Week 4 – Spectral theorem and spin

→ Spectral theorem (wrap up)

- For an operator A, you can write:

$$A = AI = A\sum_i |v_i\rangle\langle v_i| \quad \text{(for eigenvectors } |v_i\rangle\text{)}$$

$$= \sum_i A|v_i\rangle\langle v_i|$$

$$= \sum_i \lambda_i |v_i\rangle\langle v_i| \quad \text{(for eigenvalues } \{\lambda_i\} = \sigma_A\text{)}$$

If we say $P_i = |v_i\rangle\langle v_i|$, $A = \sum_i \lambda_i P_i$

- If there is degeneracy:

λ has many eigenkets $|\lambda, j\rangle$ for index j

e.g.: $\lambda_a = 1 \longrightarrow |v_1\rangle, |v_2\rangle \longrightarrow |\lambda_a, 1\rangle$ and $|\lambda_a, 2\rangle$

$\lambda_b = 2 \longrightarrow |v_3\rangle, |v_4\rangle, |v_5\rangle \rightarrow |\lambda_{b}, 1\rangle, |\lambda_{b}, 2\rangle, |\lambda_b, 3\rangle$

(note this shows 2D & 3D subspace)

$$P_\lambda = \Lambda_\lambda = \sum_{j=1}^{r\lambda} |\lambda, j\rangle\langle\lambda, j| \text{ is a projection to eigenspace } \lambda,$$

where $r\lambda$ = multiplicity.

- Then, $A = \sum_{\lambda\in\sigma_A} \lambda P_\lambda$

- Given eigenspace λ, $|v_1\rangle, |v_2\rangle$, you can make:

$|w_1\rangle = |v_1\rangle\cos\theta + |v_2\rangle\sin\theta\, e^{i\phi}$

$|w_2\rangle = -|v_1\rangle\sin\theta + |v_2\rangle\cos\theta\, e^{i\phi}$

$$\langle w_1|w_2\rangle = (\cos\theta\langle v_1| + \sin\theta\, e^{-i\phi}\langle v_2|)(-|v_1\rangle\sin\theta + \ldots)$$

$$= \cos\theta(-\sin\theta)\langle v_1|v_1\rangle + \sin\theta\, e^{-i\phi}\cos\theta\, e^{i\phi}\langle v_2|v_2\rangle$$

$$+ \cos\theta\langle v_1|v_2\rangle(\cos\theta\, e^{i\phi}) + \sin\theta\, e^{-i\phi}\langle v_2|v_1\rangle(-\sin\theta)$$

$$= -\cos\theta\sin\theta + \sin\theta\cos\theta + \cos^2\theta\, e^{i\phi}(0) - \sin^2\theta\, e^{-i\phi}(0)$$

$$= 0$$

$$\langle w_1|w_1\rangle = (\cos\theta\langle v_1| + \sin\theta\, e^{-i\phi}\langle v_2|)(|v_1\rangle\cos\theta + |v_2\rangle\sin\theta\, e^{i\phi})$$

$$= \cos^2\theta\langle v_1|v_1\rangle + \cos\theta\sin\theta\, e^{i\phi}\langle v_1|v_2\rangle$$

$$+ \sin\theta\cos\theta\, e^{-i\phi}\langle v_2|v_1\rangle + \sin^2\theta\, e^{-i\phi}e^{i\phi}\langle v_2|v_2\rangle$$

$$= \cos^2\theta + \sin^2\theta = 1$$

$\langle w_2|w_2\rangle = 1$ also...

(can define eigenspace with new set of orthonormal vectors)

$P = |v_1\rangle\langle v_1| + |v_2\rangle\langle v_2| = |w_1\rangle\langle w_1| + |w_2\rangle\langle w_2|$

- For eigenvector $|v\rangle$ of A, $A|v\rangle = \lambda|v\rangle$, $|v\rangle \neq 0$, $A = A^\dagger$

Then, $\langle v|A|v\rangle = \lambda \langle v|v\rangle = \lambda \langle v|v\rangle^*$, so $\langle v|v\rangle \in \mathbb{R}$.

