Homework Solution

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Contents

Homework Solution 1

Problem 1.1

(a) $L = \limsup_{k\to\infty} a_k$ if and only if

  (i) there is a subsequence $\{a_{k_j}\}$ of $\{a_k\}$ that converges to L,

  (ii) if $L' > L$, there is an integer K such that $a_k < L'$ for all $k \geqslant K$.

(b) $l = \liminf_{k\to\infty} a_k$ if and only if

  (i) there is a subsequence $\{a_{k_j}\}$ of $\{a_k\}$ that converges to l,

  (ii) if $l' < l$, there is an integer K such that $a_k > l'$ for all $k \geqslant K$.

Proof. We only need to prove (a), and (b) follows analogously.

Necessity:

(i) Notice that $L = \lim_{k\to\infty} \overline{a}_k$, where $\overline{a}_k = \sup_{l\geqslant k} a_l$. Then for every $k \in \mathbb{N}$, there is an integer n_k such that
$$|\overline{a}_{n_k} - L| < \frac{1}{k}.$$
By the definition of supremum, for a fixed k, there is an integer n_{l_k} such that
$$\overline{a}_{n_k} \leqslant a_{n_{l_k}} < \overline{a}_{n_k} + \frac{1}{k}.$$
By induction, we can choose suitable n_{l_k} such that $n_{l_1} < n_{l_2} < \cdots$ to ensure $\{a_{n_{l_k}}\}$ is a subsequence of $\{a_k\}$. Then we obtain
$$|a_{n_{l_k}} - L| \leqslant |a_{n_{l_k}} - \overline{a}_{n_k}| + |\overline{a}_{n_k} - L| \leqslant \frac{1}{k} + \frac{1}{k} = \frac{2}{k}.$$
This implies that $\{a_{n_{l_k}}\}$ converges to L.

(ii) By the definition of the limit, we obtain for $\varepsilon = \dfrac{L' - L}{2}$, there is an integer K such that for every $k \geqslant K$,
$$|\overline{a}_k - L| < \varepsilon.$$
Thus, we obtain
$$a_k \leqslant \overline{a}_k < L + \varepsilon < L', \quad \text{for all } k \geqslant K.$$

Sufficiency:

(i) On the one hand, since $a_{k_j} \leqslant \sup_{l\geqslant k_j} a_l$, we obtain
$$L = \lim_{j\to\infty} a_{k_j} = \lim_{k_j\to\infty} a_{k_j} \leqslant \lim_{k_j\to\infty} \sup_{l\geqslant k_j} a_l =: L_0.$$
Since $\{a_{k_j}\}$ is a subsequence of $\{a_k\}$, for every $k \geqslant k_1$, there is a unique k_j such that $k_j \leqslant k < k_{j+1}$. Define $\phi(k) = k_j, \psi(k) = k_{j+1}$, we obtain
$$\sup_{l\geqslant\phi(k)} a_l \geqslant \sup_{l\geqslant k} a_l \geqslant \sup_{l\geqslant\psi(k)} a_l.$$
Taking the limit as $k \to \infty$, we get
$$\limsup_{k\to\infty} a_l = L_0 \geqslant L.$$

(ii) On the other hand, we use the order-preserving property of the supremum and limit, and obtain for every $L' > L$,

$$\limsup_{k\to\infty} a_k \leqslant L'.$$

Since $L' > L$ is arbitrary, we obtain

$$\limsup_{k\to\infty} a_k \leqslant L.$$

$\square$

Remark. Part (a) shows that the L is the maximum of all limits of convergent subsequences of $\{a_k\}$.

Problem 1.2

A sequence $\{a_k\}$ in $\mathbb{R}^1$ converges to a, $-\infty \leqslant a \leqslant +\infty$, if and only if

$$\limsup_{k\to\infty} a_k = \liminf_{k\to\infty} a_k = a.$$

Proof. Necessity: Any subsequence of a convergent sequence $\{a_j\}$ converges to the same element, which is the limit of $\{a_k\}$.
Sufficiency: For every $\varepsilon > 0$, by Problem 1.1(i), there is an integer K_1 such that $a_k \leqslant L+\varepsilon$ for every $k \geqslant K_1$. And by Problem 1.1(ii), there is an integer K_2 such that $a_k \geqslant L-\varepsilon$ for every $k \geqslant K_1$. Therefore, let $K = \max\{K_1, K_2\}$ and obtain for $k \geqslant K$ we have $|a_k - L| < \varepsilon$. $\square$

Problem 1.3

Find $\limsup E_k$ and $\liminf E_k$ if $E_k = \left[-\frac{1}{k}, 1\right]$ for k odd and $E_k = \left[-1, \frac{1}{k}\right]$ for k even.

Solution. Recall that $\limsup E_k$ is the set of points which appear in infinitely many E_k, and $\liminf E_k$ is the set of points which appear in all but finitely many E_k. If $x \in [-1, 0]$, then $x \in E_k$ for all even k, and if $x \in [0, 1]$, then $x \in E_k$ for all odd k, so $\limsup E_k = [-1, 1]$. If $x \in [-1, 0)$, then $x \notin E_k$ for odd k when $x < -\frac{1}{k}$, which must eventually happen for large enough k.If $x \in (0, 1]$, then $x \notin E_k$ for even k when $x > \frac{1}{k}$ analogously. Thus, every point except $x = 0$ fails to appear in infinitely many E_k, so $\liminf E_k = \{0\}$. $\square$

Problem 1.4

Show that $\limsup_{k\to\infty}(a_k + b_k) \leqslant \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k$, provided that the expression on the right does not have the form $\infty + (-\infty)$ or $-\infty + \infty$.

Proof 1. We only need to observe that

$$a_l + b_l \leqslant \sup_{l\geqslant k} a_l + \sup_{l\geqslant k} b_l.$$

Take the supremum over $l \geqslant k$ and take the limit as $k \to \infty$. $\square$

Proof 2.

(i) Assume that the sum on the right hand side is not of the form $\infty+\infty$ or $-\infty-\infty$, otherwise we clearly have the equality.
Assume that one of the terms of the sum of the right hand side is ∞. Then without loss of generality consider $\limsup\limits_{k\to\infty} a_k = \infty$ and $\limsup\limits_{k\to\infty} b_k = b$. So $\limsup\limits_{k\to\infty} a_k + \limsup\limits_{k\to\infty} b_k = \infty$ and the inequality is trivially satisfied.

(ii) Assume $\limsup\limits_{k\to\infty} a_k = -\infty$, and $\limsup\limits_{k\to\infty} b_k = b \in (-\infty,\infty)$. Let $x \in \mathbb{R}$ be such that $a_{k_j} \to x$ for some subsequence $\{a_{k_j}\}$ of $\{a_k\}$.Thus $x \leqslant \limsup\limits_{k\to\infty} a_k$. Therefore one must have that $x = -\infty$. Since $\limsup\limits_{k\to\infty} b_k = b \in (-\infty,\infty)$, $\{b_k\}$ is bounded, so one sees that $\limsup\limits_{k\to\infty}(a_k+b_k) = -\infty$ and the inequality is clearly testified.

(iii) Assume that $\limsup\limits_{k\to\infty} a_k = a \in (-\infty,\infty)$ and $\limsup\limits_{k\to\infty} b_k = b \in (-\infty,\infty)$.
Let $y \in \mathbb{R}$ be such that $a_{k_j}+b_{k_j} \to y$ for some subsequence $\{a_{k_j}+b_{k_j}\}$ of $\{a_k+b_k\}$. Since $\{a_{k_j}\}$ and $\{b_{k_j}\}$ are bounded, there is convergent subsequences $\{a_{k_{j_l}}\}$ and $\{b_{k_{j_l}}\}$ such that $a_{k_{j_l}} \to s$, $b_{k_{j_l}} \to t$. Then

$$y = \lim_{j\to\infty}(a_{k_j}+b_{k_j}) = \lim_{l\to\infty}(a_{k_{j_l}}+b_{k_{j_l}}) = s+t,$$

and we have

$$y = s+t \leqslant \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k.$$

Since y is an arbitrary real number, we have that

$$\limsup_{k\to\infty}(a_k+b_k) \leqslant \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k.$$

□

Problem 1.5

If $\{a_k\}$ and $\{b_k\}$ are non-negative, bounded sequences, show that

$$\limsup_{k\to\infty}(a_k b_k) \leqslant \Big(\limsup_{k\to\infty} a_k\Big)\Big(\limsup_{k\to\infty} b_k\Big).$$

Proof. We only need to observe that

$$a_l b_l \leqslant \Big(\sup_{l\geqslant k} a_l\Big)\Big(\sup_{l\geqslant k} b_l\Big).$$

Take the supremum over $l \geqslant k$ and take the limit as $k \to \infty$. □

Problem 1.6

Give examples for which the inequalities in Problem 1.4 and Problem 1.5 are not equalities. Show that if either $\{a_k\}$ or $\{b_k\}$ converges, equality holds in Problem 1.4 and Problem 1.5.

Solution 1. Counterexample: For both Problem 1.4 and Problem 1.5, take

$$\{a_k\} = \{1,2,1,2,\cdots\}, \quad \{b_k\} = \{2,1,2,1,\cdots\}.$$

If $\{a_k\}$ converges to A, then for every $\varepsilon > 0$, there is an integer N such that for $k \geqslant N$,

$$|a_k - A| < \varepsilon.$$

Then

$$A + b_k = A + b_k - \varepsilon + \varepsilon \leqslant (a_k + b_k) + \varepsilon \leqslant \sup_{k \geq n}(a_k + b_k) + \varepsilon.$$

By taking the supremum over $k \geqslant n$, we obtain

$$A + \sup_{k \geqslant n} b_k \leq \sup_{k \geq n}(a_k + b_k) + \varepsilon.$$

Let $n \to \infty$ and obtain

$$A + \limsup_{k \to \infty} b_k \leqslant \limsup_{k \to \infty}(a_k + b_k) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\lim_{k \to \infty} a_k + \limsup_{k \to \infty} b_k \leq \limsup_{k \to \infty}(a_k + b_k).$$

Note that we do not necessarily require both sequences to converge. $\square$

Solution 2. Let $\lim\limits_{k \to \infty} a_k = \alpha$. If $\limsup\limits_{k \to \infty} b_k = +\infty$ (or $-\infty$), then the conclusion obviously holds.

Therefore, let $\limsup\limits_{k \to \infty} b_k = \beta$ be a finite number.

Since $\limsup\limits_{k \to \infty} b_k = \beta$, there exists a subsequence $\{b_{k_j}\}$ of $\{b_k\}$ such that $\lim\limits_{j \to \infty} b_{k_j} = \beta$ and β is the maximum of all limits of convergent subsequences.

Also, since $\lim\limits_{k \to \infty} a_k = \alpha$, then $\lim\limits_{j \to \infty} a_{k_j} = \alpha$, so $\lim\limits_{j \to \infty}(a_{k_j} + b_{k_j}) = \alpha + \beta$, $\lim\limits_{j \to \infty}(a_{k_j} \cdot b_{k_j}) = \alpha\beta$.

Next, we prove that $\alpha + \beta$ is the maximum of all limits of convergent subsequences of $\{a_k + b_k\}$ (by contradiction).

Assume that there is a convergent subsequence $\{a_{k_{j'}} + b_{k_{j'}}\}$ of $\{a_k + b_k\}$ such that $\lim\limits_{j' \to \infty}(a_{k_{j'}} + b_{k_{j'}}) = \gamma > \alpha + \beta$.

Then $\lim\limits_{j' \to \infty} b_{k_{j'}} = \lim\limits_{j' \to \infty}(a_{k_{j'}} + b_{k_{j'}}) - \lim\limits_{j' \to \infty} a_{k_{j'}} = \gamma - \alpha > \beta$.

This contradicts the fact that β is the maximum of all limits of convergent subsequences of $\{b_k\}$.

Therefore, $\alpha + \beta$ is the maximum of all limits of convergent subsequences of $\{a_k + b_k\}$.

It can be proved that when $\alpha > 0$, $\alpha + \beta$ is the maximum of all limits of convergent subsequences of $\{a_k + b_k\}$ analogously.

Thus,

$$\limsup_{k \to \infty}(a_k + b_k) = \alpha + \beta = \limsup_{k \to \infty} a_k + \lim_{k \to \infty} b_k,$$
$$\limsup_{k \to \infty}(a_k \cdot b_k) = \alpha\beta = \limsup_{k \to \infty} a_k \cdot \lim_{k \to \infty} b_k$$

$\square$

Homework Solution 2-3

Problem 2.1

(i) $\overline{B(x;\delta)} = \{y : |x-y| \leqslant \delta\}$.
(ii) E is closed if and only if $E = \overline{E}$; that is, E is closed if and only if it contains all its limit points.
(iii) $\overline{E}$ is closed, and $\overline{E}$ is the smallest closed set containing E; that is, if F is closed and $E \subset F$, then $\overline{E} \subset F$.

Proof.

(i) On the one hand, every point $y \in \partial B(x;\delta) = \{y \in \mathbb{R}^n : |x-y| = \delta\}$ is a limit point of $B(x;\delta)$. We can choose a sequence $\left\{\frac{1}{k}x + \left(1 - \frac{1}{k}\right)y\right\}_k$. On the other hand, every point $y \in CB(x;\delta)$ is not a limit point of $B(x;\delta)$, since $|y-x| > \delta$ and we can choose $\varepsilon_0 = \dfrac{|y-x|-\delta}{2}$. Then every point $z \in B(x,\delta)$ satisfies $|y-z| > \varepsilon_0$.

(ii) *Necessity:* Suppose that there is a limit point of E such that $x \in CE$. Since CE is open, there is an open ball $B(x;\delta) \subset CE$. But x is a limit point of E, which implies that there is $y \neq x$ and $y \in B(x;\delta) \cap E = \varnothing$, a contradiction.
Sufficiency: Suppose E is not closed; we need to prove that E does not contain all its limit points. Since CE is not open, there is $x \in CE$ such that $CE \cap B(x;\delta) \neq \varnothing$ for all $\delta > 0$. Then $x \in \overline{E} \cap CE$.

(iii) *$\overline{E}$ is closed:* We need to prove that $\overline{E}' \subset \overline{E}$. Take any $x \in \overline{E}'$, where E' consists of all limit points of E. We will prove that $x \in E' \subset \overline{E}$. Take any $\varepsilon > 0$. Our goal is to show that $B(x;\delta) \setminus \{x\} \cap E \neq \varnothing$. From the fact that $x \in \overline{E}'$, there is some $y \in B(x;\delta) \setminus \{x\} \cap \overline{E}$. We see that $y \in E \cup E'$:

a) If $y \in E$, then we are done.
b) If $y \in E'$, then there is z in a set $B(y,\delta_1) \setminus \{y\} \cap E$ for any $\delta_1 > 0$. Since $B(x;\delta) \setminus \{x\}$ is open and y belongs to it, we have $z \in B(y;\delta_1) \subset B(x;\delta) \setminus \{x\}$ for some small δ_1. Therefore, $z \in B(x;\delta) \setminus \{x\} \cap E \neq \varnothing$.

Let $\mathscr{F}_E$ consist of all closed sets containing E. We will prove $\overline{E} = \bigcap\limits_{F \in \mathscr{F}_E} F$.

a) Taking the closure of every $E \subset F$, we have $\overline{E} \subset \overline{F} = F$ for every $F \in \mathscr{F}_E$. Then,
$$\overline{E} \subset \bigcap_{F \in \mathscr{F}_E} F.$$
b) Let F be any closed set such that $E \subseteq F$. We need to show that $\overline{E} \subseteq F$. Because $E \subseteq F$, every limit point of E is also a limit point of F. But F is closed, so it contains all its limit points. Therefore, $E' \subseteq F$. Since $\overline{E} = E \cup E'$, and $E \subseteq F$ by assumption, we have:
$$\overline{E} = E \cup E' \subseteq F \cup F = F.$$
Thus, $\overline{E} \subseteq F$.

$\square$

Problem 2.2

A set $E_1 \subset E$ is relatively closed with respect to E if and only if $E_1 = E \cap \overline{E}_1$, that is, if and only if every limit point of E_1 that lies in E is in E_1.

Proof.

(i) If E_1 is relatively closed in E, then there is a closed set F such that $E_1 = E \cap F$. The closure $\overline{E}_1$ is the smallest closed set containing E_1, thus $\overline{E}_1 \subset F$. By intersecting with E:
$$E \cap \overline{E}_1 \subset E \cap F = E_1.$$
Conversely, $E_1 \subset \overline{E}_1$ and $E_1 \subset E$, thus
$$E_1 \subset E \cap \overline{E}_1.$$
Therefore, $E_1 = E \cap \overline{E}_1$.

(ii) If $E_1 = E \cap \overline{E}_1$, since $\overline{E}_1$ is closed, by the definition of the relatively closed set, E_1 is relatively closed in E.

□

Problem 2.3

Any closed (open) set in $\mathbb{R}^n$ is of type $G_\delta (F_\sigma)$. (If F is closed, consider the sets $\left\{x : \operatorname{dist}(x,F) < \frac{1}{k}\right\}, k = 1,2,\cdots$.)

Proof. Since the complement of a G_δ set is an F_σ set, we only need to prove that any closed set $F \subset \mathbb{R}^n$ is of type G_δ. Define
$$f(x) = \operatorname{dist}(x,A) := \inf_{a \in A} |x-a|.$$
Since f is continuous, i.e., the pre-image of any (relatively) open set is open, and
$$A = \bigcap_{k=1}^{\infty} f^{-1}((-1/k, 1/k)),$$
we obtain that A is an intersection of some open sets, thus of type G_δ. □

Problem 2.4

For a set $E \subset \mathbb{R}^n$, the following statements are equivalent:

(i) E is compact.

(ii) E is closed and bounded.

(iii) Every sequence of points of E has a subsequence that converges to a point of E.

Proof.

(i) $\Rightarrow$ (ii): [Lee00] $\{B(0;n)\}_{n\in\mathbb{Z}}$ is an open cover of E. By the compactness of E, there are finitely many open sets $B(0;n_k)$ such that

$$E \subset \bigcup_{k=1}^{K} B(0;n_k).$$

Thus, E is bounded.

Fix a point $y \notin E$. We will prove that there is an open set containing y and disjoint with E. For every point $x \in E$, there are disjoint open subsets $U_x = B(x,|x-y|/3)$ and $V_{y,x} = B(y;|x-y|/3)$. The collection $\{U_x : x \in E\}$ is an open cover of E, so it has a finite subcover $\{U_{x_k}\}_{k=1}^{K}$. Let

$$\mathbb{U} = \bigcup_{k=1}^{K} U_{x_k}, \quad \mathbb{V} = \bigcap_{k=1}^{K} V_{y,x_k}.$$

Then $\mathbb{U},\mathbb{V}$ are disjoint open subsets with $E \subset \mathbb{U}$ and $y \in \mathbb{V}$. Thus, y is exterior to E, and E is closed.

(ii) $\Rightarrow$ (iii): The standard proof follows from the diagonal argument and the one-dimensional Bolzano-Weierstrass theorem.

(iii) $\Rightarrow$ (i): We claim: *any open cover $\mathscr{U}$ of E has a countable subcover $\{U_n\}_{n=1}^{\infty}$*. Suppose that no finite subcollection of U_n's covers E. This means that for each n, there is $x_n \in E$ such that $x_n \notin U_1 \cup \cdots \cup U_n$. By hypothesis, the sequence $\{x_n\}$ has a convergent subsequence $x_{n_k} \to x$. Now, $x \in U_m$ for some m because the U_n's cover E, and then convergence of the subsequence means that there is some N such that $x_{n_k} \in U_m$ whenever $k \geqslant N$. But by construction, $x_{n_k} \notin U_1 \cup \cdots \cup U_m$ as soon as $n_k \geqslant m$, which is a contradiction. This proves that finitely many of the U_n's cover E. Therefore, E is compact.

□

Proof of the claim:[Lee00]. Let $\mathscr{U}$ be an arbitrary open cover of $E \subset \mathbb{R}^n$. Let

$$\mathscr{B} = \{B(x;1/n) : x \in \mathbb{Q}^n, n \in \mathbb{N}^+, \exists U \in \mathscr{U}, U \supset B(x;1/n)\}.$$

We see that $\mathscr{B}$ is a countable set.

Now, for each element $B \in \mathscr{B}$, choose an element $U_B \in \mathscr{U}$ such that $B \subset U_B$. The collection $\mathscr{U}_1 = \{U_B : B \in \mathscr{B}\}$ is a countable subset of $\mathscr{U}$; we will show that $\mathscr{U}_1$ still covers E.

If $x \in E$ is arbitrary, then $x \in U_0$ for some open set $U_0 \in \mathscr{U}$. Then there is some $B(x;1/n_x) \subset U_0$ such that $n_x \in \mathbb{N}^+$. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. This means that there is some $y \in \mathbb{Q}^n$ such that $|y-x| < 1/(4n_x)$, and therefore by $x \in B(y;1/(2n_x)) \subset U_0$, there is a set $U_{B(y;1/(2n_x))} \in \mathscr{U}'$ such that $x \in B(y;1/(2n_x)) \subset U_{B(y;1/(2n_x))}$. This shows that $\mathscr{U}'$ is a cover. □

Problem 2.5

The intersection of a countable sequence of decreasing, nonempty, compact sets $\{K_n\}_{n=1}^{\infty}$ is nonempty.

