INTERLACING OF ZEROS OF ODD PERIOD POLYNOMIALS

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Abstract. The work of Conrey, Farmer and Imamoglu shows that all nontrivial zeros of the odd period polynomial of a Hecke eigenform of level one lie on the unit circle. We further show that nontrivial zeros of odd period polynomials of Hecke eigenforms of different weights share certain interlacing property.

1. Introduction

Let $f \in S_k(1)$ be a normalized Hecke eigenform of weight $k \geq 12$ and level one. Suppose the Fourier expansion of f is given by $f(z) = \sum_{n\geq 1} a_f(n)q^n$. Let $L(f,s) = \sum_{n\geq 1} a_f(n)n^{-s}$ be its associated L-function and [4, (1.1)]

$$r_f^-(X) := \sum_{n=1,\, 2\nmid n}^{w-1} (-1)^{\frac{n-1}{2}} \binom{w}{n} n!(2\pi)^{-n-1} L(f, n+1) X^{w-n}$$

be the odd period polynomial of f, where $w = k-2$. It follows from the functional equation of $L(f,s)$ that [4, (2.2)]

$$r_f^-(X) = X^w r_f^-(1/X). \tag{1.1}$$

For convenience, we normalize the odd period polynomial as [4, (2.6)]

$$\begin{aligned} p_f^-(X) := \frac{2\pi r_f^-(X)}{(-1)^{w/2}(2\pi)^{-w}(w-1)!} &= \sum_{n=1,\, 2\nmid n}^{w-1} (-1)^{(n-1)/2} \frac{(2\pi X)^n}{n!} L(f, w-n+1) \\ &= \sum_{j=0}^{w/2-1} \frac{(-1)^j (2\pi X)^{2j+1}}{(2j+1)!} L(f, w-2j). \end{aligned}$$

Note that $p_f^-(X)$ shares the same zeros as $r_f^-(X)$ and the same functional equation (1.1).

In order to study the zeros of $p_f^-(X)$ we break $p_f^-(X)$ into higher and lower degree parts, and write [4, (2.7)]

$$p_f^-(X) = q_f(X) + X^w q_f(1/X), \tag{1.2}$$

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where the lower degree part is given by [4, (2.8)]

$$q_f(X) = \sum_{j=0}^{[(w-6)/4]} \frac{(-1)^j (2\pi X)^{2j+1}}{(2j+1)!} L(f, w-2j) + \frac{L(f,(w+2)/2)(2\pi X)^{w/2}}{2(w/2)!}, \tag{1.3}$$

and both are real polynomials.

For any Hecke eigenform $f \in S_k(1)$, Conrey, Farmer and Imamoglu [4] showed that, except for a fixed set of nine trivial zeros (counting multiplicity), the remaining $w-10$ nontrivial zeros of $r_f^-(f)$ are all located on the unit circle. Several generalizations of [4] have appeared since. For example, in the work of [8], it was shown that all zeros of the full period polynomials of newforms of level N lie on the circle $|z| = 1/\sqrt{N}$. When $N = 1$, this was also proved by El-Guindy and Raj [5]. Choi [2] showed the nontrivial zeros of odd period polynomials associated to newforms of level two all lie on the circle given by $|z| = 1/\sqrt{2}$, and in [3] demonstrated that almost all zeros of even period polynomials of level N lie on $|z| = 1/\sqrt{N}$.

This work is inspired by and closely follows [1], where interlacing properties of the full period polynomials associated to newforms were exploited. It should be noted that [1] was motivated by [11], [7] and [6], where it was shown that the zeros of Eisenstein series satisfy various interlacing properties.

We begin by recalling the definition of Stieltjes interlacing, following the exposition of [1, Definition 1.2]. Note that the notion of Stieltjes interlacing has its origin in the theory of orthogonal polynomials.

Definition 1.1. ([1, Definition 1.2]) Let $I \subset \mathbb{R}$ be an interval. Let $m, n \in \mathbb{N}$ with $m \geq n-1$. Suppose $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ are strictly increasing ordered sets of points in I. Then

(1) if there exists at least one element of X strictly between any two elements of Y, then X is said to Stieltjes interlace with Y, and
(2) if the condition (1) is satisfied and $x_1 < y_1$, $x_m > y_n$, then X is said to strongly Stieltjes interlace with Y.

Before presenting our results, we introduce the notion of sample angles on the upper half unit circle, following the ideas of [8, Section 6] and [1, Section 1].

Definition 1.2. Let $k = 2m+2$ with $m \geq 7$ and let $\theta_{k,\ell}$ denote the (unique) solution in the interval $[0, 2\pi)$ to the equation

$$m\theta - a(\theta) = \left(\ell + \frac{1}{2}\right)\pi \tag{1.4}$$

for $0 \leq \ell \leq m-6$. We call these solutions the sample angles.
We present the derivation of this equation in Section 2.2 and define $a(\theta)$ in (2.2).

We next introduce the notion of angles for zeros of $r_f^-(z)$ on the upper half unit circle.

Definition 1.3. Let $k = 2m + 2$ with $m \geq 7$. For $0 \leq \ell \leq m - 6$, we denote the nontrivial zeros of $r_f^-(z)$ on the upper unit circle by $e^{i\theta_{k,\ell}^*}$, where each $\theta_{k,\ell}^*$ denotes the solution angle closest to the sample angle $\theta_{k,\ell}$ in Definition 1.2. For each Hecke eigenform $f \in S_k(1)$, we denote the set of actual angles on the upper unit circle as

$$A_f := \{\theta_{k,\ell}^*\}_{\ell=0}^{m-6}.$$

Remark 1.4. (1) We only consider the nontrivial zeros on the upper half of the unit circle because (1.1) implies that if $e^{i\theta_{k,\ell}^*}$ is a zero, then so is $e^{i(2\pi-\theta_{k,\ell}^*)}$. We also exclude the trivial zeros at ± 1, corresponding to solutions $\ell = -1$ and $\ell = m - 5$ in (1.4), respectively; see Remark 2.4 for more detail.
(2) Later in Section 2.4 and Lemma 4.3 we will show that for $k \geq 98$ and $0 \leq \ell \leq m - 6$, the actual angle $\theta_{k,\ell}^*$ is well-defined and is indeed the closest actual angle to the sample angle $\theta_{k,\ell}$ given by (1.4). This demonstrates that the elements of A_f are well ordered by ℓ.

Our main goal is to establish the following interlacing property between the nontrivial zeros of odd period polynomials associated to Hecke eigenforms of different weights.

Theorem 1.5. *Suppose $k' > k \geq 16$. Let $f \in S_k(1)$ and $f' \in S_{k'}(1)$ be Hecke eigenforms. Then the set $A_{f'}$ strongly Stieltjes interlaces with A_f.*

Theorem 1.5 is achieved by combining the next three results.

Proposition 1.6. *Suppose $k' > k \geq 98$. Let $f \in S_k(1)$ and $f' \in S_{k'}(1)$ be Hecke eigenforms. Then the set $A_{f'}$ strongly Stieltjes interlaces with A_f.*

Proposition 1.7. *Suppose $16 \leq k < 98$, $k' \geq 108$. Let $f \in S_k(1)$ and $f' \in S_{k'}(1)$ be Hecke eigenforms. Then $A_{f'}$ strongly Stieltjes interlaces with A_f.*

Proposition 1.8. *Suppose $16 \leq k < 98$, $k' < 108$. Let $f \in S_k(1)$ and $f' \in S_{k'}(1)$ be Hecke eigenforms. Then $A_{f'}$ strongly Stieltjes interlaces with A_f.*

Note that Proposition 1.8 has only finitely many cases and is verified numerically in [9, check-int.py]. Hence, we only need to prove Proposition 1.6 and Proposition 1.7.

Using an argument similar to that of [1, Corollary 1.9], we arrive at a conclusion regarding the indivisibility between odd period polynomials.

