Week 3 – More !! Linear algebra

→ Inner product space

- $(V, \langle\cdot|\cdot\rangle)$ is an inner prod. space. IF over $\mathbb{R}^n$, for example, $\langle \vec{v}|\vec{w}\rangle = \sum v_i w_i$, which is the dot product.
- The set of continuous functions $f: [a,b] \to \mathbb{C}$, $\mathcal{C}_{[a,b]}$ is also an inner prod. space:
  - Since continuous and finite (fcns take all input $[a,b]$) $\langle f|g\rangle$ is defined as:

$$\langle f|g\rangle = \int_a^b f^*(x)g(x)\,dx$$

To show property 1 of inner prod. spaces, find $\langle f|f\rangle$:

$$\langle f|f\rangle = \int_a^b f^*(x)f(x)\,dx = \int_a^b |f(x)|^2\,dx$$

$$\langle f|f\rangle \geq 0 \checkmark$$

- To show linearity in the second entry: $\langle h|\alpha f+\beta g\rangle$ must be equal to $\alpha\langle h|f\rangle + \beta\langle h|g\rangle$.

$$\langle h|\alpha f+\beta g\rangle = \int_a^b h^*(x)\left(\alpha f(x)+\beta g(x)\right)dx$$

$$= \int_a^b h^*(x)\alpha f(x) + h^*(x)\beta g(x)\,dx$$

$$= \alpha\int_a^b h^*(x)f(x)\,dx + \int_a^b h^*(x)\beta g(x)\,dx$$

$$= \alpha\langle h|f\rangle + \beta\langle h|g\rangle \checkmark$$

- To show anti-linear in first entry:

$$\langle \alpha f_1 + \beta f_2|g\rangle = \int_a^b (\alpha f_1(x) + \beta f_2(x))^* g(x)\,dx$$

$$= \int_a^b \alpha^* f_1^*(x)g(x) + \beta^* f_2^*(x)g(x)\,dx$$

$$= \alpha^*\langle f_1|g\rangle + \beta^*\langle f_2|g\rangle \checkmark$$

- Can show $\langle \vec{v}|\vec{w}\rangle = \langle \vec{w}|\vec{v}\rangle^*$ similarly...
- Thus it is an inner product space.

- $\vec{v}$ and $\vec{w}$ in an inner product space are orthogonal if $\langle \vec{v} | \vec{w} \rangle = 0$ (then they are considered normal or perpendicular)
- a set of vectors $\{v_i\}$ in an inner product space is an orthonormal set if $\langle \vec{v_i} | \vec{v_j} \rangle = \delta_{ij}$
  - if $i \neq j$, the inner product is 0 (orthogonal)
  - if $i = j$, " " " " 1 (normalized)

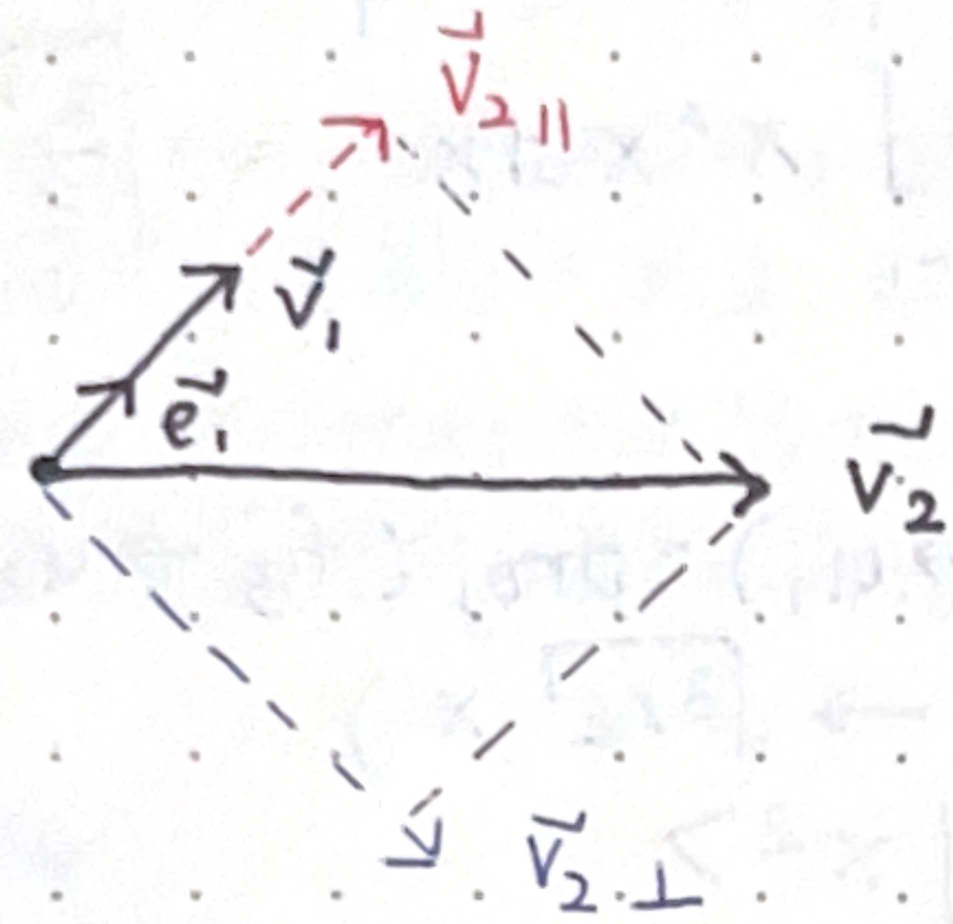
→ Gram - Schmidt process of orthonormalization

