WEEK 1 - ANGULAR MOMENTUM

Recap from last semester:

In Classical Mech., Rotations are canonical transformations
In Quantum Mech., Rotations are unitary transformations

$\hat{n}$, ϕ is $\hat{R}_{\hat{n}}(\phi)$. Two rotations can be combined: $\hat{R}_{\hat{n}}(\phi_1) + \hat{R}_{\hat{n}}(\phi_2)$ is the same as $\hat{R}_{\hat{n}}(\phi_1 + \phi_2)$.

They are unitary, so $\hat{R}^{\dagger}_{\hat{n}}(\phi) = \hat{R}^{-1}_{\hat{n}}(\phi) = \hat{R}_{\hat{n}}(-\phi)$, and of course, $\hat{R}_{\hat{n}}(0)$ is just the identity $\hat{1}$.

→ Rotations are a strongly-continuous one-parameter group of unitary transformations
abelian

We can use Stone's theorem:

$$\hat{R}_{\hat{n}}(\phi) = e^{-i\phi\hat{A}}, \quad \hat{A} = \hat{A}^{\dagger}, \text{ where } \hat{A} = \hat{n}\cdot\hat{\vec{J}}/\hbar$$

For a spinless particle, $\hat{\vec{J}} = \hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$, and for a particle with spin, $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$ where $\hat{\vec{S}}$ is the intrinsic spin.

Algebraic Properties of $\hat{\vec{J}}$:

$[\hat{J}_i, \hat{J}_j] = \varepsilon_{ijk}\, i\hbar\, \hat{J}_k$ (and the same for $\hat{L}$)

for scalar $\hat{O}$ (invariant under rotation), $[\hat{J}_i, \hat{O}] = 0$ - they commute.
So, $[\hat{J}_i, \hat{J}^2] = 0$

unit vector, not operator

for any vector $\hat{\vec{A}} = \sum_i x_i \hat{A}_i$, $[\hat{J}_i, \hat{A}_j] = i\hbar\, \varepsilon_{ijk} \hat{A}_k$, and the scalar product btwn two vector operators is a scalar, e.g. $\hat{J}^2 = \hat{\vec{J}}\cdot\hat{\vec{J}}$

Spectral properties:

$\hat{J}^2 |kjm\rangle = \hbar^2 j\, |kjm\rangle$ and $\hat{J}_z |kjm\rangle = \hbar m |kjm\rangle$

The value of m can be ± 1 between $-j$ and j (inclusive) using the ladder operators, so there are $2j+1$ possible values (multiplicity)

$\hat{J}_{\pm} \equiv \hat{J}_x \pm i\hat{J}_y$ and $\hat{J}_{\pm} |kjm\rangle \propto |kjm\pm1\rangle$

j can be an integer or half integer. $|m|$ cannot be more than j:

$\hat{J}_+ |kjj\rangle = \hat{J}_- |kj-j\rangle = 0$

More spectral properties (?):
If the eigenvectors are normalized so that $\langle kjm|k'j'm'\rangle = \delta_{kk'}\delta_{jj'}\delta_{mm'}$
then:

$$\hat{J}_\pm |kjm\rangle = \hbar\sqrt{j(j+1)-m(m\pm1)}\,|kjm\pm1\rangle$$

(there are a bunch of other properties, e.g. $[\hat{J}_z, \hat{J}_\pm] = \pm\hbar\hat{J}_\pm$, see last semester)

Rotations
Operators $\hat{J}_z$ and $\hat{J}_\pm$ (and $\hat{J}_x$, $\hat{J}_y$) leave k and j unaltered.
This means that $(2j+1)$ dimensional space:

$$V_{kj} = \text{Span}\{|kjm\rangle, m=-j, -j+1, \ldots, j-1, j\}$$

is invariant under action of $\hat{J}_i$ - No matter how much you apply $\hat{J}_i$, you never leave V_{kj}. One can say $\hat{J}_i V_{kj} = V_{kj}$.

