2.25 a) $E[\text{call arrival}] = 1\text{ hr} = \frac{1}{\lambda}$ or

$$\Rightarrow \lambda = \frac{1}{60}$$

$E[\text{service time}] = 6\text{ min} \Rightarrow \mu = \frac{1}{6}$.

states:

0 → phone idle

1 → phone busy

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \end{tikzcd}$$

$P_0 = P(\text{phone idle})$

$P_1 = P(\text{phone busy})$

in steady state:

$$P_1 = \frac{\lambda}{\mu} P_0$$

and $P_1 + P_0 = 1$

Solving for P_0:

$$P_0 + \frac{\lambda}{\mu} P_0 = 1$$

$$P_0\left(1 + \frac{\lambda}{\mu}\right) = 1$$

$$P_0\left(1 + \frac{1/60}{1/6}\right) = 1$$

$$P_0\left(1 + \frac{6}{60}\right) = 1$$

$$P_0 = \frac{1}{\frac{66}{60}} = \frac{10}{11}$$

$P(\text{phone idle}) = 10/11$ $\quad P(\text{phone busy}) = \frac{1}{11}$

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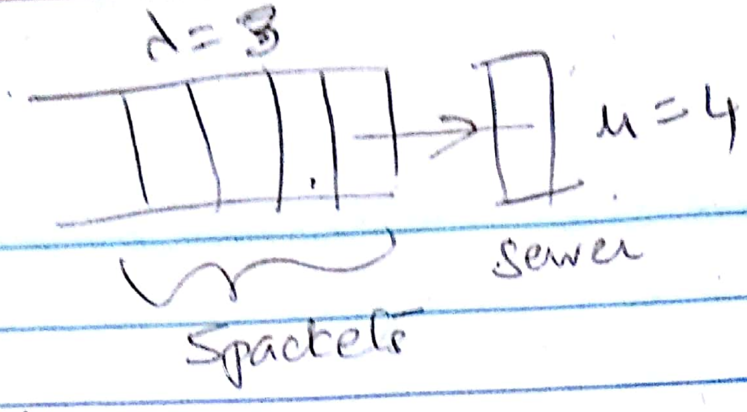
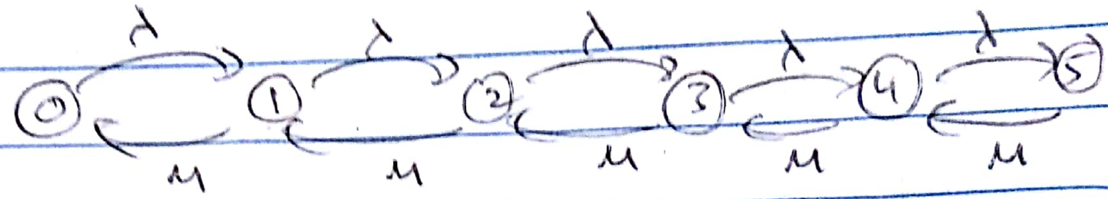
(b)



States :



Steady State Probabilities :

$$P_1 = \frac{\lambda}{\mu} P_0$$

$$P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$P_3 = \left(\frac{\lambda}{\mu}\right)^3 P_0$$

$$P_4 = \left(\frac{\lambda}{\mu}\right)^4 P_0$$

$$P_5 = \left(\frac{\lambda}{\mu}\right)^5 P_0$$

and $P_0 + P_1 + P_2 + P_3 + P_4 + P_5 = 1$

$$\Rightarrow \quad P_0\left(1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3 + \left(\frac{\lambda}{\mu}\right)^4 + \left(\frac{\lambda}{\mu}\right)^5\right) = 1$$

$$P_0 = \frac{1}{1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \left(\frac{3}{4}\right)^5}$$