Corollary 1.9. *Suppose $k' > k \geq 16$. Let $f \in S_k(1)$, $f' \in S_{k'}(1)$ be Hecke eigenforms. If $2k > k' + 10$, then $r_f^-(z) \nmid r_{f'}^-(z)$.*

Proof. Suppose to the contrary that $r_f^-(z) \mid r_{f'}^-(z)$. Then, we have $A_f \subset A_{f'}$. On the other hand, by the Stieltjes interlacing between $A_{f'}$ and A_f, $A_{f'}$ must have at least another $|A_f| + 1$

elements strictly between elements of A_f together with the two extreme points. Therefore, $|A_{f'}| \geq |A_f| + |A_f| + 1$, which is equivalent to $m' - 5 \geq 2(m-5)+1$, or $m' \geq 2m-4$. Thus, $k' \geq 2k-10$, a contradiction. $\square$

The main idea to prove the strong Stieltjes interlacing of zeros is as follows. We begin by explicitly bounding the distances between sample and actual angles, and use these bounds to demonstrate that the distance between any two consecutive elements of $A_{f'}$ is smaller than the distance between the corresponding two elements in A_f. This will demonstrate the Stieltjes interlacing between $A_{f'}$ and A_f. Finally, if the first and last element of $A_{f'}$ are, respectively, less than and greater than the first and last element of A_f, then we have the strong Stieltjes interlacing.

We now provide an outline of the paper. In Section 2.1 and Section 2.3, we recall facts about trivial zeros and approximations of odd period polynomials as given by [4], and we give a different proof for the fact that all nontrivial zeros of odd period polynomials lie on the unit circle. This different proof is of independent interest. In Section 2.2, we introduce the derivation of $a(\theta)$ mentioned in Definition 1.2. We establish explicit bounds for distances between the sample and actual angles in Section 2.4. Bounds and ordering of sample and actual angles are detailed in Section 3, and these are used throughout the paper to prove the interlacing property. Specifically, in Section 4 and Section 5, they are used to complete the proof of Proposition 1.6 and Proposition 1.7, respectively.

2. Zeros of odd period polynomials and their approximations

The goal of this section is two-fold. First, we will present a slightly different (or simpler) proof of [4, Theorem 2.8]. Furthermore, the methods involved will serve as the basis for the remainder of this paper. We begin by revisiting what is proven in [4] and recall some results therein. Then we derive a smooth representation of the argument of our sample function resulting in the equation (1.4) presented in Definition 1.2. Finally, we set up a bound for the distances between sample and actual angles. Throughout the way, we present a new proof of [4, Theorem 2.8].

2.1. **Approximating odd period polynomials**

In this section, we recall some facts about trivial zeros and an approximation of odd period polynomials. First, recall the following result from [4, Lemma 2.1] about the "trivial zeros" of $r_f^-(X)$.

Lemma 2.1 ([4, Lemma 2.1]). *Let $f \in S_k(1)$ be a Hecke eigenform. Then its odd period polynomial $r_f^-(X)$ has simple zeros at ± 2, $\pm 1/2$, 0, and double zeros at ± 1.*

The remaining $(w-1)-9 = w-10$ zeros are called the "nontrivial zeros" of $r_f^-(X)$. Theorem 1.1 (or Theorem 2.8) of [4] shows that all the nontrivial zeros are located on the unit circle. Our goal is to present a slightly different proof of this fact that suits our later application. First of all, by symmetry, it suffices to find $\frac{1}{2}(w-10) = m-5$ zeros in the upper half of the unit circle.

Recall $p_f^-(X)$ in (1.2) and let $k-2=w=2m$. The following function

$$X^{-m}p_f^-(X) = X^{-m}\left(q_f(X) + X^w q_f(1/X)\right) = X^{-m}q_f(X) + X^m q_f(1/X),$$

is real for $X = e^{i\theta}$ on the unit circle, and has the same zeros on the unit circle as $r_f^-(X)$.

On the other hand, since $X^{-m}p_f^-(X)$ is real on the unit circle, its zeros on the unit circle are exactly the zeros of the real part of $X^{-m}q_f(X)$ (or equivalently of $X^m q_f(1/X)$).

As seen from (1.3), $q_f(X)$ can be approximated by $\sin(2\pi X)$, so the function $X^{-m}p_f^-(X)$ can be approximated by

$$X^{-m}\sin(2\pi X) + X^m \sin(2\pi/X).$$

As

$$X^{-m}\sin(2\pi X) + X^m\sin(2\pi/X) = \mathrm{Re}(X^m\sin(2\pi/X))$$

for $X = e^{i\theta}$, the zeros of $X^{-m}\sin(2\pi X) + X^m\sin(2\pi/X)$ on the unit circle are the same as those of $\mathrm{Re}(X^m\sin(2\pi/X))$. Therefore, we need to take a closer look at the zeros of $\mathrm{Re}(X^m\sin(2\pi/X))$ on the unit circle, which are called the "sample zeros" or "sample angles".

2.2. **The smooth representation of the argument**

In this section, we derive the function $a(\theta)$ in Definition 1.2, which is a smooth representation of the argument of $\sin(2\pi X)$ for $X = e^{i\theta}$. For $X = e^{i\theta}$ direct computation reveals

$$\sin(2\pi X) = \sin(2\pi\cos(\theta))\frac{e^{2\pi\sin\theta} + e^{-2\pi\sin\theta}}{2} + i\cos(2\pi\cos(\theta))\frac{e^{2\pi\sin\theta} - e^{-2\pi\sin\theta}}{2}.$$

This means that the argument of $\sin(2\pi X)$ has its tangent given by

$$\frac{\cos(2\pi\cos(\theta))(e^{2\pi\sin\theta} - e^{-2\pi\sin\theta})}{\sin(2\pi\cos(\theta))(e^{2\pi\sin\theta} + e^{-2\pi\sin\theta})}. \tag{2.1}$$

To create a continuous representation of the argument of $\sin(2\pi X)$ in terms of θ, we define $a(\theta)$ by modifying the inverse tangent of (2.1) in the following way (see Figure 2.1):

$$a(\theta) = \begin{cases} \arctan\left(\frac{\cos(2\pi\cos(\theta))(e^{2\pi\sin\theta}-e^{-2\pi\sin\theta})}{\sin(2\pi\cos(\theta))(e^{2\pi\sin\theta}+e^{-2\pi\sin\theta})}\right) + \pi, & \text{for } 0 \le \theta < \frac{\pi}{3}, \\ \arctan\left(\frac{\cos(2\pi\cos(\theta))(e^{2\pi\sin\theta}-e^{-2\pi\sin\theta})}{\sin(2\pi\cos(\theta))(e^{2\pi\sin\theta}+e^{-2\pi\sin\theta})}\right) + 2\pi, & \text{for } \frac{\pi}{3} \le \theta < \frac{\pi}{2}, \\ \arctan\left(\frac{\cos(2\pi\cos(\theta))(e^{2\pi\sin\theta}-e^{-2\pi\sin\theta})}{\sin(2\pi\cos(\theta))(e^{2\pi\sin\theta}+e^{-2\pi\sin\theta})}\right) + 3\pi, & \text{for } \frac{\pi}{2} \le \theta < \frac{2\pi}{3}, \\ \arctan\left(\frac{\cos(2\pi\cos(\theta))(e^{2\pi\sin\theta}-e^{-2\pi\sin\theta})}{\sin(2\pi\cos(\theta))(e^{2\pi\sin\theta}+e^{-2\pi\sin\theta})}\right) + 4\pi, & \text{for } \frac{2\pi}{3} \le \theta \le \pi. \end{cases} \tag{2.2}$$

Here $\arctan(x)$ as usual takes values in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, such that $\arctan(\infty) = \frac{\pi}{2}$ and $\arctan(-\infty) = -\frac{\pi}{2}$. Notice that the shift by multiples of π in these equations is made to ensure that the resulted angle matches the argument of $\sin(2\pi X)$ in those subintervals.

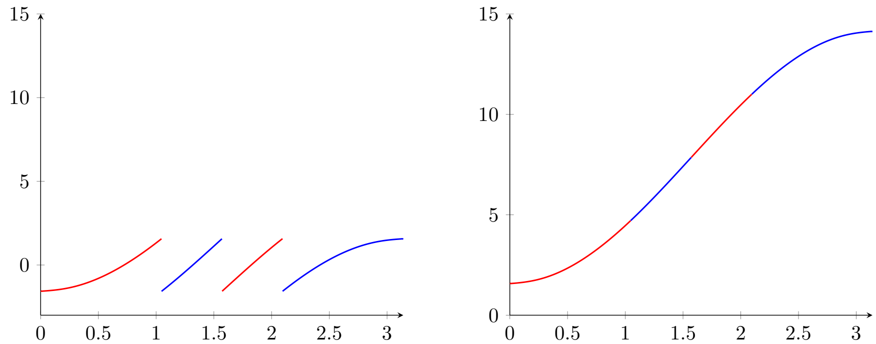

FIGURE 2.1. To account for the discontinuities of arctan of (2.1) (left), we let $\arctan(\pm\infty) = \pm\frac{\pi}{2}$ and shift vertically by factors of π as needed. This results in the smooth, continuous function $a(\theta)$ (right) as defined in (2.2).