Also:

$$\lambda = \frac{\langle v|A|v\rangle}{\langle v|v\rangle} = \frac{\langle v|A^\dagger|v\rangle^*}{\langle v|v\rangle} = \frac{\langle v|A|v\rangle^*}{\langle v|v\rangle}, \text{ so } \langle v|A|v\rangle \in \mathbb{R}.$$

- Say $A|v\rangle = \lambda|v\rangle$ and $A|w\rangle = \mu|w\rangle$:

$\lambda \neq \mu$, $\langle w|\underline{A|v\rangle} = \lambda\langle w|v\rangle$

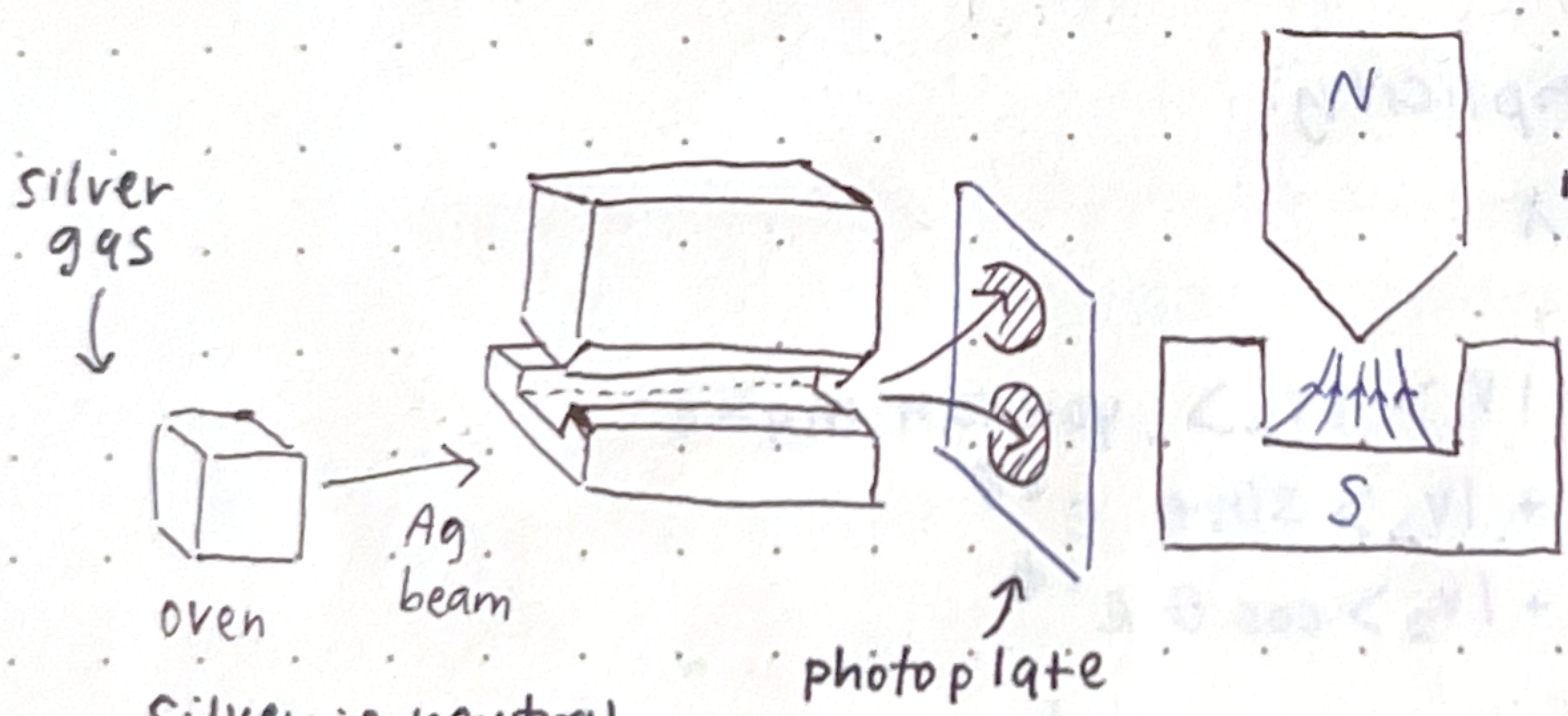
$$\begin{aligned}\langle v|A^\dagger|w\rangle^* &= \langle v|A|w\rangle^* = (\langle v|(\mu|w\rangle))^* \\ &= (\mu \langle v|w\rangle)^* \\ &= \mu \langle v|w\rangle^* \quad (\text{since } \lambda, \mu \in \mathbb{R})\end{aligned}$$

