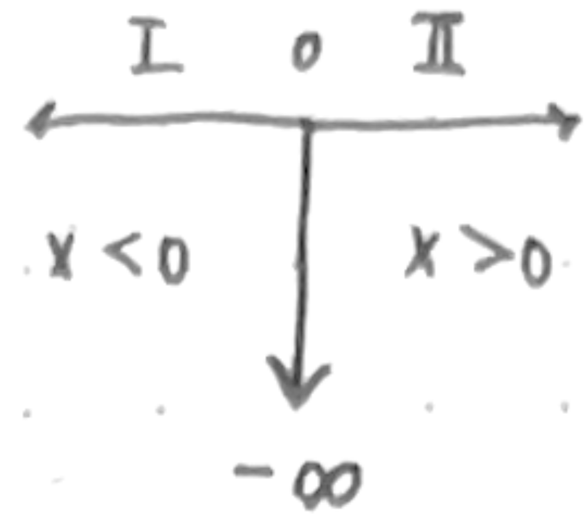
Week 13 - Attractive Delta potential in 1D

Potential $V(x) = -\gamma\,\delta(x)$ for $\gamma > 0$:



sometimes useful if

$\lambda_{\text{de Broglie}} = \frac{2\pi}{k} \gg$ width of well

$$\hat{H}|\psi\rangle = E|\psi\rangle \rightarrow -\frac{\hbar^2}{2m}\psi''(x) - \gamma\,\delta(x)\psi(x) = E\psi(x)$$

Bound states, $E<0$: in I and II, $V(x)=0$, so:

$$-\frac{\hbar^2}{2m}\psi''(x) = E\psi(x)$$

$$\psi''(x) = -\frac{2mE}{\hbar^2}\psi(x), \quad E<0 \text{ so } \beta = \sqrt{\frac{-2mE}{\hbar^2}} \in \mathbb{R}$$

$\psi(x) = Ae^{\beta x} + Be^{-\beta x}$ splitting into regions:

$$\psi(x) = \begin{cases} \psi_I(x) = A_I e^{\beta x} + B_I e^{-\beta x} & x<0 \\ \psi_{II}(x) = A_{II} e^{\beta x} + B_{II} e^{-\beta x} & x>0 \end{cases}$$

To ensure $\psi(x)$ is normalizable, $A_{II} = B_I = 0$. Thus $\psi(x)$ is bounded and we can do:

$$\left|\int_{-\varepsilon}^{\varepsilon} \psi(x)\,dx\right| \le \int_{-\varepsilon}^{\varepsilon} |\psi(x)|\,dx \le 2\varepsilon \max_{x\in(-\varepsilon,\varepsilon)} |\psi(x)| = O(\varepsilon) \rightarrow 0$$

thus

$$\int_{-\varepsilon}^{\varepsilon} -\frac{\hbar^2}{2m}\psi'' - \gamma\,\delta(x)\psi(x)\,dx = E\int_{-\varepsilon}^{\varepsilon}\psi(x)\,dx = O(\varepsilon)$$

$$-\frac{\hbar^2}{2m}\left(\psi'(\varepsilon) - \psi'(-\varepsilon)\right) - \gamma\int_{-\varepsilon}^{\varepsilon}\delta(x)\psi(x)\,dx = O(\varepsilon)$$

RHS is infinitesimal in ε, LH 2nd term is finite, so we know that $\psi'(\varepsilon) - \psi'(-\varepsilon)$ must be finite. $\therefore$ $\psi(x)$ is continuous, even though delta potential causes $\psi'(x)$ to be discontinuous.

$$-\frac{\hbar^2}{2m}\left(\psi'(0^+) - \psi'(0^-)\right) - \gamma\psi(0) = 0 \quad \text{as } \varepsilon \rightarrow 0$$

Now constraints are:

$$\psi(0^+) = \psi(0^-) \longrightarrow \psi_I(0) = \psi_{II}(0)$$

$$\psi'(0^+) = \psi'(0^-) - \frac{2m\gamma}{\hbar^2}\psi(0) \rightarrow \psi'_{II}(0) = \psi'_I(0) - \frac{2m\gamma}{\hbar^2}\psi(0)$$