The following result collects some important characteristics of the function $a(\theta)$.

Lemma 2.2. *On $[0, \pi]$, the functions $a(\theta)$ and $a'(\theta)$ are continuously differentiable and satisfy*

$$\frac{\pi}{2} \leq a(\theta) \leq 4.5\pi, \quad 0.5 \leq a'(\theta) < 6.5, \quad \text{and} \quad -2\pi < a''(\theta) < 2\pi.$$

Proof. We note that $a(0) = \frac{\pi}{2}$ and $a(\pi) = 4.5\pi$, $a''(\theta) > 0$ on $(0, \frac{\pi}{2})$, $a''(\pi/2) = 0$, and $a''(\theta) < 0$ on $(\frac{\pi}{2}, \pi)$. Furthermore, $a'(\theta)$ achieves a maximum (approx. 6.2833) at $\frac{\pi}{2}$ and minima ($= 0.5$) at 0 and π. These can be seen from straightforward but tedious calculus computations. $\square$

The argument of $X^m \sin(2\pi/X)$ for $X = e^{i\theta}$ is given by $m\theta - a(\theta)$ and satisfies the following.

Lemma 2.3. *When $m \geq 7$, or equivalently when $k \geq 16$, the function $m\theta - a(\theta)$ is strictly increasing on the interval $[0, \pi]$ and takes values in $[-\frac{1}{2}\pi, (m - 5 + \frac{1}{2})\pi]$.*

Proof. It follows from Lemma 2.2. $\square$

Remark 2.4. Observe that each zero for $\mathrm{Re}(X^m \sin(2\pi/X))$ on the upper unit circle occurs when the argument is of the form

$$m\theta - a(\theta) = \left(\ell + \frac{1}{2}\right)\pi,$$

where $-1 \leq \ell \leq m - 5$ is an integer (Lemma 2.3). Also note $\theta = 0, \pi$ are exactly given by $\ell = -1, m - 5$ in the above equation and correspond to when the radius is zero. We need to remove them in Definition 1.2 because they correspond to the trivial zeros ± 1.

2.3. **Zeros of odd period polynomials revisited**

In this section, we present a slightly different proof of [4, Theorem 2.8], by following more closely the argument in [8]. We write

$$Q_f(X) := X^m q_f(1/X), \tag{2.3}$$

whose real part has the same zeros as those of $p_f^-(X)$ (or equivalently $r_f^-(X)$). As done in the proof of [4, Lemma 2.5], we write

$$Q_f(X) = X^m \sin(2\pi/X) + S_1(X) + S_2(X) + S_3(X),$$

with

$$S_1(X) = X^m \sum_{j \le w/8} \frac{(-1)^j (2\pi/X)^{2j+1}}{(2j+1)!} \left(L(f, w-2j) - 1\right), \tag{2.4}$$

$$S_2(X) = X^m \sum_{w/8 < j \le [(w-6)/4]} \frac{(-1)^j (2\pi/X)^{2j+1}}{(2j+1)!} L(f, w-2j) + \frac{L(f, m+1)(2\pi)^m}{2(m!)}, \tag{2.5}$$

$$S_3(X) = -X^m \sum_{j > w/8} \frac{(-1)^j (2\pi/X)^{2j+1}}{(2j+1)!}. \tag{2.6}$$

For $X = e^{i\theta}$ on the unit circle, we have

$$|Q_f(X) - X^m \sin(2\pi/X)| \le |S_1(X)| + |S_2(X)| + |S_3(X)|. \tag{2.7}$$

We need to estimate these terms separately. First, recall the following corrected version of [4, Lemma 2.4]; see [5, Remark 3.3].

Lemma 2.5. *Let f be a normalized Hecke eigenform in $S_k(1)$ and let $L(f, s) = \sum_{n \ge 1} a_f(n) n^{-s}$ be its L-function. Then for real number $\sigma \ge \frac{3k}{4}$, we have*

$$|L(f, \sigma) - 1| \le 5 \cdot 2^{-k/4}$$

and for σ an integer with $\sigma \ge k/2$, we have

$$|L(f, \sigma)| \le 2\sqrt{k}\log(2k) + 1.$$

Next, we estimate the error terms (2.4), (2.5) and (2.6) for $X = e^{i\theta}$ on the unit circle.

Lemma 2.6. *For $|X| = 1$ and $k \ge 12$ we have*

$$|S_1(X)| + |S_2(X)| + |S_3(X)| < 5 \cdot 2^{-k/4} e^{2\pi} + (2\sqrt{k}\log(2k) + 2) \frac{(2\pi e)^{k/2+1}}{(k/2)^{k/2}(k/2 + 1 - 2\pi)}.$$

Proof. By Lemma 2.5, we get

$$|S_1(X)| \le \left(\sum_{j \le w/8} \frac{(2\pi)^{2j+1}}{(2j+1)!} \right) \cdot 5 \cdot 2^{-k/4} < 5 \cdot 2^{-k/4} e^{2\pi}.$$

For $S_2(X)$ we obtain

$$\begin{aligned} |S_2(X)| &< \sum_{w/8 < j < w/4} \frac{(2\pi)^{2j+1}}{(2j+1)!} L(f, w-2j) \\ &< (2\sqrt{k}\log(2k) + 1) \sum_{j > w/8} \frac{(2\pi)^{2j+1}}{(2j+1)!}. \end{aligned}$$

Also, clearly

$$|S_3(X)| \le \sum_{j>w/8} \frac{(2\pi)^{2j+1}}{(2j+1)!}.$$

Now, recall the following inequality at the bottom of [4, p. 4768]:

$$\sum_{j=r+1}^{\infty} \frac{x^j}{j!} < \frac{(ex)^{r+1}}{r^r(r+1-x)}.$$

This, together with the three estimates above, completes the proof. $\square$

For future applications, we define

$$B_k := 5 \cdot 2^{-k/4} e^{2\pi} + (2\sqrt{k}\log(2k)+2) \frac{(2\pi e)^{k/2+1}}{(k/2)^{k/2}(k/2+1-2\pi)}. \tag{2.8}$$

Now, we are ready to prove that all the nontrivial zeros of $r_f^-(X)$ are located on the unit circle.

Theorem 2.7 (cf. [4, Theorem 2.8])**.** *Let $f \in S_k(1)$ be a Hecke eigenform with $k \ge 52$. Then all $w-10$ nontrivial zeros of its odd period polynomial $r_f^-(X)$ lie on the unit circle.*

Proof. It suffices to locate $m-5$ zeros of $r_f^-(X)$ on the open upper unit circle. As observed above, $r_f^-(X)$ and the real part of $Q_f(X) = X^m q_f(1/X)$ share the same zeros on the unit circle. Also, by (2.7), the real part of $Q_f(X)$ for $X = e^{i\theta}$ is approximated by

$$\operatorname{Re}\left(X^m \sin\left(2\pi/X\right)\right) = \cos\left(m\theta - a(\theta)\right) \cdot \frac{1}{2}\sqrt{e^{4\pi\sin\theta} + e^{-4\pi\sin\theta} - 2\cos(4\pi\cos\theta)}, \tag{2.9}$$

where the first factor is the cosine of the argument of $X^m \sin(2\pi/X)$, and the second factor is the radius of $X^m \sin(2\pi/X)$. The argument function $m\theta - a(\theta)$ is monotonically increasing for $\theta \in [0,\pi]$ and takes every value in $[-\frac{\pi}{2}, (m-\frac{9}{2})\pi]$, see Lemma 2.3. In particular, $\cos(m\theta - a(\theta))$ takes values of ± 1 alternating at θ such that $m\theta - a(\theta) \in \{0, \pi, 2\pi, \cdots, (m-5)\pi\}$, which results in exactly $m-5$ sign changes in $\operatorname{Re}(X^m \sin(2\pi/X))$.

To warrant $m-5$ sign changes in $\operatorname{Re}(Q_f(X))$, we just need to make sure that the radius of $X^m \sin(2\pi/X))$ at each of these theta values is greater than the bound in Lemma 2.6 or B_k in (2.8). Since the radius function of $X^m \sin(2\pi/X)$ is symmetric about $\theta = \frac{\pi}{2}$ and is strictly increasing in $[0, \frac{\pi}{2}]$, it suffices to show

$$\frac{1}{2}\sqrt{e^{4\pi\sin x_1} + e^{-4\pi\sin x_1} - 2\cos(4\pi\cos x_1)} > B_k \tag{2.10}$$

for the smallest $x_1 \in [0,\pi]$ such that $mx_1 - a(x_1) = 0$.