- Say inner product space V has vectors $\{\vec{v_i}\}_{i=1\ldots n}$. The goal is to produce orthonormal basis $\{\vec{e_i}\}_{i=1\ldots n}$

1. Form $\vec{e}_1$

   (since so far $\vec{e}_1$ is the only vector we have, just worry about normalization) $\longrightarrow$ $\vec{e}_1 = \vec{v}_1 / \sqrt{\langle \vec{v}_1 | \vec{v}_1 \rangle}$
   - Since $\sqrt{\langle \vec{v}_1 | \vec{v}_1 \rangle}$ is the length of $\vec{v}_1$.

2. Form $\vec{e}_2$
   - Orthoganality: $\vec{e}_2$ must be orthogonal/perpendicular to $\vec{e}_1$:



- We want $\vec{v}_{2\perp}$, not $\vec{v}_{2\parallel}$
- $\vec{v}_2 = \vec{v}_{2\perp} + \vec{v}_{2\parallel} \rightarrow \vec{v}_{2\perp} = \vec{v}_2 - \vec{v}_{2\parallel}$
- We can find $\vec{v}_{2\parallel}$:

  $\vec{e}_1 \underline{\langle \vec{e}_1 | \vec{v}_2 \rangle}$

  in the direction of $\vec{e}_1$ — with this length
- This is a version of projecting one vector to another, proj $(\vec{v_2} \longrightarrow \vec{e}_1)$

- So we can say that after accounting for orthogonality, we have vector: $\vec{v}_2 - \vec{e}_1 \langle \vec{e}_1 | \vec{v}_2 \rangle$

- Normalization: The normalized vector is then

$$\vec{e}_2 = \frac{\vec{v}_2 - \vec{e}_1 \langle \vec{e}_1 | \vec{v}_2 \rangle}{\sqrt{\langle \vec{v}_2 - \vec{e}_1 \langle \vec{e}_1 | \vec{v}_2 \rangle | \vec{v}_2 - \vec{e}_1 \langle \vec{e}_1 | \vec{v}_2 \rangle \rangle}}$$

Where $\langle \vec{v}_2 - \vec{e}_1 \langle \vec{e}_1 | \vec{v}_2 \rangle | \vec{v}_2 - \vec{e}_1 \langle \vec{e}_1 | \vec{v}_2 \rangle \rangle$ is

(linear wrt 2nd) $= \langle \vec{v}_2 - \vec{e}_1 \langle \vec{e}_1 | \vec{v}_2 \rangle | \vec{v}_2 \rangle + \langle \vec{v}_2 - \vec{e}_1 \langle \vec{e}_1 | \vec{v}_2 \rangle | -\vec{e}_1 \langle \vec{e}_1 | \vec{v}_2 \rangle \rangle$

(anti linear) $= \langle \vec{v}_2 | \vec{v}_2 \rangle + (-\langle \vec{e}_1 | \vec{v}_2 \rangle)^* \langle \vec{e}_1 | \vec{v}_2 \rangle + (-\langle \vec{e}_1 | \vec{v}_2 \rangle) \langle \vec{v}_2 | \vec{e}_1 \rangle + (-\langle \vec{e}_1 | \vec{v}_2 \rangle)^* (-\langle \vec{e}_1 | \vec{v}_2 \rangle) \langle \vec{e}_1 | \vec{e}_1 \rangle$

$= \langle \vec{v}_2 | \vec{v}_2 \rangle - \langle \vec{v}_2 | \vec{e}_1 \rangle \langle \vec{e}_1 | \vec{v}_2 \rangle - \langle \vec{e}_1 | \vec{v}_2 \rangle \langle \vec{v}_2 | \vec{e}_1 \rangle + \langle \vec{v}_2 | \vec{e}_1 \rangle \langle \vec{e}_1 | \vec{v}_2 \rangle (1)$

$= \langle \vec{v}_2 | \vec{v}_2 \rangle - \langle \vec{v}_2 | \vec{e}_1 \rangle \langle \vec{e}_1 | \vec{v}_2 \rangle = \langle \vec{v}_2 | \vec{v}_2 \rangle - |\langle \vec{e}_1 | \vec{v}_2 \rangle|^2$

$= (\text{length of } \vec{v}_2)^2 - (\text{length of } \vec{v}_{2\parallel})^2.$

- for $\vec{e}_i$, first construct $v_i - \text{proj}(v_i \to u_1) - \text{proj}(v_i \to u_2) - \dots - \text{proj}(v_i \to u_{i-1})$, the orthogonal vector, then normalize.

Ex. Gram-Schmidt

- given $\{f : [-1,1] \to \mathbb{C}, \text{ continuous}\}$, say we take set $\{x^n, n \in \mathbb{N}\}$ to create a basis with.

