WEEK 2 : Tensor Products ...? & Addition of Angular Momenta

Now a more formal definition of the tensor product. Vector spaces V and W have cartesian product $V \times W = \{(v,w),\ v \in V,\ w \in W\}$.

From this, define $E_{V\times W} \equiv \left\{ \sum_{(v,w)\in S} (v,w)\, c_{v,w},\ S \subset V\times W,\ \underline{S \text{ is finite}},\ c_{vw} \in \mathbb{C} \right\}$.

which is called a vector space "freely generated" by $V \times W$.

To enforce bilinearity, we introduce this equivalence relation:

$$\begin{cases} \forall \alpha,\beta \in \mathbb{C},\ w\in W,\ v_1, v_2 \in V : (\alpha v_1 + \beta v_2, w) \sim \alpha(v_1, w) + \beta(v_2, w) \\ \forall \alpha,\beta \in \mathbb{C},\ v \in V,\ w_1, w_2 \in W : (v, \alpha w_1 + \beta w_2) \sim \alpha(v, w_1) + \beta(v, w_2) \end{cases}$$

to $E_{V\times W}$.

Now given basis $\{v_i\}_{i=1\ldots N}$ for V and $\{w_j\}_{j=1\ldots M}$ for W, arbitrary vectors v and w are:

$$v = \sum_i v_i a_i \quad \text{and} \quad w = \sum_j w_j b_j$$

which means we can describe our ordered pair (v,w) as:

$$(v,w) = \left(\sum_i v_i a_i,\ \sum_j w_j b_j\right)$$

$$= \sum_i \left(v_i,\ \sum_j w_j b_j\right) a_i = \sum_{ij} (v_i, w_j)\, a_i b_j$$

so any arbitrary pair is a linear combo of pairs of basis vectors. Now, an element in $E_{V\times W}$ with our above equivalence relation is:

$$\sum_{(v,w)\in S} (v,w)\, c_{v,w} = \sum_{(v,w)\in S} \left(\sum_{i=1}^{N} v_i a_i(v),\ \sum_{j=1}^{M} w_j b_j(w) \right) c_{v,w}$$

$$\sim \sum_{(v,w)\in S} \sum_{ij} (v_i, w_j)\, a_i(v)\, b_j(w)\, c_{v,w}$$

same process

$$\sim \sum_{ij} (v_i, w_j) \underbrace{\sum_{(v,w)\in S} c_{v,w}\, a_i(v)\, b_j(w)}_{c_{ij} \in \mathbb{C}}$$

$$\sim \sum_{ij} (v_i, w_j)\, c_{ij}$$

So elements in $E_{V\times W}$, not just $V\times W$, can be written as a linear combo of basis vector pairs. Not only that, it is clear we can add elements of $E_{V\times W}$ and have associative/commutative properties, there are inverse and identity elements, etc. (Closed under scalar mult./vector add.)

$\therefore E_{V\times W}$ is a vector space / linear space.

The set of its equivalence classes, $E_{V\times W}/\sim$ is also a vector space.

The tensor product of two <u>vectors</u> $v\otimes w \equiv [(v,w)]$ is such an equivalence class:

$$v\otimes w = \sum_i v_i a_i \otimes \sum_j w_j b_j$$

$$= \sum_i a_i \left(v_i \otimes \sum_j w_j b_j\right)$$

$$= \sum_i a_i \sum_j b_j \left(v_i \otimes w_j\right) = \sum_{ij} a_i b_j \left(v_i \otimes w_j\right)$$

because an arbitrary $v\otimes w$ can be written as a linear combination of $v_i\otimes w_j$, the space <u>of $v\otimes w$'s</u> has basis $v_i\otimes w_j$ and is $N\times M$ dimensional.

$\rightarrow E_{V\times W}/\sim = V\otimes W = \mathrm{Span}\{v_i\otimes w_j\} = \left\{\sum_{ij} v_i \otimes w_j\, a_{ij},\ a_{ij}\in\mathbb{C}\right\}$

because these classes are defined by our prior equivalence relation, we know they're closed under scalar multiplication and vector addition, so $V\otimes W$ is a vector space.

Notes: $\dim(V\otimes W) = \dim(V)\dim(W)$ and $V_1\otimes(V_2\otimes V_3) = (V_1\otimes V_2)\otimes V_3 = V_1\otimes V_2\otimes V_3$

<u>Implications:</u>

- For a particle in 1D, the Hilbert space is $L^2(\mathbb{R})$, hence For a particle in 3D, $H = L^2(\mathbb{R}^3) = L^2(\mathbb{R})\otimes L^2(\mathbb{R})\otimes L^2(\mathbb{R})$.
- Generalized $|\vec{r}\rangle \equiv |x\rangle\otimes|y\rangle\otimes|z\rangle$
- $|\vec{r},\sigma\rangle \equiv |\vec{r}\rangle\otimes|\sigma\rangle$

- And, $H_{I+II} = H_I \otimes H_{II}$. If they both have dimension > 1, it is possible to have $|\Psi\rangle = |\psi_1\rangle \otimes |\phi_1\rangle + |\psi_2\rangle \otimes |\phi_2\rangle$, which is <u>impossible</u> to write as $|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle$.

