第9周实变作业答案

第1题(9th week)

$m(E)<\infty, \{f_k\}\subset L(E), \{f_k\}$ 依测度收敛于 f 于 E. 若 $\{f_k(x)\}$ 为 E 上积分等度连续的函数列, 即 $\forall\varepsilon>0, \exists\delta=\delta(\varepsilon)$, $s.t.$, 对 $\forall A\subset E$, 若 $m(A)<\delta$, 则 $\int_A |f_k(x)|\,dx<\varepsilon$, 对所有$k$都成立. 则 $f\in L(E)$. 且

$$\lim_{k\to\infty}\int_E f_k(x)dx=\int_E f(x)\,dx.$$

$m(E) < \infty, \{f_k\} \subset L(E), \{f_k\}$ 依测度收敛于 f 于 E. 若 $\{f_k(x)\}$ 为 E 上积分等度连续的函数列, 即 $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon)$, s.t. , 对 $\forall A \subset E$, 若 $m(A) < \delta$, 则 $\int_A |f_k(x)|\, dx < \varepsilon$, 对所有$k$都成立. 则 $f \in L(E)$. 且

$$\lim_{k\to\infty} \int_E f_k(x)dx = \int_E f(x)\, dx.$$

Proof:

(i) f_k is bounded in $L(E)$. It follows from $\{f_k\}$ being a sequence of functions of equicontinuity of integrals that, $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 0$, such that, if $m(A) < \delta$, $A \subset E$, we have

$$\int_A |f_k| < \varepsilon, \quad \forall k \in \mathbb{N}^+.$$

Take $\varepsilon = 1$ above and divide E into $n = [\frac{|E|}{\delta}] + 1$ parts. Then $\frac{|E|}{n} \leq \delta$, and $E = \bigcup_{i=1}^{n} A_i$ with $A_i \cap A_j = \emptyset$ if $i \neq j$ and $m(A_i) \leq \delta$. Then we have,

$$\int_{A_i} |f_k| < 1 \quad \forall k, i.$$

$$\int_E |f_k| = \sum_{i=1}^{n} \int_{A_i} |f_k| < n.$$

Thus, f_k is uniformly bounded in $L(E)$.

(ii) $f \in L(E)$. f_k converges to f in measure $\Rightarrow$ there exists a subsequence $\{f_{k_j}\}$ of $\{f_k\}$ such that

$$f_{k_j} \to f \quad \text{a.e. in } E.$$

Then by Fatou's lemma,

$$\int_E |f| = \int_E \liminf_{j\to\infty} |f_{k_j}| \leq \liminf_{j\to\infty} \int_E |f_{k_j}| \leq n.$$

So, $f \in L(E)$.

(iii) The proof for $\lim_{k\to\infty} \int_E f_k = \int_E f$.
It follows from the equicontinuity that, $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon)$, such that

$$\text{if} \quad A \subset E, \qquad m(A) < \delta, \qquad \int_A |f_k(x)| dx \leq \frac{\varepsilon}{3}.$$

Since $f \in L(E)$, it follows integral's absolute continuity that: $\forall \varepsilon > 0$, $\exists \delta_0 > 0$, if $m(A) \leq \delta_0$, then

$$\int_A |f| < \frac{\varepsilon}{3}.$$

第1题(9th week)

We take $\eta = \min\{\delta,\ \delta_0\}$.
It follows from $f_k \xrightarrow{m} f$ that, for the above ε and η, $\exists N > 0$ such that if $k > N$, we have

$$m\{|f_k - f| > \frac{\varepsilon}{3|E|}\} < \eta.$$

Denote $E_k := \{x \in E : |f_k(x) - f(x)| > \frac{\varepsilon}{3|E|}\}$.

Then

$$\begin{aligned}
\int_E |f_k - f| &= \int_{E_k} |f_k - f| + \int_{E \setminus E_k} |f_k - f| \\
&\le \int_{E_k} |f_k - f| + \frac{\varepsilon}{3} \\
&\le \int_{E_k} |f| + \int_{E_k} |f_k| + \frac{\varepsilon}{3} \\
&\le \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \qquad \text{for } k > N.
\end{aligned}$$

So, $\lim_{k\to\infty} \int_E f_k = \int_E f$.

