Week 9 – Classical Mechanics

→ Deriving Hamitonian / canonical equations of Motion

We know that $\dot{q} = \frac{\partial H}{\partial p}$ and $\dot{p} = -\frac{\partial H}{\partial q}$, but why?

$$dH = \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp$$ since H is a function of p & q.

$$H = \dot{q}(q,p)p - L(q,\dot{q}(q,p))$$ by definition.

$$dH = \left[\frac{\partial \dot{q}}{\partial q} dq + \frac{\partial \dot{q}}{\partial p} dp\right] p + \dot{q}\, dp$$
$$- \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial \dot{q}} \left[\frac{\partial \dot{q}}{\partial q} dq + \frac{\partial \dot{q}}{\partial p} dp\right]$$

Since $\frac{\partial L}{\partial q} = \dot{p}$ and $\frac{\partial L}{\partial \dot{q}} = p$:

$$dH = \dot{q}\, dp - \dot{p}\, dq$$

so $\dot{q}\, dp - \dot{p}\, dq = \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp \Rightarrow \dot{q} = \frac{\partial H}{\partial p},\ \dot{p} = -\frac{\partial H}{\partial q}$

→ Poisson Brackets

- In general,

$$\frac{d(f(x))}{dx} = g(x) \longleftrightarrow f(x) + dx\, g(x) + O(dx^2) = f(x+dx)$$

(first order Taylor Expansion)

So, for an infinitessimal time interval dt,

$$\begin{cases} q \mapsto q' = q + dt \frac{\partial H}{\partial p} + o(dt^2) \\ p \mapsto p' = p - dt \frac{\partial H}{\partial q} + o(dt^2) \end{cases}$$

e.g. $\frac{dq}{dt} = \dot{q} = \frac{\partial H}{\partial p}$, so "$f(x)$" $= q$, "x" $= t$, $g(x) = \frac{\partial H}{\partial p}$

so $q(t+dt) = q + dt \frac{\partial H}{\partial p} + o(dt^2) = q'$

(We're saying this is how q & p are different after time dt has passed...)

- If we have some $A(q,p) \mapsto A(q',p') = A(q + dt\frac{\partial H}{\partial p}, p - dt\frac{\partial H}{\partial q})$ we can use the same expansion:

$$A(q(t)+dq, p(t)+dp) = A(q(t), p(t)) + dt \frac{dA}{dt} + O(dt^2)$$

where:

$$\frac{dA}{dt} = \frac{\partial A}{\partial q}\left(q, p - dt \frac{\partial H}{\partial q}\right)\frac{\partial H}{\partial p} - \frac{\partial H}{\partial q}\frac{\partial A}{\partial p}\left(q + dt\frac{\partial H}{\partial p}, p\right)$$

since generally $dA = \frac{\partial A}{\partial q} dq + \frac{\partial A}{\partial p} dp$ since A is a function of q and p,

$$\frac{dA}{dt} = \frac{\partial A}{\partial q}\frac{dq}{dt} + \frac{\partial A}{\partial p}\frac{dp}{dt} = \frac{\partial A}{\partial q}\dot{q} + \frac{\partial A}{\partial p}\dot{p}$$

$$= \frac{\partial A}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial H}{\partial q}$$

If we only care about orders under dt^2:

$$A(q',p') = A(q,p) + dt\left(\frac{\partial A}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial H}{\partial q}\right)$$

which is similar to the form:

$f(x+dx) = f(x) + dx\,g(x) + o(dt^2)$, which tells us:

$$\text{"}g(x)\text{"} = \frac{\partial A}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial H}{\partial q} = \text{"}\frac{df}{dx}\text{"} = \frac{dA(q,p)}{dt}$$

$$= \{A, H\}_{P.B.}$$

This is the poisson bracket in 1 DOF.

- For n degrees of freedom,

$$\{A, B\}_{PB} = \sum_{i=1}^{n\ DOF} \frac{\partial A}{\partial q_i}\frac{\partial B}{p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}$$

- Properties of the Poisson bracket:
  1. $\{A, B\} = -\{B, A\}$ (so $\{A, A\} = 0$)
  2. $\{\alpha A + \beta B, C\} = \alpha\{A, C\} + \beta\{B, C\}$ (also linear in 2nd arg.)
  3. $\{A, \{B, C\}\} + \{B, \{C, A\}\}, \{C, \{A, B\}\} = 0$ (Jacobi)

  These are also true for the QM commutator!
- $\{q_i, q_j\} = \{p_i, p_j\} = 0$

and

$$\{q_i, p_j\} = \sum_k \frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \underbrace{\frac{\partial q_i}{\cancel{\partial p_k}}\frac{\partial p_j}{\partial q_k}}_{=0} = \delta_{ij}$$

are the canonical Poisson brackets.

