

# Knowledge's Power: Unveiling the Imprints of History through Staircase Relics

## Summary

The staircase is a common structural element in historical buildings. As the wear on stairs provides crucial evidence for archaeologists and helps reveal the historical evolution of the building, in this paper, we use **Archard's theory** and **Archie's law** to establish the **Original Wear Value Model**, the **Impact Probability Model**, and the **Internal Wear Value Model**, for deducing specific historical details from the wear on stairs.

First, we considered the smoothness of the staircase surface and the volume of material lost, and based on **Archard's theory**, we established the **Original Wear Value Model**. During the analysis, we also derived the **Single Wear Formula**, which gave an explicit formula for analyzing the stair usage frequency.

Considering the different landing points when people go up and down the stairs, we proposed the **Impact Probability Model**. Taking the single wear formula as a bridge, we established the connection between the Wear value model and the impact probability model. This helps us answer the question of using the stairs in a specific direction. Further analysis of the impact of several people using the stairs simultaneously on the impact probability model led us to improve it, which well describes the situation of several people using the stairs at the same time.

However, the research did not stop there. We developed deeply into Archard's theory and used the **proportional constant  $K/H$**  to obtain ideas for determining the materials used for the stairs. For the remaining questions, we found that the original wear value model could hardly correctly reflect the real situation. Therefore, we turned to studying the internal wear degree of the stairs. We introduced **Archie's law** to successfully build the **Internal Wear Value Model**. Finally, we considered the **Complete Wear Value Model** from different angles and gradually revealed more historical information hidden in the stairs.

Finally, the sensitivity analysis of the model is carried out to verify the stability of the model, and the strengths and weaknesses of the model are analyzed.

**Keywords:** Archard's theory; Archie's law; Wear Value Model

# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
1.1	Background . . . . .	3
1.2	Problem Restatement . . . . .	3
1.3	Problem Analysis and Our Work . . . . .	3
<b>2</b>	<b>Assumptions and Justifications</b>	<b>5</b>
<b>3</b>	<b>Notations</b>	<b>6</b>
<b>4</b>	<b>Models</b>	<b>7</b>
4.1	Problem 1 . . . . .	8
4.1.1	Problem Analysis and Preliminary Ideas . . . . .	8
4.1.2	Model Preparation . . . . .	8
4.1.3	Model Construction . . . . .	8
4.2	Problem 2 . . . . .	10
4.2.1	Problem Analysis and Preliminary Ideas . . . . .	10
4.2.2	Model Preparation . . . . .	10
4.2.3	Model Construction . . . . .	12
4.3	Problem 3 . . . . .	12
4.3.1	Problem Analysis and Preliminary Ideas . . . . .	12
4.3.2	Model Preparation . . . . .	12
4.3.3	Model Construction . . . . .	13
4.4	Model Extension and Application . . . . .	13
4.4.1	Deeper Considerations . . . . .	13
4.4.2	Model Preparation . . . . .	14
4.4.3	Enhancement of the Wear Value Model . . . . .	15
4.4.4	Answers to Additional Problems . . . . .	15
<b>5</b>	<b>Sensitivity Analysis</b>	<b>21</b>

5.1	The Sensitivity Analysis of the Wear Value Model . . . . .	21
5.1.1	Sensitivity Factors . . . . .	22
5.1.2	Sensitivity of Wear Value $W$ to Volume $V_{\text{total}}$ and Resistivity Ratio $\frac{\rho}{\rho_0}$ . . .	23
<b>6</b>	<b>Model Evaluation and Future Work</b>	<b>23</b>
6.1	Model Advantages . . . . .	23
6.2	Model Innovations . . . . .	23
6.3	Limitations . . . . .	24
6.4	Future Work . . . . .	24

# 1 Introduction

## 1.1 Background

In historical architecture, stone steps are a common structural element. It is evident that even the most durable stone materials will experience wear and tear with prolonged use. This wear is not only a physical alteration but also reflects the historical usage of the steps and the traces of human activity. Therefore, studying this wear can provide archaeologists with valuable evidence regarding the building's history, usage patterns, and human interactions. The degree of wear on the stone steps reveals information such as the length of use, pedestrian traffic in different historical periods, the direction of foot traffic, and the frequency of use. Archaeologists are well aware that inferring specific historical details from these physical changes is a complex and challenging task.

## 1.2 Problem Restatement

In order to assist archaeologists in utilizing step wear levels for their research, we will address the following series of questions:

Firstly, three fundamental questions:

- 1. What is the frequency of stair usage?
- 2. Do people who use the stairs prefer a specific direction for walking?
- 3. How many people typically use the stairs simultaneously? (Is it one person or two people walking side by side?)

In addition to these basic questions, there are more challenging ones that can further aid archaeologists' studies:

- 4. Is the wear consistent with the existing information?
- 5. What is the estimated age of the stairs, and how reliable is this estimation?
- 6. What repairs or renovations have been made to the stairs?
- 7. Can the source of the materials be determined? If stone was used, can archaeologists assume that the wear corresponds to materials from the original quarry?
- 8. How can we determine whether the wear is a result of heavy use by a large number of people in a short period, or the result of use by a few individuals over a long period of time?

## 1.3 Problem Analysis and Our Work

In this study, we developed a comprehensive wear value model by progressively addressing a series of problems, from basic to complex.

First, we established a basic wear value model based on Archard's theory, incorporating stair smoothness and wear volume to reflect the wear process. We derived a formula for the wear value from a single use of the stairs, enabling us to estimate pedestrian traffic over time and address Problem 1: stair usage frequency.

Next, recognizing the variability in footfall points during ascent and descent, we proposed a footfall probability model, integrating it into the wear value model along with the single-use wear formula. This addressed Problem 2: whether pedestrians show a directional preference when using the stairs, with the varying footfall points leading to distinct wear patterns.

To account for simultaneous stair use by multiple pedestrians, we enhanced the footfall probability model, recognizing that different footfall models correspond to varying wear values. This development addressed Problem 3: how many people use the stairs simultaneously.