$$= \frac{1}{3.28} = 0.3048$$

utilization: fraction of time server is busy

$$= \sum_{i=0}^{5} P(\text{server busy} / N=i) \cdot P(N=i)$$

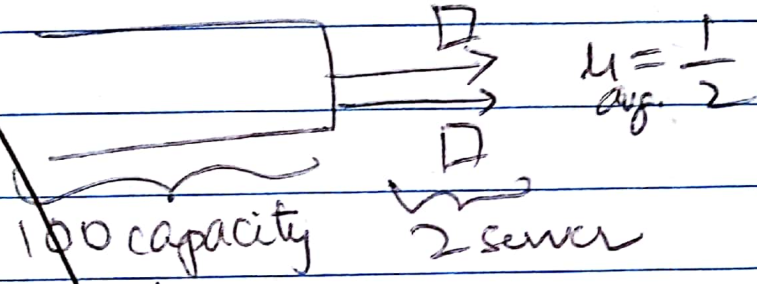
$$= 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 + \cdots + 5 \cdot P_5$$

$$= 1 - P_0$$

$$= 1 - 0.3048$$

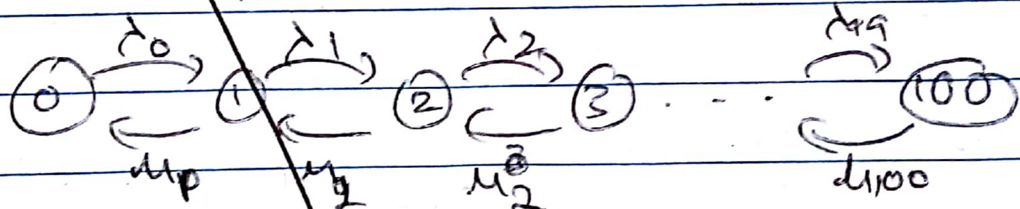
$$= 0.6952$$

(c) 100 phones fed in 2 lines



$\lambda_{avg} = \frac{1}{60}$

States:



$$\lambda_0 = \lambda_1 \cdots \lambda_{99} = \lambda = \frac{1}{60}$$

$$\mu_1 = \mu_2 = \cdots = \mu_{100} = \frac{1}{2}$$

Steady State Probabilities:

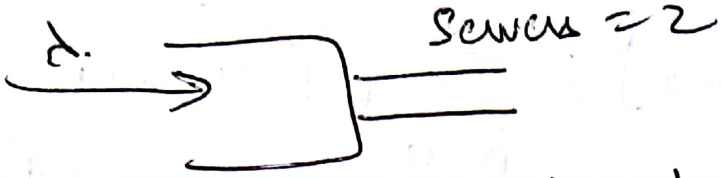
$$P_1 = \left(\frac{\lambda}{\mu}\right) P_0$$

$$P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0 \qquad \Rightarrow P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

$$\Rightarrow \sum_{n=0}^{100} P_n = 1$$

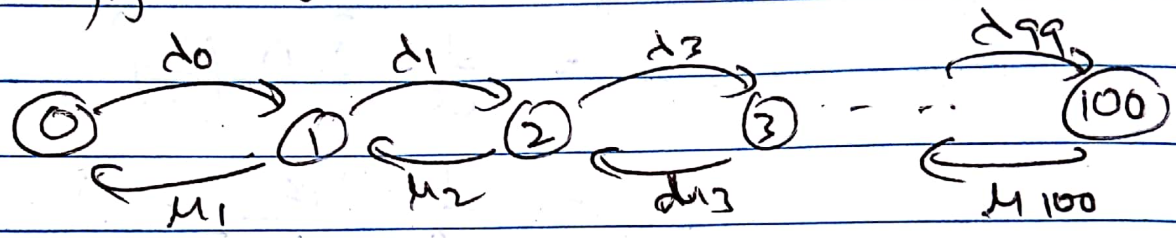
$$\Rightarrow P_0 \left(\sum_{n=0}^{100} \left(\frac{\lambda}{\mu}\right)^n \right) = 1$$

2.25 c:



population capacity = 100 $\quad \lambda_{avg} = \frac{1}{60} \quad \mu_{avg} = \frac{1}{2}$.

