Hw4

1. (a) $\frac{n(n-1)}{2}$ is $O(n^2)$

Prove: let $\frac{n(n-1)}{2} \le Cn^2$. $\therefore n[(2C-1)n-1] \ge 0$

$\therefore$ let $C=1$, then $n \ge 1 \Rightarrow N_0 = 1$

$\therefore \exists C \ge 1\ N_0 \ge 1$, for $\forall n \ge N_0$. $Cn^2 \ge \frac{n(n-1)}{2}$

$\therefore$ According to formal definition of big-O

$O(n^2) = \frac{n(n-1)}{2}$

(b) $\max(n^3, 10n^2)$ is $O(n^3)$

Prove: $\max(n^3, 10n^2) = \begin{cases} 10n^2 & n \le 10 \\ n^3 & n > 10 \end{cases} = f(n)$

let $f(n) \le Cn^3$.

let $C=1$. when $n \le 10$, $10n^2 \le n^3$, $n \ge 10$

when $n > 10$ $n^3 \le n^3$. $n > 10$

$\therefore N_0 \ge 10$

$\therefore \exists C \ge 1\ N_0 \ge 10$ for $\forall n \ge N_0$ $Cn^3 \ge f(n)$

$\therefore$ According to formal definition of big-O

$O(n^3) = \max(n^3, 10n^2)$

(c) $\sum_{i=1}^{n} i^k$ is $O(n^{k+1})$ and $\Omega(n^{k+1})$ $k \in N$ and $k > 0$

Prove: ① for $\sum_{i=1}^{n} i^k \approx \int_{i=0}^{n} i^k di = \frac{1}{k+1} n^{k+1}$

$\because \int_{i=m}^{m+1} i^k di \le \int_{i=m}^{m+1} (m+1)^k di = (m+1)^k$

$\therefore \sum_{i=1}^{n} i^k \ge \int_{i=0}^{n} i^k di = \frac{1}{k+1} n^{k+1}$

$\therefore (k+1) \sum_{i=1}^{n} i^k \ge n^{k+1}$. $\therefore$ According to the formal definition of big-Ω $\therefore \exists C = \frac{1}{k+1}$, $N_0 \ge 1$, let $\forall n \ge N_0$ $\sum_{i=1}^{n} i^k \ge n^{k+1}$

$\therefore \sum_{i=1}^{n} i^k = \Omega(n^{k+1})$

②. $\because \sum_{i=1}^{n} i^k \le \sum_{i=1}^{n} n^k = n^{k+1}$ $\therefore$ According to the big-O's definition, $\exists C = 1$, $N_0 \ge 1$, let $\forall n \ge N_0$, $\sum_{i=1}^{n} i^k \le n^{k+1}$ $\therefore \sum_{i=1}^{n} i^k = O(n^{k+1})$

(d) if $p(x)$ is kth polynomial with positive coefficient, the $p(n) = \Theta(n^k)$

Prove:

① let $p(x) = a_k x^k + a_{k-1} x^{k-1} + \dots a_1 x + a_0 \quad a_n > 0$

$$= \sum_{n=0}^{k} a_n x^n$$

$\because a_n > 0$ and $k \geq 0$

$$\therefore \sum_{n=0}^{k} a_n x^n \leq \sum_{n=0}^{k} a_n x^k = C \cdot x^k$$

where $C = \sum_{n=0}^{k} a_n$

$\therefore \exists\, C = \sum_{n=0}^{k} a_k,\ N_0 \geq 0$ let $\forall n \geq N_0,\ p(x) \leq C x^k$.

$\therefore$ According to big-O's definition. $p(n) = O(n^k)$

② let $p(x) = \sum_{n=0}^{k} a_n x^n \quad a_n > 0$

$$p(x) = a_k x^k + \sum_{n=0}^{k-1} a_n x^n$$

let $p(x) \geq C \cdot x^k$. where $C = a_k$.

$\therefore$ According to the definition of big-Omega.

for $\exists\, C = a_k,\ N_0 \geq 0$. let $\forall n \geq N_0\ p(x) \geq C x^k$.

$\therefore p(n) = \Omega(n^k)$

2. Which function grows faster?

(a) $(\log(n))^n$

(b) $(\log n)^k \quad (k > 1)$

(c) $(\log n)!$

(d) n^n.

The proof is as follows

2. (a) $n^{\log n}$; $(\log n)^n$.

Proof: let $g(n) = n^{\log n}$ $f(n) = (\log n)^n$.