For Problem 7, involving proportional constants in Archard's theory, we utilized least squares algorithms to determine the most suitable constant, comparing it with specific material coefficients to identify the material used for the stairs.

Finally, to address Problems 5, 6, 7, and 8, we expanded our study to the internal sections of the stairs, where the original model is insufficient. By incorporating Archie's law, which links porosity and resistivity to material wear, we improved the model to account for resistivity changes over time due to oxidation, thereby enhancing the model's accuracy.

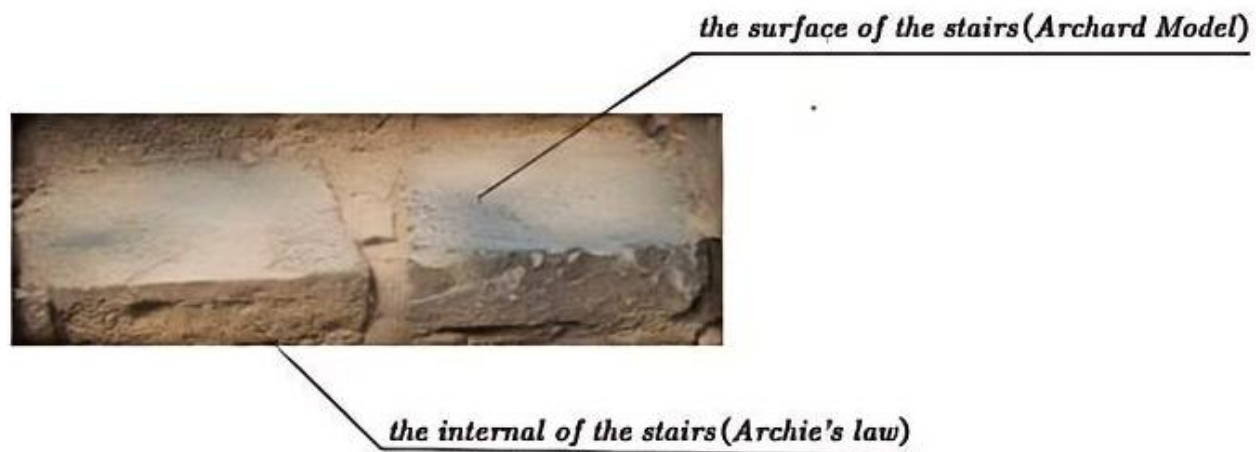


Figure 1: Staircase Surface and Internal Structure Diagram

## 2 Assumptions and Justifications

After thoroughly analyzing the background and the issues, in order to simplify our model, we will make the following reasonable assumptions.

**Assumption and Justification 1:** The degree of wear on the stair surface is only related to the smoothness of the stairs and the volume lost due to friction. The degree of wear in the deeper layers of the stairs is solely related to the material's porosity. These factors can largely reflect the wear degree, and at the same time, they do not make the model overly complex or difficult to apply in practice.

**Assumption and Justification 2:** Although the wear caused by each person's step on the stairs varies, we are considering the situation under a large number of steps. Therefore, we can assume that the wear caused by each step is consistent and equals the average value.

**Assumption and Justification 3:** In different walking directions, the footfall position on the stairs varies, leading to a different overall footfall pattern. Typically, when walking on stairs, people tend to step more on the right side of the stairs.

**Assumption and Justification 4:** The number of people using the stairs simultaneously also affects the footfall pattern. This is obvious: when walking alone, people are more likely to step in the middle of the stairs, while when walking in pairs, they tend to step on the two sides of the stairs.

**Assumption and Justification 5:** Archaeologists on-site can provide the necessary data for us. The data we need can be measured using easy-to-operate instruments and will not cause any damage to the stairs.

**Assumption and Justification 6:** We assume that the wear caused by each step on the staircase is linearly additive. This is because the wear caused by each step is proportional to the applied force, which simplifies our model.

### 3 Notations

To facilitate readers' understanding, we have listed the main symbols used in the model, as shown in Table 1

Table 1: Notations used in this literature

Symbol	Description
$W$	Wear value: Describes the wear on the stairs.
$V$	Wear volume: Material lost due to friction.
$S$	Contact surface area: Area of contact between two objects.
$F$	Normal load: Vertical force applied between two objects.
$K$	Wear constant: Rate of wear per unit contact area.
$H$	Hardness: Material's resistance to deformation.
$\mu$	Friction factor.
$N$	Total number of people passing through the stairs within a given period.
$\alpha$	Proportional coefficient in $W = \alpha NF$ .
$p$	Probability of each person's foot stepping on a particular location on the stairs.
$h$	Depth of wear at a particular point on the stairs.
$\rho$	A physical quantity that measures the resistance of a material to electrical current.
$n$	Exponent in Archie's law.
$\phi$	Proportion of the void volume in the material relative to the total volume.
$r$	Proportional coefficient introduced to the wear model through Archie's law.

## 4 Models

All the models above we can get is the Weierstrass - Mandelbrot function method (M - W)<sup>[10]</sup> : After most material components in nature are processed mechanically or manually, the surface profile exhibits fractal characteristics such as self - affine nature and scale independence. The W - M model can be used to represent the rough profile. For a two - dimensional profile curve, which give us the steps' model more correctly and more interesting.

So for the model, we can get the direct graph about it to get more correct computation. All the models established are based on the assumptions mentioned above. Therefore, we will not restate these assumptions in the process of model construction.

In addition, we will systematically organize the connections between the theories, models, and formulas we have used or developed and the series of problems addressed.

There are many methods to get the picture about the road, then we take the method about the W-M functional method to make more and more correct model for the staircase. There is the formula for it.

$$z(x) = G^{(D-1)} \sum_{n=n_{min}}^a \frac{\cos(2\pi\gamma^n x)}{\gamma^{(2-D)n}} \quad [10]$$

The constant of it must be the sum of frequency and smoothness's relative constant. We just use it for the drawing of the model<sup>[10]</sup>.

D: dimension of the model. x: coordinate of the edge.  $\gamma$ : is the frequency of space. n: the index of the frequency.

Finally, we will reiterate the entire model's approach. This has already been explained in the 'Our Work', so this time we will represent it using a flowchart.

