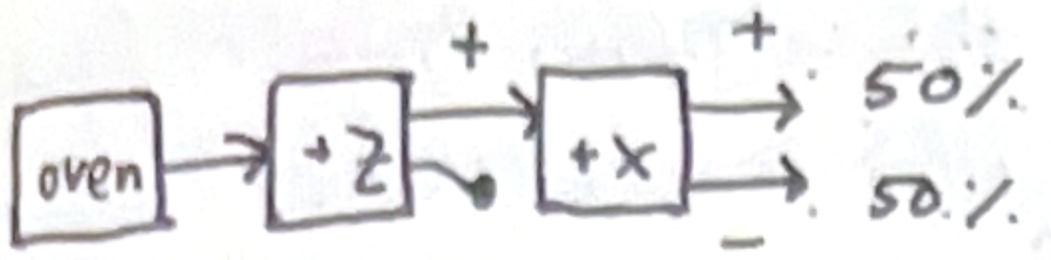
Week 5 – Spin, measurements

→ Measurement is a stochastic process



(for a beam of atoms)

We know "flux" out, $\phi^{out}_{S_{x},+}$ / flux in, $\phi^{in}_{S_{z},+}$ $= \dfrac{\|\hat{P}_{S_x,+}|\Psi\rangle\|^2}{\||\Psi\rangle\|^2}$
where the projector $\hat{P}_{S_x,+}$ describes the action of $\boxed{+x}$: $|S_x,+\rangle\langle S_x,+|$ and $|\Psi\rangle$ is the incoming wave. In this case, $|\Psi\rangle$ is the output of $\boxed{+Z}$, $|+\rangle$, so we have:

$$\frac{\| |S_x,+\rangle\langle S_x,+|+\rangle\|^2}{\||+\rangle\|^2} = |\langle S_x,+|+\rangle|^2$$

(since $|S_x,+\rangle$ and $|+\rangle$ are normalized, $\||+\rangle\| = 1$, $\||S_x,+\rangle\| = 1$.)
$\llcorner \; \||v\rangle\| = 1$

$|+\rangle = \binom{1}{0}$ and $|S_x,+\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$

Essentially we're just calculating:

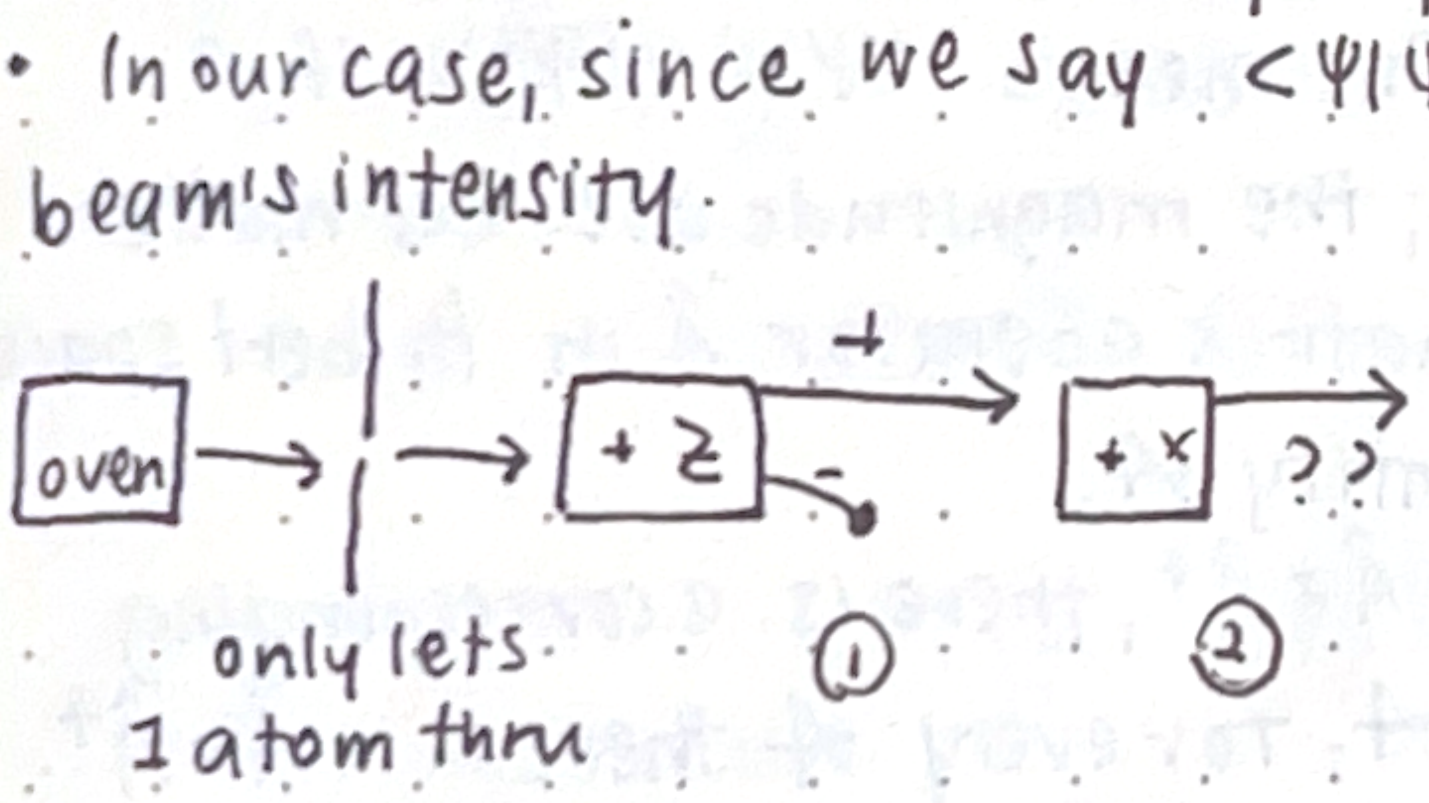
$$\frac{\langle \hat{P}_{S_x,+}\Psi \,|\, \hat{P}_{S_x,+}\Psi\rangle}{\langle\Psi|\Psi\rangle} — \langle\Psi|\hat{P}^{+}\hat{P}|\Psi\rangle$$

