10-12周实变作业答案

Page125 第1题

- *(a) Let E be a measurable subset of $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}^1$, $\{y : (x, y) \in E\}$ has $\mathbb{R}^1$-measure zero. Show that E has measure zero and that for almost every $y \in \mathbb{R}^1$, $\{x : (x, y) \in E\}$ has measure zero.*
- *(b) Let $f(x, y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x \in \mathbb{R}^1$, $f(x, y)$ is finite for almost every y. Show that for almost every $y \in \mathbb{R}^1$, $f(x, y)$ is finite for almost every x.*

- *(a) Let E be a measurable subset of $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}^1$, $\{y : (x, y) \in E\}$ has $\mathbb{R}^1$-measure zero. Show that E has measure zero and that for almost every $y \in \mathbb{R}^1$, $\{x : (x, y) \in E\}$ has measure zero.*
- *(b) Let $f(x, y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x \in \mathbb{R}^1$, $f(x, y)$ is finite for almost every y. Show that for almost every $y \in \mathbb{R}^1$, $f(x, y)$ is finite for almost every x.*

Proof. **(a)** First, for almost every $x \in E$, define $E_x := \{y : (x, y) \in E\}$, Then we have $|E_x| = 0$.
Let

$$\chi_E(x, y) = \begin{cases} 1, & (x, y) \in E \\ 0, & (x, y) \notin E \end{cases}$$

be E's characteristic function. It follows from Tonelli's theorem that:

$$|E| = \iint_{\mathbb{R}^2} \chi_E(x, y)\, dx\, dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_E(x, y)\, dy \right) dx = \int_{\mathbb{R}} |E_x|\, dx = 0.$$

Also, we have

$$0 = |E| = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_E(x, y)\, dx \right) dy = \int_{\mathbb{R}} |E_y|\, dx,$$

in which $E_y = \{x : (x, y) \in E\}$, therefore $|E_y| = 0$ for almost every $y \in \mathbb{R}$.

Also, we have

$$0 = |E| = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_E(x,y)\, dx \right) dy = \int_{\mathbb{R}} |E_y|\, dx,$$

in which $E_y = \{x : (x,y) \in E\}$, therefore $|E_y| = 0$ for almost every $y \in \mathbb{R}$.

For (b), let $E = \{(x,y) : f(x,y) = +\infty\}$. It follows from the fact $f(x,y)$ is measurable in $\mathbb{R}^2$ that, E is measurable.
Because for almost every $x \in \mathbb{R}$, $f(x,y)$ is finite for almost every $y \in \mathbb{R}$, then we have $\{y : (x,y) \in E\}$ has measure zero.
Therefore, by (a) above, for a.e. $y \in \mathbb{R}$, $\{x : (x,y) \in E\}$ has measure zero.

Page125 第2题

If f and g are measurable in $\mathbb{R}^n$, show that the function $h(x, y) = f(x)g(y)$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$. Deduce that if E_1 and E_2 are measurable subsets of $\mathbb{R}^n$ then their Cartesian product $E_1 \times E_2$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$, and $|E_1 \times E_2| = |E_1||E_2|$.

Page125 第2题

If f and g are measurable in $\mathbb{R}^n$, show that the function $h(x,y)=f(x)g(y)$ is measurable in $\mathbb{R}^n\times\mathbb{R}^n$. Deduce that if E_1 and E_2 are measurable subsets of $\mathbb{R}^n$ then their Cartesian product $E_1\times E_2$ is measurable in $\mathbb{R}^n\times\mathbb{R}^n$, and $|E_1\times E_2|=|E_1||E_2|$.

Proof. **(1)** Since f is measurable in $\mathbb{R}^n$, let $F(x,y)=f(x)$, in which $y\in\mathbb{R}^n$, it follows as in Lemma 6.15 that $F(x,y)$ is measurable in $\mathbb{R}^n\times\mathbb{R}^n$. In fact,

$$\{(x,y):F(x,y)>a\}=\{(x,y):f(x)>a,y\in\mathbb{R}^n\},$$

and it is a cylinder type set with measurable base $\{x:f(x)>a\}$ in $\mathbb{R}^n$. So is $g(y)=G(x,y)$.
It follows from Theorem 4.10 that $h(x,y)=f(x)g(y)=F(x,y)G(x,y)$ is measurable in $\mathbb{R}^{2n}$.

