WEEK 6 - TDSE and the RABI Model

Last semester we saw that for time evolution, $i\hbar\partial_t|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$ where $\hat{H}$ is time independent.

Say we have some quantum system in an external field that is dependent on time, $\vec{F}(t)$ which is, in reality, from a larger quantum system, with ~~some~~ different degrees of freedom.

Total state $|\Psi\rangle = |\Psi_{sys}\rangle \otimes |\Psi_{rad}\rangle$ (this is a product state, but could be entangled) where $|\Psi_{rad}(t)\rangle$ is time dependent and V_{rad} does not "care" about the quantum system because it's very small.

$\hat{H} = \hat{H}_{sys} \otimes \hat{1} + \hat{1} \otimes \hat{H}_{Rad} + \underbrace{\hat{O} \otimes \hat{F}}_{\text{Some interaction term.}}$ is also time independent.

Also say $\langle\Psi_{rad}(t)|\Psi_{rad}(t)\rangle = 1$.

$$i\hbar\partial_t|\Psi(t)\rangle = i\hbar\,\partial_t|\Psi_{sys}(t)\rangle \otimes |\Psi_{rad}(t)\rangle + |\Psi_{sys}(t)\rangle \otimes i\hbar\partial_t|\Psi_{rad}(t)\rangle$$

and

$$\hat{H}|\Psi(t)\rangle = H_{sys}|\Psi_{sys}(t)\rangle \otimes |\Psi_{rad}(t)\rangle + |\Psi_{sys}(t)\rangle \otimes \hat{H}_{rad}|\Psi_{rad}(t)\rangle$$
$$+ \hat{O}|\Psi_{sys}(t)\rangle \otimes \hat{F}|\Psi_{rad}(t)\rangle$$

Now project using $\hat{1} \otimes \langle\Psi_{rad}(t)| : V \to V_{sys}$, $|\phi\rangle \otimes |\chi\rangle \to |\phi\rangle \underbrace{\langle\Psi_{rad}(t)|\chi\rangle}_{\text{a scalar!}}$

The LHside, $i\hbar\partial_t|\Psi(t)\rangle$ becomes:

$$i\hbar\partial_t|\Psi_{sys}(t)\rangle + \underline{|\Psi_{sys}(t)\rangle\langle\Psi_{rad}(t)|i\hbar\partial_t|\Psi_{rad}(t)\rangle}$$

The RH side, $\hat{H}|\Psi(t)\rangle$ becomes:

$$\hat{H}_{sys}|\Psi_{sys}(t)\rangle + \underline{|\Psi_{sys}(t)\rangle\langle\Psi_{rad}(t)|\hat{H}_{rad}|\Psi_{rad}(t)\rangle}$$
$$+ \hat{O}|\Psi_{sys}(t)\rangle\langle\Psi_{rad}(t)|\hat{F}|\Psi_{rad}(t)\rangle$$

Remember that $i\hbar\partial_t|\Psi_{rad}(t)\rangle = \hat{H}_{rad}|\Psi_{rad}\rangle$, so the underlined terms on the LHS and RHS cancel! Also call $\langle\Psi_{rad}(t)|\hat{F}|\Psi_{rad}(t)\rangle$ field $F(t)$.

$$i\hbar\partial_t|\Psi_{sys}(t)\rangle = \hat{H}_{sys}|\Psi_{sys}(t)\rangle + \hat{O}F(t)|\Psi_{sys}(t)\rangle$$
$$= (H_{sys} + \hat{O}F)|\Psi_{sys}(t)\rangle = \hat{H}(t)|\Psi_{sys}(t)\rangle$$

This effective Hamiltonian $\hat{H}(t) = \hat{H}_{sys} + \hat{O}F(t)$ is time dependent (we also assume no entanglement!) What does this Hamiltonian look like?

We have particles with mass, charge, position $\{m_i, q_i, r_i\}$, in fields $\{\vec{E}, \vec{B}\}$. We can bust out:

$$\hat{H} = \underbrace{\sum_i \frac{1}{2m_i}\left(\vec{p}_i - \frac{q_i}{c}\vec{A}(\hat{\vec{r}}_i, t)\right)^2}_{\text{Kinetic energy is not ONLY } p^2/2m \text{ anymore!}} + \sum_{i,j>i} \frac{q_i q_j}{|\hat{\vec{r}}_i - \hat{\vec{r}}_j|} + \frac{1}{8\pi}\int d^3 r \left(|\vec{E}_\perp(r,t)|^2 + |\vec{B}(\ldots)|^2\right)$$

where we have vector potential $\vec{A}$, and transverse/perpendicular component of the electric field, $\vec{E}_\perp$.

Here, we treat all fields as classical, use Gauss units, and assume the kinetic energy is well under the relativistic limit $m_i c^2$.

The Hamiltonian can be broken down into components:

$$\hat{H}_{matter} = \sum_i \frac{1}{2m_i} p_i^2 + \sum_{i,j>i} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

no hat, classical

$$H_{rad} = \frac{1}{8\pi}\int d^3 r \,(E_\perp^2 + B^2)$$ (will become operators if we quantize)

$$\hat{H}_{interaction} = \sum_i \frac{1}{2m_i}\left(-\frac{q_i}{c}\underbrace{\hat{\vec{p}}\cdot\vec{A}}_{\text{order does not matter.}} + \frac{q_i^2}{c^2}A^2\right)$$

What is the Vector potential? Maxwell's equations in Gauss units:

$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ (divergence of electric field)

$\vec{\nabla}\cdot\vec{B} = 0$ (divergence of magnetic field)

$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\partial_t\vec{B}$ (curl!)

linear in the field / Homog. eqns that constrain field

$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\partial_t E$ ($\vec{j}$ is density of charge)

Since $\vec{\nabla}\cdot\vec{B} = 0$, we can find some $\vec{A}$ where $\vec{B} = \vec{\nabla}\times A$ and $\vec{\nabla}\cdot\vec{\nabla}\times\vec{A} = 0$.
(You can verify this with $\partial_i B_i = \partial_i \varepsilon_{ijk}\partial_j A_k = \varepsilon_{ijk}\partial_i\partial_j A_k = 0$)
Thus $\vec{B}$ is the curl of vector potential $\vec{A}$.