The systems corresponding to H_I and H_{II} are <u>entangled</u> just as a result of the tensor product. You <u>cannot</u> just specify one unique state for each system, and measuring one subsystem affects latter measurements of the other system.

- For $H = H_I \otimes H_{II}$, there is a $H^* = H_I^* \otimes H_{II}^*$ <u>dual space</u> (or space of linear functionals

for $|\chi\rangle \otimes |\theta\rangle \in H$ and $\langle\psi| \otimes \langle\phi| \in H^*$,

$$(\langle\psi| \otimes \langle\phi|)(|\chi\rangle \otimes |\theta\rangle) = \langle\psi|\chi\rangle\langle\phi|\theta\rangle$$

- $|\psi\rangle \in L^2(\mathbb{R}^3) = |\varphi_x\rangle \otimes |\varphi_y\rangle \otimes |\varphi_z\rangle$ then wave function is

$$\langle\vec{r}|\psi\rangle = (\langle x| \otimes \langle y| \otimes \langle z|)(|\varphi_x\rangle \otimes |\varphi_y\rangle \otimes |\varphi_z\rangle)$$
$$= \varphi_x(x)\varphi_y(y)\varphi_z(z)$$

- If $\hat{A}: H_1 \to H_1$ (it acts on subsystem 1 only) then:

$\hat{A} \otimes \hat{1}: H \to H$ and total system $\sum_{ij} \underbrace{|\varphi_i\rangle}_{H_I} \otimes \underbrace{|\psi_j\rangle}_{H_{II}} c_{ij} \mapsto$

$\sum_{ij} (\hat{A}|\varphi_i\rangle) \otimes (\hat{1}|\psi_j\rangle) c_{ij}$ (and similar for $\hat{B}: H_{II} \to H_{II}$)

- $\hat{A} \otimes \hat{B}: H \to H$, then $\sum_{ij} |\varphi_i\rangle \otimes |\psi_j\rangle c_{ij} \mapsto \sum_{ij} (\hat{A}|\varphi_i\rangle) \otimes (\hat{B}|\psi_j\rangle) c_{ij}$

- $\hat{L} \cdot \hat{S} = \sum_{i=1}^{3} \hat{L}_i \otimes \hat{S}_i$

Adding Angular Momentum

Say we have two particles, S_I and S_2. We only care about internal degrees of freedom in this case, and say both are spin 1/2.

$$H_I = \text{span}\{|1/2, \pm 1/2\rangle\} = \text{span}\{|\pm\rangle\} \quad (\text{same for } H_{II})$$

$$H_S = H_I \otimes H_{II}, \quad \dim(H_S) = 4$$

$$= \text{span}\{|+\rangle_1 \otimes |+\rangle_2, |+\rangle_1 \otimes |-\rangle_2, |-\rangle_1 \otimes |+\rangle_2, |-\rangle_1 \otimes |-\rangle_2\}$$

$$= \text{span}\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$$

(Since we're only concerned with intrinsic spin, $\hat{\vec{J}}$ and $\hat{\vec{S}}$ are interchangable)

Total spin / angular momentum $\hat{\vec{J}} = \hat{\vec{J}}_1 + \hat{\vec{J}}_2 = \hat{\vec{J}}_1 \otimes \hat{1} + \hat{1} \otimes \hat{\vec{J}}_2$

$\hat{J}_1^2, \hat{J}_{1z}, \hat{J}_{2z}, \hat{J}_2^2$ all commute, and we want to find a basis for H_S where they are all well-defined (?).

Because $\hat{J}^2$ for systems 1 and 2 are the same, they don't divide up the eigenspace at all. $\hat{J}_2^2|\psi\rangle = \hbar(\frac{1}{2})(\frac{1}{2}+1)|\psi\rangle = \frac{3}{4}\hbar|\psi\rangle$ (total angular momentum j is always positive) and $\hat{J}_2^2|\psi\rangle = \frac{3}{4}\hbar|\psi\rangle$.

$\hat{J}_z$ on the other hand does. In fact, $\{\hat{J}_{1z}, \hat{J}_{2z}\}$ is a CSCO — their eigenvalues evidently describe unique states.

$$\hat{J}_z = \hat{J}_{1z} + \hat{J}_{2z} = \hat{J}_z \otimes \hat{1} + \hat{1} \otimes \hat{J}_z$$

$$\hat{J}^2 = \sum_{i=1}^{3} \hat{J}_i^2 = \sum_{i=1}^{3} (\hat{J}_i \otimes \hat{1} + \hat{1} \otimes \hat{J}_i)^2$$