so $\lambda\langle w|v\rangle = \mu\langle v|w\rangle^* = \mu\langle w|v\rangle$

$$\underbrace{(\lambda-\mu)}_{\substack{\lambda\neq\mu, \text{ so} \\ \lambda-\mu\neq 0}} \underbrace{\langle w|v\rangle}_{} = 0$$

$\langle w|v\rangle = 0$: Eigenvectors of different eigenvalues are orthonormal!

→ The Stern-Gerlach Experiment



Inhomogenous magnetic field means if a magnetic dipole ↑ enters, it accelerates up. If it is unaligned with the field (in this case ↓), it accelerates downwards.

- $V = -\mu_m \cdot \vec{B}$
- $\vec{F}(\vec{\pi}) = -\vec{\nabla} V(\vec{\pi})$
  $= \vec{\nabla}(\vec{\mu}_m \cdot \vec{B}(\vec{\pi}))$
- $F_z = \mu_z \cdot \frac{\partial B}{\partial z}$

- Silver is neutral so resistant to abberations in charge
- Silver is heavy, e^- is light : less change in speed to preserve momentum
- Silver is a noble atom, so safe on equipment lol

$$[Ag] = [Kr]\underbrace{4d^{10}}_{\substack{4^{th}\ e^-\text{-shell}}}\underbrace{5s^1}_{5^{th}\ e^-\text{ shell has a single } e^-}$$

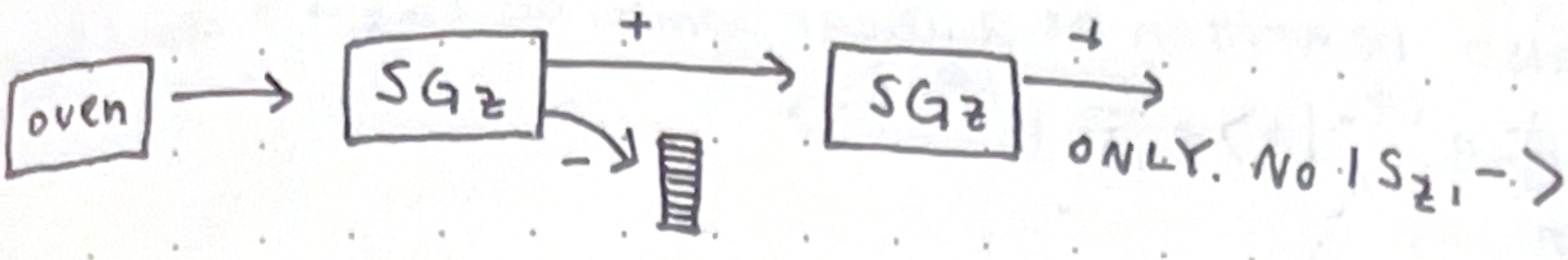
in Kr has s & p orbitals full.

Ag additionally has d orbital ful

- The single e^- that is unpaired (no opposite spin e^- to make pair) contributes all the angular momentum since all other shells are closed (so symmetric): $\vec{\mu}_m \propto$ spin of this e^-, $\vec{S}$.