First, by Lemma 2.2, we know $a'(x) \ge \frac{1}{2}$ and $a(0) = \frac{\pi}{2}$, implying $a(x) \ge \frac{x}{2} + \frac{\pi}{2}$ for $x \in [0,\pi]$. This, coupled with $0 = mx_1 - a(x_1)$, gives

$$x_1 \ge \frac{\pi}{2m-1} = \frac{\pi}{k-3}.$$

At the same time, since $a'(x) < 6.5$ and $a(0) = \frac{\pi}{2}$, we have $a(x) < 6.5x + \frac{\pi}{2}$ and thus

$$x_1 < \frac{\pi}{2(m - 6.5)} \leq \frac{\pi}{4}, \quad \text{for} \quad m \geq 9, \quad \text{or} \quad k \geq 20.$$

On the other hand, a direct calculus computation shows that for $x \in (0, \frac{\pi}{4}]$:

$$\frac{d}{dx}\left(\frac{1}{2}\sqrt{e^{4\pi \sin x} + e^{-4\pi \sin x} - 2\cos(4\pi \cos x)}\right) \geq 2\pi.$$

Therefore, since the radius at 0 is 0, we conclude that

$$\frac{1}{2}\sqrt{e^{4\pi \sin x_1} + e^{-4\pi \sin x_1} - 2\cos(4\pi \cos x_1)} \geq 2\pi x_1 \geq \frac{2\pi^2}{k-3}. \tag{2.11}$$

At last, we just need to note that for $k \geq 52$, we have

$$\frac{2\pi^2}{k-3} > 5 \cdot 2^{-k/4} e^{2\pi} + (2\sqrt{k}\log(2k) + 2)\frac{(2\pi e)^{k/2+1}}{(k/2)^{k/2}(k/2 + 1 - 2\pi)},$$

completing the proof of (2.10). □

For [4, Theorem 1.1], Theorem 2.7 together with numerical verification for $k \leq 50$ proves that all the nontrivial zeros of $r_f^-(X)$ for Hecke eigenform $f \in S_k(1)$ with $k \geq 12$ lie on the unit circle.

Remark 2.8. In a forthcoming work [10] we will employ the method developed above to study the location of zeros of even and odd period polynomials of newforms of any level and large weight.

2.4. **Distances between sample and actual angles**

For large weight k, we now establish an upper bound on the distances between the actual angles (Definition 1.3) and the sample angles (Definition 1.2). For $k \geq 98$, these bounds together with Lemma 3.2 show that the actual angle $\theta_{k,\ell}^*$ is indeed the closest element in A_f to the sample angle $\theta_{k,\ell}$, justifying Definition 1.3.

Recall that for $X = e^{i\theta}$, by (2.3) and Lemma 2.6 we have

$$|Q_f(X) - X^m \sin(2\pi/X)| = |E(X)| < B_k,$$

where $E(X) = S_1(X) + S_2(X) + S_3(X)$, and B_k is defined in (2.8).

Observe that by (2.9)

$$\mathrm{Re}(Q_f(X)) = \cos(m\theta - a(\theta)) \cdot \frac{1}{2}\sqrt{e^{4\pi \sin\theta} + e^{-4\pi \sin\theta} - 2\cos(4\pi\cos\theta)} + \mathrm{Re}(E(X)). \tag{2.12}$$

Define

$$\delta_k := \frac{(m - 0.5)B_k}{4\pi(m - 6.5)}.$$

If $k = 2m + 2 \geq 98$, then

$$0 < \delta_k \leq \frac{(1.1285 \times 10^{-4}) \cdot (m - 0.5)}{4\pi(m - 6.5)}.$$

Now we want to show that for all $0 \le \ell \le m-6$

$$|\theta_{k,\ell} - \theta^*_{k,\ell}| < \delta_k. \tag{2.13}$$

It is sufficient to show that if we plug $\theta_{k,\ell} \pm \delta_k$ into (2.12), there is a sign change.

At the same time, for $m \ge 7$ and $0 \le \ell \le m-6$, we know from the fact $m - a'(\theta) > 0$ and Lemma 3.2 part (3) that

$$\theta_{k,\ell} \ge \theta_{k,0} \ge \frac{\pi}{m-0.5}.$$

By (2.11),

$$\frac{1}{2}\sqrt{e^{4\pi \sin\theta_{k,\ell}} + e^{-4\pi\sin\theta_{k,\ell}} - 2\cos(4\pi\cos\theta_{k,\ell})} \ge 2\pi\theta_{k,\ell} \ge \frac{2\pi^2}{m-0.5}.$$

Thus, we find that $\cos(m\theta - a(\theta)) \cdot \frac{1}{2}\sqrt{e^{4\pi\sin\theta} + e^{-4\pi\sin\theta} - 2\cos(4\pi\cos\theta)}$ has the same sign as $\frac{2\pi^2}{m-0.5}\cos(m\theta - a(\theta))$ but has larger absolute value than the latter one. Consequently, it suffices to show that there is a sign change if we plug $\theta_{k,\ell} \pm \delta_k$ into

$$\frac{2\pi^2}{m-0.5}\cos(m\theta - a(\theta)) + \mathrm{Re}(E(X)).$$

We begin with $\theta_{k,\ell} + \delta_k$:

$$\begin{aligned}
&\frac{2\pi^2}{m-0.5} \cdot \cos\big(m(\theta_{k,\ell}+\delta_k) - a(\theta_{k,\ell}+\delta_k)\big)\\
=&\frac{2\pi^2}{m-0.5} \cdot \cos\big((m\theta_{k,\ell} - a(\theta_{k,\ell})) + m\delta_k + a(\theta_{k,\ell}) - a(\theta_{k,\ell}+\delta_k)\big)\\
=&\frac{2\pi^2}{m-0.5} \cdot \cos\left(\left(\ell + \frac{1}{2}\right)\pi + m\delta_k + a(\theta_{k,\ell}) - a(\theta_{k,\ell}+\delta_k)\right) \qquad \text{(by (1.4))}\\
=&\pm\frac{2\pi^2}{m-0.5} \cdot \sin\big(m\delta_k + a(\theta_{k,\ell}) - a(\theta_{k,\ell}+\delta_k)\big).
\end{aligned}$$

As $0.5 \le a'(\theta) < 6.5$, the mean value theorem yields $0.5\delta_k \le a(\theta_{k,\ell}+\delta_k) - a(\theta_{k,\ell}) < 6.5\delta_k$. This implies

$$(m-6.5)\delta_k < m\delta_k + a(\theta_{k,\ell}) - a(\theta_{k,\ell}+\delta_k) \le (m-0.5)\delta_k. \tag{2.14}$$

As $0 \le (m-0.5)\delta_k < \frac{\pi}{2}$ for $k \ge 98$, we can apply the fact that $\sin(x) \ge \frac{2}{\pi}x$ for $0 \le x < \frac{\pi}{2}$ to get

$$\begin{aligned}
&\left|\frac{2\pi^2}{m-0.5} \cdot \cos\big(m(\theta_{k,\ell}+\delta_k) - a(\theta_{k,\ell}+\delta_k)\big)\right| \qquad (2.15)\\
=&\frac{2\pi^2}{m-0.5} \cdot \sin\big(m\delta_k + a(\theta_{k,\ell}) - a(\theta_{k,\ell}+\delta_k)\big)\\
\ge&\frac{2\pi^2}{m-0.5} \cdot \frac{2}{\pi}(m-6.5)\delta_k\\
=&B_k > |E(X)| \ge |\,\mathrm{Re}(E(X))|.
\end{aligned}$$

This means that $\mathrm{Re}(Q_f(X))$ at $X = e^{i(\theta_{k,\ell}+\delta_k)}$ and $\cos\left(m(\theta_{k,\ell}+\delta_k) - a(\theta_{k,\ell}+\delta_k)\right)$ share the same sign. Repeating this process with $\theta_{k,\ell}-\delta_k$, we obtain

$$\begin{aligned}&\frac{2\pi^2}{m-0.5}\cdot\cos\big(m(\theta_{k,\ell}-\delta_k) - a(\theta_{k,\ell}-\delta_k)\big)\\ =&\frac{2\pi^2}{m-0.5}\cdot\cos\left(\left(\ell+\frac{1}{2}\right)\pi - m\delta_k + a(\theta_{k,\ell}) - a(\theta_{k,\ell}-\delta_k)\right)\\ =&\mp\frac{2\pi^2}{m-0.5}\cdot\sin\big(m\delta_k - a(\theta_{k,\ell}) + a(\theta_{k,\ell}-\delta_k)\big).\end{aligned}$$

The same argument as (2.14) reveals

$$(m-6.5)\delta_k < m\delta_k - a(\theta_{k,\ell}) + a(\theta_{k,\ell}-\delta_k) \le (m-0.5)\delta_k.$$

Thus, $\cos\big(m(\theta_{k,\ell}-\delta_k) - a(\theta_{k,\ell}-\delta_k)\big)$ and $\cos\big(m(\theta_{k,\ell}+\delta_k) - a(\theta_{k,\ell}+\delta_k)\big)$ have opposite signs, and (2.15) also holds true for $\theta_{k,\ell}-\delta_k$.