Use Egorov's theorem to prove the bounded convergence theorem.

Proof:

First, we recall the Egorov theorem and Bounded convergence theorem.
Egorov theorem for measurable sequence f_k,

$$\begin{cases} f_k \to f, a.e. \text{in } E \\ f \text{ is finite a.e. in } E, |E| < \infty. \end{cases} \Rightarrow \begin{cases} \forall \varepsilon > 0, \exists\, F \subset E \text{ s.t. } |E \setminus F| < \varepsilon, \text{ and} \\ f_k \to f \text{ uniformly in } F. \end{cases}$$

Bounded convergence theorem:

$$\begin{cases} f_k \to f, a.e. \text{in } E \\ |f_k| \le M \text{ a.e. in } E, |E| < \infty. \end{cases} \Rightarrow \int_E f_k \to \int_E f.$$

It follows from $f_k \to f$ a.e. in E and $|f_k| \le M$ a.e. in E that

$$|f(x)| \le M \text{ a.e. } \quad \text{in} \qquad E.$$

By Egorov's theorem, $\forall \varepsilon > 0$, $\exists$ a closed set $F \subset E$ such that

$$m(E \setminus F) < \frac{\varepsilon}{4M} \quad \text{and} \quad f_k \to f \qquad \text{uniformly in } F.$$

Therefore, for the above ε, $\exists N \in \mathbb{N}^+$, such that for all $k \ge N$,

$$|f_k(x) - f(x)| < \frac{\varepsilon}{2m(E)} \quad \forall x \in F.$$

Then we have,

$$\begin{aligned} \left| \int_E f_k - \int_E f \right| \le \int_E |f_k - f| &= \int_{E \setminus F} |f_k - f| + \int_F |f_k - f| \\ &\le 2M \cdot \frac{\varepsilon}{4M} + \frac{\varepsilon}{2m(E)} \cdot m(F) < \varepsilon. \end{aligned}$$

第9题(9th week)

If $p > 0$ *and* $\int_E |f - f_k|^p \to 0$ *as* $k \to \infty$, *show that* $f_k \xrightarrow{m} f$ *on* E *(and thus that there is a subsequence* $f_{k_j} \to f$ *a.e. in* E*).*

第9题(9th week)

If $p > 0$ and $\int_E |f - f_k|^p \to 0$ as $k \to \infty$, show that $f_k \xrightarrow{m} f$ on E (and thus that there is a subsequence $f_{k_j} \to f$ a.e. in E).

Proof: Suppose f_k does not converge to f in measure. Then $\exists \varepsilon_0 > 0, \eta_0 > 0$, such that $\forall N \in \mathbb{N}^+$, $\exists k_N \geq N$ with

$$m\{|f_{k_N}(x) - f(x)| > \varepsilon_0\} \geq \eta_0.$$

Thus, for the subsequence $\{f_{k_N}\}$ of $\{f_k\}$, we have

$$\int_E |f_{k_N} - f|^p \geq \int_{\{|f_{k_N} - f| > \varepsilon_0\}} |f_{k_j} - f|^p \geq \varepsilon_0^p \eta_0 .$$

This contradicts with $\int_E |f_k - f|^p \to 0$.

第10题(9th week)

If $p > 0$, $\int_E |f - f_k|^p \to 0$, *and* $\int_E |f_k|^p \le M$ *for all* k, *show that* $\int_E |f|^p \le M$.

第10题(9th week)

If $p > 0$, $\int_E |f - f_k|^p \to 0$, and $\int_E |f_k|^p \le M$ for all k, show that $\int_E |f|^p \le M$.