As a result, rotation $\hat{R}_{\hat{n}}(\phi) = e^{-i\phi\hat{n}\cdot\hat{\vec{J}}/\hbar}$ of a vector in V_{kj} will always result in another vector in V_{kj} (V_{kj} is invariant under rotation):

$$\forall|\psi\rangle \in V_{kj},\ \hat{R}_{\hat{n}}(\phi)|\psi\rangle \in V_{kj}$$

Because V_{kj} is a vector space with basis $|kjm\rangle$, the rotated $|kjm\rangle$ can be written as linear combo:

$$\hat{R}_{\hat{n}}(\phi)|kjm\rangle = \underbrace{\sum_{m'=-j}^{j} |kjm'\rangle\langle kjm'|}_{\text{res. of identity}}\hat{R}_{\hat{n}}(\phi)|kjm\rangle$$

$$= \sum_{m'=-j}^{j} |kjm'\rangle D^j_{m'm}(\hat{R})$$

where coefficients $\langle kjm'|\hat{R}_{\hat{n}}(\phi)|kjm\rangle$ can be written/encoded? in a $(2j+1)\times(2j+1)$ Matrix, $D^j_{m'm}(\hat{R})$.

Thus rotating a quantum state with a defined k & j yields a linear combination according to matrix $D^j_{m'm}(\hat{R})$.

$D^j_{m'm}(\hat{R})$ is the representation of the rotation group. Say we have a group, $\mathcal{G}$, and application $\Gamma: \mathcal{G} \to \text{Aut}(V): \hat{R} \longmapsto D(\hat{R})$

where Aut(V) is an automorphism. IDK what that means.
Anyway if $\mathbb{D}(\hat{a}\cdot\hat{b})$ where $\hat{a}$ and $\hat{b}$ are different rotations, is the same as $\mathbb{D}(\hat{a})\mathbb{D}(\hat{b})$ — that is, the matrix for $\hat{a}\cdot\hat{b}$ is the matrix product of the individual respective matrices for $\hat{a}$ and $\hat{b}$, that is called a "TRUE REPRESENTATION".

Basically, in this case, $\hat{R}_3 = \hat{R}_2\hat{R}_1$, and the action of $\hat{R}_3$ and the action of performing $\hat{R}_1$, then $\hat{R}_2$ are physically indistinguishable. They yield the same state.

HOWEVER: In QM, a state is a ray, not a vector. "The same state" could be referring to the same ray, but different vectors, with different phase factors:

$$\hat{R}_2\hat{R}_1|jm\rangle = \hat{R}_2\left(\sum_{m'=-j}^{j}|jm'\rangle D^j_{m'm}(\hat{R}_1)\right)$$

$$= \sum_{m''=-j}^{j}|jm''\rangle \sum_{m'} D_{m''m'}(\hat{R}_2) D_{m'm}(\hat{R}_1)$$

$$\hat{R}_3|jm\rangle = \sum_{m''=-j}^{j}|jm''\rangle \mathbb{D}^j_{m''m}(\hat{R}_3)$$

True repr.: $\sum_{m'} D^j_{m''m'}(\hat{R}_2)D^j_{m'm}(\hat{R}_1) = \left[\mathbb{D}^j(\hat{R}_2)\mathbb{D}^j(\hat{R}_1)\right]_{m''m}$

(element m'',m of matrix $\mathbb{D}^j(\hat{R}_2)\mathbb{D}^j(\hat{R}_1)$) is the same as matrix elt. $\mathbb{D}^j_{m'm}(\hat{R}_3)$.

Projective representation: $\mathbb{D}(\hat{R}_3 = \hat{R}_2\hat{R}_1) = \mathbb{D}(\hat{R}_2)\mathbb{D}(\hat{R}_1)e^{i\omega(\hat{R}_2,\hat{R}_1)}$

A concrete example:

$e^{-i\phi\hat{J}_z/\hbar}|jm\rangle = e^{-im\phi}|jm\rangle$ (recall $\hat{J}_z|jm\rangle = \hbar m|jm\rangle$ and so $f(\hat{J}_z)|jm\rangle = f(\hbar m)|jm\rangle$)

if $\phi = 0$:
$e^{-im\phi}|jm\rangle = 1\cdot|jm\rangle$

if $\phi = 2\pi$:
$e^{-im\phi}|jm\rangle = e^{-i2\pi m}|jm\rangle$. A rotation of 2π should be the same as a rotation of 0, yet if m is a half-integer, $e^{-2\pi im} = -1$, not 1 !!

