CS590 Algorithm Fall 2014

1. **The degree of a node in a tree is the number of children the node has. If a tree has n1 nodes of degree 1, n2 nodes of degree 2, ... , nm nodes of degree m, compute the number of leaves in the tree in terms of n1,n2,...,nm.**

   Anwser:
   Step1:
   First we suppose there are X leaves in the tree, as we know the degree of each leaf is zero. According to the description:
   Total degree = 0*X+1*n1+2*n2+…+n*nm.
   Total nodes = X+n1+n2+…+nm.
   Step2:
   As we know, every child node has only one parent node, and root of the tree has no parent node. So, the formula to calculate total degree can be written as:
   Total degree = Total node -1
   Total degree = X+n1+n2+…+nm-1
   Step3:
   So, X = (0*X+1*n1+2*n2+…+n*nm) - (n1+n2+…+nm-1)
   = 1*n2+2*n3+…+(n-1)*nm+1

2. **Consider a complete binary tree with an odd number of nodes. Let n be the number of internal nodes (non-leaves) in the tree. Define the internal path length, I, as the sum, taken over all the internal nodes of the tree, of the depth of each node. Likewise, define the external path length, E, as the sum, taken over all the leaves of the tree, of the depth of each leaf. Show that E = I +2n.**

   Answer:

   Define
   E = external path length = the total distance of all external nodes from the root
   I = internal path length = the total distance of all internal nodes from the root

Then, $E = I + 2n$ where n is the number of internal nodes

Proof (by induction)

Step 1.
Show that $P(n)$: $E = I + 2n$ is true for the base case ($n= 1$)
Then the tree consists of a single internal node, the root, and its two(external node) children.
$I = 0$, $n = 1$, and $E = 2$ and the statement $P(1)$ is correct.

Step 2.
The inductive hypothesis – show $P(n)$ à $P(n+1)$
Assume $P(n)$ is true.
Replace any one of the external nodes – say the one at a distance d from the root with an internal node (and its two external children).
Then E' the external path length of the new tree (with n+1 nodes) is $E' = E - d + 2(d + 1)$ since the original tree lost an external node at depth d but gained two new ones at depth $d + 1$.
And similarly, $I' = I + d$ Since one additional internal node was added at depth d. Then

$$E' = E - d + 2(d+1) = I' + 2(n + 1)$$

$$E + d + 2 = I + d + 2(n + 1)$$

Which reduces to $E = I + 2n$
Which is assumed true.