MF20006: Introduction to Computer Science

Lecture 1: Numbers and Computation

Hui Xu

xuh@fudan.edu.cn

Outline

1. Binary System

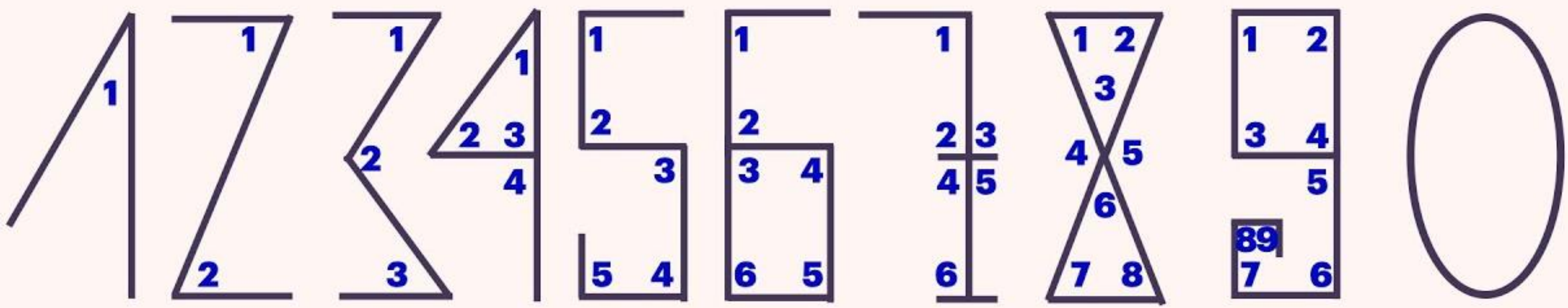
2. Computation

1. Binary System

How to Represent Numbers?

- **Decimal numbers are commonly used in daily life.**
  - Each digit represents a power of 10
  - e.g., $123 = 10^3 + 2 * 10^2 + 3 * 10^1$
- **In computer, we represent numbers with binary (base-2) numbers.**
  - Each binary digit, or bit, represents a power of 2
  - e.g., $(1011)_2 = 1 * 2^3 + 0 * 2^2 + 1 * 2^1 + 1 * 2^0 = 11$
- **We can generalize this idea to any base *n*.**
  - Each position represents a power of *n*.
  - e.g., $(1011)_n = 1 * n^3 + 0 * n^2 + 1 * n^1 + 1 * n^0$

Arabic Numbers are Based on The Numbers of Angles



Binary-based Numbers in Computer

- ❑**Computer storage consists of consecutive bits.**
- ❑**Which position is the start of a number?**
  - ➢Generally via the memory address.
  - ➢Granularity: 1-byte or 8 bit.
- ❑**Which position is the end of a number?**
  - ➢We need a fixed number of bits of to represent a number.
  - ➢E.g., 8-bit, 32-bit, *etc.*

...0000000000000001000000100000001100000100000000101...

Memory Address: p

Represent Natural Numbers with 8 Bits (1 Byte)

Decimal	**Fixed-length** (8 bit/1 byte)
0	00000000
1	00000001
2	00000010
3	00000011
4	00000100
5	00000101
6	00000110
7	00000111
8	00001000
9	00001001
10	00001010
11	00001011
12	00001100
13	00001101
14	00001110
15	00001111
16	00010000
…	…

Simplified Representations in Hexadecimal Format

Decimal	**Fixed-length** (8 bit/1 byte)	**Hex** (base 16)
0	00000000	0
1	00000001	1
2	00000010	2
3	00000011	3
4	00000100	4
5	00000101	5
6	00000110	6
7	00000111	7
8	00001000	8
9	00001001	9
10	00001010	A
11	00001011	B
12	00001100	C
13	00001101	D
14	00001110	E
15	00001111	F
16	00010000	10
…	…	…

Exercise

- What is the binary representation for:
  - $(100)_{10}$
  - $(1024)_{10}$ (also known as 1 kilo)
- Given any decimal number X, how to calculate the binary representation step by step?

More Quantifiers

- 1 mega (M): 2^{20}
- 1 giga (G): 2^{30}
- 1 tera (T): 2^{40}
- 1 peta (P): 2^{50}
- 1 exa (E): 2^{60}
- 1 zetta (Z): 2^{70}
- 1 yotta (Y): 2^{80}
- ...

