Week 11

→ Generalized Eigenstates

- For $\hat{x}|x\rangle = x|x\rangle$ and $\hat{p}|p\rangle = p|p\rangle$, we know that $|x\rangle$ and $|p\rangle$ are generalized eigenstates and not in a Hilbert space.

However, they let us map some $|\psi\rangle$ in a Hilbert space to a $\langle x|\psi\rangle$ or $\langle p|\psi\rangle$, functions that are square integrable. Basically, $H \to L^2(\mathbb{R})$.

So, inserting the identity into $|\psi\rangle$, we see that:

$$\int_{-\infty}^{\infty} dx\ |x\rangle\langle x|\psi\rangle = \int_{-\infty}^{\infty} dx\ |x\rangle \psi(x) \in H. \text{ (even if } \psi(x) \text{ is not)}$$

- If we want to change from position to momentum space, and get $\phi_\psi(p) = \phi(p) = \langle p|\psi\rangle$, we can also insert the identity:

$$\langle p|\psi\rangle = \int_{-\infty}^{\infty} dx\ \langle p|x\rangle\langle x|\psi\rangle = \int_{-\infty}^{\infty} dx\ \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx/\hbar}\ \psi(x)$$

↑ found this previously!

which is the Fourier transform of $\psi(x)$ to momentum space.

- The inverse can take us from position to momentum:

$$\psi_\psi(x) = \psi(x) = \langle x|\psi\rangle = \int_{-\infty}^{\infty} dp\ \langle x|p\rangle\langle p|\psi\rangle$$

$$= \int_{-\infty}^{\infty} dp\ \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}\ \phi(x)$$

so, the inverse fourier transform.

- We can substitute $\phi(x)$ into the last above equation:

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp\ e^{ipx/\hbar} \int_{-\infty}^{\infty} dx' \frac{1}{\sqrt{2\pi\hbar}} e^{-ipx'/\hbar}\ \psi(x')$$

$$= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx'\ e^{ip(x-x')/\hbar}\ \psi(x')$$

We can demonstrate this is true without using any of what we showed above by working with the RHS:

- It would be convenient if we swapped the two integrals, so we had:

$$\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dx'\ \psi(x') \int_{-\infty}^{\infty} dp\ e^{ip(x-x')/\hbar}$$

but $e^{ip(x-x')/\hbar}$ is not integrable. so, introduce some normalization factor that makes the integrand 0 at ∞ and $-\infty$:

$e^{-\lambda|p|/\hbar}$ and then at the end, we just take limit as $\lambda \to 0$ to get rid of the term. We get:

$$\int_a^b e^{\alpha x}\,dx = \frac{1}{\alpha}\left(e^{\alpha b} - e^{\alpha a}\right) \quad \text{so}$$

split integral because of abs value.

$$\int_{-\infty}^{\infty} dp\, e^{ip(x-x')/\hbar - \lambda|p|/\hbar} = \int_0^{\infty} dp\, e^{ip(x-x')/\hbar - \lambda p/\hbar} + \int_{-\infty}^{0} dp\, e^{\cdots + \lambda p/\hbar}$$

$$= \frac{1}{\frac{i(x-x')}{\hbar} - \frac{\lambda}{\hbar}}\left(e^{\infty} - e^{0}\right) + \frac{1}{\frac{i(x-x')}{\hbar} + \frac{\lambda}{\hbar}}\left(e^{0} - e^{-\infty}\right)$$

$$= \frac{\hbar}{i(x-x')-\lambda}(-1) + \frac{\hbar}{i(x-x')+\lambda}(1) = \frac{2\hbar\lambda}{(x-x')^2+\lambda^2}$$

so overall, RHS after swapping order of integrals:

$$\int_{-\infty}^{\infty} dx'\,\Psi(x') \int_{-\infty}^{\infty} dp\, \frac{1}{2\pi\hbar} e^{ip(x-x')/\hbar}$$

