omega}} =$

$= \frac{1}{\sqrt{m\hbar\omega}}(-i\hbar)\frac{d}{dx}\frac{1}{c}\langle x| = \frac{1}{\sqrt{m\hbar\omega}}\sqrt{\frac{m\omega}{\hbar}}(-i\hbar)\frac{d}{dQ}\langle Q| = -i\frac{d}{dQ}\langle Q|$

either way, this allows us to continue:

$\hat{a}|0\rangle = 0 = \frac{1}{\sqrt{2}}\left(Q + i\left(-i\frac{d}{dQ}\right)\right)\langle Q|0\rangle$, where $\langle Q|0\rangle = \psi_0(Q)$

$0 = \frac{1}{\sqrt{2}}(Q\psi_0(Q) + \psi_0'(Q))$

$\psi_0'(Q) = -Q\psi_0(Q)$ has solution $\frac{d}{dQ}(\ln\psi_0(Q)) = -\frac{d}{dQ}\frac{Q^2}{2}$

so $\ln\psi_0(Q) = -\frac{1}{2}Q^2 + C \rightarrow \psi_0(Q) = Ne^{-Q^2/2}$

Enforce $\langle 0|0\rangle = 1$: $\int_{-\infty}^{\infty} dQ\,|\psi_0(Q)|^2 = N^2\int_{-\infty}^{\infty} e^{-Q^2}dQ = N^2\sqrt{\pi} = 1$

so, $N = \frac{1}{\sqrt[4]{\pi}}$

Thus the ground state in Q representation is $\frac{1}{\sqrt[4]{\pi}}e^{-Q^2/2}$ and as mentioned earlier, is the gaussian wavepacket with minimum uncertainty.

We already found $|n\rangle = \frac{(\hat{a}^+)^n}{\sqrt{n!}}|0\rangle$, so we can find $\psi_n(Q)$ now:

$\psi_n(Q) = \frac{1}{\sqrt{n!}}\left((\hat{Q}-i\hat{P})/\sqrt{2}\right)^n\psi_0(Q)$

$= \frac{1}{\sqrt{n!}}\left(Q - \frac{d}{dQ}\right)^n/2^{n/2}\,\frac{1}{\sqrt[4]{\pi}}e^{-Q^2/2}$ } add $\underbrace{e^{-Q^2/2}e^{Q^2/2}}_{=1}$

$= \frac{e^{-Q^2/2}e^{Q^2/2}}{\sqrt{2^n n!}\sqrt[4]{\pi}}\left(Q - \frac{d}{dQ}\right)^n e^{-Q^2/2}$

We can, of course write $\left(Q-\frac{d}{dQ}\right)^n$ as $\prod_{i=1}^{n}\left(Q-\frac{d}{dQ}\right)$, so part of $\psi_n(Q)$

is $e^{Q^2/2}\prod_{i=1}^{n}\left(Q-\frac{d}{dQ}\right)e^{-Q^2/2}$. Because $e^{Q^2/2}e^{-Q^2/2} = 1$, we can pull them into $\prod$ and still get the same result. You can also think of it as:

$$(e^{Q^2/2} a e^{-Q^2/2})(e^{Q^2/2} a e^{-Q^2/2}) = e^{Q^2/2} a (e^{-Q^2/2} e^{Q^2/2}) a e^{-Q^2/2}$$

$$= e^{Q^2/2} a (1) a e^{-Q^2/2} = e^{Q^2/2} a^2 e^{-Q^2/2}$$ & so on...

Basically, $e^{Q^2/2}(Q - \frac{d}{dQ})^n e^{-Q^2/2} = \prod_{i=1}^{n} e^{Q^2/2}(Q - \frac{d}{dQ}) e^{-Q^2/2}$

$= (e^{Q^2/2}(Q - \frac{d}{dQ}) e^{-Q^2/2})^n$ and we plug this into $\Psi_n(Q)$:

$$\Psi_n(Q) = \frac{e^{-Q^2/2}}{\sqrt{2^n n!}\,\sqrt[4]{\pi}} (e^{Q^2/2}(Q - \frac{d}{dQ}) e^{-Q^2/2})^n$$