How to Represent Negative Numbers?

- **Option 1: Simply employ the first bit as the sign bit.**
  - $(0011)_2 = 3, (1011)_2 = -3$
  - Issue: 0 has two representations.
- **Option 2: Complement-based approach by flipping all bits.**
  - e.g., one's complement: $x + y = 2^n - 1$
  - $(1000)_2 = -(0111)_2 = -7$
  - $(1011)_2 = -(0100)_2 = -4$
  - $(1111)_2 = -(0000)_2 = -0$
  - Issue: seems no benefit; 0 still has two representations.
- **Two's complement:** $x + y = 2^n$
  - $(1000)_2 = -(0111 + 1)_2 = -(1000)_2 = -8$
  - $(1111)_2 = -(0000 + 1)_2 = -1$

Representing Negative Integers with Two's Complement

- **Employ the leftmost bit as the sign bit.**
  - 0 for positive numbers.
  - 1 for negative numbers.
  - Padding 0 or 1 to a fixed size, *e.g.*, 4 bits
- **Obtain the two's complement of numbers.**
  - Invert all the bits of the positive number, then add one.
  - Or sub one first, then invert all bits.

$$-(0001)_2 \xrightarrow{\text{invert}} 1110 \xrightarrow{\text{add 1}} \boxed{1111}$$

Decimal	Binary
-8	1000
-7	1001
-6	1010
-5	1011
-4	1100
-3	1101
-2	1110
-1	1111
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111

Benefits of Two’s Complement

- Turn all subtractions in to adds => Reuse the adders.

$$7 - 5 = 7 + (-5)$$

$$\begin{array}{rr} & 0111 \\ + & 1011 \\ \hline & \cancel{1}0010 \end{array}$$

$$2 - 5 = 2 + (-5)$$

$$\begin{array}{rr} & 0010 \\ + & 1011 \\ \hline & 1101 \end{array}$$

Why?

To Prove: $x + (-y) \bmod 2^n = x - y \bmod 2^n$

According to the definition of two's complement:

$$-y = 2^n - y$$

Therefore,

$$x + (-y) \bmod 2^n$$

$$= x + 2^n - y \bmod 2^n$$

$$= x - y \bmod 2^n$$

Exercise

- What is the largest signed integer that 16 bits can represent?
- What is the smallest signed integer that 16 bits can represent?

Questions

- What if the the number is very large?
- How to represent fractional numbers?

Representing Decimals

- **Fixed-point method: Allocate a fixed number of bits for the integer part and the fractional part.**
  - e.g., $(1000.0100)_2 = 2^3 + 2^{-2} = 8.25$
  - Limitation: limited range and fixed precision
- **Floating-point method: shift the decimal point based on the number.**
  - If the number is large, use more bits for the integer part
  - If the number is small, use more bits for the fractional part

Floating-point Numbers Defined in IEEE 754

❑Meaning of bits for 32-bit single-precision floating-point numbers:

- ➢32 (leftmost): sign bit.
- ➢24-31: exponent bits, to represent a wide range of values.
- ➢1-23: mantissa bits.

❑Evaluation: $2^{exp-bias} * mantissa$

$$0\underbrace{10000110}_{\text{exponent (8 bits)}}\underbrace{10010000000000000000000}_{\text{mantissa (23 bits)}}$$

$$2^7 + 2^2 + 2^1 - 127 = 7$$

$$1 + 2^{-1} + 2^{-4} = 1.5625$$

$$2^7 * 1.5625 = 200$$

Why do we subtract a bias 127?

Exercise

- What is the value of the following floating-point numbers
  - 01111111100000000000000000000000
  - 01000000001100000000000000000000
  - 11000000101100000000000000000000

Conversion to Floating-point Numbers

❑ **What is the floating-point representation for 11.25?**

Integer Part	**Fractional Part**
`11/2 = 5 + 1`	`0.25 * 2 = 0.5 + 0`
`5/2  = 2 + 1`	`0.50 * 2 = 0.0 + 1`
`2/2  = 1 + 0`	
`1/2  = 0 + 1`	