Plug this into the third equation:

$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\partial_t \vec{\nabla}\times\vec{A} \rightarrow \vec{\nabla}\times\vec{E} + \frac{1}{c}\partial_t \vec{\nabla}\times\vec{A} = 0$

If the curl of something is 0, that thing can be written as $\vec{\nabla}\phi$ where ϕ is a scalar field.

negative by convention

So, $\vec{E} + \frac{1}{c}\partial_t \vec{A} = -\vec{\nabla}\phi$ and we now know a new parameterization

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\partial_t \vec{A}, \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

Gauge: The auxiliary fields ϕ and $\vec{A}$ are NOT uniquely defined by $\vec{E}$ and $\vec{B}$. They are not physical, observable quantities, as opposed to being able to observe $\vec{E}$ and $\vec{B}$ using $\vec{F} = q(\vec{E} + \vec{v}/c \times \vec{B})$

You can, in fact, define new vector potential $\vec{A}'$, where:

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi \text{ (where } \chi \text{ is a scalar field)}$$

This means the magnetic field realized by $\vec{A}'$ is:

$$\vec{B}' = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times(\vec{A} + \vec{\nabla}\chi) = \vec{\nabla}\times\vec{A} = \vec{B}$$ the same magnetic field! Thus we can pick from a gazillion vector potentials.

We can show the same is true for ϕ: say we have new vector potential $\vec{A}'$ that defines a new electric field:

$$\vec{E}' = -\vec{\nabla}\phi' - \frac{1}{c}\partial_t \vec{A}' = -\vec{\nabla}\phi' - \frac{1}{c}\partial_t(\vec{A} + \vec{\nabla}\chi)$$

$$= -\vec{\nabla}(\phi' + \frac{1}{c}\partial_t \chi) - \frac{1}{c}\partial_t \vec{A}$$

Thus if $\phi' = \phi - \frac{1}{c}\partial_t \chi$, then $\vec{E}' = -\vec{\nabla}\phi - \frac{1}{c}\partial_t \vec{A} = \vec{E}$ the same electric field.

So, for many possible $\vec{A}$ and ϕ, $\vec{E}$ and $\vec{B}$ are fixed, since $\vec{E}$ and $\vec{B}$ are physical and not $\vec{A}$ and ϕ.

Gauge transformation: $\begin{cases} \vec{A}' = \vec{A} + \vec{\nabla}\chi \\ \phi' = \phi - \frac{1}{c}\partial_t \chi \end{cases}$

Since we can always find some field $\vec{A}$ such that:

$$\vec{\nabla}\cdot\vec{A}' = 0 = \vec{\nabla}\cdot\vec{A} + \nabla^2\chi$$

but what is χ? It turns out that $\nabla^2\chi = -\vec{\nabla}\cdot\vec{A}$ is a Poisson equation which has a closed form solution. Thus χ is...

$$\chi(\vec{r},t) = \frac{1}{4\pi}\int d^3r' \frac{\vec{\nabla}\cdot\vec{A}(\vec{r}',t)}{|\vec{r}-\vec{r}'|} + \psi(\vec{r}) \text{ where } \nabla^2\psi = 0.$$

and these are spherical harmonics, (and $\vec{A}$ is instantaneous, which some people don't like...)

$\vec{\nabla}\cdot\vec{A} = 0$ is the Coulomb gauge, under which Maxwell's equations become:

$$\vec{\nabla}\cdot\left(-\vec{\nabla}\phi - \frac{1}{c}\partial_t\vec{A}\right) = 4\pi\rho = -\vec{\nabla}^2\phi - \frac{1}{c}\partial_t\overbrace{\vec{\nabla}\cdot\vec{A}}^{=0}$$

using Poisson equation again!

so $\vec{\nabla}^2\phi = -4\pi\rho$ and $\phi(\vec{r},t) = \int d^3r' \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$ which depends on the position/motion of charged particles: $\rho(\vec{r},t) = \langle\psi_{sys}|\sum_i q_i\,\delta(\vec{r}-\hat{\vec{r}}_i)|\psi_{sys}\rangle$ depends on the quantum system only, <u>not</u> the external field.

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\partial_t\left(-\vec{\nabla}\phi - \frac{1}{c}\partial_t\vec{A}\right)$$

$$= \vec{\nabla}(\cancel{\vec{\nabla}\cdot\vec{A}}^{=0}) \underline{-\nabla^2\vec{A}} = \frac{4\pi}{c}\vec{j} - \frac{1}{c}\partial_t\vec{\nabla}\phi \underline{- \frac{1}{c^2}\partial_t^2\vec{A}}$$

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\vec{A} = \frac{4\pi}{c}\vec{j} - \frac{1}{c}\partial_t\vec{\nabla}\phi$$

We can split the RHS into transverse ($\perp$) and longitudinal ($\parallel$) parts. In general: $\vec{F} = \vec{F}_\perp + \vec{F}_\parallel$ where:

$$\vec{\nabla}\cdot\vec{F}_\perp = 0 \text{ and } \vec{\nabla}\times\vec{F}_\parallel = 0$$

we can split $\vec{j}$ this way. There is also a $-\frac{1}{c}\partial_t\vec{\nabla}\phi$ term whose curl must be 0, so it is a longitudinal component. Furthermore, the LHS is purely transverse:

$$\nabla\cdot\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\vec{A} = \left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\underbrace{(\nabla\cdot\vec{A})}_{0 \text{ in this gauge}} = 0$$

The transverse equation:

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\vec{A} = \frac{4\pi}{c}\vec{j}_\perp \rightarrow \Box A = \frac{4\pi}{c}\vec{j}_\perp$$

The longitudinal equation:

$$0 = \frac{4\pi}{c}\vec{j}_\parallel - \frac{1}{c}\partial_t\vec{\nabla}\phi$$

$$\vec{j}_\parallel = \frac{1}{4\pi}\partial_t\vec{\nabla}\phi \quad (\text{so } \vec{j}_\parallel \text{ is uniquely determined (?)})$$

WEEK 6 – TDSE and RABI continued.

