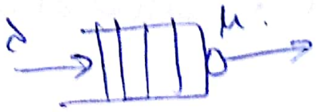
2.19:



given $\frac{\lambda}{\mu} = 0.3 = \rho$.

equilibrium state probability
$P_n(t) = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$.

$P(N(t) \leq m) = \frac{97}{100}$.

$\Rightarrow \sum_{n=0}^{m} P_n(t) = \frac{97}{100}$

$\Rightarrow \sum_{n=0}^{m} \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) = 0.97$

$\Rightarrow \sum_{n=0}^{m} (0.3)^n (0.7) = 0.97$

$\Rightarrow 0.7 \left(\sum_{n=0}^{m} (0.3)^n\right) = 0.97$

$\Rightarrow \cancel{0.7} \times \frac{1 - (0.3)^{m+1}}{\cancel{0.7}} = 0.97$

$\rightarrow (0.3)^{m+1} = 0.03$.

$m+1 = \log_{0.3} 0.3/10$

$m+1 = 1 - \log_{0.3} 10$

$m = -\log_{0.3} 10$

$m = 1.91 \simeq 2.$

2.20: $P_0(t) = 0.415$ $P_1(t) = 0.277$ $P_2(t) = 0.185$ $P_3(t) = 0.123$

$P_4(t) = 0 \ldots$

State Probabilities:

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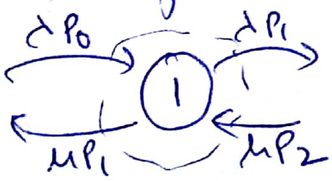
At equilibrium, inflow is equal to out flow at any state



$$\Rightarrow \quad \lambda P_0 + \mu P_2 = \lambda P_1 + \mu P_1$$

$$\Rightarrow \quad \lambda (P_0 - P_1) = \mu (P_1 - P_2)$$

$$\lambda = \frac{\mu (P_1 - P_2)}{P_0 - P_1}$$

$$= \frac{10 (0.277 - 0.185)}{0.415 - 0.277}$$

$$= \frac{10 (0.092)}{0.138}$$

$$= 0.66 \times 10$$

$$= 6.66.$$

Mean throughput:

$$\sum_{n=1}^{3} \mu(n) \cdot P_n.$$

$$= 10 \cdot P_1 + 10 P_2 + 10 P_3$$

$$= 10 \times 0.277 + 10 \times 0.185 + 10 \times 0.123$$

$$= 2.77 + 1.85 + 1.23.$$

$$= 4 + 1.85$$

$$= 5.85$$

Mean no. of customers: $E[N] = \sum_{n=0}^{3} n \cdot P_n(t).$

$$= \sum_{n=1}^{3} n P_n(t).$$

$$= 1 \cdot P_1 + 2 \cdot P_2 + 3 \cdot P_3.$$

therefore given probabilities are not realizable for finite buffer state-independent system.

2.36: $P(z) = \sum_{n=0}^{\infty} z^n P_n(t)$ $\qquad P_n(t) = \rho^n P_0.$

$$= \sum_{n=0}^{\infty} z^n \rho^n P_0.$$

$$= P_0 \sum_{n=0}^{\infty} (z\rho)^n.$$

$$= \frac{P_0}{1-z\rho} \quad \forall\ z\rho < 1$$

we know that $\left.\frac{d}{dz} P(z)\right|_{z=1}$ is $E[P_n(t)]$.

$E[P_n(t)]$ is $\frac{\rho}{1-\rho}$ for M/M/1 queueing system.

$$\frac{\rho}{1-\rho} = \left.\frac{d}{dz}\left(\frac{P_0}{1-z\rho}\right)\right|_{z=1}$$

$$\frac{\cancel{\rho}}{\cancel{1-\rho}} = \left.\frac{P_0 \cancel{\rho}}{(1-z\rho)^2}\right|_{z=1}$$

$$\Rightarrow \quad P_0 = 1 - z\rho \Big|_{z=1}$$

$$\boxed{P_0 = 1-\rho.}$$

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$= 1 \times 0.277 + 2 \times 0.185 + 3 \times 0.123$

$= 0.277 + 0.370 + 0.369$

$= 1.016$

Mean delay : $E[T] = \frac{E[N]}{\lambda}$ as per little's law.

$E[T] = \frac{1.016}{6.66} = 0.152$

(b)

at equilibrium local state equilibrium is zero

$\Rightarrow$ in flow = out flow.

