WEEK 14 – ANGULAR MOMENTUM

We've seen the CM Rotation operator $\hat{R}_{\hat{n}}(d\phi)\vec{v} = \vec{v} + \hat{n}\times\vec{v}\,d\phi$ for any vector $\vec{v}$. The infinitesimal generator is $G(\vec{r},\vec{p}) = \hat{n}\cdot(\vec{r}\times\vec{p})$, and we say:

$\vec{L} = \vec{r}\times\vec{p}$ = orbital angular momentum

Classically, we also know generators have the following property:

$$\frac{dA(\vec{r},\vec{p})}{d\phi} = \{A(\vec{r},\vec{p}),\, \hat{n}\cdot\vec{L}\}_{PB}$$ for a function A paramaterized by the same quantities as generator G.

This means a scalar quantity (invariant under rotation) s, has, by definition:

$$\frac{ds}{d\phi} = 0, \text{ so } \{s, \hat{n}\cdot\vec{L}\} = 0$$

This is true for any axis $\hat{n}$ you want to try and rotate around, so

$$\{s, \hat{n}\cdot\vec{L}\} = \{s, n_i L_i\} = n_i\{s, L_i\} = 0 \rightarrow \{s, L_i\} = 0$$

$$= \{s, \sum_i n_i L_j\} = \{s, n_i L_i\} + \{s, n_j L_j\} + \{s, n_k L_k\}$$

$$= n_i\{s, L_i\} + n_j\{s, L_j\} + n_k\{s, L_k\}$$

n_i, n_j, and n_k can be <u>anything</u>, since this is for <u>any</u> $\hat{n}$, as long as (n_i, n_j, n_k) has length 1.

So, the only solution is that $\{s, L_i\}$, $\{s, L_j\}$, and $\{s, L_k\}$ are all 0.

That's why we can say any $\{s, L_i\} = 0$.

So, we've seen the derivatives wrt ϕ of functions and scalars, what about a vector $\vec{A}$?

Without the generator property: $$\frac{d\vec{A}}{d\phi} = \lim_{\Delta\phi\to 0}\frac{\hat{R}_{\hat{n}}(\Delta\phi)\vec{A} - \vec{A}}{\Delta\phi}$$

(def of a derivative) which is $\hat{n}\times\vec{A}$. (Change in $\vec{A}$ as it is rotated gets very close to a vector perpendicular to $\hat{n}$ and $\vec{A}$. $\|n\times A\| \approx \|\vec{A}\|\theta$ for small θ)

With generator property: $$\frac{d\vec{A}}{d\phi} = \{\vec{A}, \hat{n}\cdot\vec{L}\}, \text{ so } \{\vec{A}, \hat{n}\cdot\vec{L}\} = \hat{n}\times\vec{A}$$

Because this is true for all $\hat{n}$ again,

$\{\vec{A}, \hat{n}\cdot\vec{L}\} = \{(A_i, A_j, A_k), \sum_\ell n_\ell L_\ell\}$

$= (\{A_i, \sum_\ell n_\ell L_\ell\}, \{A_j, \sum_\ell n_\ell L_\ell\}, \{A_k, \sum_\ell n_\ell L_\ell\})$

$= \hat{n}\times\vec{A} = ((n\times A)_1, (n\times A)_2, (n\times A)_3)$

$= (\sum_{jk} \varepsilon_{1jk} n_j A_k, \sum_{jk} \varepsilon_{2jk} n_j A_k, \sum_{jk} \varepsilon_{3jk} n_j A_k)$

implying $\{A_i, \sum_\ell n_\ell L_\ell\} = \sum_{jk} \varepsilon_{ijk} n_j A_k$, etc.