1. $f_1(x) = x^0 = 1$

   $u_1(x) = \alpha 1$, such that $\int_{-1}^{1} |\alpha|^2 dx = 1 = |\alpha|^2 [x]_{-1}^{1} = |\alpha|^2 \cdot 2$

   So, $\alpha = 1/\sqrt{2}$

   $u_1(x) = 1/\sqrt{2}$

2. $f_2(x) = x^1 = x$

   $u_2(x) = \alpha x$. Find $\vec{v}_2 - \text{proj}(\vec{v}_2 \to \vec{u}_1)$: $\vec{v}_2$ is f_2, $\vec{u}_1$ is $u_1(x)$

   $$\vec{v}_2 - \text{proj}(\vec{v}_2 \to \vec{u}_1) = x - \text{proj}(x \to 1/\sqrt{2})$$
   $$= x - \langle 1/\sqrt{2} \,|\, x \rangle$$
   $$= x - \int_{-1}^{1} \frac{1}{\sqrt{2}} x\,dx = x - \frac{1}{\sqrt{2}}\left[\frac{1}{2}x^2\right]_{-1}^{1} = x$$

   Normalize: Find length of $x = \sqrt{\int_{-1}^{1} x^* x\,dx} = \sqrt{\frac{2}{3}}$

   so, $\alpha = 1/\sqrt{2/3} = \sqrt{3/2}$

   $u_2(x) = \sqrt{3/2}\, x$

3. $f_3(x) = x^2$. Find $f_3 - \text{proj}(f_3 \to u_1) - \text{proj}(f_3 \to u_2)$

   $$x^2 - \text{proj}(x^2 \to 1/\sqrt{2}) - \text{proj}(x^2 \to \sqrt{3/2}\, x)$$
   $$= x^2 - \langle 1/\sqrt{2} \,|\, x^2 \rangle - \langle \sqrt{3/2}\, x \,|\, x^2 \rangle$$
   $$= x^2 - \int_{-1}^{1} \frac{1}{\sqrt{2}} x^2 dx - \int_{-1}^{1} \sqrt{3/2}\, x \cdot x^2 dx = x^2 - \frac{1}{3}$$

   Normalize: $\int_{-1}^{1} (x^2 - \frac{1}{3})^2 dx = \text{length}^2$

   and so on...

   (Apparently you approach a series proportional to Legendre's polynom.)

→ Norms

- The norm induced by the scalar product for $\vec{v}$ is $\|\vec{v}\|$.
- $\|\vec{v}\| = \sqrt{\langle \vec{v} | \vec{v} \rangle}$ = magnitude of $\vec{v}$.
- in $\mathbb{R}$ and $\mathbb{C}$, $\|\cdot\|$ coincides with absolute value.

→ Schwartz Inequality

- $|\langle \vec{v} | \vec{w} \rangle| \leq \|\vec{v}\| \cdot \|\vec{w}\|$
- Proof: Consider $\langle \vec{v} + \lambda \vec{w} \,|\, \vec{v} + \lambda \vec{w} \rangle \geq 0$

  $$= \langle \vec{v} | \vec{v} \rangle + |\lambda|^2 \langle \vec{w} | \vec{w} \rangle + 2\text{Re}\{\lambda \langle \vec{v} | \vec{w} \rangle\} \geq 0$$

  Pick $\lambda = -\frac{\langle \vec{w} | \vec{v} \rangle}{\langle \vec{w} | \vec{w} \rangle}$. Then:

$$\langle\vec{v}|\vec{v}\rangle + \left|-\frac{\langle\vec{w}|\vec{v}\rangle}{\langle\vec{w}|\vec{w}\rangle}\right|^2 \langle\vec{w}|\vec{w}\rangle + 2Re\left\{-\frac{\langle\vec{w}|\vec{v}\rangle}{\langle\vec{w}|\vec{w}\rangle}\langle\vec{v}|\vec{w}\rangle\right\} \geq 0$$

$$\|\vec{v}\|^2 + \frac{|\langle\vec{w}|\vec{v}\rangle|^2}{(\|\vec{w}\|^2)^2}\|\vec{w}\|^2 - 2Re\left\{\frac{\langle\vec{w}|\vec{v}\rangle\langle\vec{v}|\vec{w}\rangle}{\|\vec{w}\|^2}\right\} \geq 0$$

$$\|\vec{v}\|^2 + \frac{|\langle\vec{w}|\vec{v}\rangle|^2}{\|\vec{w}\|^2} - 2\frac{|\langle\vec{w}|\vec{v}\rangle|^2}{\|\vec{w}\|^2} \geq 0$$

$$\|\vec{v}\|^2 - \frac{|\langle\vec{w}|\vec{v}\rangle|^2}{\|\vec{w}\|^2} \geq 0$$

$$\|\vec{v}\|^2\|\vec{w}\|^2 \geq |\langle\vec{w}|\vec{v}\rangle|^2$$

$$\|\vec{v}\|\|\vec{w}\| \geq |\langle\vec{w}|\vec{v}\rangle| \checkmark$$

→ Triangular Inequality
- $\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$
- Proof:

$\|\vec{v}+\vec{w}\|^2 \leq (\|\vec{v}\| + \|\vec{w}\|)^2$

$\langle\vec{v}+\vec{w}|\vec{v}+\vec{w}\rangle \leq (\|\vec{v}\|+\|\vec{w}\|)^2$

$\langle\vec{v}|\vec{v}\rangle + \langle\vec{w}|\vec{w}\rangle + 2Re\{\langle\vec{v}|\vec{w}\rangle\} \leq (\|\vec{v}\|+\|\vec{w}\|)^2$

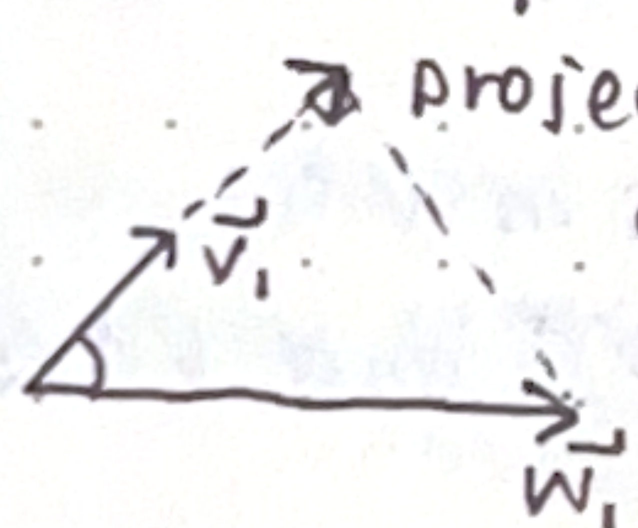
$\|\vec{v}\|^2 + \|\vec{w}\|^2 + 2|\langle\vec{v}|\vec{w}\rangle| \leq (\|\vec{v}\|+\|\vec{w}\|)^2$

(using Schwartz's inequality):

$\|\vec{v}\|^2 + \|\vec{w}\|^2 + 2|\langle\vec{v}|\vec{w}\rangle| \leq \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\|\|\vec{w}\|$

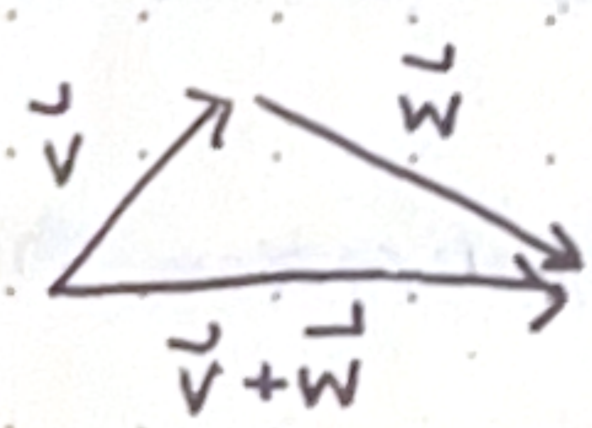
$\leq (\|\vec{v}\|+\|\vec{w}\|)^2 \checkmark$

→ Visually
- Schwartz's Inequality (if $\vec{v}$ is a unit vector)



projection $\langle\vec{v}_1|\vec{w}_1\rangle$ has to be smaller than $\vec{w}$ (magnitude-wise)

- Triangle Inequality



Obviously, $\|\vec{v}+\vec{w}\|$ is not longer than $\|\vec{v}\|+\|\vec{w}\|$

→ "Obtaining" bra-ket notation
- We know how to use $\hat{L}: V \to W$ as a matrix of L_{ij} values. How about wrt to a specifically orthonormal basis?
- Say $\dim(V)=n$, $\dim(W)=m$, and we have: $\{\vec{e}_i\}_{i=1\cdots n}$ so $\langle\vec{e}_i|\vec{e}_j\rangle = \delta_{ij}$ and $\{\vec{u}_i\}_{j=1\sim m}$ so $\langle\vec{u}_i|\vec{u}_j\rangle = \delta_{ij}$ (they are orthonormal).
- Then, $\hat{L}\vec{e}_i = \sum_j \vec{u}_j L_{ji}$

This $L_{ji} = \langle \vec{u}_j | \hat{L}\vec{e}_i \rangle$, aka the length of output $\hat{L}\vec{e}_i$ along vector $\vec{u}_j$. (So, if vector $\vec{e}$ is written as a linear combo of basis vectors ($\vec{e}_i$'s), $\hat{L}\vec{e} = \Sigma \hat{L}\vec{e}_i \cdot c$. $\hat{L}\vec{e}_i$ is $\sum_j \vec{u}_j \cdot$ length of $\hat{L}\vec{e}_i$ on $\vec{u}_j$, for all output basis vectors $\vec{u}_j$, hence transforming $V \to W$.)

- Anyway. $L_{ji} = \langle \vec{u}_j | \hat{L}\vec{e}_i \rangle$

$$= \langle \vec{u}_j | \sum_{k=1}^{m} \vec{u}_k L_{ki} \rangle$$

$$= \sum_{k=1}^{m} L_{ki} \langle \vec{u}_j | \vec{u}_k \rangle = \sum_{k=1}^{m} L_{ki}\, \delta_{jk}$$

The significance of this is lost on me...