`1011.01`

$\Downarrow$

`1+.01101` with `exp = 3`

$$0\,\underbrace{10000010}_{3+127}\,011010000000000000000000000$$

Exercise

- Which of the following numbers can be represented by floating-point numbers without precision loss?
  - 0.1
  - 0.2
  - 0.3
  - 0.4
  - 0.5

Beside Numbers, All Data are Stored in Computers as Bits

- Text/Documents
- Images
- Videos
- Sounds
- Code
- Neural Networks
- …

Encoding Characters in Bits (Bytes)

❑ASCII (American Standard Code for Information Interchange)

- ➢Using 8 bits (7 useful bits) to represent a character.
- ➢0 can be represented as 0x30 in hex or 0011 0000 in binary, or 48.
- ➢A can be represented as 0x41 in hex or 0100 0001 in binary, or 65.

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**A**	**B**	**C**	**D**	**E**	**F**
0x00	NUL	SOH	STX	ETX	EOT	ENQ	ACK	BEL	BS	HT	LF	VT	FF	CR	SO	SI
0x10	DLE	DC1	DC2	DC3	DC4	NAK	SYN	ETB	CAN	EM	SUB	ESC	FS	GS	RS	US
0x20	SP	!	"	#	$	%	&	'	(	)	*	+	,	-	.	/
0x30	0	1	2	3	4	5	6	7	8	9	:	;	<	=	>	?
0x40	@	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0x50	P	Q	R	S	T	U	V	W	X	Y	Z	[	\	]	^	_
0x60	`	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
0x70	p	q	r	s	t	u	v	w	x	y	z	{	|	}	~	DEL

2. Computation

Computation Unit

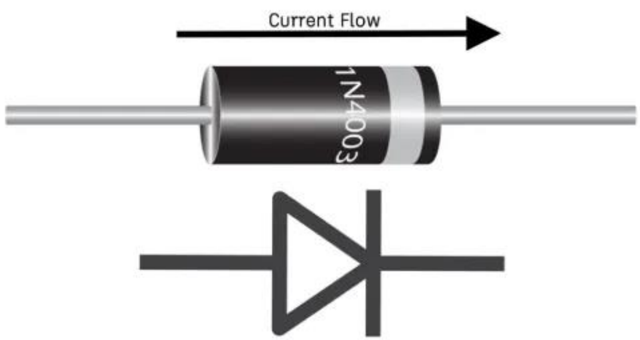
- **CPUs** are made with ALU (Arithmetic Logic Unit) and others.
- The **ALU** is built using logic gates.
- **Logic gates** are implemented with transistors.
- **Transistors** are made from **semiconductors**, primarily silicon.

Semiconductor

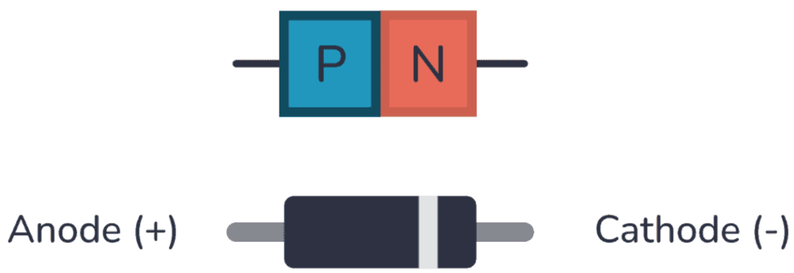
❑ **Pure materials like silicon (Si) with limited conductivity.**

❑ **Doping: Adding small amounts of impurities to change conductivity.**

- P-type: Add atoms with 3 valence electrons *e.g.,* boron.
  - Create holes (positive charge carriers).
- N-type: Add atoms with 5 valence electrons (*e.g.,* phosphorus)
  - Add free electrons (negative charge carriers).



Diode
(one-direction)



Periodic Table of Elements

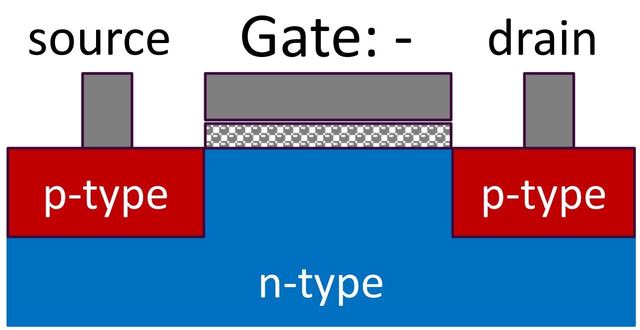
- Non-metal
- Alkali metal
- Alkaline earth metal
- Transition metal
- Metal
- Metalloid
- Halogen
- Noble gas
- Lanthanide
- Actinide