Proof. This is a simple version of the Cantor's intersection theorem. Use the relative topology of K_1, and assume $\bigcap_{n\in\mathbb{N}} K_n = \varnothing$. Then $C(\bigcap_{n\in\mathbb{N}} K_n) = C\varnothing = K_1$. So the collection $\{CK_n\}_{n=1}^{\infty}$ is an open cover of K_1. Since K_1 is compact, there is a finite subcover of the collection $\{CK_{n_j}\}_{j=1}^{N}$. Thus, $C(\bigcup_{j=1}^{N} CK_n) = CK_1 = \varnothing$, which means $\bigcap_{j=1}^{\infty} K_{n_j} = \varnothing$. By the decreasing property of K_n, the smallest set in the collection $\{K_{n_j}\}_{j=1}^{\infty}$ is K_{n_j}. Thus, $K_{n_j} = \bigcap_{j=1}^{N} K_n = \varnothing$, which is a contradiction. □

Problem 2.6

Compare $\limsup_{k\to\infty} a_k$ and $\limsup(-\infty, a_k)$.

Proof. We claim that

$$\big(-\infty, \limsup_{k\to\infty} a_k\big) = \limsup(-\infty, a_k).$$

(i) If $x \in \big(-\infty, \limsup_{k\to\infty} a_k\big)$, we can choose a subsequence $\{a_{k_j}\}$ such that $\lim_{j\to\infty} a_{k_j} > x$. Then there is an integer K such that for $j > K$, $x < a_{k_j}$. Then for any integer j,

$$x < \begin{cases} a_{k_K}, & j \leqslant K, \\ a_{k_j}, & j > K. \end{cases}$$

Thus, $x \in \limsup(-\infty, a_k)$.

(ii) If $x \in \limsup(-\infty, a_k)$, we get that for any integer j, there is $k_j \geqslant j$ such that $x \leqslant a_k$. By choosing $j = 1, k_1, k_{k_1}, \cdots$, we can obtain a subsequence a_{k_j} such that $x \leqslant a_{k_j}$, and it has a convergent subsequence, converging to $a_0 \leqslant \limsup_{k\to\infty} a_k$. Then $x \in \big(-\infty, \limsup_{k\to\infty} a_k\big)$.

□

Problem 2.7

Show that $A^\circ \cap B^\circ = (A\cap B)^\circ$ and $A^\circ \cup B^\circ \subset (A\cup B)^\circ$. Give an example when $A^\circ \cup B^\circ \neq (A\cup B)^\circ$.

Proof.

(i) If $x \in (A\cap B)^\circ$, then there is $\varepsilon > 0$ such that $B(x;\varepsilon) \subseteq A\cap B$. Since $A\cap B \subseteq A$, it follows that $B(x;\varepsilon) \subset A$, which implies $x \in A^\circ$. Similarly, $x \in B^\circ$.

(ii) We claim: $A \subset B \implies A^\circ \subset B^\circ$. Then

$$A\cap B \subset A, \quad A\cap B \subset B \implies (A\cap B)^\circ \subset A^\circ, \quad (A\cap B)^\circ \subset B^\circ.$$

Therefore, $(A\cap B)^\circ \subset A^\circ \cap B^\circ$.

(iii) Since $A \subset A\cup B$ and $B \subset A\cup B$, it follows that $A^\circ \subset (A\cup B)^\circ$ and $B^\circ \subset (A\cup B)^\circ$.

(iv) Counterexample: $A = \mathbb{Q}, B = \mathbb{R} - \mathbb{Q}$.

□

Proof of the claim: If x is an interior point of A, there is an open set U such that $x \in U \subseteq A$. Since $A \subseteq B$, we have $U \subseteq B$, so x is also an interior point of B. Thus, all interior points of A are interior points of B, and $A^\circ \subseteq B^\circ$. □

Problem 2.8

(a) Give an example of two disjoint, nonempty, closed sets E_1 and E_2 in $\mathbb{R}^n$ for which $d(E_1, E_2) = 0$.

(b) Let E_1, E_2 be nonempty sets in $\mathbb{R}^n$ with E_1 closed and E_2 compact. Show that there are points $x_1 \in E_1$ and $x_2 \in E_2$ such that $d(E_1, E_2) = |x_1 - x_2|$. Deduce that $d(E_1, E_2)$ is positive if such E_1, E_2 are disjoint.

Proof. [Zho16]

(a) Let

$$E_1 = \{(x,y) : xy = 1\}, \quad E_2 = \{(x,y) : y = 0, x \text{ arbitrary}\},$$

then E_1 and E_2 are both closed sets and disjoint. For any $\varepsilon > 0$, take the points $(2/\varepsilon, \varepsilon/2)$ in E_1 and $(2/\varepsilon, 0)$ in E_2. It is easy to see that the distance between these two points is $\varepsilon/2 < \varepsilon$. Thus,

$$d(E_1, E_2) = \inf_{\substack{p \in E_1 \\ q \in E_2}} d(p,q) = 0.$$

(b) We need the following theorems:

Theorem 1 If $F \subseteq \mathbb{R}^n$ is a non-empty closed set, and $x_0 \in \mathbb{R}^n$, then there is $y_0 \in F$ such that

$$|x_0 - y_0| = d(x_0, F).$$

Proof Consider the closed ball $B = \overline{B(x_0; \delta)}$ such that $B \cap F$ is not empty. Clearly,

$$d(x_0, F) = d(x_0, B \cap F).$$

Since $B \cap F$ is a bounded closed set, and $|x_0 - y|$ as a function of y defined on $B \cap F$ is continuous, it attains its minimum on $B \cap F$, i.e., there is $y_0 \in B \cap F$ such that

$$|x_0 - y_0| = \inf\{|x_0 - y| : y \in B \cap F\},$$

thus $|x_0 - y_0| = d(x_0, F)$. □

Theorem 2 If E is a non-empty set in $\mathbb{R}^n$, then $d(x, E)$ as a function of x is (uniformly) continuous on $\mathbb{R}^n$.

Proof Consider two points x, y in $\mathbb{R}^n$. By the definition of $d(y, E)$, for any given $\varepsilon > 0$, there exists $z \in E$ such that $|y - z| < d(y, E) + \varepsilon$, and thus

$$d(x, E) \leqslant |x - z| \leqslant |x - y| + |y - z| < |x - y| + d(y, E) + \varepsilon.$$

By the arbitrariness of ε, we have

$$d(x, E) - d(y, E) \leqslant |x - y|.$$

We can prove $d(y, E) - d(x, E) \leqslant |x - y|$ analogously. This shows that

$$|d(x, E) - d(y, E)| \leqslant |x - y|.$$

□

Since E_1 is compact and $d(\cdot, E_2)$ is continuous,

$$d(E_1, E_2) = \sup_{x \in E_1} d(x, E_2) = \max_{x \in E_1} d(x, E_2).$$

By Theorem 1, we obtain the conclusion.

$\square$

Remark. In (i), let E_1 and E_2 be two non-empty closed sets, and at least one of them is bounded. Then, if $d(E_1,E_2)=0$, E_1 and E_2 must intersect. The above counterexample is possible because the given two closed sets are both unbounded.

Problem 2.9

If f is defined and uniformly continuous on E, show that there is a function $\overline{f}$ defined and continuous on $\overline{E}$ such that $\overline{f}=f$ on E.

Proof. Define $\overline{f}=f$ in E and for any point $x\in\partial E=\overline{E}-E$, choose any sequence $\{x_k\}\to x$ where $x_k\in E$ and define $\overline{f}(x)=\lim\limits_{k\to\infty}f(x_k)$.

(i) The sequence $\{f(x_k)\}$ converges: since $\{x_k\}$ is a Cauchy sequence in E and f is uniformly continuous on E, then $\{f(x_k)\}$ is also a Cauchy sequence in $\mathbb{R}$, and converges to some number.

(ii) $\overline{f}$ is well-defined: choose two sequences $\{x_k\},\{y_k\}\to x$ where $x_k,y_k\in E$, then there is an integer K such that for $k>K$,
$$|x_k-x|<\frac{\delta}{2},\quad |y_k-x|<\frac{\delta}{2}$$
for some given $\delta>0$. By the uniform continuity of f on E, for any $\varepsilon>0$, there is $\delta>0$ such that if $|x_k-y_k|=|x_k-x|+|y_k-x|<\delta$, then
$$|f(x_k)-f(y_k)|<\varepsilon.$$
Thus, $\lim\limits_{k\to\infty}f(x_k)=\lim\limits_{k\to\infty}f(y_k)$.

(iii) $\overline{f}$ is (uniformly) continuous on $\overline{E}$: for any $\varepsilon>0$, and $x,y\in\overline{E}$, choose any $\{x_k\}\to x,\{y_k\}\to y$ and $x_k,y_k\in E$. Since
$$|\overline{f}(x)-\overline{f}(y)|=|\overline{f}(x_k)-\overline{f}(x)|+|\overline{f}(x_k)-\overline{f}(y_k)|+|\overline{f}(y_k)-\overline{f}(y)|,$$
by (ii), we can choose an integer K such that for $k>K$,
$$|f(x_k)-\overline{f}(x)|<\frac{\varepsilon}{3},\quad |f(y_k)-\overline{f}(y)|<\frac{\varepsilon}{3}.$$
By the uniform continuity of f on E, for $\varepsilon/3>0$, there is $\delta>0$ and $|x-y|<\delta$ such that $|f(x)-f(y)|<\varepsilon$. We obtain
$$|f(x_k)-f(y_k)|<\frac{\varepsilon}{3}.$$
unless
$$|x_k-y_k|\leqslant|x_k-x|+|x-y|+|y_k-y|<\delta,$$
which holds when k is sufficiently large, and $|x-y|<\dfrac{\delta}{3}$.

Thus, for any $\varepsilon>0$, if $x,y\in\overline{E}$ such that $|x-y|<\dfrac{\delta}{3}$ where $\delta>0$ is chosen above, then we have
$$|\overline{f}(x)-\overline{f}(y)|<\varepsilon.$$

$\square$

Remark. The construction above also shows that $\overline{f}$ is the unique continuous extension of f. If not, there is some $x\in\overline{E}$ such that for another extension f_1, $f_1(x)\neq\overline{f}(x)=\lim\limits_{k\to\infty}f(x_k)$, where $\{x_k\} \subset E$ converges to x. Thus, f_1 is not continuous at $x \in \overline{E}$.

Problem 2.10

If f is defined and uniformly continuous on a bounded set E, show that f is bounded on E.

Proof. By Problem 2.9, there is a unique continuous function $\overline{f}$ such that $\overline{f}|_E = f$. By Cantor's uniform continuity criteria, $\overline{f}$ is bounded on $\overline{E}$, so is E. □

Homework Solution 4-5

Problem 3.1

Construct a subset of $[0,1]$ in the same manner as the Cantor set, except that at the kth stage, each interval removed has length $\delta 3^{-k}$, where $0 < \delta < 1$. Show that the resulting set is perfect, has measure $1-\delta$, and contains no intervals.

Proof. Let D_k denote the union of the intervals left at the kth stage, so that the resultant set is $D = \bigcap_{k=1}^{\infty} D_k$. At each stage k, the length of the intervals removed is $2^{k-1}\delta 3^{-k}$. Thus,

$$|D_k| = 1 - \sum_{j=1}^{k} 2^{j-1}\delta 3^{-j}.$$

Since $D_k \searrow D$ and $|D_1| < \infty$, by the Monotone Convergence Theorem for measure,

$$|D| = \lim_{k\to\infty} |D_k| = 1 - \delta.$$

Observe that D_1 cannot contain an interval of length greater than $\frac{1}{2}$, since the interval removed is in the middle. Inductively, D_k cannot contain an interval of length greater than $\frac{1}{2^k}$. Thus, D cannot contain an interval of length greater than $\frac{1}{2^k}$ for all k, so D contains no intervals.

Since each D_k is closed, D is closed. To show D is perfect, consider $x \in D$. x lies in an interval I_k in D_k for every k. Let $x_k \in D \setminus \{x\}$ be an endpoint of I_k. Then

$$|x - x_k| \leqslant |I_k| \leqslant \frac{1}{2^k} \to 0 \text{ as } k \to \infty.$$

Thus, $\{x_k\}$ is a sequence in $D \setminus \{x\}$ that converges to x. Hence, D is perfect. $\square$

Problem 3.2

If $\{E_k\}_{k=1}^{\infty}$ is a sequence of sets with $\sum |E_k|_e < +\infty$, show that $\limsup E_k$ (and so also $\liminf E_k$) has measure zero.

Proof. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} \sum_{k=1}^{n} |E_k|_e = \sum_{k=1}^{\infty} |E_k|_e < \infty$, there exists N such that for $n \geqslant N$,

$$\sum_{k=n+1}^{\infty} |E_k|_e = \sum_{k=1}^{\infty} |E_k|_e - \sum_{k=1}^{n} |E_k|_e < \varepsilon.$$

Write $U_j = \bigcup_{k=j}^{\infty} E_k$, so that $\limsup E_k = \bigcap_{j=1}^{\infty} U_j$. Since $\limsup E_k \subseteq U_{N+1}$,

$$|\limsup E_k|_e \leqslant |U_{N+1}|_e \leqslant \sum_{k=N+1}^{\infty} |E_k|_e < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $|\limsup E_k|_e = 0$, so $\limsup E_k$ has measure zero.
Since $\liminf E_k \subseteq \limsup E_k$, $|\liminf E_k|_e = 0$ as well, so $\liminf E_k$ has measure zero. $\square$

Problem 3.3

If $\{I_k\}_{k=1}^N$ is a finite collection of nonoverlapping intervals, then $\bigcup I_k$ is measurable and $\left|\bigcup I_k\right| = \sum |I_k|$.

Proof. We may assume that the intervals I_k $(k = 1, 2, \cdots, N)$ are closed and do not intersect each other. The former is because the open interval and the closed interval differ only by a zero measure set, and the latter is because the measure of two disjoint open intervals is the sum of the measures of the two intervals. We have

$$|I| = \left|\bigcup_{k=1}^N I_k\right| \leqslant \sum_{k=1}^N |I_k|,$$

and need to prove the reverse inequality. First, as a finite union of closed intervals in $\mathbb{R}$, I is compact. For any $\varepsilon > 0$, let $\{J_n\}_{n=1}^\infty$ be a cover of I by open intervals satisfying:

$$\sum_{n=1}^\infty |J_n| < (1+\varepsilon)|I|.$$

For each J_n, define an open interval J_n^* such that

$$J_n \subset \operatorname{int}(J_n^*), \quad |J_n^*| < (1+\varepsilon)|J_n|.$$

This ensures $\{J_n^*\}$ is an open cover of I. Then by compactness of I, there exists a finite subcover $\{J_1^*, \cdots, J_M^*\}$. Since the non-overlapping intervals I_k are contained in the union of J_n^*:

$$\sum_{k=1}^N |I_k| \leqslant \sum_{n=1}^M |J_n^*| \leqslant \sum_{n=1}^\infty |J_n^*| \leqslant (1+\varepsilon)\sum_{n=1}^\infty |J_n| \leqslant (1+\varepsilon)^2|I|.$$

As $\varepsilon > 0$ is arbitrary, taking $\varepsilon \to 0$ gives

$$\sum_{k=1}^N |I_k| \leqslant |I|.$$

□

Problem 3.4

If E_1 and E_2 are measurable, show that $|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$.

Proof. Since E_1 and E_2 are measurable, $E_1 \cap E_2$, $E_1 \cup E_2$, $E_1 - E_2 = E_1 \cap CE_2$, and $E_2 - E_1 = E_2 \cap CE_1$ are all measurable. Then by

$$E_1 \cup E_2 = (E_1 \cap E_2) \cup (E_1 - E_2) \cup (E_2 - E_1),$$

and $(E_1 \cap E_2) \cup (E_1 - E_2) \cup (E_2 - E_1)$ are disjoint from each other, we have

$$|E_1 \cup E_2| = |E_1 \cap E_2| + |E_1 - E_2| + |E_2 - E_1|.$$

Since

$$E_1 = (E_1 \cap E_2) \cup (E_1 - E_2), \quad E_2 = (E_1 \cap E_2) \cup (E_2 - E_1),$$

we have

$$|E_1| + |E_2| = 2|E_1 \cap E_2| + |E_1 - E_2| + |E_2 - E_1|.$$

Thus,

$$|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|.$$

□

Problem 3.5

Suppose that $|E|_e < +\infty$. Then E is measurable if and only if given $\varepsilon > 0$, $E = (S \cup N_1) - N_2$, where S is a finite union of nonoverlapping intervals and $|N_1|_e, |N_2|_e < \varepsilon$.

Proof. If $E = (S \cup N_1) - N_2$ for some S, N_1, N_2 above, then E is measurable since S is measurable. Conversely, assume E is measurable. Let $\varepsilon > 0$. There exists an open set G such that $E \subseteq G$ and $|G \setminus E|_e < \varepsilon$.
Since G is open, it can be written as a countable union of nonoverlapping (closed) intervals, say $G = \bigcup_{k=1}^{\infty} I_k$.
We have

$$\sum_{k=1}^{\infty} |I_k| = \left| \bigcup_{k=1}^{\infty} I_k \right| = |G| \leqslant |E|_e + |G \setminus E|_e < \infty.$$

Thus, there exists N such that

$$\left| \bigcup_{k=N+1}^{\infty} I_k \right| = \sum_{k=N+1}^{\infty} |I_k| < \varepsilon.$$

Let $S = \bigcup_{k=1}^{N} I_k$, $N_1 = \bigcup_{k=N+1}^{\infty} I_k$, and $N_2 = G \setminus E$.
Then $E = (S \cup N_1) \setminus N_2$ with $|N_1|_e, |N_2|_e < \varepsilon$ as desired. □

Problem 3.6

If E_1 and E_2 are measurable subsets of $\mathbb{R}^1$, show that $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^2$ and $|E_1 \times E_2| = |E_1||E_2|$.

Proof. We first claim:

Lemma. *If $A \subseteq \mathbb{R}$ and $Z \subseteq \mathbb{R}$ with $|Z| = 0$, then $|A \times Z| = 0$. Similarly, $|Z \times A| = 0$.*

Proof. Let $\varepsilon > 0$. Since $|Z| = 0$, there exists intervals $\{I_k\}$ such that $Z \subseteq \bigcup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} |I_k| < \varepsilon$.
Write $A_n = A \cap [-n, n]$. Then $A = \bigcup_{n=1}^{\infty} A_n$. Note that

$$A_n \times Z \subseteq [-n, n] \times \bigcup_{k=1}^{\infty} I_k = \bigcup_{k=1}^{\infty} ([-n, n] \times I_k),$$

so

$$|A_n \times Z|_e \leqslant \sum_{k=1}^{\infty} 2n|I_k| = 2n\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $|A_n \times Z| = 0$ for each n. So

$$|A \times Z|_e = \left| \bigcup_{n=1}^{\infty} (A_n \times Z) \right|_e \leqslant \sum_{n=1}^{\infty} |A_n \times Z|_e = 0.$$

Thus, $|A \times Z| = 0$ as desired. $\square$

Now, since E_1 is measurable, $E_1 = H_1 \cup Z_1$, where H_1 is of type F_σ and $|Z_1| = 0$. Similarly, write $E_2 = H_2 \cup Z_2$ where H_2 is of type F_σ and $|Z_2| = 0$. Then

$$E_1 \times E_2 = (H_1 \times H_2) \cup (H_1 \times Z_2) \cup (Z_1 \times H_2) \cup (Z_1 \times Z_2).$$

Note that $H_1 \times H_2$ is of type F_σ, while the other terms have measure zero by the previous lemma. Thus, $E_1 \times E_2$ is measurable.

(i) Suppose $|E_1|$ and $|E_2|$ are both finite. Since E_1, E_2 are measurable, for each $k \in \mathbb{N}$, there are open sets $S_k \supseteq E_1, T_k \supseteq E_2$ such that

$$|S_k \setminus E_1| < \frac{1}{k}, |T_k \setminus E_2| < \frac{1}{k}.$$

We may assume $S_{k+1} \subseteq S_k, T_{k+1} \subseteq T_k$ (if $S_{k+1} \not\subseteq S_k$, then define $S'_{k+1} = S_{k+1} \cap S_k$ instead).

Since S_k is open, $S_k = \bigcup_{i \in \mathbb{N}} I_i$ for some nonoverlapping closed intervals. Similarly, $T_k = \bigcup_{j \in \mathbb{N}} J_j$ for some nonoverlapping closed intervals.