But $L_{ij} = \langle \vec{u}_i | \hat{L}\vec{e}_j \rangle$ in an orthonormal basis

- We can extend this further: Say we have linear functional $f: V \to \mathbb{C}$, linear & Continuous

vector $\vec{u} \in V$ gives $\underline{\underline{f_{\vec{u}}}}: V \to \mathbb{C}$, $\vec{v} \in V \to \underline{\underline{\langle \vec{u} | \vec{v} \rangle}}$

We can write as $f_{\vec{u}} = \langle \vec{u} |$ and $\vec{v} = | \vec{v} \rangle$ so $\langle \vec{u} | \vec{v} \rangle = \langle \vec{u} | (| \vec{v} \rangle)$

- When we use this notation with L_{ij}:

$L_{ij} = \langle \vec{u}_i | \hat{L} | \vec{e}_j \rangle$

- This is bra-ket notation.

→ Dual Spaces

- Dual Spaces are the space of all linear functionals:

$$V^* = \{ f: V \to \mathbb{C}, \text{ linear (and continuous in } \infty\text{-dim)} \}$$

(or, $V^* = \mathrm{Hom}(V, \mathbb{C})$, where $\mathrm{Hom}(V, \mathbb{C})$ is the set of all linear applications from V to $\mathbb{C}$.)

- Functionals with form $\langle u |$ are elements in V^*.
- Each vector space V has a dual space V^*, and V^* is itself a vector space, since:

$$\forall f, g \in V^*, \forall \alpha, \beta \in \mathbb{C}, (\alpha f + \beta g)|v\rangle = \alpha f |v\rangle + \beta g |v\rangle$$

(from how + and $\cdot$ by scalar are defined for V^*.)

→ Dual Correspondence

- Dual Correspondence $C: V \to V^*$ is a linear application that maps a vector in V to a functional in V^* or:

$$|v\rangle \longmapsto f_v = \langle v |$$

(flips a ket to a bra)

- Is every bra in V^* the dual of a ket in V? Yes, Riesz theorem for <u>finite</u>-dimensional case:
  - A functional $f \in V^*$, $f: V \to \mathbb{C}$ acts totally on an orthonormal basis $\{e_i\}_{i=1...n}$ of V
  - Basically, we can write:

$\forall |v\rangle \in V$, $|v\rangle = \sum_{i=1}^{n} |e_i\rangle c_i$ ($|v\rangle$ is a linear combo of $|e_i\rangle$)

$$f|v\rangle = f\left(\sum_{i=1}^{n} |e_i\rangle c_i\right) = \sum_{i=1}^{n} f|e_i\rangle c_i = \sum_{i=1}^{n} \alpha_i^* c_i$$

(if we define $\alpha_i = (f|e_i\rangle)^*$, α_i^* and c_i are both complex numbers.)

- The set of α_i^* (so vector of α_i^*'s) describes f's action
- A vector $|w\rangle \equiv \sum_{i=1}^{n} |e_i\rangle \alpha_i$ has a dual $\langle w| = C|w\rangle$, which is:

$$\langle \sum_{i=1}^{n} \alpha_i e_i | = \sum_{i=1}^{n} \alpha_i^* \langle e_i|$$

$\langle w|v\rangle$ then is $\langle \sum_{i=1}^{n} \alpha_i e_i | v\rangle$, or $\sum_{i=1}^{n} \alpha_i^* \langle e_i|v\rangle$

which means $\langle w|v\rangle = \sum_{i=1}^{n} \alpha_i^* c_i = \underline{\underline{f|v\rangle}}$

So, there is a $\langle w|$ in V^*, which is the dual of $|w\rangle$ in V, which represents f.

- Note this takes advantage of the fact that $c_i = \langle e_i|v\rangle$ which is clear from:

$$|v\rangle = \sum_{i=1}^{n} |e_i\rangle c_i = \sum_{i=1}^{n} |e_i\rangle\langle e_i|v\rangle = |v\rangle$$

$\sum_{i=1}^{n} |e_i\rangle\langle e_i| = I$ because $|e_i\rangle\langle e_i|v\rangle$ is $\underbrace{\vec{e_i}}_{\text{direction of basis } |e_i\rangle \text{ (unit/norm)}} \underbrace{\langle e_i|v\rangle}_{\text{length of the projection.}}$, or a projection of $|v\rangle$ onto $|e_i\rangle$.

If you sum the components of $|v\rangle$ along $\underbrace{\text{each basis}}_{\text{all } |e_i\rangle\text{s}}$, you just get $|v\rangle$, hence $\sum_{i=1}^{n} |e_i\rangle\langle e_i| = I$

(This is the closure relation / resolution of identity!)

— Anyway, we know now that $\forall f \in V^*$:

$$f = C\Bigg[\underbrace{\sum_{i=1}^{n} |e_i\rangle (f|e_i\rangle)^*}_{\text{our } |w\rangle}\Bigg]$$

- This means dual correspondence C is invertible (since all elements in V^* have a corresponding element in V, C is 1 to 1):

$$C^{-1}: V^* \to V,\ f \mapsto \sum_{i=1}^{n} |e_i\rangle \alpha_i = \sum_{i=1}^{n} |e_i\rangle (f|e_i\rangle)^*$$

and $\dim(V^*) = \dim(V)$.

