13-14周实变作业答案

Exercise 18(13-14weeks)

Prove the following set-theoretic result related to the simple Vitali covering lemma. If $\mathcal{C} = \{Q\}$ is a collection of cubes all contained in a fixed bounded set in $\mathbb{R}^n$, then there is a countable subcollection $\{Q_k\}$ of disjoint cubes in $\mathcal{C}$ such that every $Q \in \mathcal{C}$ is contained in some Q_k^*, where Q_k^* denotes the cube concentric with Q_k of edge length 5 times that of Q_k.
Deduce the measure-theoretic consequence (cf. Lemma 7.4) that if a set E is covered by such a collection $\mathcal{C}$ of cubes, then there exist $\beta > 0$, depending only on n, and a finite number of disjoint cubes $Q_1, \dots, Q_N$ in $\mathcal{C}$ such that $\beta|E|_e \leq \sum_{k=1}^{N} |Q_k|$.
Formulate analogues of these facts for a collection of balls in $\mathbb{R}^n$.

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Proof: Since $\mathcal{C} = \{Q\}$ is contained in a fixed bounded set in $\mathbb{R}^n$, the following cases in Lemma 7.4 would not happen:

(1) $t_j^* = \infty$.

(2) $t_j^* \geq \delta > 0, \ \forall j \in \mathbb{N}^+$.

Therefore, either there exists $N > 0$, s.t. $t_j^* = 0$, for $j \geq N+1$ or $t_j^* \to 0$ as $j \to \infty$. In any case, the results follows from the same statement in Lemma 7.4.
Recall that: The idea is to pick a relatively large cube to cover $\mathcal{C}$ in Lemma 7.4.
Step 1: Define $\mathcal{C}_1 = \mathcal{C}$, $t_1^* \triangleq \sup\{t : Q(t) \in \mathcal{C}_1\}$ and $t_1^* < \infty$. Choose $Q_1(t_1) \in \mathcal{C}_1$ with $t_1 > \frac{1}{2}t_1^*$, and let $Q_1^* = 5Q_1$ (the cube concentric with Q_1 and 5 times its edge length).
Step 2: Rewrite $\mathcal{C}_1 = \mathcal{C}_2 \cup \mathcal{C}_2'$, where $\mathcal{C}_2$ consists of cubes in $\mathcal{C}_1$ that are disjoint from Q_1 and $\mathcal{C}_2'$ consists of those that intersect Q_1. In the next, we chose $Q_2 \in \mathcal{C}_2$ and $\mathcal{C}_2'$ can be covered by Q_1^*.
Step 3: Repeat the process for $\mathcal{C}_j$, selecting $Q_j(t_j) \in \mathcal{C}_j$ with $t_j > \frac{1}{2}t_j^*$ and $Q_j^* = 5Q_j$. We get $\{Q_j^*\}$ and $\{t_j^*\}$. We continue the proof in two cases .

Exercise 18(13-14weeks)

(1) If $\exists\, N > 0$, s.t. $t_j^* = 0$, for $\forall\, j \geq N+1$. It is obvious .
(2) If $t_j^* > 0$ and $t_j^* \to 0$ as $j \to \infty \Rightarrow K \subseteq \bigcup\limits_j Q_j^*$. Else we can find a cube $Q(t) \in \mathcal{C}$ such that $Q \not\subset \bigcup\limits_j Q_j^* \;\Rightarrow Q(t) \in \mathcal{C}_j$ for $\forall\, j \;\Rightarrow t \leq t_j^*$ for $\forall\, j \Rightarrow\; t = 0$.
Now we find $\{Q_j^*\}$ which can cover $\mathcal{C}$.

Exercise 19(13-14weeks)

Use Exercise 18 to prove Lemma 7.9.

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Proof:

Fix $\alpha > 0$ and let $E = \{f^* > \alpha\}$. If $x \in E$, then by the definitions of E and f^*, there is a cube Q_x with center x such that $|Q_x|^{-1} \int_{Q_x} |f| > \alpha$. Equivalently,

$$|Q_x| < \frac{1}{\alpha} \int_{Q_x} |f|.$$

The collection of such Q_x covers E. For $k = 1, 2, \ldots$, the sets E_k defined by $E_k = E \cap \{x : |x| < k\}$ are also covered and have finite measure. By Exercise 18 applied to each E_k, there exist $\beta > 0$ (depending only on n) and a finite number of points $\{x_j^{(k)}\}_j \subset E$ such that the cubes $Q_{x_j^{(k)}}$ are disjoint in j (for each k) and

$$|E_k| < \beta^{-1} \sum_j |Q_{x_j^{(k)}}|.$$

Exercise 19(13-14weeks)