(For our purposes, we pick coordinate q so the Hamiltonian is total energy, so we're always using canonical coordinates?)

The implication of this is:

$$\{q', p'\} = \{q + dt\,\partial_p H, p - dt\,\partial_q H\}$$
$$= \{q, p\} + dt[\{\partial_p H, p\} - \{q, \partial_q H\}] + o(dt^2)$$
$$= \{q, p\} + dt\left[\frac{\partial^2 H}{\partial q \partial p}\frac{\partial p}{\partial p} - \frac{\partial^2 H}{\partial p^2}\frac{\partial p}{\partial q} - \frac{\partial^2 H}{\partial p \partial q}\frac{\partial q}{\partial q} \ldots\right] + o(dt^2)$$
$$= \{q, p\} + dt[0] + o(dt^2)$$
$$= \{q, p\}$$

so, $\{q', p'\}$ where q' and p' are defined as @ the very beginning is equal to $\{q, p\}$

$\therefore$ Infinitessimal time translations <u>preserve</u> the canonical Poisson bracket.

(Note we assume H is continuous here, so $\frac{\partial^2 H}{\partial q \partial p} - \frac{\partial^2 H}{\partial p \partial q} = 0 \ldots$)

- This is a significant result because it means we can define a <u>canonical transformation</u>:

$$\begin{cases} q \longrightarrow Q(q, p) \\ p \longrightarrow P(q, p) \end{cases} \text{satisfies } \{Q_i, P_j\} = \{q_i, p_j\} = \delta_{ij} \text{ and } \{Q_i, Q_j\} = \{P_i, P_j\} = 0$$

where you can define a Hamiltonian K for Q and P where <u>equations of motion</u> are <u>preserved</u>:

$$K(Q, P) \equiv H(q(Q, P), p(Q, P)), \quad \begin{cases} \dot{Q}_i = \partial_{p_i} K \\ \dot{P}_i = -\partial_{Q_i} K \end{cases}$$

(not taking time as a changing variable... ?)

→ Canonical Transformations

$$\begin{cases} q \mapsto q' = q + dt \frac{\partial H}{\partial p} + o(dt^2) \\ p \mapsto p' = p - dt \frac{\partial H}{\partial q} + o(dt^2) \end{cases}$$ is a canonical transformation

called <u>Hamiltonian Flow</u>

- we can actually generalize this to:

$$\begin{cases} q \mapsto q' = q + d\alpha \frac{\partial G(q, p)}{\partial p} \\ p \mapsto p' = p - d\alpha \frac{\partial G(q, p)}{\partial q} \end{cases}$$ The generic infinitessimal transformation,

where we call $G(q, p)$ the infinitessimal generator of the canonical transformation, where

$$\frac{dq}{d\alpha} = \frac{\partial G}{\partial p} \quad \text{and} \quad \frac{dp}{d\alpha} = -\frac{\partial G}{\partial q}$$

- Again say we have some $A(q, p) \rightarrow A(q', p')$.

$$A(q', p') = A\left(q + d\alpha \frac{\partial G}{\partial p}, p - d\alpha \frac{\partial G}{\partial q}\right)$$

$$= A(q, p) + d\alpha \left(\frac{\partial A}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial G}{\partial q}\right)$$

$$= A(q, p) + d\alpha \{A, G\}_{PB} \rightarrow \frac{dA}{d\alpha} = \{A, G\}_{PB}$$

(as expected). If $G = H$ as in our previous formulation, and $\{A, H\}_{PB} = 0$, then $\frac{dA}{dt} = 0$, and A is preserved under time translations, or A is time invariant.

(Here, H is the infinitesimal generator of time translation defined at the beginning of this section...)

We can "reverse" this and use A as the infinitesimal generator to get: $\frac{dH}{d\alpha} = \{H, A\}_{PB} = 0$ (if $\{A, H\} = 0$), so the Hamiltonian is invariant w.r.t. transformation A over $d\alpha$ also.