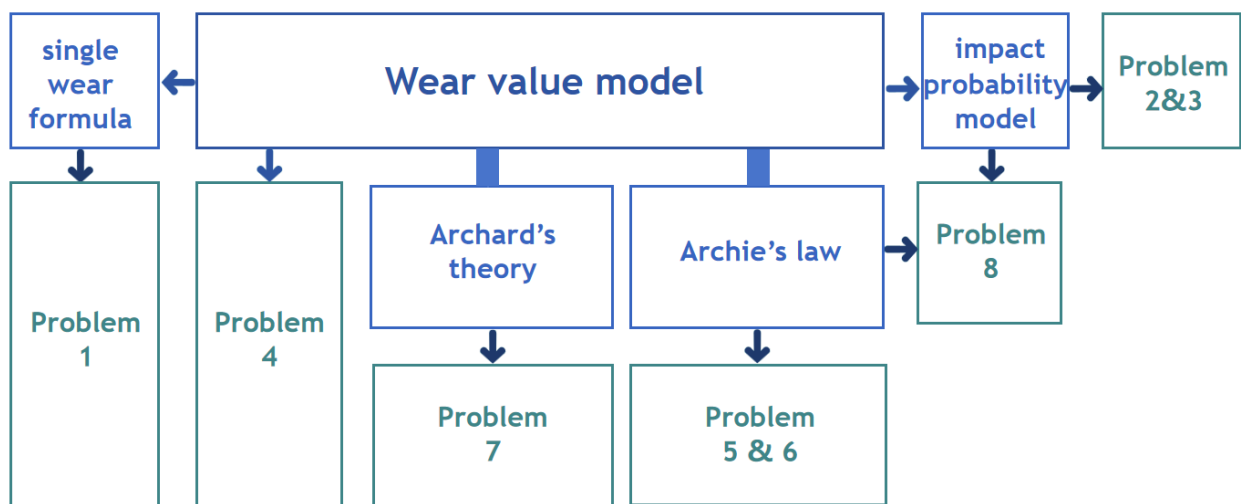


Figure 2: Model Flowchart



## 4.1 Problem 1

### 4.1.1 Problem Analysis and Preliminary Ideas

The issue we need to address is the frequency of stairway usage. It is evident that both high and low usage frequencies will manifest in the degree of wear on the surface of the stairs. When a large number of people use the stairs within a certain period, the wear on the stairs during this time is relatively severe. Conversely, when the number of users decreases, the wear on the stairs is relatively minor.

This observation inspires us to construct a model that describes the degree of stairway wear. Additionally, we can represent the usage frequency by dividing the total number of people passing through the stairs during a certain period by the duration of that period. Furthermore, to establish a relationship between usage frequency and wear, we need to develop a formula for the wear caused by a single individual using the stairs. By integrating the model with this formula, we can derive the usage frequency.

### 4.1.2 Model Preparation

The original wear model we constructed is based on Archard's theory<sup>[5]</sup>, which we will briefly introduce. Archard's theory, proposed by John F. Archard in 1953, is a tribological theory primarily used to describe the wear behavior when two objects are in contact. Its core idea is that the wear volume is directly proportional to the contact pressure, directly proportional to a power function of the sliding distance, and related to the hardness of the material. Specifically, the Archard wear equation can be expressed as:

$$V = K \cdot \frac{F \cdot S}{H}^{[5]}$$

where: V: Represents the wear volume, which indicates the amount of material lost due to friction. k: The Archard wear coefficient, which is an empirical constant dependent on the properties of the materials. F: The normal force applied on the contact surface. S: Represents the actual contact area between the objects. H: The hardness of the material. Generally, the higher the material's hardness, the smaller the wear.

The specific value of V can be measured, because according to the definition of V, we only need to measure the original height and the height after loss h at each point on the surface of the staircase, and then multiply it by the area dS at that point to obtain V. At this point, we have the function h(x,y), and the formula is:

$$V = \iint h(x, y) dx dy^{[5]}$$

### 4.1.3 Model Construction

Based on the aforementioned Archard theory, we take into account that the degree of wear is not only related to the volume loss due to friction on the staircase but also to the surface smoothness. Specifically, a larger volume loss indicates a higher degree of wear, and a smoother staircase surface also corresponds to a higher degree of wear. The smoothness of the staircase can be represented by the physical quantity of the friction factor  $\mu$ . (Similarly, the coefficient of friction  $\mu$  can

also be easily measured, which only requires some fundamental knowledge of classical mechanics.) Therefore, we construct the following wear model<sup>[5]</sup>:

$$W = \frac{V}{\mu}^{[5]}$$

In addition, we need to consider the formula for the wear caused by a single use of the staircase: the single-use wear formula. It is evident that the wear caused by a single use is directly proportional to the pressure exerted by a person. Therefore, we have:

$$W = \alpha F^{[5]}$$

Of course, according to the assumptions,  $F=mg$ , where  $m$  is the average body weight of the people. We may assume that  $m=62$  kg, which was derived by Roberts et al. in 2012 based on the data published by the World Health Organization (WHO). Based on the assumption that  $N$  people pass through the staircase in time  $t$ , the total wear value caused is given by:

$$W = \alpha F \cdot N$$

<sup>[5]</sup> To determine the wear, we measure the wear value  $W$  before and after the time period. Thus, we can derive:

$$N = \frac{W(t + \Delta t) - W(t)}{\alpha F}$$

After obtaining  $N$ , to represent the usage frequency of the staircase, we simply need to calculate.

$$\frac{N}{\Delta t}$$

A larger value indicates a higher usage frequency of the staircase during this period, and vice versa.

And also, for us studying, we must have the research for Bowden's research: Conduct further research based on the elastic or plastic deformation of the surface which is the multiply of the force, which can let us have the studying about what the  $V$  must be.