So

$$\begin{aligned} |S_k \times T_k| &= \left| \bigcup_{(i,j) \in \mathbb{N} \times \mathbb{N}} (I_i \times J_j) \right| = \sum_{i,j \in \mathbb{N}} |I_i \times J_j| \\ &= \sum_{i,j \in \mathbb{N}} |I_i||J_j| = \left(\sum_{i \in \mathbb{N}} |I_i| \right) \left(\sum_{j \in \mathbb{N}} |J_j| \right) = |S_k||T_k|. \end{aligned}$$

Write

$$S = \bigcap_{k=1}^{\infty} S_k, T = \bigcap_{k=1}^{\infty} T_k.$$

Then $|S \setminus E_1| = |T \setminus E_2| = 0$. Hence

$$|E_1 \times E_2| = |S \times T| = \lim_{k \to \infty} |S_k \times T_k| = \lim_{k \to \infty} |S_k||T_k| = |E_1||E_2|,$$

where the second equality follows by the Monotone Convergence Theorem for measure, since $S_k \times T_k \searrow S \times T$ and $|S_k \times T_k| < \infty$ for some k since $|E_1|, |E_2|$ are both finite. The last equality also follows by the Monotone Convergence Theorem for measure.

(ii) Suppose one of $|E_1|, |E_2|$ is infinite. If $|E_1| = \infty$ and $|E_2| > 0$, then write $E_1^n = E_1 \cap [-n, n]$.

$$|E_1 \times E_2| = \lim_{n \to \infty} |E_1^n \times E_2| = \lim_{n \to \infty} |E_1^n||E_2| = |E_1||E_2| = \infty,$$

where the first equality follows by the Monotone Convergence Theorem for measure, since $E_1^n \times E_2 \nearrow E_1 \times E_2$.

If $|E_1| = \infty$ and $|E_2| = 0$, $|E_1 \times E_2| = 0$ by our first lemma.

$\square$

Problem 3.7

Motivated by (3.7), define the inner measure of E by $|E|_i = \sup |F|$, where the supremum is taken over all closed subsets F of E. Show that

(i) $|E|_i \leqslant |E|_e$.

(ii) if $|E|_e < +\infty$, then E is measurable if and only if $|E|_i = |E|_e$.

Proof.

(i) Let G be an open set containing E, and let F be a closed set contained in E. Then $F \subset G$. We have $|F| \leqslant |G|$. Therefore,
$$|E|_e \leqslant |G|_e = |G|.$$
This inequality holds for all open sets G containing E, so we obtain
$$|E|_i = \sup |F|_i = \sup |F| \leqslant \inf |G| = |E|_e.$$

(ii) By the characterizations of measurability (Theorem 3.28), we can find a set K of type F_σ such that
$$E = K \cup N \quad \text{ where } |N| = 0$$
For each integer n, by the definition of inner measure, we can obtain a closed set $F_n \subset E$ such that
$$|F_n| > |E|_e - \frac{1}{n}.$$
Define $K := \bigcup_{n\in\mathbb{N}} F_n$ and we have $|K| = |E|_e$.
Now let $N = E - K$. We have
$$|E-K|_e = \left| E - \bigcup_{n\in\mathbb{N}} K_n \right|_e = \left| \bigcap_{n\in\mathbb{N}} (E-K_n) \right|_e \leqslant \lim_{n\to\infty} |E-K_n|_e$$
Where $K_n = \bigcup_{k=1}^{n} F_k$ is an increasing sequence of closed sets and $K = \bigcup_{n\in\mathbb{N}} K_n$.
For each integer n, by the definition of outer measure, there is an open set $G_n \supseteq E$ such that
$$|G| < |E|_e + \frac{1}{n}.$$
Thus,
$$|E-K_n|_e \leqslant |G-K_n| = |G| - |K_n| < |E|_e + \frac{1}{n} - |K_n|.$$
Then take limit about $n \to \infty$, since $\lim_{n\to\infty} |K_n| = |E|_e$, we get $\lim_{n\to\infty} |E-K_n|_e = 0$.

□

Remark. For more information about inner measure, one can consult [Kna07, Chapter V. section 5] and [MA09, Chapter 3, section 1].

Problem 3.8

Show that the conclusion of part (ii) of the previous exercise is false if $|E|_e = +\infty$.

Proof. Take any non-measurable set $E_1 \subset [0,1]$, and let $E_2 = [2,+\infty]$. Denote $E = E_1 \cup E_2$. Then by the monotone property of inner and outer measure, we obtain $|E|_i = |E|_e = +\infty$, but E is non-measurable. □

Problem 3.9

If E is measurable and A is any subset of E, show that $|E| = |A|_i + |E-A|_e$.

Proof. [MA09] We first prove that: for any set $A \subset \mathbb{R}^n$, there are two measurable sets F, G such that

(i) $F \subset A \subset G$.
(ii) $|F| = |A|_i, |G| = |A|_e$.
(iii) $|G - A|_i = 0$.

We only need to prove the existence of G, and F follows analogously. Consider first the case where $|A|_e < +\infty$. For each integer n, there is an open set G_n such that $A \subset G_n$ and
$$|G_n| \leqslant |A|_e + \frac{1}{2^n}.$$
Put
$$G = \bigcap_{n=1}^{\infty} G_n.$$
Then
$$|G| \leqslant |G_n| \leqslant |A|_e + \frac{1}{2^n}, \quad n \in \mathbb{N}.$$
As $n \to \infty$, we obtain $|G| \leqslant |A|_e$. On the other hand, $A \subset G$, whence $|A|_e \leqslant |G|_e = |G|$. Suppose that $|G - A|_i = 2\eta > 0$. Then there is a closed set $K \subset G - A$ such that $|K| \geqslant \eta$. For each integer n, the set $G_n - K$ is open, $A \subset G_n - K$, whence
$$|A|_e \leqslant |G_n - K| = |G_n| - |K| \leqslant |G_n| - \eta.$$
Thus
$$|G_n| \leqslant |G_n| + \frac{1}{2^n} - \eta,$$
or
$$\frac{1}{2^n} - \eta \geqslant 0.$$
Let $n \to \infty$, we obtain a contradiction.
If $|A|_e = +\infty$, then do the decomposition as follows: define $A_k = A \cap (B(0,k) - B(0,k-1))$, and
$$|G_{k,n}| \leqslant |A_k|_e + \frac{1}{2^n}, \quad \forall k \in \mathbb{N}.$$
Put
$$G = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} G_{k,n}.$$
And do the same argument as $|A|_e < +\infty$.
Next, for A and $E - A$, there are measurable sets $F \subset A$ and $G \supset E - A$ such that $|F| = |A|_i$ and $|G| = |E - A|_e$. Since
$$E - F - G \subset G - (E - A),$$
we obtain $|E - F - G| \leqslant |G - (E - A)|_i = 0$. Thus,
$$|E| = |E - F - G| + |G| + |F| = |G| + |F| = |E - A|_e + |A|_i.$$
□

Remark. [NB16] For A bounded case, we can prove $|E - A|_e + |A|_i \leqslant |E|$ as follows: For any $\varepsilon > 0$, take a closed set F such that
$$F \subset A, \quad |F| > |A|_i - \varepsilon.$$
Let $G = E - F$, then G is an open set containing $E - A$. Thus, we obtain
$$|E - A|_e \leqslant |G| = |E| - |F| < |E| - |A|_i + \varepsilon.$$
Since ε is arbitrary, we have
$$|A|_i + |E - A|_e \leqslant |E|.$$

Problem 3.10

Show that there exist sets $E_1, E_2, \cdots, E_k, \cdots$ such that $E_k \downharpoonleft E$, $|E_k|_e < +\infty$, and $\lim_{k\to\infty} |E_k|_e > |E|_e$ with strict inequality.

Proof. The idea is to use a non-measurable set as a counterexample to the "continuity from above" property that holds for measurable sets. Vitali set given by Theorem 3.38 can be used to construct such a counterexample[Wan13, p.143].

Classify all points in $[-1/2, 1/2]$ as follows: two points x and y are said to belong to the same class if and only if $x-y$ is a rational number. For $x \in [-1/2, 1/2]$, the set of all points of the form $x+r$ (where r is a rational number) is grouped into a class $K(x)$. Thus, for each x, there is a corresponding class $K(x)$, and $x \in K(x)$.

Next, we can prove that different classes $K(x)$ and $K(y)$ are disjoint. If they intersect, then there exists $z \in K(x) \cap K(y)$. Therefore,

$$z = x + r_x = y + r_y,$$

where r_x and r_y are rational numbers, so

$$y = x + r_x - r_y.$$

Now, assume $t \in K(y)$, then

$$t = y + r = x + (r_x - r_y + r) = x + r',$$

which implies $t \in K(x)$, and thus $K(y) \subset K(x)$. Similarly, we can show $K(x) \subset K(y)$, so $K(x) = K(y)$. This contradicts the assumption.

After classifying $[-1/2, 1/2]$ as above, we select one representative element from each class, and the set of all such points is denoted by A. We now prove that A is a non-measurable set.

Let the set of all rational points in $[-1, 1]$ be

$$r_0 = 0, r_1, r_2, r_3, \cdots.$$

Let A be translated by

$$\varphi_k(x) = x + r_k$$

to obtain the set A_k. If $x \in A$, then $\varphi_k(x) \in A_k$; and if $x \in A_k$, then $x - r_k \in A$. In particular, $A_0 = A$. By the invariance of inner and outer measures under translation, we have

$$|A_k|_i = |A|_i = \alpha, \quad |A_k|_e = |A|_e = \beta.$$

First, we prove

$$\beta > 0. \tag{1}$$

To do this, we first prove

$$[-1/2, 1/2] \subset \bigcup_{k=0}^{\infty} A_k. \tag{2}$$

Indeed, when $x \in [-1/2, 1/2]$, x must belong to one of the classes defined above. Let x_0 be the representative element of this class. Then $x - x_0$ is a rational number and must lie in $[-1, 1]$, so

$$x - x_0 = r_k,$$

and $x \in A_k$. Thus, (2) is proved. Since

$$1 = |[-1/2, 1/2]|_e \leqslant \left| \bigcup_{k=0}^{\infty} A_k \right|_e \leqslant \sum_{k=0}^{\infty} |A_k|_e,$$

we have

$$1 \leqslant \beta + \beta + \beta + \cdots ,$$

which shows that (1) is true.
On the other hand, We can prove

$$\alpha = 0. \tag{3}$$

Indeed, when $n \neq m$,

$$A_n \cap A_m = \varnothing. \tag{4}$$

If there is a point $z \in A_n \cap A_m$, then

$$x_n = z - r_n \quad \text{and} \quad x_m = z - r_m$$

(which are clearly different) both belong to A but to different classes. This is impossible since

$$x_n - x_m = r_n - r_m$$

is a rational number. Thus, we obtain (4). Moreover, for any k,

$$A_k \subset [-3/2, 3/2]$$

(because $x \in A_k$ implies $x = x_0 + r_k$, where $|x_0| \leqslant 1/2, |r_k| \leqslant 1/2$). Therefore,

$$\bigcup_{k=0}^{\infty} A_k \subset [-3/2, 3/2]. \tag{5}$$

From (5) and (4), we obtain

$$3 = |[-3/2, 3/2]|_i \geqslant \left| \bigcup_{k=0}^{\infty} A_k \right|_i \geqslant \sum_{k=0}^{\infty} |A_k|_i,$$

Thus,

$$\alpha + \alpha + \alpha + \cdots \leqslant 3,$$

which implies $\alpha = 0$. This is (3). Combining (1) and (3), we obtain

$$|A|_i < |A|_e,$$

so A is a non-measurable set.
Then for each integer n, define

$$E_n = \bigcup_{k \geqslant n} A_k,$$

then E_n is decreasing and by $A_m \cap A_n = \varnothing$ for any $m \neq n$, we obtain

$$E = \lim_{n \to \infty} E_n = \bigcap_{n \geqslant 1} E_n = \varnothing.$$

However,

$$|E_n|_e = \left| \bigcup_{n \geqslant k} A_k \right|_e \geqslant |A_n|_e = |A|_e > 0.$$

$\square$

Homework Solution 6-7

Problem 4.1

Let f be defined and measurable in $\mathbb{R}^n$. If T is a nonsingular linear transformation of $\mathbb{R}^n$, show that $f(Tx)$ is measurable. (If $E_1 = \{x : f(x) > a\}$ and $E_2 = \{x : f(Tx) > a\}$, show that $E_2 = T^{-1}E_1$.)

Proof. For any $x \in E_2$, there is an element $x' \in \mathbb{R}^n$ such that $x = T^{-1}x'$ and $f(x') = f(TT^{-1}x') = f(Tx) > a$. Hence, $E_2 \subset T^{-1}E_1$. And $E_2 \supset T^{-1}E_1$ follows analogously. By Theorem 3.35, we obtain E_2 is measurable. And so $f \circ T$ is measurable. □

Problem 4.2

Let f be upper semicontinuous and less than $+\infty$ on a compact set E. Show that f is bounded above on E. Show also that f assumes its maximum on E, that is, that there exists $x_0 \in E$ such that $f(x_0) \geqslant f(x)$ for all $x \in E$.

Proof. [Xio20] Let $F_n = \{x \in E : f(x) < n\}$ where n is any integer. Since f is upper semicontinuous, $\{F_n\}_{n\in\mathbb{Z}}$ is an open cover of E. Then there is a finite subcover $\{F_{n_i}\}_{i=1}^{k}$ such that

$$F_{n_1} \cup F_{n_2} \cdots \cup F_{n_k} \supset E.$$

Then

$$f(E) \subset f(F_{n_1}) \cup f(F_{n_2}) \cdots \cup f(F_{n_k}) \subset F_N, \quad N = \max_{1\leqslant i\leqslant k} n_i.$$

So f is bounded above on E.
Let $M = \sup_{x\in E} f(x)$. Then for any integer m there is some $x_m \in E$ such that

$$M - \frac{1}{m} < f(x_m) \leqslant M.$$

Then by the upper semicontinuity of f, the set $G_m := f^{-1}([M - 1/m, M])$ is a non-empty bounded closed set. Then by Cantor's intersection theorem,

$$G = \bigcap_{m=1}^{\infty} G_m \neq \varnothing.$$

That is, there is some $x_0 \in G$ such that $f(x_0) = M$, as required. □

Problem 4.3

(i) If $\{f_k\}$ is a sequence of functions that are upper semicontinuous at x_0, show that $\inf_k f_k(x)$ is upper semicontinuous at x_0.

(ii) If $\{f_k\}$ is a sequence of functions that are upper semicontinuous at x_0 and converge uniformly near x_0, show that $\lim f_k$ is upper semicontinuous at x_0.

Proof. [Fol99, p.218]

(i) Let $f = \inf_k f_k$, we need to prove

$$f^{-1}([-\infty, a)) = \bigcup_{k=1}^{\infty} f_k^{-1}([-\infty, a)), \quad \forall a > -\infty.$$

On the one hand, for any $x \in f^{-1}([-\infty, a))$, i.e. $\inf_k f_k(x) = f(x) < a$. For any $n \in \mathbb{N}$, there is an integer $k = k(n)$ such that

$$f(x) \leqslant f_k(x) < f(x) + \frac{1}{n}.$$

Let $n = \left[\frac{1}{a-f(x)}\right] + 1$ and we can find k such that $f_k(x) < a$. On the other hand, for some k_0 such that $f_{k_0}(x) < a$, we obtain $f(x) = \inf_k f_k(x) \leqslant f_{k_0}(x) < a$, i.e. $x \in f^{-1}([-\infty, a))$.

(ii) Let $\{f_n\}$ converge uniformly on $B(x_0, \delta)$. Choose any $\alpha > f(x_0)$, and $\varepsilon := \dfrac{\alpha - f(x_0)}{3}$. Since $\{f_n\}$ converges uniformly to f, there is an integer N such that $|f_N(x) - f(x)| < \varepsilon$ for any $x \in B(x_0, \delta)$. Since f_N is upper semicontinuous, there is $r \in (0, \delta)$ such that $f_N(x) < f_N(x_0) + \varepsilon$ for every $x \in B(x_0, r)$. Hence, for any $x \in B(x_0, r)$, we obtain

$$f(x) < f_N(x) + \varepsilon < f_N(x_0) + 2\varepsilon < f(x_0) + 3\varepsilon = \alpha.$$

Since α is arbitrary, f is upper semicontinuous at x_0.

□

Problem 4.4

(i) Show that the limit of a decreasing (increasing) sequence of functions upper semicontinuous (lower semicontinuous) at x_0 is upper semicontinuous (lower semicontinuous) at x_0. In particular, the limit of a decreasing (increasing) sequence of functions continuous at x_0 is upper semicontinuous (lower semicontinuous) at x_0.

(ii) Let f be upper semicontinuous and less than $+\infty$ on $[a, b]$. Show that there exist continuous f_k on $[a, b]$ such that $f_k \searrow f$. (First show that there are upper semicontinuous step functions $f_k \searrow f$.)

Proof.

(i) Since for any decreasing sequence $\{f_k\}$, we have $\lim_k f_k(x) = \inf_k f_k(x)$. So the conclusion follows from Problem 4.3.

(ii) By Problem 4.2, we can assume f is non-positive. For any $x \in [a, b]$, since

$$f(x) = \lim_{m \to \infty} \left(f(x) - \frac{1}{m} \right),$$

we can assume $f(x) < 0$. Then for any integer n, $F_{x,n} := \{y : f(y) < f(x) - 1/n\}$ is an open neighborhood of x. By the compactness of $[a, b]$, there are finitely many open sets $\{F_{x_i,n}\}_{i=1}^{k_n}$ covering $[a, b]$. On each set $F_{x_i,n}$ we can construct a continuous function $g_{x_i,n}$ such that $g_{x_i,n}(x_i) = f(x_i) - \dfrac{1}{n}$, $g_{x_i,n}(x) < 0$ on a compact subset of $F_{x_i,n}$ and $g_{x_i,n}(x) = 0$ otherwise (this can be done by representing $F_{x_i,n}$ as a union of at most countable disjoint open intervals, and constructing on every open intervals). Let

$$g_n = \sum_{i=1}^{k} g_{x_i,n} \geqslant f.$$

To ensure $g_n \searrow f$, we can use induction and construct g_n based on $f - g_{n-1}$ (which is also non-positive and upper semicontinuous).

□

Problem 4.5

Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable E with $|E| < +\infty$. If $|f_k(x)| \leqslant M_x < +\infty$ for all k for each $x \in E$, show that given $\varepsilon > 0$, there is a closed $F \subset E$ and a finite M such that $|E \setminus F| < \varepsilon$ and $|f_k(x)| \leqslant M$ for all k and all $x \in F$.

Proof. Let $\varepsilon > 0$. For each $n \in \mathbb{N}$, define

$$E_n := \{x \in E : |f_k(x)| \leqslant n \text{ for all } k\} = \bigcap_{k=1}^{\infty} \{x \in E : |f_k(x)| \leqslant n\}.$$

Note that each E_n is measurable since each f_k is measurable.
Since every $M_x < \infty$, we have that $E_n \nearrow E$. By the Monotone Convergence Theorem for measure, $\lim_{n\to\infty} |E_n| = |E| < \infty$. Thus, there is N such that

$$|E| - |E_N| = |E \setminus E_N| < \varepsilon/2.$$

Let F be a closed set contained in E_N such that $|E_N \setminus F| < \varepsilon/2$. Then $|E \setminus F| = |E \setminus E_N| + |E_N \setminus F| < \varepsilon$ and $|f_k(x)| \leqslant N$ for all k and all $x \in F$. □

Problem 4.6

Suppose that $f_k \xrightarrow{m} f$ and $g_k \xrightarrow{m} g$ on E. Show that $f_k + g_k \xrightarrow{m} f + g$ on E and, if $|E| < +\infty$, that $f_k g_k \xrightarrow{m} fg$ on E. If, in addition, $g_k \to g$ on E, $g \neq 0$ a.e., and $|E| < +\infty$, show that $f_k/g_k \xrightarrow{m} f/g$ on E. (For the product $f_k g_k$, write $f_k g_k - fg = (f_k - f)(g_k - g) + f(g_k - g) + g(f_k - f)$. Consider each term separately, using the fact that a function that is finite on E, $|E| < +\infty$, is bounded outside a subset of E with small measure.)

Proof.

(i) Assume that $f_k \xrightarrow{m} f$ and $g_k \xrightarrow{m} g$ on E. Let $\varepsilon > 0$. There is K_1 such that $|\{|f - f_k| > \varepsilon/2\}| < \varepsilon/2$ if $k > K_1$. Similarly, there is K_2 such that $|\{|g - g_k| > \varepsilon/2\}| < \varepsilon/2$ if $k > K_2$. Since

$$\{|(f+g) - (f_k + g_k)| > \varepsilon\} \subseteq \{|f - f_k| > \varepsilon/2\} \cup \{|g - g_k| > \varepsilon/2\},$$

thus

$$|\{|(f+g) - (f_k + g_k)| > \varepsilon\}| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

if $k > \max\{K_1, K_2\}$. This implies $f_k + g_k \xrightarrow{m} f + g$ on E.

