CSC240 Lecture Notes

Max Xu

'25 Winter Semester

Contents

§1 Day 1: Course Admin & Predicate Logic (Jan. 4, 2025)

§1.1 About this Course

1. Material is Chapter 4 of MIT textbook "Mathematics for Computer Science", and Chapter 0 of the **CSC236/240** course notes.
2. **CSC240** is for students that are *"very good at mathematics"*[1], and comfortable with abstract concepts.

Topics are harder, covered in more detail, or may not have appeared at all in **CSC236**.

- Propositional and Predicate Logic
- Proofs
- Correctness and Analysis of Algorithms
- Language and Automata Theory

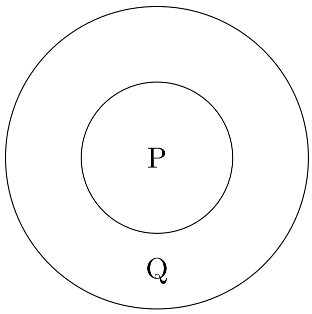
2 tutorials happen at the same time during tutorial hours, with one focused on giving examples and the other on "fun" and advanced topics. Email Prof. Ellen to make an appointment. Rigour is expected for all assignments. Quizzes are due 1 hour before the tutorial. Completing peer reviews is mandatory?

Problem 1.1

Write the following in predicate logic: You won't get a good mark in CSC240 (P) unless you hand in all of the homework assignments (Q).

If P is true (you got a good mark), then Q must be true (you handed in all homework). If P is false (you didn't get a good mark), Q can be either true or false (whether you handed in all homework or not doesn't determine P). Handing in all of the homework doesn't guarantee a good mark.[a]. The Euler diagram illustrates this fact.

[a] The connective for that is *if and only if*



Proposition 1.2

$\neg P$ UNLESS Q means the same thing as P IMPLIES Q.

Using English to express logical statements is both ambiguous and confusing. For this reason, we prefer using predicate logic instead.

[1] remember that 'good' is subjective... kinda

Problem 1.3

Consider a circuit with three Boolean inputs, x_0, x_1, and b, and three Boolean outputs, c, y_1, and y_0, where the string cy_1y_0 represents the sum of b and the number represented by the string x_1x_0.

$$\begin{array}{cccc} & & x_1 & x_0 \\ + & & & b \\ \hline & c & y_1 & y_0 \end{array}$$

Find the values of c, y_1, y_0 in terms of the inputs. Once you're done, find c and y_i.

When there's a carry into the next bit, both x_0 and b are 1. We can detect this through $x_0 \wedge b$. We know that if there is a carry, the y_0 must be zero, so $\neg(x_0 \wedge b)$ will likely be a part of the final expression. If both x_0 and b are zero, then y_0 must be 0. Otherwise, it will be a 1, and we can capture this using $x_0 \vee b$. When both of these conditions are satisfied, y_0 will be a 1, or written in predicate logic

$$(\neg(x_0 \wedge b)) \wedge (x_0 \vee b)$$

Convince yourself using a truth table that this is equivalent to $x_0 \oplus b$. Generalizing this, y_i can be found through the sum of x_i and the carry that comes before, which will be 1 only when $x_0, \ldots, x_{i-1}$ and b are 1. The carry c for the entire operation is subsequently $x_0 \wedge x_1 \wedge \cdots \wedge x_i \wedge b$.

Definition 1.4 (Associative Property)**.** A binary operator $\odot$ on a set S is called associative **if and only if**

$$(x \odot y) \odot z = x \odot (y \odot z), \forall x, y, z \in S$$

Over $S = \{\text{True, False}\}$, AND, OR, and XOR are associative, yet $\implies$ isn't. You would typically derive these facts using a truth table.

Problem 1.5. Write predicates : $\mathbb{R} \times \mathbb{Z} \to \{T, F\}$ such that for $x \in \mathbb{R}$ and $y \in \mathbb{Z}$,

- floor(x, y) is T *if and only if* y is the largest integer lesser than equal to x.
- round(x, y) is T *if and only if* y is the "closest"[2] integer to x.

Valid solutions include:

- For $x \in \mathbb{R}$ and $y \in \mathbb{Z}$, let $\text{floor}(x, y) = (y \leq x) \wedge (x < y + 1)d$.
- For $x \in \mathbb{R}$ and $y \in \mathbb{Z}$, let $\text{round}(x, y) = \forall z \in \mathbb{Z}, (|y - x| \leq |z - x|)$.

With this implementation of round, round$(0.5, 1)$ and round$(0.5, 0)$ are both true. As an exercise, suppose instead that round follows typical (yet more pathological) rounding rules, where $\forall a \in \mathbb{Z}, \text{round}(a + 0.5, b) \iff b = a + 1$. (Hint: prove that round(x, y) and floor$(x + 0.5, y)$ are equivalent)

[2] euclidean distance