The transverse portion, $\Box A = \frac{4\pi}{c}\vec{j}_\perp$ has know solution of the form:

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int d^3r' \, \frac{\vec{j}_\perp(\vec{r}',t')}{|\vec{r}-\vec{r}'|} \quad \text{where } t' = t - \frac{|\vec{r}-\vec{r}'|}{c}$$

this t' is apparently "the time when you reach the detector".

The Longitudinal portion, $\vec{j}_\parallel = \frac{1}{4\pi}\partial_t \vec{\nabla}\phi$, expresses conservation of Charge:

$$\vec{\nabla}\cdot\vec{j}_\parallel = \frac{1}{4\pi}\partial_t \nabla^2\phi = -\partial_t \rho \quad \text{(continuity equation)}$$

INTERACTION REPRESENTATION

We can often separate the TDSE's Hamiltonian into a time independent term and a time dependent perturbation.

Thus $i\hbar\partial_t|\Psi(t)\rangle = \hat{H}(t)|\Psi(t)\rangle$ becomes: $(\hat{H}_0 + \hat{H}'(t))|\Psi(t)\rangle$

where $\hat{H}_0|\varphi_i\rangle = E_i|\varphi_i\rangle$

and $|\Psi(t)\rangle = \sum_{i=1}^{\infty} |\varphi_i\rangle \underbrace{c_i(0)}_{\substack{\text{coeff @}\\ t=0}} \underbrace{e^{-iE_it/\hbar}}_{\substack{\text{time}\\ \text{evolution}}} = e^{-i\hat{H}_0t/\hbar}|\Psi(0)\rangle$

($|\varphi_i\rangle$ forms a basis, so any $|\Psi(t)\rangle$ can be written as a linear combo of $|\varphi_i\rangle$'s.)

We define interaction representation where:

$|\Psi_I(t)\rangle \equiv e^{i\hat{H}_0t/\hbar}|\Psi(t)\rangle$ is our wave function. Then we can write

$|\Psi(t)\rangle = e^{-i\hat{H}_0t/\hbar}|\Psi_I(t)\rangle$ and plug into TDSE:

$$i\hbar\partial_t\left(e^{-i\hat{H}_0t/\hbar}|\Psi_I(t)\rangle\right) = (\hat{H}_0 + \hat{H}'(t))e^{-i\hat{H}_0t/\hbar}|\Psi_I(t)\rangle$$

$$\underline{i\hbar\left(-\frac{i\hat{H}_0}{\hbar}e^{-i\hat{H}_0t/\hbar}|\Psi_I(t)\rangle\right)} + \underline{e^{-i\hat{H}_0t/\hbar}}\, i\hbar\partial_t|\Psi_I(t)\rangle$$

$$= \underline{\hat{H}_0 e^{-i\hat{H}_0t/\hbar}|\Psi_I(t)\rangle} + \hat{H}'(t)\underline{e^{-i\hat{H}_0t/\hbar}}|\Psi_I(t)\rangle$$

$$i\hbar\partial_t|\Psi_I(t)\rangle = e^{i\hat{H}_0t/\hbar}H'(t)e^{-i\hat{H}_0t/\hbar}|\Psi_I(t)\rangle$$

The Hamiltonian in the interaction representation is:

$$\hat{H}_I(t) = e^{i\hat{H}_0 t/\hbar}\,\hat{H}'(t)\,e^{-i\hat{H}_0 t/\hbar}$$

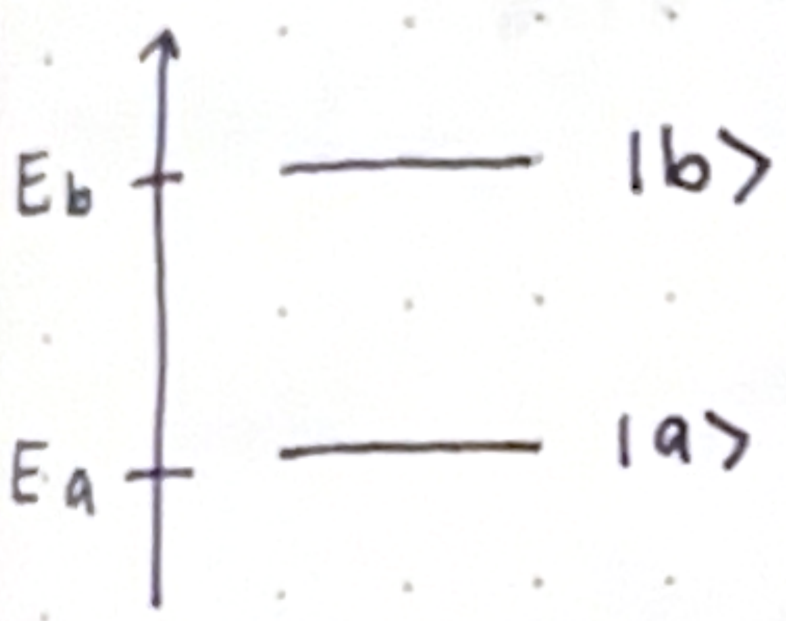
where we can think of the term $e^{-i\hat{H}_0 t/\hbar}$ and $e^{i\hat{H}_0 t/\hbar}$ as field-free propagators to time t and to time $t = 0$ respectively.