This is a consequence of the rotation group, called SO(3).

SO(3) and SU(2)

A rotation $\hat{R}_{\hat{n}}(\phi)$ is defined by the "direction" of rotation, $\hat{n}$, and the magnitude of rotation ϕ.

(of course, if $\hat{n}$ is defined by $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$, we can write $\hat{n} = \sin\theta(\cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}) + \cos\theta\,\hat{z}$).

We can thus represent a rotation as a vector, and the space of all possible rotations is a ball (solid sphere), where magnitude and direction is determined by ϕ and $\hat{n}$.

What does this ball look like? Because a rotation $\hat{R}_{\hat{n}}(\phi)$ is the same as $\hat{R}_{-\hat{n}}(2\pi - \phi)$, we can restrict $\phi \in [0, \pi]$.

Any rotation of $\phi > \pi$ around $\hat{n}$ can be written as a rotation of $2\pi - \phi < \pi$ around $-\hat{n}$ instead. Since $-\hat{n}$ is well defined if $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$, this works out well.

$\therefore$ The vector space is all vectors with length $\in [0, \pi]$ in direction bounded by θ and φ — a solid sphere/ ball with radius π.

Say now we have a <u>sequence of rotations</u> whose endpoints create a path through the sphere. Say this forms a loop that starts and ends at the origin. This is a rotation of <u>2π</u>.

single rotation $\hat{R}_{\hat{n}}(\phi)$ as vector $\hat{n}\cdot\phi$.

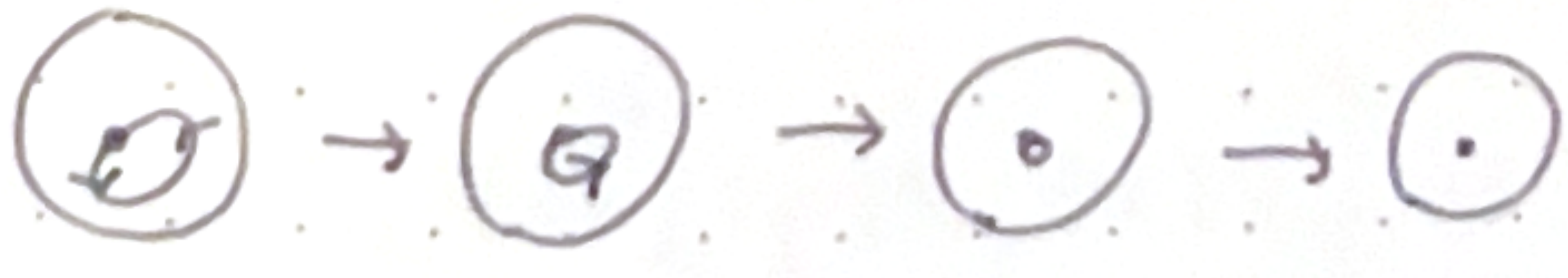
sequence of rotations thru origin.

(I assume this a continuous set of rotations, each relative to the rotated object's <u>initial</u> orientation, so this loop returns to object to normal)

There are two such loops: I  II 

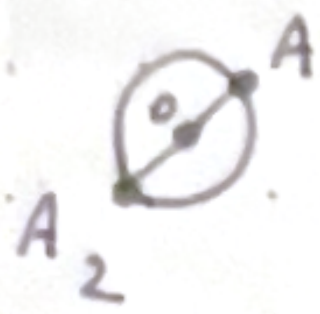
in the latter, the path exits on one side of the ball and enters on the opposite side ☹..

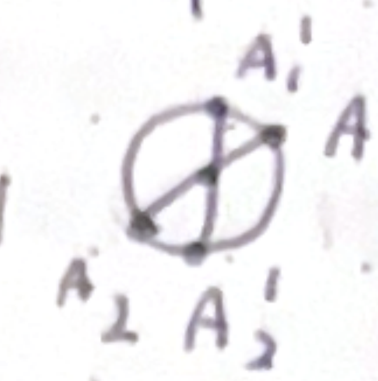
I can be deformed / contracted to a single point (the origin):



which is a single, equivalent rotation to the whole loop.