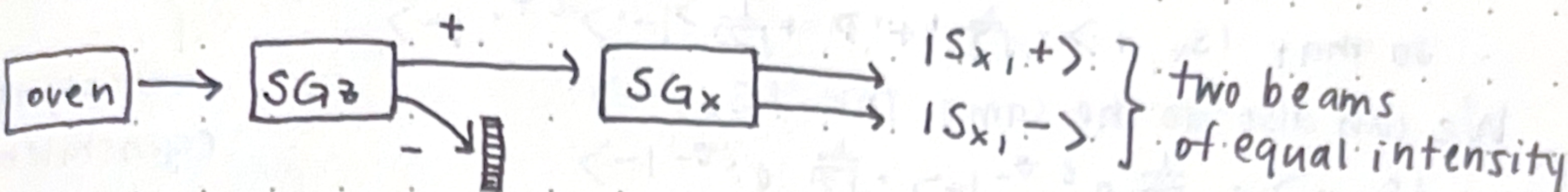
- can represent states of the two groups of atoms as:
  $|S_z, +\rangle$ where $S_z = \hbar/2$ and
  $|S_z, -\rangle$ where $S_z = -\hbar/2$

→ Variation 1: two SG_z's.



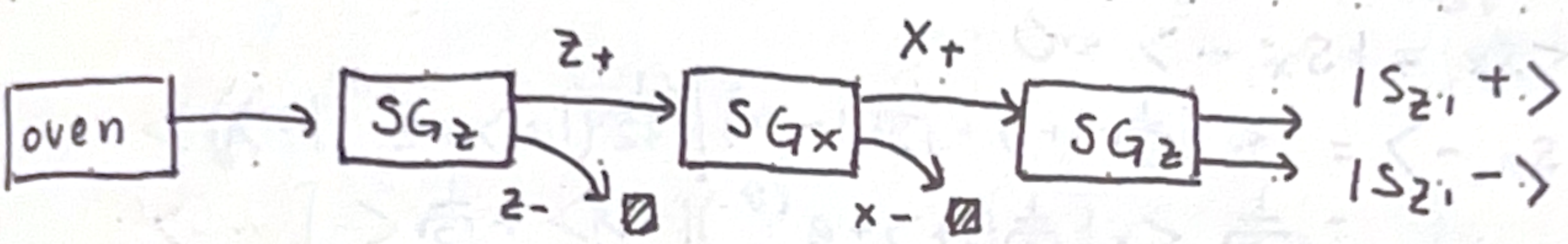
$|S_z, +\rangle$ is unaltered, so we can say $|S_z, +\rangle$ is an eigenvector of the system SG_z.
(Note atoms travel along Y axis)

→ Variation 2: SG_z, SG_x



(classically expected result)

→ Variation 3: SG_z, SG_x, SG_z



(uncertainty principle? measurement on x destroys previous z!!)