You can find $e^{Q^2/2}(Q - \frac{d}{dQ}) e^{-Q^2/2}$ as you would any other expression:

$$e^{Q^2/2}(Qe^{-Q^2/2} - (-Qe^{-Q^2/2})) - \frac{d}{dQ} = 2Q - \frac{d}{dQ}$$

differential operator still needs to operate on any following terms

so:

$$\Psi_n(Q) = \frac{e^{-Q^2/2}}{\sqrt{2^n n!}\,\sqrt[4]{\pi}} (2Q - \frac{d}{dQ})^n$$

<u>Furthermore</u> we call $(2Q - \frac{d}{dQ})^n 1$ Hermite polynomial $H_n(Q)$:

$$H_{n=0}(Q) = (2Q - \frac{d}{dQ})^0 1 = 1$$

$$H_{n=1}(Q) = (2Q - \frac{d}{dQ}) H_0 = 2Q$$

$$H_{n=2}(Q) = (2Q - \frac{d}{dQ}) H_1 = 4Q^2 - 2$$

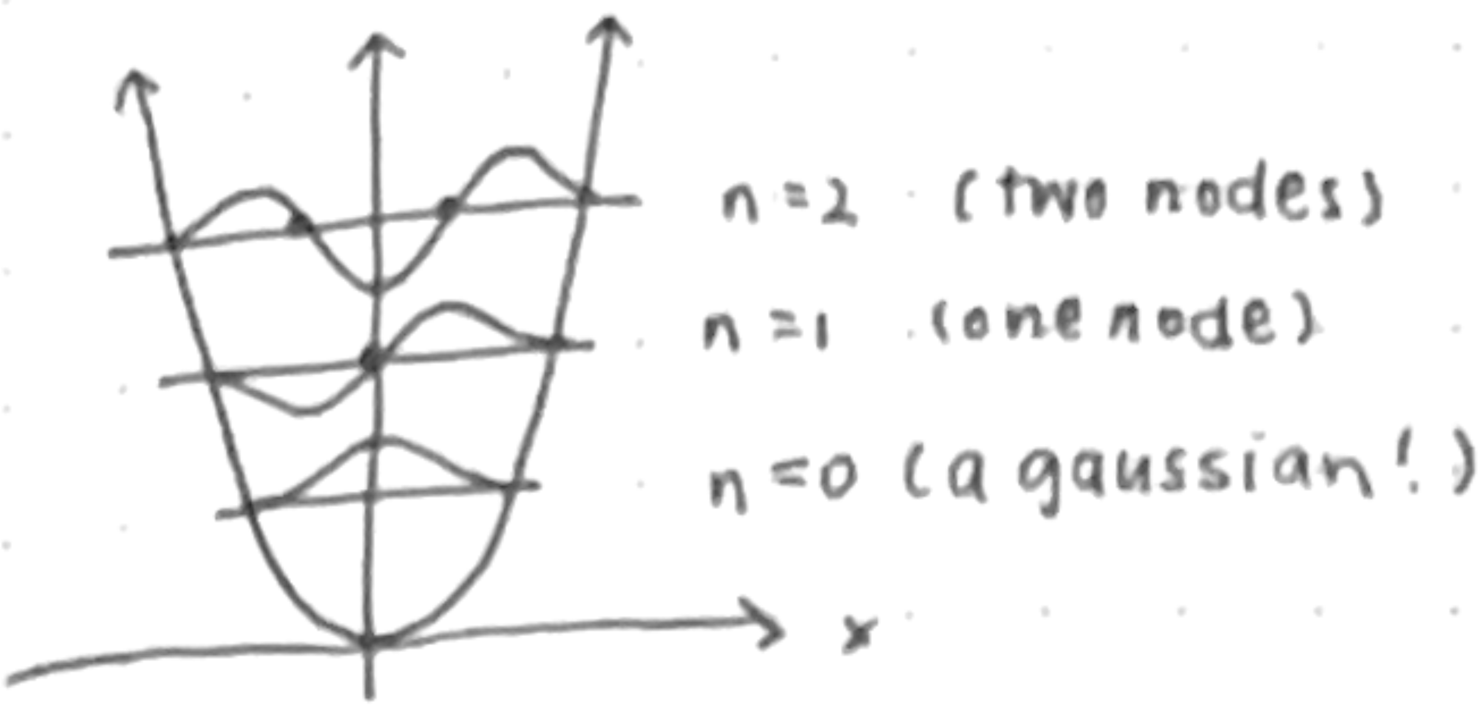
$$H_{n=3}(Q) = (2Q - \frac{d}{dQ}) H_2 = 8Q^3 - 4Q - 8Q = 8Q^3 - 12Q$$

etc. We use this $H_n(Q)$ notation:

$\Psi_n(Q) = \frac{e^{-Q^2/2}}{\sqrt{2^n n!}\,\sqrt[4]{\pi}} H_n(Q)$ yay. In terms of x, we have

$\phi_n(x) = \sqrt[4]{\frac{m\omega}{\pi\hbar}} \frac{1}{\sqrt{2^n n!}} e^{-\frac{m\omega}{2\hbar}x^2} H_n(\sqrt{\frac{m\omega}{\hbar}}x)$ yayy

These eigenfunctions of the Harmonic oscillator behave as expected:



WEEK 14 - COHERENT STATES & QM HARMONIC OSCILLATOR

During our previous discussion on the Harmonic oscillator we were introduced to operators $\hat{Q}$, $\hat{P}$, $\hat{a}$, and $\hat{a}^{+}$:

$$\hat{Q} = \sqrt{\frac{m\omega}{\hbar}}\,\hat{x}, \quad \hat{P} = \frac{1}{\sqrt{m\hbar\omega}}\,\hat{p} \qquad [\hat{Q},\hat{P}] = i \quad \text{so } \hat{H} = \frac{\hbar\omega}{2}(\hat{Q}^2 + \hat{P}^2)$$ and

any transformation in the Q-P plane leaves H invariant, and

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{Q} + i\hat{P}), \quad \hat{a}^{+} = \frac{1}{\sqrt{2}}(\hat{Q} - i\hat{P}), \quad \hat{n} = \hat{a}^{+}\hat{a} \text{ so } \hat{H} = \hbar\omega(\hat{n} + 1/2)$$

where $\sigma(\hat{n}) = \mathbb{N}_0$ and $\hat{n}$ is non-degenerate, $|n\rangle$ are eigenvectors of $\hat{H}$ also. $\hat{H}$'s eigenvalues are evenly spaced by $\hbar\omega$.

$\hat{a}$ is not self-adjoint or normal, so we can't expect the Spectral theorem to guarantee anything. <u>However</u>, there are solutions to $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, and these are <u>COHERENT STATES</u>, which reproduce <u>classical</u> limits.

(Cat states can be a linear combination of coherent states; coherent states describe classical limit of monochromatic fields, e.g. light plane waves, etc.)

Let's find $|\alpha\rangle$: $\quad |\alpha\rangle = \sum_{n=0}^{\infty} |n\rangle C_n$

$$\hat{a}|\alpha\rangle = \sum_{n=0}^{\infty} \hat{a}|n\rangle C_n = \alpha\sum_{n=0}^{\infty}|n\rangle C_n$$

$$= \sum_{n=1}^{\infty} \sqrt{n}\,|n-1\rangle C_n = \sum_{n=1}^{\infty} \alpha\,|n-1\rangle C_{n-1}$$

so, $$\sum_{n=1}^{\infty} \sqrt{n}\,|n-1\rangle C_n - \alpha|n-1\rangle C_{n-1} = 0$$

$$\sum_{n=1}^{\infty} |n-1\rangle(\sqrt{n}\,C_n - \alpha C_{n-1}) = 0 \rightarrow \sqrt{n}\,C_n = \alpha C_{n-1} \quad \forall n > 0$$

Then $$C_n = \frac{\alpha}{\sqrt{n}}C_{n-1} = \frac{\alpha}{\sqrt{n}}\frac{\alpha}{\sqrt{n-1}}C_{n-2} = \dots = \frac{\alpha^n}{\sqrt{n!}}C_0$$

So: $|\alpha\rangle = \sum_{n=0}^{\infty} |n\rangle C_n = C_0\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$. We can check to make sure $\| |\alpha\rangle \|$ doesn't diverge:

$$\| |\alpha\rangle \|^2 = \langle\alpha|\alpha\rangle = |C_0|^2 \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\alpha^*)^n}{\sqrt{n!}} \underbrace{\langle n|m\rangle}_{\delta_{nm}} \frac{\alpha^m}{\sqrt{m!}}$$

$$= |C_0|^2 \sum_{n=0}^{\infty} \frac{(|\alpha|^2)^n}{n!} = |C_0|^2 e^{|\alpha|^2} \text{ is finite } \checkmark$$

Thus the <u>normalized</u> state is $|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$, and this is valid for all $\alpha \in \mathbb{C}$, so $\sigma(\hat{a}) = \mathbb{C}$.