$\downarrow$

$$\int_{-\infty}^{\infty} dx'\,\Psi(x') \frac{1}{2\pi\hbar}\frac{2\hbar\lambda}{(x-x')^2+\lambda^2} = \frac{1}{\pi}\int_{-\infty}^{\infty} dx'\,\Psi(x')\frac{\lambda}{(x-x')^2+\lambda^2}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} dx'/\lambda\ \Psi(x')\frac{\lambda^2}{(x-x')^2+\lambda^2} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dx'}{\lambda}\Psi(x')\frac{1}{\left(\frac{x'-x}{\lambda}\right)^2+1}$$

$u = \frac{x'-x}{\lambda}$ $du = dx'/\hbar$ and $x' = x + u\lambda$, so:

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} du\,\Psi(x+\lambda u)\frac{1}{u^2+1}$$

then: $$\lim_{\lambda\to 0}\frac{1}{\pi}\int_{-\infty}^{\infty} du\,\Psi(x+\lambda u)\frac{1}{u^2+1} = \frac{1}{\pi}\Psi(x)\int_{-\infty}^{\infty} du\,\frac{1}{u^2+1}$$

$$= \frac{1}{\pi}\Psi(x)\Big[\arctan(u)\Big]_{-\infty}^{\infty} = \Psi(x)\ \checkmark$$

From $\Psi(x) = \int_{-\infty}^{\infty} dx'\,\Psi(x')\underbrace{\int_{-\infty}^{\infty} dp\,\frac{1}{2\pi\hbar}e^{ip(x-x')/\hbar}}$ it also follows that this must be $\delta(x-x')$.

• swapping the order of integrals and adding $e^{-\lambda|p|/\hbar}$ is very hand-wavey. To clarify:

We can only swap the order of integration if the integrand is absolutely integrable, or:

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, f(x,y) = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx\, f(x,y) \quad \text{IFF} \quad \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, |f(x,y)| < \infty$$

In our case, $f(x',p) = e^{ip(x-x')/\hbar}\,\psi(x')$ and so, $|f(x',p)|$ is $|\psi(x')|$ which means:

$$\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx'\, \psi(x') = \infty,$$ since $\psi(x')$ is not in terms of p.

We can't swap the order unless we recast this integral so that the integrand is in terms of p and absolutely integrable... ?

$$f(x',p) \rightarrow f_\lambda(x',p) \equiv e^{-\lambda|p|/\hbar} f(x',p)$$

$$f(x'p) = \lim_{\lambda \to 0^+} e^{-\lambda|p|/\hbar} f(x',p)$$ pointwise. We substitute this in:

$$\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx'\, e^{ip(x-x')/\hbar}\, \psi(x')$$

$$= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \lim_{\lambda \to 0^+} e^{ipx/\hbar} e^{-\lambda|p|/\hbar} \int_{-\infty}^{\infty} dx'\, e^{-ipx'/\hbar}\, \psi(x'),$$ but we still can't swap the order of integration because we have a limit! If we can pull out the limit, we can then swap & solve as we did before. To do this, we need DCT:

According to the Dominated Convergence Theorem (DCT), you can swap integral and limit: $\int \lim_{n\to\infty} F_n(x)\,dx = \lim_{n\to\infty} \int F_n(x)\,dx$ under the following conditions.

- Say we have a family of functions $F_n(x): \mathbb{R} \to \mathbb{C}$ which is continuous and parameterized by n.
- $\exists F \mid \lim_{n\to\infty} F_n = F$ pointwise.
- $\exists G$ which is positive and integrable (so absolutely integrable) such that $|F_n(x)| \leq G(x)\ \forall x \in \mathbb{R}$ (domain of F_n ...)