$\hat{J}_z|\sigma_1, \sigma_2\rangle = \hbar\sigma_1|\sigma_1, \sigma_2\rangle + \hbar\sigma_2|\sigma_1, \sigma_2\rangle$ or more verbosely:

$$= (\hat{J}_z \otimes \hat{1} + \hat{1} \otimes \hat{J}_z)(|\sigma_1\rangle \otimes |\sigma_2\rangle)$$

$$= (\hat{J}_z|\sigma_1\rangle) \otimes (\hat{1}|\sigma_2\rangle) + (\hat{1}|\sigma_1\rangle) \otimes (\hat{J}_z|\sigma_2\rangle)$$

$$= \hbar\sigma_1|\sigma_1\rangle \otimes |\sigma_2\rangle + |\sigma_1\rangle \otimes \hbar\sigma_2|\sigma_2\rangle$$

$$= \hbar\sigma_1(|\sigma_1\rangle \otimes |\sigma_2\rangle) + \hbar\sigma_2(|\sigma_1\rangle \otimes |\sigma_2\rangle) = \hbar\underline{(\sigma_1 + \sigma_2)}(|\sigma_1\rangle \otimes |\sigma_2\rangle)$$

so "total m", $M = \sigma_1 + \sigma_2$

$M = \sigma_1 + \sigma_2$, and σ_1 can be $\pm 1/2$, σ_2 can be $\pm 1/2$, so M can be 1, 0, or -1.

Obviously $M=1$ is for $|1/2\ 1/2\rangle$, $M=-1$ for $|-1/2\ -1/2\rangle$, and $M=0$ is for $|1/2\ -1/2\rangle$ and $|-1/2\ 1/2\rangle$. This latter eigenspace is 2D.

Because $[\hat{J}^2, \hat{J}_z] = 0$, each eigenspace of $\hat{J}_z$ has eigenvector(s) of $\hat{J}^2$ also. The 1D eigenspaces are straightforward — $|++\rangle$ (or $|1/2\ 1/2\rangle$) is also an eigenvector of $\hat{J}^2$, same for $|--\rangle$.

Oh also $\hat{J}_z$ is diagonal in the basis $|\sigma_1, \sigma_2\rangle$ because it's eigenvectors are the basis. Anyway, $|++\rangle$ and $|--\rangle$ are eigenvectors of $\hat{J}^2$. We can use ladder operators to find the eigenvector of $\hat{J}^2$ in the $M=0$ eigenspace:

$$\hat{J}_- = \hat{J}_{1-} \otimes \hat{1} + \hat{1} \otimes \hat{J}_{2-} \quad \text{and} \quad |++\rangle = |1/2, 1/2; \overset{J}{1}, \overset{M}{1}\rangle$$

$$\hat{J}_-|++\rangle \to \frac{1}{\hbar\sqrt{2 - M(M-1)}}\hat{J}_-|++\rangle \text{ normalized.}$$

$$= \frac{1}{\hbar\sqrt{2}}\left(\hat{J}_{1-} \otimes \hat{1} + \hat{1} \otimes \hat{J}_{2-}\right)(|+\rangle \otimes |+\rangle)$$

$$= \frac{1}{\hbar\sqrt{2}}\left(\hat{J}_{1-}|+\rangle \otimes \hat{1}|+\rangle + \hat{1}|+\rangle \otimes \hat{J}_{2-}|+\rangle\right)$$

$$= \frac{\hbar}{\hbar\sqrt{2}}\left(|-\rangle \otimes |+\rangle + |+\rangle \otimes |-\rangle\right) = \frac{\hbar}{\hbar\sqrt{2}}\left(|-+\rangle + |+-\rangle\right)$$

Thus the three eigenvectors of $\hat{J}^2$ are:

$$\left.\begin{aligned} |1/2, 1/2; 1, 1\rangle &= |++\rangle \\ |1/2, 1/2; 1, 0\rangle &= \frac{|+-\rangle + |-+\rangle}{\sqrt{2}} \\ |1/2, 1/2; 1, -1\rangle &= |--\rangle \end{aligned}\right\}$$

Same even if we flip particle order.

These are called <u>Triplet Functions</u>.

You may notice that we assume $J=1$ because M ranges from -1 to 1. We can verify there are no greater values of M by applying $\hat{J}_+$:

$$\hat{J}_+|++\rangle = (\hat{J}_{1+}|+\rangle \otimes \hat{1}|+\rangle) + (\hat{1}|+\rangle \otimes \hat{J}_{2+}|+\rangle)$$

the coefficient on $\hat{J}_+|+\rangle$ (in our case) where $j = m = 1/2$ $= \hbar\sqrt{\frac{1}{2}(\frac{1}{2}+1) - \frac{1}{2}(\frac{3}{2}}$ which is 0. $0 \otimes |+\rangle = |+\rangle \otimes 0 = 0$, so $\hat{J}_+|++\rangle = 0$.

Because we know $|{}^1\!/_2, {}^1\!/_2; 1, 0\rangle$ where $J = 1$, we can also find the eigenvector for $J=0$, $|{}^1\!/_2, {}^1\!/_2; 0, 0\rangle$.

The space H_S is 4D - 2D per particle. We only found three eigenvectors, which are orthogonal, so there must be another. It is in the $M=0$ eigenspace.