States: let 'n' be the no. of calls in the system trying to use asian call lines.



transition rates:

$\lambda_0 = 100\lambda \qquad \mu_1 = \mu$

$\lambda_1 = 99\lambda \qquad \mu_2 = 2\mu$

$\lambda_2 = 98\lambda \qquad \mu_3 = 2\mu$

$\vdots \qquad \vdots$

$\lambda_{99} = \lambda \qquad \mu_{100} = 2\mu$.

$\lambda_{100} = 0$

Steady State Probabilities:

$$P_1 = \frac{100\lambda}{\mu} P_0$$

$$P_2 = \frac{99 \cdot 100}{2\mu^2} \lambda^2 P_0$$

$$P_3 = \frac{98 \cdot 99 \cdot 100}{2^2} \cdot \frac{\lambda^3}{\mu^3} P_0$$

$$P_k = (100-k+1)(100-k+2) \ldots 99 \cdot 100 \left(\frac{\lambda}{\mu}\right)^k P_0$$

$$= 100 P_k \left(\frac{\lambda}{\mu}\right)^k P_0 \,/\, 2^{k-1}$$

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$$\sum_{k=0}^{\alpha} P_k = 1$$

$$P_0 \sum_{k=0}^{\alpha} \left[100 \rho_k \left(\frac{\lambda}{\mu}\right)^k \right] = 1$$

$$P_0 = \frac{1}{\sum_{k=0}^{\cancel{100}} 100 \rho_k \left(\frac{\lambda}{\mu}\right)^k / 2^{k-1}} = \frac{1}{2 \sum_{k=0}^{\alpha} 100 \rho_k \left(\frac{\lambda}{2\mu}\right)^k}$$

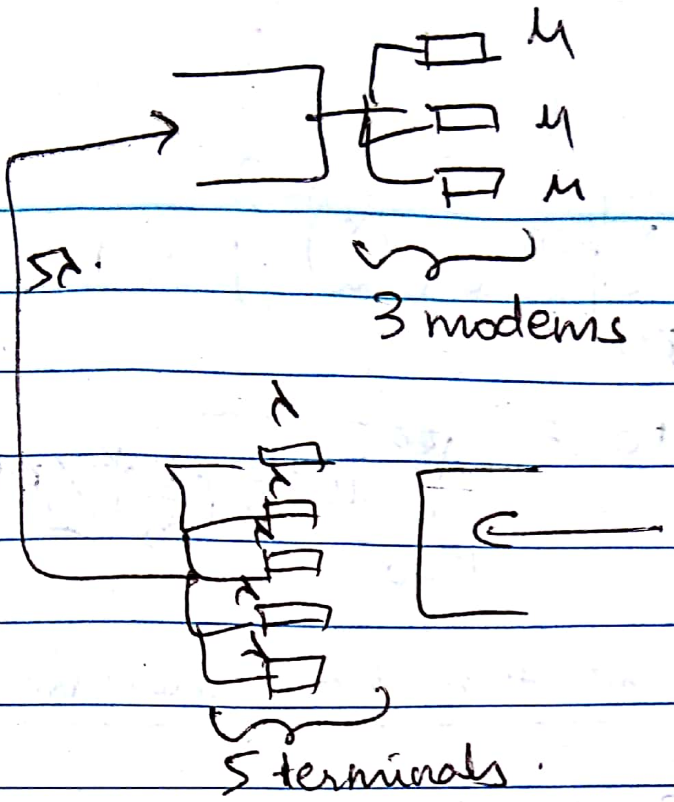
<u>fraction of time that both lines are busy:</u>

= 1 − [fraction time that zero lines are busy + fraction time that one line is busy]

$$= 1 - \left[1 \times P_0 + \frac{1}{2} \cdot P_1\right]$$

$$= 1 - \frac{1}{2 \sum_{k=0}^{100} 100 \rho_k \left(\frac{\lambda}{2\mu}\right)^k} - \frac{100\lambda}{2\mu} \left(\frac{1}{2 \sum_{k=0}^{100} 100 \rho_k \left(\frac{\lambda}{2\mu}\right)^k} \right)$$

2.27a:



→ State variable 'n' is the no. of active modems.