$\Rightarrow$ at state 1

$\lambda P_0 + \mu P_2 = \lambda P_1 + \mu P_1$

$\Rightarrow \frac{\lambda}{\mu} = \frac{P_1 - P_2}{P_0 - P_1}$

$= \frac{0.3 - 0.2}{0.4 - 0.3} = 1$

we also know that $P_1 = \frac{\lambda}{\mu} P_0$ $P_2 = \frac{\lambda}{\mu} P_1$ $P_3 = \frac{\lambda}{\mu} P_2$

$\Rightarrow$ $P_1 = 1 \cdot P_0$ $P_2 = 1 \cdot P_1$ $P_3 = 1 \cdot P_2$

which is not satisfied with the given probabilities.

2.37:

(a) $$P(z) = \frac{P_0}{1-z\rho} \qquad P_0 = 1-\rho$$

$$= \frac{1-\rho}{1-z\rho}$$

$$E[n] = \frac{d}{dz}(P(z))\Big|_{z=1} = \frac{d}{dz}\left(\frac{1-\rho}{1-z\rho}\right)\Big|_{z=1}$$

$$= \cancel{(1-\rho)}\frac{\rho}{(1-z\rho)^{2}}\Big|_{z=1}$$

$$= \frac{\rho}{1-\rho}$$

(b) $$\frac{d^2}{dz^2}P(z)\Big|_{z=1} + E[n] = \sigma^2$$

$$\Rightarrow \sigma^2 = \frac{d^2}{dz^2}\left(\frac{1-\rho}{(1-z\rho)}\right)\Big|_{z=1} + \frac{\rho}{1-\rho}$$

$$= \frac{d}{dz}\left(\frac{\rho(1-\rho)}{(1-z\rho)^2}\right)\Big|_{z=1} + \frac{\rho}{1-\rho}$$

$$= \rho(1-\rho)\frac{d}{dz}\left(\frac{1}{(1-z\rho)^2}\right)\Big|_{z=1} + \frac{\rho}{1-\rho}$$

$$= \frac{\rho}{(1-z\rho)^2}\Big|_{z=1} + \frac{\rho}{1-\rho}$$

$$= \frac{\rho^2}{(1-\rho)^2}$$

(2) $N=3$ files ,

download of file 'i' $\Rightarrow$ download of file given file 'i' is selected.

let 't' be r.v. for download of file

and 'x' be r.v. for file selection

't' $\Rightarrow$ continuous r.v.

'x' $\Rightarrow$ discrete r.v.

given $E[t/x=i] = \alpha_i$

$$\Rightarrow f(t/x=i) = \frac{1}{\alpha_i} e^{-\frac{t}{\alpha_i}}$$

(a) $x=1 \Rightarrow f(t|x=1) = \frac{1}{\alpha_1} e^{-t/\alpha_1}$

$$\Rightarrow f(t\leq 2|x=1) = \int_0^2 f(t|x=1)dt$$

$$= \int_0^2 \frac{1}{\alpha_1} e^{-\frac{t}{\alpha_1}} dt.$$

(b) downloading randomly selected file $= f(t)$

$$\Rightarrow f(t) = f(t|x=1) P(x=1) + f(t|x=2)P(x=2) + f(t|x=3) P(x=3)$$

$$= \frac{1}{\alpha_1} e^{-\frac{t}{\alpha_1}} . P_1 + \frac{1}{\alpha_3} e^{-\frac{t}{\alpha_2}} . P_2 + \frac{1}{\alpha_3} e^{-t/\alpha_3} P_3$$

$$= \frac{P_1}{\alpha_1} e^{-\frac{t}{\alpha_1}} + \frac{P_2}{\alpha_2} e^{-t/\alpha_2} + \frac{P_3}{\alpha_3} e^{-t/\alpha_3}.$$

Expected value of download of randomly selected file:

$$E[f(t)] = \int_{-\alpha}^{\alpha} t\, f(t)\, dt.$$

$$= \int_0^{\alpha} t \left[\frac{P_1}{\alpha_1} e^{-t/\alpha_1} + \frac{P_2}{\alpha_2} e^{-t/\alpha_2} + \frac{P_3}{\alpha_3} e^{-t/\alpha_3}\right] dt.$$

(c) Variance $= E\left[(f(t))^2\right] - \left(E[f(t)]\right)^2$

$$= \int_0^{\alpha} t^2 \sum_{i=1}^{3} \frac{P_i}{\alpha_i} e^{-t/\alpha_i}\, dt. - \left[\int_0^{\alpha} t \sum_{i=1}^{3} \frac{P_i}{\alpha_i} e^{-t/\alpha_i}\, dt\right]^2$$