↳ The only perpendicular vector to $\frac{|+-\rangle + |-+\rangle}{\sqrt{2}}$ is $\frac{|+-\rangle - |-+\rangle}{\sqrt{2}}$

We can verify this corresponds to $J=0$:

$$\hat{J}_+\left(\frac{|+-\rangle - |-+\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\hat{J}_+\left(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\hat{J}_{1+} \otimes \hat{1} + \hat{1} \otimes \hat{J}_{2+}\right)\left(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\hat{J}_{1+}|+\rangle \otimes \hat{1}|-\rangle + \hat{1}|+\rangle \otimes \hat{J}_{2+}|-\rangle - (\hat{J}_{1+}|-\rangle \otimes \hat{1}|+\rangle + |-\rangle \otimes \hat{J}_{2+}|+\rangle)\right)$$

$$= \frac{1}{\sqrt{2}}\left(0 \otimes \hat{1}|-\rangle + \underline{|+\rangle \otimes |+\rangle - |+\rangle \otimes |+\rangle} - |-\rangle \otimes 0\right) = 0$$

and similar for $\hat{J}_-$ ...

Note for specific values j_1, j_2, $|j_1 - j_2| \leq J \leq j_1 + j_2$. In this case, $j_1 = j_2 = {}^1\!/_2$ (j is always positive), so $J=0$ or $J=1$.

Clebsch-Gordan Coefficients

Notation-wise, $\{|{}^1\!/_2, \sigma_1\rangle \otimes |{}^1\!/_2, \sigma_2\rangle\}$ is sometimes written as $\{|{}^1\!/_2, {}^1\!/_2; \sigma_1, \sigma_2\rangle\}$ but this notation sucks.

$$|{}^1\!/_2, {}^1\!/_2; J, M\rangle = \sum_{\sigma_1, \sigma_2} |{}^1\!/_2, \sigma_1\rangle \otimes |{}^1\!/_2, \sigma_2\rangle \, C^{JM}_{{}^1\!/_2\sigma_1, {}^1\!/_2\sigma_2}$$

$$= \sum_{\sigma_1, \sigma_2} |{}^1\!/_2\, {}^1\!/_2; \underline{\sigma_1}\underline{\sigma_2}\rangle \, C^{JM}_{\underline{{}^1\!/_2\sigma_1}, {}^1\!/_2\sigma_2}$$

where Clebsch-Gordan coefficient $C^{JM}_{{}^1\!/_2\sigma_1, {}^1\!/_2\sigma_2}$ is:

$$\langle {}^1\!/_2, {}^1\!/_2; \sigma_1, \sigma_2 | {}^1\!/_2, {}^1\!/_2; J, M\rangle$$

For example, we found that $C^{1\,1}_{{}^1\!/_2{}^1\!/_2, {}^1\!/_2{}^1\!/_2} = 1$

Recall $|+\rangle = |1/2, 1/2\rangle$, so $|++\rangle = |1/2, 1/2\rangle \otimes |1/2, 1/2\rangle$. Here, J and $M = 1/2 + 1/2 = 1$.

$$|1/2, 1/2; 1, 1\rangle = \sum_{\sigma_1 \sigma_2} |1/2, \underline{\sigma_1}\rangle \otimes |1/2, \underline{\sigma_2}\rangle C^{JM}_{1/2\sigma_1, 1/2\sigma_2}$$

$$= |++\rangle = |1/2, 1/2\rangle \otimes |1/2, 1/2\rangle = |1/2, 1/2, \underline{1/2, 1/2}\rangle$$

Hence when $\sigma_1 = \sigma_2 = 1/2$, $C = 1$ and $C = 0$ for all other σ_1 and σ_2.

Similarly with $|--\rangle = |1/2, -1/2\rangle \otimes |1/2, -1/2\rangle$, $M = -1$ and $J = 1$, we found that:

$$C^{1-1}_{1/2-1/2, 1/2-1/2} = 1 \text{ also, and } C = 0 \text{ for all other } \sigma_1 \text{ and } \sigma_2$$

We also found $C^{10}_{1/2 1/2, 1/2 -1/2} = C^{10}_{1/2-1/2, 1/2 1/2} = \frac{1}{\sqrt{2}}$

and: $C^{00}_{1/2-1/2, 1/2 1/2} = -C^{00}_{1/2 1/2, 1/2-1/2} = -\frac{1}{\sqrt{2}}$

You can look up these values in a table; often they suck to compute.

Essentially, a state of a whole system with defined j_1, j_2, J, and M can be written as a linear combination of possible (whole-system) states with varying m_1, m_2.