Proof:

- It follows from Exercise 9, there exists a subsequence $f_{k_j} \to f$ a.e. in E. By Fatou's lemma,

$$\int_E |f|^p = \int_E \liminf_{j\to\infty} |f_{k_j}|^p \le \liminf_{j\to\infty} \int_E |f_{k_j}|^p \le M.$$

第13题(9th week)

(a) Let $\{f_k\}$ *be a sequence of measurable functions on* E*. Show that* $\sum f_k$ *converges absolutely a.e. in* E *if* $\sum \int_E |f_k| < +\infty$*. (Use Theorems 5.16 and 5.22.)*

(b) If $\{r_k\}$ *denotes the rational numbers in* $[0,1]$ *and* $\{a_k\}$ *satisfies* $\sum |a_k| < +\infty$*, show that* $\sum a_k |x - r_k|^{-1/2}$ *converges absolutely a.e. in* $[0,1]$*.*

(a) Let $\{f_k\}$ be a sequence of measurable functions on E. Show that $\sum f_k$ converges absolutely a.e. in E if $\sum \int_E |f_k| < +\infty$. (Use Theorems 5.16 and 5.22.)
(b) If $\{r_k\}$ denotes the rational numbers in $[0,1]$ and $\{a_k\}$ satisfies $\sum |a_k| < +\infty$, show that $\sum a_k|x-r_k|^{-1/2}$ converges absolutely a.e. in $[0,1]$.

Proof of (a)
Recall that $\sum f_k$ converges absolutely a.e. in E means that:

$$\sum_{k=1}^{\infty} |f_k(x)| < \infty, \quad \text{a.e.} \quad \text{in} \quad \text{E.}$$

It follows from Theorem 5.16 that:

$$\int_E \left(\sum_{k=1}^{\infty} |f_k| \right) = \sum_{k=1}^{\infty} \int_E |f_k| < \infty.$$

Thus, by Theorem 5.22, we have $\sum_{k=1}^{\infty}|f_k| < \infty$ a.e. in E.

Proof of (b)

Thanks to (a), we only need to prove:

$$\int_0^1 \sum_{k=1}^{\infty} |a_k||x-r_k|^{-1/2} < \infty.$$

For any k,

$$\begin{aligned}\int_0^1 |a_k||x-r_k|^{-1/2}dx &= |a_k|\int_{-r_k}^{1-r_k} |t|^{-1/2}dt \\ &\le |a_k|\int_{-1}^{1} t^{-\frac{1}{2}}dt = 4|a_k|.\end{aligned}$$

Thus,

$$\int_0^1 \sum_{k=1}^{\infty} |a_k||x-r_k|^{-1/2} = \sum_{k=1}^{\infty}\int_0^1 |a_k||x-r_k|^{-1/2} \le 4\sum_{k=1}^{\infty}|a_k| < \infty.$$

第23题(9th week)

Prove the following fact, sometimes referred to as the Sequential (or Generalized) Version of the Lebesgue Dominated Convergence Theorem. Let $\{f_k\}$ and $\{\phi_k\}$ be sequences of measurable functions on E satisfying $f_k \to f$ a.e. in E, $\phi_k \to \phi$ a.e. in E, and $|f_k| \leq \phi_k$ a.e. in E. If $\phi \in L(E)$ and $\int_E \phi_k \to \int_E \phi$, then $\int_E |f_k - f| \to 0$. (In case $f = 0$ and all $f_k \geq 0$, apply Fatou's lemma to $\{\phi_k - f_k\}$.) An application is given in Exercise 12 of Chapter 8; for example, if $f_k \geq 0$, $f_k \to f$ a.e. in E, $f \in L(E)$, and $\int_E f_k \to \int_E f$, then $\int_E |f_k - f| \to 0$.

Prove the following fact, sometimes referred to as the Sequential (or Generalized) Version of the Lebesgue Dominated Convergence Theorem. Let $\{f_k\}$ and $\{\phi_k\}$ be sequences of measurable functions on E satisfying $f_k \to f$ a.e. in E, $\phi_k \to \phi$ a.e. in E, and $|f_k| \le \phi_k$ a.e. in E. If $\phi \in L(E)$ and $\int_E \phi_k \to \int_E \phi$, then $\int_E |f_k - f| \to 0$. (In case $f = 0$ and all $f_k \ge 0$, apply Fatou's lemma to $\{\phi_k - f_k\}$.) An application is given in Exercise 12 of Chapter 8; for example, if $f_k \ge 0$, $f_k \to f$ a.e. in E, $f \in L(E)$, and $\int_E f_k \to \int_E f$, then $\int_E |f_k - f| \to 0$.

Proof:

Let $g_k = |f_k - f|$, then we know from the assumption that,
(1) $g_k \ge 0$ on E;
(2) $g_k \to 0$ a.e. in E;
(3) $g_k \le \phi_k + |f|$ a.e. in E.