(ii) Further assume $|E| < \infty$. Write

$$f_k g_k - fg = (f_k - f)(g_k - g) + f(g_k - g) + g(f_k - f).$$

There is K_3 such that $|\{|f-f_k|>\sqrt{\varepsilon}\}|<\varepsilon/2$ if $k>K_3$. There is K_4 such that $|\{|g-g_k|>\sqrt{\varepsilon}\}|<\varepsilon/2$ if $k>K_4$. If $k>\max\{K_3,K_4\}$, then

$$|\{|(f_k-f)(g_k-g)|>\varepsilon\}|\leqslant|\{|f-f_k|>\sqrt{\varepsilon}\}|+|\{|g-g_k|>\sqrt{\varepsilon}\}|<\varepsilon.$$

Thus $(f_k-f)(g_k-g)\xrightarrow{m}0$. There is a closed $F\subseteq E$ and a finite M such that $|E\setminus F|<\varepsilon/2$ and $|f(x)|\leqslant M$ for all $x\in F$. We may assume $M\neq 0$, as the case $M=0$ is trivially true.

Thus

$$\begin{aligned}&|\{|f(g_k-g)|>\varepsilon\}|\\=&|\{x\in F:|f(g_k-g)|>\varepsilon\}|+|\{x\in E\setminus F:|f(g_k-g)|>\varepsilon\}|\\\leqslant&|\{x\in F:|g_k-g|>\varepsilon/M\}|+|E\setminus F|<\varepsilon/2+\varepsilon/2\end{aligned}$$

for sufficiently large k. Thus $f(g_k-g)\xrightarrow{m}0$. Similarly $g(f_k-f)\xrightarrow{m}0$. By (i), $f_kg_k-fg\xrightarrow{m}0$, that is, $f_kg_k\xrightarrow{m}fg$ on E.

(iii) Assume in addition that $g_k\to g$ on E, $g\neq 0$ a.e., and $|E|<\infty$. Note that since we have proved (ii), it suffices to show that $1/g_k\xrightarrow{m}1/g$ on E.
We use Theorem 4.21 (a.e. convergence and $|E|<\infty$ implies convergence in measure). Note that since $g\neq 0$ a.e., $1/g$ is measurable and finite a.e. on E. Since $g_k\to g$ on E, for sufficiently large k, $g_k\neq 0$ a.e. so that $1/g_k$ is also measurable and finite a.e. in E. By Theorem 4.21, since $1/g_k\to 1/g$ a.e. on E and $|E|<\infty$, then $1/g_k\xrightarrow{m}1/g$ on E.

□

Problem 4.7

If f is measurable on E, define $\omega_f(a)=|\{f>a\}|$ for $-\infty<a<+\infty$. If $f_k\nearrow f$, show that $\omega_{f_k}\nearrow\omega_f$. If $f_k\xrightarrow{m}f$, show that $\omega_{f_k}\to\omega_f$ at each point of continuity of ω_f. (For the second part, show that if $f_k\xrightarrow{m}f$, then $\limsup_{k\to\infty}\omega_{f_k}(a)\leqslant\omega_f(a-\varepsilon)$ and $\liminf_{k\to\infty}\omega_{f_k}(a)\geqslant\omega_f(a+\varepsilon)$ for every $\varepsilon>0$.)

Proof. Since $f_k\nearrow f$, we have

$$\omega_{f_k}(a)=|\{f_k>a\}|\leqslant|\{f_{k+1}>a\}|=\omega_{f_{k+1}}(a).$$

By the monotonicity of $\{f_k>a\}$ and the Monotone Convergence Theorem for measure, we obtain $\omega_{f_k}\nearrow\omega_f$.

Now assume $f_k\xrightarrow{m}f$. Let $\varepsilon>0$ and define $E_k=\{|f_k-f|>\varepsilon\}$. For any $\delta>0$, there is K such that $|E_k|<\delta$ for all $k\geqslant K$. From the decomposition $f_k=(f_k-f)+f$, we derive

$$\{f_k>a\}\subseteq E_k\cup\{f>a-\varepsilon\}.$$

Thus for all $k\geqslant K$:

$$\omega_{f_k}(a)\leqslant|E_k|+\omega_f(a-\varepsilon)<\delta+\omega_f(a-\varepsilon).$$

This establishes

$$\limsup_{k\to\infty}\omega_{f_k}(a)\leqslant\omega_f(a-\varepsilon).$$

Analogously, from $f=(f-f_k)+f_k$ we obtain:

$$\{f>a+\varepsilon\}\subseteq E_k\cup\{f_k>a\},$$

which implies for all $k \geqslant K$

$$\omega_f(a+\varepsilon) \leqslant |E_k| + \omega_{f_k}(a) < \delta + \omega_{f_k}(a).$$

Hence

$$\liminf_{k\to\infty} \omega_{f_k}(a) \geqslant \omega_f(a+\varepsilon).$$

At continuity points a of ω_f, taking $\varepsilon \to 0$ yields

$$\limsup_{k\to\infty} \omega_{f_k}(a) \leqslant \omega_f(a) \leqslant \liminf_{k\to\infty} \omega_{f_k}(a).$$

Therefore

$$\lim_{k\to\infty} \omega_{f_k}(a) = \omega_f(a).$$

$\square$

Lemma. *For $f_k \nearrow f$, the following set equality holds*

$$\bigcup_{k=1}^{\infty} \{f_k > a\} = \{f > a\}.$$

Proof. The inclusion $\bigcup_{k=1}^{\infty} \{f_k > a\} \subseteq \{f > a\}$ is immediate from monotonicity. For the reverse inclusion: given $x \in \{f > a\}$, there is K such that $f_K(x) > a$ since $f_k(x) \nearrow f(x)$. Thus $x \in \{f_K > a\}$. $\square$

Problem 4.8

If f is measurable and finite a.e. on $[a,b]$, show that given $\varepsilon > 0$, there is a continuous g on $[a,b]$ such that $|\{x : f(x) \neq g(x)\}| < \varepsilon$. (See Exercise 18 of Chapter 1.) Formulate and prove a similar result in $\mathbb{R}^n$ by combining Lusin's theorem with the Tietze extension theorem.

Proof. According to Lusin's Theorem, there is a compact set $K \subseteq A$ such that $|A - K| < \varepsilon$ and the restriction $f|_K$ is continuous. Then by Tietze extension theorem, there is a continuous function $\bar{f} : \mathbb{R}^n \to \mathbb{R}$ that agrees with f on K, satisfying

$$|\{x \in A \mid \bar{f}(x) \neq f(x)\}| \leqslant |A - K| < \varepsilon.$$

$\square$

Theorem (Tietze Extension Theorem)**.** *[EG18, Chapter 1 Section 2], see also [Zho16]. Suppose $K \subseteq \mathbb{R}^n$ is compact and $f : K \to \mathbb{R}$ is continuous. Then there is a continuous extension $\bar{f} : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\bar{f} = f \text{ on } K.$$

Proof.
(1) Let $U := \mathbb{R}^n - K$. For $x \in U$ and $s \in K$, define

$$u_s(x) := \max\left\{2 - \frac{|x-s|}{\operatorname{dist}(x,K)}, 0\right\}.$$

This satisfies

$$\begin{cases} x \mapsto u_s(x) \text{ is continuous on } U, \\ 0 \leqslant u_s(x) \leqslant 1, \\ u_s(x) = 0 \text{ when } |x-s| \geqslant 2\operatorname{dist}(x,K). \end{cases}$$

Take a countable dense subset $\{s_j\}_{j=1}^{\infty}$ of K and define

$$\sigma(x) := \sum_{j=1}^{\infty} 2^{-j} u_{s_j}(x) \quad (x \in U)$$

where $0 < \sigma(x) \leqslant 1$. Then define

$$v_k(x) := \frac{2^{-k} u_{s_k}(x)}{\sigma(x)} \quad (x \in U, k = 1, 2, \dots)$$

The family $\{v_k\}_{k=1}^{\infty}$ forms a partition of unity on U. Set

$$\bar{f}(x) := \begin{cases} f(x), & x \in K \\ \sum_{k=1}^{\infty} v_k(x) f(s_k), & x \in U. \end{cases}$$

By the Weierstrass M-test, $\bar{f}$ is continuous on U.

(2) We now show that for any $a \in K$,

$$\lim_{\substack{x \to a \\ x \in U}} \bar{f}(x) = f(a).$$

Fix $\varepsilon > 0$. There exists $\delta > 0$ such that $|f(a) - f(s_k)| < \varepsilon$ when $|a - s_k| < \delta$. For $x \in U$ with $|x - a| < \delta/4$, if $|a - s_k| \geqslant \delta$, then

$$\delta \leqslant |a - s_k| \leqslant |a - x| + |x - s_k| < \frac{\delta}{4} + |x - s_k|,$$

hence

$$|x - s_k| > \frac{3}{4}\delta > 2|x - a| \geqslant 2\mathrm{dist}(x, K).$$

Thus $v_k(x) = 0$ when $|x - a| < \delta/4$ and $|a - s_k| \geqslant \delta$. Since $\sum_{k=1}^{\infty} v_k(x) = 1$, we have for $|x - a| < \delta/4$ and $x \in U$:

$$\bar{f}(x) - f(a)| \leqslant \sum_{k=1}^{\infty} v_k(x) |f(s_k) - f(a)| < \varepsilon.$$

$\square$

Problem 4.9

Show that the necessity part of Lusin's theorem is not true for $\varepsilon = 0$, that is, find a measurable set E and a finite measurable function f on E such that f is not continuous relative to $E \setminus Z$ for any Z with $|Z| = 0$. (Consider, e.g., χ_E for the set E in Exercise 25 of Chapter 3.)

Proof. [Zho07] Construct a Cantor-like set H in $[0,1]$ with $m(H) = 1/2$. Define the function

$$f(x) = \begin{cases} 1, & x \in H, \\ -1, & x \in [0,1] \setminus H. \end{cases}$$

It is easy to see that $([0,1] \setminus Z) \cap H \neq \varnothing$. Therefore, for any $x_0 \in ([0,1] \setminus Z) \cap H$, in the interval $(x_0 - \delta, x_0 + \delta)$ there are always points from $[0,1] \setminus (Z \cup H)$. This shows that $f(x)$ is discontinuous at $x = x_0$. $\square$

Problem 4.10

Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions defined on a measurable set E. Show that the sets

$\{x : \lim f_k(x) \text{ exists and is finite}\}, \quad \{x : \lim f_k(x) = +\infty\}, \quad \{x : \lim f_k(x) = -\infty\}$

are measurable.

Proof. [Zho16] We first prove that: for any real-valued function f defined on E. The set D of all points x where $f_k(x)$ does not converge to $f(x)$ can be expressed as

$$D = \bigcup_{n=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \left\{ x \in E : |f_k(x) - f(x)| \geqslant \frac{1}{n} \right\} =: D'. \tag{*}$$

Then let

$$E_{k,j}^n = \left\{ x \in E : |f_k(x) - f_j(x)| < \frac{1}{n} \right\}.$$

The convergence set of $\{f_k(x)\}$ is

$$E = \bigcap_{n=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} \bigcap_{j=N}^{\infty} E_{k,j}^n.$$

Hence it is measurable.

Proof of (*): on the one hand, for any $x_0 \in D$, there is $\varepsilon_0 = \varepsilon_0(x_0) > 0$ such that for any integer $K > 0$, there is some $k \geqslant N$ such that

$$|f_k(x_0) - f(x_0)| \geqslant \varepsilon_0.$$

Let $n = [1/\varepsilon_0] + 1$ (which is depend on x_0), we obtain $x_0 \in D'$ by the definition of the upper limit set. On the other hand, we only need to observe that the sets

$$\bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \left\{ x \in E : |f_k(x) - f(x)| \geqslant \frac{1}{n} \right\}$$

is monotone. For $x_0 \in D'$, then there is some integer n such that x_0 is in the upper limit set of $\left\{ x \in E : |f_k(x) - f(x)| \geqslant \frac{1}{n} \right\}$.

For $\{x \in E : \limsup f_k(x) = \limsup f_k(x) = +\infty\}$, we only need to notice that

$$\bigcap_{M=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} \{|f_k(x)| > M\}$$

$\square$

Problem 4.11

If $f(x), x \in \mathbb{R}^1$, is continuous at almost every point of an interval $[a,b]$, show that f is measurable on $[a,b]$. Generalize this to functions defined in $\mathbb{R}^n$. (For a constructive proof, use the subintervals of a sequence of partitions to define a sequence of simple measurable functions converging to f a.e. in $[a,b]$. Use Theorem 4.12. See also the proof of Theorem 5.54.)

Proof. Assume f is continuous at almost every point of $[a,b]$. Define the sets:

$$E = \{x \in [a,b] : f \text{ is continuous at } x\}, \quad Z = [a,b] \setminus E.$$

By hypothesis, $|Z| = 0$. Both Z and $E = [a,b] \setminus Z$ are measurable.

For any $\alpha \in \mathbb{R}$, decompose the superlevel set as
$$\{f > \alpha\} = \underbrace{\{x \in E : f(x) > \alpha\}}_{\text{(I)}} \cup \underbrace{\{x \in Z : f(x) > \alpha\}}_{\text{(II)}}.$$

(I) The restriction $f|_E$ is continuous, hence measurable on E. Thus $\{x \in E : f(x) > \alpha\}$ is relatively open in E and therefore measurable.

(II) Since $\{x \in Z : f(x) > \alpha\} \subseteq Z$ and $|Z| = 0$, this subset is measurable with measure zero.

The union of two measurable sets remains measurable, proving that f is Lebesgue measurable on $[a,b]$. □

Problem 4.12

One difficulty encountered in trying to extend the proof of Egorov's theorem to the continuous parameter case $f_y(x) \to f(x)$ as $y \to y_0$ is showing that the analogues of the sets E_m in Lemma 4.18 are measurable. This difficulty can often be overcome in individual cases. Suppose, for instance, that $f(x,y)$ is defined and continuous in the square $0 \leqslant x \leqslant 1, 0 < y \leqslant 1$ and that $f(x) = \lim_{y \to 0} f(x,y)$ exists and is finite for x in a measurable subset E of $[0,1]$. Show that if ε and δ satisfy $0 < \varepsilon, \delta < 1$, the set $E_{\varepsilon,\delta} = \{x \in E : |f(x,y) - f(x)| \leqslant \varepsilon \text{ for all } y < \delta\}$ is measurable. (If $y_k, k = 1,2,\ldots,$ is a dense subset of $(0,\delta)$, show that $E_{\varepsilon,\delta} = \bigcap_k \{x \in E : |f(x,y_k) - f(x)| \leqslant \varepsilon\}$.)

Proof. Let $E'_{\varepsilon,\delta} = \bigcap_k \{x \in E : |f(x,y_k) - f(x)| \leqslant \varepsilon\}$. We only need to prove $E'_{\varepsilon,\delta} \subset E_{\varepsilon,\delta}$. Given fixed $x_0 \in E'_{\varepsilon,\delta}$, Since $f(x_0,\cdot)$ is continuous on $(0,\delta)$. For any integer $n > 0$, there are a subsequence y_{k_j} convergence to y and a integer $J > 0$ such that for any $k_j > k_J$,
$$|f(x_0,y) - f(x_0,y_{k_j})| < \frac{1}{n}.$$
Then
$$|f(x_0,y) - f(x_0)| \leqslant |f(x_0,y) - f(x_0,y_{k_j})| + |f(x_0,y_{k_j}) - f(x_0)| \leqslant \varepsilon + \frac{1}{n}.$$
This holds for any $n > 0$. Hence we have $x \in E_{\varepsilon,\delta}$. □

Problem 4.13

Let $f(x,y)$ be as in Problem 4.12. Show that given $\varepsilon > 0$, there exists a closed $F \subset E$ with $|E - F| < \varepsilon$ such that $f(x,y)$ converges uniformly for $x \in F$ to $f(x)$ as $y \to 0$. (Follow the proof of Egorov's theorem, using the sets $E_{\varepsilon,1/m}$ defined in Problem 4.12 in place of the sets E_m in the proof of Lemma 4.18.)

Proof. We only need to prove a similar conclusion to Lemma 4.18. Let
$$E_{\varepsilon,1/m} = \{x \in E : |f(x) - f(x,y_k)| < \varepsilon \text{ for all } y_k < 1/m\}.$$
Thus, E_m is measurable. Clearly, $E_{\varepsilon,1/m} \subset E_{\varepsilon,1/(m+1)}$. Take $\varepsilon = \dfrac{1}{n}$ where $n > 0$ is any integer. Since $f_k \to f$ a.e. in E and f is finite, $E_{1/n,1/m} \nearrow E - Z_n$ where $|Z_n| = 0$.

Hence, by Theorem 3.26, $|E_{1/n,1/m}| \to |E - Z_n| = |E|$. Since $|E| < +\infty$, it follows that $|E - E_{1/n,1/m}| \to 0$ when $m \to \infty$. Choose $m_0 = m_0(n)$ so that $|E - E_{1/n,1/m_0}| < \dfrac{1}{2^{n+1}}\eta$, and let F be a closed subset of $E_{1/n,1/m_0}$ with $|E_{1/n,1/m_0} - F_n| < \dfrac{1}{2^{n+1}}\eta$. Then $|E - F_n| < \dfrac{1}{2^n}\eta$, and $|f(x) - f_k(x)| \leqslant \dfrac{1}{n}$ in F_n for all $k > m_0$. Let $F = \bigcup_n F_n$ and we have $|F| < \eta$. $\square$

Homework Solution 8

Problem 5.1

Let $f \in C([a,b])$. If there exists a function $g(x)$ defined on $[a,b]$ such that $g(x) = f(x)$ a.e. $x \in [a,b]$, must g be almost everywhere continuous on $[a,b]$? If yes, provide a proof; if no, give a counterexample.

Proof. No. Consider the Dirichlet function

$$g(x) = \begin{cases} 1, & x \in [0,1] \cap \mathbb{Q}, \\ 0, & x \in [0,1] - \mathbb{Q}. \end{cases}$$

It equals to zero almost everywhere on $[0,1]$, but it is discontinuous everywhere. $\square$

Problem 5.2

Let $f(x)$ be an almost everywhere continuous function on $\mathbb{R}$. Does there exist $g \in C(\mathbb{R})$ such that

$$g(x) = f(x), \quad \text{a.e.} \quad x \in \mathbb{R}?$$

Proof. No. Consider the sign function

$$f(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

Let $g \in C(\mathbb{R})$ be such that $f = g$ a.e. on $\mathbb{R}$. Since $f = g$ everywhere in $(0,\delta) - Z$ where Z is a measure-zero set, we can choose a sequence $\{x_k\}_k \subset (0,\delta) - Z$ which converges to zero such that

$$g(x_k) \to g(0), \quad k \to \infty.$$

From $g(x_k) = f(x_k) = 1$, we get $g(0) = 1$. But analogously, by choosing $x_k \to 0^-$, we get $g(0) = -1$, a contradiction. $\square$

Problem 5.3

Let $\{f_k\}_{k\in\mathbb{N}}$ be real-valued measurable functions on $E \subset \mathbb{R}^n$. If $\{f_k\}$ converges in measure to f on E, prove that there exists a subsequence $\{f_{k_i}\}$ such that

$$\lim_{i\to\infty} f_{k_i}(x) = f(x), \quad \text{a.e.} \quad x \in E.$$

(Do not use the approach that establishes convergence in measure via Cauchy sequences in measure.)

Proof. [XWYS10, p.132] By the definition of the convergence in measure, for any positive integer i, there is some positive integer k_i such that as long as $k \geqslant k_i$,

$$\left| \left\{ x \in E : |f_k(x) - f(x)| > \frac{1}{2^i} \right\} \right| < \frac{1}{2^i}.$$

Denote $E_i = \left\{x \in E : |f_{k_i}(x) - f(x)| > 2^{-i}\right\}$. By induction, we can choose proper k_i such that $k_1 < k_2 < \cdots$, and let

$$F_i = \bigcap_{j=i}^{\infty}(E - E_j).$$

Then $\{f_{k_i}\}$ converge to f everywhere on F_i, so as on $F := \bigcup_{i=1}^{\infty} F_i$. Then the key point is to prove $E - F$ is a measure-zero set. This comes from

$$E - F = \bigcap_{i=1}^{\infty}(E - F_i) = \bigcap_{i=1}^{\infty}\bigcup_{j=i}^{\infty} E_j = \limsup_{i\to\infty} E_i,$$

and

$$\sum_{i=1}^{\infty} |E_i| \leqslant \sum_{i=1}^{\infty} \frac{1}{2^i} = 1 < +\infty.$$

□

Problem 5.4

Suppose $\{f_k(x)\}$ converges to $f(x)$ almost everywhere on E, and converges to $g(x)$ in measure. Is the following relation true:

$$g(x) = f(x), \quad \text{a.e.} \quad x \in E\,?$$

Proof. Yes. By Problem 5.3, there is some subsequence $\{f_{k_i}\}$ such that f_{k_j} converge to f everywhere on $E - Z$ where $|Z| = 0$. So by the uniqueness of the everywhere limit we have

$$f = g \quad \text{on } E - (Z \cup Z')$$

where $|Z'| = 0$ and f_k converge to g everywhere on $E - Z'$. Since $|Z \cup Z'| = 0$, we obtain the conclusion. □

Problem 5.5

Let $\{f_k\}$ converge to zero in measure on E, and let g be a real-valued measurable function on E. If $|E| = +\infty$, show that $\{g(x)f_k(x)\}$ does not necessarily converge to zero in measure on E.