II cannot be deformed/ contracted in the same way. For the loop to stay continuous + connected, and through the origin, one point on the there must be antipodal to a second point : 180° away.

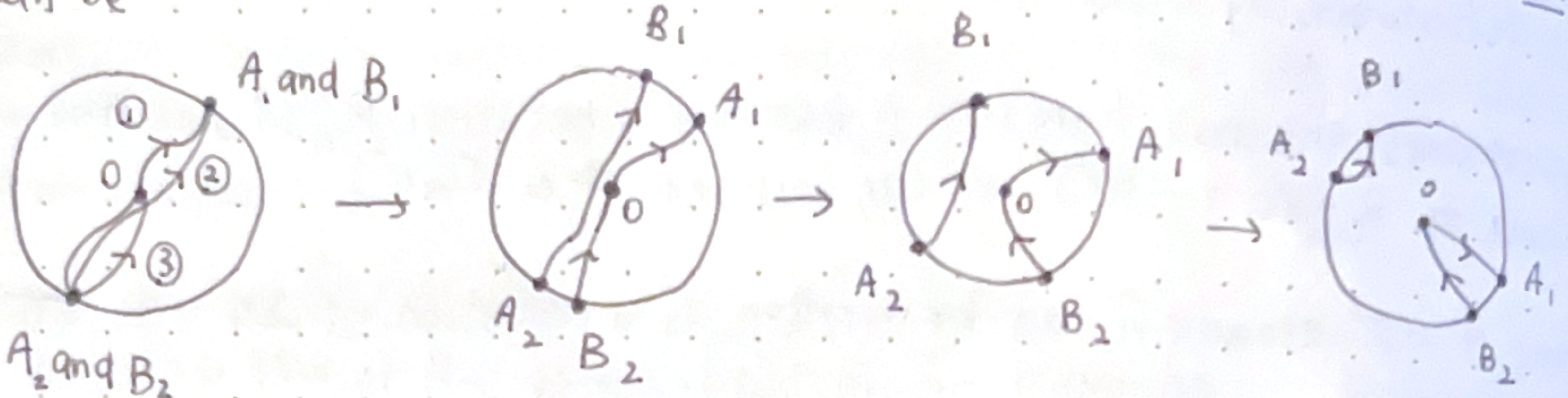


if we move A_1 CCW, A_2 also moves CCW


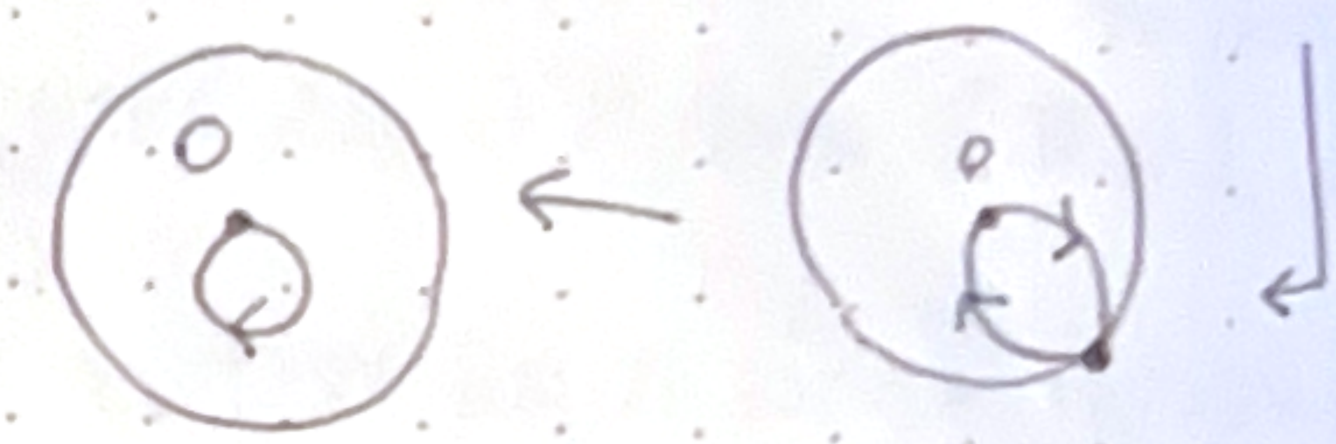
hence the loop never contracts.