So, in there we must take the way of model. Hertz theory: the method of application about the force theory about one point's contact.

$$a = \sqrt[3]{\frac{3W}{4} \cdot \frac{R_1 R_2}{R_1 + R_2} \left( \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right)}^{[10]}$$

$$p = \sqrt[3]{\frac{6}{\pi^3} \cdot \frac{R_1 + R_2}{R_1 R_2} \cdot \frac{W}{\left( \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right)}}^{[10]}$$

$$\delta = \sqrt[3]{\frac{9}{16} \cdot \frac{R_1 + R_2}{R_1 R_2} \cdot \left( \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right) W^2}^{[10]}$$

$R$ : the radical.  $E$ : the Elastic Modulus  $\nu$ : poisson's ratio.

## 4.2 Problem 2

### 4.2.1 Problem Analysis and Preliminary Ideas

We consider the differences in footfall locations on the staircase when people are moving in different directions (upward or downward). There is a common consensus that people tend to walk on the right side of the staircase, which is consistent with the "keep right" principle in traffic regulations. This habit not only helps maintain order but also provides a clear path for those in a hurry on the left side. Based on this, we propose a footfall probability model that describes the probability of foot contact at different locations on the staircase under various conditions. The footfall locations vary depending on the direction of movement (upward or downward), and thus the footfall probability functions are different. Naturally, this will determine the differences in  $N$  at each location on the staircase surface when it is used. Given the wear model

$$W = \alpha F \cdot N$$

, it is evident that this will also result in different  $W$  values at each location. This is reflected in the wear model and will assist us in addressing Question 2.

### 4.2.2 Model Preparation

Firstly, from the model of personal action on one thing, we can get a model of probability about the actions' probability function is:

$$r(x, y) = \frac{1}{\exp\left(\frac{1}{\kappa}\right) \sigma(r(t), f(t)) \sqrt{2\pi}} \exp\left(-\frac{(\tau - T)^2}{2\left(\exp\left(\frac{1}{\kappa}\right) \sigma(r(t), f(t))\right)^2}\right) [?]$$

$\kappa$  : is the constant about personality, and the  $\sigma$  is the constant depend on time.  $\tau$  : just be time from now.  $T$  : is the constant the starting time.

Here, we get theory about personal habits function, then if we want, we can take the change about the formula for two persons about their habits probability model. That is the formula which from two normal distribution also is an normal distribution.

So, we then can assume that all constant in our paper is easy than the facts constant. We then establish the footfall probability model:  $p(x, y)$ , where  $x$  and  $y$  represent the position information at each point on the staircase surface, and  $p$  denotes the probability of a foot contact at that point during a single instance. Clearly, different directions of movement will yield different footfall probability functions. We assume that the probability density function at each position on the staircase is given by:

$$p(x, y) = \frac{1}{\sigma_x \sigma_y \sqrt{2\pi}} \exp\left(-\frac{(x - \mu_x)^2}{2\sigma_x^2} - \frac{(y - \mu_y)^2}{2\sigma_y^2}\right)$$

Here,  $\mu_x$  and  $\mu_y$  represent the expected positions of the footfalls (with  $\mu_x$  biased towards the right side when ascending and possibly towards the left side or unchanged when descending), while  $\sigma_x$  and  $\sigma_y$  control the width of the distribution.

To account for the directional bias in walking, a directional correction factor  $\beta$  can be introduced. When ascending,  $\beta$  increases the rightward bias, and when descending,  $\beta$  is relatively small or has no effect. The corrected probability density functions can be expressed as:

$$p(x, y | \text{up}) = \frac{1}{\sigma_x \sigma_y \sqrt{2\pi}} \exp\left(-\frac{(x - \mu_x - \beta)^2}{2\sigma_x^2} - \frac{(y - \mu_y)^2}{2\sigma_y^2}\right)$$

$$p(x, y | \text{down}) = \frac{1}{\sigma_x \sigma_y \sqrt{2\pi}} \exp\left(-\frac{(x - \mu_x + \beta)^2}{2\sigma_x^2} - \frac{(y - \mu_y)^2}{2\sigma_y^2}\right)$$

where:

$\mu_x$  is the expected horizontal position, which varies with the direction of movement.

$\mu_y$  is the expected vertical position (assumed to be unaffected by direction).

$\beta$  is the correction factor that adjusts the bias towards the right side, typically  $\beta > 0$  when ascending and  $\beta$  may be negative or smaller when descending.

We will now present a differentiated footfall probability map for ascending and descending the stairs.

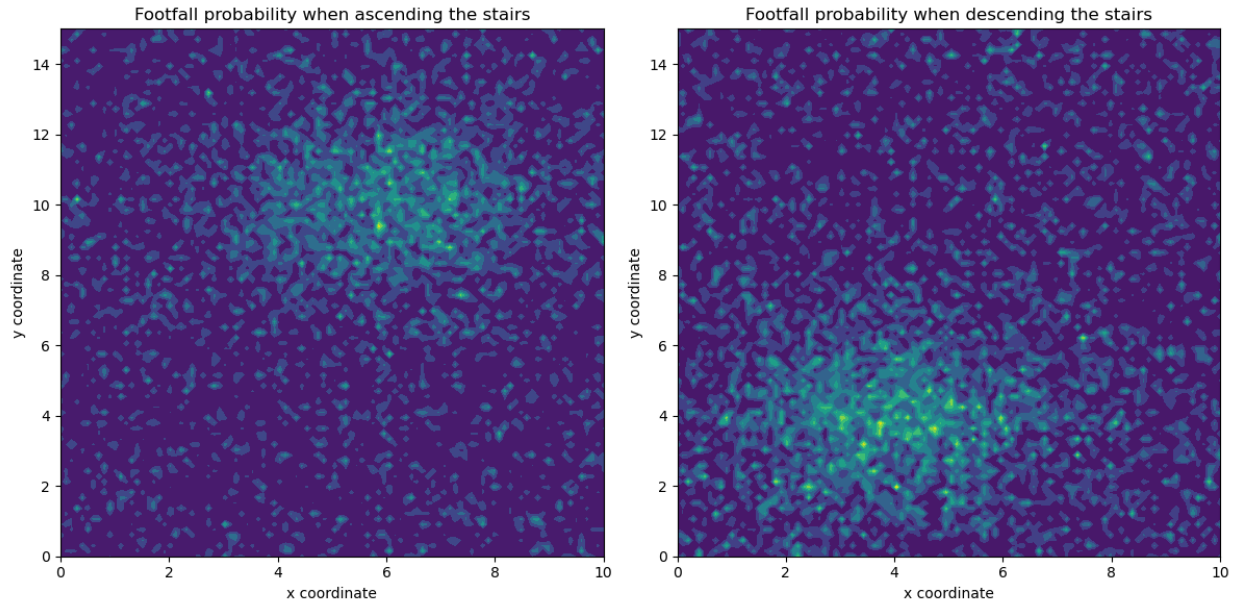


Figure 3: Footfall probability model for ascending and descending the stairs

Considering the number of contacts at each point after  $N$  people have passed, it is not difficult to derive that:

$$N(x, y) = N \cdot p(x, y)$$

### 4.2.3 Model Construction

Given that we can determine  $h(x,y)$ , we can then use the formula:

$$W(x,y) = \frac{\iint h(x,y) dx dy}{\mu}$$

We can also determine  $W(x,y)$ . The above method is used to determine the actual measured  $W(x,y)$ . We can also derive the  $W(x,y)$  corresponding to theoretical assumptions.

$$W(x,y) = \alpha FN \cdot p(x,y)$$

By comparing the measured  $W(x,y)$  with the  $W(x,y)$  corresponding to specific directions of movement (upward or downward), we can infer whether the staircase has been used more frequently for ascending or descending in the past.

## 4.3 Problem 3

### 4.3.1 Problem Analysis and Preliminary Ideas

It is evident that the footfall probability models corresponding to different numbers of people using the staircase simultaneously are distinct. Thus, we can continue to employ the approach from Problem 2. We can refine the footfall probability model. Starting from this point, we can make a judgment for Problem 3: How many people are using the staircase at the same time? (Is it two people or a single-file line?)

### 4.3.2 Model Preparation

We only consider the three footfall probability models for one person, two people, and multiple people using the staircase at the same time. According to the above, we have:

The footfall probability function for a single person :

$$p(x,y) = \frac{1}{\sigma_x \sigma_y \sqrt{2\pi}} \exp\left(-\frac{(x - \mu_x)^2}{2\sigma_x^2} - \frac{(y - \mu_y)^2}{2\sigma_y^2}\right)$$

The footfall probability function for two people :

$$p(x,y | \text{two people}) = \frac{1}{\sigma_x \sigma_y \sqrt{2\pi}} \left( \exp\left(-\frac{(x - \mu_x^1)^2}{2\sigma_x^2} - \frac{(y - \mu_y)^2}{2\sigma_y^2}\right) + \exp\left(-\frac{(x - \mu_x^2)^2}{2\sigma_x^2} - \frac{(y - \mu_y)^2}{2\sigma_y^2}\right) \right)$$

The probability density function for multiple people :

$$p(x,y | \text{multiple people}) = \frac{1}{\sigma_x \sigma_y \sqrt{2\pi}} \sum_{i=1}^n \exp\left(-\frac{(x - \mu_x^i)^2}{2\sigma_x^2} - \frac{(y - \mu_y)^2}{2\sigma_y^2}\right)$$

Next, we will present some examples with customized parameters:

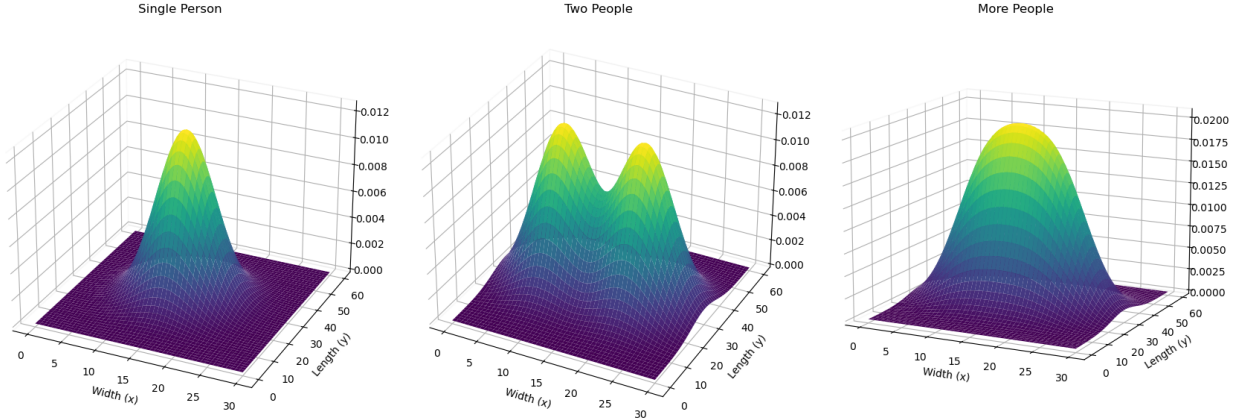


Figure 4: The probability density function of landing points under three different scenarios.

### 4.3.3 Model Construction

We can continue to employ the model and approach used in Question 2: The actual measured  $W(x,y)$ :

$$W(x,y) = \frac{\iint h(x,y) dx dy}{\mu}$$

The theoretically assumed  $W(x,y)$ :

$$W(x,y) = \alpha FN \cdot p(x,y)$$

Similarly, by comparing the actual model with several assumed models, we can determine which one is closer to reality and thus infer how many people are most likely using the staircase simultaneously.

## 4.4 Model Extension and Application

In this section, we will delve deeper into the model and make further improvements to address the additional questions 4 to 8.

### 4.4.1 Deeper Considerations

In-depth Analysis of Archard's Theory We consider Archard's formula:

$$V = K \cdot \frac{F \cdot S}{H}$$

Given that  $K/H$  is a constant and varies with different materials, we can derive the actual  $K/H$  value for the staircase from  $V$  using algorithms such as the least squares method. By comparing this value with the  $K/H$  of specific materials, archaeologists can determine the material used for the staircase. This provides a potential approach for addressing Question 7.