- C is also anti-linear:

$$C(\alpha|v\rangle + \beta|w\rangle) = \alpha^* \langle v| + \beta^* \langle w|$$

→ The Adjoint

- The adjoint of linear application $T: V \to V$ in finite dimensional Hilbert space V is $T^\dagger$ (T-dagger)
- $T^\dagger$ is the complex conjugate transpose of T (or $(T^T)^*$)
  or, $T^\dagger_{ij} = T^*_{ji}$
- If $T = T^\dagger$, T is <u>self-adjoint</u>.
- Conceptually:
  $\forall$ functional $\langle v| \in V^*$, $\langle v|T$ is also a functional,
  $\langle v|T : V \to \mathbb{C}$, $|w\rangle \mapsto \langle v|T|w\rangle$, $\langle v|T \in V^*$
  (implying $T|w\rangle \in V$.)
  - Since $\langle v|T \in V^*$, $\exists\, |u\rangle \in V$ such that $\langle v|T = \langle u|$
    (we learned from Riesz that any functional can be represented as the dual of some $|u\rangle \in V$, in V^* ...)
  - Then we can say:
    $$\left.\begin{aligned} C(|v\rangle) &= \langle v| \\ \langle v|T &= \langle u| \\ C^{-1}(\langle u|) &= |u\rangle \end{aligned}\right\} \begin{aligned} &C^{-1}C(|v\rangle)T = |u\rangle \\ &(|v\rangle \neq |u\rangle \text{ also}) \end{aligned}$$
    (C and C^{-1} are anti-linear, T is linear, so $C^{-1}TC$ is linear)
    Or, there is a <u>mapping $|v\rangle \to |u\rangle$, $T^\dagger$</u>
  - So, the adjoint $T^\dagger$ acts on a $|v\rangle$ like:
    $T^\dagger|v\rangle = |T^\dagger v\rangle$ (just notation, ways to write "$T^\dagger \vec{v}$")
    $= |u\rangle$
    follows from "C" both sides → $= C^{-1}[\langle v|T]$
- An element in the matrix representation of T is $\langle e|T|v\rangle$ (see earlier in week 3)
  $\langle e|T = \langle T^\dagger e|$, so an element in $T = \langle T^\dagger e|v\rangle$
  or, $(\langle v|T^\dagger e\rangle)^* = (\langle v|T^\dagger|e\rangle)^*$
  Then:
  $$\langle e|T|v\rangle = \langle v|T^\dagger|e\rangle^*$$
  which means for a $T_{ij} = \langle e_i|T|e_j\rangle$ and (independently) a $T^\dagger_{ij} = \langle e_i|T^\dagger|e_j\rangle$, elements of T and $T^\dagger$ respectively, we can say:
  $$T_{ij} = \langle e_i|T|e_j\rangle = \langle e_j|T^\dagger|e_i\rangle^* = (T^\dagger_{ji})^*$$
  (since by definition, $T^\dagger|e_i\rangle = C^{-1}(\langle e_i|T)$
  $C(T^\dagger|e_i\rangle) = \langle e_i|T$
  $\langle T^\dagger e_i| = \langle e_i|T$
  and then $\langle e_i|T|e_j\rangle = \langle T^\dagger e_i|e_j\rangle = \langle e_j|T^\dagger e_i\rangle^*$)
  $T_{ij} = (T^\dagger_{ji})^*$
  $T^\dagger_{ij} = T_{ji}^*$ (elements in $T^\dagger$ are corresp. elements in $(T^T)^*$)

- The Matrix representation of $T^\dagger$ is the <u>Hermitian conjugate</u> (complex transpose) of the matrix representation of T.

→ Projectors

- Linear application $P: V \rightarrow V$ is a projection if:

  $P^\dagger = P$ (self-adjoint)

  $PP = P$ (Idempotent)

Ex: Orthonormal basis $\{|e_i\rangle\}_{i=1..n}$ and $A = |e_1\rangle\langle e_1|$

- A is self-adjoint:

$$A^\dagger|v\rangle = C^{-1}C(|v\rangle)|e_1\rangle\langle e_1| = C^{-1}(\langle v|A)$$
$$= C^{-1}(\langle v|e_1\rangle\langle e_1|)$$
$$= \langle v|e_1\rangle^* |e_1\rangle \quad \text{(since } \langle v|e_1\rangle \text{ is a scalar)}$$
$$= \langle e_1|v\rangle|e_1\rangle = |e_1\rangle\langle e_1|v\rangle$$

and C^{-1} is anti-linear!

$A|v\rangle = |e_1\rangle\langle e_1|v\rangle$, hence $A^\dagger = A$, and A is self-adjoint.

- A is Idempotent:

$$A^2|v\rangle = (|e_1\rangle\langle e_1|)(|e_1\rangle\langle e_1|)|v\rangle$$
$$= |e_1\rangle\langle e_1|e_1\rangle\langle e_1|v\rangle \leftarrow \langle e_1|e_1\rangle = 1 \text{ since } \vec{e}_1 \text{ is from an orthonormal set}$$
$$= |e_1\rangle\langle e_1|v\rangle. \checkmark$$

Ex: Less "restricted" projector $B = \sum_{k=1}^{m} |e_k\rangle\langle e_k|$ (not necessarily $m=n$)

- For the same reason as A, $B^\dagger = B$. ✓
- Idempotent:

$$B^2 = \sum_{k=1}^{m}\sum_{j=1}^{m} |e_k\rangle\langle e_k|e_j\rangle\langle e_j| = \sum_k\sum_j |e_k\rangle \underbrace{\delta_{kj}}_{=1 \text{ if } j=k\ldots} \langle e_j|$$
$$= \sum_k |e_k\rangle\langle e_k|. \checkmark$$

- Here, B is the projection on the subspace of V generated by $\{|e_1\rangle \ldots |e_m\rangle\}$, not just onto $|e_1\rangle$ like A.