These eigenstates are not orthogonal: $\langle\alpha|\beta\rangle = e^{-(|\alpha|^2+|\beta|^2)/2} \sum_n \frac{(\alpha^*\beta)^n}{n!}$
$= e^{-1/2(|\alpha|^2+|\beta|^2-2\alpha^*\beta)} \neq 0$

But, $|\langle\alpha|\beta\rangle|^2 = e^{-(|\alpha|^2+|\beta|^2-\alpha^*\beta-\alpha\beta^*)} = e^{-|\alpha-\beta|^2}$ which means the overlap between two eigenstates gets very small when α and β are very far from each other and $|\alpha-\beta| \gg 1$.

PROBABILITY. The probability of measuring any state $|n\rangle$, so of measuring E_n when in state $|\alpha\rangle$ is:

$$P_n = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} = |\langle n|\alpha\rangle|^2 = \left| e^{-|\alpha|^2/2} \sum_{n'=0}^{\infty} \frac{\alpha^{n'}}{\sqrt{n'!}} \langle n|n'\rangle \right|^2$$

This is called the Poisson Distribution, and is used as sort of a measure of "classicalness" of light, as it describes the distribution of classical/non-entangled light.

(Also note $|\alpha=0\rangle = e^0 \sum_{n=0}^{\infty} \frac{0^n}{\sqrt{n!}}|n\rangle$, $0^n = 1$ only if $n=0$, $0^n = 0$ otherwise, so $|\alpha=0\rangle = |n=0\rangle$, which is the ground state of $\hat{H}$. No other coherent state corresponds to an eigenstate of $\hat{H}$.)

EXPECTED VALUE / UNCERTAINTY

$$\langle\hat{H}\rangle_\alpha = \hbar\omega\langle\alpha|\hat{a}^\dagger\hat{a}+1/2|\alpha\rangle = \hbar\omega\left(\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle + \tfrac{1}{2}\langle\alpha|\alpha\rangle\right)$$
$$= \hbar\omega\left(|\alpha|^2 + 1/2\right)$$

$$\langle\hat{n}\rangle_\alpha = \langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle = |\alpha|^2 \quad \text{"average excitation"}$$

$$\langle\hat{H}^2\rangle_\alpha = \hbar^2\omega^2\langle\alpha|(\hat{a}^\dagger\hat{a}+1/2)(\hat{a}^\dagger\hat{a}+1/2)|\alpha\rangle$$
$$= \hbar^2\omega^2\langle\alpha|\hat{a}^\dagger\underline{\hat{a}\hat{a}^\dagger}\hat{a} + \hat{a}^\dagger\hat{a} + \tfrac{1}{4}|\alpha\rangle$$
$$= \hbar^2\omega^2\langle\alpha|\hat{a}^\dagger(\hat{a}^\dagger\hat{a}+1)\hat{a} + \hat{a}^\dagger\hat{a} + \tfrac{1}{4}|\alpha\rangle$$
$$= \hbar^2\omega^2\langle\alpha|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + 2\hat{a}^\dagger\hat{a} + \tfrac{1}{4}|\alpha\rangle = \hbar^2\omega^2\left(|\alpha|^4 + 2|\alpha|^2 + \tfrac{1}{4}\right)$$

$$\sigma_H^2 \equiv \langle \hat{H}^2 \rangle_\alpha - \langle \hat{H} \rangle_\alpha^2$$

$$= (\hbar\omega)^2 \left(|\alpha|^4 + 2|\alpha|^2 + \tfrac{1}{4} - |\alpha|^4 - |\alpha|^2 - \tfrac{1}{4} \right)$$

$$= \hbar^2 \omega^2 |\alpha|^2$$

$\sigma_H = \hbar\omega|\alpha| \rightarrow \dfrac{\sigma_H}{\langle \hat{H} \rangle_\alpha} = \dfrac{1}{|\alpha| + \frac{1}{2|\alpha|}}$ so the relative uncertainty of the energy decreases as $|\alpha|$ gets larger and larger.