1 H HYDROGEN 1.0079																	2 He HELIUM 4.0026
3 Li LITHIUM 6.941	4 Be BERYLLIUM 9.0122											5 B BORON 10.811	6 C CARBON 12.011	7 N NITROGEN 14.007	8 O OXYGEN 15.999	9 F FLUORINE 18.998	10 Ne NEON 20.1797
11 Na SODIUM 22.989	12 Mg MAGNESIUM 24.305											13 Al ALUMINIUM 26.981	14 Si SILICON 28.085	15 P PHOSPHORUS 30.974	16 S SULFUR 32.066	17 Cl CHLORINE 35.453	18 Ar ARGON 39.948
19 K POTASSIUM 39.098	20 Ca CALCIUM 40.078	21 Sc SCANDIUM 44.955	22 Ti TITANIUM 47.867	23 V VANADIUM 50.9415	24 Cr CHROMIUM 51.9961	25 Mn MANGANESE 54.938	26 Fe IRON 55.845	27 Co COBALT 58.933	28 Ni NICKEL 58.6934	29 Cu COPPER 63.546	30 Zn ZINC 65.38	31 Ga GALLIUM 69.723	32 Ge GERMANIUM 72.63	33 As ARSENIC 74.921	34 Se SELENIUM 78.971	35 Br BROMINE 79.904	36 Kr KRYPTON 83.798
37 Rb RUBIDIUM 85.467	38 Sr STRONTIUM 87.62	39 Y YTTRIUM 88.9058	40 Zr ZICRONIUM 91.224	41 Nb NIOBIUM 92.9063	42 Mo MOLYBDENUM 95.95	43 Tc TECHNETIUM (98)	44 Ru RUTHENIUM 101.07	45 Rh RHODIUM 102.90	46 Pd PALLADIUM 106.42	47 Ag SILVER 107.8682	48 Cd CADMIUM 112.414	49 In INDIUM 114.818	50 Sn TIN 118.710	51 Sb ANTIMONY 121.760	52 Te TELLURIUM 127.60	53 I IODINE 126.90	54 Xe XENON 131.293
55 Cs CAESIUM 132.905	56 Ba BARIUM 137.327	57-71*	72 Hf HAFNIUM 178.49	73 Ta TANTALUM 180.94	74 W TUNGSTEN 183.84	75 Re RHENIUM 186.207	76 Os OSMIUM 190.23	77 Ir IRIDIUM 192.217	78 Pt PLATINUM 195.084	79 Au GOLD 196.96	80 Hg MERCURY 200.59	81 Tl THALLIUM 204.38	82 Pb LEAD 207.2	83 Bi BISMUTH 208.98	84 Po POLONIUM (209)	85 At ASTATINE (210)	86 Rn RADON (222)
87 Fr FRANCIUM (223)	88 Ra RADIUM (226)	89-103**	104 Rf RUTHERFORDIUM (267)	105 Db DUBNIUM (268)	106 Sg SEABORGIUM (271)	107 Bh BOHRIUM (272)	108 Hs HASSIUM (270)	109 Mt MEITNERIUM (276)	110 Ds DARMSTADTIUM (281)	111 Rg ROENTGENIUM (280)	112 Cn COPERNICIUM (285)	113 Uut UNUNTRIUM (284)	114 Fl FLEROVIUM (289)	115 Uup UNUNPENTIUM (288)	116 Lv LIVERMORIUM (293)	117 Uus UNUNSEPTIUM (294)	118 Uuo UNUNOCTIUM (294)