$\log(g) = \log n \log n$ $\log(f) = n \log(\log n)$

$\log(\log(g)) = 2\log\log n$ ①, $\log(\log(f)) = \log n + \log\log\log n$ ②

let $k = \log n$

$\therefore$ ① $= 2\log k$ ② $= k + \log\log k$.

Apparently ② grows faster than ① $\therefore f(n) = (\log n)^n$ grows faster.

(b) $\log n^k$, $(\log n)^k$.

Proof: let $g(n) = \log n^k = k\log n$. $f(n) = (\log n)^k$

$\log(g(n)) = \log(k\log n) = \log k + \log\log n$.

$\log(f(n)) = k\log\log n$.

$\lim_{n\to+\infty} \frac{\log(g(n))}{\log(f(n))} = \lim_{n\to+\infty} \frac{\log k + \log\log n}{k\log\log n} = \frac{1}{k}$. $\therefore$ if $k > 1$ then $g(n) = \log(n)^k$ grows faster. if $0<k<1$, then $f(n)$ grows faster.

$\therefore$ if $k = 0, 1$ then $g(n)$ and $f(n)$ share the same growth rate.

(c) $n^{\log\log\log n}$, $(\log n)!$

Proof: let $g(n) = n^{\log\log\log n}$ $f(n) = (\log n)!$

$\log(g(n)) = \log n \log\log\log n$ $\log(f(n)) = \log((\log n)!)$

let $k = \log n$.

then $\log(g(n)) = k\log\log k$ $\log(f(n)) = \log(k!)$

$\because \lim_{k\to\infty} \frac{\log(k!)}{k\log k} = \lim_{k\to\infty} \frac{\log((k+1)!) - \log(k!)}{(k+1)\log(k+1) - k\log k}$

$= \lim_{k\to\infty} \frac{\log((k+1))}{\log(k+1) + k[\log(k+1) - \log k]} = \lim_{k\to\infty} \frac{\log(k+1)}{\log(k+1) + [\log(1+\frac{1}{k})^k]} = 1$

$\therefore \log(k!)$ and $k\log k$ have the same growth rate

Apparently, $k\log\log k$ grows slower than $k\log k$.

$\therefore$ $\log(k!)$ grows faster than $k\log\log k$.

$\therefore$ $f(n) = (\log n)!$ grows faster than $g(n) = n^{\log\log\log n}$

(d) n^n, $n!$

let $f(n) = n^n$ $g(n) = n!$

~~$\log(f(n)) = n\log n$ $\log(g(n)) = \sum_{k=1}^{n} \log k = \log n!$~~

~~According to the proof in (c)~~

~~$n\log n$ and $\log n!$ have the same growth rate~~

~~$\therefore$ n^n and $n!$ have the same growth rate.~~

$f(n)$ grows faster.

3\. Proof:

let $f_1(n) = O(g_1(n))$ $f_2(n) = O(g_2(n))$

$f_1(n) > 0$ $f_2(n) > 0$

$\therefore \exists C_1, N_1$, let $\forall n \geq N_1$ $f_1(n) \leq g_1(n) \times C_1$

$\exists C_2, N_2$, let $\forall n \geq N_2$ $f_2(n) \leq g_2(n) \times C_2$

$\therefore$ then $f_1(n) + f_2(n) \leq C_1 g_1(n) + C_2 g_1(n)$

$\because N_1, N_2 > 0$; $f_1, f_2 > 0$ $\therefore n \geq \max(N_1, N_2)$

Then $\because$ $C_1 g_1(n) + C_2 g_1(n) \leq (C_1 + C_2) \max(g_1(n), g_2(n))$

$\therefore \exists N_0 = \max(N_1, N_2)$, $C = (C_1 + C_2)$ let $\forall n \geq N_0$

$f_1(n) + f_2(n) \leq C \max(g_1(n), g_2(n))$

According to the definition of Big-O

$f_1(n) + f_2(n) = O(\max(g_1(n), g_2(n)))$

4: Proof:

let $n \leq C\frac{n}{2}$, then $C \geq 2$. $(n \geq 0)$

$\therefore \exists C \geq 2$, $N_0 \geq 0$, let $\forall n \geq N_0$, $n \leq C\frac{n}{2}$

$\therefore$ According to the definition of Big-O

$n = O(\frac{n}{2})$

5\. Proof:

let. $3^n \leq C2^n$ $\because C > 0$,

then let $\log 3^n \leq \log C2^n$. $\therefore n \log 3 \leq \log C + n \log 2$.

$\therefore n \leq \dfrac{\log C}{\log 3 - \log 2}$.

So there is no constant N_0 let $n \geq N_0$

to make $3^n \leq C2^n$.

$\therefore$ $3^n \neq O(2^n)$.