So, the TDSE in interaction representation becomes:

$$i\hbar\partial_t |\Psi_I(t)\rangle = \hat{H}_I(t)|\Psi_I(t)\rangle$$

(if $\hat{H}_I(t)$ exists...)

<u>RABI MODEL</u>: a two-level system in a monochromatic external field



$$\hat{H}(t) = \hat{H}_0 + \hat{H}'(t)$$

$$\hat{H}_0|a\rangle = E_a|a\rangle \qquad \langle a|a\rangle = \langle b|b\rangle = 1$$

$$\hat{H}_0|b\rangle = E_b|b\rangle \qquad \langle a|b\rangle = 0$$

$\hat{H}'(t) = -\mathcal{E}(t)\hat{\mu}$ with electric dipole moment operator $\hat{\mu}$ and time dependent electric field:

$$\mathcal{E}(t) = \mathcal{E}_0 \sin(\omega t), \quad \mathcal{E}_0 \text{ is positive (we assume)}$$

which is <u>strictly monochromatic</u>, meaning the field is always in phase with itself.

Operator $\hat{\mu}$ describes how charge is distributed / which dipole transitions are allowed. Matrix elements of $\hat{\mu}$ follow:

$$\mu_{aa} = \langle a|\hat{\mu}|a\rangle = 0, \quad \mu_{bb} = \langle b|\hat{\mu}|b\rangle = 0$$

and for simplicity we assume $\mu_{ab} = \mu_{ba} \in \mathbb{R}^+$.

If $|\Psi(t=0)\rangle = |a\rangle$, what is $|\Psi(t)\rangle$? To determine this, we start with the TDSE in interaction representation, and also say that <u>any</u> state $|\Psi_I(t)\rangle$ in this system can be written as:

$$|\Psi_I(t)\rangle = c_a(t)|a\rangle + c_b(t)|b\rangle \quad (|a\rangle \text{ and } |b\rangle \text{ form a basis!})$$

TDSE becomes $i\hbar\partial_t\left(c_a(t)|a\rangle + c_b(t)|b\rangle\right)$ which can be written as $i\hbar\left(\dot{c}_a(t)|a\rangle + \dot{c}_b(t)|b\rangle\right)$ on the LHS and

$$e^{i\hat{H}_0 t/\hbar}\left(-\mathcal{E}_0\hat{\mu}\sin(\omega t)\right)e^{-i\hat{H}_0 t/\hbar}\left(c_a(t)|a\rangle + c_b(t)|b\rangle\right)$$

$$= -e^{i\hat{H}_0 t/\hbar}\mathcal{E}_0\hat{\mu}\sin(\omega t)e^{-i\hat{H}_0 t/\hbar}\left(c_a(t)|a\rangle + c_b(t)|b\rangle\right)$$

We can obtain two different equations by projecting onto $|a\rangle$ and $|b\rangle$.

Onto $|a\rangle$, we get $i\hbar\dot{c}_a(t) = -e^{+iE_a t/\hbar}\mathcal{E}_0\langle a|\hat{\mu}|b\rangle\sin(\omega t)c_b(t)e^{-iE_b t/\hbar}$

because $\langle a|b\rangle = 0$, $\langle a|\hat{\mu}|a\rangle = 0$

Likewise, onto $|b\rangle$, we get:

$$i\hbar\dot{c}_b(t) = -\mathcal{E}_0\sin(\omega t)e^{iE_b t/\hbar}\langle b|\hat{\mu}|a\rangle c_a(t)e^{-i\hat{E}_a t/\hbar}$$

or, simplified:

$$\begin{cases} i\hbar\dot{c}_a(t) = -\mathcal{E}_0\mu_{ab}e^{i(E_a-E_b)t/\hbar}\sin(\omega t)c_b(t) \\ i\hbar\dot{c}_b(t) = -\mathcal{E}_0\mu_{ab}e^{i(E_b-E_a)t/\hbar}\sin(\omega t)c_a(t) \end{cases}$$

If we define frequency $\omega_{ba} \equiv (E_b - E_a)/\hbar$ (where $E_b > E_a$) and write $\sin(\omega t)$ as $(e^{i\omega t} - e^{-i\omega t})/2i$, we get:

$$\begin{cases} i\hbar\dot{c}_a(t) = -\dfrac{\mathcal{E}_0\mu_{ab}}{2i}\left(e^{i(\omega-\omega_{ba})t} - e^{-i(\omega+\omega_{ba})t}\right)c_b(t) \\ i\hbar\dot{c}_b(t) = -\dfrac{\mathcal{E}_0\mu_{ab}}{2i}\left(e^{i(\omega+\omega_{ba})t} - e^{-i(\omega-\omega_{ba})t}\right)c_a(t) \end{cases}$$

Here, ω_{ba} is the frequency associated with the energy needed to go from state $|a\rangle$ to $|b\rangle$ (so, I assume the frequency of the monochromatic field?).

The system above is not analytically solvable (or it wasn't until the 2010s, and the process is extremely complicated). Instead, we typically use the <u>rotating wave approximation</u>

We assume that the actual frequency of the field, ω, is <u>very</u> close to ω_{ba}, or $\delta = \omega - \omega_{ba} \ll \omega_{ba}$. We call δ the "detuning" parameter (because it slightly "detunes" you/nudges you away from ω_{ba}.)

This assumption means that wave components with frequency δ oscillate <u>MUCH</u> slower than those with frequency linear in ω_{ba}.