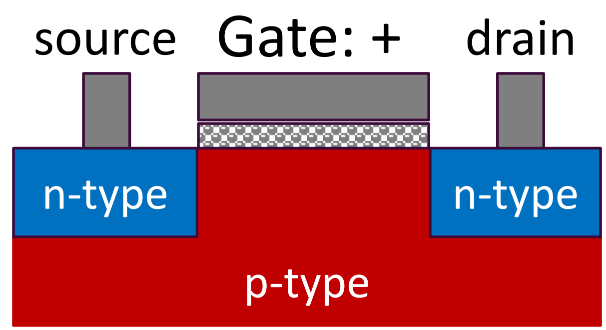
*	57 La LANTHANUM 138.90	58 Ce CERIUM 140.116	59 Pr PRASEODYMIUM 140.90	60 Nd NEODYMIUM 144.242	61 Pm PROMETHIUM (145)	62 Sm SAMARIUM 150.36	63 Eu EUROPIUM 151.964	64 Gd GADOLINIUM 157.25	65 Tb TERIBIUM 158.92	66 Dy DYSPROSIUM 162.500	67 Ho HOLMIUM 164.93	68 Er ERBIUM 167.259	69 Tm THULIUM 168.93	70 Yb YTTERBIUM 173.054	71 Lu LUTETIUM 174.9668
**	89 Ac ACTINIUM (227)	90 Th THORIUM 232.0377	90 Pa PROTACTINIUM 231.03	92 U URANIUM 238.02	93 Np NEPTUNIUM (237)	94 Pu PLUTONIUM (244)	95 Am AMERICIUM (243)	96 Cm CURIUM (247)	97 Bk BERKELIUM (247)	98 Cf CALIFORNIUM (251)	99 Es EINSTEINIUM (252)	100 Fm FERMIUM (257)	101 Md MENDELEVIUM (258)	102 No NOBELIUM (259)	103 Lr LAWRENCIUM (262)

Transistors

- ❑ **A transistor is an electronic switch that controls the flow of current.**
- ❑ **Types of Transistors:**
  - ➢ PMOS (P-type MOSFET): Conducts when gate is low.
  - ➢ NMOS (N-type MOSFET): Conducts when gate is high.



PMOS



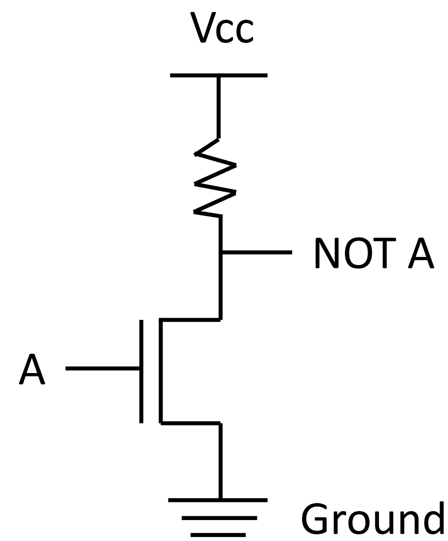
NMOS

Transistor => Logical Gate: NOT

- When a voltage is applied to a semiconductor, it can become conductive.

Truth Table:

A	Not A
0	1
1	0



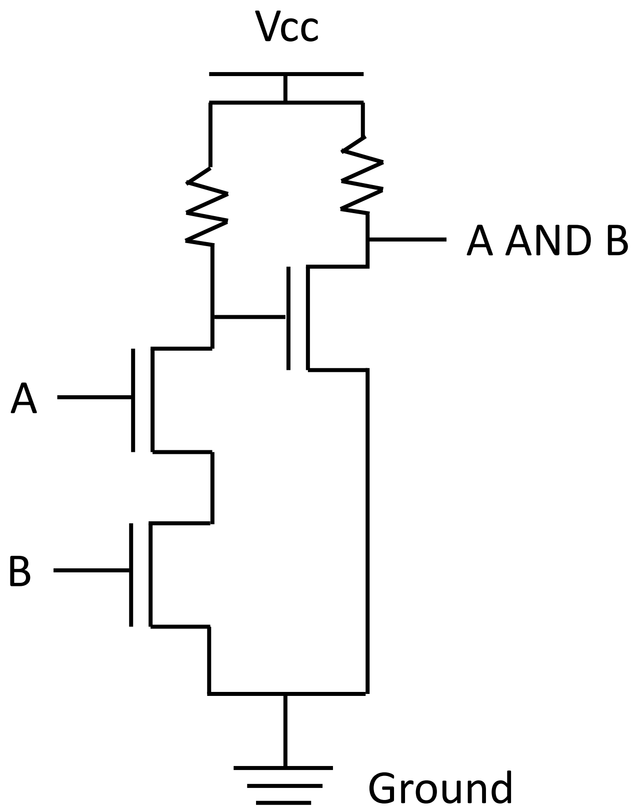
NMOS NOT

Vcc: Voltage Common Collector (+ terminal)