- Other canonical transformations
  - Space translations

$$\begin{cases} q \mapsto q + dq \\ p \mapsto p \end{cases} \quad \text{is canonical if } \exists G \text{ so} \quad \begin{cases} q' = q + d\alpha \frac{\partial G}{\partial p} \\ p' = p - d\alpha \frac{\partial G}{\partial q} \end{cases}$$

we want $\frac{\partial G}{\partial q} = 0$ and $\frac{\partial G}{\partial p} = 1$ (α is some small distance)

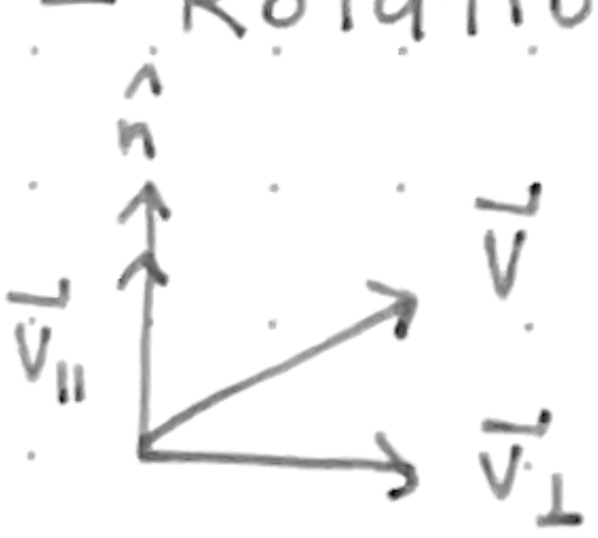
Does such a G exist? Yes. $G = p$.

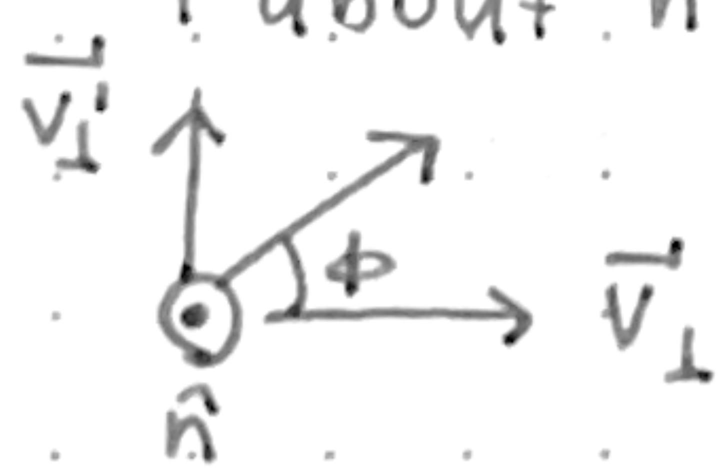
So it is canonical and p is the infinitesimal generator of space translations.

  - Momentum boosts (translations)

$$\begin{cases} q \mapsto q \\ p \mapsto p + dp \end{cases}$$ is canonical with $G = -q$ (note negative sign as change of frame of reference)

  - Rotations: ϕ about $\hat{n}$





$$\vec{v}_{\parallel} = \hat{n}(\hat{n} \cdot \vec{v})$$
$$\vec{v}_{\perp} = \vec{v} - \vec{v}_{\parallel}, \quad \vec{v}_{\perp}' = \hat{n} \times \vec{v}$$

$$\hat{R}_{\hat{n}}(\phi)\vec{v} = \hat{R}_{\hat{n}}(\phi)\vec{v}_{\parallel} + \hat{R}_{\hat{n}}(\phi)\vec{v}_{\perp}$$

$$= \vec{v}_{\parallel} + \cos\phi \, \vec{v}_{\perp} + \sin\phi \, \vec{v}_{\perp}'$$
$$= \hat{n}(\hat{n} \cdot \vec{v})(1 - \cos\phi) + \cos\phi \, \vec{v} + \sin\phi (\hat{n} \times \vec{v})$$

using small angle approx:

$$\hat{R}_{\hat{n}} = \vec{v} + d\phi (\hat{n} \times \vec{v}) + O(d\phi^2)$$

Now,

$\hat{R}_{\hat{n}}(d\phi)\vec{r} = \vec{r} + d\phi(\hat{n}\times\vec{r})$, $\hat{R}_{\hat{n}}(d\phi)\vec{p} = \vec{p} + d\phi(\hat{n}\times\vec{p})$

or,

$$\begin{cases} \vec{r} \mapsto \vec{r}\,' = \vec{r} + d\phi\,(\hat{n}\times\vec{r}) \\ \vec{p} \mapsto \vec{p}\,' = \vec{p} + d\phi\,(\hat{n}\times\vec{p}) \end{cases}$$

if this is canonical : $\begin{cases} \vec{r} \mapsto \vec{r} + d\alpha\,\vec{\nabla}_p G(\vec{r},\vec{p}) \\ \vec{p} \mapsto \vec{p} + d\alpha\,\vec{\nabla}_r G(\vec{r},\vec{p}) \end{cases}$

forsome G:

$\vec{\nabla}_p G(\vec{r},\vec{p}) = \hat{n}\times\vec{r}$ so $G = (\hat{n}\times\vec{r})\cdot\vec{p}$

since $\vec{\nabla}_{\vec{r}} f = \vec{a}$ means f is linear in $\vec{r}$ (we're working in multiple dimensions now!)