- Projectors are mutually orthogonal: Say P_1, P_2 are projectors to V_1, V_2. $P_1P_2 = \vec{0}$. <u>Any</u> $|v\rangle$ gives $P_1(P_2|v\rangle) = \vec{0}$, meaning subspaces V_1 and V_2 are orthogonal, or $V_1 \cap V_2 = \{\vec{0}\}$.
  - if $|v\rangle$ was in V_1 <u>and</u> V_2 (so they intersect/not orthogonal), $P_1(P_2|v\rangle) = P_1|v\rangle = |v\rangle$

→ Spectral Theorem

- Since we work (for now) in the finite dimensions, being self-adjoint implies <u>Hermitian</u>, and an inner product space is necessarily <u>complete</u> (and a Hilbert space)

- For a self-adjoint operator in a finite dimensional inner product space, the operator has n eigenvectors which form a basis for the space (dim $= n$.)
- Say we have operator A so $A|v\rangle = \lambda |v\rangle$.

In some orthonormal basis we can write A as matrix $\mathbb{A}$ and $|v\rangle$ as column vector $\mathbb{C}$:

$$\mathbb{A}\mathbb{C} = \lambda \mathbb{C}$$

$$(\mathbb{A} - \lambda I)\mathbb{C} = 0, \text{ say } \mathbb{B} = \text{matrix } \mathbb{A} - \lambda I$$

$$\mathbb{B}\mathbb{C} = \begin{bmatrix} B_{11} & B_{12} \cdots & B_{1n} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mn} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} = \begin{bmatrix} \underline{\underline{B_{11} C_1}} + \underline{\underline{B_{12} C_2}} + \cdots + \underline{\underline{B_{1n} C_n}} \\ \vdots \\ \underline{\underline{B_{m1} C_1}} + \underline{\underline{B_{m2} C_2}} + \cdots + \underline{\underline{B_{mn} C_n}} \end{bmatrix}$$

Since each element in the output vector has to be $\vec{0}$,

$B_{11}C_1 = 0$, $B_{12}C_2 = 0$, $\cdots$, $B_{mn}C_n = \vec{0}$

Passing the trivial case where $\mathbb{C} = \vec{0}$, this means that we can write the result as:

$$\begin{bmatrix} B_{11} \\ B_{21} \\ \vdots \\ B_{m1} \end{bmatrix} C_1 + \begin{bmatrix} B_{12} \\ B_{22} \\ \vdots \\ B_{m2} \end{bmatrix} C_2 + \cdots + \begin{bmatrix} B_{1n} \\ B_{2n} \\ \vdots \\ B_{mn} \end{bmatrix} C_n$$

(a linear combo of columns in $\mathbb{B}$ ◡̈)

<u>and</u> that the colums in $\mathbb{B}$ are <u>all</u> linearly dependent.

- By extension this means $\det(\mathbb{B}) = 0$:

  Intuitively, if the colums are scaled by elements of $\mathbb{C}$, where each element in $\mathbb{C}$ dictates how much a column contributes to the result — if all colums are the same scaled vector, the result lies along that vector, too — dimensions are "lost".

  det = scale of volume, so would be 0.

- For eigenvalue λ, $|A - \lambda I| = 0$, where $|A - \lambda I|$ is the characteristic polynomial (because cols are linearly dependent... ?)

  If a polynomial $P(\lambda) = 0$ and $P(\lambda) = \sum_{i=0}^{n} c_i \lambda^i$, there are $\underline{\underline{n}}$ roots (n = multiplicity)
- All this to say there is at least one eigenvalue/vector but <u>not</u> necessarily the same # of eigenvalues as dimensions:

  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has $\lambda = 1$ only, so not a basis

  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has $\lambda = \pm 1$, and eigen vecs for basis!

- Spectral theorem says if $A \in \mathbb{R}^{n\times n}$, $A^\dagger = A$ (A is Hermitian matrix), it has eigenvecs = basis in $\mathbb{R}^n$. Demonstration:

$A: V \to V$ and Hermitian, $\mathbb{A}^\dagger = \mathbb{A}$, $V \in \mathbb{C}^n$

$f: \underline{\{|v\rangle \in V,\ \||v\rangle\| = 1\}} \longrightarrow \mathbb{R},\ |v\rangle \longmapsto \langle v|A|v\rangle$

- (e.g. vectors of a sphere, circle ...)
- closed (no holes) + bounded (radius = 1) = "compact"

– since f maps to $\mathbb{R}$: $\langle v|A|v\rangle = \mathbb{v}^\dagger \mathbb{A}\mathbb{v}$