$$\sum_\ell \{A_i, n_\ell L_\ell\} = \sum_{jk} \varepsilon_{1jk} n_j A_k$$

$i=1$ $\quad \ell=j$

$\{A_i, n_i L_i\} + \{A_i, n_j L_j\} + \{A_i, n_k L_k\} = \varepsilon_{11k} n_1 A_k + \varepsilon_{12k} n_2 A_k + \varepsilon_{13k} n_3 A_k$

so some $\{A_i, n_j L_j\} = \varepsilon_{ijk} n_j A_k$

$$\{A_i, L_j\} = \varepsilon_{ijk} A_k$$

Both the poisson bracket and ε_{ijk} are anti-symmetric, so

$$+\{L_i, A_j\} = +\varepsilon_{ijk} A_k$$

if $\vec{A}$ is $\vec{p}$, $\vec{x}$, or $\vec{L}$:

$\{L_i, p_j\} = \varepsilon_{ijk} p_k$, $\{L_i, x_j\} = \varepsilon_{ijk} x_k$, $\{L_i, L_j\} = \varepsilon_{ijk} L_k$

and because $L^2 = \vec{L}\cdot\vec{L}$ is a scalar, we can use our previous result and say $\{L^2, L_i\} = 0$

QM Rotation / angular momentum

Rotation is a unitary transformation, so we can write rotation operator $\hat{D}(R)$ as

$\hat{D}(R) = e^{-i\phi\hat{G}} = e^{-i\phi(\hat{n}\cdot\hat{J})}$ where $\hat{G}$ is the self-adjoint operator and ϕ is the single parameter of the rotation operator (Stone's theorem). I guess the units work so $/\hbar$ is not needed as in the other canonical quantizations.

we also previously showed $\{A, B\}_{PB} = [\hat{A}, \hat{B}]/i\hbar$ so we can say:

$[\hat{J}_i, \hat{J}_j] = i\hbar \, \varepsilon_{ijk} \hat{J}_k$

$[\hat{J}_i, S] = 0$ (scalars)

$[\hat{J}_i, \hat{A}_j] = i\hbar \, \varepsilon_{ijk} \hat{A}_k$ (vectors)

$[\hat{J}^2, \hat{J}_i] = 0$

Note $\hat{\vec{J}}$ does not correspond exactly to classical L, or even to QM $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$.. $\hat{\vec{J}}$ is <u>total</u> angular momentum, not just orbital angular momentum:

$$\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$$

orbital ↑ intrinsic spin! ofc not in CM but shown via experiment to contribute.

Demonstrating $[\hat{L}_i, \hat{L}_j] = i\hbar \, \varepsilon_{ijk} \hat{L}_k$ for some reason:

- we know that $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$, so individual $\hat{L}_i$'s are $\varepsilon_{ijk} \hat{x}_j \hat{p}_k$

$$[\hat{L}_i, \hat{L}_j] = [\varepsilon_{ilm} \hat{x}_l \hat{p}_m, \varepsilon_{jko} \hat{x}_k \hat{p}_o]$$

$$= \varepsilon_{ilm} \varepsilon_{jko} [\hat{x}_l \hat{p}_m, \hat{x}_k \hat{p}_o]$$

$$= \varepsilon_{ilm} \varepsilon_{jko} \left(\hat{x}_l [\hat{p}_m, \hat{x}_k] \hat{p}_o + \hat{x}_k [\hat{x}_l, \hat{p}_o] \hat{p}_m \right)$$

$$= \varepsilon_{ilm} \varepsilon_{jko} \left(\hat{x}_l (-i\hbar \, \delta_{mk}) \hat{p}_o + \hat{x}_k (+i\hbar \, \delta_{lo}) \hat{p}_m \right)$$

$$= i\hbar \, \varepsilon_{ilm} \varepsilon_{jkl} \hat{x}_k \hat{p}_m - i\hbar \, \varepsilon_{ilk} \varepsilon_{jko} \hat{x}_l \hat{p}_o$$

$$= i\hbar \, \varepsilon_{lmi} \varepsilon_{ljk} \hat{x}_k \hat{p}_m - i\hbar \, \varepsilon_{kil} \varepsilon_{koj} \hat{x}_l \hat{p}_o$$