Proof. [Wan13, p.180] for instance, define the following functions on $E = [0, +\infty)$:

$$g(x) = x,$$

$$f_k(x) = \begin{cases} 0, & x \in [0, k), \\ 1/x, & x \in [k, +\infty), \end{cases}$$

Then, for any positive number σ, when n is sufficiently large, we have

$$\{x \in E : |f_k(x)| \geqslant \sigma\} = \varnothing.$$

Thus, $\{f_k\}$ converges in measure to zero on E. However, $\{gf_k\}$ does not converge in measure to zero on E, because for any positive integer k, it holds that

$$\{x \in E : |g(x)f_k(x) - 0| \geqslant 1\} = [k, +\infty),$$

and consequently,

$$\lim_{k\to\infty} |\{x \in E : |g(x)f_k(x)| \geqslant 1\}| = +\infty.$$

□

Problem 5.6

Let $|E| < +\infty$. If for every subsequence $\{f_{k_i}\}$ of $\{f_k\}$ there exists a further subsequence $\{f_{k_{i_j}}\}$ that converges to f on E, prove that f_k converges to f in measure on E.

Proof. [XWYS10][Zho07] Assume for contradiction that there is some $\varepsilon_0 > 0$ such that the numerical sequence $\{\mu(|f_k - f| > \varepsilon_0)\}_k$ does not converge to zero. Without loss of generality, we may assume that this sequence is bounded along a subsequence. Consequently, there is a convergent sub-subsequence, and thus a subsequence $\{f_{k_j}\}$ such that

$$\lim_{j\to\infty} \mu(|f_{k_j} - f| > \varepsilon_0) = \delta_0 > 0,$$

which shows that $\{f_{k_j}\}$ does not converge almost everywhere. □

Problem 5.7

Let $\{f_k\}$ converge to f in measure on E, and $\{g_k\}$ converge to g in measure on E. Prove:

1. $\{f_k \pm g_k\}$ converges to $f \pm g$ in measure on E
2. If $|E| < \infty$, then $\{f_k(x)g_k(x)\}$ converges to $f(x)g(x)$ in measure on E

Proof.

1. For any $\varepsilon > 0$, we only need to observe that
   $\{x \in E : |f_k + g_k - f - g| > \varepsilon\} \subset \{x \in E : |f_k - f| > \varepsilon/2\} \cup \{x \in E : |g_k - g| > \varepsilon/2\}$.
   Then the conclusion follows from the definition.
2. Take any subsequence $\{f_{k_i}g_{k_i}\}$ of $\{f_k g_k\}$, For $\{f_{k_i}\}$ and $\{g_{k_i}\}$, we use Riesz theorem, there is a subsequence k_{i_j} of k such that $f_{k_{i_j}} \to f$ and $g_{k_{i_j}} \to g$ a.e. on E. Then $\{f_{k_{i_j}} g_{k_{i_j}}\} \to fg$ a.e. on E. Thus by Problem 5.6, $\{f_k g_k\}$ converge to fg in measure.

□

Problem 5.8

Prove the Fatou's Lemma for convergence in measure: If $\{f_k\}$ is a sequence of non-negative measurable functions converging to f in measure on E, then

$$\int_E f(x)\mathrm{d}x \leqslant \liminf_{k\to\infty} \int_E f_k(x)\mathrm{d}x.$$

Proof. By Problem 1.1, let $a_k = \int_E f_k \mathrm{d}x$, then there is a subsequence $a_{k_i} \to \liminf \int_E f_k \mathrm{d}x$. The sequence $\{f_{k_i}\}$ converges to f in measure. Then there is a subsequence of this subsequence such that $f_{k_{i_j}} \to g$ a.e. on E. Then

$$\int_E g\mathrm{d}x = \int_E \liminf f_{k_{i_j}} \mathrm{d}x \leqslant \liminf \int_E f_{k_{i_j}} \mathrm{d}x = \lim \int_E f_{k_{i_j}} \mathrm{d}x = \lim \int_E f_{k_i} \mathrm{d}x = \liminf \int_E f_k \mathrm{d}x.$$

□

Homework Solution 9

Problem 6.1 (Lebesgue-Vitali Theorem, Finite Measure Case)

Let $|E| < \infty$, $\{f_k\} \subset L(E)$, and $\{f_k\}$ converge to f in measure on E. If $\{f_k(x)\}$ is equicontinuously integrable on E, i.e., for any $\varepsilon > 0$, there is $\delta = \delta(\varepsilon)$ such that for all measurable $A \subset E$ with $|A| < \delta$, we have

$$\int_A |f_k(x)|\mathrm{d}x < \varepsilon \quad \text{for all } k,$$

then $f \in L(E)$ and

$$\lim_{k\to\infty} \int_E f_k(x)\mathrm{d}x = \int_E f(x)\mathrm{d}x.$$

Proof. [Bog07] We can consider the numerical sequence

$$\left\{a_k := \int_E f_k(x)\mathrm{d}x\right\}_k$$

In fact, it suffices to show that each subsequence $\{b_k\}$ of $\{a_k\}$ contains a convergent subsequence $\{b_{k_j}\}$. Then the conclusion follows by the fact that the Remark 2 behind. Choose any subsequence g_k of f_k. Since g_k also converges to f in measure. By Riesz theorem, there is some subsequence g_{k_j} converging to f a.e. on E. For any $\varepsilon > 0$, there is $\delta = \delta(\varepsilon)$ such that for all measurable $A \subset E$ with $|A| < \delta$, we have

$$\int_A |g_{k_j}(x)|\mathrm{d}x < \varepsilon \quad \text{for all } j,$$

By Fatou's lemma, we have

$$\int_A |f(x)|\mathrm{d}x \leqslant \varepsilon.$$

For such A, we use Egoroff' s theorem, then we have g_k converge to f uniformly on $E - A$. So

$$\begin{aligned}\int_E |g_{k_j} - f|\mathrm{d}x &= \int_{E-A} |g_{k_j} - f|\mathrm{d}x + \int_A |g_{k_j} - f|\mathrm{d}x \\ &\leqslant |E| \sup_{\substack{Z\subset E:|Z|=0 \\ x\in E-Z}} |g_{k_j} - f| + \int_A |f|\mathrm{d}x + \int_A |g_{k_j}|\mathrm{d}x \\ &\leqslant |E|\|g_{k_j} - f\|_\infty + 2\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\lim_{k\to\infty} \int_E |g_{k_j} - f|\mathrm{d}x = 0.$$

Then by the remark of Problem 6.6, we have

$$\lim_{k\to\infty} \int_E g_{k_j}(x)\mathrm{d}x = \int_E f(x)\mathrm{d}x.$$

□

Remark 1.

(i) If $|E| < +\infty$, the same argument tell us that convergence a.e. up to subsequence + equi-continuously integrable $\iff$ strong convergence, the inverse is called Vainberg's theorem[Bré10, Chapter 4].

(ii) If $|E| = +\infty$, the conclusion may not hold. for instance, we can choose $E = [0, +\infty)$ and define

$$f_k = \frac{1}{k}\chi_{[0,k]}.$$

But a similar equivalent condition of strong convergence still exists, see [Bog07, pp.267-270], [Hal71] and [Hal74, p.108].

Remark 2. We claim that the numerical sequence a_k converge to $a \in (-\infty, +\infty)$ if any subsequence of it has a further subsequence which converges to a. We suppose a_k dose not converge to a. If a_k is unbounded, then it has a subsequence which diverges to $+\infty$ or $-\infty$, it can not converge to a. If a_k is bounded, choose two subsequences b_k, c_k converging to $\limsup a_k$ and $\liminf a_k$, resp. So

$$\limsup a_k = \liminf a_k = a.$$

We can get a_k converge to a.

Problem 6.2

Use Egorov's theorem to prove the bounded convergence theorem: Let $f_k \subset L(E)$ converge to f a.e. on E and $|E| < +\infty$, if there is some $M > 0$ satisfying $|f(x)| \leqslant M$ for a.e. $x \in E$, then $f \in L(E)$ and

$$\lim_{k\to\infty} \int_E |f_k - f| \mathrm{d}x = 0.$$

Proof. First, by Fatou's lemma we obtain $f \in L(E)$. By Egorov's theorem, there is a measurable subset $F \subset E$ with

$$|E \setminus F| < \delta = \frac{\varepsilon}{4M},$$

and $\{f_k\}$ converges uniformly to f on F. Thus, there is $k_0 \in \mathbb{N}$ such that for all $k > k_0$,

$$|f(x) - f_k(x)| < \frac{\varepsilon}{2|E|}, \quad \forall x \in F.$$

Consequently, for all $k > k_0$, we have

$$\int_{E-F} |f - f_k| \mathrm{d}x < \frac{\varepsilon}{2}.$$

Then for all $k > k_0$ we obtain

$$\int_E |f_k - f| \mathrm{d}x = \int_{E\setminus F} |f_k - f| \mathrm{d}x + \int_F |f_k - f| \mathrm{d}x < 2M\frac{\varepsilon}{4M} + \frac{\varepsilon}{2} = \varepsilon,$$

as required. □

Remark 1. The bounded convergence theorem is a special case of a more general theorem in L^p space concerning the weak convergence.

Theorem (A). *Let E be a measurable set, $1 < p < +\infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. If $\{f_k\} \subset L^p(E)$ converges to f almost everywhere, and $K = \sup_{k\in\mathbb{N}} \|f_k\|_p < +\infty$, then for any $g \in L^q(E)$,*

$$\lim_{k\to\infty} \int_E f_k g \mathrm{d}x = \int_E f g \mathrm{d}x$$

Notice that

$$\frac{1}{q/p} + \frac{1}{r/p} = 1,$$

so we can expect the following result from the theorem above

Theorem (B). *Let E be a measurable set and $1 < r \leqslant +\infty$. If a sequence of functions $\{f_k\} \subset L^r(E)$ converges to f almost everywhere, and $K = \sup_{k\in\mathbb{N}} \|f_k\|_r < +\infty$, then for any $1 \leqslant p < r$ and any $g \in L^q(E)$ where $\frac{1}{q} + \frac{1}{r} = \frac{1}{p}$, we have*

$$\lim_{k\to\infty} \|f_k g - fg\|_p = 0.$$

When E satisfies $|E| < +\infty$, $r = +\infty$, $p = q = 1$, and $g = \chi_E$, we recover the classical form of the bounded convergence theorem. All these theorems can be proved using the same manner of above we used to prove the bounded convergence theorem, except using the absolutely continity of the L^p integral of g to choose proper set A depending on ε-δ (and we may need to compose the desired integral into three parts)[Ben13]. Also see Problem 10.9.

Remark 2. The bounded convergence theorem does not hold for infinite measure spaces. for instance, consider the sequence $\{\chi_{[k,k+1]}\}$ in $\mathbb{R}$.

Remark 3. Theorem (A) does not hold when $p = 1$. Consider the sequence in $[0,1]$:

$$f_k(x) = \begin{cases} k - k^2 x, & x \in [0, 1/k], \\ 0, & x \in (1/k, 1], \end{cases}$$

This sequence converges almost everywhere to $f \equiv 0$ but does not converge weakly to its pointwise limit f in $L^1([0,1])$.

Remark4. Theorem (B) does not hold for the case $r = p$. For instance, in $L^r([0,1])$, consider

$$f_k(x) = \begin{cases} k^{1/r}, & x \in [0, 1/k], \\ 0, & x \in (1/k, 1], \end{cases}$$

This sequence is bounded in L^r-norm and converges almost everywhere, but fails to satisfy the conclusion of Theorem (B) when $r = p = 1$ [Wad74][Wan14, p.220].

Remark 5. By the reflexivity of L^p spaces ($1 < p < +\infty$), any norm-bounded sequence $\{f_k\}$ in Theorem (A) has a weakly convergent subsequence. However, this alone does not imply weak convergence of the original sequence. The significance of Theorem (A) lies in showing that adding the almost everywhere convergence condition is sufficient to guarantee weak convergence of the entire sequence.

Problem 6.3

If $p > 0$ and $\int_E |f - f_k|^p \, d\mu \to 0$ as $k \to \infty$, show that: $f_k \xrightarrow{m} f$ on E (convergence in measure) and so there is a subsequence $\{f_{k_j}\}$ such that $f_{k_j} \to f$ a.e. in E.

Proof. For any $\varepsilon > 0$ we have

$$\begin{aligned} |\{x \in E : |f_k(x) - f(x)| > \varepsilon\}| &= \int_{\{x\in E:|f_k(x)-f(x)|>\varepsilon\}} 1 \mathrm{d}x \\ &\leqslant \frac{1}{\varepsilon^p} \int_E |f_k - f|^p \mathrm{d}x \to 0. \end{aligned}$$

So the conclusion holds. □

Remark. Here the inequality we derive is sometimes called Chebyshev's inequality.

Problem 6.4

If $p>0$, $\int_E |f-f_k|^p \mathrm{d}x \to 0$, and $\int_E |f_k|^p \mathrm{d}x \leqslant M$ for all k, show that $\int_E |f|^p \mathrm{d}x \leqslant M$.

Proof. Assume $p>0$, $\int_E |f-f_k|^p \mathrm{d}x \to 0$, and $\int_E |f_k|^p \mathrm{d}x \leqslant M$ for all k.
By Problem 6.3, there is a subsequence $\{f_{k_j}\}$ such that $f_{k_j} \to f$ a.e. on E. Since the function $t \mapsto |t|^p$ is continuous, we have $|f_{k_j}|^p \to |f|^p$ a.e. on E. Applying Fatou's Lemma, we obtain

$$\int_E |f|^p \mathrm{d}x \leqslant \liminf_{j\to\infty} \int_E |f_{k_j}|^p \mathrm{d}x \leqslant M.$$

□

Problem 6.5

1. Let $\{f_k\}$ be a sequence of measurable functions on E. Show that $\sum f_k$ converges absolutely a.e. in E if $\sum \int_E |f_k| \mathrm{d}x < +\infty$. (Use Theorems 5.16 and 5.22.)
2. If $\{r_k\}$ denotes the rational numbers in $[0,1]$ and $\{a_k\}$ satisfies $\sum |a_k| < +\infty$, show that $\sum a_k |x-r_k|^{-1/2}$ converges absolutely a.e. in $[0,1]$.

Proof.

1. By Theorems 5.16, we have
$$\int_E \sum |f_k(x)| \mathrm{d}x = \sum \int_E |f_k(x)| \mathrm{d}x < +\infty.$$
Then by Theorems 5.22, we have $\sum |f_k(x)|$ is finite a.e. in E. So $\sum f_k(x)$ converges absolutely a.e. in E.
2. Since
$$\begin{aligned}\sum |a_k| \int_0^1 |x-r_k|^{-1/2} \mathrm{d}x &= 2\sum |a_k| \left(|1-r_k|^{1/2} + |r_k|^{1/2}\right)\\ &\leqslant 2\sum |a_k| < +\infty.\end{aligned}$$
Then by (i) we get the conclusion.

□

Remark 1. Once we prove the completeness of $L(E)$ space, then the conclusion of (i) can be strengthened to $\sum f_k$ converge. In fact, For any normed vector space $(V, \|\cdot\|)$, it is complete if and only if every absolutely summable sequence f_k is summable, and

$$\left\|\sum f_k\right\| \leqslant \sum \|f_k\|.$$

Remark 2. The example in (ii) shows us that the function in $L(E)$ can be extremely discontinuous and unbounded[Eva10, p.260]. The derivative in some weak sense of such function (the power can be different but still is negtive) can still integrable on E. This function is not Riemann integrable but is Lebesgue integrable, and it is a conterexample of the following statement

$$\forall a,b \in [0,1], r>0, |\{x\in(a,b) : |f(x)|>r\}| > 0 \overset{?}{\Longrightarrow} \int_{[0,1]} f(x)\mathrm{d}x = +\infty.$$

See [Ben13, p.90][Wan13, p.199].

Problem 6.6

Prove the *Sequential Lebesgue Dominated Convergence Theorem*: Let $\{f_k\}$ and $\{\phi_k\}$ be measurable function sequences on E satisfying:

1. $f_k \to f$ a.e. in E,
2. $\phi_k \to \phi$ a.e. in E,
3. $|f_k| \leqslant \phi_k$ a.e. in E.

If $\phi \in L(E)$ and $\int_E \phi_k \to \int_E \phi$, then $\int_E |f_k - f| \to 0$.
(Hint: For the case $f = 0$ with $f_k \geqslant 0$, apply Fatou's Lemma to $\{\phi_k - f_k\}$. An application appears in Chapter 8, Exercise 12. For instance, when $f_k \geqslant 0$, $f_k \to f$ a.e., $f \in L(E)$, and $\int_E f_k \to \int_E f$, then $\int_E |f_k - f| \to 0$.)

Proof. The integrability of f follows from the inequality $|f(x)| \leqslant \phi(x)$ a.e. Applying Fatou's Theorem yields

$$\begin{aligned}\int_E f \mathrm{d}x + \int_E \phi \mathrm{d}x &= \int_E \lim_{k\to\infty} (f_k + \phi_k)\mathrm{d}x \\ &\leqslant \liminf_{k\to\infty} \int_E (f_k + \phi_k)\mathrm{d}x \\ &= \liminf_{k\to\infty} \int_E f_k \mathrm{d}x + \int_E \phi \mathrm{d}x,\end{aligned}$$

from which we derive

$$\int_E f \mathrm{d}x \leqslant \liminf_{k\to\infty} \int_E f_k \mathrm{d}x.$$

Analogously, by considering $-\phi_k$, we obtain

$$\int_E f \mathrm{d}x \geqslant \limsup_{k\to\infty} \int_E f_k \mathrm{d}x.$$

□

Remark. [Bog07, pp.134-135] This theorem is due to Young[You11, p.315], while the functions f_k may not converge to f strong (in L^1 space). A direct corollary is the Vitali-Scheffé Theorem (historically developed by Vitali, Young, and Fichtenholz for Lebesgue measure, later generalized by Scheffé):

Theorem. For non-negative integrable functions f_k converging a.e. to an integrable function f, if

$$\lim_{k\to\infty} \int_E f_k \mathrm{d}x = \int_E f \mathrm{d}x,$$

then

$$\lim_{k\to\infty} \int_E |f - f_k| \mathrm{d}x = 0.$$

For signed functions $f_k \to f$ a.e., strong convergence occurs if and only if

$$\int_E |f_n| \mathrm{d}x \to \int_E |f| \mathrm{d}x$$

.

Proof. The inequality $0 \leqslant |f_n - f| \leqslant |f_n| + |f|$ allows application of Problem 6.6. □

Problem 6.7

Prove this variant: If $\{f_k\}$ converges in measure ($f_k \xrightarrow{m} f$) on E with $|f_k| \leqslant \phi \in L(E)$, then $f \in L(E)$ and

$$\int_E f_k \to \int_E f.$$

(Hint: Show every subsequence of $\{f_k\}$ has a further subsequence $\{f_{k_j}\}$ with $\displaystyle\int_E f_{k_j} \to \int_E f$.)

Proof. We can consider the numerical sequence

$$\left\{a_k := \int_E f_k(x)\mathrm{d}x\right\}_k$$

In fact, it suffices to show that each subsequence $\{b_k\}$ of $\{a_k\}$ contains a convergent subsequence $\{b_{k_j}\}$. Choose any subsequence g_k of f_k. Since g_k also converges to f in measure. By Riesz theorem, there is some subsequence g_{k_j} converging to f a.e. on E. Then by Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{k\to\infty}\int_E |g_{k_j} - f|\mathrm{d}x = 0.$$

Then by the remark of Problem 6.6, we have

$$\lim_{k\to\infty}\int_E g_{k_j}(x)\mathrm{d}x = \int_E f(x)\mathrm{d}x.$$

The proof of this conclusion is similar to that of Problem 6.1, or even simpler. □

Homework Solution 10-12

Problem 7.1

(a) Let E be a measurable subset of $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}^1$,
$$E_x := \{y : (x,y) \in E\} \text{ has } \mathbb{R}^1\text{-measure zero.}$$
Show that E has measure zero and that for almost every $y \in \mathbb{R}^1$, $E^y := \{x : (x,y) \in E\}$ has measure zero.

(b) Let $f(x,y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x \in \mathbb{R}^1$, $f(x,y)$ is finite for almost every y. Show that for almost every $y \in \mathbb{R}^1$, $f(x,y)$ is finite for almost every x.

Proof.