$\dfrac{\sigma_n}{\langle n \rangle_\alpha} = \dfrac{1}{|\alpha|}$ exactly. In both cases, as $\alpha \rightarrow \infty$, precision increases to the classical limit.

$$\hat{x} = \sqrt{\frac{\hbar}{m\omega}}\, \hat{Q} = \sqrt{\frac{\hbar}{m\omega}} \frac{1}{\sqrt{2}} (\hat{a} + \hat{a}^\dagger)$$

$$\langle \hat{x} \rangle_\alpha = \sqrt{\frac{\hbar}{2m\omega}} \langle \alpha | \hat{a} + \hat{a}^\dagger | \alpha \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\langle \alpha | \hat{a} | \alpha \rangle + \langle \alpha | \hat{a}^\dagger | \alpha \rangle \right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} (\alpha + \alpha^*) = \sqrt{\frac{2\hbar}{m\omega}}\, \mathrm{Re}\{\alpha\}$$

indicating that the real part of α is the center of the distribution in position space.

$$\hat{p} = \sqrt{m\hbar\omega}\, \hat{P} = \sqrt{\frac{m\hbar\omega}{2}} \frac{1}{i} (\hat{a} - \hat{a}^\dagger)$$

$$\langle \hat{p} \rangle_\alpha = \sqrt{\frac{m\hbar\omega}{2}} \frac{1}{i} \langle \alpha | \hat{a} - \hat{a}^\dagger | \alpha \rangle = \sqrt{2m\hbar\omega} \frac{\alpha - \alpha^*}{2i} = \sqrt{2m\hbar\omega}\, \mathrm{Im}\{\alpha\}$$

indicating imaginary part of α is the center of the distribution in momentum space.

$$\langle \hat{x}^2 \rangle = \frac{\hbar}{2m\omega} \langle \alpha | \hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger \hat{a} | \alpha \rangle$$

$$= \frac{\hbar}{2m\omega} \left(\alpha^2 + (\alpha^*)^2 + 2|\alpha| + 1 \right) \rightarrow \sigma_x = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\langle \hat{p}^2 \rangle = -\frac{m\hbar\omega}{2} \langle \alpha | \hat{a}^2 + \hat{a}^{\dagger 2} - \hat{a}^\dagger \hat{a} - \hat{a}\hat{a}^\dagger | \alpha \rangle$$

$$= -\frac{m\hbar\omega}{2} \left(\alpha^2 + (\alpha^*)^2 - 2|\alpha| - 1 \right) \rightarrow \sigma_p = \sqrt{\frac{m\hbar\omega}{2}}$$

so, $\sigma_x \sigma_p = \frac{\hbar}{2}$ for EVERY coherent state (neither σ_x or σ_p depend on α...) so every coherent state is a minimum uncertainty wave packet.

TIME DEPENDENCE

$|\psi(t=0)\rangle = |\alpha\rangle$

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\alpha\rangle = e^{-i\hat{H}t/\hbar} e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

$$= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}} e^{-i\hbar\omega(n+1/2)t/\hbar}|n\rangle$$

$$= e^{-i\omega t/2} e^{-|\alpha|^2/2}\sum_{n=0}^{\infty} e^{-i\omega t n}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

$$= e^{-i\omega t/2} e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}(\alpha e^{-i\omega t})^n|n\rangle$$

since $|\alpha e^{-i\omega t}| = |\alpha|$, this can be written as:

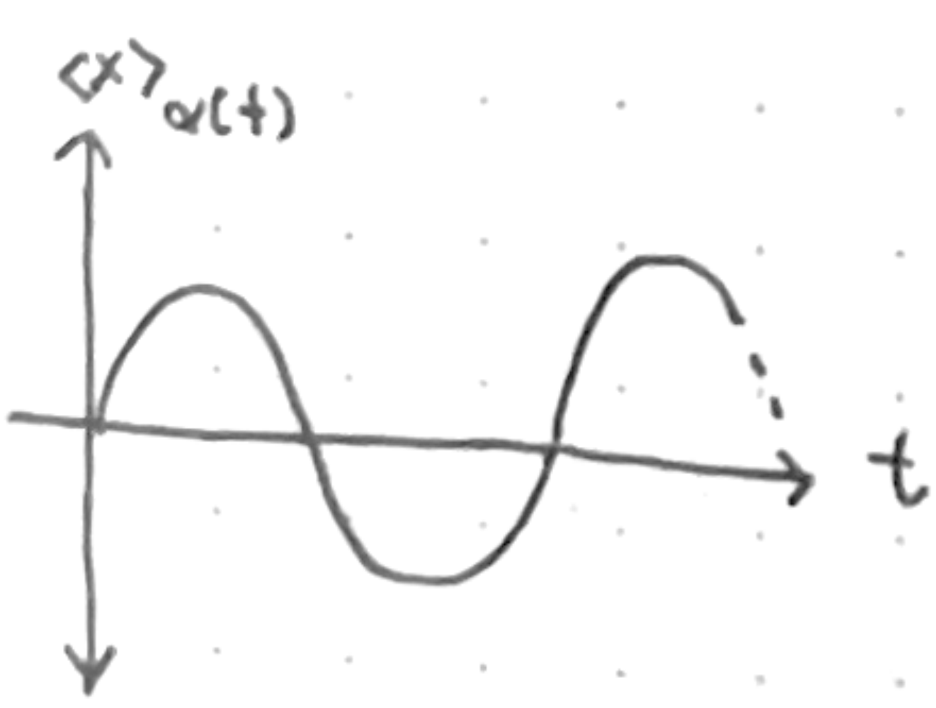
$$|\psi(t)\rangle = e^{-i\omega t/2} e^{-|\alpha e^{-i\omega t}|^2/2}\sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}(\alpha e^{-i\omega t})^n|n\rangle$$

$= e^{-i\omega t/2}|\alpha e^{-i\omega t}\rangle$ where $|\alpha e^{-i\omega t}\rangle$ is a totally valid coherent state. So, global phase factor aside, coherent states stay coherent states over time, where $\alpha(t) = \alpha e^{-i\omega t}$.

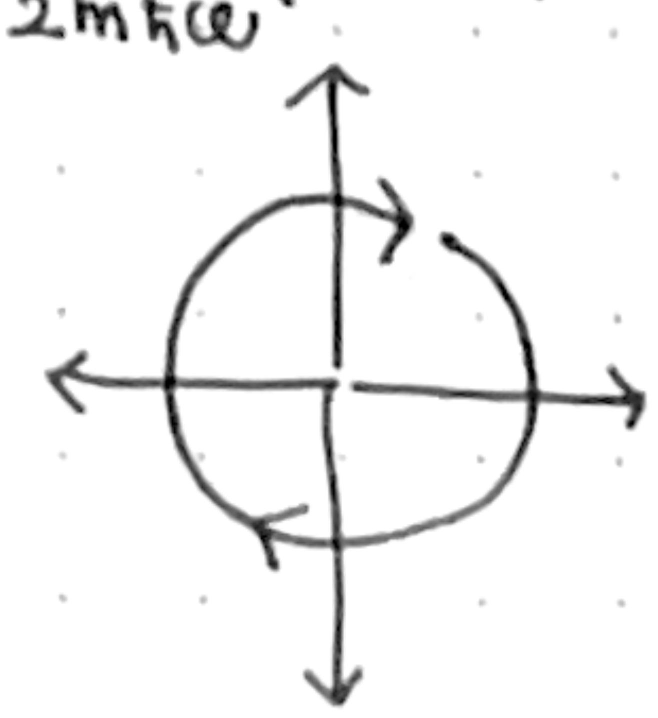
$\langle\hat{x}\rangle_{\alpha(t)} = \sqrt{\frac{2\hbar}{m\omega}}\,\mathrm{Re}\{\alpha e^{-i\omega t}\} = \sqrt{\frac{2\hbar}{m\omega}}|\alpha|\cos(\phi-\omega t)$, where

$\alpha = |\alpha|e^{i\phi}$

$\langle\hat{p}\rangle_{\alpha(t)} = \sqrt{2m\hbar\omega}\,|\alpha|\sin(\phi-\omega t)$



$\frac{\langle p\rangle}{\sqrt{2m\hbar\omega}} = \mathrm{Im}(\alpha)$



$\sqrt{\frac{m\omega}{2\hbar}}\langle x\rangle = \mathrm{Re}(\alpha)$

DISPLACEMENT OPERATOR: Coherent states are translated (pos) and boosted (momentum) versions of the harmonic oscillator's ground state:

We define the displacement operator $\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}}$ which can be rewritten as:

$$\hat{D}(\alpha) = e^{-i(i\alpha\hat{a}^\dagger - i\alpha^*\hat{a})}$$ where $i\alpha\hat{a}^\dagger - i\alpha^*\hat{a}$ is self-adjoint,

meaning $\hat{D}(\alpha)$ is a unitary operator.