(d) $P(x=2 \mid t \leq 4) = \dfrac{f(t \leq 4 \mid x=2)\, P(x=2)}{f(t \leq 4)}$

$$= \frac{\left[\int_0^4 \frac{1}{\alpha_2} e^{-t/\alpha_2} dt\right] \cdot P_2}{\int_0^4 \sum_{i=1}^{3} \frac{P_i}{\alpha_i} e^{-t/\alpha_i}\, dt}$$

③ ⓐ $\lambda = 10$

M = arrivals in 9:00 to 9:10

$= \lambda[9{:}10 - 9{:}00]$

$= 10\lambda$.

N = arrivals in 9:30 to 9:35

$= \lambda[9{:}35 - 9{:}30]$

$= 5\lambda$.

distribution of $M+N$

M(Poisson)

N(Poisson)

M+N: (Poisson).

$\Rightarrow M+N = 15\lambda$.

$\Rightarrow$ pmf $(M+N)$ = Poisson with parameter 15λ.

$$= \frac{(15\lambda t)^n e^{-15\lambda t}}{n!}$$

$$= \frac{(150t)^n e^{-150t}}{n!}$$

ⓑ given poisson process, Inter arrival time is exponential

0   4

$\hookrightarrow$ random period of time 'T' $\rightarrow$ follows uniform distribution.

$$P(N(T)=n \mid x=T) = P(N(T)=1)$$

$$= \frac{(\lambda T)^1 e^{-\lambda T}}{1!}$$

$$= \frac{\lambda T e^{-\lambda T}}{1}$$

$$= \lambda T e^{-\lambda T} = \underline{\underline{10Te^{-10T}}}$$

Expectation: $E\left[P(N(T)=n \mid x=T)\right]$

$$= \sum_{n=0}^{\infty} \frac{(\lambda T)^n e^{-\lambda T}}{n!}$$

$$= \lambda T$$

$$= \underline{\underline{10T}}.$$

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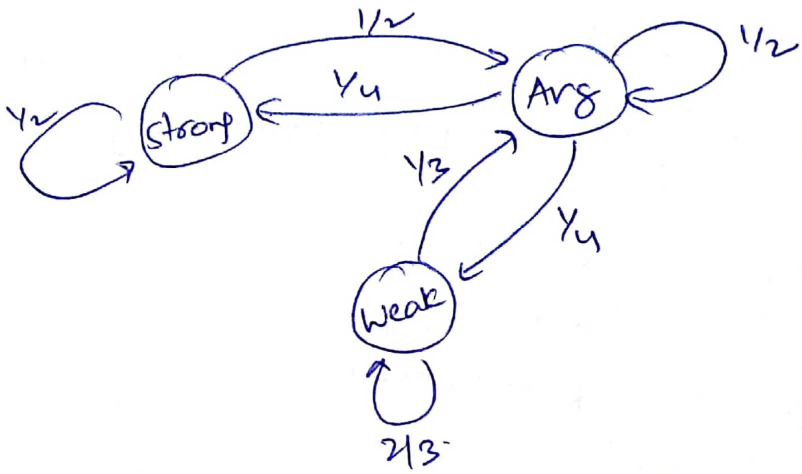
Q4 (a)



$$P_0 = \begin{array}{c|ccc} & S & A & W \\ \hline S & 1/2 & 1/2 & 0 \\ A & 1/4 & 1/2 & 1/4 \\ W & 0 & 1/3 & 2/3 \end{array}$$

Modelling this as per discrete time markov chain:
discrete time ~~since~~ since no. of years is discrete

To find fraction of time company is in avg performance:
In steady state, probability of avg state is the fraction of time company stays in avg state.