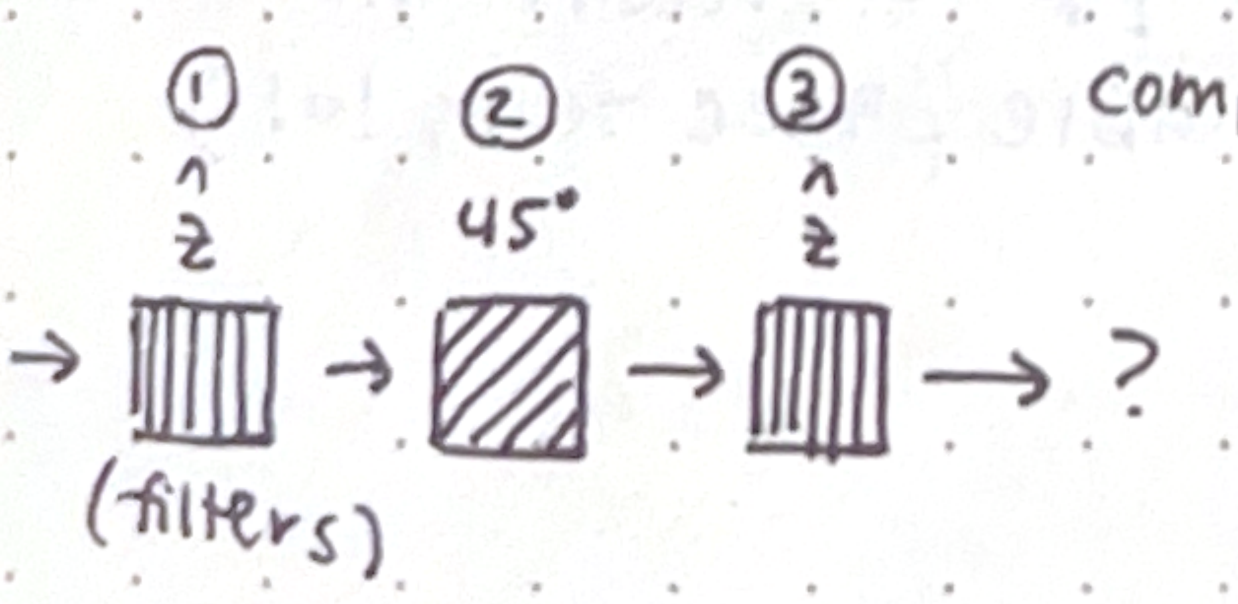
→ Analog: polarized light

- Say we have Monochromatic plane wave $E_x \cdot \hat{x} \cdot \cos(ky - \omega t + \phi) + E_z \cdot \hat{z} \cdot \cos(ky - \omega t)$ — a linear space!

can write it as:

$$\mathrm{Re}\left\{ E_x \cdot \hat{x} \cdot e^{i(ky - \omega t + \phi)} + E_z \cdot \hat{z}\, e^{i(ky - \omega t)} \right\}$$

$$= \mathrm{Re}\left\{ e^{i(ky - \omega t)} \underbrace{\left[E_x \cdot \hat{x} \cdot e^{i\phi} + E_z \cdot \hat{z} \right]}_{\text{complex vector } \in \mathbb{C}^2} \right\}$$



$\hat{z}' = (\hat{x} - \hat{z})/\sqrt{2}$ and $\hat{x}' = (-\hat{x} - \hat{z})/\sqrt{2}$

$\hat{x} = (\hat{z}' - \hat{x}')/\sqrt{2}$ and $\hat{z} = (\hat{x}' + \hat{z}')/\sqrt{2}$

So, light entering is along $\hat{z}$ only, light exiting is along $\hat{z}'$ only
($\hat{x}$ is blocked by ①, $\hat{x}'$ is blocked by ②)

② Output: $E_z \cdot \hat{z} \cdot (\hat{x} - \hat{z})/\sqrt{2}\ \hat{z}' = E_z/\sqrt{2}\,(-1)\,\hat{z}' = -E_z/\sqrt{2}\ \hat{z}'$

③ Output: $(-E_z/\sqrt{2}\ \hat{z}') \cdot (\hat{x}' + \hat{z}')/\sqrt{2}\ \hat{z} = -\frac{1}{2} E_z \hat{z}$
$= \frac{1}{2\sqrt{2}} E_z (\hat{z} - \hat{x})$

- has $\hat{z}$ and $\hat{x}$ components!

- Polarized light is acted on by a polar filter P:

$P_{\hat{n}} = 1|\hat{n}\rangle\langle\hat{n}| + 0|\hat{n}_\perp\rangle\langle\hat{n}_\perp|$

- The point of this analogy: Matter acts like a wave, which can be, e.g. a linear combination of $|S_{z},+\rangle$ and $|S_{z},-\rangle$

$|S_x, \pm\rangle$ can also be written as a linear combo of $|S_z,+\rangle$ and $|S_z,-\rangle$:

$$|S_x,+\rangle = \frac{1}{\sqrt{2}} e^{i\phi_+}|+\rangle + \frac{1}{\sqrt{2}} e^{i\phi_-}|-\rangle$$

both 'beams' have equal intensity

phase factor

we can arbitrarily say $|S_x,+\rangle = e^{-i\phi_+}|+\rangle$ and $|-\rangle' = e^{-i\phi_-}|-\rangle$

so that $|S_x,+\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$

multiplying eigenstate by constant phase = still eigenstate

We can also do the same for $|S_x,-\rangle$:

$$|S_x,-\rangle = \frac{1}{\sqrt{2}} e^{i\theta_+}|+\rangle + \frac{1}{\sqrt{2}} e^{i\theta_-}|-\rangle$$