For $F_\lambda(p) = e^{-\lambda|p|/\hbar} e^{ipx/\hbar} \int_{-\infty}^{\infty} dx' e^{-ipx'/\hbar} \psi(x')$, we have that

$|F_\lambda(p)| < |\phi(p)| c$, where c is some constant, so we can swap the limit and integral:

$$\lim_{\lambda\to0^+} \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx' \underbrace{e^{-\lambda|p|/\hbar} e^{ip(x-x')/\hbar} \psi(x')}_{\text{is now absolutely integrable}} \text{ and swap!}$$

$$= \lim_{\lambda\to0^+} \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dx' \psi(x') \int_{-\infty}^{\infty} dp \, e^{-\lambda|p|/\hbar} e^{ip(x-x')/\hbar}$$

we can solve this exactly as we did earlier and show this expression is $\psi(x)$, getting to:

$$RHS = \lim_{\lambda\to0^+} \frac{1}{\pi} \int_{-\infty}^{\infty} \psi(x+\lambda u) \frac{du}{u^2+1}$$, where we can justify moving the limit inside using DCT again, $\psi(x+\lambda u)/u^2+1$ is parameterized by λ and is always less than/equal to some positive integrable fcn:

$\psi(x+\lambda u) \leq \max|\psi(x)|/u^2+1$

So: $RHS = \frac{1}{\pi}\int_{-\infty}^{\infty} \lim_{\lambda\to0^+} \psi(x+\lambda u) \frac{du}{u^2+1} = \psi(x)$ as before, and again, this tells us that $\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dp \, e^{ip(x-x')/\hbar} e^{-\lambda|p|/\hbar} = \delta(x-x')$

- Conceptually, say $\hat{L}f = f(x)$ (evaluates f @ a specific x) and $\hat{L}$ is linear, so $\hat{L}(\alpha f + \beta g) = \alpha f(x) + \beta g(x)$. You can use

$F(x) = \int K(x,x') f(x') dx'$ as this $\hat{L}f(x)$, where $K(x,x')$ is $\delta(x-x')$ (as we saw!). I don't know anything about functional analysis, though... Anyway:

$RHS = \int dx' \psi(x') \delta(x-x')$ fits this form and so, I guess gives us $\psi(x)$ as we showed (twice...)

→ More about generalized eigenstates
- For non-generalized eigenstates, $|x\rangle\langle x|$ is a projector if $\hat{x}|x\rangle = x|x\rangle$. For generalized eigenstates, however:

$|x\rangle\langle x|\psi\rangle$ for $|\psi\rangle \in H = |x\rangle \psi(x) \notin H$, so $|x\rangle\langle x|$ is not a projector.

instead we can define a projector on an interval of the continuous spectra:

$$\hat{P}_{[a,b]} = \int_a^b |x\rangle\langle x|\, dx \,, \quad \hat{P}_{[a,b]}|\psi\rangle = \int_a^b \overset{dx}{\hat{}} |x\rangle\langle x|\psi\rangle = \int_a^b \overset{dx}{\hat{}} |x\rangle\, \psi(x) \in H$$

since (or maybe this is a result, not a cause) $\int dx\, |x\rangle\langle x| = \hat{1}$?

Probability $P_{[a,b]} = \| \hat{P}_{ab} |\psi\rangle \|^2 = \langle \psi | \hat{P}_{ab}^2 |\psi\rangle$ for a normalized $|\psi\rangle$

$$\langle \psi | \hat{P}_{ab}^2 |\psi\rangle = \langle \psi | \hat{P}_{ab} |\psi\rangle = \langle \psi | \int_a^b |x\rangle\langle x|\psi\rangle\, dx$$

$$= \int_a^b |\psi(x)|^2\, dx$$

so $P_{[a-\frac{\Delta x}{2}, a+\frac{\Delta x}{2}]} = \int_{a-\Delta x}^{a+\Delta x} |\psi(x)|^2\, dx = |\psi(a)|^2 \cdot \Delta x + O(\Delta x^3)$

as we make Δx smaller:

$$\lim_{\Delta x \to 0} P_{[a-\frac{\Delta x}{2}, a+\frac{\Delta x}{2}]} = \lim_{\Delta x \to 0} |\psi(a)|^2 \Delta x + \overset{\sim 0}{\cancel{O(\Delta x^3)}}$$

$$\lim_{\Delta x \to 0} \frac{P_{[a-\frac{\Delta x}{2}, a+\frac{\Delta x}{2}]}}{\Delta x} = \lim_{\Delta x \to 0} |\psi(a)|^2 = |\psi(a)|^2$$

meaning that any $|\psi(x)|^2$ tells us about probability! $|\psi(x)|^2$ is a probability density (some prob $/ \Delta x$) like $|x\rangle\langle x|$ is a projector density.