So, $G = (\hat{n}\times\vec{r})\cdot\vec{p} = \hat{n}\cdot(\vec{r}\times\vec{p}) = \hat{n}\cdot\vec{L}$, where

orbital angular momentum $\vec{L} = \vec{r}\times\vec{p}$

From our previous generic demonstration:

$$\frac{dA}{d\alpha} = \{A, G\}_{PB} \longrightarrow \frac{dA}{d\phi} = \{A, \hat{n}\cdot\vec{L}\}_{PB}$$

if $\frac{dA}{d\phi} = 0$, or A is invariant upon rotation, we call it a scalar, and for every vector, $\{A, \vec{L}\}_{PB} = 0$.

if instead, $\frac{dA}{d\phi} = \hat{n}\times\vec{A} = \{A, \hat{n}\cdot\vec{L}\}_{PB}$, and:

$$\{A_i, n_j L_j\} = \sum_{jk} \varepsilon_{ijk}\, n_j A_k$$

$$\{A_i, \vec{L}_j\} = \sum_k \varepsilon_{ijk} A_k$$

since this is true for any n_j.

and if $A = \vec{L}$, $\{\vec{L}_i, \vec{L}_j\} = \sum_k \varepsilon_{ijk}\vec{L}_k$

but, if not, we still know $\{\vec{A}_i, \vec{L}_j\} = \varepsilon_{ijk}\vec{A}_k$ for any vector $\vec{A}_i$.

→ Connecting classical x and p to $\hat{x}$ and $\hat{p}$

$\hat{x}|x\rangle = x|x\rangle$ and under the spectral theorem, $\langle x|x'\rangle = \delta(x-x')$

and $\hat{1} = \int dx |x\rangle\langle x|$ also

. What is the physical meaning of p w.r.t. x?

Classically, p is the infinitesimal generator of space translation.

In QM, we have a space translation operator $\hat{T}_{\Delta x}$ (week 8)

- Via Stone's thm,

$$\hat{T}_{\Delta x} = e^{-i\Delta x \hat{O}}, \quad \hat{O} = \hat{O}^{\dagger}$$

For an infinitesimal translation, we can use the Taylor expansion for e: $e^{A} = \hat{1} + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$

we're only interested in up to order Δx, so:

$$\hat{T}_{\Delta x} = e^{-i\Delta x \hat{O}} = \hat{1} - i\Delta x \hat{O} + O(\Delta x^2)$$

we assume $\hat{O} \propto \hat{p}$ because of classical mechanics, but $\hat{1}$ is dimensionless so $\hat{O}$ has units $1/\text{length}$ (Δx has units of length).

Then: $\hat{O} = \hat{p}/\beta$, $[\beta] = [\text{ACTION}] = [L][MLT^{-1}]$

- By experiment, $\beta = \hbar = \frac{h}{2\pi}$. Now we can use $\hat{T}_{\Delta x}$ to learn about $\hat{p}$:

$$\begin{aligned}
\hat{T}_{\Delta x}\hat{x}\hat{T}_{-\Delta x}|x\rangle &= \hat{T}_{\Delta x}\hat{x}|x-\Delta x\rangle \\
&= \hat{T}_{\Delta x}(x-\Delta x)|x-\Delta x\rangle \\
&= (x-\Delta x)|x\rangle = x|x\rangle - \Delta x|x\rangle = \hat{x}|x\rangle - \Delta x|x\rangle \\
&= (\hat{x}-\Delta x)|x\rangle
\end{aligned}$$

$$\begin{aligned}
\hat{T}_{\Delta x}\hat{x}\hat{T}_{-\Delta x} &= \hat{x} - \Delta x \\
&= e^{-i\Delta x\hat{p}/\hbar}\,\hat{x}\,e^{i\Delta x \hat{p}/\hbar} \\
&= (\hat{1} - i\Delta x\hat{p}/\hbar + O(\Delta x^2))\,\hat{x}\,(\hat{1} + i\Delta x\hat{p}/\hbar + O(\Delta x^2))
\end{aligned}$$

$$\hat{x} - \Delta x = \hat{x} + i\Delta x/\hbar\,\hat{x}\hat{p} - i\Delta x/\hbar\,\hat{p}\hat{x} + O(\Delta x^2)$$