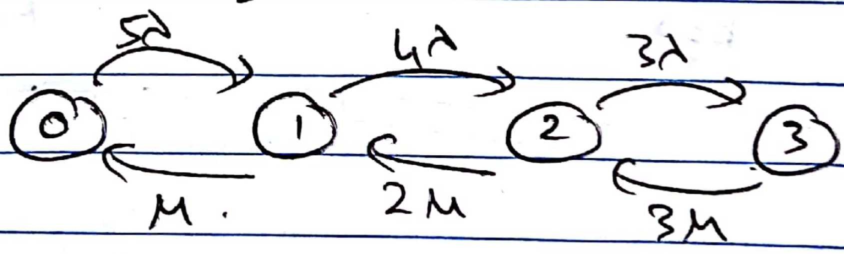
State description:

$\lambda_0 = 5\lambda \qquad \mu_1 = \mu$

$\lambda_1 = 4\lambda \qquad \mu_2 = 2\mu$

$\lambda_2 = 3\lambda \qquad \mu_3 = 3\mu$

$\lambda_3 = 0$



Steady State Probabilities:

$$P_1 = \frac{5\lambda}{\mu} P_0$$

$$P_2 = \left(\frac{4\lambda}{2\mu}\right)\left(\frac{5\lambda}{\mu}\right) P_0$$

$$P_3 = \left(\frac{3\lambda}{3\mu}\right)\left(\frac{4\lambda}{2\mu}\right)\left(\frac{5\lambda}{\mu}\right) P_0.$$

IIb) $P_0 + P_1 + P_2 + P_3 = 1$

$$\Rightarrow \quad P_0 \left[1 + \frac{5\lambda}{\mu} + \frac{10\lambda^2}{\mu^2} + \frac{10\lambda^3}{\mu^3} \right] = 1$$

$$\Rightarrow \quad P_0 = \frac{1}{1 + \frac{5\lambda}{\mu} + \frac{10\lambda^2}{\mu^2} + \frac{10\lambda^3}{\mu^3}}$$

$$P_1 = \frac{\frac{5\lambda}{\mu}}{1 + \frac{5\lambda}{\mu} + \frac{10\lambda^2}{\mu^2} + \frac{10\lambda^3}{\mu^3}}$$

$$P_2 = \frac{\frac{10\lambda^2}{\mu^2}}{1 + \frac{5\lambda}{\mu} + \frac{10\lambda^2}{\mu^2} + \frac{10\lambda^3}{\mu^3}}$$

$$P_3 = \frac{\frac{10\lambda^3}{\mu^3}}{1 + \frac{5\lambda}{\mu} + \frac{10\lambda^2}{\mu^2} + \frac{10\lambda^3}{\mu^3}}.$$

① 2.25 ⓓ infinite no. of servers.

$$\frac{\lambda}{\mu} = 0.7$$

$$\begin{tikzcd}
0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "2\mu"] \arrow[r, bend left, "\lambda"] & 3 \arrow[l, bend left, "3\mu"] & \cdots
\end{tikzcd}$$

Steady State Probabilities:

$$P_1 = P_0 \cdot \frac{\lambda}{\mu}$$

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0$$

$$\text{and } \sum_{n=0}^{\infty} P_n = 1$$

$$\Rightarrow \quad P_0 = \frac{1}{\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n}$$

$$= \frac{1}{e^{\lambda/\mu}} = e^{-\lambda/\mu}$$

fraction of time more than 2 server busy = 1 − [fraction of time 0 server busy + fraction of time 1 server busy + fraction of time 2 server busy].