(and $\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$)

$$= i\hbar (\delta_{mj} \delta_{ik} - \delta_{mk} \delta_{ij}) \hat{x}_k \hat{p}_m - i\hbar (\delta_{io} \delta_{lj} - \delta_{ij} \delta_{lo}) \hat{x}_l \hat{p}_o$$

$$= i\hbar (\hat{x}_i \hat{p}_j - \hat{x}_k \hat{p}_k \delta_{ij} - \hat{x}_j \hat{p}_i + \hat{x}_l \hat{p}_l \delta_{ij} \}$$

$$= i\hbar (\hat{x}_i \hat{p}_j - \hat{x}_j \hat{p}_i) = i\hbar (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \hat{x}_l \hat{p}_m$$

$$= i\hbar \, \varepsilon_{ijk} \varepsilon_{klm} \hat{x}_l \hat{p}_m = i\hbar \, \varepsilon_{ijk} \hat{L}_k$$

The spectrum of $\hat{J}_i$

Say we use $\hat{J}_z$. $[\hat{J}^2, \hat{J}_z] = 0$. To find the spectra of $\hat{J}_x$ and $\hat{J}_y$ we can rotate:

$$\underset{\substack{\uparrow \\ \text{align} \\ \text{back to} \\ x\text{-axis}}}{\hat{J}_x = e^{-i\hat{J}_y \pi/2}} \; \underset{\substack{\uparrow \\ \text{measure} \\ \ldots?}}{\hat{J}_z} \; \underset{\substack{\uparrow \\ \text{align to} \\ z\text{-axis}}}{e^{i\hat{J}_y \pi/2}} \quad \text{and} \quad \hat{J}_y = e^{i\hat{J}_x \pi/2} \hat{J}_z e^{-i\hat{J}_x \pi/2}$$

so $\hat{J}_z |\lambda\rangle = \lambda |\lambda\rangle \rightarrow e^{i\hat{J}_y \pi/4} \hat{J}_x e^{-i\hat{J}_y \pi/4} |\lambda\rangle = \lambda |\lambda\rangle$

$$\hat{J}_x \underbrace{e^{-i\hat{J}_y \pi/4} |\lambda\rangle}_{\text{eigenvector of } \hat{J}_x} = \lambda \underline{e^{-i\hat{J}_y \pi/4} |\lambda\rangle}$$

Anyway. The best way to determine the spectrum of $\hat{J}_z$ is using non-self-adjoint ladder operators again:

$$\hat{J}_\pm \equiv \hat{J}_x \pm i\hat{J}_y, \quad \hat{J}_\pm^\dagger = \hat{J}_\mp$$

that commute as follows:

$$[\hat{J}^2, \hat{J}_\pm] = 0$$

$$[\hat{J}_z, \hat{J}_\pm] = [\hat{J}_z, \hat{J}_x \pm i\hat{J}_y] = [\hat{J}_z, \hat{J}_x] \pm i[\hat{J}_z, \hat{J}_y]$$

$$= i\hbar \hat{J}_y \pm i(-i\hbar)\hat{J}_x$$

$$= \pm\hbar(\hat{J}_x \pm i\hat{J}_y) = \pm\hbar \hat{J}_\pm$$

$$[\hat{J}_+, \hat{J}_-] = [\hat{J}_x + i\hat{J}_y, \hat{J}_x - i\hat{J}_y]$$

$$= [\hat{J}_x, \hat{J}_x] + [\hat{J}_x, -i\hat{J}_y] + [i\hat{J}_y, \hat{J}_x] + [i\hat{J}_y, -i\hat{J}_y]$$

$$= 0 - i[\hat{J}_x, \hat{J}_y] + i[\hat{J}_y, \hat{J}_x] + 0$$

$$= -i(i\hbar)\hat{J}_z + i(-i\hbar)\hat{J}_z$$

$$= 2\hbar \hat{J}_z$$

And applying $\hat{J}_+ \hat{J}_-$ behaves as:

$$\hat{J}_+\hat{J}_- = (\hat{J}_x + i\hat{J}_y)(\hat{J}_x - i\hat{J}_y)$$
$$= \hat{J}_x^2 + \hat{J}_y^2 + i\hat{J}_y\hat{J}_x - i\hat{J}_x\hat{J}_y$$
$$= \hat{J}_x^2 + \hat{J}_y^2 + i[\hat{J}_y, \hat{J}_x]$$
$$= \hat{J}_x^2 + \hat{J}_y^2 + i(-i\hbar)\hat{J}_z = \hat{J}_x^2 + \hat{J}_y^2 + \hbar\hat{J}_z$$

and, $\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$, so $\hat{J}_x^2 + \hat{J}_y^2 = \hat{J}^2 - \hat{J}_z^2$

$$\hat{J}_+\hat{J}_- = \hat{J}^2 - \hat{J}_z^2 + \hbar\hat{J}_z$$

so similarly, $\hat{J}_-\hat{J}_+ = \hat{J}^2 - \hat{J}_z^2 - \hbar\hat{J}_z$

In addition to defining $\hat{J}_\pm$, we re-define/re-write the eigenvalues of $\hat{J}^2$ and $\hat{J}_z$:

$[\hat{J}^2, \hat{J}_z] = 0$ (they commute), so there is a basis of eigenvectors where:

$$\hat{J}^2|k,\lambda,\mu\rangle = \lambda|k,\lambda,\mu\rangle, \quad \hat{J}_z|k,\lambda,\mu\rangle = \mu|k,\lambda,\mu\rangle$$

$\langle\psi|\hat{J}^2|\psi\rangle$ for self-adjoint $\hat{J}$ is $\|\hat{J}|\psi\rangle\|^2 \geq 0$, so all expected values are positive or $0 \longrightarrow$ all eigenvalues ≥ 0.

So we can define $\lambda = \hbar^2 j(j+1)$.

We also define $\mu = \hbar m$, where j and m are dimensionless.

Then $|k,\lambda,\mu\rangle \rightarrow |k,j,m\rangle$:

$$\hat{J}^2|kjm\rangle = \hbar^2 j(j+1)|kjm\rangle$$

$$\hat{J}_z|kjm\rangle = \hbar m|kjm\rangle$$

<u>Finally</u> we can find the spectrum of $\hat{J}_z$.

$$\| \hat{J}_+ |kjm\rangle \|^2 = \langle kjm | \hat{J}_- \hat{J}_+ | kjm\rangle$$

$$= \langle kjm | \hat{J}^2 - \hat{J}_z^2 - \hbar \hat{J}_z | kjm\rangle$$

$$= \langle kjm | \hat{J}^2 | kjm\rangle - \langle kjm | \hat{J}_z^2 | kjm\rangle - \hbar \langle kjm | \hat{J}_z | kjm\rangle$$

$$= \hbar^2 j(j+1) - \hbar^2 m^2 - \hbar(\hbar m)$$

$$= \hbar^2 (j(j+1) - m(m+1))$$

Norm must be ≥ 0, so:

$$\hbar^2 (j(j+1) - m(m+1)) \geq 0$$

$$j(j+1) \geq m(m+1) \rightarrow j^2 + j \geq m^2 + m$$

$\therefore$ if $m \geq 0$, $j \geq m$ (if $j < m$, $j^2 < m^2$, $j^2 + j < m^2 + m \ldots$)

$$\| \hat{J}_- |kjm\rangle \|^2 = \langle kjm | \hat{J}_+ \hat{J}_- | kjm\rangle$$

$$= \langle kjm | \hat{J}^2 - \hat{J}_z^2 + \hbar \hat{J}_z | kjm\rangle$$

$$= \hbar^2 j(j+1) - \hbar^2 m^2 + \hbar(\hbar m)$$

$$0 \leq \hbar^2 (j(j+1) - m(m-1))$$

$$m(m-1) \leq j(j+1) \rightarrow m^2 - m \leq j^2 + j$$

$\therefore$ The magnitude $|m|$ must be less than or equal to j, if $m < 0$:

$$m^2 - m = m^2 + |m| \leq j^2 + j$$

No matter what, $|m| \leq j$ (which can be positive or negative), so

$$-j \leq m \leq j$$

What do $\hat{J}_\pm$ actually do to $|kjm\rangle$?