Transistor => Logical Gate: AND

Truth Table:

A	B	A AND B
0	0	0
0	1	0
1	0	0
1	1	1

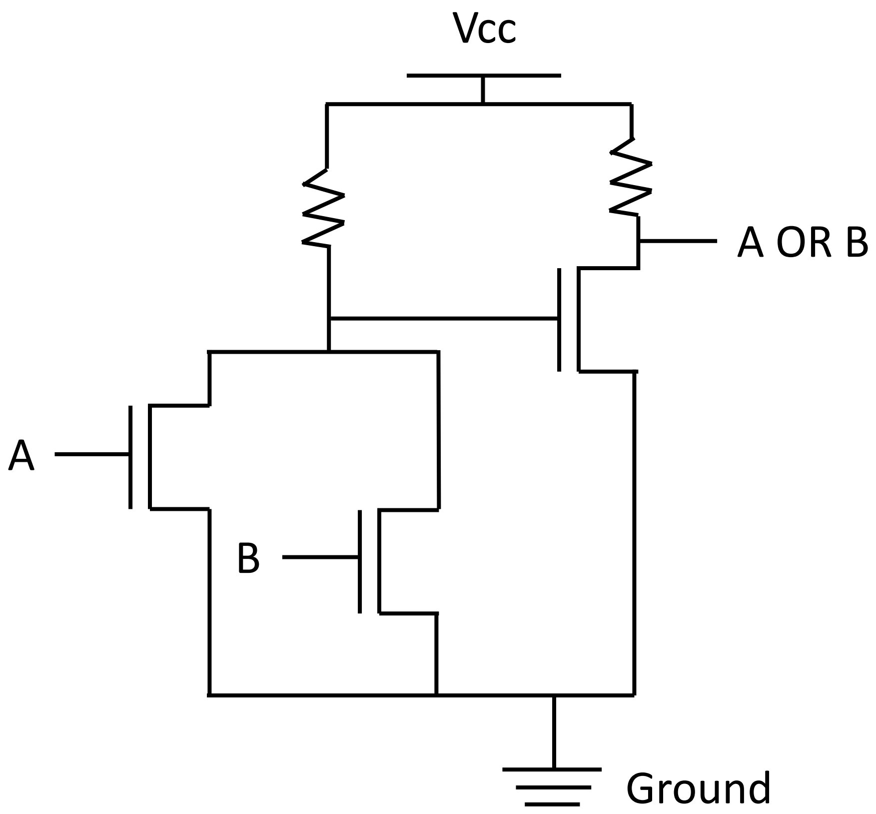


NMOS AND

Transistor => Logical Gate: OR

Truth Table:

A	B	A OR B
0	0	0
0	1	1
1	0	1
1	1	1

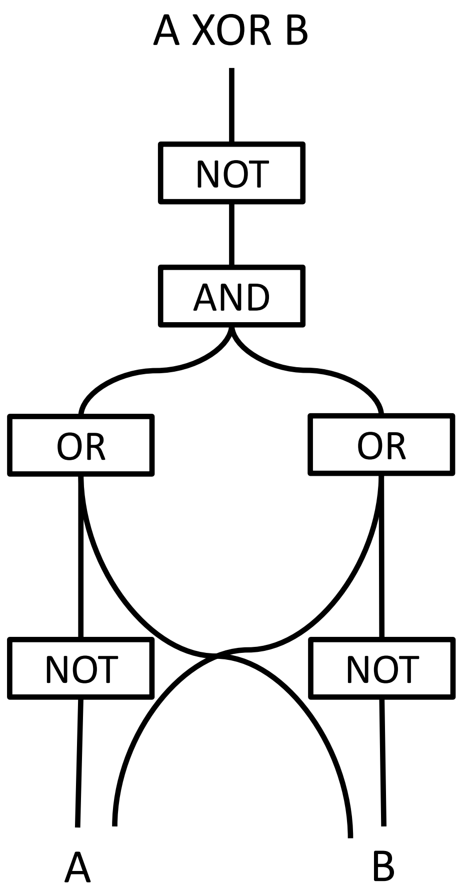


NMOS OR

Transistor => Logical Gate: XOR

Truth Table:

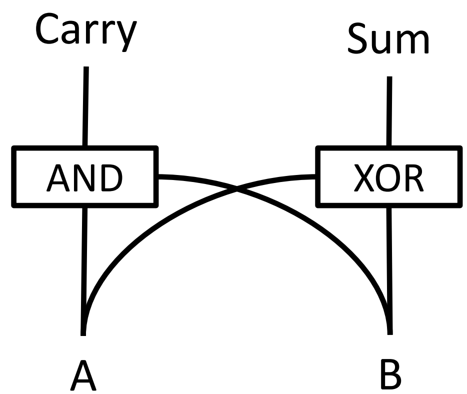
A	B	A XOR B
0	0	0
0	1	1
1	0	1
1	1	0



Compose a Half Adder with Logical Gates

- Half adder has no carry input.

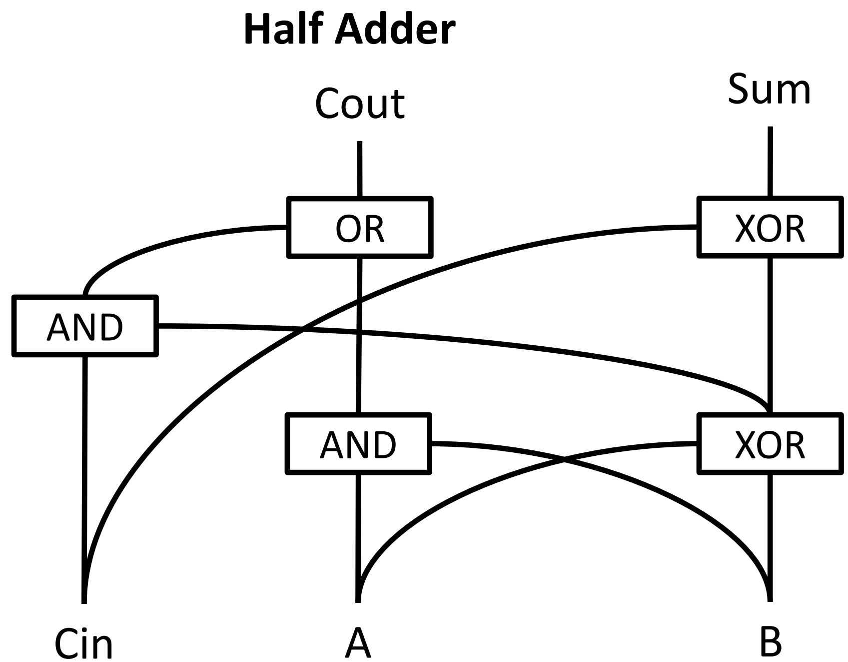
Inputs		Outputs	
A	**B**	**C_{out}**	**S**
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0



Half Adder

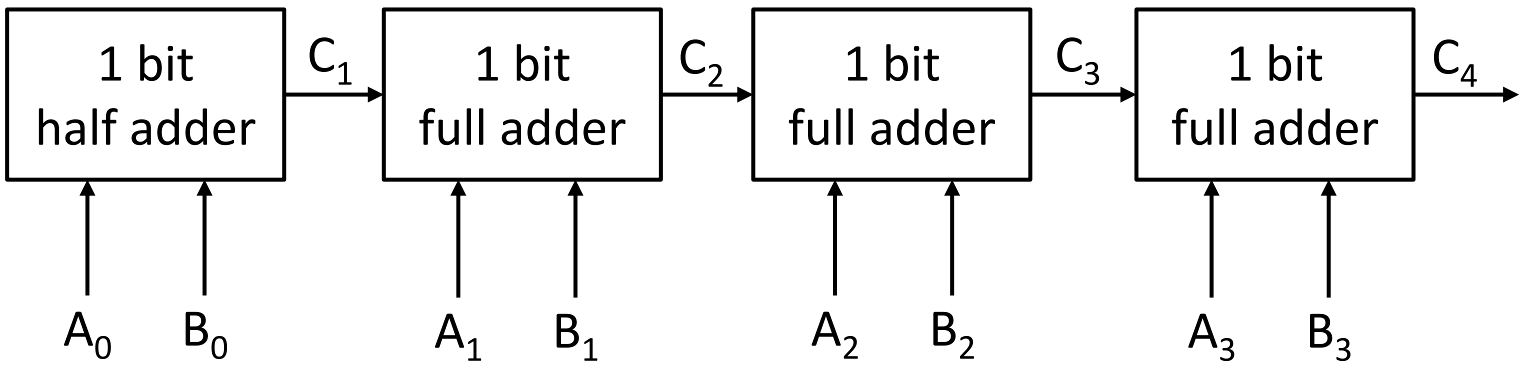
Full Adder

Inputs			Outputs	
A	B	C_{in}	C_{out}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1