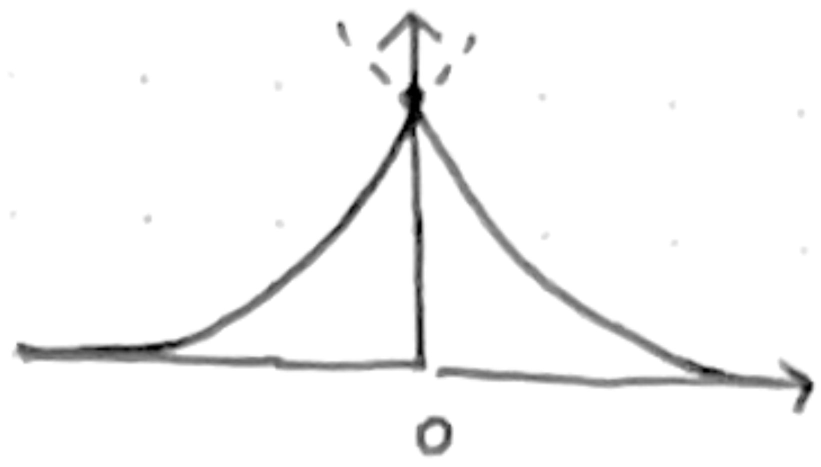
First constraint: $A_I = B_{II}$, call this A. Then the second constraint becomes:

$-\beta A = \beta A - 2m\gamma/\hbar^2 A$

$-2A\beta = -\frac{2m\gamma}{\hbar^2} A \rightarrow \beta = \frac{m\gamma}{\hbar^2} = \sqrt{-\frac{2mE}{\hbar^2}} \rightarrow E = -\frac{m\gamma^2}{2\hbar^2}$

There is only one unique E, so there is only one bound state:

$$\psi(x) = A \begin{cases} e^{-\beta x} & x > 0 \\ e^{\beta x} & x < 0 \end{cases} = A e^{-\beta |x|}$$



Continuous states, $E > 0$, doubly degenerate.

Again, $\psi''(x) = -\frac{2mE}{\hbar^2}\psi(x)$ in I and II, but $E > 0$ now, so:

$\psi(x) = Ae^{ikx} + Be^{-ikx}$ for $k = \sqrt{\frac{2mE}{\hbar^2}}$, k not quantized.

splitting over regions:

$$\psi(x) = \begin{cases} \psi_I(x) = A_I e^{ikx} + B_I e^{-ikx} & x < 0 \\ \psi_{II}(x) = A_{II} e^{ikx} + B_{II} e^{-ikx} & x > 0 \end{cases}$$

and we still have the same constraints as before. Plus, $V(x)$ is even, so $[\hat{\Pi}, \hat{H}] = 0$ and we can classify eigenstates into even/odd:

ODD: $\psi_o(-x) = -\psi_o(x) \rightarrow \psi_o(0) = 0$, so the second constraint is simply $\psi'_{II}(0) = \psi'_I(0)$ ($\psi'_o(x)$ is also continuous now, unlike in bound state!)

First constraint: $A_I + B_I = A_{II} + B_{II}$
second constraint: $A_I - B_I = A_{II} - B_{II}$
$\Big\}$ $A_I = A_{II} = A$, $B_I = B_{II} = B$

$\psi_o(-x) = -\psi_o(x) \rightarrow Ae^{-ikx} + Be^{ikx} = -(Ae^{ikx} + Be^{-ikx})$

$-(Ae^{-ikx} + Be^{ikx}) = \psi_o(x)$

using Euler's:

$\psi_o(x) = -(A\cos(kx) - Ai\sin(kx) + B\cos(kx) + Bi\sin(kx))$

to ensure $\psi_o(0) = 0$, $A + B = 0$, so:

$\psi_o(x) = -(\cos(kx)(A+B) + i\sin(kx)(B-A)) = (A-B)i\sin(kx)$

$= 2Ai\sin(kx)$

(which you can normalize ...)

EVEN: First constraint says $A_I + B_I = A_{II} + B_{II}$.
Even fcn follows $\psi_e(x) = \psi_e(-x)$ which means $\psi_{e,II}(x) = \psi_{e,I}(-x)$
for $x > 0$:

$$A_{II} e^{ikx} + B_{II} e^{-ikx} = A_I e^{-ikx} + B_I e^{ikx}$$

$$A_{II}(\cos(kx) + i\sin(kx)) + B_{II}(\cos(kx) - i\sin(kx))$$

$$= A_I(\cos(kx) - i\sin(kx)) + B_I(\cos(kx) + i\sin(kx))$$

$$\cos(kx)(A_{II} + B_{II}) + i\sin(kx)(A_{II} - B_{II})$$

$$= \cos(kx)\underbrace{(A_I + B_I)}_{= A_{II} + B_{II}\text{, first constraint!}} + i\sin(kx)(-A_I + B_I)$$