<u>Generalized Case</u>

If we can't make any claims about S_I and S_2, we can only say:

$$H_1 = \text{span}\{|k_1, j_1, m_1\rangle\}_{k_1 j_1 m_1} = \bigoplus_{k_1 j_1} H_{k_1 j_1}$$

$\llcorner H_{k_1 j_1} = \text{span}\{|k_1 j_1 m_1\rangle\}_{m_1}$ with $\dim = 2j_1 + 1$

and similarly,

$H_2 = \bigoplus_{k_2 j_2} H_{k_2 j_2}$ so that space $S_1 + S_2$ has $H = H_1 \otimes H_2$

thus $H = \left(\bigoplus_{k_1 j_1} H_{k_1 j_1}\right) \otimes \left(\bigoplus_{k_2 j_2} H_{k_2 j_2}\right) = \bigoplus_{k_1 j_1 k_2 j_2} \left(\underbrace{H_{k_1 j_1}}_{\dim = 2j_1+1} \otimes \underbrace{H_{k_2 j_2}}_{\dim = 2j_2+1}\right)$

where $H_{k_1 j_1} \otimes H_{k_2 j_2} = \text{span}\{|j_1 m_1\rangle \otimes |j_2 m_2\rangle\}$ (the k's are implicit.)

$\dim (2j_1+1)(2j_2+1)$

What are the common eigenstates of $\hat{J}^2$ and $\hat{J}_z$ in this space?
Now, we can't guarantee $j_1 = j_2$.

As before: $\hat{J}_z = \hat{J}_{1z} \otimes \hat{1} + \hat{1} \otimes \hat{J}_{2z}$, etc. and $\hat{J}_1^2, \hat{J}_2^2, \hat{J}_{1z}, \hat{J}_{2z}$ all commute.

$$\hat{J}_z(|j_1 m_1\rangle \otimes |j_2 m_2\rangle) = (\hat{J}_z |j_1 m_1\rangle \otimes |j_2 m_2\rangle) + (|j_1 m_1\rangle \otimes \hat{J}_z |j_2 m_2\rangle)$$

$$= \hbar (m_1 + m_2) |j_1 m_1\rangle \otimes |j_2 m_2\rangle$$

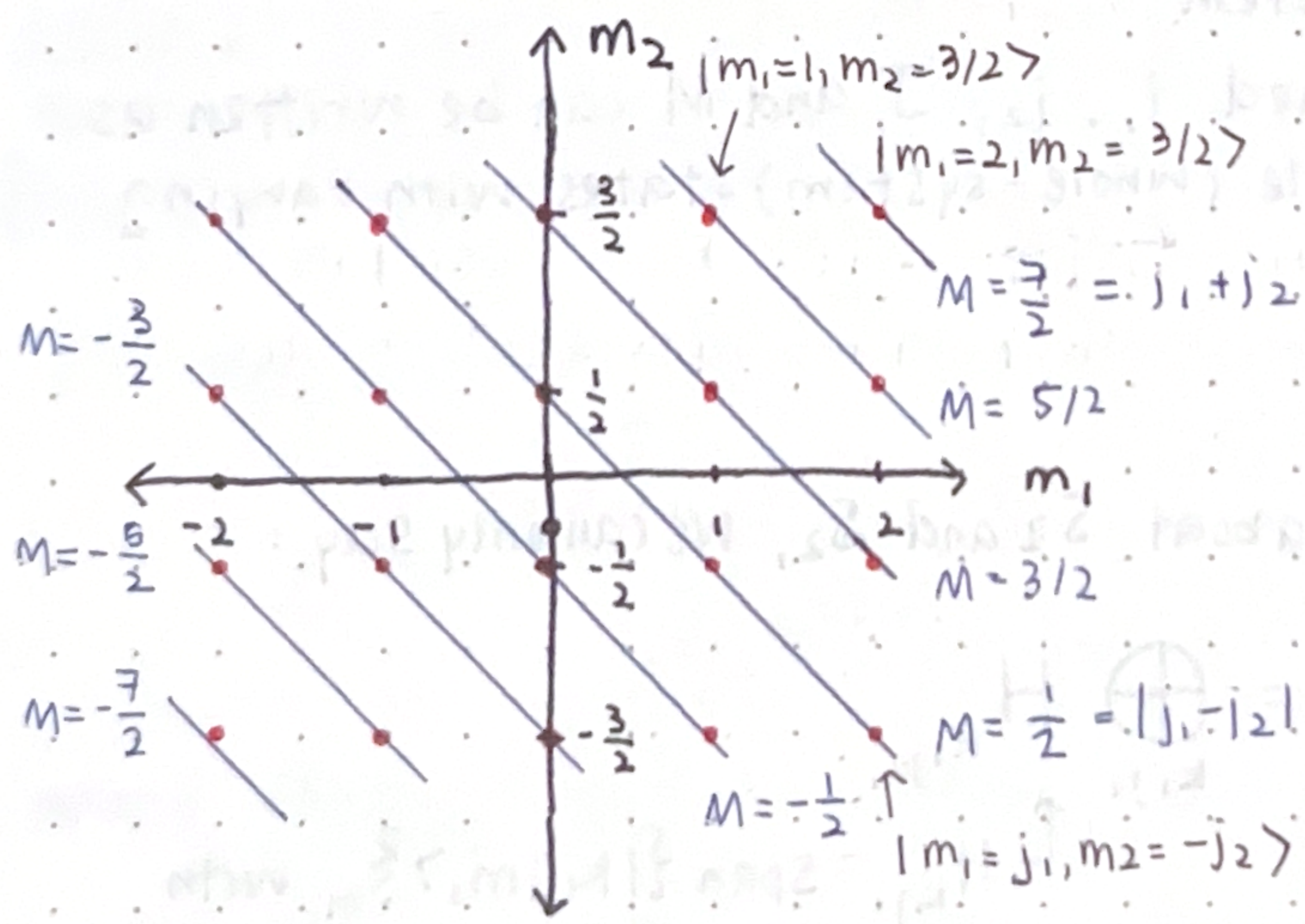
$$= \hbar (m_1 + m_2) |j_1 j_2 ; m_1 m_2\rangle$$ notation