$$= 1 - \left[\sum_{n=0}^{\infty} P(0 \text{ server busy} \mid N=n)P(N=n) + \sum_{n=0}^{\infty} P(1 \text{ server busy} \mid N=n)P(N=n) + \sum_{n=0}^{\infty} P(2 \text{ server busy} \mid N=n)P(N=n)\right]$$

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$$= 1 - \left[P_0 + \frac{1}{\infty} P_1 + \frac{2}{\infty} P_2 \right]$$

$$\cong 1 - P_0$$

$$\cong 1 - e^{-\lambda/\mu}$$

given $\frac{\lambda}{\mu} = 0.7$

$\Rightarrow$ fraction of time more than 2 servers busy

$$= 1 - e^{-0.7}$$

$$= 1 - 0.4965$$

$$= 0.5035$$

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2.39a: M/D/1 queing system

$$\lambda = 10$$

Deterministic system :

$$E[S] = 20$$

$$\rho = \lambda \cdot E[S]$$

$$= 10 \times 20 = 200$$

and for deterministic systems $\sigma^2 = 0$.

$$\Rightarrow E[N] = \frac{2\rho + \lambda^2\sigma^2 - \rho^2}{2 - 2\rho}$$

$$= \frac{2\rho + 0 - \rho^2}{2 - 2\rho}$$

$$= \frac{2\rho - \rho^2}{2(1-\rho)} = \frac{2\rho(\cancel{1-\rho})}{\cancel{2(1-\rho)}}$$

$$= \frac{\rho + \rho - \rho^2}{2(1-\rho)}$$

$$= \frac{\rho + \rho(1-\rho)}{2(1-\rho)}$$

$$= \frac{\rho}{2(1-\rho)} + \frac{\rho}{2}$$

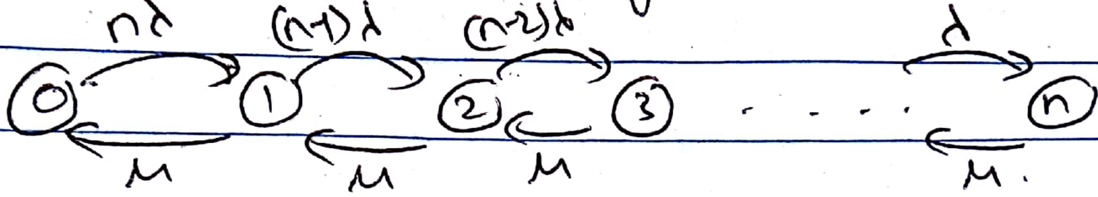
$$= 100 + \frac{200}{2(1-200)}$$

$$= 100 - \frac{200}{199} = 99.497$$

② factory has 'n' machines.

$$\lambda = \frac{1}{45} \quad \mu = \frac{1}{4}$$

<u>States</u>: $N(t) = n \Rightarrow$ no. of machines in the repair center any time.



$$\lambda_0 = n\lambda \qquad \mu_1 = \mu$$
$$\lambda_1 = (n-1)\lambda \qquad \mu_2 = \mu$$
$$\lambda_2 = (n-2)\lambda \qquad \mu_3 = \mu$$
$$\vdots \qquad \vdots$$
$$\lambda_{n-1} = \lambda$$
$$\lambda_n = 0 \qquad \mu_n = \mu$$

<u>Steady State Equations</u>:

$$P_1 = \left(\frac{n\lambda}{\mu}\right) P_0$$

$$P_2 = (n-1)(n)\left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$P_3 = (n-2)(n-1)(n)\left(\frac{\lambda}{\mu}\right)^3 P_0$$

$$\Downarrow$$

$$P_k = {}^nP_k \left(\frac{\lambda}{\mu}\right)^k P_0$$

$$\Rightarrow \sum_{k=0}^{n} P_k = 1$$

$$\Rightarrow P_0 \sum_{k=0}^{n} {}^nP_k \left(\frac{\lambda}{\mu}\right)^k = 1$$

$$\Rightarrow P_0 = \frac{1}{\sum_{k=0}^{n} {}^nP_k \left(\frac{\lambda}{\mu}\right)^k} = \frac{1}{\sum_{k=0}^{n} {}^nP_k \left(\frac{4}{45}\right)^k}$$