$\hat{P}$ is a projector, so $\hat{P}^{+}\hat{P} = \hat{P}\hat{P} = \hat{P}$

$\langle\Psi|\hat{P}|\Psi\rangle$

where $\hat{P}_{S_x,+}$ is a projector to $|S_x,+\rangle$'s eigenspace. (It spans the space.) $\langle\Psi|\hat{P}|\Psi\rangle = \langle\Psi|S_x,+\rangle\langle S_x,+|\Psi\rangle$, and we call $\langle S_x,+|\Psi\rangle$ the probability amplitude that $|\Psi\rangle$ has for $|S_x,+\rangle$.

- In our case, since we say $\langle\Psi|\Psi\rangle = 1$, this is ultimately the outgoing beam's intensity.



- Only one atom at a time!
- At ①, atoms going to ② is known to be in $|+\rangle$
- At ②, even though the input is always the same... we can't predict the outcome (and there's more than one possible!)

The probability is well defined:
if N = # atoms into ② total,
$n_{\pm}(N)$ = # $|S_x,\pm\rangle$ atoms out,

$$\lim_{N\to\infty}\frac{n_+(N)}{N} = \lim_{N\to\infty}\frac{n_-(N)}{N} = 0.5$$

But the outcome is truly random.

So, measuring spin is a stochastic process.

→ Postulates about spin so far (+ one extra) - Copenhagen

- ONTOLOGICAL POSTULATES (defining fundamental objects)

1. The state of a system is a vector $|\Psi\rangle$ in the complex Hilbert Space $\mathbb{C}^2$. The global phase and normalization of $|\Psi\rangle$ is irrelevant.
2. An observable is a Hermitian operator $\hat{A}$ that represents a spin physical quantity, e.g. $S_i = \frac{\hbar}{2}\sigma_i$, where σ_i in the basis of $|t\rangle$ is: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. This quantity is $\mathcal{A}$.

- MEASUREMENT POSTULATES

3. The measurement of a physical quantity $\mathcal{A}$ yields an eigenvalue of $\hat{A}$, the observable matrix, e.g. $\hbar/2$
4. After measuring an eigenvalue λ, the state vector $|\Psi\rangle$ becomes $\hat{P}_\lambda|\Psi\rangle$, the projection of $|\Psi\rangle$ into λ's eigenspace. The original $|\Psi\rangle$ is destroyed.
5. The probability of measuring $\lambda = \underline{\underline{\|\hat{P}(\lambda)\|^2}} / \||\Psi\rangle\|^2$, or the probability of the function after / before measurement.
6. (Not covered yet) While not measuring an isolated system, $|\Psi\rangle$ evolves over time as: $|\Psi(t)\rangle = U(t)|\Psi(t=0)\rangle$, where U is a strongly continuous one-parameter group of unitary transformations. Schroedinger's $i\hbar\partial_t|\Psi\rangle = H|\Psi\rangle$ is a special (unbounded) case.