$A_{II} - B_{II} = B_I - A_I$. With $A_I + B_I = A_{II} + B_{II}$, we see that
$A_{II} = B_I$ and $B_{II} = A_I$, call these C_1 and C_2 respectively.

$$\text{Thus } \psi_e(x) = \begin{cases} C_2 e^{ikx} + C_1 e^{-ikx} & x < 0 \\ C_1 e^{ikx} + C_2 e^{-ikx} & x > 0 \end{cases}$$

Using the second constraint, we can find C_2/C_1:

$$\psi_{II}'(0) = \psi_I'(0) - \frac{2m\gamma}{\hbar^2}\psi(0)$$

$$ikC_1 - ikC_2 = ikC_2 - ikC_1 - \frac{2m\gamma}{\hbar^2}(C_1 + C_2)$$

$$C_1 - C_2 = C_2 - C_1 - \frac{2m\gamma}{ik\hbar^2}(C_1 + C_2)$$

$$2C_1 = 2C_2 - \frac{2m\gamma}{ik\hbar^2}(C_1 + C_2)$$

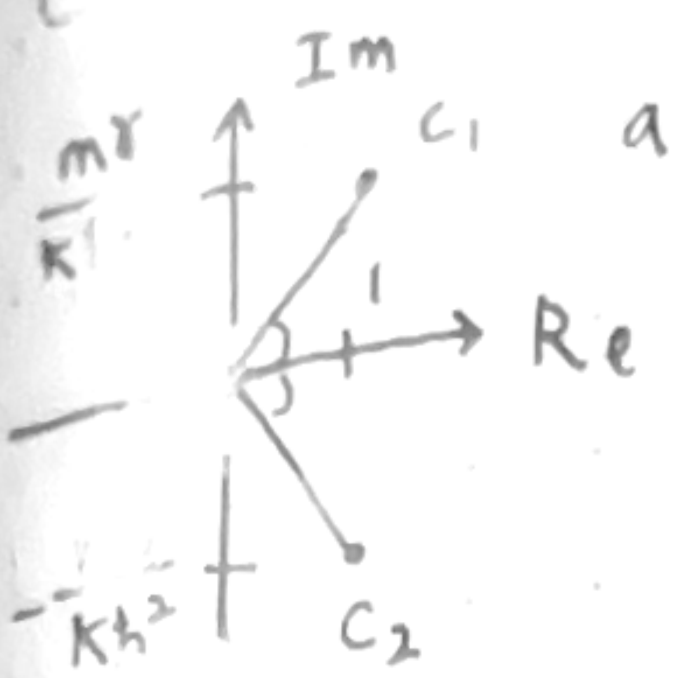
$$C_1 = C_2 - \frac{m\gamma}{ik\hbar^2}C_1 - \frac{m\gamma}{ik\hbar^2}C_2 \rightarrow C_1\left(1 + \frac{m\gamma}{ik\hbar^2}\right) = C_2\left(1 - \frac{m\gamma}{ik\hbar^2}\right)$$

$C_2/C_1 = \dfrac{1 + \frac{m\gamma}{ik\hbar^2}}{1 - \frac{m\gamma}{ik\hbar^2}}$ which has a square modulus of 1:

$$\left(\frac{ik\hbar^2 + m\gamma}{ik\hbar^2 - m\gamma}\right)\left(\frac{-ik\hbar^2 + m\gamma}{-ik\hbar^2 - m\gamma}\right) = \left(\frac{ik\hbar^2 + m\gamma}{ik\hbar^2 - m\gamma}\right)\left(\frac{-1(ik\hbar^2 - m\gamma)}{-1(ik\hbar^2 + m\gamma)}\right) = \frac{-1}{-1} = 1$$

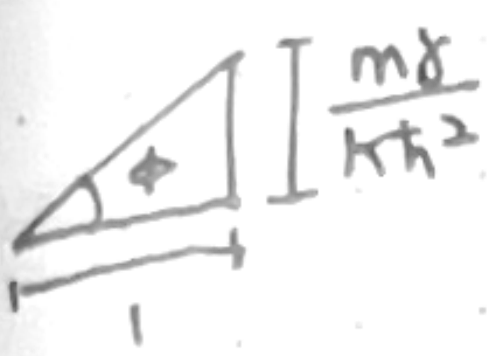
You can write the numerator C_2 as $\ell e^{i\theta}$ where ℓ is the modulus of C_2:

Evidently $|C_1| = \ell$ also, else $|C_2/C_1|^2$ would not be 1. Also, because C_1 and C_2 have opposite imaginary parts, $\pm \frac{m\gamma}{k\hbar^2}$, and the same real part, 1, C_1 and C_2 look something like:



and clearly then $C_1 = \ell e^{i\phi}$, $C_2 = \ell e^{i(-\phi)}$ and the ratio C_2/C_1 can be written as

$$\frac{\ell e^{-i\phi}}{\ell e^{i\phi}} = e^{-2i\phi} \quad \text{where } \phi = \arctan\left(\frac{m\gamma}{\hbar^2 k}\right)$$



so $\tan\phi = \frac{m\gamma}{k\hbar^2}$. For convenience, say instead that

$$\frac{C_2}{C_1} = e^{2i\phi} \quad \text{where } \phi = -\arctan\left(\frac{m\gamma}{\hbar^2 k}\right).$$

Now, $$\psi(x) = C_1 \begin{cases} C_2/C_1\, e^{ikx} + e^{-ikx} & x<0 \\ e^{ikx} + C_2/C_1\, e^{-ikx} & x>0 \end{cases}$$

$$= C_1 \begin{cases} e^{2i\phi} e^{ikx} + e^{-ikx} & x<0 \\ e^{ikx} + e^{2i\phi} e^{-ikx} & x>0 \end{cases}$$

$$= C_1 e^{i\phi} \begin{cases} e^{i\phi} e^{ikx} + e^{-ikx} e^{-i\phi} & x<0 \\ e^{ikx} e^{-i\phi} + e^{i\phi} e^{-ikx} & x>0 \end{cases}$$

$$= C_1 e^{i\phi} \begin{cases} e^{i(\phi+kx)} + e^{-i(\phi+kx)} & x<0 \\ e^{i(kx-\phi)} + e^{-i(kx-\phi)} & x>0 \end{cases}$$

And if we apply Euler's identity:

$$\psi(x) = C_1 e^{i\phi} \begin{cases} 2\cos(\phi+kx) & x<0 \\ 2\cos(kx-\phi) & x>0 \end{cases}$$

because $\cos(a) = \cos(-a)$, $2\cos(\phi+kx) = 2\cos(-kx-\phi)$ for $x<0$. When $x<0$, obviously $-kx$ is positive: $-k(-|x|) = k|x|$. We can also trivially write $2\cos(k|x|-\phi)$ for $x>0$, since here, x is always positive. Thus:

$\psi(x) = C_1 e^{i\phi} 2\cos(k|\phi| - \phi)$ for all x. This makes sense, as the $|x|$ ensures $\psi(x)$ is even.

We said $\phi = -\arctan\left(\frac{m\gamma}{\hbar^2 k}\right)$, so $\psi_e(x) = A\cos\left(k|x| + \arctan\frac{m\gamma}{\hbar^2 k}\right)$

Not sure why I set ϕ to negative arctan(...), tbh.

A absorbs all coeffs. is this the same as the earlier A's? no. sorry...