Application of Archie's Law To delve deeper into the wear within each step, it is evident that  $V$  and  $\mu$  cannot be accurately measured using conventional methods, as doing so would compromise the integrity of the staircase, which is undesirable. Therefore, we introduce Archie's law from the field of materials chemistry, which establishes a relationship between the material's resistivity and the volume of wear. Utilizing this theory can significantly enhance the functionality of our model.

#### 4.4.2 Model Preparation

Description of Archie's Law: Archie's law is a classic theory used to describe the relationship between the porosity and resistivity of rocks. Naturally, we can adapt it by adjusting the corresponding coefficients to examine the relationship between resistivity and porosity in other materials. The basic expression of Archie's law is as follows:

$$\frac{\rho}{\rho_0} = \phi^{-n}$$

<sup>[9]</sup> where

$\rho$  represents the actual resistivity

$\rho_0$  represents the reference resistivity, which is the initial resistivity of the original material.

$\phi$  denotes the porosity, which is the ratio of the volume of voids  $V_{\text{voids}}$  to the total volume of the material  $V_{\text{total}}$ . That is to say:  $\phi = \frac{V_{\text{voids}}}{V_{\text{total}}}$ .

$n$  is Exponent in Archie's law.

Since the total volume of the material and its resistivity are easily measurable, we can determine the volume of voids, which will reflect the wear value  $W$  deep within the staircase.

There are so many ways to get the resistivity.

Using the Wenner formula<sup>[4]</sup>, the current is applied to the outer electrodes, and the potential difference between the inner electrodes is measured. To simulate the current and potential distributions, we need the governing equations. The current density vector is related to the electric field vector, and their ratio defines the material's electrical conductivity according to Ohm's law, with resistivity being its reciprocal. The electric field vector  $E$  is the gradient of the electric potential. Since charge does not accumulate, the law of conservation of current holds, meaning the current flux entering the conductive material equals the flux leaving. Assuming homogeneity and isotropy of the material, the resistivity gradient term in equation (5) can be eliminated, and the governing equation for charge transfer becomes Laplace's equation. Given that current only flows inside the material, all surfaces can be treated as insulated.

Thus, the following formula <sup>[4]</sup> can be derived:

$$\mathbf{J} = \sigma \mathbf{E} = \rho^{-1} \mathbf{E} \quad (1)$$

$$\mathbf{E} = -\nabla V \quad (2)$$

$$\mathbf{J} = -\rho^{-1} \nabla V \quad (3)$$

$$\nabla \cdot \mathbf{J} = 0 \quad (4)$$

$$\nabla(\rho^{-1}) \cdot \nabla V + \rho^{-1} \nabla^2 V = 0 \quad (5)$$

$$\rho^{-1} \nabla^2 V = 0 \quad (6)$$

$$\mathbf{n} \cdot \mathbf{J} = 0 \quad (7)$$

$$\rho_s = 2\pi s \frac{U_{23}}{I} \quad (I = 1\text{mA}) \quad \text{where } \rho_s \text{ is the simulated - value of the resistivity.} \quad (8)$$

$$\rho = k\rho_s, \text{ where } \rho \text{ is the reference value of the concrete resistivity, and } \rho = 130\Omega \cdot m. \quad (9)$$

Above model can be used to do some analysis for resistivity about the different materials. And also for us. we take the method and know some materials for us using. So, for next diagram is what we get what we want to.

material	concrete	calcium carbonate	wood	soil	breccia	Wutong
resistivity( $\Omega \cdot m$ )	1	701	200 (average)	102	299	430

#### 4.4.3 Enhancement of the Wear Value Model

Our wear value model

$$W = \frac{V}{\mu}$$

can be enhanced to:

$$W = \begin{cases} \frac{V}{\mu} & \text{if considering the surface layer} \\ rV_{\text{total}} \cdot \phi & \text{if considering the deeper layer} \end{cases}$$

Here,  $r$  is a proportionality coefficient used to relate  $W$  to the void volume in the deeper layers.

#### 4.4.4 Answers to Additional Problems

Next, we will provide insights into these questions by drawing from both Archard's theory and Archie's law.

Problem 7:

The approach to Question 7 is derived from Archard's theory. Following the aforementioned methodology, we simply need to look up the constant  $K/H$  for specific materials and utilize the formula:

$$\frac{K}{H} = \frac{V}{F \cdot S}$$

The volume  $V$  can be measured on-site using instruments. Meanwhile, based on our assumptions,  $F$  and  $S$  can also be determined through research.



The average human mass  $m$  and the gravitational acceleration  $g$  at the equator can be used to calculate  $F$ :  $F=mg$ . And  $S$  represents the average sole area of shoes worldwide. By calculating the  $K/H$  value for the material composing the staircase and comparing it with known values, we can identify the material used for the staircase. For all above, we can take the  $K/H$  to be the constant for Wear coefficient. Then, we get the static for the different condition of the different material :

Table 2: Wear coefficient under dry - friction conditions<sup>[5]</sup>

Load(N)	Wear coefficient $mm^3/(N \cdot mm)$
50	$3.628 \times 10^{-7}$
60	$3.379 \times 10^{-7}$
70	$4.236 \times 10^{-7}$
80	$4.37 \times 10^{-7}$
90	$4.22 \times 10^{-7}$

Table 3: Wear - coefficient values under water - lubrication conditions<sup>[5]</sup>

Load(N)	Wear coefficient $mm^3/(N \cdot mm)$
50	$3.55 \times 10^{-7}$
60	$3.65 \times 10^{-7}$
70	$2.83 \times 10^{-7}$
80	$2.75 \times 10^{-7}$
90	$1.056 \times 10^{-7}$

And below all of the material we get, the center of the ideal is the concrete, so, we get the concrete number<sup>[2]</sup> of the analysis is  $1.2610 \hat{(-12)}Pa$ , which is the staircase state which we want.