so $M = m_1 + m_2$ again.

If we want to find the eigenvalues of $\hat{J}^2$ by looking at the eigenspaces of $\hat{J}_z$, we need to know how M varies.

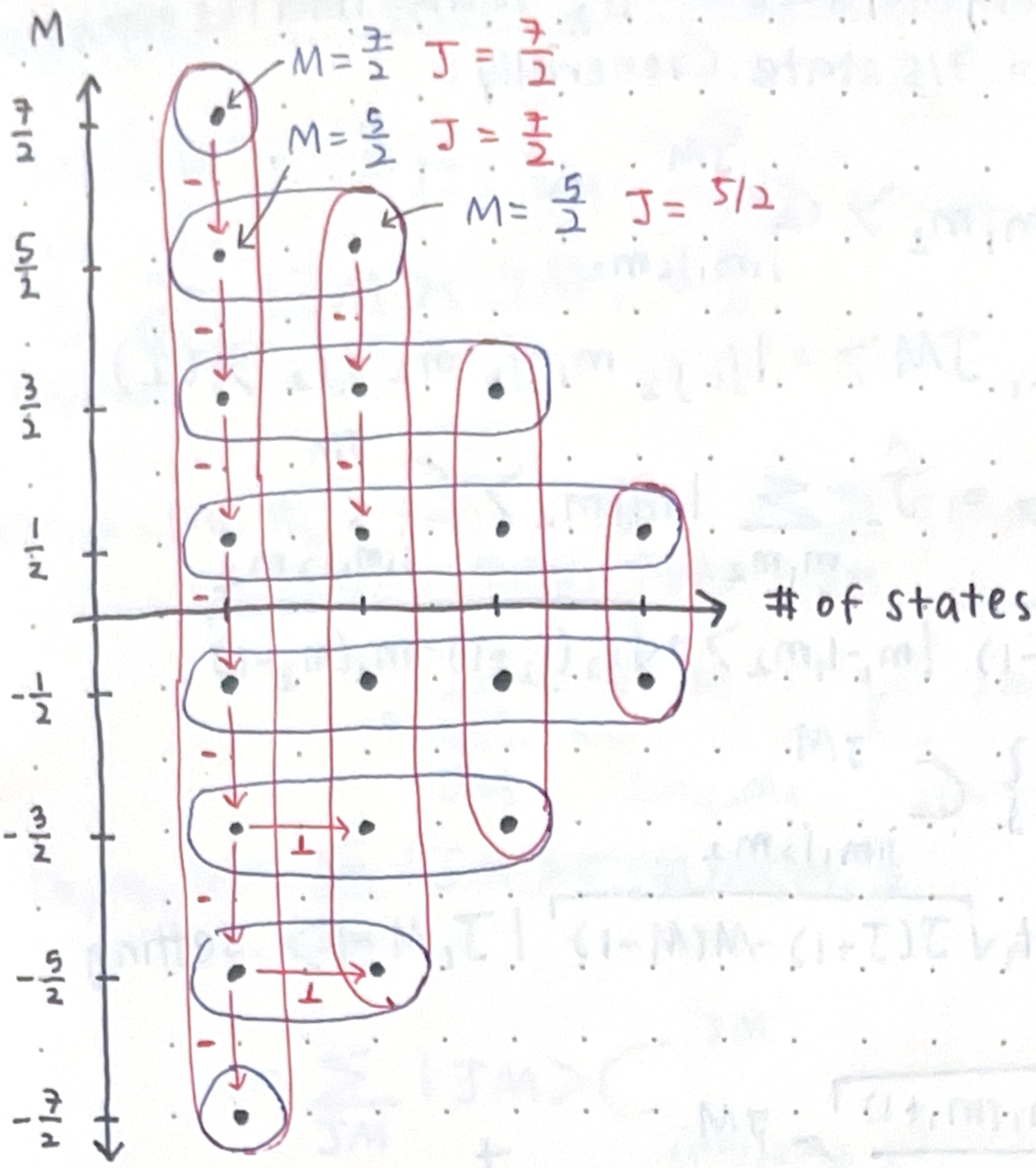
ex. $j_1 = 2$, so $m_1 = -2, -1, 0, 1, 2$ and H_1 is 5 dimensional
$j_2 = 3/2$, so $m_2 = -3/2, -1/2, 1/2, 3/2$ and H_2 is 4 dimensional

$H = H_1 \otimes H_2$ is then 20-dimensional. M also varies as shown:



The value of M ranges from $j_1 + j_2$ to $-(j_1 + j_2)$. The dimensionality of each eigenspace is smaller for the more extreme values of M, and largest at the values of M with the smallest magnitude, $|j_1 - j_2|$.