(say $|S_x,-\rangle = e^{-i\theta_+}|S_x,-\rangle$)

$$|S_x,-\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{2} e^{i\theta_-}|-\rangle$$

Because $\langle S_x,+|S_x,-\rangle = 0$:

$$\langle S_x,+|S_x,-\rangle = \langle (\tfrac{1}{\sqrt{2}}|+\rangle + \tfrac{1}{\sqrt{2}}|-\rangle) | (\tfrac{1}{\sqrt{2}}(|+\rangle + e^{i\theta_-}|-\rangle))\rangle$$

$$= \frac{1}{\sqrt{2}}\langle x|\frac{1}{\sqrt{2}}(|+\rangle + e^{i\theta_-}|-\rangle)\rangle + \frac{1}{\sqrt{2}}\langle -| \dots\rangle$$

$$= \frac{1}{2}\langle +|+\rangle + \frac{1}{\sqrt{2}} e^{i\theta_-} \overset{0}{\cancel{\langle x|-\rangle}} + \frac{1}{2}\overset{0}{\cancel{\langle -|+\rangle}} + \frac{1}{2} e^{i\theta_-}\langle -|-\rangle$$

$$= \frac{1}{2} + \frac{1}{2} e^{i\theta_-}$$

so, pick $\theta = \pi$, so that $e^{i\theta_-} = -1$, then:

$$|S_x,-\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$$

- It's totally okay to redefine $|-\rangle$ and use the new $|-\rangle$ when we define $|S_x,-\rangle$ and then to add a phase coefficient in front of it (even if it's a specific and not variable phase term lol)

→ Matrix representations of S_z, S_x, S_y

- $|S_x, \pm\rangle$ can be written as $\frac{|+\rangle \pm |-\rangle}{\sqrt{2}}$, $|S_z, \pm\rangle$ as $|\pm\rangle$.

we know (in hindsight) spin $= \hbar/2$ in $|S_z,+\rangle$ and $-\hbar/2$ in $|-\rangle$.

$\hat{S}_z$ as a projector: $\frac{\hbar}{2}|+\rangle\langle+| - \frac{\hbar}{2}|-\rangle\langle-|$

(if $\hat{S}_z$ acts on a $|+\rangle$ state, you measure spin $= \hbar/2$, etc.)

So, the matrix representation (in $|\pm\rangle$ basis) is:

$$\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2}\sigma_3$$

Can think of the elements of this matrix as:

$$S_{z,++} = \langle+|\hat{S}_z|+\rangle, \quad S_{z,--} = \langle-|\hat{S}_z|-\rangle, \text{ etc.}$$

where $\langle+|+\rangle = \langle-|-\rangle = 1$, $\langle+|-\rangle = \langle-|+\rangle = 0$

Similarly for S_x:

$$\hat{S}_x = \frac{\hbar}{2}|S_x,+\rangle\langle S_x,+| - \frac{\hbar}{2}|S_x,-\rangle\langle S_x,-|$$

$$= \frac{\hbar}{2}\left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right)\left(\frac{\langle+| + \langle-|}{\sqrt{2}}\right) - \frac{\hbar}{2}\left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right)\left(\frac{\langle+| - \langle-|}{\sqrt{2}}\right)$$

$$= \frac{\hbar}{4}(|+\rangle + |-\rangle)(\langle+| + \langle-|) - \frac{\hbar}{4}(|+\rangle - |-\rangle)(\langle+| - \langle-|)$$

$$= \frac{\hbar}{4}\Big(\underline{|+\rangle\langle+|} + |+\rangle\langle-| + |-\rangle\langle+| + |-\rangle\langle-| - \underline{|+\rangle\langle+|} + |+\rangle\langle-|$$
$$+ |-\rangle\langle+| - |-\rangle\langle-|\Big)$$

$$= \frac{\hbar}{4}\left(2|+\rangle\langle-| + 2|-\rangle\langle+|\right) = \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+|\right)$$

So:

$$\begin{array}{ccc} & |+\rangle \;\; |-\rangle & \\ \frac{\hbar}{2} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \begin{array}{c} |+\rangle \\ |-\rangle \end{array} \end{array}$$

$$= \frac{\hbar}{2}\sigma_1$$

(since $\langle-|\frac{\hbar}{2}(|+\rangle\langle-| + |-\rangle\langle+|)|+\rangle$
$= \langle-|\frac{\hbar}{2}(|-\rangle\langle+|+\rangle)$
$= \frac{\hbar}{2}$, or you can think of them as rows/cols labelled as shown...)