For atleast 5 machines to be operational, no. of machines in the repairshop should not exceed $(n-6)$.

$$\Rightarrow \quad P_0 + P_1 + P_2 + \cdots + P_{n-6} = 0.95$$

$$\Rightarrow \quad 1 - \left[P_{n-5} + P_{n-4} + P_{n-3} + P_{n-2} + P_{n-1} + P_n\right] = 0.95$$

$$\Rightarrow \quad P_{n-5} + P_{n-4} + P_{n-3} + P_{n-2} + P_{n-1} + P_n = 0.05$$

$$P_0\left[{}^nP_{n-5}\left(\frac{\lambda}{\mu}\right)^{n-5} + {}^nP_{n-4}\left(\frac{\lambda}{\mu}\right)^{n-4} + {}^nP_{n-3}\left(\frac{\lambda}{\mu}\right)^{n-3} + {}^nP_{n-2}\left(\frac{\lambda}{\mu}\right)^{n-2} + {}^nP_{n-1}\left(\frac{\lambda}{\mu}\right)^{n-1} + {}^nP_n\left(\frac{\lambda}{\mu}\right)^{n}\right] = 0.05$$

'n' has to be chosen such that following equation is satisfied.

$$\Rightarrow \frac{1}{\sum_{k=0}^{n} {}^nP_k\left(\frac{4}{45}\right)^k}\left[{}^nP_{n-5}\left(\frac{4}{45}\right)^{n-5} + {}^nP_{n-4}\left(\frac{4}{45}\right)^{n-4} + {}^nP_{n-3}\left(\frac{4}{45}\right)^{n-3} + {}^nP_{n-2}\left(\frac{4}{45}\right)^{n-2} + {}^nP_{n-1}\left(\frac{4}{45}\right)^{n-1} + {}^nP_n\left(\frac{4}{45}\right)^{n}\right] = 0.05.$$

③ Given 2 processor system

arrival rate $= \lambda$ jobs per second

$\mu = \frac{1}{500m}$

<u>States</u>: let 'n' be the no. of jobs in the system

$P(N(t)=n)$ = probability of state being 'n' where 'n' is the no. of jobs in the system.

<u>transition rates</u>:

$\lambda_0 = \lambda$ $\qquad\qquad$ $\mu_1 = \mu$

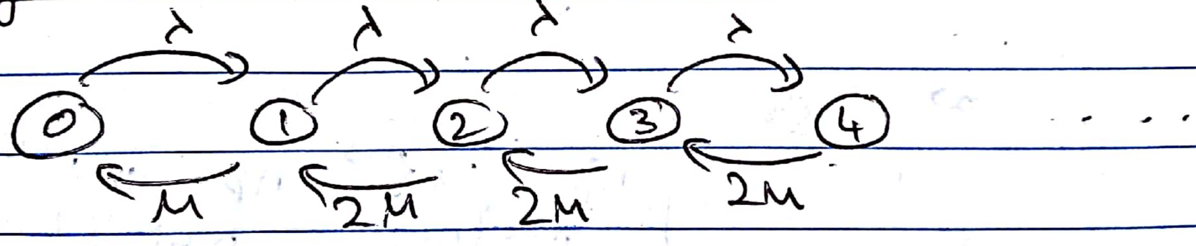
$\lambda_1 = \lambda$ $\qquad\qquad$ $\mu_2 = 2\mu$

$\vdots = \lambda$ $\qquad\qquad$ $\mu_3 = 2\mu$

$\lambda_n = \lambda$ $\qquad\qquad$ $\mu_n = 2\mu$.