This means SO(3) is not simply connected.

Notice, however, that while a rotation of 2π can't be reduced, one of 4π can be:



(start @ origin, go to A_1, appear on A_2. Go to B_1, appear on B_2. Go to origin. End.)



(same as I now!)

(SO(3) is doubly connected — two non-contractible loops composed tog. yields a contractible loop!) — Its representation of rotations can be projective

All non-simply connected groups have a locally identical group that is simply connected. Looks identical near origin. For SO(3), the universal covering group is SU(2), and all its rotation representations are true.

Euler rotations/angles

Mentioned earlier, $\hat{n} = \sin\theta \cos\varphi\, \hat{x} + \sin\theta \sin\varphi\, \hat{y} + \cos\theta\, \hat{z}$, and three parameters, ϕ, θ, φ are required to specify rotation $\hat{R}_{\hat{n}}(\phi)$.

Because $\hat{R}_{\hat{n}}(\phi) = e^{-i\phi \hat{n}\cdot\hat{\vec{J}}}$, this is inconvenient — a single operator depends on more than one parameter, so we use Euler angles:

1) rotate about $\hat{z}$: $\hat{R}_{\hat{z}}(\alpha)$ which yields:

$$\hat{x} \to \hat{x}' = \hat{R}_{\hat{z}}(\alpha)\hat{x} = \hat{x}\cos\alpha + \hat{y}\sin\alpha$$

$$\hat{y} \to \hat{y}' = \hat{R}_{\hat{z}}(\alpha)\hat{y} = \hat{y}\cos\alpha - \hat{x}\sin\alpha$$

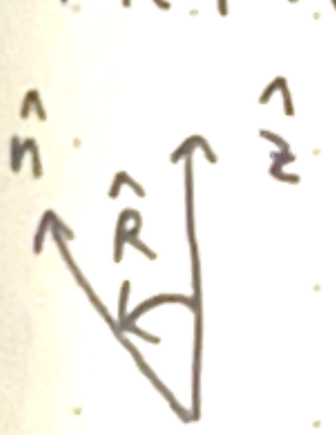
$$\hat{z} \to \hat{z}' = \hat{z}$$

2) rotate around $\hat{y}'$: $\hat{R}_{\hat{y}'}(\beta)$, yields $\hat{x}_i'' = \hat{R}_{\hat{y}'}(\beta)\hat{x}_i'$

3) rotate around $\hat{z}''$: $\hat{R}_{\hat{z}''}(\gamma)$, yields $\hat{x}_i''' = \hat{R}_{\hat{z}''}(\gamma)\hat{x}_i''$

$\hat{R}(\alpha,\beta,\gamma) = \hat{R}_{\hat{z}''}(\gamma)\hat{R}_{\hat{y}'}(\beta)\hat{R}_{\hat{z}}(\alpha)$ — we have one parameter per operator. BUT each rotation depends on the previous. We can separate even more:

Any rotation $\hat{R}_{\hat{n}}(\phi)$ can be performed by rotating the entire system so $\hat{n}$ aligns with $\hat{z}$ (or any axis), rotating by ϕ around $\hat{z}$, then rotating so $\hat{n}$ is put back where it was before:



$$\hat{n} = \underline{\hat{R}}\hat{z} \to \hat{R}_{\hat{n}}(\phi) = \underline{\hat{R}}\hat{R}_{\hat{z}}(\phi)\underline{\hat{R}^{-1}}$$

The $\hat{z}''$ axis is produced by applying $\hat{R}_{\hat{z}}(\alpha)$, then $\hat{R}_{\hat{y}'}(\beta)$ to $\hat{z}$. A rotation around $\hat{z}''$ is then:

$$\underline{\hat{R}_{\hat{z}''}(\gamma)} = \hat{R}_{\hat{y}'}(\beta)\hat{R}_{\hat{z}}(\alpha)\underline{\hat{R}_{\hat{z}}(\gamma)}\hat{R}^{-1}_{\hat{z}}(\alpha)\hat{R}^{-1}_{\hat{y}'}(\beta)$$