We can look at the eigenspaces another way, as a histogram displaying the number of states that correspond to a single M (the states along the diagonals above).



○ = an eigenspace of $\hat{J}_z$

○ = an eigenspace of $\hat{J}^2$

Starting at a known state, e.g $|M=7/2, J=7/2\rangle$, use ladder operators to find other states in $\hat{J}^2$'s eigenspace for J

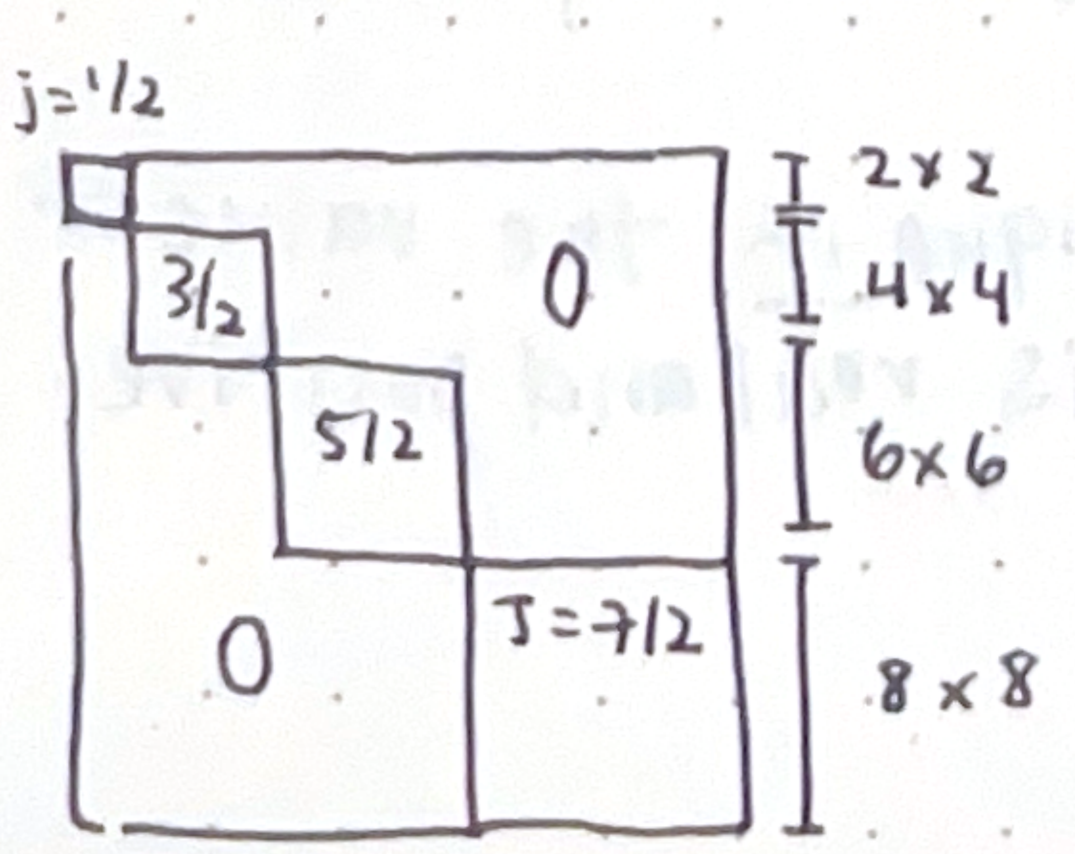
use orthogonality to move sideways into a new eigenspace of $\hat{J}^2$ since eigenstates in M's eigenspace form an orthogonal basis for that space.

Recall that ladder operators increment/decrement the value of m of a system from -j to j in integer intervals WITHOUT changing the value of j.

This means for an eigenspace of some J, there are 2J+1 dimensions (or basis states), one for each possible M, and the possible values of J are evidently the positive values of M : $|j_1-j_2|, |j_1-j_2|+1, \ldots, j_1+j_2$.