In summary, we have proved that there is a sign change in $\mathrm{Re}(Q_f(X))$ at $X = e^{i(\theta_{k,\ell}\pm\delta_k)}$, and hence completed the proof of (2.13).

Proposition 2.9. *Retaining the notation and assumptions of Definitions 1.2 and 1.3. If $k \ge 98$, then for all $0 \le \ell \le m-6$*

$$|\theta^*_{k,\ell} - \theta_{k,\ell}| < \frac{C(k)}{2^{k/4}}, \qquad \text{where} \qquad C(k) := \frac{2^{k/4}(m-0.5)B_k}{4\pi(m-6.5)}. \tag{2.16}$$

By considering the derivative of $C(k)$, we observe the following.

Lemma 2.10. *If $k \ge 16$, then $C(k) > C(k+2)$. Therefore, for $k \ge 98$, and $0 \le \ell \le m-6$*

$$|\theta^*_{k,\ell} - \theta_{k,\ell}| < \frac{C(98)}{2^{k/4}} < \frac{243.8700}{2^{k/4}}.$$

3. Bounds and ordering of sample and actual angles

In this section, we establish several bounds for various sample and actual angles.

3.1. Bounds and ordering of sample angles

We begin by considering the extreme sample zeros of different weights.

Lemma 3.1. *Suppose that $k' = 2m'+2 > k = 2m+2$ and $m \ge 7$. Then*

(1) $\theta_{k',0} < \theta_{k,0}$, and furthermore $\theta_{k,0} - \theta_{k',0} \ge \frac{(m'-m)\pi}{(m-0.5)m'}$.

(2) $\theta_{k,m-6} < \theta_{k',m'-6}$, and furthermore $\theta_{k',m'-6} - \theta_{k,m-6} \ge \frac{(m'-m)\pi}{(m-0.5)m'}$.

Proof. We only present the proof of part (1) because part (2) follows in a similar fashion. First, suppose on the contrary that $\theta_{k,0} \le \theta_{k',0}$. Applying (1.4) when k, k' and $\ell = \ell' = 0$, we get

$$m\theta_{k,0} - a(\theta_{k,0}) = m'\theta_{k',0} - a(\theta_{k',0}),$$

$$a(\theta_{k',0}) - a(\theta_{k,0}) = m(\theta_{k',0} - \theta_{k,0}) + (m' - m)\theta_{k',0}. \tag{3.1}$$

The assumption $\theta_{k,0} \le \theta_{k',0}$ paired with the mean value theorem and the fact $a'(\theta) < 6.5$ implies

$$a(\theta_{k',0}) - a(\theta_{k,0}) < 6.5(\theta_{k',0} - \theta_{k,0}).$$

Since $m \ge 7$, $m' - m > 0$ and $\theta_{k',0} > 0$, we obtain a contradiction. This shows that $\theta_{k',0} < \theta_{k,0}$. Now, since $a'(\theta) \ge 0.5$ (Lemma 2.2), the mean value theorem and (3.1) yield

$$m(\theta_{k,0} - \theta_{k',0}) + (m - m')\theta_{k',0} = a(\theta_{k,0}) - a(\theta_{k',0}) \ge 0.5(\theta_{k,0} - \theta_{k',0}),$$

which implies

$$\theta_{k,0} - \theta_{k',0} \ge \frac{(m' - m)\theta_{k',0}}{m - 0.5}. \tag{3.2}$$

Applying (1.4) to $k', \ell' = 0$ and noting that $a(\theta) \ge \pi/2$, we find

$$m'\theta_{k',0} = \frac{\pi}{2} + a(\theta_{k',0}) \ge \pi.$$

Combining this with (3.2), we obtain

$$\theta_{k,0} - \theta_{k',0} \ge \frac{(m' - m)\pi}{(m - 0.5)m'},$$

as desired. □

In the following lemma, we obtain bounds for the distances between consecutive sample angles.

Lemma 3.2. *Let $m \ge 7$. Then*
(1) for $0 \le \ell \le m - 7$, there exists some $\phi_{k,\ell} \in (\theta_{k,\ell}, \theta_{k,\ell+1})$ such that

$$\theta_{k,\ell+1} - \theta_{k,\ell} = \frac{\pi}{m - a'(\phi_{k,\ell})};$$

(2) for $0 \le \ell \le m - 7$, we have

$$\frac{\pi}{m - 0.5} \le \theta_{k,\ell+1} - \theta_{k,\ell} < \frac{\pi}{m - 6.5};$$

(3) we have

$$\frac{\pi}{m - 0.5} \le \theta_{k,0} < \frac{\pi}{m - 6.5}, \qquad \frac{\pi}{m - 0.5} \le \pi - \theta_{k,m-6} < \frac{\pi}{m - 6.5}.$$

Proof. (1) Taking the difference of (1.4) evaluated at $\theta_{k,\ell}$ and $\theta_{k,\ell+1}$, we obtain

$$m(\theta_{k,\ell+1} - \theta_{k,\ell}) - (a(\theta_{k,\ell+1}) - a(\theta_{k,\ell})) = \pi.$$

Thus,

$$m - \frac{\pi}{\theta_{k,\ell+1} - \theta_{k,l}} = \frac{a(\theta_{k,\ell+1}) - a(\theta_{k,\ell})}{\theta_{k,\ell+1} - \theta_{k,l}}. \tag{3.3}$$

Applying the mean value theorem to $a(\theta)$ and (3.3) yields the existence of $\phi_{k,\ell} \in (\theta_{k,\ell}, \theta_{k,\ell+1})$ such that

$$m - \frac{\pi}{\theta_{k,\ell+1} - \theta_{k,l}} = \frac{a(\theta_{k,\ell+1}) - a(\theta_{k,\ell})}{\theta_{k,\ell+1} - \theta_{k,l}} = a'(\phi_{k,\ell}).$$

(2) It follows from (1) and the fact that $0.5 \leq a'(\theta) < 6.5$ (Lemma 2.2).

(3) They follow from (2) by noticing that $\theta_{k,-1} = 0$ and $\theta_{k,m-5} = \pi$, see Remark 2.4. □

3.2. **Bounds and ordering of actual angles**

Now we compare the extreme actual angles of differing weights.

Lemma 3.3. *Let $f' \in S_{k'}(1)$, $f \in S_k(1)$ be Hecke eigenforms with $k' > k \geq 16$ (or $m \geq 7$). Then*

(1) if $\frac{(m'-m)\pi}{(m-0.5)m'} > \frac{C(k)}{2^{k/4}} + \frac{C(k')}{2^{k'/4}}$, then $\theta^*_{k',0} < \theta^*_{k,0}$ and $\theta^*_{k,m-6} < \theta^*_{k',m'-6}$;

(2) if $\frac{\pi}{m-0.5} > \frac{C(k)}{2^{k/4}}$, then $\theta^*_{k,0} > 0$ and $\theta^*_{k,m-6} < \pi$.