#### Answer to Problem 5 and 6 – One Solution to Multiple Problems

##### Problem 5:

The relationship between the service life of the staircase and its wear value is intrinsically linked. However, it is challenging to determine the service life of the staircase based solely on the surface wear value, as we cannot exclude the impact of renovations, deliberate damage, wars, and large-scale festivals on the surface wear value. Therefore, we must consider the wear conditions in the deeper layers of the staircase to accurately assess its service life. Thus, we naturally turn to the deep-layer wear value model:

$$W = rV_{\text{total}} \cdot \phi$$

For this, we are aware that the resistivity in Archie's law is measurable, and that resistivity is related to the service time of the material and is proportional to the wear value. Thus, based on publicly available data, we can derive a function:

$$W(t, m) = r_{t,m} \cdot V_{\text{total}} \cdot \left( \frac{\rho_0}{\rho_{t,m}} \right)^n$$

Here,  $t$  represents the service time of the material, and  $m$  is the index corresponding to different materials. Clearly, different values of  $m$  and  $t$  will result in different values of resistivity  $\rho$  and different coefficients  $r$ .

The following will present the function plots of  $W(t, m)$  with respect to different values of  $m$  and  $t$ .

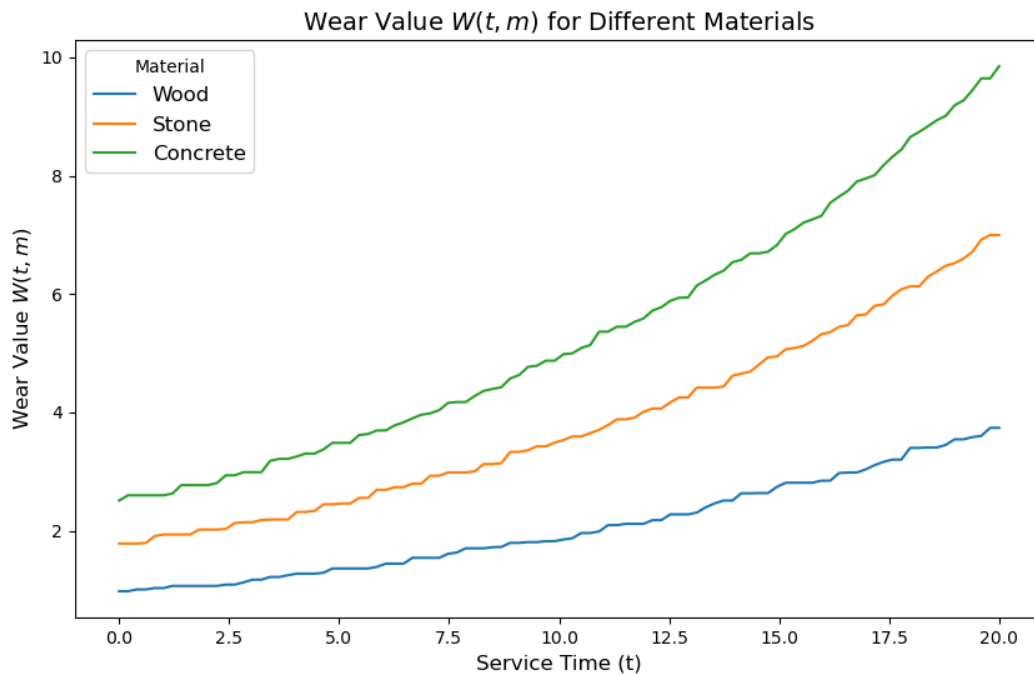


Figure 5: The sample plot of  $W(t, m)$ .

We can measure the resistivity of the material at the base of the staircase and determine the material index  $m$  (which can evidently be obtained from the answer to Problem 7). Of course, the data for  $\rho_0$  is publicly available. Now, by substituting the resistivity and  $m$  into the wear value model, we can derive  $t$ , that is, the service life of the staircase. Thus, we have provided a corresponding answer to Problem 5.

The reason why the base layer is chosen is quite straightforward: The service life of the staircase refers to the time elapsed from its initial construction to the present. During this period, the staircase must have undergone multiple renovations and damages. Except for the base layer, the materials above it may correspond to the  $n$ -th repair or damage event. The time associated with these materials is not the service life but rather the time of the specific repair or damage event.

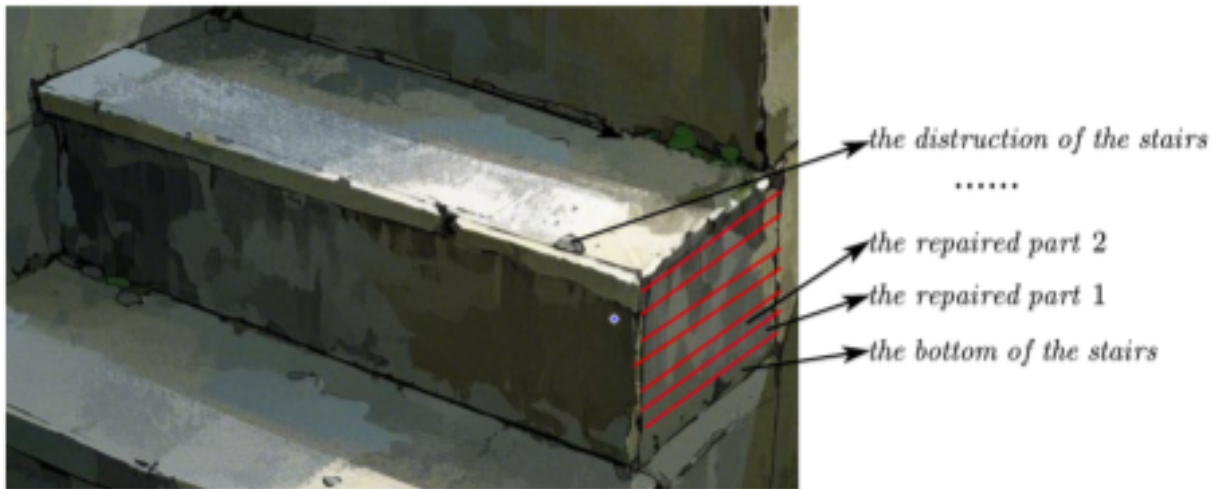


Figure 6: Staircase Section Diagram

## Problem 6:

Based on this analysis, we gain valuable insights for addressing Problem 6. The resistivity corresponding to each phase (each repair or damage event) is different, which will help us determine how many times the staircase has been repaired or damaged. Since repairs or damages can be considered as short-term changes to the surface wear value at the time (which means we do not need to account for the effects of oxidation or other wear-related impacts), the current surface layer of the staircase has now moved to a corresponding deeper position due to these events. Thus, we conclude that different depths correspond to materials built or damaged in different eras, which in turn correspond to different wear values  $W$ . Clearly, there is a corresponding  $W(d)$  model:

$$W(d, m) = r_{d,m} \cdot V_{\text{total}} \cdot \left( \frac{\rho_0}{\rho_{d,m}} \right)^n$$