The total system's Hilbert space $\bigoplus (H_{j_1} \otimes H_{j_2})$ is clearly made up of subspaces/eigenspaces of $\hat{J}^2$ that slowly increase; in our example, the eigenspace for $J=1/2$ has a dimension of 2, $J=3/2$ a dimension of 4, $J=5/2$ of 6, and $J=7/2$ of 8:

$$H = \bigoplus_{j_1 j_2} (H_{j_1} \otimes H_{j_2}), \quad H_{j_1} \otimes H_{j_2} = \bigoplus_{J=|j_1-j_2|}^{j_1+j_2} H_J$$



As described, find the 1-D eigenspaces of $\hat{J}_z$ to also find the eigenve-
tor of $\hat{J}^2$, e.g. the $M = J = 7/5$ state. Generally:

$$|j_1 j_2, JM\rangle = \sum_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle C^{JM}_{j_1 m_1 j_2 m_2}$$

$J = M = j_1 + j_2$, so $|j_1, j_2, JM\rangle = |j_1\ j_2\ m_1 = j_1\ m_2 = j_2\rangle$ (1)

Then use $\hat{J}_- |j_1 j_2, JM\rangle = \hat{J}_- \sum_{m_1 m_2} |m_1 m_2\rangle C^{JM}_{j_1 m_1 j_2 m_2}$

$$= \hbar \sum_{m_1 m_2} \left\{ \sqrt{j_1(j_1+1) - m_1(m_1-1)}\ |m_1 - 1, m_2\rangle + \sqrt{j_2(j_2+1) - m_2(m_2-1)}\ |m_1, m_2 - 1\rangle \right\} C^{JM}_{j_1 m_1 j_2 m_2}$$

which we know must equal $\hbar\sqrt{J(J+1) - M(M-1)}\ |J, M-1\rangle$. Setting them equal leads to:

$$|J, M-1\rangle = \sum_{m_1 m_2} \left\{ \sqrt{\frac{j_1(j_1+1) - m_1(m_1+1)}{J(J+1) - M(M-1)}}\ C^{JM}_{j_1 m_1+1\ j_2 m_2} + \sqrt{\frac{j_2(j_2+1) - m_2(m_2+1)}{J(J+1) - M(M-1)}}\ C^{JM}_{j_1 m_1 j_2 m_2+1} \right\} |m_1, m_2\rangle$$

(and similarly for $|J, M+1\rangle$...)
which all condenses down to:

$$|J, M-1\rangle = \sum_{m_1 m_2} C^{JM-1}_{j_1 m_1 j_2 m_2} |m_1, m_2\rangle$$

Which defines the recurrence relation for the Clebsch-Gordan coefficients (think dynamic programming).

Finally, just as with our simple example, use orthogonality to move "sideways" into a new eigenspace of new J:

$$|J', J'\rangle = \sum_{m_1 m_2} |m_1 m_2\rangle C^{J'J'}_{j_1 m_1 j_2 m_2} \quad \text{where } J > J',\ M = J'.$$

The orthonormalization to find $|J', J'\rangle$ is unique if the value of $C^{J'J'}_{j_1 m_1 j_2 m_2}$ for the state with the largest m_1 is real and positive.

Properties of the Clebsch-Gordan coefficients

$$\langle JM|J'M'\rangle = \delta_{JJ'}\delta_{MM'} = \sum_{m_1m_2} C^{JM}_{j_1m_1j_2m_2}\, C^{J'M'}_{j_1m_1j_2m_2}$$ (orthonormality 1)

$$\sum_{J=|j_1-j_2|}^{j_1+j_2}\sum_{M=-J}^{J} |JM\rangle\langle JM| = \hat{1}$$

$$\langle m_1m_2|m_1'm_2'\rangle = \delta_{m_1m_1'}\delta_{m_2m_2'} = \sum_{JM}\langle m_1m_2|JM\rangle\langle JM|m_1'm_2'\rangle$$

$$= \sum_{JM} C^{JM}_{j_1m_1j_2m_2}\, C^{JM}_{j_1m_1'j_2m_2'}$$

$$|m_1m_2\rangle = \sum_{JM}|JM\rangle\langle JM|m_1m_2\rangle$$

$$= \sum_{JM}|JM\rangle C^{JM}_{j_1m_1j_2m_2}$$

$C^{c\gamma}_{a\alpha b\beta} = 0$ when $\gamma = \alpha+\beta$

$$C^{c\gamma}_{a\alpha b\beta} = \underbrace{(-1)^{a-b+\gamma}}_{\text{phase}}\ \underbrace{\sqrt{2c+1}}_{\text{norm}}\ \underbrace{\begin{pmatrix} a & b & c\\ \alpha & \beta & -\gamma\end{pmatrix}}_{\text{3-j coefficients}}$$

3-j coefficients, invariant under cyclic permutations of columns.

$\begin{pmatrix} a & b & c\\ \alpha & \beta & \gamma\end{pmatrix} = 0$ if $\alpha+\beta+\gamma = 0$

flip changes sign

$$\begin{pmatrix} a & b & c\\ \alpha & \beta & \gamma\end{pmatrix} = \begin{pmatrix} b & c & a\\ \beta & \gamma & \alpha\end{pmatrix} = \begin{pmatrix} c & a & b\\ \gamma & \alpha & \beta\end{pmatrix} = (-1)^{a+b+c}\begin{pmatrix} a & c & b\\ \alpha & \gamma & \beta\end{pmatrix} = \begin{pmatrix} a & b & c\\ -\alpha & -\beta & -\gamma\end{pmatrix}$$