第23题(9th week)

First, applying Fatou's lemma to $|f_k|$, we have

$$\int_E |f| = \int_E \liminf_{k\to\infty} |f_k| \le \liminf_{k\to\infty} \int_E |f_k|$$
$$\le \liminf_{k\to\infty} \int_E \phi_k = \int_E \phi,$$

Then we have $|f| \in L(E)$.
Apply Fatou's lemma to $\phi_k + |f| - g_k$:

$$\int_E \liminf_{k\to\infty}(\phi_k + |f| - g_k) \le \liminf_{k\to\infty} \int_E (\phi_k + |f| - g_k).$$

$$LHS = \int_E \phi + |f|, \qquad RHS = \int_E \phi + |f| - \limsup_{k\to\infty} \int_E g_k.$$

Thus, $\limsup_{k\to\infty} \int_E g_k = 0$, i.e., $\int_E |f_k - f| \to 0$.

第26题(9th week)

Prove the following variant of Lebesgue' s dominated convergence theorem: if $\{f_k\}$ *satisfies* $f_k \xrightarrow{m} f$ *on* E *and* $|f_k| \le \phi \in L(E)$, *then* $f \in L(E)$ *and* $\int_E f_k \to \int_E f$. *(Show that every subsequence of* $\{f_k\}$ *has a subsequence* $\{f_{k_j}\}$ *such that* $\int_E f_{k_j} \to \int_E f$.*)*

Proof:

Method 1

Suppose that $\int_E f_k \nrightarrow \int_E f$, then $\exists \varepsilon_0 < 0$, $\{f_{k_j}\}$ such that

$$|\int_E f_{k_j} - \int_E f| \geq \varepsilon_0.$$

Because $f_k \xrightarrow{m} f \ \Rightarrow f_{k_j} \xrightarrow{m} f \ \Rightarrow f_{k_{ji}} \longrightarrow f$ a.e. in E. Combing $|f_{k_{ji}}| \leq \phi$ a.e. in E, $\phi \in L(E)$. It follows from Lebesgue Dominated convergence theorem that:

$$\int_E f = \lim_{i \to \infty} \int_E f_{k_{ji}}.$$

Contradicts with the assumption.

Method 2

First , we prove that $f \in L(E)$ and $|f| \le \phi$ a.e. in E.
It follows from $f_k \xrightarrow{m} f$ on E that, there exists a subsequence $f_{k_j} \to f$ a.e. in E and then $|f_{k_j}| \to |f|$ a.e. in E.
Applying Fatou's lemma for $|f_{k_j}|$,

$$\int_E |f| = \int_E \liminf_{j\to\infty} |f_{k_j}| \le \liminf_{j\to\infty} \int_E |f_{k_j}| \le \int_E \phi < \infty,$$

and $|f| \le \phi$ a.e. in E follows from $|f_k| \le \phi$ a.e. in E.
Denote $g_k = |f_k - f|$ and we have $g_k \le 2\phi$.
It follows from absolutely continuity of integral that: $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$, such that if $A \subset E$ with $|A| \le \delta_\varepsilon$, we have

$$\int_A g_k \le \int_A 2\phi \le \varepsilon.$$

It follows from property of integral that, for the above ε, there exists a bounded set $F \subset E$ with such that

$$\int_{E\setminus F} g_k \le \int_{E\setminus F} 2\phi \le \varepsilon.$$

Take $\varepsilon' = \frac{\varepsilon}{|F|}$, $\delta = \delta_\varepsilon$, due to $f_k \xrightarrow{m} f$ on E, there exists $N_0 > 0$ such that,

$$|\{x : g_k(x) > \varepsilon'\}| \le \delta, \text{ for all } k > N_0.$$

Thus, we have

$$\begin{aligned}\int_E g_k = \int_{E\setminus F} g_k + \int_{F\bigcap\{g_k>\varepsilon'\}} g_k + \int_{F\bigcap\{g_k\le\varepsilon'\}} g_k \\ \le \varepsilon + \varepsilon + \varepsilon' \cdot |F| = 3\varepsilon \quad \text{for } k > N_0.\end{aligned}$$