When we match by order, $O(\Delta x^2) = 0$ because there is no Δx^2 on LHS, $\hat{x}$ cancels, and

$$\begin{aligned}
-\Delta x &= i\Delta x/\hbar\,(\hat{x}\hat{p} - \hat{p}\hat{x}) = i\Delta x/\hbar\,[\hat{x},\hat{p}] \\
-1 &= i/\hbar\,[\hat{x},\hat{p}] \\
[\hat{x},\hat{p}] &= i\hbar
\end{aligned}$$

- We can also start with $\psi(x) = \langle x|\psi\rangle$ to get a different result

$\langle x|\hat{T}_{\Delta x}|\psi\rangle$ shifts $|\psi\rangle$, then evaluates at a specific x.

$= \langle \hat{T}^{\dagger}_{\Delta x} x|\psi\rangle = \langle \hat{T}_{-\Delta x} x|\psi\rangle$

$= \langle x-\Delta x|\psi\rangle$

$= -\Delta x \frac{d}{dx}\langle x|\psi\rangle + O(\Delta x^2) + \langle x|\psi\rangle$

we can alternatively do:

$\langle x|\hat{T}_{\Delta x}|\psi\rangle = \langle x|\hat{1} - i\Delta x\hat{p}/\hbar + O(\Delta x^2)|\psi\rangle$

$= \langle x|\psi\rangle - i\Delta x/\hbar \langle x|\hat{p}|\psi\rangle + O(\Delta x^2)$

so:

$$\underline{\langle x|\psi\rangle} - \Delta x\frac{d}{dx}\langle x|\psi\rangle + \underline{O(\Delta x^2)} = \underline{\langle x|\psi\rangle} - i\Delta x/\hbar\langle x|\hat{p}|\psi\rangle + \underline{O(\Delta x^2)}$$

$$-\Delta x\frac{d}{dx}\langle x|\psi\rangle = -i\Delta x/\hbar\langle x|\hat{p}|\psi\rangle$$

$$\frac{d}{dx}\langle x|\psi\rangle = i/\hbar\langle x|\hat{p}|\psi\rangle$$

$$-i\hbar\frac{d}{dx}\langle x|\psi\rangle = \langle x|\hat{p}|\psi\rangle$$

we now know what $\hat{p}$ is on $\langle x|\psi\rangle = \psi(x)$:

$$\langle x|\hat{p} = -i\hbar\frac{d}{dx}\langle x|$$

- We can do even more! In more than 1D:

$\begin{cases} \vec{r} \to \vec{r}\,' = \vec{r} + \vec{a} \\ \vec{p} \to \vec{p} \end{cases}$ has generator $\vec{a}\cdot\hat{\vec{p}}$, which can form a one-param group (param = length of $\vec{a}$).

so, $\hat{T}_{\vec{a}} = e^{-i\vec{a}\cdot\hat{\vec{p}}/\hbar}$ is this translation. We want to enforce that $\{\hat{T}_{\vec{a}}\}$ is an abelian group $\hat{T}_{\vec{a}}\hat{T}_{\vec{b}} = \hat{T}_{\vec{b}}\hat{T}_{\vec{a}}$ for two different vectors: $\vec{a} = a\cdot\hat{e}_i$, $\vec{b} = b\cdot\hat{e}_i$ ($\hat{e}_i$ is some unit vector)

$$\hat{T}_{\vec{a}}\hat{T}_{\vec{b}} = (\hat{1} - ia\hat{p}_i/\hbar + O(a^2))(\hat{1} - ib\hat{p}_j/\hbar + O(b^2))$$
$$= (\hat{1} - ib\hat{p}_j/\hbar + O(b^2))(\hat{1} - ia\hat{p}_i/\hbar + O(a^2)) = \hat{T}_{\vec{b}}\hat{T}_{\vec{a}}$$

$$\hat{1} - ia\hat{p}_i/\hbar - ib\hat{p}_j/\hbar - ab\hat{p}_i\hat{p}_j/\hbar^2 + O(b^2) + O(a^2)$$
$$= \hat{1} - ia\hat{p}_i/\hbar - ib\hat{p}_j/\hbar - ab\,\hat{p}_j\hat{p}_i/\hbar^2 + O(b^2) + O(a^2)$$
$$\hat{p}_i\hat{p}_j = \hat{p}_j\hat{p}_i$$

which tells us that $[\hat{p}_i, \hat{p}_j] = 0$.