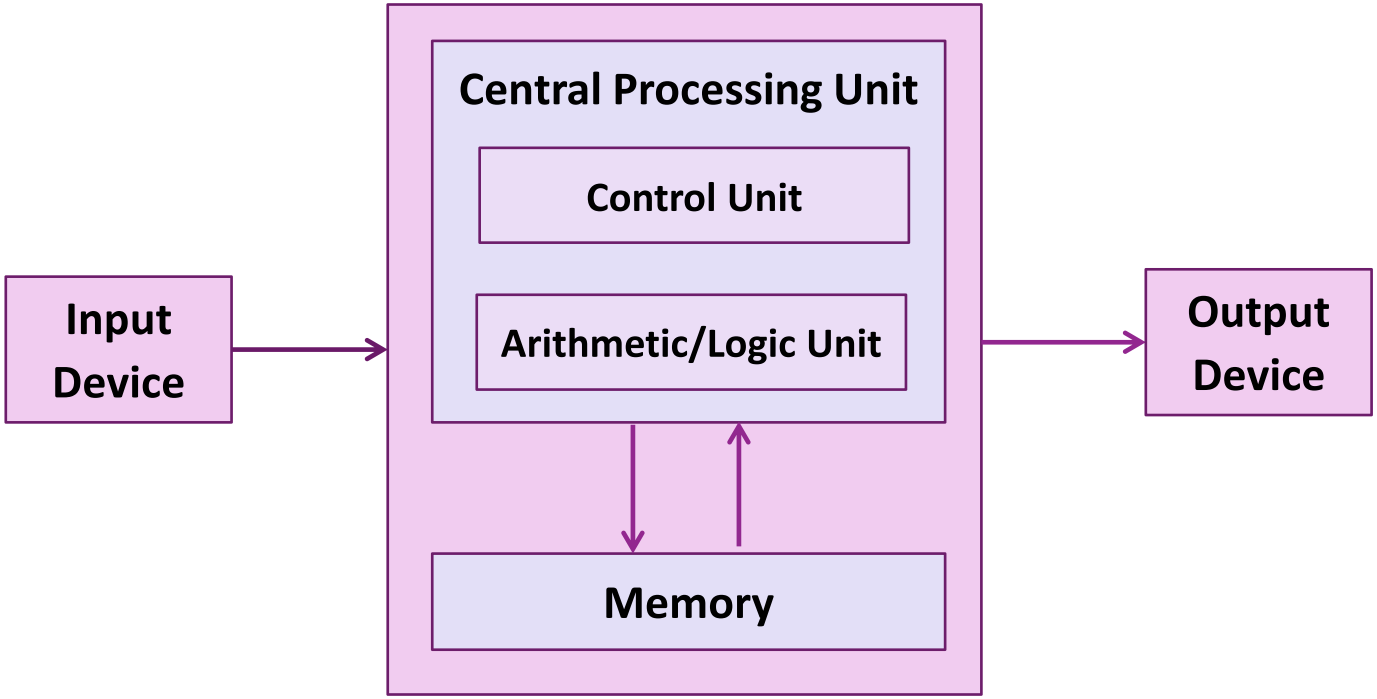
Full Adder (with carrier input)

Adder of 4 Bits



CPU and Von Neumann Model

- Control unit: fetch instructions from memory at the address specified by the program counter.
- ALU: perform the operation specified by the instruction, write the results to memory or registers.



Summary

- **All data are stored in computers as bits.**
  - Integers: Base-2 system.
  - Negative integers: Two’s complement.
  - Decimals: Floating-point numbers with IEEE 754.
- **Computers are made of transistors that accept bit inputs.**
  - Composing of logic gates with transistors.
  - Composing arithmetic units with logical gates.