(a) Consider the characteristic function $\chi_E : \mathbb{R}^2 \to \{0,1\}$ defined by
$$\chi_E(x,y) = \begin{cases} 1 & \text{if } (x,y) \in E \\ 0 & \text{otherwise.} \end{cases}.$$
Since E is measurable, χ_E is non-negative and measurable.
By the Tonelli theorem, we have
$$|E|_2 = \iint_{\mathbb{R}^2} \chi_E(x,y)\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_E(x,y)\mathrm{d}y \right) \mathrm{d}x.$$
The inner integral $\displaystyle\int_{\mathbb{R}} \chi_E(x,y)\mathrm{d}y = |E_x| = 0$, for almost every x. Therefore, $|E|_e = 0$.
Moreover, we have
$$\iint_{\mathbb{R}^2} \chi_E(x,y)\mathrm{d}x\mathrm{d}y = 0.$$
Applying the Tonelli theorem again with the order of integration reversed,
$$\iint_{\mathbb{R}^2} \chi_E(x,y)\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_E(x,y)\mathrm{d}x \right) \mathrm{d}y = \int_{\mathbb{R}} |E^y|\mathrm{d}y = 0,$$
where $|E^y|$ is the Lebesgue measure of E^y in $\mathbb{R}$. Since $|E^y| \geqslant 0$ and the integral is zero, it follows that $|E^y| = 0$ for almost every y.

(b) Define the set
$$E = \{(x,y) \in \mathbb{R}^2 : f(x,y) = \infty\}.$$
Since f is measurable, E is measurable. The assumption states that for almost every x, $f(x,y) < \infty$ for almost every y, which is equivalent to
$$|E_x| = 0 \quad \text{for almost every } x,$$
where $E_x = \{y : (x,y) \in E\}$ is the x-section of E. Then we can use the first part of this problem and get the desired conclusion.

$\square$

Problem 7.2

If f and g are measurable in $\mathbb{R}^n$, show that the function $h(x,y) = f(x)g(y)$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$. Deduce that if E_1 and E_2 are measurable subsets of $\mathbb{R}^n$, then their Cartesian product $E_1 \times E_2 = \{(x,y) : x \in E_1, y \in E_2\}$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$, and $|E_1 \times E_2| = |E_1||E_2|$. As usual in measure theory, $0 \cdot \infty$ and $\infty \cdot 0$ are interpreted as 0.

Proof. Let $F(x,y) := f(x)$ and $G(x,y) := g(y)$ for all $x,y \in \mathbb{R}^n$. Observe that

$$\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \mid F(x,y) > \alpha\} = \{x \in \mathbb{R}^n \mid f(x) > \alpha\} \times \mathbb{R}^n,$$

which is measurable by repeated application of Lemma 5.2 (which states that $E \times \mathbb{R}$ is measurable for measurable $E \subseteq \mathbb{R}^m$). Thus, $F(x,y)$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$.
Similarly,

$$\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \mid G(x,y) > \alpha\} = \mathbb{R}^n \times \{y \in \mathbb{R}^n \mid g(y) > \alpha\}$$

is a measurable set. Thus, $G(x,y)$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$.
Hence, the product $F(x,y)G(x,y) = f(x)g(y)$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$.
Let E_1 and E_2 be measurable subsets of $\mathbb{R}^n$. Then the characteristic functions χ_{E_1} and χ_{E_2} are measurable in $\mathbb{R}^n$. By the previous result, $\chi_{E_1}(x)\chi_{E_2}(y)$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$.
Note that $\chi_{E_1}(x)\chi_{E_2}(y) = \chi_{E_1 \times E_2}(x,y)$, so $E_1 \times E_2$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$.
The measure of the product set is given by:

$$\begin{aligned}
|E_1 \times E_2| &= \iint_{\mathbb{R}^n \times \mathbb{R}^n} \chi_{E_1 \times E_2}(x,y)\mathrm{d}x\mathrm{d}y \\
&= \iint_{\mathbb{R}^n \times \mathbb{R}^n} \chi_{E_1}(x)\chi_{E_2}(y)\mathrm{d}x\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \chi_{E_1}(x) \left[\int_{\mathbb{R}^n} \chi_{E_2}(y)\mathrm{d}y\right] \mathrm{d}x \quad \text{(by Tonelli's Theorem)} \\
&= \int_{\mathbb{R}^n} \chi_{E_1}(x)\mathrm{d}x \cdot |E_2| \\
&= |E_1||E_2|.
\end{aligned}$$

□

Problem 7.3

Let f be measurable and finite a.e. on $[0,1]$. If $f(x) - f(y)$ is integrable over the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, show that $f \in L[0,1]$.

Proof. Assume by contradiction, $f \notin L[0,1]$. Then

$$\int_0^1 |f(x)|\mathrm{d}x = +\infty.$$

Then we consider the integral over the square

$$\int_0^1 \int_0^1 |f(x) - f(y)|\mathrm{d}x\mathrm{d}y < +\infty.$$

By Tonelli's theorem,

$$\int_0^1 \left(\int_0^1 |f(x) - f(y)|\mathrm{d}y\right) \mathrm{d}x < +\infty.$$

This implies

$$\int_0^1 |f(x)-f(y)|\mathrm{d}y < +\infty \quad \text{a.e. } x \in [0,1]$$

Let x_0 be such that $f(x_0)$ is finite and

$$\int_0^1 |f(x_0)-f(y)|\mathrm{d}y < +\infty.$$

For this x_0, we have

$$|f(y)| \leqslant |f(x_0)| + |f(x_0)-f(y)|.$$

Integrating both sides yields

$$\int_0^1 |f(y)|\mathrm{d}y \leqslant |f(x_0)| + \int_0^1 |f(x_0)-f(y)|\mathrm{d}y < +\infty.$$

This contradicts our initial assumption that $\int_0^1 |f| = +\infty$. □

Problem 7.4

(a) If f is nonnegative and measurable on E and $\omega(y) = |\{x \in E : f(x) > y\}|$, $y > 0$, use Tonelli's theorem to prove that

$$\int_E f\mathrm{d}x = \int_0^\infty \omega(y)\mathrm{d}y.$$

(By definition of the integral, $\int_E f\mathrm{d}x = |R(f,E)| = \iint_R \mathbf{1}_{\{f(x)>y\}}\mathrm{d}x\mathrm{d}y$. Use the observation that $\{x \in E : f(x) \geqslant y\} = \{x : (x,y) \in R(f,E)\}$, and recall that $\omega(y) = |\{x \in E : f(x) \geqslant y\}|$ unless y is a point of discontinuity of ω.)

(b) Deduce from this special case the general formula

$$\int_E f^p\mathrm{d}x = p\int_0^\infty y^{p-1}\omega(y)\mathrm{d}y \quad (f \geqslant 0, 0 < p < \infty).$$

Proof. We prove a general formula. Let $\varphi \in C^1([0,+\infty))$ be non-decreasing and f be a measurable function on measurable set $E \subset \mathbb{R}^n$, then

$$\int_E \varphi(|f|)\mathrm{d}x = \int_0^{+\infty} \varphi'(y)\omega(y)\mathrm{d}y.$$

In fact,

$$\begin{aligned}
\int_0^{+\infty} \varphi'(y)\omega(y)\mathrm{d}y &= \int_0^{+\infty} \varphi'(y)\int_E \chi_{\{x\in E:|f(x)|>y\}}(x)\mathrm{d}x\mathrm{d}y \\
&= \int_E \int_0^{+\infty} \varphi'(y)\chi_{\{x\in E:|f(x)|>y\}}(x)\mathrm{d}y\mathrm{d}x \\
&= \int_E \int_0^{|f(x)|} \varphi'(y)\mathrm{d}y\mathrm{d}x \\
&= \int_E \varphi(|f|)\mathrm{d}x.
\end{aligned}$$

□

Problem 7.5

For $f \in L(\mathbb{R}^1)$, define the Fourier transform $\hat{f}$ of f by

$$\hat{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)\mathrm{e}^{-\mathrm{i}xt}\mathrm{d}t \quad (x \in \mathbb{R}^1).$$

(For a complex-valued function $F = F_0 + \mathrm{i}F_1$ whose real and imaginary parts F_0 and F_1 are integrable, we define $\int F = \int F_0 + \mathrm{i}\int F_1$). Show that if f and g belong to $L(\mathbb{R}^1)$, then

$$\widehat{(f*g)}(x) = 2\pi\hat{f}(x)\hat{g}(x).$$

Proof. By definition, the Fourier transform of the convolution is:

$$\widehat{(f*g)}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(f*g)(t)\mathrm{e}^{-\mathrm{i}xt}\mathrm{d}t.$$

Since $f, g \in L^1(\mathbb{R})$, Theorem 6.14 guarantees that

$$\int_{-\infty}^{\infty}|(f*g)(t)|\mathrm{d}t \leqslant \left(\int_{-\infty}^{\infty}|f(t)|\mathrm{d}t\right)\left(\int_{-\infty}^{\infty}|g(t)|\mathrm{d}t\right) < \infty.$$

Hence, the integral converges absolutely

$$\left|\int_{-\infty}^{\infty}(f*g)(t)e^{-ixt}\mathrm{d}t\right| \leqslant \int_{-\infty}^{\infty}|(f*g)(t)|\mathrm{d}t < \infty.$$

We compute the Fourier transform using Fubini's Theorem

$$\begin{aligned}
\widehat{(f*g)}(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t-u)g(u)\,du\right)\mathrm{e}^{-\mathrm{i}xt}\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(u)\left(\int_{-\infty}^{\infty} f(t-u)\mathrm{e}^{-\mathrm{i}xt}\mathrm{d}t\right)\mathrm{d}u \quad \text{(Fubini's Theorem)} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(u)\mathrm{e}^{-\mathrm{i}xu}\left(\int_{-\infty}^{\infty} f(t-u)\mathrm{e}^{-\mathrm{i}x(t-u)}\mathrm{d}t\right)\mathrm{d}u \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(u)\mathrm{e}^{-\mathrm{i}xu}\left(\int_{-\infty}^{\infty} f(v)\mathrm{e}^{-\mathrm{i}xv}\mathrm{d}v\right)\mathrm{d}u \quad (\text{let } v = t-u) \\
&= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\mathrm{e}^{-\mathrm{i}xv}\mathrm{d}v\right)\left(\int_{-\infty}^{\infty} g(u)\mathrm{e}^{-\mathrm{i}xu}\mathrm{d}u\right) \\
&= 2\pi\hat{f}(x)\hat{g}(x),
\end{aligned}$$

as required.

For a complex-valued function $F = F_0 + \mathrm{i}F_1$, if $\int |F| < \infty$, then

$$\int |F_0| \leqslant \int\sqrt{F_0^2 + F_1^2} = \int |F| < \infty,$$

and similarly $\int |F_1| < \infty$. This justifies applying Fubini's Theorem to the real and imaginary parts separately. $\square$

Problem 7.6

Let v_n be the volume of the unit ball in $\mathbb{R}^n$. Show by using Fubini's theorem that

$$v_n = 2v_{n-1}\int_0^1 (1-t^2)^{(n-1)/2}\mathrm{d}t.$$

(We also observe that by setting $w=t^2$, the integral is a multiple of a classical β-function and so can be expressed in terms of the Γ-function:

$$\Gamma(s)=\int_0^\infty \mathrm{e}^{-t}t^{s-1}\mathrm{d}t, \quad s>0.)$$

Proof. We prove by induction on n that the volume v_n of the unit ball in $\mathbb{R}^n$ satisfies

$$v_n = 2v_{n-1}\int_0^1 (1-t^2)^{(n-1)/2}\mathrm{d}t$$

and can be expressed in terms of the Γ-function. First, the conclusion is trivial when $n=1$ or 2. Assume the formula holds for $n-1$. Let

$$B^n := \{x\in\mathbb{R}^n : x_1^2+\cdots+x_n^2 \leqslant 1\}$$

$$\begin{aligned}
v_n &= \int\cdots\int_{B^n} 1\mathrm{d}x_1\cdots\mathrm{d}x_n\\
&= \int_{-1}^1\left(\int\cdots\int_{\{x_2^2+\cdots+x_n^2\leqslant 1-x_1^2\}} 1\mathrm{d}x_2\cdots\mathrm{d}x_n\right)\mathrm{d}x_1 \quad \text{(Tonelli's Theorem)}
\end{aligned}$$

Make the change of variables $u_j=\dfrac{x_j}{\sqrt{1-x_1^2}}$ for $j=2,\cdots,n$.

$$\begin{aligned}
v_n &= \int_{-1}^1\left(\int\cdots\int_{\{u_2^2+\cdots+u_n^2\leqslant 1\}}(1-x_1^2)^{(n-1)/2}\mathrm{d}u_2\cdots\mathrm{d}u_n\right)\mathrm{d}x_1\\
&= \int_{-1}^1 v_{n-1}(1-x_1^2)^{(n-1)/2}\mathrm{d}x_1\\
&= 2v_{n-1}\int_0^1(1-t^2)^{(n-1)/2}\mathrm{d}t \quad \text{(even integrand)}
\end{aligned}$$

Let $w=t^2$,

$$\begin{aligned}
\int_0^1(1-t^2)^{(n-1)/2}\mathrm{d}t &= \frac{1}{2}\int_0^1 w^{-1/2}(1-w)^{(n-1)/2}\mathrm{d}w\\
&= \frac{1}{2}\mathrm{B}\left(\frac{1}{2},\frac{n+1}{2}\right)\\
&= \frac{1}{2}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2}+1)}
\end{aligned}$$

where $\mathrm{B}(x,y)$ is the beta function and Γ is the gamma function.
Thus we obtain the closed form

$$v_n=\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}.$$

□

Remark. This is a general case of the Dirichlet repeated integral[WS00]. Define

$$I := \int\cdots\int_E f(t_1+t_2+\cdots+t_n)t_1^{a_1-1}t_2^{a_2-1}\cdots t_n^{a_n-1}\mathrm{d}t_1\mathrm{d}t_2\cdots\mathrm{d}t_n,$$

where $f(t)$ is continuous and Re $a_k > 0$,

$$E = \left\{ (t_1, t_2, \cdots, t_n) : t_k \geqslant 0, k = 1, 2, \cdots, n; \sum_{k=1}^{n} t_k = 1 \right\}.$$

I can be reduced as

$$I = \frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_n)}{\Gamma(a_1 + a_2 + \cdots + a_n)} \int_0^1 f(\tau)\tau^{a_1+a_2+\cdots+a_n-1} \mathrm{d}\tau.$$

Then we can deduce from this formula the volume of the "L^p ball"

$$B_{p_1,p_2,\cdots,p_n} := \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : |x_1|^{p_1} + \cdots |x_n|^{p_n} = 1\}$$

is

$$V_{B_{p_1,p_2,\cdots,p_n}} = 2^n \frac{\Gamma\left(1 + \frac{1}{p_1}\right)\Gamma\left(1 + \frac{1}{p_2}\right)\cdots\Gamma\left(1 + \frac{1}{p_n}\right)}{\Gamma\left(1 + \frac{1}{p_1} + \frac{1}{p_2} \cdots \frac{1}{p_n}\right)}.$$

Problem 7.7

Use Fubini's theorem to prove that

$$\int_{\mathbb{R}^n} \mathrm{e}^{-|x|^2} \mathrm{d}x = \pi^{n/2}.$$

(For $n = 1$, write

$$\left(\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2} \mathrm{d}x\right)^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathrm{e}^{-x^2-y^2} \mathrm{d}x\mathrm{d}y$$

and use polar coordinates. For $n > 1$, use the formula $\mathrm{e}^{-|x|^2} = \mathrm{e}^{-x_1^2} \cdots \mathrm{e}^{-x_n^2}$ and Fubini's theorem to reduce to the case $n = 1$.)

Proof. Consider the square of the integral

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} \mathrm{d}x\mathrm{d}y.$$

By Fubini's theorem, switching to polar coordinates ($r^2 = x^2 + y^2$, $\mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta$) yields

$$\begin{aligned}\int_0^{2\pi} \int_0^{\infty} \mathrm{e}^{-r^2} r\mathrm{d}r\mathrm{d}\theta &= 2\pi \left[-\frac{1}{2}\mathrm{e}^{-r^2}\right]_0^{\infty} \\ &= 2\pi \cdot \frac{1}{2} = \pi.\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^n} \mathrm{e}^{-|x|^2} \mathrm{d}x = \prod_{i=1}^{n} \left(\int_{\mathbb{R}} \mathrm{e}^{-x_i^2} \mathrm{d}x_i\right) = \pi^{\frac{n}{2}}.$$

□

Problem 7.8

Let $E \subset \mathbb{R}^n$ be a measurable set and $f : E \to \mathbb{R}$ be a nonnegative measurable function. Show that the graph of f,

$$\Gamma(f, E) \triangleq \{(x, f(x)) \in \mathbb{R}^{n+1} : x \in E\},$$

has Lebesgue measure zero in $\mathbb{R}^{n+1}$, i.e., $|\Gamma(f, E)|_{(n+1)} = 0$.

Proof. (Lemma 5.3 on book) Given $\varepsilon > 0$ and for each $k = 0, 1, \cdots$, define the sets

$$E_k = \{x \in E : k\varepsilon \leqslant f(x) < (k+1)\varepsilon\}.$$

Then $\{E_k\}$ is a sequence of measurable sets and its disjoint union is $\{x \in E : f(x) < +\infty\}$.

Thus, the subgraph satisfies $\Gamma(f,E) = \bigcup_{k=0}^{\infty} \Gamma(f,E_k)$.

By Lemma 5.2 on book, we have the estimate

$$|\Gamma(f,E_k)|_e \leqslant \varepsilon |E_k|,$$

where $|\cdot|_e$ denotes the exterior measure. Therefore,

$$|\Gamma(f,E)|_e \leqslant \sum_{k=0}^{\infty} |\Gamma(f,E_k)|_e \leqslant \varepsilon \sum_{k=0}^{\infty} |E_k| \leqslant \varepsilon |E|.$$

We consider two cases:

1. If $|E| < +\infty$, then letting $\varepsilon \to 0$ shows that $\Gamma(f,E)$ has measure zero.
2. If $|E| = +\infty$, we can write E as a countable union of disjoint measurable sets with finite measure $E = \bigcup_{j=1}^{\infty} F_j$ where $|F_j| < \infty$. Then

   $$\Gamma(f,E) = \bigcup_{j=1}^{\infty} \Gamma(f,F_j)$$

   is a countable union of sets of measure zero (by the previous case), and thus has measure zero.

□

Problem 7.9

Let $E \subset \mathbb{R}^n$, $f \in L(E)$. Then for any $\varepsilon > 0$, there exists a measurable simple function h with compact support in $\mathbb{R}^n$ such that

$$\int_E |f(x) - h(x)| \mathrm{d}x < \varepsilon.$$

Proof. Without loss of generality, assume first that $f \geqslant 0$. By [Zho16, Theorem 3.9] and the monotone convergence theorem, there is a measurable simple function φ with $0 \leqslant \varphi(x) \leqslant f(x)$ satisfying

$$\int_E (f(x) - \varphi(x)) \mathrm{d}x < \frac{\varepsilon}{2}.$$

Define

$$\varphi_N(x) = \varphi(x) \cdot \chi_{B_N(0)}(x)$$

where $B_N(0)$ is the ball of radius N. As $N \to \infty$,

$$\int_E |\varphi(x) - \varphi_N(x)| \mathrm{d}x \to 0,$$

so for N sufficiently large,

$$\int_E |\varphi(x) - \varphi_N(x)| \mathrm{d}x < \frac{\varepsilon}{2}.$$

Taking $h = \varphi_N$, we obtain

$$\int_E |f(x) - h(x)|\mathrm{d}x \leqslant \int_E |f(x) - \varphi(x)|\mathrm{d}x + \int_E |\varphi(x) - h(x)|\mathrm{d}x$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

For general $f \in L^1(E)$, apply the above to both f^+ and f^-. □

Problem 7.10

Let f be an a.e. finite measurable function on $E \subset \mathbb{R}^n$. For any $\delta > 0$, there exists a continuous function g on $\mathbb{R}^n$ such that:

$$|\{x \in E : f(x) \neq g(x)\}| < \delta.$$

If E is bounded, g can be chosen with compact support.