For example, the first equation in our system is:

$$i\hbar \dot{c}_a(t) = -\frac{\mathcal{E}_0 \mu_{ab}}{2i}\Big(\underbrace{e^{i\delta t}}_{\text{VERY SLOW}} - \underbrace{e^{-i(2\omega_{ba}+\delta)t}}_{\text{VERY FAST}}\Big) c_b(t)$$

And the second is:

$$i\hbar \dot{c}_b(t) = -\frac{\mathcal{E}_0 \mu_{ab}}{2i}\Big(\underbrace{e^{i(2\omega_{ba}+\delta)t}}_{\text{FAST}} - \underbrace{e^{-i\delta t}}_{\text{SLOW}}\Big) c_a(t)$$

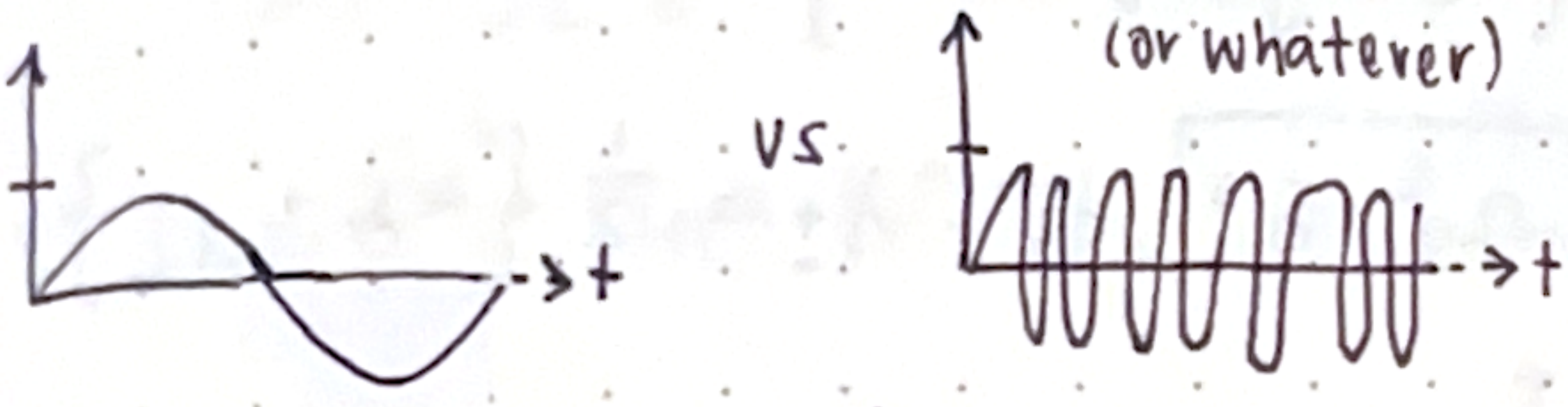
We can, of course, write expressions for $\dot{c}_a(t)$ and $\dot{c}_b(t)$ now, e.g.:

$$\dot{c}_a(t) = +\frac{\mathcal{E}_0 \mu_{ab}}{2\hbar}\left(e^{i\delta t} - e^{-i(2\omega_{ba}+\delta)t}\right) c_b(t)$$

to find $c_a(t)$, we can integrate from some t_0 to t:

$$c_a(t) = c_a(t_0) + \frac{\mathcal{E}_0 \mu_{ab}}{2\hbar}\int_{t_0}^{t} e^{i\delta t'} c_b(t')\,dt' - \frac{\mathcal{E}_0 \mu_{ab}}{2\hbar}\int_{t_0}^{t} e^{-i(2\omega_{ba}+\delta)t'} c_b(t')\,dt'$$

the terms $e^{i\delta t'}$ and $e^{-i(2\omega_{ba}+\delta)t'}$ clearly have the same amplitude, but one oscillate much more; e.g.:



We also assume $c_a(t)$ and $c_b(t)$ change <u>slowly</u> and smoothly on the timescale of $2\pi/\omega_{ba}$ or at least, I'm assuming, because I don't understand Dr. Argenti's explanation at all lol.

"clearly", the rapidly oscillating term then "cancels" out much of its contribution to the integral on time interval t, while the slower oscillation has an appreciable effect. So...

The RWA <u>discards</u> the $e^{-i(2\omega_{ba}+\delta)t'}$ integral altogether, which makes the system exactly solvable.

We have now:

$$\begin{cases} i\hbar \dot{c}_a = -\dfrac{\mathcal{E}_0 \mu_{ab}}{2i} e^{i\delta t} c_b(t) \\ i\hbar \dot{c}_b = +\dfrac{\mathcal{E}_0 \mu_{ab}}{2i} e^{-i\delta t} c_a(t) \end{cases}$$

We solve for $C_b(t)$ first. Define the fundamental Rabi frequency:

$$\Omega_0 = \frac{\mathcal{E}_0 \mu_{ab}}{\hbar} \quad \text{(we'll see what this is later)}$$

Then, $\dot{C_a} = \frac{\Omega_0}{2} e^{i\delta t} C_b(t)$, $\dot{C_b} = -\frac{\Omega_0}{2} e^{-i\delta t} C_a(t)$ and

$$\ddot{C_b} = \underbrace{-\frac{\Omega_0}{2}}(-i\delta)\underbrace{e^{-i\delta t} C_a(t)} - \frac{\Omega_0}{2} e^{-i\delta t} \underline{\dot{C_a}(t)}$$

$$= -i\delta \dot{C_b} - \frac{\Omega_0}{2} e^{-i\delta t}\left(\frac{\Omega_0}{2} e^{i\delta t} C_b(t)\right)$$

$$= -i\delta \dot{C_b} - \frac{\Omega_0^2}{4} C_b(t)$$

$$0 = \ddot{C_b} + i\delta \dot{C_b} + \frac{\Omega_0^2}{4} C_b(t)$$

i here so that $\lambda \in \mathbb{R}$

This is a second-order differential with solution $C_b(t) = e^{i\lambda t}$

Solve for λ:

$$\ddot{C_b} = -\lambda^2 e^{i\lambda t} = (-i\delta) i\lambda e^{i\lambda t} - \frac{\Omega_0^2}{4} e^{i\lambda t}$$

$$-\lambda^2 = \delta\lambda - \frac{\Omega_0^2}{4} \rightarrow \lambda^2 + \delta\lambda - \frac{\Omega_0^2}{4} = 0$$

$$\lambda_\pm = -\frac{\delta}{2} \pm \sqrt{\frac{\delta^2}{4} + \frac{\Omega_0^2}{4}} = \frac{1}{2}\left\{-\delta \pm \sqrt{\Omega_0^2 + \delta^2}\right\}$$