(2) Let $\chi_{E_1}(x)$ and $\chi_{E_2}(y)$ be the characteristic function of E_1 and E_2, so $\chi_{E_1}(x)$ and $\chi_{E_2}(y)$ are measurable functions in $\mathbb{R}^n$.
It follows from the above result that $\chi_{E_1\times E_2}(x,y)=\chi_{E_1}(x)\chi_{E_2}(y)$ is measurable in $\mathbb{R}^n\times\mathbb{R}^n$.
Because $\chi_{E_1\times E_2}(x,y)$ is characteristic function of $E_1\times E_2$, then we have $E_1\times E_2$ is measurable in $\mathbb{R}^n\times\mathbb{R}^n$. Apply Tonelli's theorem, we have

$$\begin{aligned}|E_1\times E_2| &= \iint_{\mathbb{R}^n\times\mathbb{R}^n}\chi_{E_1\times E_2}(x,y)\,dx\,dy\\ &= \int_{\mathbb{R}^n}\chi_{E_1}(x)\,dx\int_{\mathbb{R}^n}\chi_{E_2}(y)\,dy=|E_1||E_2|.\end{aligned}$$

Page125 第3题

Let f be measurable and finite a.e. on $[0, 1]$. If $f(x) - f(y)$ is integrable over the square $0 \le x \le 1, 0 \le y \le 1$, show that $f \in L[0, 1]$.

Let f be measurable and finite a.e. on $[0,1]$. If $f(x)-f(y)$ is integrable over the square $0\le x\le 1, 0\le y\le 1$, show that $f\in L[0,1]$.

Proof. Since $f(x)-f(y)$ is integrable over $[0,\ 1]\times[0,\ 1]$, it follows by Fubini's theorem that,

$$\forall \text{ a.e. } y\in[0,\ 1], f(x)-f(y) \text{ is integrable with respect to } x \text{ on } [0,\ 1].$$

Also, we know that f is finite a.e. on $[0,\ 1]$, then we can find $y_0\in[0,\ 1]$, s.t.

$$(1)\ f(x)-f(y_0) \text{ is integrable on } [0,\ 1],$$
$$(2)\ f(y_0) \text{ is finite.}$$

Now, we rewrite $f(x)=f(x)-f(y_0)+f(y_0)$, due to $f(x)-f(y_0)$ is integrable on $[0,\ 1]$ and $f(y_0)$ is finite, we deduce that f is integrable on $[0,\ 1]$.

- *(a) If f is nonnegative and measurable on E and $\omega(y) = |\{x \in E : f(x) > y\}|$, $y > 0$, use Tonelli' s theorem to prove that $\int_E f = \int_0^\infty \omega(y)\,dy$. (By definition of the integral, $\int_E f = |R(f,E)| = \int_{\mathbb{R}^2} \chi_{R(f,E)}(x,y)\,dx\,dy$. Use the observation in the proof of Theorem 6.11 that $\{x \in E : f(x) \geq y\} = \{x : (x,y) \in R(f,E)\}$, and recall that $\omega(y) = |\{x \in E : f(x) \geq y\}|$ unless y is a point of discontinuity of ω.)*
- *(b) Deduce from this special case the general formula*

$$\int_E f^p = p\int_0^\infty y^{p-1}\omega(y)\,dy \quad (f \geq 0,\ 0 < p < \infty).$$

Page125 第5题

- (a) If f is nonnegative and measurable on E and $\omega(y) = |\{x \in E : f(x) > y\}|$, $y > 0$, use Tonelli' s theorem to prove that $\int_E f = \int_0^\infty \omega(y)\,dy$. (By definition of the integral, $\int_E f = |R(f,E)| = \int_{\mathbb{R}^2} \chi_{R(f,E)}(x,y)\,dx\,dy$. Use the observation in the proof of Theorem 6.11 that $\{x \in E : f(x) \geq y\} = \{x : (x,y) \in R(f,E)\}$, and recall that $\omega(y) = |\{x \in E : f(x) \geq y\}|$ unless y is a point of discontinuity of ω.)
- (b) Deduce from this special case the general formula

$$\int_E f^p = p\int_0^\infty y^{p-1}\omega(y)\,dy \quad (f \geq 0,\ 0 < p < \infty).$$