First, we show $|\alpha\rangle = \hat{D}(\alpha)|n=0\rangle$.

Glauber's Formula says if $[\hat{A},\hat{B}]$ commutes with $\hat{A}$ and $\hat{B}$,

$$e^{\hat{A}+\hat{B}} = e^{-[\hat{A},\hat{B}]/2} e^{\hat{A}} e^{\hat{B}}$$

in our case, $[\hat{a}^\dagger,\hat{a}] = -1$, and -1 commutes with $\hat{a}^\dagger$ and $\hat{a}$.

Thus we can write:

$$\hat{D}(\alpha) = e^{\frac{-[\alpha\hat{a}^\dagger, -\alpha^*\hat{a}]}{2}} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}}$$

$$\text{and } \hat{D}(\alpha)|0\rangle = e^{\frac{-[\alpha\hat{a}^\dagger, -\alpha^*\hat{a}]}{2}} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}}|0\rangle$$

$$= e^{\frac{-[\alpha\hat{a}^\dagger, -\alpha^*\hat{a}]}{2}} e^{\alpha\hat{a}^\dagger} \left(1 - \alpha^*\hat{a} + \frac{(-\alpha^*)^2}{2!}\hat{a}^2 + \dots\right)|0\rangle$$

because $\hat{a}^n|0\rangle = 0$ (see video lecture notes), we have:

$$\hat{D}(\alpha)|0\rangle = e^{\frac{-[\alpha\hat{a}^\dagger, -\alpha^*\hat{a}]}{2}} e^{\alpha\hat{a}^\dagger}|0\rangle$$

$$= e^{\frac{-[\alpha\hat{a}^\dagger, -\alpha^*\hat{a}]}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}(\hat{a}^\dagger)^n|0\rangle$$

because $\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$,

$$\hat{D}(\alpha)|0\rangle = e^{\frac{-[\alpha\hat{a}^\dagger, -\alpha^*\hat{a}]}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!}(\hat{a}^\dagger)^0\sqrt{n!}\,|n\rangle$$

$$= e^{\frac{-[\alpha\hat{a}^\dagger, -\alpha^*\hat{a}]}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

and $-[\alpha\hat{a}^\dagger, -\alpha^*\hat{a}] = +|\alpha|^2[\hat{a}^\dagger,\hat{a}] = |\alpha|^2(-1)$

$$\text{so } \hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle = |\alpha\rangle \checkmark$$

Second, we show $\hat{D}(\alpha) = e^{i\theta}\hat{B}_{\langle p\rangle_\alpha}\hat{T}_{\langle \hat{x}\rangle_\alpha}$, which, paired with the first, shows all $|\alpha\rangle$ are translated and boosted versions of $|0\rangle$.

This time, use $\hat{a} = \frac{1}{\sqrt{2}}(\hat{Q}+i\hat{P})$ and $\hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{Q}-i\hat{P})$, $\hat{Q} = \sqrt{\frac{m\omega}{\hbar}}\,\hat{x}$ and $\hat{P} = \frac{\hat{p}^2}{\sqrt{m\hbar\omega}}$ to write $\hat{D}(\alpha)$ as:

$$\hat{D}(\alpha) = e^{(\alpha-\alpha^*)\sqrt{\frac{m\omega}{2\hbar}}\hat{x} - i\frac{\alpha+\alpha^*}{\sqrt{2m\hbar\omega}}\hat{p}}$$, then use Glauber's again.