Where Rabi frequency $\Omega = \sqrt{\Omega_0^2 + \delta^2}$, so $\lambda_\pm = \frac{1}{2}\{-\delta \pm \Omega\}$

$$\text{So, } C_b(t) = Ae^{i\lambda_+ t} + Be^{i\lambda_- t}$$

$$= e^{-i\frac{\delta}{2}t}\left\{Ae^{i\Omega t} + B^{-i\Omega t}\right\}$$

at $t = 0$, the initial state, we're purely in $|a\rangle$, so $C_b(0) = 0$, and so $0 = A + B \rightarrow B = -A$, and:

$$C_b(t) = Ae^{-i\delta t/2}\left(e^{i\Omega t} - e^{-i\Omega t}\right)$$

$$= 2iAe^{-i\delta t/2} \sin(\Omega t/2)$$

(we determine what A is later)

using $\dot{C_b} = -\frac{\Omega_0}{2} e^{-i\delta t} C_a(t)$, we know that:

$$C_a(t) = -\frac{2}{\Omega_0} e^{i\delta t} \dot{C_b}$$

$$= -\frac{2}{\Omega_0} e^{i\delta t} \partial_t \left(2iAe^{-i\delta t/2} \sin(\Omega t/2)\right)$$

$$= -\frac{2}{\Omega_0} e^{i\delta t}\, 2iA \left(-i\frac{\delta}{2} e^{-i\delta t/2} \sin\left(\frac{\Omega t}{2}\right) + e^{-i\delta t/2}\frac{\Omega}{2}\cos\left(\frac{\Omega t}{2}\right)\right)$$

$$= -2iA\frac{\Omega}{\Omega_0} e^{i\delta t/2}\left(-i\frac{\delta}{\Omega}\sin\left(\frac{\Omega t}{2}\right) + \cos\left(\frac{\Omega t}{2}\right)\right)$$

$C_a(0) = 1$, so $-2iA\frac{\Omega}{\Omega_0} = 1 \rightarrow 2iA = -\frac{\Omega_0}{\Omega}$. We plug this in!

$$C_a(t) = (1)\, e^{i\delta t/2}\left(\cos\left(\frac{\Omega t}{2}\right) - i\frac{\delta}{\Omega}\sin\left(\frac{\Omega t}{2}\right)\right)$$

Yay. Now that we know $C_a(t)$ and $C_b(t)$, we can finally see what happens to initial state $|\Psi(t=0)\rangle = |a\rangle$ over time!

$P_a(t) + P_b(t) = |C_a(t)|^2 + |C_b(t)|^2 = 1$ (of course)