Proof for (a) By the definition of integral,

$$\int_E f = |R(f,\ E)| = \iint_{R(f,\ E)} dxdy.$$

It follows from Tonelli's theorem that,

$$\int_E f = \int_0^\infty |\{x \in E : f(x) \geq y\}| dy.$$

Because $w(y) = |\{x \in E : f(x) > y\}|$ is monotonous, then we know the set of discontinuous points of $w(y)$ has measure 0.
Since at the continuous points y of w, $w(y) = |\{x \in E : f(x) \geq y\}|$, it follows that,

$$\text{for } a.e.\ y \in [0,\ \infty),\ w(y) = |\{x \in E : f(x) \geq y\}|.$$

Therefore, $\int_E f = \int_0^\infty w(y) dy$.

Proof for (b)

Since $f \geq 0$, $0 < p < \infty$, it follows from (a) that:

$$\begin{aligned}\int_E f^p &= \int_0^\infty |\{x \in E : f(x)^p \geq y\}| dy \\ &= \int_0^\infty |\{x \in E : f(x) \geq y^{\frac{1}{p}}\}| dy = p \int_0^\infty t^{p-1} w(t) dt.\end{aligned}$$

Page126 第6题

For $f \in L(\mathbb{R}^1)$, define the Fourier transform $\hat{f}$ of f by

$$\hat{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)e^{-ixt}\,dt \quad (x \in \mathbb{R}^1).$$

(For a complex-valued function $F = F_0 + iF_1$ whose real and imaginary parts F_0 and F_1 are integrable, we define $\int F = \int F_0 + i\int F_1$.) Show that if f and g belong to $L(\mathbb{R}^1)$, then

$$\widehat{(f * g)}(x) = 2\pi\hat{f}(x)\hat{g}(x).$$

Page126 第6题

For $f \in L(\mathbb{R}^1)$, define the Fourier transform $\hat{f}$ of f by

$$\hat{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)e^{-ixt}\,dt \quad (x \in \mathbb{R}^1).$$

(For a complex-valued function $F = F_0 + iF_1$ whose real and imaginary parts F_0 and F_1 are integrable, we define $\int F = \int F_0 + i\int F_1$.) Show that if f and g belong to $L(\mathbb{R}^1)$, then

$$\widehat{(f * g)}(x) = 2\pi\hat{f}(x)\hat{g}(x).$$

Proof. Because $f, g \in L(\mathbb{R})$, it follows from Tonelli's theorem that $f(t-y)g(y) \in L(\mathbb{R}^2)$.
Recall that $f * g(t) = \int_{\mathbb{R}} f(t-y)g(y)dy$,

Applying Fubini's theorem, we have

$$
\begin{aligned}
\widehat{(f*g)}(x) &= \frac{1}{2\pi}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(t-y)g(y)dy\right)e^{-ixt}dt \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} f(t-y)e^{-ixt}dt\right)g(y)dy \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(s)e^{-ix(s+y)}ds\, g(y)dy \\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(s)e^{-ixs}ds\cdot g(y)e^{-ixy}dy \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} f(s)e^{-ixs}ds\int_{\mathbb{R}} g(y)e^{-ixy}dy \\
&= 2\pi\hat{f}(x)\hat{g}(x).
\end{aligned}
$$

Let v_n be the volume of the unit ball in $\mathbb{R}^n$. Show by using Fubini's theorem that

$$v_n = 2v_{n-1}\int_0^1 (1-t^2)^{(n-1)/2}\,dt.$$

(We also observe that by setting $w = t^2$, the integral is a multiple of a classical β-function and so can be expressed in terms of the Γ-function: $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt$, $s > 0$.)

Page126 第10题

Let v_n be the volume of the unit ball in $\mathbb{R}^n$. Show by using Fubini’s theorem that

$$v_n = 2v_{n-1} \int_0^1 (1 - t^2)^{(n-1)/2}\, dt.$$

(We also observe that by setting $w = t^2$, the integral is a multiple of a classical β-function and so can be expressed in terms of the Γ-function: $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt$, $s > 0$.)