→ General case postulates

- ONTOLOGICAL POSTULATES

1. The state of a system is a ray in a complex separable (so it has a countable basis) Hilbert Space. This means $|\Psi\rangle \sim |\Psi'\rangle$ if $\exists c \in \mathbb{C}/\{0\}$ where $|\Psi\rangle = c|\Psi'\rangle$; the magnitude does not matter.
2. An observable is a self-adjoint matrix operator $\hat{A}$ in Hilbert space that corresponds to a physical quantity $\mathcal{A}$.
   - The converse is true: for every $\hat{A} = \hat{A}^\dagger$, there is a corresponding measurable physical quantity $\mathcal{A}$, for every $\mathcal{A}$ there is a $\hat{A} = \hat{A}^\dagger$.
   - In an infinite dimensional space, Hermitian does not guarantee self adjointness:

     $A: H \to H$ is not guaranteed.

     A: Domain$(A) \to H$ & $\forall \Psi, \Phi \in D(A)$, $\langle\Psi|\hat{A}|\Phi\rangle = \langle\Phi|\hat{A}\Psi\rangle^*$ (A is Hermitian) still allows Domain$(A) \neq$ Domain$(A^\dagger)$ >:0

   Then you can't use the spectral theorem to guarantee anything about eigenvectors. A is not even a measurement.

3. Measuring physical quantity $\mathcal{A}$ yields some eigenvalue λ of $\hat{A}$, $\lambda \in \sigma_A$
4. Measuring $\mathcal{A}$ and getting λ makes initial state $|\psi\rangle \to \hat{P}_\lambda|\psi\rangle$
5. Probability of observing λ, $P(\lambda) = \|\hat{P}_\lambda|\psi\rangle\|^2 / \||\psi\rangle\|^2$
6. State of an isolated system evolves according to a time unitary : $|\psi(t)\rangle = \hat{u}(t)|\psi(0)\rangle$.

→ Spectral theorem according to eigenspaces ...

- if $\hat{A} = \hat{A}^\dagger$, the spectral theorem holds : $\hat{A} = \sum_{\lambda \in \sigma_A} \lambda \hat{P}_\lambda$
  say each eigenspace is denoted by $H_{\lambda i}$, then $H = H_{\lambda_1} \oplus H_{\lambda_2} \oplus \cdots H_{\lambda n} = \bigoplus_{\lambda \in \sigma_A} H_\lambda$ where H is a Hilbert space so $\hat{A} : H \to H$.
- Essentially, $\hat{A}$ will map vectors in $H_{\lambda i}$ to the same $H_{\lambda i}$, since in that space $\hat{A}$ acts like a scaled $\mathbb{I}$.

→ Compatible observables

- observables are self-adjoint operators in $\mathcal{H}$. Say $\hat{A} = \hat{A}^\dagger$ and $\hat{B} = \hat{B}^\dagger$, and $[\hat{A}, \hat{B}] = 0$. $\hat{A}$ and $\hat{B}$ are compatible.
  - (This rarely happens)
- Th: You can define a common basis of eigenvectors for $\hat{A}$ & $\hat{B}$.
  - Starting from the spectral theorem : $\hat{A} = \sum_{\lambda \in \sigma_A} \lambda \hat{P}_\lambda$
    Since $[\hat{A}, \hat{B}] = 0$, $\hat{P}_\mu [\hat{A}, \hat{B}] \hat{P}_\lambda = 0$
    where $\hat{P}_\mu$ projects to some eigenspace of $\hat{A}$ and $\hat{P}_\lambda$ projects into some eigenspace of $\hat{A}$ also.

$$\hat{P}_\mu [\hat{A}, \hat{B}] \hat{P}_\lambda = \underline{\hat{P}_\mu \hat{A}} \hat{B} \hat{P}_\lambda - \hat{P}_\mu \hat{B} \underline{\hat{A} \hat{P}_\lambda} = 0$$

$$= \mu \hat{P}_\mu \hat{B} \hat{P}_\lambda - \hat{P}_\mu \hat{B} \lambda \hat{P}_\lambda = 0$$

$$= (\mu - \lambda) \hat{P}_\mu \hat{B} \hat{P}_\lambda$$

- This is because:
  $\hat{P}_\mu \hat{A} = \hat{P}_\mu (\sum_{\lambda \sigma_A} \lambda \hat{P}_\lambda)$ and a $\hat{P}_\mu (\lambda \hat{P}_\lambda)$ term is :