Proof. (1) By Lemma 3.1 part (1) and Proposition 2.9,

$$\begin{aligned}
\theta^*_{k,0} - \theta^*_{k',0} &= (\theta^*_{k,0} - \theta_{k,0}) + (\theta_{k,0} - \theta_{k',0}) + (\theta_{k',0} - \theta^*_{k',0}) \\
&\geq (\theta_{k,0} - \theta_{k',0}) - |\theta^*_{k,0} - \theta_{k,0}| - |\theta_{k',0} - \theta^*_{k',0}| \\
&> \frac{(m'-m)\pi}{(m-0.5)m'} - \frac{C(k)}{2^{k/4}} - \frac{C(k')}{2^{k'/4}} \\
&> 0.
\end{aligned}$$

Similarly, by Lemma 3.1 part (2) we have

$$\theta^*_{k',m'-6} - \theta^*_{k,m-6} > \frac{(m'-m)\pi}{(m-0.5)m'} - \frac{C(k)}{2^{k/4}} - \frac{C(k')}{2^{k'/4}} > 0.$$

(2) By Lemma 3.2 part (3) and Proposition 2.9 we obtain

$$\theta^*_{k,0} \geq \theta_{k,0} - |\theta^*_{k,0} - \theta_{k,0}| > \frac{\pi}{m-0.5} - \frac{C(k)}{2^{k/4}} > 0.$$

Likewise, Lemma 3.2 part (3), Proposition 2.9, and the assumption imply

$$\pi - \theta^*_{k,m-6} > \frac{\pi}{m-0.5} - \frac{C(k)}{2^{k/4}} > 0,$$

as desired. □

4. Interlacing of zeros for $k \geq 98$

In this section, we shall establish the strong Stieltjes interlacing for $k' > k \geq 98$, hence completing the proof of Proposition 1.6. We begin with the following observation.

Lemma 4.1. *For all $m' > m \geq 48$ with $k' = 2m' + 2$ and $k = 2m + 2$,*

$$\frac{2\pi^2}{(m-0.5)m'}\left(\frac{m'-m}{2\pi} - \frac{\pi}{m-6.5} - \frac{\pi}{m'-6.5}\right) - 3\cdot C(98)\left(\frac{1}{2^{k/4}} + \frac{1}{2^{k'/4}}\right) > 0.$$

Proof. Note that the left hand side of this inequality is increasing on m' for any fixed $m \geq 48$. Thus, it suffices to show its positivity for the case when $m' = m + 1$.

By calculations and choice of k, we know for $m = 48$, $m' = m + 1 = 49$ $(k = 98, k' = 100)$,

$$\frac{2\pi^2}{(m-0.5)(m+1)}\left(\frac{1}{2\pi} - \pi\left(\frac{1}{m-6.5} + \frac{1}{m-5.5}\right)\right) > 3\cdot C(98)\left(\frac{1}{2^{k/4}} + \frac{1}{2^{k+2/4}}\right). \tag{4.1}$$

Finally, we note that the left hand side of (4.1) decreases quadratically while the right hand side decreases exponentially. Thus, for all $m' > m \geq 48$, the desired inequality holds. □

Lemma 4.2. *For all $k' > k \geq 98$, or equivalently $m' > m \geq 48$, we have*

(1) $\theta_{k,0} - \theta_{k',0} > \frac{3C(98)}{2^{k/4}} + \frac{3C(98)}{2^{k'/4}}$,
(2) $\theta_{k',m'-6} - \theta_{k,m-6} > \frac{3C(98)}{2^{k/4}} + \frac{3C(98)}{2^{k'/4}}$,
(3) $\theta^*_{k',0} < \theta^*_{k,0}$, *and* $\theta^*_{k,m-6} < \theta^*_{k',m'-6}$.

Proof. (1) By Lemma 4.1, we have

$$\frac{2\pi^2}{(m-0.5)m'}\left(\frac{m'-m}{2\pi} - \frac{\pi}{m'-6.5} - \frac{\pi}{m-6.5}\right) > \frac{3C(98)}{2^{k/4}} + \frac{3C(98)}{2^{k'/4}},$$

implying that the first term on the left side satisfies

$$\frac{\pi(m'-m)}{(m-0.5)m'} > \frac{3C(98)}{2^{k/4}} + \frac{3C(98)}{2^{k'/4}}.$$

Now, Lemma 3.1 part (1) shows that

$$\theta_{k,0} - \theta_{k',0} \geq \frac{\pi(m'-m)}{(m-0.5)m'} > \frac{3C(98)}{2^{k/4}} + \frac{3C(98)}{2^{k'/4}}. \tag{4.2}$$

(2) Similarly, by Lemma 3.1 part (2), we see

$$\theta_{k,m'-6} - \theta_{k,m-6} \geq \frac{\pi(m'-m)}{(m-0.5)m'} > \frac{3C(98)}{2^{k/4}} + \frac{3C(98)}{2^{k'/4}}. \tag{4.3}$$

Part (3) is proved by applying Lemma 3.3 part (1) and using the fact $C(98) \geq C(k) > C(k')$ (Lemma 2.10), in conjunction with (4.2) and (4.3). □

The next lemma establishes the ordering of actual angles, justifying Definition 1.3.

Lemma 4.3. *Suppose $k \geq 98$, or equivalently $m \geq 48$. Then for $0 \leq \ell \leq m-7$, $\theta^*_{k,\ell+1} - \theta^*_{k,\ell} > 0$.*

Proof. By Lemma 3.2 part (2) and Proposition 2.9,

$$\begin{aligned}\theta^*_{k,\ell+1} - \theta^*_{k,\ell} &= (\theta^*_{k,\ell+1} - \theta_{k,\ell+1}) + (\theta_{k,\ell+1} - \theta_{k,\ell}) + (\theta_{k,\ell} - \theta^*_{k,\ell}) \\ &\geq (\theta_{k,\ell+1} - \theta_{k,\ell}) - |\theta^*_{k,\ell+1} - \theta_{k,\ell+1}| - |\theta_{k,\ell} - \theta^*_{k,\ell}| \\ &> \frac{\pi}{m-0.5} - \frac{2C(k)}{2^{k/4}},\end{aligned}$$

which is positive for $m \geq 48$. □

Following [1, Definition 5.4], we now define a set that will allow us to determine some $\theta_{k',\widehat{\ell}}$ that is both close to and less than $\theta_{k,\ell}$.

Definition 4.4. Suppose $m' > m \geq 48$. For each $0 \leq \ell \leq m-7$, we define the following set:

$$U_\ell := \left\{ 0 \leq \ell' \leq m' - 7 : \theta_{k',\ell'} < \theta_{k,\ell+1} - \frac{C(98)}{2^{k/4}} - \frac{C(98)}{2^{k'/4}} \right\}.$$

We define $\widehat{\ell}$ to be the largest element of U_ℓ.

Remark 4.5. From Lemma 4.2 part (1) and Lemma 3.2, for $m \geq 48$, we have

$$\theta_{k',0} < \theta_{k,0} - \frac{C(98)}{2^{k/4}} - \frac{C(98)}{2^{k'/4}} < \theta_{k,\ell+1} - \frac{C(98)}{2^{k/4}} - \frac{C(98)}{2^{k'/4}}.$$

That is, the set U_ℓ is nonempty and $\widehat{\ell}$ is well defined, and as $\widehat{\ell} \leq m'-7$, we know $\theta_{k',\widehat{\ell}+1}$ is defined.

Lemma 4.6. *Suppose $k' > k \geq 98$, or equivalently $m' > m \geq 48$. Then for all $0 \leq \ell \leq m-7$,*

$$\frac{C(98)}{2^{k/4}} + \frac{C(98)}{2^{k'/4}} + \frac{\pi}{m'-6.5} > \theta_{k,\ell+1} - \theta_{k',\widehat{\ell}} > \frac{C(98)}{2^{k/4}} + \frac{C(98)}{2^{k'/4}}.$$

Proof. From the definition of $\widehat{\ell}$, it is clear

$$\theta_{k,\ell+1} - \theta_{k',\widehat{\ell}} > \frac{C(98)}{2^{k/4}} + \frac{C(98)}{2^{k'/4}}.$$

To see the other side of the inequality, we suppose to the contrary that

$$\frac{C(98)}{2^{k/4}} + \frac{C(98)}{2^{k'/4}} + \frac{\pi}{m'-6.5} \leq \theta_{k,\ell+1} - \theta_{k',\widehat{\ell}}. \tag{4.4}$$

By Lemma 3.2 part (2) and our assumption (4.4), we arrive at

$$\theta_{k',\widehat{\ell}+1} - \theta_{k',\widehat{\ell}} < \frac{\pi}{m'-6.5} \leq \theta_{k,\ell+1} - \theta_{k',\widehat{\ell}} - \frac{C(98)}{2^{k/4}} - \frac{C(98)}{2^{k'/4}}.$$

Thus,

$$\theta_{k',\widehat{\ell}+1} < \theta_{k,\ell+1} - \frac{C(98)}{2^{k/4}} - \frac{C(98)}{2^{k'/4}}.$$

This shows $\widehat{\ell} + 1 \in U_\ell$, contradicting the maximality of $\widehat{\ell}$. The proof is complete. □

We now demonstrate this ordering for actual angles.