Proof. Recall that the volume of the ball with radius r in $\mathbb{R}^n$ is $r^n V_n$.

$$V_n = \int_{x_1^2+\cdots+x_n^2 \le 1} dx_1 \cdots dx_n = \int_{-1}^1 \left(\int_{x_1^2+\cdots+x_{n-1}^2 \le 1-x_n^2} dx_1 \cdots dx_{n-1} \right) dx_n$$

$$= \int_{-1}^1 (1 - x_n^2)^{\frac{n-1}{2}} V_{n-1} dx_n = 2V_{n-1} \int_0^1 (1 - t^2)^{\frac{n-1}{2}} dt.$$

Page127 第11题

Use Fubini' s theorem to prove that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\,dx = \pi^{n/2}.$$

(For $n = 1$*, write* $\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}\,dx\,dy$ *and use polar coordinates. For* $n > 1$*, use the formula* $e^{-|x|^2} = e^{-x_1^2}\cdots e^{-x_n^2}$ *and Fubini' s theorem to reduce to the case* $n = 1$*.)*

Use Fubini's theorem to prove that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\, dx = \pi^{n/2}.$$

(For $n = 1$, write $\left(\int_{-\infty}^{+\infty} e^{-x^2}\, dx\right)^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}\, dx\, dy$ and use polar coordinates. For $n > 1$, use the formula $e^{-|x|^2} = e^{-x_1^2}\cdots e^{-x_n^2}$ and Fubini's theorem to reduce to the case $n = 1$.)

Proof. For $n = 1$,

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dxdy = \int_0^{2\pi} d\theta \int_0^{\infty} re^{-r^2} dr = \pi.$$

So, we have

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

For $n > 1$, use Fubini's theorem,

$$\int_{\mathbb{R}^n} e^{-|x|^2} dx = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^n = \pi^{n/2}.$$

第8题(10-12 weeks)

Assume that $E \subset \mathbb{R}^n$ *and* $f : E \to \mathbb{R}$ *is nonnegative and measurable, show* $|\Gamma(f, E)| = 0$， *where*

$$\Gamma(f, E) \triangleq \{(x, f(x)) : x \in E\}.$$

第8题(10-12 weeks)

Assume that $E \subset \mathbb{R}^n$ *and* $f : E \to \mathbb{R}$ *is nonnegative and measurable, show* $|\Gamma(f, E)| = 0$， *where*

$$\Gamma(f, E) \triangleq \{(x, f(x)) : x \in E\}.$$

Proof. See Lemma 5.3 on page 83.

Assume that $E \subset \mathbb{R}^n$ *and* $f : E \to \mathbb{R}$ *is nonnegative and measurable, show* $|\Gamma(f, E)| = 0$， *where*

$$\Gamma(f, E) \triangleq \{(x, f(x)) : x \in E\}.$$

Proof. See Lemma 5.3 on page 83.

For any $\varepsilon > 0$, define $E_k := \{x \in E : (k-1)\varepsilon \le f(x) < k\varepsilon\}$, where $k \in \mathbb{N}^+$. So $E = \bigcup\limits_{k=1}^{\infty} E_k$ and $E_i \bigcap E_j = \emptyset$, if $i \ne j$. Therefore, $\Gamma(f, E) = \bigcup\limits_{k=1}^{\infty} \Gamma(f, E_k)$. It follows that,

$$\begin{aligned} |\Gamma(f, E)| \le \Sigma_{k=1}^{\infty} |\Gamma(f, E_k)| \le \Sigma_{k=1}^{\infty} |E_k \times |[(k-1)\varepsilon,\ k\varepsilon)| = \Sigma_{k=1}^{\infty} |E_k| \cdot \varepsilon \\ = \varepsilon |E|. \end{aligned}$$

第8题(10-12 weeks)

- In the case of $|E| < \infty$, we have, $\forall\, \varepsilon > 0$,

$$|\Gamma(f, E)| < \varepsilon |E|.$$

Therefore, $|\Gamma(f, E)| = 0$.