- Finally, for S_y, first define $|S_y, \pm\rangle$
  - Since $|S_y,+\rangle$ has equal parts $|S_x,+\rangle$ and $|S_x,-\rangle$, we know $|\langle S_x,+|S_y,\pm\rangle|^2 = 1/2$
  - We can redefine $|S_y,+\rangle$ and $|S_y,-\rangle$ as usual so:
    $|S_y,\pm\rangle = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{1}{\sqrt{2}}e^{i\delta}|-\rangle$. Then:
    $\langle S_x,+|S_y,\pm\rangle$ is...

$$\left(\frac{\langle+| + \langle-|}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}(|+\rangle \pm e^{i\delta}|-\rangle)\right) = \frac{1}{2}(1 \pm e^{i\delta})$$

And $|\langle S_{x}, + | S_{y}, \pm \rangle|^2 = |\frac{1}{2}(1 \pm e^{i\delta})|^2$

$= \frac{1}{4}(1 \pm e^{i\delta})(1 \pm e^{-i\delta})$

(since $\cos(\theta)$ is the same as $\cos(-\theta)$ ...) $= \frac{1}{4}(1+1 \pm (e^{i\delta} + e^{-i\delta}))$

$= \frac{1}{4}(2 \pm 2\cos\delta)$

$= \frac{1}{2} \pm \frac{1}{2}\cos\delta$

So, $\cos\delta = 0$, meaning $\delta = \pm\pi/2$

Picking $\delta = \pi/2$:

$$|S_{y}, \pm\rangle = \frac{|+\rangle \pm e^{i\pi/2}|-\rangle}{\sqrt{2}}$$, and $e^{i\theta} = \cos\theta + i\sin\theta$, so:

$$e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = i$$

$$|S_{y}, \pm\rangle = \frac{|+\rangle \pm i|-\rangle}{\sqrt{2}}$$

Note that $|S_{y}, \pm\rangle$ has an imaginary term as a result of us choosing real coefficients for $|S_{x}, \pm\rangle$.

Finally, we can write matrix:

$$\hat{S_y} = \frac{\hbar}{2}|S_{y}, +\rangle\langle S_{y}, +| - \frac{\hbar}{2}|S_{y}, -\rangle\langle S_{y}, -|$$

$$= \frac{\hbar}{2}\left(\frac{|+\rangle + i|-\rangle}{\sqrt{2}}\right)\left(\frac{\langle +| - i\langle -|}{\sqrt{2}}\right) - \frac{\hbar}{2}\left(\frac{|+\rangle - i|-\rangle}{\sqrt{2}}\right)\left(\frac{\langle +| + i\langle -|}{\sqrt{2}}\right)$$

$$= \frac{\hbar}{4}\left((|+\rangle + i|-\rangle)(\langle +| - i\langle -|) - (|+\rangle - i|-\rangle)(\langle +| + i\langle -|)\right)$$

$$= \frac{\hbar}{4}\Big(|+\rangle\langle +| - i|+\rangle\langle -| + i|-\rangle\langle +| + |-\rangle\langle -| - |+\rangle\langle +| - i|+\rangle\langle -| + i|-\rangle\langle +| - |-\rangle\langle -|\Big)$$

$$= \frac{\hbar}{4}(-2i|+\rangle\langle -| + 2i|-\rangle\langle +|) = \frac{\hbar}{2}(-i|+\rangle\langle -| + i|-\rangle\langle +|)$$

So: $\frac{\hbar}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_2$