$[\hat{J}^2, \hat{J}_\pm] = 0$, so $\hat{J}^2 \hat{J}_\pm |kjm\rangle = \hat{J}_\pm \hat{J}^2 |kjm\rangle = \hbar^2 j(j+1) \hat{J}_\pm |kjm\rangle$

Thus $\hat{J}_\pm |kjm\rangle$ is still an eigenvector of $\hat{J}^2$, which has the same eigenvalue.

$\hat{J}_z \hat{J}_\pm |kjm\rangle = ?$

$$\hat{J}_z \hat{J}_\pm - \hat{J}_\pm \hat{J}_z = [\hat{J}_z, \hat{J}_\pm] = \pm\hbar \hat{J}_\pm$$

$$\hat{J}_z \hat{J}_\pm = \hat{J}_\pm \hat{J}_z \pm \hbar \hat{J}_\pm$$

$$= (\hat{J}_\pm \hat{J}_z \pm \hbar \hat{J}_\pm)|kjm\rangle$$

$$= \hat{J}_\pm (\hat{J}_z \pm \hbar)|kjm\rangle = \hat{J}_\pm (m\hbar \pm \hbar)|kjm\rangle$$

$$= \hbar(m \pm 1)\hat{J}_\pm |kjm\rangle$$

so $\hat{J}_\pm |kjm\rangle$ is still an eigenvector of $\hat{J}_z$, with eigenvalue $\hbar(m \pm 1)$ where m is raised/lowered by 1.

If $m = \pm j$, applying $\hat{J}_\pm$ yields the null vector:

$$\| \hat{J}_+ |kjj\rangle \|^2 = \langle kjj | \hat{J}_- \hat{J}_+ | kjj\rangle$$

$$= \langle kjj | \hat{J}^2 - \hat{J}_z^2 - \hbar \hat{J}_z | kjj\rangle$$

$$= \hbar^2 j(j+1) - j^2\hbar^2 - \hbar^2 j = \hbar^2(j(j+1) - j^2 - j)$$

$$= 0$$

$$\| \hat{J}_- |kj{-j}\rangle \|^2 = \langle kj{-j} | \hat{J}_+ \hat{J}_- | kj{-j}\rangle$$

$$= \langle kj{-j} | \hat{J}^2 - \hat{J}_z^2 + \hbar \hat{J}_z | kj{-j}\rangle$$

$$= \hbar^2 j(j+1) - \hbar^2 j^2 - \hbar^2 j = 0$$

So, if you start at $m < j$ and apply $\hat{J}_+$ until $m = j$, you get the null vector, and no longer get an eigenvector of $\hat{J}_z$.

Similarly, if you start at $m > -j$ and apply $\hat{J}_-$ until $m = -j$, you get the null vector.

Additionally, if you start at non-integer $j \pm m$, you can increase or decrease m until it is under $-j$ or over j, which we showed is not valid.

Altogether: $j+m \in \mathbb{N}_0$ else we get $m < -j$, and $j-m \in \mathbb{N}_0$ else we get $m > j$ $\Rightarrow$ $2j \in \mathbb{N}_0$: j is an integer or half-integer.

Also, $m = -j, -j+1, -j+2, \cdots, j-1, j$ ($2j+1$ poss. values)

(Note $m-(-j)$ is the distance from m to $-j$, alway positive or 0 since $|m| \leq j$, and $j-m$ is pos. distance from m to j...)