$$\hat{D}(\alpha) = e^{+[(\alpha-\alpha^*)\sqrt{\frac{m\omega}{2\hbar}}\hat{x},\, +i\frac{\alpha+\alpha^*}{\sqrt{2m\hbar\omega}}\hat{p}]/2}\, e^{(\alpha-\alpha^*)\sqrt{\frac{m\omega}{2\hbar}}\hat{x}}\, e^{-i\frac{\alpha+\alpha^*}{\sqrt{2m\hbar\omega}}\hat{p}}$$

$$= e^{(\alpha-\alpha^*)(\alpha+\alpha^*)\frac{i}{\sqrt{4\hbar^2}}\underbrace{[\hat{x},\hat{p}]}_{i\hbar}/2}\, e^{(\alpha-\alpha^*)\sqrt{\frac{m\omega}{2\hbar}}\hat{x}}\, e^{-i\frac{\alpha+\alpha^*}{\sqrt{2m\hbar\omega}}\hat{p}}$$

$$= e^{-\frac{(\alpha-\alpha^*)(\alpha+\alpha^*)}{4}}\, e^{(\alpha-\alpha^*)\sqrt{\frac{m\omega}{2\hbar}}\hat{x}}\, e^{-i\frac{\alpha+\alpha^*}{\sqrt{2m\hbar\omega}}\hat{p}}$$

recall that $\langle\hat{x}\rangle_\alpha = \sqrt{\frac{\hbar}{2m\omega}}(\alpha+\alpha^*)$ and $\langle\hat{p}\rangle_\alpha = \sqrt{2m\hbar\omega}\,\frac{\alpha-\alpha^*}{2i}$
We can write exponents as:

$$(\alpha-\alpha^*)\sqrt{\frac{m\omega}{2\hbar}}\hat{x} = \frac{\langle\hat{p}\rangle_\alpha 2i}{\sqrt{2m\hbar\omega}}\sqrt{\frac{m\omega}{2\hbar}}\hat{x} = \langle\hat{p}\rangle_\alpha i\hat{x}/\hbar$$

and,

$$-i\frac{\alpha+\alpha^*}{\sqrt{2m\hbar\omega}}\hat{p} = -i\frac{\langle\hat{x}\rangle_\alpha\sqrt{\frac{2m\omega}{\hbar}}}{\sqrt{2m\hbar\omega}}\hat{p} = -i\langle\hat{x}\rangle_\alpha\hat{p}/\hbar$$

so:

$$\hat{D}(\alpha) = e^{(\alpha^{*2}-\alpha^2)/4}\, e^{i\langle\hat{p}\rangle_\alpha\hat{x}/\hbar}\, e^{-i\langle\hat{x}\rangle_\alpha\hat{p}/\hbar}$$

$$= \underbrace{e^{(\alpha^{*2}-\alpha^2)/4}}_{\text{global phase factor}}\ \underbrace{\hat{B}_{\langle\hat{p}\rangle_\alpha}}_{\text{momentum boost}}\ \underbrace{\hat{T}_{\langle\hat{x}\rangle_\alpha}}_{\text{space translation}} \quad \checkmark$$

∴ A coherent state is a shifted and boosted wave packet.

Let's see this... $\psi_0(x) = \langle x|0\rangle = \sqrt[4]{\frac{m\omega}{\pi\hbar}}\, e^{-\frac{m\omega}{2\hbar}x^2}$ (see video lecture)

so $\langle x|\alpha\rangle = \psi_\alpha(x) = \langle x|\hat{D}(\alpha)|0\rangle$

$$= e^{i\theta_\alpha}\sqrt[4]{\frac{m\omega}{\pi\hbar}}\, e^{i\langle\hat{p}\rangle_\alpha\hat{x}/\hbar}\, e^{-i\langle\hat{x}\rangle_\alpha\hat{p}/\hbar}\, e^{-\frac{m\omega}{2\hbar}x^2}$$

(for $\theta_\alpha = (\alpha^{*2}-\alpha^2)/4i$ ..)

$$= e^{i\theta_\alpha}\sqrt[4]{\frac{m\omega}{\pi\hbar}}\, e^{i\langle\hat{p}\rangle_\alpha\hat{x}/\hbar}\, \overbrace{e^{-\frac{m\omega}{2\hbar}(x-\langle\hat{x}\rangle_\alpha)^2}}^{\psi_0(x-\langle\hat{x}\rangle_\alpha)}$$

The shape / probability distribution over x does not change over time:

$$|\psi_{\alpha(t)}(x)|^2 = |\psi_0(x-\langle\hat{x}\rangle_{\alpha(t)})|^2$$ ← translation only (shape is constant)