$= \mathbb{v}^\dagger \mathbb{A}^\dagger v$ (since $\mathbb{A} = \mathbb{A}^\dagger$)

$= (\mathbb{v}^\dagger \mathbb{A}\mathbb{v})^*$ (so $\langle v|A|v\rangle$ is real :))

– The continuous function f realizes min & max of its <u>compact</u> set (Bolzano–Wierstrauss)

– Say that $\mathbb{v}_1$ produces f's maximum

we can use Gram-Schmidt to create $\{\mathbb{w}_2, \dots, \mathbb{w}_n\}$, which is orthonormal (also to $\mathbb{v}_1$):

$\langle w_i | v_1\rangle = 0$, $\langle w_i | w_j\rangle = \delta_{ij}$

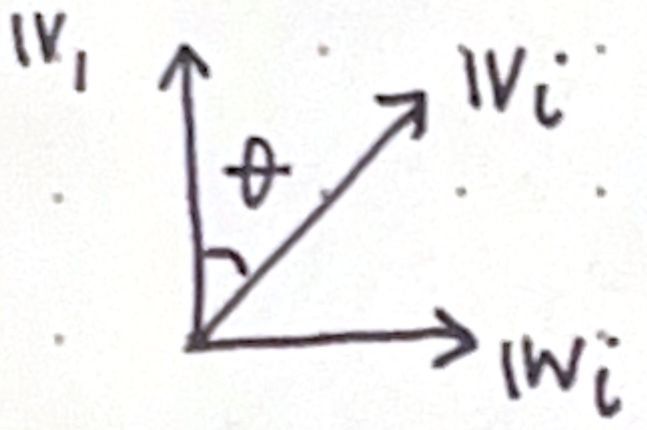
$|w_2\rangle$ w/ a diff. "phase"

– We can guarantee if $\langle w_2 | A | v_1\rangle = i$, we can use $|w_2'\rangle = |w_2\rangle i$

$\langle w_2'| = |w_2'\rangle^* = -i$, so $\langle w_2'|A|v_1\rangle = -i\langle w_2|A|v_1\rangle = 1$

(so it is real...)

– How to define all other $\mathbb{w}_i$: $\mathbb{v}(i,\theta) = \mathbb{v}_1\cos\theta + \mathbb{w}_i \sin\theta$



Cool. Now we want to determine some relationship between $\mathbb{A}$, $\mathbb{v}_1$, and $\mathbb{w}_i$

Since we picked $\mathbb{v}_1$ to have $\theta = 0$, there's a turning point in $f(\mathbb{v}(i,\theta))$ at $\theta = 0$ (the maximum!) So we know:

$$\left[\frac{d}{d\theta} f(\mathbb{v}(i,\theta))\right]_{\theta=0} = 0, \text{ and } f(\mathbb{v}) = \mathbb{v}^\dagger \mathbb{A}\mathbb{v}$$

$\mathbb{v}(i,\theta=0) = \mathbb{v}_1$

$$= \frac{d}{d\theta}\left\{\mathbb{v}^\dagger(i,\theta)\,\mathbb{A}\,\mathbb{v}(i,\theta)\right\} = \left(\frac{d}{d\theta}\mathbb{v}(i,\theta)\right)^\dagger_{\theta=0} \mathbb{A}\mathbb{v}_1$$

not in terms of θ

$$+ \mathbb{v}_1^\dagger \mathbb{A}\left(\frac{d}{d\theta}\mathbb{v}(i,\theta)\right)_{\theta=0}$$

$$\frac{d}{d\theta}\mathbb{v}(i,\theta) = -\mathbb{v}_1 \sin\theta + \mathbb{w}_i\cos\theta, \text{ so:}$$

$$= \underline{\mathbb{w}_i} \text{ at } \theta = 0$$

$= (\mathbb{w}_i^\dagger \mathbb{A}\mathbb{v}_1)^* = \mathbb{w}_i^\dagger \mathbb{A}\mathbb{v}_1$ because real, $\mathbb{A}$ is Hermitian

$$\left[\frac{d}{d\theta} f(\mathbb{v}(i,0))\right]_{\theta=0} = \mathbb{w}_i^\dagger \mathbb{A}\mathbb{v}_1 + \mathbb{v}_1^\dagger \mathbb{A}\mathbb{w}_i = 2\mathbb{w}_i^\dagger \mathbb{A}\mathbb{v}_1 = 0$$

→ $(2\mathbb{w}_i^\dagger)(\mathbb{A}\mathbb{v}_1) = 0$, so $\mathbb{w}_i \perp \mathbb{A}\mathbb{v}_1$. $\mathbb{A}\mathbb{v}_1$ then has to be $c\mathbb{v}_1$, meaning $\mathbb{v}_1$ is an eigenvector of $\mathbb{A}$

– Take the part of $\mathbb{A}$ that is orthogonal to $\mathbb{v}_1$, $\mathbb{A}'$. We can do this because $\mathbb{A}$ is Hermitian — $\mathbb{A}'$ will be, to. Repeating the process in dim $= n-1$, $\mathbb{A}'$ to get another eigenvalue, n times...

- This process shows we can get n orthonormal eigenvectors of $\mathbb{A}^{n\times n}$ in $V \in \mathbb{C}^n$ that form a basis for V