- In the case of $|E| = \infty$. We set $E = \bigcup\limits_{k=1}^{\infty} (E \bigcap B(0\,, k)) \triangleq \bigcup\limits_{k=1}^{\infty} E_k$, in which $E \bigcap B(0\,, k) = E_k$. Hence,

$$\Gamma(f, E) \subseteq \bigcup_{k=1}^{\infty} \Gamma(f, E_k),\ |\Gamma(f, E)| \leq \Sigma_{k=1}^{\infty} |\Gamma(f, E_k)|.$$

Due to $|E_k| < \infty$, from (i), we have $|\Gamma(f, E_k)| = 0$, for all $k \in \mathbb{N}^+$. Therefore , $\Gamma(f, E) = 0$.

第9题(10-12 weeks)

State Theorem 6.11 on page 120 and give the detailed proof of it .

State Theorem 6.11 on page 120 and give the detailed proof of it .

Theorem 6.11: Let f be a nonnegative function defined on a measurable set $E \subset \mathbb{R}^n$. If $R(f, E)$, the region under f over E, is a measurable subset of $\mathbb{R}^{n+1}$, then f is measurable.

Proof. For $0 \leq y < \infty$, we have

$$\{x \in E : f(x) \geq y\} = \{x : (x,\ y) \in R(f,\ E)\} \triangleq R_y(f,\ E).$$

Since $R(f,\ E)$ is measurable, we can write $R(f,\ E) = H \bigcup Z$, in which H is of type F_σ in $\mathbb{R}^{n+1}$, and $|Z|_{n+1} = 0$. Then $R_y(f,\ E) = H_y \bigcup Z_y$, in which H_y is of type F_σ in $\mathbb{R}^n$, and $|Z_y|_n = 0$ for almost every $y \geq 0$. Hence , $R_y(f,\ E)$ is measurable for almost every $y \geq 0$. If $y < 0$, $R_y(f,\ E) = E$ is measurable. Now , for any $y \geq 0$, choose a sequence $\{y_k\}$ such that:

1. $y_k \geq y$, $\lim_{k \to 0} y_k = y$.
2. $R_{y_k}(f,\ E)$ is measurable.

Then $R_y(f,\ E) = \bigcup_{k=1}^{\infty} R_{y_k}(f,\ E)$, therfore $R_y(f,\ E)$ is measurable.

第10题(10-12 weeks)

State Theorem 6.14 on page 120-123 and give the detailed proof of it .

State Theorem 6.14 on page 120-123 and give the detailed proof of it .

Theorem 6.14: If $f \in L(\mathbb{R}^n)$ and $g \in L(\mathbb{R}^n)$, then $(f * g)(x)$ exists for almost every $x \in \mathbb{R}^n$ and is measurable. Moreover, $f * g \in L(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} |f * g|\, dx \le \left(\int_{\mathbb{R}^n} |f|\, dx \right) \left(\int_{\mathbb{R}^n} |g|\, dx \right),$$

$$\int_{\mathbb{R}^n} (f * g)\, dx = \left(\int_{\mathbb{R}^n} f\, dx \right) \left(\int_{\mathbb{R}^n} g\, dx \right).$$

Proof. First, we show that $f(x-t)g(t)$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$ .Since $f \in L(\mathbb{R}^n)$, by Exercises 2, we know f is measurable in $\mathbb{R}^n \times \mathbb{R}^n$. Set $F(x,\ t) = f(x)$, consider $x = \xi - \eta,\ t = \xi + \eta$, which is a nonsigular linear transform of $\mathbb{R}^n \times \mathbb{R}^n$, therefore by Theorem 3.33 , it follows that $F(\xi - \eta,\ \xi + \eta) = f(\xi - \eta)$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$. Then, we know $f(x-t)g(t)$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$.

第10题(10-12 weeks)

Define $I = \iint_{\mathbb{R}^n \times \mathbb{R}^n} |f(x-t)g(t)| dtdx$, then by Tonelli's theorem,

$$I = \int_{\mathbb{R}^n} |g(t)| \left(\int_{\mathbb{R}^n} |f(x-t)| dx \right) dt = \left(\int_{\mathbb{R}^n} |f(x)| dx \right) \left(\int_{\mathbb{R}^n} |g(t)| dt \right) < \infty$$

Therefore , by Fubini's theorem, $\int_{\mathbb{R}^n} f(x-t)g(t)dt$ exists for a.e. x; and

$$\begin{aligned} \iint_{\mathbb{R}^n \times \mathbb{R}^n} f(x-t)g(t)dtdx &= \int_{\mathbb{R}^n} g(t) \left(\int_{\mathbb{R}^n} f(x-t)dx \right) dt \\ &= \left(\int_{\mathbb{R}^n} f(x)dx \right) \left(\int_{\mathbb{R}^n} g(t)dt \right). \end{aligned}$$