Proof. By the Lusin theorem, there is a closed set $F \subset E$ such that $|E \setminus F| < \delta$ and $f|_F$ is continuous on F. By the extension theorem, there is $g \in C(\mathbb{R}^n)$ such that

$$g(x) = f(x), \quad \forall x \in F.$$

Thus,

$$\{x \in E : f(x) \neq g(x)\} \subset E \setminus F \implies |\{x \in E : f(x) \neq g(x)\}| < \delta.$$

Assume $E \subset B(0,k)$. Construct a cutoff function

$$\varphi(x) := \begin{cases} 1, & x \in F, \\ 0, & x \notin B(0, k+1), \\ \text{continuous interpolation}, & \text{otherwise}. \end{cases}$$

Then $h(x) := g(x)\varphi(x)$ is the required function. □

Problem 7.11

Let $f \in L(E)$. For any $\varepsilon > 0$, there is some continuous function g with compact support in $\mathbb{R}^n$ such that

$$\int_E |f(x) - g(x)|\, dx < \varepsilon.$$

Proof. By Problem 7.9, there is some measurable simple function φ with compact support satisfying

$$\int_E |f(x) - \varphi(x)|\, dx < \frac{\varepsilon}{2}.$$

Assume $|\varphi(x)| \leqslant M$ for all x. Applying Lusin's theorem, there is a continuous function g with $|g(x)| \leqslant M$ ($\forall x \in \mathbb{R}^n$) and

$$|\{x \in E : \varphi(x) \neq g(x)\}| < \frac{\varepsilon}{4M}$$

Then,

$$\int_E |\varphi(x) - g(x)|\mathrm{d}x = \int_{\{\varphi \neq g\}} |\varphi(x) - g(x)|\mathrm{d}x$$
$$\leqslant 2M|\{\varphi \neq g\}| < \frac{\varepsilon}{2}.$$

Thus,

$$\int_E |f(x) - g(x)|\mathrm{d}x \leqslant \int_E |f(x) - \varphi(x)|\mathrm{d}x + \int_E |\varphi(x) - g(x)|\mathrm{d}x < \varepsilon.$$

□

Problem 7.12

Let $f \in L^1(E)$. There exists a sequence $\{g_k\}$ of continuous functions with compact support in $\mathbb{R}^n$ such that:

(i) $\lim\limits_{k\to\infty} \int_E |f(x) - g_k(x)|\mathrm{d}x = 0$

(ii) $\lim\limits_{k\to\infty} g_k(x) = f(x)$, a.e. $x \in E$

Proof. For each $k \in \mathbb{N}$, take $\varepsilon_k = \dfrac{1}{2^k}$. Then there is some g_k with compact support satisfying

$$\int_E |f(x) - g_k(x)|dx < \frac{1}{2^k}.$$

Directly from construction, we obtain

$$\sum_{k=1}^{\infty} \int_E |f - g_k| < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 < \infty.$$

Thus,

$$\int_E |f(x) - g_k(x)|\mathrm{d}x \to 0.$$

Consider $h_k(x) := |f(x) - g_k(x)|$,

$$\int_E \sum_{k=1}^{\infty} h_k(x)dx < \infty \implies \sum_{k=1}^{\infty} h_k(x) < \infty, \qquad \text{a.e. } x \in E.$$

Hence,

$$h_k(x) \to 0 \quad \text{a.e. } x \in E.$$

□

Problem 7.13

If $f \in L(\mathbb{R}^n)$, then

$$\lim_{h\to 0} \int_{\mathbb{R}^n} |f(x+h) - f(x)|dx = 0.$$

Proof. Given $\varepsilon > 0$, we decompose f as

$$f(x) = f_1(x) + f_2(x)$$

where f_1 is a continuous function with compact support (by Problem 7.11), and

$$\int_{\mathbb{R}^n} |f_2(x)|dx < \frac{\varepsilon}{4}.$$

For f_1, since it is uniformly continuous on its compact support, there is some $\delta > 0$ such that for $|h| < \delta$,

$$\int_{\mathbb{R}^n} |f_1(x+h) - f_1(x)|\mathrm{d}x < \frac{\varepsilon}{2}.$$

Thus for $|h| < \delta$, we estimate

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(x+h) - f(x)| dx &\leqslant \int_{\mathbb{R}^n} |f_1(x+h) - f_1(x)| dx + \int_{\mathbb{R}^n} |f_2(x+h) - f_2(x)| dx \\
&\leqslant \frac{\varepsilon}{2} + \int_{\mathbb{R}^n} |f_2(x+h)| dx + \int_{\mathbb{R}^n} |f_2(x)| dx \\
&= \frac{\varepsilon}{2} + 2 \int_{\mathbb{R}^n} |f_2(x)| dx \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

□

Problem 7.14

Show that $\{\cos kx\}$ does not converge to 0 in measure on $[-\pi, \pi)$.

Proof. Let

$$E_k = \left\{ x \in [-\pi, \pi) : |\cos kx| \geqslant \frac{1}{2} \right\}.$$

We prove

$$\lim_{k \to \infty} |E_k| \neq 0.$$

Since $\cos kx$ has period $\frac{2\pi}{k}$, in each subinterval

$$I_j = \left[-\pi + \frac{2\pi j}{k}, -\pi + \frac{2\pi(j+1)}{k} \right),$$

the set where $|\cos kx| \geqslant \frac{1}{2}$ has measure $\frac{4\pi}{3k}$. Thus,

$$|E_k| = k \cdot \frac{4\pi}{3k} = \frac{4\pi}{3} \not\to 0$$

Thus no convergence in measure holds for $\{\cos kx\}$. □

Problem 7.15

For $f, f_k \in L^1(\mathbb{R}^n)$ with $\int |f_k - f| \leqslant \frac{1}{k^2}$, prove $f_k \to f$ a.e.

Proof. Let $h_k = |f_k - f|$ with $\int h_k \leqslant \frac{1}{k^2}$. Thus,

$$\sum \int h_k \leqslant \sum \frac{1}{k^2} < \infty \implies \int \sum h_k < \infty$$

We obtain $\sum h_k(x) < \infty$ a.e. $\Rightarrow h_k(x) \to 0$ a.e. $\Rightarrow f_k \to f$ a.e. □

Homework Solution 13-14

Problem 8.1

Prove the following set-theoretic result related to the simple Vitali covering lemma. If $\mathscr{C} = \{Q\}$ is a collection of cubes all contained in a fixed bounded set in $\mathbb{R}^n$, then there is a countable subcollection $\{Q_k\}$ of disjoint cubes in $\mathscr{C}$ such that every $Q \in \mathscr{C}$ is contained in some Q_k^*, where Q_k^* denotes the cube concentric with Q_k of edge length 5 times that of Q_k.
Deduce the measure-theoretic consequence (cf. Lemma 7.4) that if a set E is covered by such a collection $\mathscr{C}$ of cubes, then there exist $\beta > 0$, depending only on n, and a finite number of disjoint cubes $Q_1, \dots, Q_N$ in $\mathscr{C}$ such that $\beta |E|_e \leqslant \sum_{k=1}^{N} |Q_k|$.
Formulate analogues of these facts for a collection of balls in $\mathbb{R}^n$.

Proof. [EG18]

(i) Since all cubes are contained in a bounded set, the diameters are bounded. Let
$$D := \sup\{\operatorname{diam}(Q) \mid Q \in \mathscr{C}\} < \infty.$$
Partition the collection by size
$$\mathscr{C}_j := \left\{ Q \in \mathscr{C} \mid \frac{D}{2^j} < \operatorname{diam}(Q) \leqslant \frac{D}{2^{j-1}} \right\} \quad \text{for } j = 1, 2, \cdots .$$
We inductively construct disjoint subcollections $\mathscr{G}_j \subseteq \mathscr{C}_j$

- For $j = 1$: Consider the family of all disjoint subcollections of $\mathscr{C}_1$, partially ordered by inclusion. By Zorn's Lemma, there is a maximal disjoint subcollection $\mathscr{G}_1 \subseteq \mathscr{C}_1$.
- For $j > 1$: Having selected $\mathscr{G}_1, \cdots, \mathscr{G}_{j-1}$, define
$$\mathscr{A}_j := \left\{ Q \in \mathscr{C}_j \mid Q \cap Q' = \varnothing \text{ for all } Q' \in \bigcup_{i=1}^{j-1} \mathscr{G}_i \right\}.$$
Apply Zorn's Lemma to $\mathscr{A}_j$ to obtain a maximal disjoint subcollection $\mathscr{G}_j \subseteq \mathscr{A}_j$.

Set
$$\mathscr{G} := \bigcup_{j=1}^{\infty} \mathscr{G}_j.$$
By construction, $\mathscr{G}$ consists of disjoint cubes. While each $\mathscr{G}_j$ exists, we must show $\mathscr{G}$ is countable. Let $S = \{d_m\}_{m=1}^{\infty}$ collect those points whose coordinates are rational numbers. Each cube $Q \in \mathscr{G}$ contains some $d_{m(Q)} \in D$ in its interior. Since the cubes are disjoint, the map $Q \mapsto m(Q)$ is injective. Thus $\mathscr{G}$ is countable, and we can enumerate it as $\{Q_k\}_{k=1}^{\infty}$.
For any $Q \in \mathscr{C}$, let j be such that $Q \in \mathscr{C}_j$. By maximality of $\mathscr{G}_j$, there exists
$$Q' \in \bigcup_{i=1}^{j} \mathscr{G}_i \quad \text{with} \quad Q \cap Q' \neq \varnothing.$$

Let $s = \text{edge length}(Q')$ and $t = \text{edge length}(Q)$. Then
$$t \leqslant \frac{D}{2^{j-1}} = 2 \cdot \frac{D}{2^j} < 2 \cdot s$$
Let c be the center of Q'. For any $x \in Q$
$$\|x-c\|_\infty \leqslant \underbrace{\|x-q\|_\infty}_{\leqslant t/2} + \underbrace{\|q-c\|_\infty}_{\leqslant s/2} \quad \text{for some } q \in Q \cap Q'$$
$$\leqslant \frac{t}{2} + \frac{s}{2} < \frac{2s}{2} + \frac{s}{2} = \frac{3s}{2} < \frac{5s}{2}.$$
Thus $x \in Q'^*$, so $Q \subseteq Q'^*$.

We have constructed a countable disjoint subcollection $\{Q_k\}$ such that every $Q \in \mathscr{C}$ is contained in some Q_k^* with 5 times the edge length.

(ii) First, by the Vitali Covering Lemma for cubes, there is a countable disjoint subcollection $\{Q_k\}_{k=1}^\infty \subset \mathscr{C}$ such that every $Q \in \mathscr{C}$ is contained in some expanded cube Q_k^*, where Q_k^* has the same center as Q_k but edge length multiplied by 5.

Since $\{Q_k\}$ covers E up to expansion,
$$E \subset \bigcup_{k=1}^\infty Q_k^*.$$
Taking Lebesgue outer measure yields
$$|E|_e \leqslant \sum_{k=1}^\infty |Q_k^*| = 5^n \sum_{k=1}^\infty |Q_k|.$$
Now, since the sum $\sum_{k=1}^\infty |Q_k|$ converges (as all Q_k are disjoint and contained in a bounded set), there is a finite subcollection $Q_1, \ldots, Q_N$ such that
$$\sum_{k=1}^N |Q_k| \geqslant \frac{1}{5^n} \sum_{k=1}^\infty |Q_k| \geqslant \frac{1}{5^{2n}} |E|_e.$$
Thus, setting $\beta = \frac{1}{5^{2n}}$, we obtain
$$\beta |E|_e \leqslant \sum_{k=1}^N |Q_k|.$$
□

Problem 8.2

Use Problem 8.1 to prove: If $f \in L(\mathbb{R}^n)$, then f^* belongs to weak $L^1(\mathbb{R}^n)$. Moreover, there is a constant c independent of f and α such that
$$|\{x \in \mathbb{R}^n : f^*(x) > \alpha\}| \leqslant \frac{c}{\alpha} \int_{\mathbb{R}^n} |f|, \quad \alpha > 0.$$

Proof. Let $f \in L(\mathbb{R}^n)$ and $\alpha > 0$ be given. We need to show
$$|\{x \in \mathbb{R}^n : f^*(x) > \alpha\}| \leqslant \frac{c}{\alpha} \|f\|_{L^1}.$$
Let $E_\alpha := \{x : f^*(x) > \alpha\}$. By definition of the maximal function, for each $x \in E_\alpha$, there is a cube Q_x centered at x such that
$$\frac{1}{|Q_x|} \int_{Q_x} |f(y)| \mathrm{d}y > \alpha.$$

The collection $\{Q_x\}_{x\in E_\alpha}$ covers E_α. By Problem 8.1, there is a countable disjoint subcollection $\{Q_k\}_{k=1}^\infty$ such that

$$E_\alpha \subseteq \bigcup_{k=1}^{\infty} \widehat{Q_k}$$

where $\widehat{Q_k}$ is the cube with the same center as Q_k but 5 times the side length. Thus,

$$\begin{aligned}
|E_\alpha| &\leqslant \sum_{k=1}^{\infty} |\widehat{Q_k}| = 5^n \sum_{k=1}^{\infty} |Q_k| \\
&< 5^n \sum_{k=1}^{\infty} \frac{1}{\alpha} \int_{Q_k} |f(y)| \mathrm{d}y \quad (\text{since } \alpha |Q_k| < \int_{Q_k} |f|) \\
&= \frac{5^n}{\alpha} \sum_{k=1}^{\infty} \int_{Q_k} |f(y)| \mathrm{d}y.
\end{aligned}$$

Because $\{Q_k\}$ are disjoint,

$$\sum_{k=1}^{\infty} \int_{Q_k} |f(y)| \mathrm{d}y \leqslant \int_{\mathbb{R}^n} |f(y)| \mathrm{d}y = \|f\|_{L^1}.$$

Combining these estimates yields

$$|E_\alpha| < \frac{5^n}{\alpha} \|f\|_{L^1}.$$

□

Problem 8.3

(a) Let $f(x)$ be defined for all $x \in \mathbb{R}^n$ by $f(x) = 0$ if every coordinate of x is rational, and $f(x) = 1$ otherwise. Describe the set of all x at which $\frac{1}{|Q|}\int_Q f$ has a limit as $Q \searrow x$ and describe all Lebesgue points of f.

(b) Give an example of a bounded function f on $(-\infty, \infty)$ with the following properties: f is continuous except at a single point x_0; $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f(y)\mathrm{d}y = f(x)$ for all x (in particular when $x = x_0$); x_0 is not a Lebesgue point of f.

Proof.

(a) Since for any $x \in \mathbb{R}^n$ and any cube Q_x centered at x,
$$|\{y \in Q_x : f(y) = 1\}| = 0,$$
we have
$$\frac{1}{Q_x} \int_{Q_x} f(y) \mathrm{d}y = 0.$$
Thus, the desired set is $\{f = 0\}$, also the set of Lebesgue points.

(b) [Wan13] Define
$$f(x) = \begin{cases} 1, & |x| \in \left[\frac{1}{n+1}, \frac{1}{n+1} + \frac{1}{2n(n+1)}\right), \\ -1, & |x| \in \left[\frac{1}{n+1} + \frac{1}{2n(n+1)}, \frac{1}{n}\right), \\ 0, & x = 0 \text{ or } |x| \geqslant 1, \end{cases} \qquad \text{for } n = 1, 2, \cdots.$$

Let
$$\varphi(x) = \int_0^x f(t)\mathrm{d}t.$$
We analyze $\varphi'(0+)$ only. For $0 < h < 1$, choose n_0 such that
$$\frac{1}{n_0+1} < h \leqslant \frac{1}{n_0}.$$
Then
$$\int_0^h f = \sum_{n=n_0}^{\infty} \int_{1/(n+1)}^{1/n} f + \int_{1/(n_0+1)}^{h} f.$$
By construction,
$$\int_{1/(n+1)}^{1/n} f = 1 \cdot \frac{1}{2n(n+1)} + (-1) \cdot \left(\frac{1}{n} - \frac{1}{n+1} - \frac{1}{2n(n+1)} \right) = 0.$$
Thus,
$$\left| \frac{\varphi(h)}{h} \right| \leqslant \frac{1}{n_0(n_0+1)} \cdot \frac{1}{h} \leqslant \frac{1}{n_0} \to 0, \quad \text{as } h \to 0^+.$$
However,
$$\lim_{h\to 0^+} \frac{1}{2h} \int_{-h}^{h} |f(y) - f(0)|\mathrm{d}y = 1 \neq 0,$$
which yields $x_0 = 0$ is not a Lebesgue point.

□

Remark. In (b), the function
$$f(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0 \end{cases}$$
is not a desired example, since its indefinite integral is not differentiable at $x = 0$. And the function
$$f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$
is also not a desired example.

Problem 8.4

Let f be a measurable function on $\mathbb{R}^n$ that is non-zero on a set of positive measure. Then there exists a constant $c > 0$ such that the Hardy-Littlewood maximal function satisfies
$$f^*(x) \geqslant \frac{c}{|x|^n} \quad \text{for all } |x| \geqslant 1.$$

Proof. Let $f_j = f\chi_{[-j,j]^n}$. Then $f_j \to f$ in L^1, so there exists j_0 such that $f_{j_0} \not\equiv 0$. For $|x| \geqslant 1$, define Q_x as following: its center is $\frac{x}{2}$ and side length is $2(|x| + j_0\sqrt{n})$ Then we have
$$[-j_0, j_0]^n \subset Q_x \quad \text{and} \quad x \in Q_x$$

since $\|y\|_\infty \leqslant j_0 \Rightarrow y \in Q_x$ and $\|x - \frac{x}{2}\|_\infty \leqslant \frac{|x|}{2} < |x| + j_0\sqrt{n}$. Then

$$\begin{aligned} f^*(x) &\geqslant \frac{1}{|Q_x|}\int_{Q_x} |f(y)|dy \\ &\geqslant \frac{1}{(2(|x|+j_0\sqrt{n}))^n}\int_{[-j_0,j_0]^n} |f_{j_0}(y)|dy \\ &= \frac{c_n}{(|x|+j_0\sqrt{n})^n}\int_{[-j_0,j_0]^n} |f(y)|dy \end{aligned}$$

where $c_n = 2^{-n}$. For $|x| \geqslant 1$,

$$|x| + j_0\sqrt{n} \leqslant (1 + j_0\sqrt{n})|x|$$

yielding

$$f^*(x) \geqslant \frac{c}{(1+j_0\sqrt{n})^n|x|^n} = \frac{c'}{|x|^n}$$

with $c' = c_n\|f_{j_0}\|_{L^1}/(1+j_0\sqrt{n})^n > 0$. $\square$

Remark. This proposition shows that the maximal operator $f \mapsto f^*$ can not be a bounded operator on L^1 space (compare with Problem 8.2).

Homework Solution 15-16

Problem 9.1

Let f be monotone increasing and finite on an finite interval $(a,b) \subset \mathbb{R}$. Denote the set

$$A = \{x \in (a,b) : D^+ f(x) > D_- f(x)\},$$
$$B = \{x \in (a,b) : D^- f(x) > D_+ f(x)\}.$$

If $|A| = 0$, show that $|B| = 0$.

Problem 9.2

If $f \in BV([a,b])$, and $\bigvee_a^b(f) = f(b) - f(a)$, show that f is montone increasing in $[a,b]$.

Proof. Define

$$F(x) := \bigvee_a^x(f) - f(x) + f(a).$$

Then $F(a) = F(b) = 0$. We claim that $F(x)$ is increasing since for any $a \leqslant x_1 \leqslant x_2 \leqslant b$, from the definition of variation

$$|f(x_2) - f(x_1)| \leqslant \bigvee_{x_1}^{x_2}(f).$$

Therefore $F(x) = 0$ for all $a \leqslant x \leqslant b$, which implies $f(x) = f(a) + \bigvee_a^x(f)$, which is increasing. $\square$

Problem 9.3

Assume $\{f_n\} \subset AC([a,b])$ and each f_n is montone increasing in $[a,b]$, if $\sum_{n=1}^{\infty} f_k(x)$ converges in $[a,b]$. Prove that $\sum_{n=1}^{\infty} f_k \in AC([a,b])$.

Proof. Let $S(x) = \sum_{n=1}^{\infty} f_n(x)$ $(a \leqslant x \leqslant b)$. Then $S(x)$ is an increasing function. By Fubini's Theorem, we have

$$S'(x) = \sum_{n=1}^{\infty} f_n'(x) \quad \text{a.e. } x \in [a,b],$$

where $f_n'(x) \geqslant 0$ for a.e. $x \in [a,b]$. Consequently,

$$\int_a^b S'(x)\mathrm{d}x = \sum_{n=1}^{\infty} \int_a^b f_n'(x)\mathrm{d}x$$
$$= \sum_{n=1}^{\infty} [f_n(b) - f_n(a)] = S(b) - S(a).$$

This shows that $S \in AC([a,b])$ (see Theorem 7.29). □

Problem 9.4

Assume $f \in BV([0,1])$, if for each $\varepsilon > 0$, $f \in AC([\varepsilon,1])$ and f is continuous at 0. Prove that $f \in AC([0,1])$.

Proof. For the decreasing sequence $\varepsilon_n = \dfrac{1}{n}$ converging to 0 and any $x \in (0,1]$, we have

$$\int_0^x f'(t)\mathrm{d}t = \lim_{n\to\infty} \int_{1/n}^x f'(t)\mathrm{d}t = \lim_{n\to\infty} \left[f(x) - f\left(\tfrac{1}{n}\right)\right] = f(x) - f(0).$$

This shows that $f' \in L([0,1])$, and thus the conclusion holds. □

Remark. This conclusion does not hold when we suppose $f \in C([0,1])$ only. for instance, define

$$f(x) = \begin{cases} x\sin\dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Problem 9.5 (Riesz's Rising Sun Lemma)

Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Define the rising set:

$$E = \{x \in (a,b) | \exists y > x \text{ such that } f(y) > f(x)\}.$$

Then E is an open subset of (a,b) and can be written as a countable union of disjoint open intervals (a_k, b_k), where for each k,

$$f(a_k) \leqslant f(b_k).$$

Proof. [Rie32] The openness of E follows from continuity: given any $x \in E$ with corresponding $y > x$ satisfying $f(y) > f(x)$, there exists $\delta > 0$ such that $f(y) > f(z)$ for all $z \in (x-\delta, x+\delta)$, making E open. As an open subset of (a,b), E decomposes into a countable union of disjoint open intervals $E = \bigcup_k (a_k, b_k)$.