Therefore,

$$|E_k| < \frac{1}{\beta} \sum_j \frac{1}{\alpha} \int_{Q_{x_j^{(k)}}} |f| = \frac{1}{\beta\alpha} \int_{\bigcup_j Q_{x_j^{(k)}}} |f| \leq \frac{1}{\beta\alpha} \int_{\mathbb{R}^n} |f|.$$

Since $E_k \nearrow E$ as $k \to \infty$, it follows from Le that

$$|E| \leq \frac{1}{\beta\alpha} \int_{\mathbb{R}^n} |f|,$$

which proves the lemma with $c = \beta^{-1}$.

Exercise 20(13-14weeks)

(a) Let $f(x)$ be defined for all $x \in \mathbb{R}^n$ by $f(x) = 0$ if every coordinate of x is rational, and $f(x) = 1$ otherwise. Describe the set of all x at which $\frac{1}{|Q|}\int_Q f$ has a limit as $Q \searrow x$ and describe all Lebesgue points of f.

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The Proof for (a)

Since

$$f(x) = \begin{cases} 0, & \text{every coordinate of } x \text{ is rational} \triangleq \mathbb{Q}^n, \\ 1, & \text{otherwise,} \end{cases}$$

and $|\mathbb{Q}^n| = 0$, we have

$$\int_Q f = \int_{Q-\mathbb{Q}^n} f + \int_{\mathbb{Q}^n} f = \int_{Q-\mathbb{Q}^n} f = |Q - \mathbb{Q}^n| = |Q|.$$

Then, $\forall\, x \in \mathbb{R}^n$, $\frac{1}{|Q|}\int_Q f = 1$ for all Q with center at x.

Thus,

$$\lim_{Q \searrow x} \frac{1}{|Q|} \int_Q f = 1,$$

and

$$\mathbb{R}^n = \{x \in \mathbb{R}^n : \lim_{Q \searrow x} \frac{1}{|Q|} \int_Q f \text{ has a limit.}\}$$

Thus,

$$\lim_{Q \searrow x} \frac{1}{|Q|} \int_Q f = 1,$$

and

$$\mathbb{R}^n = \{x \in \mathbb{R}^n : \lim_{Q \searrow x} \frac{1}{|Q|} \int_Q f \text{ has a limit.}\}$$

For $\forall\ x \in \mathbb{R}^n - R_q$,

$$\lim_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - f(x)| dy = \lim_{Q \searrow x} \frac{1}{|Q|} \int_{Q - R_q} |f(y) - f(x)| dy = 0.$$

And for $x \in R_q,\ f(x) = 0$, we have

$$\lim_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - f(x)| dy = \lim_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y)| dy = 1.$$

Therefore, the set of Lebesgue points of f is $\{x : x \in \mathbb{R}^n - R_q\}$.

(b) Give an example of a bounded function f on $(-\infty, \infty)$ with the following properties: f is continuous except at a single point x_0; $(d/dx)\int_0^x f = f(x)$ for all x (in particular when $x = 0$); x_0 is not a Lebesgue point of f.

The proof for (b)

Consider the function:

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

f is bounded, continuous except at $x_0 = 0$. We need to verify $\frac{d}{dx}\int_0^x f(t)dt = f(x)$ at $x = 0$. By the definition of derivative,

$$\frac{d}{dx}\int_0^x f(t)dt = \lim_{x\to 0}\frac{1}{x}\int_0^x f(t)dt.$$

Exercise 20(13-14weeks)

$$\int_0^x f(t)dt = \int_0^x \sin(\frac{1}{t})dt = \int_{\frac{1}{x}}^{\infty} \frac{\sin u}{u^2}du$$
$$= \frac{-\cos u}{u^2}\Big|_{\frac{1}{x}}^{\infty} + \int_{\frac{1}{x}}^{\infty} \cos u d(u^{-2})$$
$$|\int_0^x f(t)dt| \leq |x^2 \cos\frac{1}{x}| + |2\int_{\frac{1}{x}}^{\infty} u^{-3}du| \leq 2|x|^2.$$

Therefore,

$$\lim_{x\to 0} |\frac{1}{x}\int_0^x f(t)dt| \leq \lim_{x\to 0}\frac{1}{x}\cdot 2|x|^2 = 0 = f(0).$$