第11题(10-12 weeks)

若 $E \subset \mathbb{R}^n$, $f \in L(E)$, 则对任给 $\varepsilon > 0$, 存在 $\mathbb{R}^n$ 上具有紧支集的可测简单函数 h, 使得

$$\int_E |f(x) - h(x)|dx < \varepsilon.$$

若 $E\subset\mathbb{R}^n$，$f\in L(E)$, 则对任给 $\varepsilon>0$, 存在 $\mathbb{R}^n$ 上具有紧支集的可测简单函数 h, 使得

$$\int_E |f(x)-h(x)|dx<\varepsilon.$$

Proof. Suppose that f is nonnegative in E , then there exists a sequence of simple and increasing functions f_k such that

$$f_k\nearrow f \quad \text{with} \quad f_k\in L(E).$$

Now we set $h_k=f_k\chi_{B(0,\ k)}$ where $\chi_{B(0,\ k)}$ is the characteristic function of $B(0,\ k)\subset\mathbb{R}^n$, $h_k\le f_k$. By Levi's theorem,

$$\int_E h_k\to\int_E f\ \ (k\to\infty).$$

Therefore , for any $\varepsilon>0$, there exists $K\in\mathbb{N}^+$ s.t. ,

$$|\int_E f-\int_E h_k|=\int_E|f-h_k|<\varepsilon, \quad \text{for any } k>K.$$

第12题(10-12 weeks)

若 f 是 $E \subset \mathbb{R}^n$ 上几乎处处有限的可测函数, 则对任给的 $\delta > 0$, 存在 $g \in C(\mathbb{R}^n)$, 使得

$$|\{x \in E : f(x) \neq g(x)\}| < \delta.$$

进一步,

(i) 若 E 是有界集, 则可使上述 g 具有紧支集.

(ii) 若 f 具有紧支集, 则可使上述 g 具有紧支集.

提示: 此题可借助于$Lusin$定理和$\mathbb{R}^n$中的闭集上的连续函数的延拓定理来证明.

若 f 是 $E \subset \mathbb{R}^n$ 上几乎处处有限的可测函数, 则对任给的 $\delta > 0$, 存在 $g \in C(\mathbb{R}^n)$，使得

$$|\{x \in E : f(x) \neq g(x)\}| < \delta.$$

进一步,

(i) 若 E 是有界集, 则可使上述 g 具有紧支集.

(ii) 若 f 具有紧支集, 则可使上述 g 具有紧支集.

提示：此题可借助于$Lusin$定理和$\mathbb{R}^n$中的闭集上的连续函数的延拓定理来证明.

Proof. By Lusin's theorem, for any $\delta > 0$, there exists a closed set $F \subset E$ such that:

$$|E - F| < \delta \text{ and } f \in C(F).$$

Then by Tietze extension theorem, there exists $g \in C(\mathbb{R}^n)$, s.t. , $f = g$ for any $x \in F$. Then the function g is what we need and $|\{x \in E : f(x) \neq g(x)\}| \leq |E - F| < \delta$.

第12题(10-12 weeks)

- If E is bounded, then there exists $k > 0$, s.t. $E \subset B(0,\ k)$. Now we construct a cut-off function $h(x)$ in $\mathbb{R}^n$, such $hg \in C_c(\mathbb{R}^n)$ and hg satisfies the same property as g. We construct h as follows:

$$h(x) = \begin{cases} 1, & x \in B(0,\ k), \\ 1 - dist(x,\ B(0,\ k)), & x \in B(0,\ k+1) - B(0,\ k), \\ 0, & else\ . \end{cases}$$

  It follows that :

$$|\{x \in E : f(x) \neq h(x)g(x)\}| = |\{x \in E : f(x) \neq g(x)\}| < \delta.$$

- If f has compact support,let $K \triangleq supp\, f$, K is compact in $\mathbb{R}^n$ and there exists a constant $k > 0$, s.t. $K \subset B(0,\ k)$. By the above statement, we know there exists $g \in C(\mathbb{R}^n)$ s.t.

$$|\{x \in E : f(x) \neq g(x)\}| < \delta.$$

  We also construct the cut-off function h as before.