Plug this into $\hat{R}(\alpha,\beta,\gamma)$ to get:

$$\hat{R}_{\hat{y}'}(\beta)\hat{R}_{\hat{z}}(\alpha)\hat{R}_{\hat{z}}(\gamma)\hat{R}^{-1}_{\hat{z}}(\alpha)\underbrace{\hat{R}^{-1}_{\hat{y}'}(\beta)\hat{R}_{\hat{y}'}(\beta)}\hat{R}_{\hat{z}}(\alpha)$$

these cancel!

$$= \hat{R}_{\hat{y}'}(\beta)\hat{R}_{\hat{z}}(\alpha)\hat{R}_{\hat{z}}(\gamma)$$

Do this again for the rotation around $\hat{y}'$:

$\hat{R}_{\hat{z}}(\alpha)\hat{y} = \hat{y}'$ so $\hat{R}_{\hat{y}'}(\beta) = \hat{R}_{\hat{z}}(\alpha)\hat{R}_{\hat{y}}(\beta)\hat{R}^{-1}_{\hat{z}}(\alpha)$ Plug in:

$$\hat{R}_{\hat{z}}(\alpha)\hat{R}_{\hat{y}}(\beta)\underbrace{\hat{R}^{-1}_{\hat{z}}(\alpha)\,\hat{R}_{\hat{z}}(\alpha)}_{\text{cancel!}}\hat{R}_{\hat{z}}(\gamma) = \hat{R}_{\hat{z}}(\alpha)\hat{R}_{\hat{y}}(\beta)\hat{R}_{\hat{z}}(\gamma)$$

This representation of rotation: $R(\alpha,\beta,\gamma) = \hat{R}_{\hat{z}}(\alpha)\hat{R}_{\hat{y}}(\beta)\hat{R}_{\hat{z}}(\gamma)$ is now three single-parameter rotations whose <u>axes</u> are <u>not</u> dependent on $\alpha,\beta,$ or γ.

Wigner D-Matrices

Recall vector space $V_{kj} = \text{span}\{|kjm\rangle\}_{|m|\leq j}$, $(2j+1)$-dim. space that is invariant under rotations.

$$\forall |\psi\rangle \in V_{kj},\ |\psi\rangle = \sum_{m=-j}^{j} |kjm\rangle \underbrace{a_m}_{\text{some coeft.}}$$

$$\hat{R}(\alpha,\beta,\gamma)|\psi\rangle = \sum_m \hat{R}(\alpha,\beta,\gamma)|kjm\rangle a_m$$

$$= \sum_{mm'} |kjm'\rangle \underbrace{\langle kjm'|\hat{R}(\alpha,\beta,\gamma)|kjm\rangle}_{\text{only depends on } j\ \&\ m.\text{ Call this Wigner D-matrices/fcns, } D^j_{m'm}(\alpha,\beta,\gamma)} a_m$$

These matrices have elements:

$$\langle jm'|\hat{R}(\alpha,\beta,\gamma)|jm\rangle = \langle jm'|\hat{R}_{\hat{z}}(\alpha)\hat{R}_{\hat{y}}(\beta)\hat{R}_{\hat{z}}(\gamma)|jm\rangle$$

$$= \langle jm'|e^{-i\alpha\hat{J}_z/\hbar}e^{-i\beta\hat{J}_y/\hbar}e^{-i\gamma\hat{J}_z/\hbar}|jm\rangle$$

$$= e^{-i(\alpha m'+\gamma m)}\underbrace{\langle jm'|e^{-i\beta\hat{J}_y/\hbar}|jm\rangle}_{d^j_{m'm}(\beta),\text{ reduced d-matrix.}}$$

Typically you use tabulated results when you need $D^j_{m'm}$ or $d^j_{m'm}$

<u>Equivalence Classes</u>: For set $S = \{a,b,c\}$, cartesian product $S\times S = \{aa, ab, ac, bb, \ldots\} = \{(a,b)\,|\,a,b\in S\}$.

A <u>relation</u> R is a subset of $S\times S$ so that if $(a,b)\in R$, a is related to b.