  $\lambda |\mu\rangle\langle\mu|\lambda\rangle\langle\lambda|$ which $= 0$ unless $\lambda = \mu$, which is:
  $\mu|\mu\rangle\langle\mu|\mu\rangle\langle\mu| = \mu \hat{P}_\mu \hat{P}_\mu = \mu \hat{P}_\mu$, since projectors are idempotent.
- When $\mu \neq \lambda$, we know that $\hat{P}_\mu \hat{B} \hat{P}_\lambda = 0$, meaning we can then write the expression $\delta_{\mu\lambda} \hat{P}_\lambda \hat{B} \hat{P}_\lambda = \hat{P}_\mu \hat{B} \hat{P}_\lambda$
  This is because we only care about $\hat{P}_\mu \hat{B} \hat{P}_\lambda$ if $\mu = \lambda$, so really $\hat{P}_\lambda \hat{B} \hat{P}_\lambda$.
- We can say $\hat{B} = (\sum_\lambda \hat{P}_\lambda) \hat{B} (\sum_\mu \hat{P}_\mu)$, since we know

$\mathbb{I} = \sum_{\lambda \in \sigma_A} \hat{P}_\lambda$. You're projecting onto all bases and summing those projections back together, yielding the same vector you started with.

Then $\hat{B} = \sum_{\lambda, \mu \in \sigma_A} \hat{P}_\lambda \hat{B} \hat{P}_\mu$. Then, given our previous result that

$\hat{P}_\mu \hat{B} \hat{P}_\lambda = \delta_{\mu\lambda} \hat{P}_\lambda \hat{B} \hat{P}_\lambda$, which we can manipulate as such:

$$(\hat{P}_\mu \hat{B} \hat{P}_\lambda)^\dagger = (\delta_{\mu\lambda} \hat{P}_\lambda \hat{B} \hat{P}_\lambda)^\dagger$$ (and all operators are self-adjoint)

$$P_\lambda B P_\mu = \delta_{\mu\lambda} \hat{P}_\lambda \hat{B} \hat{P}_\lambda$$

we write: $\hat{B} = \sum_{\lambda,\mu \in \sigma_A} \delta_{\mu\lambda} \hat{P}_\lambda \hat{B} \hat{P}_\lambda = \underbrace{\sum_{\lambda \in \sigma_A}}_{\text{all } H_\lambda} \underbrace{\hat{P}_\lambda B \hat{P}_\lambda}_{\hat{B} \text{ restricted to eigenspace } H_\lambda}$

- We see that $\hat{B}$ preserves each eigenspace of $\hat{A}$, or that $\hat{B}$ maps each eigenspace of $\hat{A}$ to itself.

" Every eigenspace H_λ of $\hat{A}$ is closed under the action of $\hat{B}$."

- $\hat{B}$ (when taking vector in H_λ): $H_\lambda \to H_\lambda$, $|\psi\rangle \to \hat{B}|\psi\rangle$

  $\hat{A}$ (" " " "): $H_\lambda \to H_\lambda$, $|\psi\rangle \to \lambda|\psi\rangle$

  also $[\hat{B}]_{H_\lambda} = [\hat{B}]^\dagger_{H_\lambda}$ — $\hat{B}$ is self-adjoint in each H_λ

- In each H_λ, then, we can choose a set of orthonormal vectors to form a basis that diagonalizes $[\hat{B}]_{H_\lambda} = \hat{P}_\lambda \hat{B} \hat{P}_\lambda$.

  $(\hat{P}_\lambda \hat{B} \hat{P}_\lambda)^\dagger = \hat{P}_\lambda \hat{B} \hat{P}_\lambda$ still. Self-adjoint operators in a H can be diagonalized

  - This means $\exists \{|v\rangle\} \in H_\lambda \mid \hat{P}_\lambda \hat{B} \hat{P}_\lambda |v\rangle = \hat{B}|v\rangle = c|v\rangle$ where c is some scalar, so $|v\rangle$ is an eigenvector of $\hat{B}$.

    $|v\rangle$ is in H_λ, so it's an eigenvector of $\hat{A}$.

  - Do this for all H_λ to get common basis of eigenvectors.