Then we claim that:

$$\{x \in E : f(x) \neq h(x)g(x)\} \subseteq \{x \in E : f(x) \neq g(x)\}.$$

For any $x_0 \in \{x \in E : f(x) \neq h(x)g(x)\}$, if $x_0 \in B(0,\ k)$, then $f(x_0) \neq g(x_0)$. If $x_0 \notin B(0,\ k)$, we know that : $f(x_0) = 0,\ f(x_0) \neq h(x_0)\ g(x_0)$. Therefore we have $h(x_0) \neq 0,\ g(x_0) \neq 0$, it follows that $f(x_0) \neq g(x_0)$, i.e. $x_0 \in \{x \in E : f(x) \neq g(x)\}$. Hence, $|\{x \in E : f(x) \neq h(x)g(x)\} \leq |\{x \in E : f(x) \neq g(x)\}| < \delta$, and gh is the function we need.

第13题(10-12 weeks)

若 $f \in L(E)$，则对任给 $\varepsilon > 0$，存在 $\mathbb{R}^n$ 上具有紧支集的连续函数 g，使得

$$\int_E |f(x) - g(x)| dx < \varepsilon.$$

第13题(10-12 weeks)

若 $f \in L(E)$，则对任给 $\varepsilon > 0$，存在 $\mathbb{R}^n$ 上具有紧支集的连续函数 g，使得
$$\int_E |f(x) - g(x)|dx < \varepsilon.$$

Proof.

(1)It follows from Exercise 11 that, for any $\varepsilon > 0$, there exists a simple function $h(x)$ with compact support such that:
$$\int_E |f(x) - h(x)|dx < \frac{\varepsilon}{2}.$$
We assume that there exist $R > 0$ and $M > 0$ such that,
$$\text{supp}(h) \subset B(0, R) \qquad \|h\|_\infty = M.$$
Then applying Exercise 12 for h on E, we have, for $\delta > 0$(determined later), there exists $g \in C_c(\mathbb{R}^n)$ such that
$$|\{x \in E, h(x) \neq g(x)\}| < \delta, \quad \text{and } \|g\|_\infty = M.$$

Thus, we have,

$$\begin{aligned}\int_E |f-g|dx &\leq \int_E |f-h|dx + \int_E |h-g|dx\\ &\leq \frac{\varepsilon}{2} + \int_E |h-g|dx,\end{aligned}$$

$$\int_E |h-g|dx = \int_{E\cap\{h\neq g\}} |h-g|dx + \int_{E\cap\{h=g\}} |h-g|dx \leq 2M\delta,$$

where we can take $\delta = \frac{\varepsilon}{4M}$, then it follows that,

$$\int_E |f-g|dx \leq \frac{\varepsilon}{2} + 2M\delta = \varepsilon.$$

第14题(10-12 weeks)

设 $f \in L(E)$, 则存在 $\mathbb{R}^n$ 上具有紧支集的连续函数列 $\{g_k\}$, 使得

(i) $\lim_{k\to\infty} \int_E |f(x) - g_k(x)| dx = 0$;

(ii) $\lim_{k\to\infty} g_k(x) = f(x)$, *a.e.* $x \in E$.

设 $f \in L(E)$, 则存在 $\mathbb{R}^n$ 上具有紧支集的连续函数列 $\{g_k\}$, 使得

(i) $\lim_{k\to\infty} \int_E |f(x) - g_k(x)|dx = 0$;

(ii) $\lim_{k\to\infty} g_k(x) = f(x)$, a.e. $x \in E$.

Proof. Take $\varepsilon = \frac{1}{k^2}$ in Exercises 13, then there exists $g_k \in C_c(\mathbb{R}^n)$, s.t.

$$\int_E |f(x) - g_k(x)|dx < \frac{1}{k^2}, \quad \text{for any } k \in \mathbb{N}^+.$$

So, we have

$$\sum_{k=1}^{\infty} \int_E |f(x) - g_k(x)|dx \le \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

It follows from Theorem 5.16 that:

$$\int_E \sum_{k=1}^{\infty} |f(x) - g_k(x)|dx = \sum_{k=1}^{\infty} \int_E |f(x) - g_k(x)|dx < \infty.$$

第14题(10-12 weeks)

Therefore, $\sum_{k=1}^{\infty}|f(x)-g_k(x)|<\infty$ for almost every $x\in E$, and we have $\lim_{k\to\infty} g_k(x)=f(x)$, $a.e.\ x\in E$.