(We can do the same for momentum boost and $\hat{B}_{\vec{a}}$ to find that $[\hat{x}_i, \hat{x}_j] = 0$ and $\langle p|\hat{x} = i\hbar\frac{d}{dp}\langle p|$ for $\hat{x}$ wrt to $\hat{p}$!)

- What about $[\hat{x}_i, \hat{p}_j]$?

we know $\langle x|\hat{p} = -i\hbar\frac{d}{dx}\langle x|$, so let's do:

$$\langle\vec{r}|[\hat{x}_i, \hat{p}_j] = \langle\vec{r}|\hat{x}_i\hat{p}_j - \langle\vec{r}|\hat{p}_j\hat{x}_i$$
$$= x_i\langle\vec{r}|\hat{p}_j + i\hbar\, d/dx_j\langle\vec{r}|\hat{x}_i$$

$$\langle\vec{r}|[\hat{x}_i,\hat{p}_j] = x_i\left(-i\hbar\frac{d}{dx_j}\right)\langle\vec{r}| + i\hbar\frac{d}{dx_j}x_i\langle\vec{r}|$$

$$= -i\hbar x_i\frac{d}{dx_j}\langle\vec{r}| + i\hbar\left(x_i\frac{d}{dx_j}\langle\vec{r}| + \delta_{ij}\langle\vec{r}|\right)$$

$$= i\hbar\,\delta_{ij}\langle\vec{r}|$$

so $\forall\,|\vec{r}\rangle$, $\langle\vec{r}|[\hat{x}_i,\hat{p}_j] = i\hbar\,\delta_{ij}\langle\vec{r}|$

$$[\hat{x}_i,\hat{p}_j] = i\hbar\,\delta_{ij}$$

– Finally: In general we have:

$$\begin{cases} q \to q' = q + d\alpha\dfrac{\partial G}{\partial p} \\ p \to p' = p - d\alpha\dfrac{\partial G}{\partial q}\end{cases} \quad \text{and } A \mapsto A' = A + d\alpha\{A,G\}_{PB}$$

If we call this canonical transformation $\hat{U}_{d\alpha}$, so $\hat{U}_{d\alpha}|\psi\rangle = |\psi'\rangle$

$$\langle\hat{A}\rangle_\psi \mapsto \langle\psi'|\hat{A}|\psi'\rangle = \langle\hat{A}\rangle_{\psi'}$$

$$= \langle\hat{U}_{d\alpha}\psi|\hat{A}|\hat{U}_{d\alpha}\psi\rangle = \langle\psi|\hat{U}^\dagger_{d\alpha}\hat{A}\hat{U}_{d\alpha}|\psi\rangle$$

since we can write:

$$\hat{U}_{d\alpha} = \hat{1} - i\,d\alpha\,\hat{G}/\hbar,$$

$$\langle\hat{A}\rangle_{\psi'} = \langle\psi|\left(\hat{1} + i\,d\alpha\frac{\hat{G}}{\hbar}\right)\hat{A}\left(\hat{1} - i\,d\alpha\frac{\hat{G}}{\hbar}\right)|\psi\rangle$$

$$= \langle\hat{A}\rangle_\psi + i\frac{d\alpha}{\hbar}\langle\psi|[\hat{G},\hat{A}]|\psi\rangle + O(d\alpha^2)$$

Knowing $A' = A + d\alpha\{A,G\}_{PB}$, we can say expectation value $\langle\hat{A}\rangle_{\psi'} = \langle\hat{A}\rangle_\psi + d\alpha\{A,G\}_{PB}$, implying that

$$i\frac{d\alpha}{\hbar}\langle\psi|[\hat{G},\hat{A}]|\psi\rangle = d\alpha\{A,G\}_{PB}$$

$$i/\hbar\,\langle\psi|[\hat{G},\hat{A}]|\psi\rangle = -\{G,A\}_{PB}$$

$$\frac{1}{i\hbar}\langle\psi|[\hat{G},\hat{A}]|\psi\rangle = \{G,A\}_{PB}$$

telling us that the "operator of $\{G,A\}_{PB}$", $\widehat{\{G,A\}}_{PB}$ is $\dfrac{[\hat{G},\hat{A}]}{i\hbar}$