(order matters!). A more specific type of relation is an <u>Equivalence</u> relation which is:

- Reflexive: $\forall a \in S$, $(a,a) \in R$ so $a \sim a$
- Symmetric: $\forall a,b \mid a \sim b \Rightarrow b \sim a$
- Transitive: if $a \sim b$, $b \sim c \Rightarrow a \sim c$

e.g. $=$, congruence, modulo equiv, N3L → mass of obj, quantity of obj, thermal equilib → temp. of obj, Rational # $m/n = m'/n'$, and so on.

Set S with equivalence relation $\sim$: class $\mathcal{C}_a$ of a is the set of all elements in S (including a) equivalent to a, $\mathcal{C}_a = [a] = \{b \in S \mid b \sim a\}$

If $\mathcal{C}_a$ and $\mathcal{C}_b$ have an element in common, they coincide; either $\mathcal{C}_a \cap \mathcal{C}_b = \emptyset$ or $\mathcal{C}_a = \mathcal{C}_b$

Every element in S has a class, so $S = \mathcal{C}_a \cup \mathcal{C}_b \cup \ldots$ The set of <u>equivalence classes</u> themselves: $\{\mathcal{C}_a, \mathcal{C}_b, \mathcal{C}_c\} = S/\sim$ or the <u>qutient</u> of S w.r.t. equivalence relation $\sim$.

if $\hat{A} = \hat{A}^{\dagger}$ and $\hat{B} = \hat{U}^{-1}\hat{A}\hat{U}$, $\hat{A}$ and $\hat{B}$ have the same spectrum (they're the same operator "at a different angle"), e.g. $\hat{J}_x, \hat{J}_y, \hat{J}_z$.

<u>Tensor Product</u>

Say system S is partitioned into subsystems S_I and S_{II}, according to some equivalence class (state, spin, whatever).

If S_I has Hilbert space H_I and S_{II} has H_{II}, what is the total H for the whole system S?

Say we prepare S_I in state ψ and S_{II} in ϕ. The total state is some $\Psi(\psi, \phi)$. If $\psi = \psi_1 a_1 + \psi_2 a_2$, we want to be able to also separate the entire state like:

$$\Psi(\psi_1 a_1 + \psi_2 a_2, \phi) = a_1 \Psi(\psi_1, \phi) + a_2 \Psi(\psi_2, \phi)$$

and vice versa:

$$\phi = \phi_1 b_1 + \phi_2 b_2 \longrightarrow \Psi = b_1 \Psi(\psi, \phi_1) + b_2 \Psi(\psi, \phi_2)$$

So that you can operate on one subsystem while keeping the other the same. Basically, Ψ must be <u>bilinear</u>

The constraint of bilinearity means (apparently) that there is only one solution: $H = H_I \otimes H_{II}$

↑ tensor product!

Say we have two vectorspaces (not specifically Hilbert Spaces)

$V_I \underset{span}{=} \{1, x, x^2\}$ and $V_{II} = \text{span}\{1, y\}$

where $f \in V_I$ (or $f(x)$) $= a_0 + a_1 x + a_2 x^2$ and $g(y) \in V_{II} = b_0 + b_1 y$

The space $M = \{ f(x)g(y) \mid f \in V_I, g \in V_{II} \}$ may seem like a good solution, but:

$$f(x)g(y) = \underline{a_0 b_0} + a_0 b_1 \underline{y} + a_1 b_0 \underline{x} + a_1 b_1 \underline{xy} + a_2 b_0 \underline{x^2} + a_2 x^2 y$$

(V_I has 3-dim, V_{II} has 2-dim, and this has 6, however...)

We can't have $\underline{\underline{x+y}}$ as a result of this diadic product $f(x)g(y)$.

→ Thus M is not closed under addition and not even a vector space and it is not bilinear.

Instead, if we take all the linear combinations of these products:

$$V = \left\{ \sum_{i=0}^{2} \sum_{j=0}^{1} x^i y^j c_{ij} \right\}$$, we can indeed get $x \cdot 1 + 1 \cdot y = x + y$. This is a 6-dimensional vector space and is a tensor product.

Note that $x \cdot 1 + 1 \cdot y$ means (f is vector x and g is vector 1) OR (f is vector 1 and g is y), or that it is impossible to represent the total state as just one set of "possible states". Tensor products then naturally induce the concept of entanglement.