→ Expectation values

$$\bar{A} = \sum_{\lambda \in \sigma_A} \lambda \underbrace{P(\lambda)}_{\text{probability}} = \sum_{\lambda \in \sigma_A} \lambda \| \underbrace{\hat{P}_\lambda}_{\text{projector}} |\psi\rangle \|^2 / \| |\psi\rangle \|^2$$

$$= \frac{\sum_{\lambda \in \sigma_A} \lambda \langle \psi \hat{P}_\lambda | \hat{P}_\lambda \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\sum_{\lambda \in \sigma_A} \lambda \langle \psi | \underbrace{\hat{P}_\lambda^\dagger \hat{P}_\lambda}_{= \hat{P}_\lambda} | \psi \rangle}{\langle \psi | \psi \rangle}$$

$$= \frac{\langle \psi | \overbrace{\sum_{\lambda \in \sigma_A} \lambda \hat{P}_\lambda}^{= \hat{A} \text{, spectral dec.}} | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{\langle \psi | \hat{A} | \psi \rangle}{\langle \psi | \psi \rangle} = \langle \hat{A} \rangle_\psi = \langle \hat{A} \rangle$$

- *Not neccesarily equal to an eigenvalue, but obviously within the same range.*

- Say we measure $\hat{A}$, then $\hat{B}$, when $[\hat{A},\hat{B}]=0$

$$\hat{A}=\sum_{\lambda\in\sigma_A}\lambda\hat{P}_\lambda,\quad \hat{B}=\sum_{\mu\in\sigma_B}\mu\hat{\Lambda}_\mu,\ \text{and}\ [\hat{P}_\lambda,\hat{\Lambda}_\mu]=0\ \text{since}$$

$\hat{A}$ and $\hat{B}$ have some common basis of eigenvectors:

$$\hat{P}_{\lambda'}|\lambda,\mu,i\rangle=\delta_{\lambda\lambda'}|\lambda,\mu,i\rangle,\ \hat{\Lambda}_{\mu'}|\lambda,\mu,i\rangle=\delta_{\mu\mu'}|\lambda,\mu,i\rangle$$

$$\hat{\Lambda}_\mu(\hat{P}_{\lambda'}|\lambda,\mu,i\rangle)=\delta_{\mu\mu'}\delta_{\lambda\lambda'}|\lambda,\mu,i\rangle=\hat{P}_\lambda(\hat{\Lambda}_\mu|\lambda,\mu,i\rangle),\ \text{so}$$
$$\hat{\Lambda}_\mu\hat{P}_\lambda=\hat{P}_\lambda\hat{\Lambda}_\mu.$$

Then:

$$|\psi\rangle\longrightarrow\boxed{A}\xrightarrow[\hat{P}_\lambda|\psi\rangle]{\lambda}\boxed{B}\xrightarrow[\hat{\Lambda}\hat{P}|\psi\rangle]{\mu}\quad(\text{Measure } \hat{A}, \text{then } \hat{B})$$

$$|\psi\rangle\longrightarrow\boxed{B}\xrightarrow[\hat{\Lambda}_\mu|\psi\rangle]{\mu}\boxed{A}\xrightarrow[\hat{P}\hat{\Lambda}|\psi\rangle]{\lambda}\quad(\text{"}\ \hat{B},\ \text{"}\ \hat{A})$$

$[\hat{\Lambda}_\mu,\hat{P}_\lambda]=0$, so $\hat{\Lambda}_\mu\hat{P}_\lambda|\psi\rangle=\hat{P}_\lambda\hat{\Lambda}_\mu|\psi\rangle$, meaning the result of the two measurements is the the same regardless of the order.

$$P(\lambda,\mu)=P(\lambda)\cdot P(\mu|\lambda)$$
$$=\frac{\|\hat{P}_\lambda|\psi\rangle\|^2}{\||\psi\rangle\|^2}\cdot\frac{\|\hat{\Lambda}_\mu\hat{P}_\lambda|\psi\rangle\|^2}{\|\hat{P}_\lambda|\psi\rangle\|^2}=\frac{\|\hat{\Lambda}_\mu\hat{P}_\lambda|\psi\rangle\|^2}{\||\psi\rangle\|^2}$$

the same!

$$P(\mu,\lambda)=P(\mu)\cdot P(\lambda|\mu)$$
$$=\frac{\|\hat{\Lambda}_\mu|\psi\rangle\|^2}{\||\psi\rangle\|^2}\cdot\frac{\|\hat{P}_\lambda\hat{\Lambda}_\mu|\psi\rangle\|^2}{\|\hat{\Lambda}_\mu|\psi\rangle\|^2}=\frac{\|\hat{P}_\lambda\hat{\Lambda}_\mu|\psi\rangle\|^2}{\||\psi\rangle\|^2}$$

The measurements <u>and</u> probabilities are the same — property of compatible observables.