$\Pi_1 \to S$
$\Pi_2 \to A$
$\Pi_3 \to W$

$\Rightarrow \quad \bar{\Pi} = \bar{\Pi} \cdot \bar{P_0}$

$$\Rightarrow \quad [\Pi_1 \quad \Pi_2 \quad \Pi_3] = [\Pi_1 \quad \Pi_2 \quad \Pi_3] \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/3 & 2/3 \end{bmatrix}$$

$\& \quad \sum \Pi_i = 1.$

$\Pi_1 + \Pi_2 + \Pi_3 = 1$

$$\Pi_1 = \frac{\Pi_1}{2} + \frac{\Pi_2}{4} \longrightarrow \Pi_1 = \frac{\Pi_2}{2}$$

$$\Pi_2 = \frac{\Pi_1}{2} + \frac{\Pi_2}{2} + \frac{\Pi_3}{3} \qquad \Pi_2 = \frac{3\Pi_2}{4} + \frac{\Pi_3}{3}$$

$$\Pi_3 = \frac{\Pi_2}{4} + \frac{2\Pi_3}{3} \qquad \Pi_3 = \frac{3\Pi_2}{4}$$

$$\Pi_1 + \Pi_2 + \Pi_3 = 1 \longrightarrow \Rightarrow \frac{\Pi_2}{2} + \frac{3\Pi_2}{4} + \Pi_2 = 1$$

$$\Pi_2\left[\frac{1}{2} + \frac{3}{4} + 1\right] = 1$$

$$\Pi_2\left[\frac{3}{2} + \frac{3}{4}\right] = 1$$

$\Pi_2 = 4/9. \Rightarrow$ company stays in avg performance for $4/9^{th}$ of time //

(b) We need to find $P(X^m = S)$. where.

$P(X^m = j)$ is probability of state in j in m^{th} year

given $P(X^1 = S) = 1$ $S \to$ strong

$P(X^1 = A) = 0$ $A \to$ average

$P(X^1 = W) = 0$ $W \Rightarrow$ weak.

$$P(X^2 = S) = P(X^1 = S) \cdot P_{SS} + P(X^1 = A) \cdot P_{AS} + P(X^1 = W) P_{WS}.$$

$$= 1 \cdot P_{SS} + 0 + 0$$

$$= \frac{1}{2}$$

Similarly $P(X^m = S) = P(X^{m-1} = S) \cdot P_{SS} + P(X^{m-1} = A) \cdot P_{AS} +$

$$P(X^{m-1} = W) P_{WS}$$

$$= \frac{P(X^{m-1} = S)}{2} + \frac{P(X^{m-1} = A)}{4} + P(X^{m-1} = W) \cdot 0$$

$$= \frac{P(X^{m-1} = S)}{2} + \frac{P(X^{m-1} = A)}{4}$$

since we need to find $P(X^m = S)$. $P(X^{m-1} = A) = 0$

continuous state strong

$$\Rightarrow P(X^m = S) = \frac{P(X^{m-1} = S)}{2}.$$

$$\Rightarrow P(X^m = S) = P(X^{m-2} = S) \left(\frac{1}{2}\right)^2$$

$$= P(X^{m-(m-1)} = S) \left(\frac{1}{2}\right)^{m-1}$$

$$= P(X^1 = S) \left(\frac{1}{2}\right)^{m-1} = \left(\frac{1}{2}\right)^{m-1}$$

Average no. of years company stays in strong state

is $E[P(X^m = S)] = \sum_{m=0}^{\infty} m \cdot \left(\frac{1}{2}\right)^{m-1}$

$$= \sum_{m=0}^{\infty} (m-1+1)\left(\frac{1}{2}\right)^{m-1}$$

$$= \sum_{m=0}^{\infty} (m-1)\left(\frac{1}{2}\right)^{m-1} + \sum_{m=0}^{\infty}\left(\frac{1}{2}\right)^{m-1}$$

$$= \frac{1/2}{(1-1/2)^2} + \frac{1}{1-1/2}$$

$$= \frac{1/2}{(1/2)^2} + \frac{1}{1/2} = 4$$

So, company stays in strong state on average of 4 years with initial state being strong.

(c) $\pi^0 = \begin{array}{ccc} S & A & W \\ \end{array}$ $[0 \quad 1 \quad 0]$ (2010)

$$P = \begin{array}{c|ccc} & S & A & W \\ S & 1/2 & 1/2 & 0 \\ A & 1/4 & 1/2 & 1/4 \\ W & 0 & 1/3 & 2/3 \end{array}$$

in 2012 $\rightarrow \pi^2 = \pi^0 . P^2$

$$= [0 \quad 1 \quad 0] \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/3 & 2/3 \end{bmatrix}$$

$$= [0.25 \quad 0.50 \quad 0.25] \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/3 & 2/3 \end{bmatrix}$$

$$= [0.\cancel{2225}\,25 \quad 0.\cancel{4575}\,4575 \quad 0.25]$$

$\Rightarrow$ company ~~reach~~ strong state probability = ~~0.2225~~ 0.25