复习第1题(10-12 weeks)

试证明函数列 $\{\cos kx\}$ 在 $[-\pi, \pi)$ 不是依测度收敛于*0*的.

复习第1题(10-12 weeks)

试证明函数列 $\{\cos kx\}$ 在 $[-\pi, \pi)$ 不是依测度收敛于*0*的.

Proof. Consider that $\{\cos kx\}$ are periodic functions , and $T_k = \frac{2\pi}{k}$ is the period. If $\{\cos kx\}$ converges to 0 in measure, we have

$$\lim_{k\to\infty} m(\{x \in [-\pi, \pi) : |\cos kx - 0| > \varepsilon\} = 0, \quad \text{for any } \varepsilon > 0.$$

Now we take $\varepsilon = \frac{1}{2}$, then

$$|\{x \in [-\pi, \pi) : |\cos kx - 0| > \frac{1}{2}\}| = \frac{2}{3} T_k \cdot k = \frac{4}{3}\pi,$$

in which we use the facts that $\cos kx$ has k periods in $[-\pi, \pi)$ and the length of points which satisfies $|\cos kx| > \frac{1}{2}$ in every period is $\frac{2}{3} T_k$. It's a contradiction !

复习第2题(10-12 weeks)

设 $f \in L(\mathbb{R}^n)$, $f_k \in L(\mathbb{R}^n)(k = 1, 2, \cdots)$, 且有

$$\int_{\mathbb{R}^n} |f_k(x) - f(x)| dx \leq \frac{1}{k^2} \quad (k = 1, 2, \cdots),$$

则

$$f_k(x) \to f(x), \ a.e. \ x \in \mathbb{R}^n.$$

复习第2题(10-12 weeks)

设 $f \in L(\mathbb{R}^n)$, $f_k \in L(\mathbb{R}^n)(k = 1, 2, \cdots)$, 且有

$$\int_{\mathbb{R}^n} |f_k(x) - f(x)| dx \le \frac{1}{k^2} \quad (k = 1, 2, \cdots),$$

则

$$f_k(x) \to f(x), \ a.e. \ x \in \mathbb{R}^n.$$

Proof. See the proof in Exercise 14.

第15题(10-12 weeks)

(平均连续性)若$f \in L(\mathbb{R}^n)$, 则有

$$\lim_{h \to 0} \int_{\mathbb{R}^n} |f(x+h) - f(x)| dx = 0.$$

(平均连续性)若$f \in L(\mathbb{R}^n)$, 则有

$$\lim_{h\to 0}\int_{\mathbb{R}^n}|f(x+h)-f(x)|dx = 0.$$

Proof. It follws from Exercise 13 that: for any $\varepsilon > 0$, there exists $g \in C_c(\mathbb{R}^n)$. s.t.

$$\int_{\mathbb{R}^n}|f(x)-g(x)|dx < \frac{\varepsilon}{3}.$$

So we know that:

$$\begin{aligned}\int_{\mathbb{R}^n}|f(x+h)-f(x)|dx &= \int_{\mathbb{R}^n}|f(x+h)-g(x+h)\\ &+g(x+h)-g(x)+g(x)-f(x)|dx\\ &\le \int_{\mathbb{R}^n}|f(x+h)-g(x+h)|dx+\int_{\mathbb{R}^n}|g(x+h)\\ &-g(x)|dx+\int\ \ |f(x)-g(x)|dx\end{aligned}$$

Consider that $g \in C_c(\mathbb{R}^n)$ and then g is uniform continuous on its support. Hence, for the above $\varepsilon > 0$, there exists $\delta > 0$, if $|h| < \delta$, we have

$$\int_{\mathbb{R}^n} |g(x+h) - g(x)| dx < \frac{\varepsilon}{3}.$$

So we can get: for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\int_{\mathbb{R}^n} |f(x+h) - f(x)| dx < \varepsilon, \quad \text{for} \quad |h| < \delta.$$