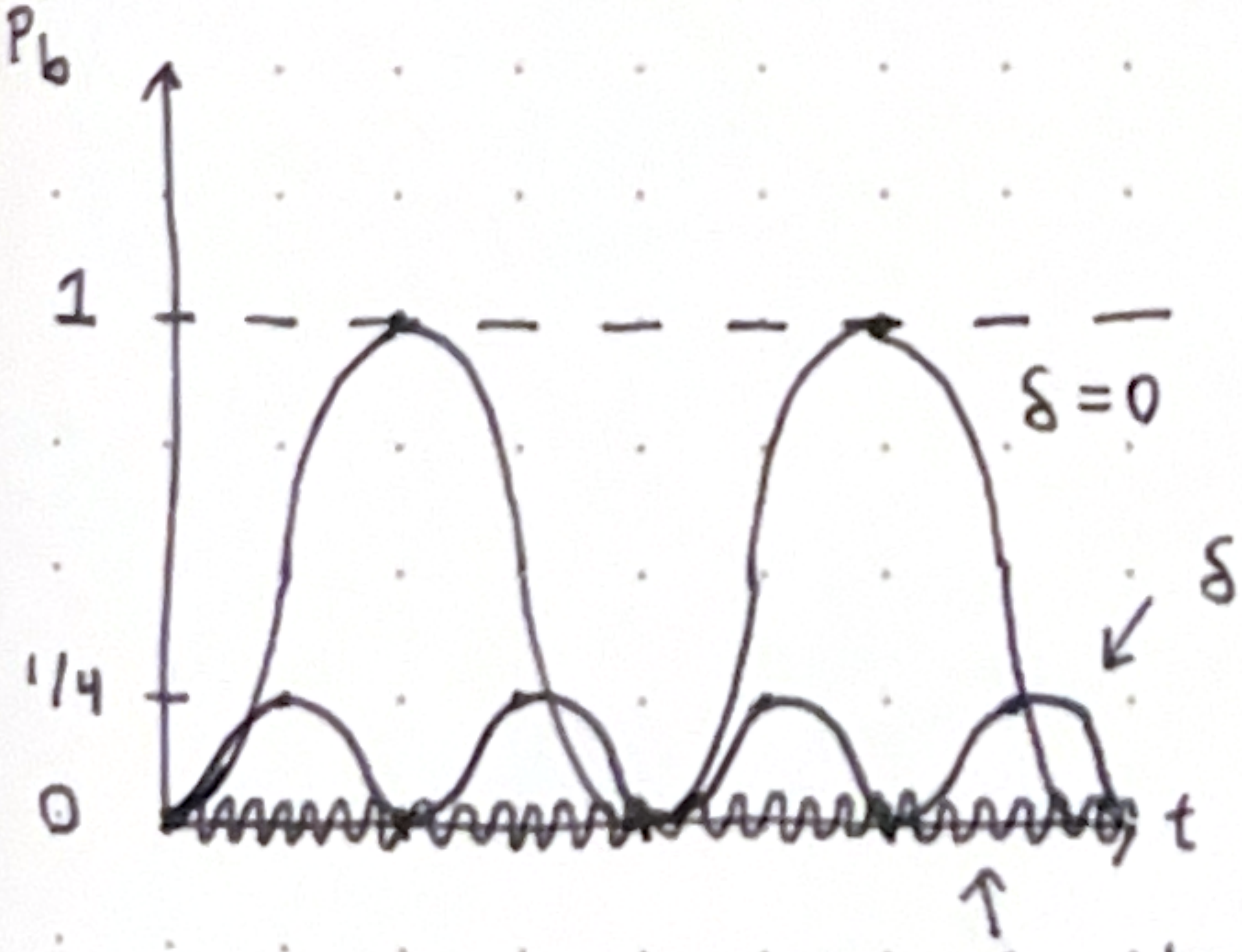
$$P_b(t) = |C_b(t)|^2$$

$$= \left|-\frac{\Omega_0}{\Omega} e^{-i\delta t/2} \sin(\Omega t/2)\right|^2$$

$$= \frac{\Omega_0^2}{\Omega^2}\sin^2(\Omega t/2) = \frac{\Omega_0^2}{\Omega_0^2+\delta^2}\sin^2(\Omega t/2)$$

This describes an oscillating probability / population that has a frequency of Ω and an amplitude of $\Omega_0^2/\Omega_0^2+\delta^2$ which is never 1 unless $\delta = 0$.

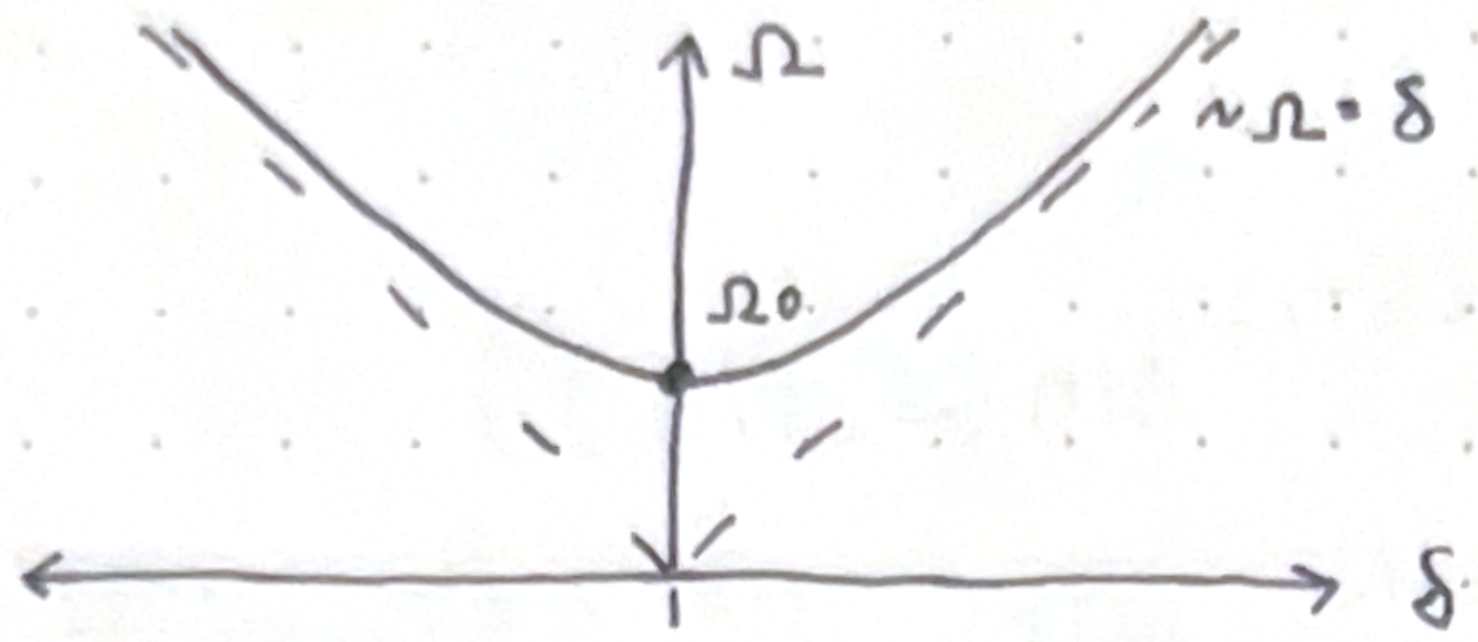
This makes sense; $\delta = \omega - \omega_{ba}$ so ONLY if $\delta = 0$, the energy imparted by the system is enough to reach $|b\rangle$ from $|a\rangle$.



very large δ, never really excited enough

For $\delta = \sqrt{3}\,\Omega_0$, $\Omega = \sqrt{3\Omega_0^2 + \Omega_0^2} = 2\Omega_0$ so the amplitude is $1/2^2 = 1/4$.

Rabi frequency Ω varies with δ sort of like this:



since $\Omega = \delta\sqrt{1+\Omega_0^2/\delta^2}$, at $\delta >> \Omega_0$, Ω is approximately $\delta(1+\Omega_0^2/2\delta^2) = \delta + \Omega_0^2/2\delta$ which is very nearly linear in δ.