Lemma 4.7. *Suppose $k' > k \geq 98$. Then for all $0 \leq \ell \leq m-7$, we have $\theta^*_{k,\ell+1} > \theta^*_{k',\widehat{\ell}}$.*

Proof. Lemma 4.6 and the error bounds (2.16)

$$|\theta^*_{k,\ell} - \theta_{k,\ell}| < \frac{C(k)}{2^{k/4}}, \quad |\theta^*_{k',\ell'} - \theta_{k',\ell'}| < \frac{C(k')}{2^{k'/4}},$$

yield

$$\begin{aligned}
\theta^*_{k,\ell+1} - \theta^*_{k',\widehat{\ell}} &= (\theta^*_{k,\ell+1} - \theta_{k,\ell+1}) + (\theta_{k,\ell+1} - \theta_{k',\widehat{\ell}}) + (\theta_{k',\widehat{\ell}} - \theta^*_{k',\widehat{\ell}}) \\
&\geq (\theta_{k,\ell+1} - \theta_{k',\widehat{\ell}}) - \Big(|\theta^*_{k,\ell+1} - \theta_{k,\ell+1}| + |\theta_{k',\widehat{\ell}} - \theta^*_{k',\widehat{\ell}}|\Big) \\
&> \left(\frac{C(98)}{2^{k/4}} + \frac{C(98)}{2^{k'/4}}\right) - \left(\frac{C(98)}{2^{k/4}} + \frac{C(98)}{2^{k'/4}}\right) \\
&= 0,
\end{aligned}$$

as desired. □

The next two results will be used to prove the key Lemma 4.10.

Lemma 4.8. *Suppose $k' > k \geq 98$, or equivalently $m' > m \geq 48$. Then for all $0 \leq \ell \leq m - 7$,*

$$\theta_{k',\widehat{\ell}+1} - \theta_{k,\ell} < \frac{\pi}{m' - 6.5} + \frac{\pi}{m - 6.5} - \frac{C(98)}{2^{k/4}} - \frac{C(98)}{2^{k'/4}}.$$

Proof. Using Lemma 3.2 part (2) and Lemma 4.6 we obtain

$$\begin{aligned}
\theta_{k',\widehat{\ell}+1} - \theta_{k,\ell} &= (\theta_{k',\widehat{\ell}+1} - \theta_{k',\widehat{\ell}}) + (\theta_{k',\widehat{\ell}} - \theta_{k,\ell+1}) + (\theta_{k,\ell+1} - \theta_{k,\ell}) \\
&< \frac{\pi}{m' - 6.5} - \frac{C(98)}{2^{k/4}} - \frac{C(98)}{2^{k'/4}} + \frac{\pi}{m - 6.5},
\end{aligned}$$

completing the proof. □

Lemma 4.9. *Suppose $k' > k \geq 98$. Then for all $0 \leq \ell \leq m - 7$,*

$$(\theta_{k,\ell+1} - \theta_{k,\ell}) - (\theta_{k',\widehat{\ell}+1} - \theta_{k',\widehat{\ell}}) > \frac{2C(k)}{2^{k/4}} + \frac{2C(k')}{2^{k'/4}}.$$

Proof. First, by Lemma 3.2 part (1), for some $\phi_{k',\widehat{\ell}} \in (\theta_{k',\widehat{\ell}}, \theta_{k',\widehat{\ell}+1})$ and some $\phi_{k,\ell} \in (\theta_{k,\ell}, \theta_{k,\ell+1})$ we have

$$(\theta_{k,\ell+1} - \theta_{k,\ell}) - (\theta_{k',\widehat{\ell}+1} - \theta_{k',\widehat{\ell}}) = \frac{\pi}{m - a'(\phi_{k,\ell})} - \frac{\pi}{m' - a'(\phi_{k',\widehat{\ell}})}. \tag{4.5}$$

Additionally, we know from Lemma 2.2 that $a''(\theta) < 2\pi$, so

$$\frac{1}{2\pi}\Big(a'(\phi_{k',\widehat{\ell}}) - a'(\phi_{k,\ell})\Big) < |\phi_{k',\widehat{\ell}} - \phi_{k,\ell}|, \tag{4.6}$$

where

$$|\phi_{k',\widehat{\ell}} - \phi_{k,\ell}| < \max(\theta_{k,\ell+1}, \theta_{k',\widehat{\ell}+1}) - \min(\theta_{k,\ell}, \theta_{k',\widehat{\ell}}). \tag{4.7}$$

Recalling the following upper bounds from Lemma 3.2 part (2), Lemma 4.6, and Lemma 4.8

$$\begin{cases}\theta_{k,\ell+1}-\theta_{k,\ell} & <\frac{\pi}{m-6.5}, \quad \theta_{k',\widehat{\ell}+1}-\theta_{k',\widehat{\ell}}<\frac{\pi}{m'-6.5},\\ \theta_{k,\ell+1}-\theta_{k',\widehat{\ell}} & <\frac{\pi}{m'-6.5}+\frac{C(98)}{2^{k/4}}+\frac{C(98)}{2^{k'/4}},\\ \theta_{k',\widehat{\ell}+1}-\theta_{k,\ell} & <\frac{\pi}{m'-6.5}+\frac{\pi}{m-6.5}-\frac{C(98)}{2^{k/4}}-\frac{C(98)}{2^{k'/4}},\end{cases}$$

we obtain

$$\max(\theta_{k,\ell+1},\theta_{k',\widehat{\ell}+1})-\min(\theta_{k,\ell},\theta_{k',\widehat{\ell}})<\frac{\pi}{m'-6.5}+\frac{\pi}{m-6.5}+\frac{C(98)}{2^{k/4}}+\frac{C(98)}{2^{k'/4}}.$$

Combining this with (4.6) and (4.7) yields

$$\frac{1}{2\pi}\left(a'(\phi_{k',\widehat{\ell}})-a'(\phi_{k,\ell})\right)<\frac{\pi}{m'-6.5}+\frac{\pi}{m-6.5}+\frac{C(98)}{2^{k/4}}+\frac{C(98)}{2^{k'/4}},$$

and thus

$$(m'-a'(\phi_{k',\widehat{\ell}}))-\left(m-a'(\phi_{k,\ell})\right)>2\pi\left(\frac{m'-m}{2\pi}-\frac{\pi}{m'-6.5}-\frac{\pi}{m-6.5}-\frac{C(98)}{2^{k/4}}-\frac{C(98)}{2^{k'/4}}\right).$$

This, together with Lemma 4.1 and the fact $\frac{2\pi^2}{(m-0.5)m'}<1$ for $m\geq 48$, yields

$$\frac{2\pi^2}{(m-0.5)m'}\left(\frac{m'-m}{2\pi}-\frac{\pi}{m-6.5}-\frac{\pi}{m'-6.5}-\frac{C(98)}{2^{k/4}}-\frac{C(98)}{2^{k'/4}}\right)>\frac{2C(98)}{2^{k/4}}+\frac{2C(98)}{2^{k'/4}}.$$

Noting $a'(\theta)\geq 0.5$ and hence $(m-a'(\phi_{k,\ell}))(m'-a'(\phi_{k',\widehat{\ell}}))<(m-0.5)m'$, by (4.5) we obtain

$$\begin{aligned}&(\theta_{k,\ell+1}-\theta_{k,\ell})-(\theta_{k',\widehat{\ell}+1}-\theta_{k',\widehat{\ell}})\\ =&\frac{\pi}{m-a'(\phi_{k,\ell})}-\frac{\pi}{m'-a'(\phi_{k',\widehat{\ell}})}\\ =&\frac{\pi\left((m'-a'(\phi_{k',\widehat{\ell}}))-(m-a'(\phi_{k,\ell}))\right)}{(m-a'(\phi_{k,\ell}))(m'-a'(\phi_{k',\widehat{\ell}}))}\\ >&\frac{2\pi^2}{(m-a'(\phi_{k,\ell}))(m'-a'(\phi_{k',\widehat{\ell}}))}\left(\frac{m'-m}{2\pi}-\frac{\pi}{m'-6.5}-\frac{\pi}{m-6.5}-\frac{C(98)}{2^{k/4}}-\frac{C(98)}{2^{k'/4}}\right)\\ >&\frac{2\pi^2}{(m-0.5)m'}\left(\frac{m'-m}{2\pi}-\frac{\pi}{m'-6.5}-\frac{\pi}{m-6.5}-\frac{C(98)}{2^{k/4}}-\frac{C(98)}{2^{k'/4}}\right)\\ >&\frac{2C(98)}{2^{k/4}}+\frac{2C(98)}{2^{k'/4}},\end{aligned}$$

completing the proof. □

The next lemma shows that the distance between two consecutive elements of A_f is larger than that of the corresponding two elements in $A_{f'}$, which is key to establishing Stieltjes interlacing.