$\vdots$ $\qquad\qquad\qquad$ $\vdots$

<u>State diagram</u>:



<u>Steady State Probabilities</u>:

$$P_1 = \frac{\lambda}{\mu} P_0$$

$$P_2 = \left(\frac{\lambda}{2\mu}\right)\left(\frac{\lambda}{\mu}\right) P_0 = \frac{1}{2}\frac{\lambda^2}{\mu^2} P_0$$

$$P_3 = \left(\frac{\lambda}{2\mu}\right)\left(\frac{\lambda}{2\mu}\right)\left(\frac{\lambda}{\mu}\right) P_0 = \frac{1}{2^2}\left(\frac{\lambda}{\mu}\right)^3 P_0$$

$$\Rightarrow \quad P_k = \left(\frac{1}{2^{k-1}}\right)\left(\frac{\lambda}{\mu}\right)^k P_0$$

$$\sum_{k=0}^{\alpha} P_k = 1$$

$$\Rightarrow \quad \sum_{k=0}^{\alpha} \left(\frac{1}{2^{k-1}}\right)\left(\frac{\lambda}{\mu}\right)^k P_0 = 1$$

$$P_0 = \frac{1}{\sum_{k=0}^{\alpha} \frac{1}{2^{k-1}}\left(\frac{\lambda}{\mu}\right)^k}$$

$$= \frac{1}{2\sum_{k=0}^{\alpha} \frac{1}{2^k}\left(\frac{\lambda}{\mu}\right)^k}$$

$$= \frac{1}{2\left(\frac{1-\cancel{0}}{1-\frac{\lambda}{2\mu}}\right)} \quad \forall \quad \frac{\lambda}{2\mu} < 1$$

$$= \frac{1}{2}\left(1-\frac{\lambda}{2\mu}\right)$$

$$\Rightarrow \quad P_k = \left(\frac{1}{2^{k-1}}\right)\left(\frac{\lambda}{\mu}\right)^k \frac{1}{2}\left(1-\frac{\lambda}{2\mu}\right) = \left(\frac{\lambda}{2\mu}\right)^k\left(1-\frac{\lambda}{2\mu}\right)$$

Condition for system to be stable :

$$\frac{\lambda}{2\mu} < 1 \Rightarrow \lambda < 2\mu$$

$$\lambda < 2 \times \frac{1}{500M}$$

$$\boxed{\lambda < \frac{1}{250M}}$$

(d) fraction of time a processor is busy:

$$= \sum_{n=0}^{\infty} P(\text{a processor busy} \mid N=n)\, P(N=n)$$

$$P(\text{a processor busy} \mid N=n) = (\text{fraction of resources used}) \times (\text{choosing the processor})$$

total resources = 3000 (1500×2)

$$P(\text{a processor busy} / N=0) = 0$$

$$P(\text{a processor busy} / N=1) = \frac{500}{3000} \times \frac{1}{2}$$

$$P(\text{a processor busy} / N=2) = \frac{2000}{3000} \times 1$$

$$P(\text{a processor busy} / N=3) = 1 \times 1$$

$$\vdots$$

fraction of time a processor is busy:

$$= 0P_0 + \frac{1}{12}P_1 + \frac{2}{3}P_2 + P_3 + P_4 + P_5 + \cdots$$

$$= \frac{1}{12}P_1 + \frac{2}{3}P_1 + (-P_0 - P_1 - P_2)$$

$$= 1 - P_0 - \frac{11}{12}P_1 - \frac{1}{3}P_2$$

where $P_0 = \frac{1}{2}\left(1 - \frac{\lambda}{2\mu}\right)$ $P_1 = \left(\frac{\lambda}{2\mu}\right)\left(1 - \frac{\lambda}{2\mu}\right)$ $P_2 = \left(\frac{\lambda}{2\mu}\right)^2\left(1 - \frac{\lambda}{2\mu}\right)$

(4) $\lambda = 4/hr = 4/60$ min

geometric distribution for server:

$$pmf(server) = P(n) = \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{n-1}$$