Exercise 20(13-14weeks)

Next , we prove 0 is not Lebesgue point. Let Q be a cube with center at 0, and the length is t,

$$\frac{1}{|Q|}\int_Q |f(y)-f(0)|dy = \frac{1}{2t}\int_{-t}^{t}\left|\sin\left(\frac{1}{y}\right)\right| dy = \frac{1}{t}\int_{0}^{t}\left|\sin\left(\frac{1}{y}\right)\right| dy = \frac{1}{t}\int_{\frac{1}{t}}^{\infty} |\sin u|\cdot u^{-2}du.$$

Take $t=\frac{1}{k}$, $k\in\mathbb{N}^+$, $Q=(-\frac{1}{k},\ \frac{1}{k})$ and refer to above equation:

$$\frac{1}{|Q|}\int_Q |f(y)-f(0)|dy = k\int_k^{\infty} |\sin u|\cdot u^{-2}du = k\sum_{n=0}^{\infty}\int_{k+n\pi}^{k+(n+1)\pi} |\sin u|\cdot u^{-2}du.$$

Thus, it holds that

$$\frac{1}{|Q|}\int_Q |f(y)-f(0)|dy \geq k\sum_{n=0}^{\infty}(k+(n+1)\pi)^{-2}\int_{k+n\pi}^{k+(n+1)\pi}|\sin u|du.$$
$$\geq 2k\sum_{n=0}^{\infty}(k+(n+1)\pi)^{-2}.$$

For the series, we have,

$$\sum_{n=0}^{\infty}(k+(n+1)\pi)^{-2} \geq \frac{1}{\pi}\int_{k+\pi}^{\infty}u^{-2}du$$
$$=\frac{1}{\pi(k+\pi)}.$$

Therefore, we have

$$\frac{1}{|Q|}\int_Q |f(y)-f(0)|dy \geq \frac{2k}{\pi(k+\pi)},$$

and

$$\lim_{k\to\infty}\frac{1}{|Q|}\int_Q |f(y)-f(0)|dy \geq \frac{2}{\pi}.$$

Therefore, 0 is not a Lebesgue point.

Exercise 21(13-14weeks)

For $x \in \mathbb{R}^n$ and $0 < \alpha < n$, define $f(x) = |x|^{-\alpha}\chi_{\{|x|<1\}}(x)$. Show that its maximal function $f^*(x)$ is bounded both above and below by positive constants (depending only on α and n) times $(|x|^{-\alpha} + |x|^{-n})$.

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Proof: We know that, for

$$\begin{cases} (|x|^\alpha + |x|^n)^{-1} \sim |x|^{-\alpha}, & \text{if} \quad |x| \le 2; \\ (|x|^\alpha + |x|^n)^{-1} \sim |x|^{-n}, & \text{if} \quad |x| \ge 2, \end{cases}$$

in which $\sim$ means that there exists constants C and c such that $c|x|^{-\alpha} \le |x|^\alpha + |x|^n)^{-1} \le C|x|^{-\alpha}$. Take cube $Q_r(x)$ with $r > 0$, recall that

$$f^*(x) = \sup_r \frac{1}{|Q_r|} \int_{Q_r} |f(y)| dy,$$

in which $\frac{1}{|Q_r|} \int_{Q_r} |f(y)| dy = \frac{C_n}{r^n} \int_{Q_r \cap B(0,\,1)} |y|^{-\alpha} dy$.

Case 1: $|x| \leq 2$.

Upper bound estimate:

If $0 < r < \frac{1}{\sqrt{n}}|x|$, we have $0 \notin Q_r(x)$ and $\forall y \in Q_r(x) \bigcap B(0,1)$,

$$|y| > |x| - \frac{\sqrt{n}}{2} r > \frac{1}{2}|x|, \ |Q_r(x) \cap B(0,1)| \leq C_n r^n.$$

Hence, we know:

$$\frac{1}{|Q_r(x)|} \int_{Q_r(x) \cap B(0,1)} |f(y)| dy \leq \frac{C_\alpha}{r^n} |x|^{-\alpha} r^n \leq C_\alpha |x|^{-\alpha}.$$

If $r \geq \frac{|x|}{\sqrt{n}}$, we have

$$Q_r(x) \cap B(0,1) \subset B(0, |x| + \frac{\sqrt{n}r}{2}).$$

Therefore we have

$$\begin{aligned}\frac{1}{|Q_r(x)|}\int_{Q_r(x)} |f(y)|dy &\leq r^{-n}\int_{B(0,|x|+\sqrt{n}r)} |y|^{-\alpha}dy = \frac{C_\alpha}{r^n}|x+\sqrt{n}r|^{n-\alpha} \\ &\leq C_\alpha |x|^{-\alpha}, \quad (\text{take } r = \frac{|x|}{\sqrt{n}}).\end{aligned}$$

Thus, for $|x| \leq 2$, $f^*(x) \leq C_\alpha |x|^{-\alpha}$.