Lemma 4.10. *Suppose $k'>k\geq 98$. Then for all $0\leq\ell\leq m-7$, we have*

$$\theta^*_{k,\ell+1}-\theta^*_{k,\ell}>\theta^*_{k',\widehat{\ell}+1}-\theta^*_{k',\widehat{\ell}}.$$

Proof. From Lemma 4.9, we have

$$(\theta_{k,\ell+1} - \theta_{k,\ell}) - (\theta_{k',\widehat{\ell}+1} - \theta_{k',\widehat{\ell}}) > \frac{2C(k)}{2^{k/4}} + \frac{2C(k')}{2^{k'/4}}.$$

This, together with (2.16), implies

$$\begin{aligned}(\theta_{k,\ell+1} - \theta_{k,\ell}) - (\theta_{k',\widehat{\ell}+1} - \theta_{k',\widehat{\ell}}) &> (\theta_{k,\ell+1} - \theta^*_{k,\ell+1}) + (\theta^*_{k,\ell} - \theta_{k,\ell}) \\ &\quad + (\theta^*_{k',\widehat{\ell}+1} - \theta_{k',\widehat{\ell}+1}) + (\theta_{k',\widehat{\ell}} - \theta^*_{k',\widehat{\ell}}),\end{aligned}$$

giving the desired inequality. □

Finally, we are ready to prove Proposition 1.6, which states the strong Stieltjes interlacing between $A_{f'}$ and A_f for $k' > k \geq 98$.

Proof of Proposition 1.6. By Lemma 4.3, for all $0 \leq \ell \leq m-7$, we have $\theta^*_{k,\ell+1} - \theta^*_{k,\ell} > 0$. Suppose for some ℓ, no element from $A_{f'}$ lies in the interval $(\theta^*_{k,\ell}, \theta^*_{k,\ell+1})$. In particular, $\theta^*_{k',\widehat{\ell}} \notin (\theta^*_{k,\ell}, \theta^*_{k,\ell+1})$. By Lemma 4.7, we have $\theta^*_{k,\ell+1} > \theta^*_{k',\widehat{\ell}}$, so we must have $\theta^*_{k,\ell} \geq \theta^*_{k',\widehat{\ell}}$.

Now, Lemma 4.10 gives $\theta^*_{k,\ell+1} - \theta^*_{k,\ell} > \theta^*_{k',\widehat{\ell}+1} - \theta^*_{k',\widehat{\ell}}$, which implies $\theta^*_{k,\ell+1} > \theta^*_{k',\widehat{\ell}+1}$. Also, since $\theta^*_{k',\widehat{\ell}+1} \notin (\theta^*_{k,\ell}, \theta^*_{k,\ell+1})$ by our assumption, we get $\theta^*_{k,\ell} \geq \theta^*_{k',\widehat{\ell}+1}$. Putting all these together,

$$\theta^*_{k,\ell+1} > \theta^*_{k,\ell} \geq \theta^*_{k',\widehat{\ell}+1} > \theta^*_{k',\widehat{\ell}}. \tag{4.8}$$

On the other hand, it is straightforward to verify that

$$\frac{\pi}{m-0.5} - \frac{C(98)}{2^{k/4}} - \frac{C(98)}{2^{k'/4}} > \frac{C(98)}{2^{k/4}} + \frac{C(98)}{2^{k'/4}},$$

which by Lemma 3.2 part (2) and Proposition 2.9 implies

$$(\theta_{k,\ell+1} - \theta_{k,\ell}) + (\theta_{k,\ell} - \theta^*_{k,\ell}) + (\theta^*_{k',\widehat{\ell}+1} - \theta_{k',\widehat{\ell}+1}) > \frac{C(98)}{2^{k/4}} + \frac{C(98)}{2^{k'/4}}.$$

This simplifies to

$$\theta_{k,\ell+1} - \theta^*_{k,\ell} + \theta^*_{k',\widehat{\ell}+1} - \theta_{k',\widehat{\ell}+1} > \frac{C(98)}{2^{k/4}} + \frac{C(98)}{2^{k'/4}}.$$

Combining this with (4.8), we get

$$\theta_{k,\ell+1} - \theta_{k',\widehat{\ell}+1} > \frac{C(98)}{2^{k/4}} + \frac{C(98)}{2^{k'/4}}.$$

Therefore, by Definition 4.4 $\widehat{\ell} + 1 \in U_\ell$ and is larger than $\widehat{\ell}$, contradicting the maximality of $\widehat{\ell}$. This proves the Stieltjes interlacing between $A_{f'}$ and A_f. Furthermore, noticing Lemma 4.2 parts (3) and (4):

$$\theta^*_{k',0} < \theta^*_{k,0} \quad \text{and} \quad \theta^*_{k',m-6} < \theta^*_{k,m'-6},$$

which thus establishes the strong Stieltjes interlacing between $A_{f'}$ and A_f for $k' > k \geq 98$. □

5. Interlacing of zeros for $16 \le k < 98$

In this section, we prove the interlacing property for $k < 98$ where some bounds, e.g. Lemma 4.1, do not hold. Following [1, Definition 8.1] we define D as the minimum distance between any two distinct zeros of weight less than 98.

Definition 5.1. We define

$$D(f) \coloneqq \min\{|x-y| : x \neq y \in A_f \cup \{0\} \cup \{\pi\}\} \text{ and } D \coloneqq \min\{D(f) : f \in S_k(1), k \le 96\}.$$

With [9, min-dist.py], we compute $D > 0.06786$.

Lemma 5.2. *Recall that $k' = 2m' + 2$. When $k' \ge 108$, we have*

$$\frac{\pi}{m' - 6.5} + \frac{2C(k')}{2^{k'/4}} < D. \tag{5.1}$$

Proof. Recall $C(k')$ is given in (2.16). As the left hand side of (5.1) is a decreasing function of m', it suffices to verify that when $m' = 53$, $\frac{\pi}{53-6.5} + \frac{2C(108)}{2^{108/4}} < 0.06756 < D$. □

Using this bound, we now prove Proposition 1.7 by demonstrating the strong Stieltjes interlacing for $16 \le k \le 98$ and $k' \ge 108$.

Proof of Proposition 1.7. By Lemma 4.3, for $k' \ge 108$ we have $\theta^*_{k',\ell'} < \theta^*_{k',\ell'+1}$ for all $0 \le \ell' \le m' - 7$. Then using (5.1), Lemma 3.2 part (2), and (2.16), we obtain

$$\theta^*_{k',\ell'+1} - \theta^*_{k',\ell'} < \theta_{k',\ell'+1} - \theta_{k',\ell'} + \frac{2C(k')}{2^{k'/4}} < \frac{\pi}{m' - 6.5} + \frac{2C(k')}{2^{k'/4}} < D.$$

With this in mind, to see the strong Stieltjes interlacing, it is sufficient to show

$$\theta^*_{k,0} > \theta^*_{k',0} \qquad \text{and} \qquad \theta^*_{k,m-6} < \theta^*_{k',m'-6}.$$

We know from Lemma 3.2 part (3) that

$$\theta_{k',0} < \frac{\pi}{m' - 6.5} \qquad \text{and} \qquad \pi - \theta_{k',m'-6} < \frac{\pi}{m' - 6.5}.$$

Thus, in combination with (2.16) and (5.1), we obtain

$$\theta^*_{k',0} < \theta_{k',0} + \frac{C(k')}{2^{k'/4}} < \frac{\pi}{m' - 6.5} + \frac{C(k')}{2^{k'/4}} < D.$$

Similarly, we have

$$\pi - \theta^*_{k',m'-6} < \pi - \theta_{k',m'-6} + \frac{C(k')}{2^{k'/4}} < \frac{\pi}{m' - 6.5} + \frac{C(k')}{2^{k'/4}} < D,$$

and this completes the proof for strong Stieltjes interlacing. □

With [9, check-int.py], we have verified strong Stieltjes interlacing holds for all $16 \le k < 98$ and $18 \le k' < 108$. With this, we have also completed the proof of Proposition 1.8.

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