For each component interval (a_k, b_k), if $a_k \neq a$ then $a_k \notin E$ implies

$$f(a_k) \geqslant \limsup_{x\to a_k^+} f(x) \geqslant f(b_k).$$

Here we prove $f(a_k) \leqslant f(b_k)$ by contradiction. Suppose $f(a_k) > f(b_k)$. Then by continuity, f attains its maximum on $[a_k, b_k]$ at some point $z \in [a_k, b_k)$. Since $z \in (a_k, b_k) \subset E$, there exists $y > z$ with $f(z) < f(y)$. If $y \leqslant b_k$, this contradicts z being the maximum on $[a_k, b_k]$. Therefore $y > b_k$, implying $f(b_k) \leqslant f(z) < f(y)$. But this would force $b_k \in E$, contradicting b_k being a right endpoint of a component interval.

Since any $x \in (a_k, b_k)$ has $f(x) < f(b_k)$, continuity gives

$$f(a_k) = \lim_{x \to a_k^+} f(x) \leqslant f(b_k),$$

forcing $f(a_k) = f(b_k)$. When $a_k = a \in E$, the existence of $y > a$ with $f(a) < f(y)$ combined with continuity yields $f(a) \leqslant f(b_k)$. Thus $f(a_k) \leqslant f(b_k)$ holds universally for the interval decomposition. $\square$

Homework Solution 17-18

Problem 10.1

Assume $0 < |E| < +\infty$, $\{p_k\} \subset \mathbb{R}$ with $1 < p_1 < p_2 < \cdots < p_k < \cdots \to \infty$. If $f \in L^{p_k}(E)$ for all $k \geqslant 1$, and $\sup\limits_{k\geqslant 1}\{\|f\|_{p_k}\} < +\infty$, then $f \in L^\infty(E)$.

Proof. For any $k > C := \sup\limits_{k\geqslant 1}\{\|f\|_{p_k}\}$, define
$$A_t := \{x \in E : f(x) \geqslant t\}.$$
We have
$$C \geqslant \|f\|_{p_k} \geqslant \left(\int_{A_t} |f(x)|^{p_k} \mathrm{d}x\right)^{\frac{1}{p_k}} \geqslant tm(A_k)^{\frac{1}{p}}.$$
This implies
$$|A_t| \leqslant \left(\frac{C}{t}\right)^{p_k}.$$
Let $p_k \to +\infty$, then we obtain
$$|A_t| = 0, \quad \text{for all } t > C.$$
Then by the difinition of L^∞ norm, we have $\|f\|_\infty \leqslant C$. □

Remark. This conclusion does not hold when $\|f\|_{p_k}$ is not unifomly bounded about k, for instance, consider $E = (0,1)$ and $f(x) = \ln x$.

Problem 10.2

Show that $(L^\infty(E), \|\cdot\|_\infty)$ is complete.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $L^\infty(E)$. For each integer $k \geqslant 1$, there is some N_k such that $\|f_m - f_n\|_\infty \leqslant \dfrac{1}{k}$ whenever $m,n \geqslant N_k$. This implies the existence of null sets Z_k satisfying
$$|f_m(x) - f_n(x)| \leqslant \frac{1}{k} \quad \text{for all } x \in \Omega \setminus Z_k \text{ and } m,n \geqslant N_k.$$
Let $Z = \bigcup\limits_k Z_k$, which remains a null set. For each $x \in E \setminus Z$, the numerical sequence $\{f_n(x)\}$ is Cauchy in $\mathbb{R}$ and therefore converges to some limit $f(x)$.
Fixing k and taking $m \to \infty$ in the inequality shows that
$$|f(x) - f_n(x)| \leqslant \frac{1}{k} \quad \text{for all } x \in E \setminus Z \text{ and } n \geqslant N_k.$$
This establishes both that $f \in L^\infty$ and that $\|f - f_n\|_\infty \leqslant \dfrac{1}{k}$ for $n \geqslant N_k$, proving $f_n \to f$ in L^∞-norm. □

Problem 10.3

For complex-valued measurable $f = f_1 + \mathrm{i} f_2$ with f_1, f_2 real-valued and measurable, prove that $\int_E f$ is finite if and only if $\int_E |f|$ is finite, and

$$\left|\int_E f\right| \leqslant \int_E |f|.$$

(Hint: Use

$$\left|\int_E f\right| = \left[\left(\int_E f_1\right)^2 + \left(\int_E f_2\right)^2\right]^{1/2}$$

and the identity

$$\left(a^2+b^2\right)^{1/2} = a\cos\alpha + b\sin\alpha$$

where $\tan\alpha = b/a$.)

Proof. Following the hint, we have

$$\begin{aligned}
\left|\int_E f(x)\mathrm{d}x\right| &= \left[\left(\int_E f_1(x)\mathrm{d}x\right)^2 + \left(\int_E f_2(x)\mathrm{d}x\right)^2\right]^{1/2} \\
&= \int_E f_1(x)\mathrm{d}x\cos\alpha + \int_E f_2(x)\mathrm{d}x\sin\alpha \\
&= \int_E f_1(x)\cos\alpha + f_2(x)\sin\alpha\mathrm{d}x \\
&\leqslant \int_E |f(x)|\mathrm{d}x,
\end{aligned}$$

where α is independent of x. □

Problem 10.4

Let f and g be real-valued and not identically zero. For $1 < p < \infty$, prove:

1. Equality holds in $\left|\int fg\right| \leqslant \|f\|_p \|g\|_{p'}$ iff fg has constant sign a.e. and $|f|^p$ is proportional to $|g|^{p'}$ a.e.
2. If $\|f+g\|_p = \|f\|_p + \|g\|_p$ with $g \neq 0$, then f is a multiple of g a.e.

Proof.

1. ($\Rightarrow$) Since we have

$$\left|\int fg\right| \leqslant \int |fg| \leqslant \|f\|_p \|g\|_{p'}.$$

Suppose $\left|\int fg\right| = \|f\|_p \|g\|_{p'}$. By the equality condition($a=b$) in Young's inequality

$$ab \leqslant \frac{a^p}{p} + \frac{b^{p'}}{p'},$$

which is used to prove Hölder's inequality, we must have

$$\frac{|f(x)|^p}{\|f\|_p^p} = \frac{|g(x)|^{p'}}{\|g\|_{p'}^{p'}} \quad \text{a.e.}$$

This shows $|f|^p$ is proportional to $|g|^{p'}$ a.e. Moreover, since we have equality in $\left|\int fg\right| = \int |fg|$, the product fg must have constant sign a.e.

($\Leftarrow$) If fg has constant sign a.e. and $|f|^p = C|g|^{p'}$ a.e. for some constant $C > 0$, then direct computation shows:

$$\left|\int fg\right| = \int |fg| = \int |f| \cdot |g| = C^{1/p} \int |g|^{p'} = \|f\|_p \|g\|_{p'}.$$

2. Assume $\|f+g\|_p = \|f\|_p + \|g\|_p$ with $g \neq 0$. The standard proof of Minkowski's inequality gives

$$\|f+g\|_p^p = \int |f+g|^p \leqslant \int (|f|+|g|)|f+g|^{p-1}.$$

Applying Hölder's inequality to each term yields

$$\int |f||f+g|^{p-1} \leqslant \|f\|_p \||f+g|^{p-1}\|_{p'} = \|f\|_p \|f+g\|_p^{p/p'},$$

$$\int |g||f+g|^{p-1} \leqslant \|g\|_p \||f+g|^{p-1}\|_{p'} = \|g\|_p \|f+g\|_p^{p/p'},$$

so

$$\|f+g\|_p^p \leqslant (\|f\|_p + \|g\|_p)\|f+g\|_p^{p/p'}.$$

Equality requires:

(i) $|f+g| = |f|+|g|$ a.e. (so f and g have the same sign a.e.)

(ii) Equality in both Hölder applications

From (ii), by part (1), there exist constants $C_1, C_2 > 0$ such that

$$|f|^p = C_1|f+g|^{(p-1)p'} = C_1|f+g|^p \quad \text{a.e.},$$

and

$$|g|^p = C_2|f+g|^p \quad \text{a.e.}.$$

Thus $|f| = C_1^{1/p}|f+g|$ and $|g| = C_2^{1/p}|f+g|$ a.e. Adding these yields

$$|f|+|g| = (C_1^{1/p} + C_2^{1/p})|f+g| \quad \text{a.e.}$$

By (i), $|f|+|g| = |f+g|$ a.e., so $C_1^{1/p} + C_2^{1/p} = 1$. Since f and g have the same sign a.e., we conclude $f = C_1^{1/p}(f+g)$ and $g = C_2^{1/p}(f+g)$ a.e. Thus

$$f = \frac{C_1^{1/p}}{1 - C_1^{1/p}} g \quad \text{a.e.},$$

so f is a multiple of g a.e.

□

Problem 10.5

Let $1 \leqslant p_i, r \leqslant \infty$ with $\sum_{i=1}^{k} \frac{1}{p_i} = \frac{1}{r}$. Prove the generalized Hölder's inequality:

$$\|f_1 \cdots f_k\|_r \leqslant \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}.$$

Proof. We prove the generalized Hölder's inequality by induction on k.
Base case ($k = 2$): This is the standard Hölder's inequality

$$\|f_1 f_2\|_r \leqslant \|f_1\|_{p_1} \|f_2\|_{p_2}$$

where $\dfrac{1}{p_1} + \dfrac{1}{p_2} = \dfrac{1}{r}$.
Inductive step: Assume the inequality holds for $k-1$ functions. Let $1 \leqslant p_i, r \leqslant \infty$ with $\sum_{i=1}^{k} \dfrac{1}{p_i} = \dfrac{1}{r}$. Define:

$$q = \left(\sum_{i=1}^{k-1} \frac{1}{p_i} \right)^{-1} = \left(\frac{1}{r} - \frac{1}{p_k} \right)^{-1}$$

Note that $\dfrac{1}{q} + \dfrac{1}{p_k} = \dfrac{1}{r}$.
Applying the standard Hölder's inequality to $f_1 \cdots f_{k-1}$ and f_k yields

$$\|f_1 \cdots f_k\|_r \leqslant \|f_1 \cdots f_{k-1}\|_q \|f_k\|_{p_k}$$

By the inductive hypothesis applied to $f_1, \ldots, f_{k-1}$ with exponents $p_1, \ldots, p_{k-1}$, we obtain

$$\|f_1 \cdots f_{k-1}\|_q \leqslant \|f_1\|_{p_1} \cdots \|f_{k-1}\|_{p_{k-1}}$$

Combining these inequalities gives

$$\|f_1 \cdots f_k\|_r \leqslant \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}$$

which completes the induction. □

Problem 10.6 (Interpolation inequality)

Let $1 \leqslant p_1 < p_2 < p_3 \leqslant \infty$ with $\dfrac{1}{p_2} = \dfrac{\theta}{p_1} + \dfrac{1-\theta}{p_3}$. If $f \in L^{p_1}(E) \cap L^{p_3}(E)$, prove

$$\|f\|_{p_2} \leqslant \|f\|_{p_1}^{\theta} \|f\|_{p_3}^{1-\theta}.$$

Proof. First consider the case where $p_3 < \infty$. Let $\alpha = \theta p_2 / p_1$ and $\beta = (1-\theta) p_2 / p_3$, noting that

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{p_1}{\theta p_2} + \frac{p_3}{(1-\theta) p_2} = \frac{1}{p_2} \left(\frac{p_1}{\theta} + \frac{p_3}{1-\theta} \right) = 1$$

by the given relationship between the p_i.
Write f as

$$|f|^{p_2} = |f|^{\theta p_2} \cdot |f|^{(1-\theta) p_2} = (|f|^{p_1})^{\alpha} \cdot (|f|^{p_3})^{\beta}$$

Applying Hölder's inequality with exponents α and β:

$$\int_E |f|^{p_2} \leqslant \left(\int_E |f|^{p_1} \right)^{\alpha/\alpha} \left(\int_E |f|^{p_3} \right)^{\beta/\beta} = \|f\|_{p_1}^{\theta p_2} \|f\|_{p_3}^{(1-\theta) p_2}$$

Taking the $1/p_2$ power of both sides yields the result.
For $p_3 = \infty$, we have $\dfrac{1}{p_2} = \dfrac{\theta}{p_1}$ and

$$\|f\|_{p_2}^{p_2} = \int_E |f|^{\theta p_2} |f|^{(1-\theta) p_2} \leqslant \|f\|_{\infty}^{(1-\theta) p_2} \int_E |f|^{\theta p_2}$$

Since $\theta p_2 = p_1$:

$$\|f\|_{p_2}^{p_2} \leqslant \|f\|_{\infty}^{(1-\theta) p_2} \|f\|_{p_1}^{p_1} = \|f\|_{\infty}^{(1-\theta) p_2} \|f\|_{p_1}^{\theta p_2}$$

Taking $1/p_2$ roots gives the inequality. □

Problem 10.7

If $f_k \to f$ in L^p $(1 \leqslant p < \infty)$, $g_k \to g$ pointwise, and $\|g_k\|_\infty \leqslant M$ for all k, prove that $f_k g_k \to fg$ in L^p.

Proof. To prove that $f_k g_k \to fg$ in L^p, i.e., $\|f_k g_k - fg\|_p \to 0$, consider the decomposition:

$$f_k g_k - fg = (f_k g_k - f g_k) + (f g_k - fg) = g_k(f_k - f) + f(g_k - g).$$

By Minkowski's inequality,

$$\|f_k g_k - fg\|_p \leqslant \|g_k(f_k - f)\|_p + \|f(g_k - g)\|_p.$$

We will show that both terms on the right-hand side converge to 0.

Prove $\|g_k(f_k - f)\|_p \to 0$. Given $\|g_k\|_\infty \leqslant M$, we have $|g_k(x)| \leqslant M$ almost everywhere. Therefore,

$$|g_k(f_k - f)| \leqslant M|f_k - f| \quad \text{a.e.}$$

Thus,

$$\|g_k(f_k - f)\|_p = \left(\int |g_k(f_k - f)|^p \mathrm{d}x\right)^{1/p} \leqslant \left(\int M^p |f_k - f|^p \mathrm{d}x\right)^{1/p} = M\|f_k - f\|_p.$$

Since $f_k \to f$ in L^p, it follows that $\|f_k - f\|_p \to 0$. Hence, $M\|f_k - f\|_p \to 0$, so $\|g_k(f_k - f)\|_p \to 0$.

Prove $\|f(g_k - g)\|_p \to 0$. Given $g_k \to g$ a.e., we have $|g_k - g|^p \to 0$ a.e. Additionally, $\|g_k\|_\infty \leqslant M$ implies $|g| \leqslant M$ a.e. Thus, $|g_k - g| \leqslant 2M$ a.e. Therefore,

$$|f(g_k - g)|^p = |f|^p |g_k - g|^p \leqslant |f|^p (2M)^p.$$

The function $|f|^p(2M)^p$ is integrable because $f \in L^p$ (since $f_k \to f$ in L^p and L^p is complete), i.e.,

$$\int |f|^p (2M)^p \mathrm{d}x = (2M)^p \|f\|_p^p < \infty.$$

By the dominated convergence theorem with dominating function $|f|^p(2M)^p \in L^1$ and $|f|^p|g_k - g|^p \to 0$ pointwise, we have

$$\int |f|^p |g_k - g|^p \mathrm{d}x \to 0 \quad \text{as } k \to \infty.$$

Consequently,

$$\|f(g_k - g)\|_p = \left(\int |f|^p |g_k - g|^p \mathrm{d}x\right)^{1/p} \to 0.$$

Combining both steps, $\|f_k g_k - fg\|_p \leqslant \|g_k(f_k - f)\|_p + \|f(g_k - g)\|_p \to 0$. Thus, $f_k g_k \to fg$ in L^p. □

Problem 10.8

Let $f, \{f_k\} \in L^p$ $(0 < p \leqslant \infty)$. Show:

1. If $\|f - f_k\|_p \to 0$, then $\|f_k\|_p \to \|f\|_p$.
2. If $f_k \to f$ a.e. and $\|f_k\|_p \to \|f\|_p$ $(0 < p < \infty)$, then $\|f - f_k\|_p \to 0$. (The converse fails for $p = \infty$.)

Proof.

1. For $p \geqslant 1$ we use the triangle inequality
$$\big|\|f_k\|_p - \|f\|_p\big| \leqslant \|f_k - f\|_p \to 0, \quad \text{as} k \to \infty.$$
For $p < 1$ we use the basic estimate
$$|a+b|^p \leqslant a^p + b^p, \quad \text{for all } a, b \geqslant 0.$$
Then,
$$\big|\|f_k\|_p^p - \|f\|_p^p\big| \leqslant \|f_k - f\|_p^p \to 0, \quad \text{as} k \to \infty.$$
2. Let $C = \min\{2^p, 1\}$, then
$$|f_k - f|^p \leqslant C(|f_k|^p + |f|^p).$$
Define
$$g_k = C(|f_k|^p + |f|^p) - |f_k - f|^p.$$
Then $g_k \geqslant 0$ and
$$g_k \to g := 2C|f|^p \quad \text{a.e.}$$
By Fatou's lemma and $\|f_k\|_p \to \|f\|_p$ we obtain
$$2C \int |f|^p \leqslant 2C \int |f|^p - \limsup \int |f_k - f|^p,$$
which yields
$$\int |f_k - f|^p \to 0.$$

□

Remark. See also Brézis-Lieb's lemma [BL83].

Problem 10.9

Suppose $f_k \to f$ a.e. with $f_k, f \in L^p$ $(1 < p \leqslant \infty)$ and $\|f_k\|_p \leqslant M < +\infty$. Prove:

1. $\displaystyle\int f_k g \to \int fg$ for all $g \in L^{p'}$ $(1/p + 1/p' = 1)$.
2. The result fails when $p = 1$. (For $p > 1$, use Egorov's theorem when $|E| < \infty$.)

Proof.

1. We first obtain $f \in L^p(\mathbb{R}^d)$ directly from Fatou's lemma. Next we prove that for any $\varepsilon > 0$ and $g \in L^{p'}(\mathbb{R}^d)$, there is $N_1 \in \mathbb{N}$ such that for all $k > N_1$ we have
$$\left|\int (f - f_k) g \,\mathrm{d}x\right| < \varepsilon.$$
Since $|g|^{p'} \in L^1(\mathbb{R}^d)$, there is $\delta > 0$ such that for any measurable set A with $|A| < \delta$,
$$\left(\int_A |g|^{p'} \mathrm{d}x\right)^{\frac{1}{p'}} < \frac{\varepsilon}{6M}.$$
Moreover, from $|g|^{p'} \in L^1(\mathbb{R}^d)$ we can find a measurable set B with $|B| < \infty$ satisfying
$$\left(\int_{\mathbb{R}^d \setminus B} |g|^{p'} \mathrm{d}x\right)^{\frac{1}{p'}} < \frac{\varepsilon}{6M}.$$
By Egorov's theorem, there is a measurable subset $E \subset B$ with $|B \setminus E| < \delta$ such that f_k converges uniformly to f on E. Thus there is $N_2 \in \mathbb{N}$ such that for all $k > N_2$ and $x \in E$,
$$|f(x) - f_k(x)| < \frac{\varepsilon}{3\|g\|_{p'} |E|^{\frac{1}{p}}}.$$

Consequently, for all $k > \max\{N_1, N_2\}$ we have
$$\left(\int_E |f - f_k|^p \mathrm{d}x\right)^{\frac{1}{p}} \|g\|_{p'} < \frac{\varepsilon}{3}.$$
Taking $A = B \setminus E$, for all $k > \max\{N_1, N_2\}$ we obtain
$$\int_{\mathbb{R}^d} |f_k - f||g|\mathrm{d}x = \int_{B\setminus E} |f_k - f||g|\mathrm{d}x + \int_{\mathbb{R}^d\setminus B} |f_k - f||g|\mathrm{d}x + \int_E |f_k - f||g|\mathrm{d}x$$
$$< 2M\frac{\varepsilon}{6M} + 2M\frac{\varepsilon}{6M} + \frac{\varepsilon}{3} = \varepsilon.$$

2. Consider the sequence in $[0,1]$:
$$f_k(x) = \begin{cases} k - k^2 x, & x \in [0, 1/k], \\ 0, & x \in (1/k, 1], \end{cases}$$
This sequence converges almost everywhere to $f \equiv 0$ but taking
$$g(x) = \chi_{[0,1]}(x)$$
yields
$$\int_{[0,1]} f_k(x)\mathrm{d}x = \frac{1}{2} \not\to 0.$$
$\square$

Remark. See Problem 6.2.

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