Continuous states in terms of scattering

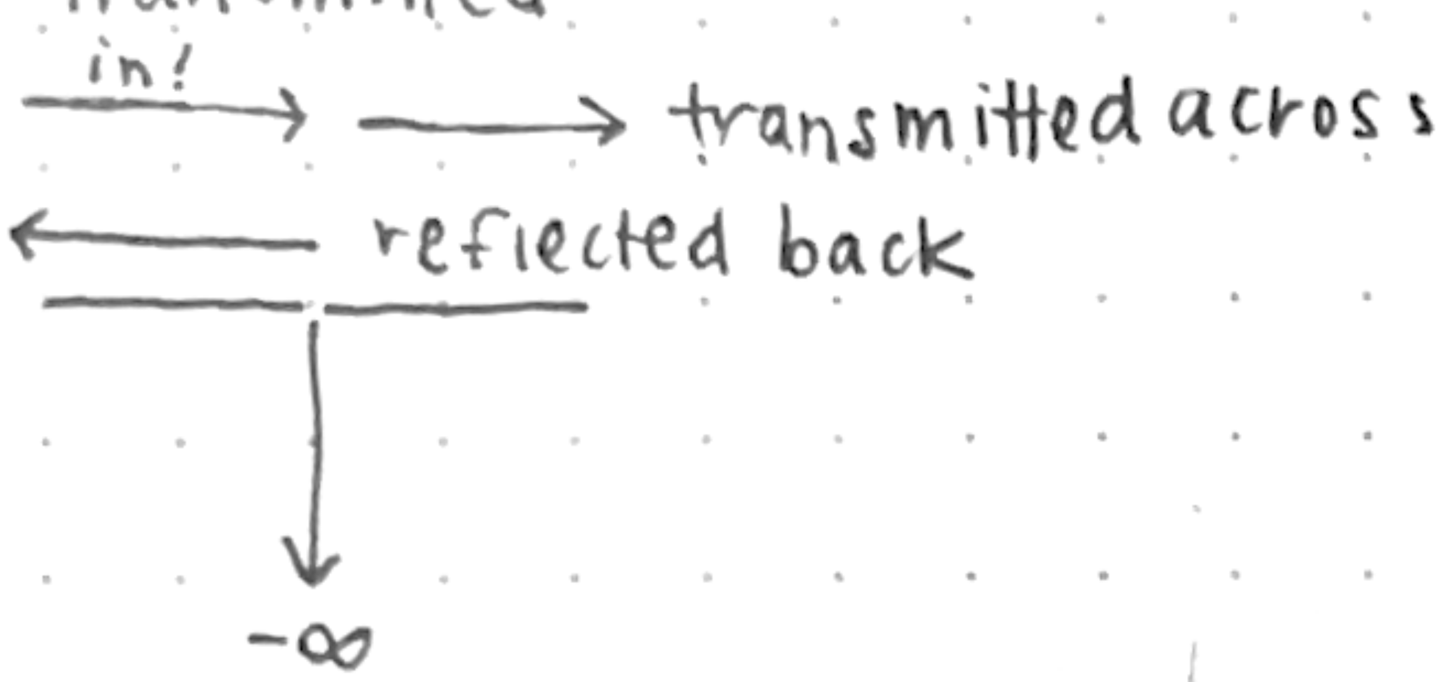
	I $(x<0)$	II $(x>0)$
ODD	$e^{ikx} - e^{-ikx}$	$e^{ikx} - e^{-ikx}$
EVEN	$e^{ikx}e^{-i\theta} + e^{-ikx}e^{i\theta}$	$e^{ikx}e^{i\theta} + e^{-ikx}e^{-i\theta}$

(ODD is $\psi_o = 2Ai\sin(kx) = A(e^{ikx} - e^{-ikx})$ and EVEN is $\psi_e = B\cos(k|x| + \theta)$ for $\theta = -\phi = \arctan(m\gamma/\hbar^2 k) = (e^{ik|x| + i\theta} + e^{-ik|x| - i\theta})B$ for norm. factors A and B... which we ignore for now.)

For Even, $x<0$: $e^{i(k(-x)+\theta)} + e^{-i(k(-x)+\theta)}$ where $-x = |x|$ for $x<0$

then we have $e^{-ikx + i\theta} + e^{ikx - i\theta}$

When we have an incoming particle from the <u>left</u>, it can be reflected or transmitted:



and obviously it does not come from the <u>right</u>.

We can write the state of this particle as $\psi_k^+ = \frac{1}{2}(e^{i\phi}\psi_e + \psi_o) = e^{ikx}$

Recall that time evolution operator $\hat{U}(t) = e^{-iEt/\hbar}$, so over time e^{ikx} becomes $e^{-i(Et/\hbar - kx)} = \cos(Et/\hbar - kx) + i\sin(Et/\hbar - kx)$, which has a square modulus of $\cos^2(Et/\hbar - kx) - \sin^2(Et/\hbar - kx)$. You can plot this and see that as you vary t, the peaks move right! (It's a plane wave...)

For $\psi_k^+ = e^{ikx}$, the flux or probability is $|e^{ikx}|^2 = 1$

Transmission: $e^{ikx}\left(\frac{1+e^{2i\theta}}{2}\right)$ so $\left|\frac{1+e^{2i\theta}}{2}\right|^2 = \cos^2\theta$ flux

Reflection: $e^{-ikx}\left(\frac{e^{2i\theta}-1}{2}\right)$ and $\left|\frac{e^{2i\theta}-1}{2}\right|^2 = \sin^2\theta$

Alternatively, transmission coeff $T \rightarrow |T|^2 = \frac{1}{1 + \left(\frac{m\gamma}{\hbar^2 k}\right)^2}$ where intuitively, an energy or $k \to \infty$ yields $|T|^2 = 1$ and $k \to 0$ yields $|T|^2 = 0$, vice versa for $|R|^2$.

Higher energy $\rightarrow$ transmitted, lower $\rightarrow$ reflected.