Lower bound estimate:

Take $r = 4|x|$, then $B(0, \frac{r}{8}) \subset Q_r(x) \cap B(0,1)$,(cubes are bigger than the circles with the same center and radius.)

$$
\begin{aligned}
\frac{1}{Q_r(x)} \int_{Q_r(x)} |f(y)| dy &= \frac{1}{r^n} \int_{Q_r(x) \cap B(0,1)} |y|^{-\alpha} dy \geq \frac{1}{r^n} \int_{B(0,\frac{r}{8})} |y|^{-\alpha} dy \\
&= \frac{C_\alpha}{r^n} r^{n-\alpha} = \frac{C_\alpha}{r^\alpha} = \frac{C_\alpha}{|x|^\alpha}, \\
\Rightarrow f^*(x) \geq \frac{C_\alpha}{|x|^\alpha}.
\end{aligned}
$$

Case 2: $|x| > 2$.
Upper bound estimate:
If $r < \frac{|x|}{\sqrt{n}}$, $Q_r(x) \cap B(0,1) = \emptyset$[1], $(r \geq 2(|x|-1) \Leftrightarrow Q_r(x) \cap B(0,1) \neq \emptyset)$

$$\frac{1}{|Q_r(x)|} \int_{Q_r(x)} |f(y)| dy = 0.$$

If $r \geq \frac{|x|}{\sqrt{n}}$, then $Q_r(x) \cap B(0,1) \subset B(0,1)$. In this case, we deduce that,

$$\begin{aligned} \frac{1}{|Q_r(x)|} \int_{Q_r(x)} |f(y)| dy &= \frac{1}{|Q_r(x)|} \int_{Q_r(x) \cap B(0,1)} |y|^{-\alpha} dy \\ &\leq \frac{1}{r^n} \int_{B(0,1)} |y|^{-\alpha} dy = \frac{C_\alpha}{r^n} \leq \frac{C_\alpha}{|x|^n}. \end{aligned}$$

[1] This is why we divide R^n by $|x| \leq 2$ and $|x| > 2$.

Lower bound estimate:
Take $r = 4|x|$, then it follows that : $B(0,1) \subset Q_r(x)$ and

$$\frac{1}{|Q_r(x)|}\int_{Q_r(x)} |f(y)|dy = \frac{1}{|Q_r(x)|}\int_{B(0,1)} |y|^{-\alpha}dy = C_\alpha |x|^{-n}.$$

Thus, $f^*(x) \geq C_\alpha |x|^{-n}$.

Show that for any measurable f that is different from zero on a set of positive measure, there is a positive constant c such that

$$f^*(x) \geq \frac{c}{|x|^n} \text{ for } |x| \geq 1.$$

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Proof:

Since f is non-zero on a set of positive measure, then

$$\exists\, \varepsilon, \delta > 0 \text{ such that } E \triangleq \{x \in \mathbb{R}^n : |f(x)| \geq \varepsilon\} \text{ satisfies } m(E) \geq \delta.$$

We take $R_0 > 0$ sufficiently large such that

$$E_{R_0} = E \cap B_{R_0}(0) \text{ has measure } \quad |E_{R_0}| \geq \delta/2.$$

For $|x| \geq 1$, let $Q_r(x)$ be a cube centered at x with side length $r = 2(|x| + R_0)$, so $E_{R_0} \subset Q_r(x)$.

Then we have:

$$f^*(x) \geq \frac{1}{|Q_r|}\int_{Q_r}|f(y)|dy \geq \frac{1}{|Q_r|}\cdot\varepsilon\cdot\frac{\delta}{2} = \frac{\varepsilon\delta}{(|x|+R_0)^n}$$
$$= \frac{\varepsilon\delta}{|x|^n(1+\frac{R_0}{|x|})^n} \geq \frac{c}{|x|^n},$$

where $|x| \geq 1$.