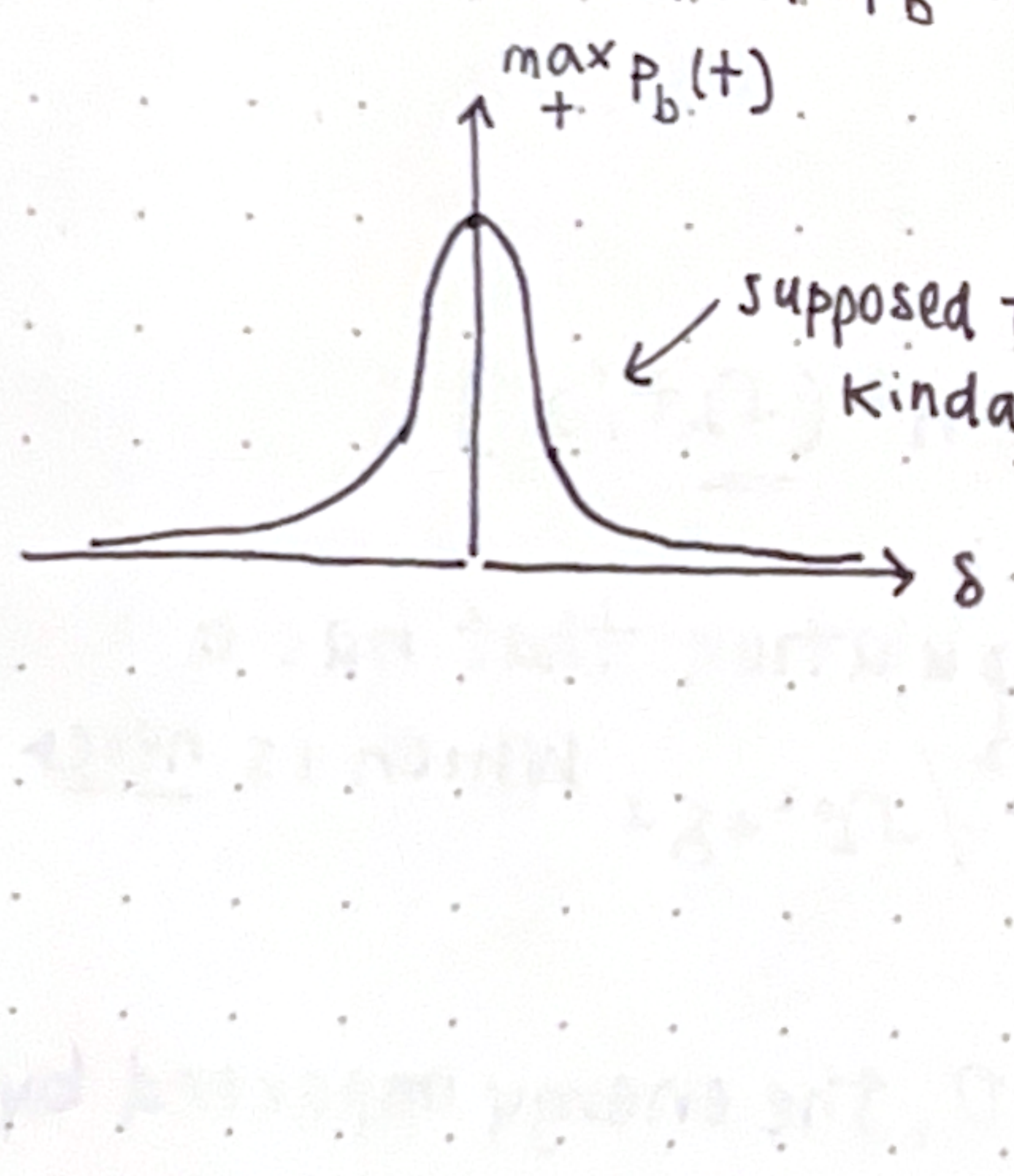
Additionally recall we defined $\Omega_0 = \frac{E_0 \mu_{ab}}{\hbar}$ where μ_{ab} is the dipole coupling between $|a\rangle$ and $|b\rangle$ or $\langle a|\hat{\mu}|b\rangle = \langle b|\hat{\mu}|a\rangle$. Thus it is a parameter of the problem ...

Also note that if detuning $\delta = 0$,

$$\begin{cases} c_a(t) = \cos(\Omega t/2) \\ c_b(t) = -\sin(\Omega t/2) \end{cases}$$

which exhibits periodic oscillations of population in $|a\rangle$ and $|b\rangle$, known as <u>Rabi oscillations</u>.

The plot of maximum $P_b(t)$ versus δ peaks at $\delta = 0$ as expected:



$$\text{Amplitude} = \frac{\Omega_0^2}{\Omega_0^2+\delta^2} = \frac{1}{1+\frac{\delta^2}{\Omega_0^2}}$$

which I guess has the form of a Lorentzian.

Note for $t \ll 2\pi/\Omega$, $P_b(t) = \frac{\Omega_0^2}{\Omega^2} \cdot \frac{\Omega^2 t^2}{4} = \frac{1}{4}\Omega_0^2 t^2$ which means at first, the probability should increase quadratically. But this is extremely difficult to see in a lab.

This is a result of the three assumptions we made:

- The field is strictly monochromatic, meaning coherence time $\gg T_{rabi}$ or $\Delta\omega \ll \Omega_0$
- States $|a\rangle$ and $|b\rangle$ are discrete and well defined, meaning $\Gamma \ll \Omega_0$ (we don't decay too quickly out of $|a\rangle$ and $|b\rangle$)
- We only have a single quantum system, or all systems are identical

Breaking any of these three leads to the atom (let's say) in our quantum system experiencing either a redshifted or blueshifted field.

We can try to control for this with cooled gas, a collimated beam, or a huge Ω_0, but if not all atoms have the same tuning (δ), δ is a distribution in instead of a single value.

distribution of δ.

$$P_b(t|\delta) \text{ has } \langle P_b \rangle = \int P_b(t|\delta)\, g(\delta)\, d\delta$$

and if we assume $g(\delta)$ is "broad over Ω_0",

$$\langle P_b \rangle \approx g(0) \int P_b(t|\delta)\, d\delta = g(0) \int_{-\infty}^{\infty} \frac{\Omega_0^2}{\Omega_0^2 + \delta^2} \sin^2\left(t\sqrt{\Omega_0^2 + \delta^2}/2\right) d\delta$$

↑ kills tail

Which (apparently) leads to an initial linear increase!