(a) $$E[S] = \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{n-1}$$

$$= \frac{1}{1/5} = 5$$

$$\sigma^2 = \left.\frac{1-q}{q^2}\right|_{q=\frac{1}{5}} = \frac{1-\frac{1}{5}}{\left(\frac{1}{5}\right)^2} = \frac{4/\cancel{5}}{\left(1/5\right)^2}$$

$$= 20.$$

(b) $$E[N] = \frac{2\rho + \lambda^2\sigma^2 - \rho^2}{2-2\rho}$$

$$\rho = \lambda E[S] = \lambda 5 = 5\lambda.$$

$$\Rightarrow E[N] = \frac{10\lambda + \lambda^2 20 - 25\lambda^2}{2-2(5\lambda)}$$

$$= \frac{10\lambda - 5\lambda^2}{2-10\lambda}$$

$$= \frac{1\cancel{0} \times 4/6\cancel{0} - 5\cdot\left(\frac{4}{60}\right)^2}{2 - 1\cancel{0}\left(4/6\cancel{0}\right)} = \frac{\cancel{4}^{2}/\cancel{6}_{3} - \frac{5 \times \cancel{16}^{4}}{\cancel{3600}_{45}}}{4/3}$$

$$= \frac{2/3 - 1/45}{4/3} = 0.35$$

Ⓒ $\underline{E[\tau_q]}$:

$$E[\tau] = E[\tau_q] + E[\tau_s]$$

$$\Rightarrow E[\tau_q] = E[\tau] - E[\tau_s]$$

$$= \frac{E[N]}{\lambda} - E[S]$$

$$= \frac{0.35}{4/60} - \cancel{20}0.5$$

$$= \frac{\overset{15}{\cancel{60}} \times 0.\overset{7}{\cancel{35}}}{\cancel{4}} - \cancel{20}5.$$

$$= 5\cancel{25} - 5. = 0.25$$

ⓓ Let 'n' be the time required to service one customer.

$$\Rightarrow \quad P(n) = \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{n-1}$$

$$E[N(t)=k \mid t=n] = \sum_{k=0}^{\infty} k\, P(N(t)=k \mid t=n) \cdot P(t=n)$$

$$= \sum_{k=0}^{\infty} k \cdot \frac{(\lambda n)^k}{k!} e^{-\lambda n} \cdot \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{n-1}$$

$$= \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{n-1} E[N(t)=k]$$

$$= \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{n-1} \cdot \lambda n. = \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{n-1}\left(\frac{1}{15}\right)n$$

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$$E[N(t)=k] = \sum_{n=0}^{\infty} E[N(t)=k \mid t=n].$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{75}\right)\left(\frac{4}{5}\right)^{n-1} \cdot n.$$

$$= \frac{8}{4} \times \frac{1}{\cancel{75}_{15}} \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^{n} n.$$

$$= \frac{1}{60} \sum_{n=0}^{\infty} n \cdot \left(\frac{4}{5}\right)^{n}.$$

$$= \frac{1}{\cancel{60}} \cdot \frac{4/\cancel{5}}{(1-4/5)^2} = \frac{1}{15} \cdot 5$$

$$= \frac{1}{3}.$$

(e) $P(N(t)=1 \mid t=n).$

$$= \frac{(\lambda n)^1 \times e^{-\lambda n}}{1}$$

$$= \lambda n e^{-\lambda n}.$$

$$P(N(t)=1) = \sum_{n=0}^{\infty} P(N(t)=1 \mid t=n) \; P(t=n)$$

$$= \sum_{n=0}^{\infty} \lambda n e^{-\lambda n} \cdot \left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{n-1}.$$

$$= \left(\frac{1}{15}\right)\left(\frac{1}{5}\right) \sum_{n=0}^{\infty} n e^{-\lambda n} \cdot (4/5)^{n-1}.$$

$$= \frac{1}{60} \sum_{n=0}^{\infty} n e^{-\lambda n} \cdot (4/5)^{n}.$$