→ Approximating eigenstates

- For generalized eigenstate $|\lambda\rangle$ and $\hat{A}|\lambda\rangle = \lambda|\lambda\rangle$, even if $|\lambda\rangle$ is not in H, there is a set/family of functions in H that approximate it:

  $\forall \lambda \in \sigma_c(\hat{A})\ \exists\, |\psi_{\lambda,n}\rangle \in H \ |\ \| \, |\psi_{\lambda,n}\rangle \| > c > 0$

  for some constant c. (For each eigenvalue of $\hat{A}$, there is an eigenvector with some length...)

  We want: not eigenvec $|\lambda\rangle$, tho.

  $$\lim_{n \to 0} \| \hat{A} |\psi_{\lambda,n}\rangle - \lambda |\psi_{\lambda,n}\rangle \| = 0$$

  (note if $\lim_{n\to\infty} |\psi_{\lambda,n}\rangle$ exists, it's already an eigenvector... so we assume it DNE ... then it's trivial to find eigen vector...)

- An example of this: Take gaussian $\psi_{\sigma,x_0}(x)$:

  $$\psi_{\sigma,x_0}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-x_0)^2}{2\sigma^2}} = \langle x | \psi_{\sigma,x_0}\rangle \text{ where } |\psi_{\sigma,x_0}\rangle \in H.$$

  (and a gaussian is in $L^2(\mathbb{R})$...)

we can use this to approximate the eigenstate for x_0:

$\int |\psi_{\sigma,x_0}(x)|^2 dx > C > 0$, also $\int \psi_{\sigma,x_0}(x)\, dx = 1$

and,

$$\| \hat{x}|\psi_{x_0,\sigma}\rangle - x_0|\psi_{x_0,\sigma}\rangle \|^2 = \langle(\hat{x}-x_0)\psi_{x_0,\sigma}|(\hat{x}-x_0)\psi_{x_0,\sigma}\rangle$$

$$= \langle\psi_{x_0,\sigma}|(\hat{x}-x_0)^2|\psi_{x_0,\sigma}\rangle = \int dx \langle\psi_{x_0,\sigma}|(\hat{x}-x_0)^2|x\rangle\langle x|\psi_{x_0,\sigma}\rangle$$

$$= \frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty} dx\, (x-x_0)^2 \langle\psi_{x_0,\sigma}|x\rangle\langle x|\psi_{x_0,\sigma}\rangle$$

$$= \frac{1}{2\pi\sigma^2}\int_{-\infty}^{\infty} (x-x_0)^2 e^{-\frac{(x-x_0)^2}{\sigma^2}} dx, \text{ say } u = \frac{x-x_0}{\sigma}:$$

$$= \frac{\sigma^3}{2\pi\sigma^2}\int_{-\infty}^{\infty} \frac{(x-x_0)^2}{\sigma^2} e^{-\frac{(x-x_0)^2}{\sigma^2}} \frac{dx}{\sigma} = \frac{\sigma}{2\pi}\int_{-\infty}^{\infty} u^2 e^{-u^2} du$$

= some fixed value

so, as $\sigma \to 0$, $\| \hat{x}|\psi_{x_0,\sigma}\rangle - x_0|\psi_{x_0,\sigma}\rangle \| = 0$, like we want.

so, a gaussian works if $\sigma \to 0$, which approaches the Dirac Delta

 etc.

We start getting $\hat{x}|\psi_{x_0,\sigma}\rangle \simeq x_0|\psi_{x_0,\sigma}\rangle$ where: 

since $\int \psi_{\sigma,x_0}(x)\, dx = 1$ also, we can say:

→ $\langle x|x'\rangle = \delta(x-x')$ (since $\langle x|\psi_{x_0,\sigma}\rangle = \int dx' \langle x|x'\rangle\langle x'|\psi_{x_0,\sigma}\rangle$
$= \int dx' \langle x|x'\rangle \psi_{\sigma,x_0}(x')$, etc.)

→ probability of measuring a value very close to x_0 = 1