Evidently, the  $W$  values at each stage are discontinuous, while the  $W$  values within a period that has not undergone repair or destruction are continuous. Therefore, we only need to identify the breakpoints in the  $W(d, m)$  function to determine how many times the staircase has been repaired and damaged. Moreover, we must consider that a repair will cause  $W$  to "instantly" decrease, and a destruction will cause  $W$  to "instantly" increase. By considering only the breakpoints where  $W$  decreases, we can determine the total number of repairs and renovations the staircase has undergone.

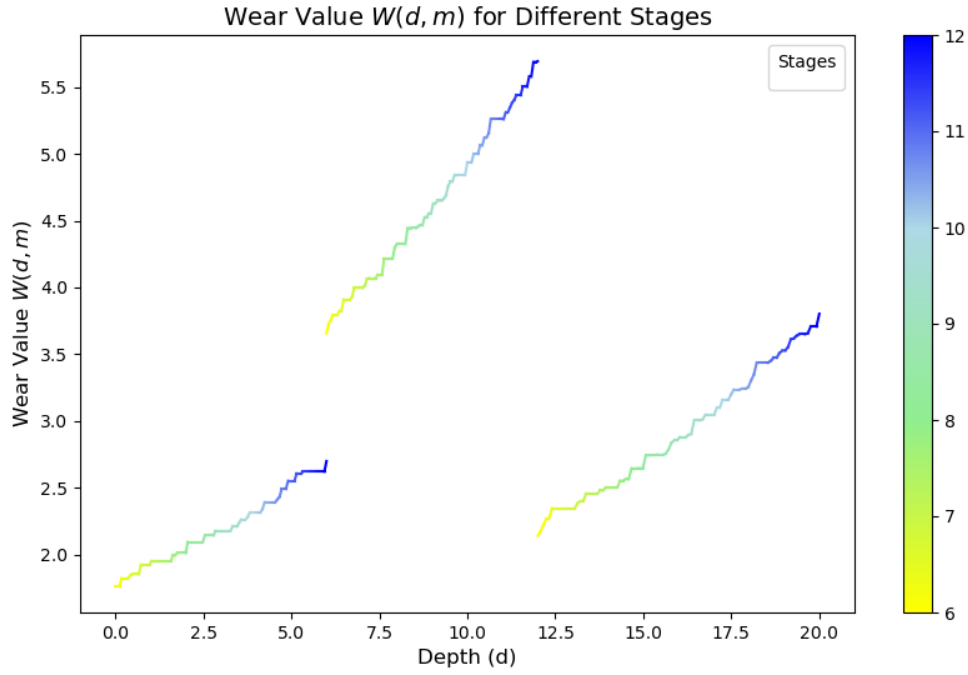


Figure 7: The sample plot of  $W(d, m)$ .

As shown in the figure, the unit of  $d$  is cm. It is evident that the staircase has undergone human-induced damage and subsequent repairs.

#### Answer to Problem 8 – One Problem with Multiple Solutions

We aim to determine the usage pattern of the stairs: whether it involves a large crowd using the stairs in a short period or a few individuals using the stairs over an extended period. To address this issue, we can approach it from two directions using the loss value model:

**Method 1:**  $W(x, y) = \alpha FN \cdot p(x, y)$

We assume that the probability models  $p(x, y)$  corresponding to the two situations are different, and we can apply the ideas from Problems 2 and 3 as follows:

We assume that for a large crowd using the stairs in a short time, the corresponding probability model  $p_1(x, y)$  is relatively uniform with no significant peak. In contrast, for a few individuals using the stairs over a longer time, the corresponding probability model  $p_2(x, y)$  should be non-uniform with a prominent peak near the center.

The corresponding probability models  $p(x, y)$  are given by:

$$p_1(x, y) = \frac{1}{\sigma_x \sigma_y \sqrt{2\pi}} \sum_{i=1}^n \exp\left(-\frac{(x - \mu_x^i)^2}{2\sigma_x^2} - \frac{(y - \mu_y)^2}{2\sigma_y^2}\right)$$

$$p_2(x, y) = \frac{1}{\sigma_x \sigma_y \sqrt{2\pi}} \exp\left(-\frac{(x - \mu_x)^2}{2\sigma_x^2} - \frac{(y - \mu_y)^2}{2\sigma_y^2}\right)$$

Archaeologists can measure the actual loss value model  $W(x, y)$  and use a single loss formula to derive  $p(x, y)$ . By comparing this with the hypothesized probability models for different situations, we can make a determination regarding the question in Problem 8.

**Method 2:**  $W(t, m) = r_{t,m} \cdot V_{\text{total}} \cdot \left(\frac{\rho_{t,m}}{\rho_0}\right)^n$

We consider the similarities and differences between the two situations described above: the commonality between the two is that both will cause significant wear on the surface. The difference is that the impact of foot traffic on the deeper layers of the stairs' wear is negligible in both cases, and the deeper wear value is more influenced by  $t$ . In one case, the time is short, and the wear value changes less; in the other, the time is long, and the wear value changes more significantly.

Therefore, when measuring the actual resistivity of the stairs, we can use the above research to distinguish between the two situations with  $W(t, m)$ : when the time  $t$  is long and the wear value changes significantly, it corresponds to the case of a few individuals using the stairs over a long period; when the time is short and the wear value changes minimally, it corresponds to the case of a large crowd using the stairs over a short period.

Here is a case to better help with understanding:

It is not difficult to see that the surface wear values of the two are not significantly different, making it hard to distinguish between the two. However, the internal wear value of Stairs 2 is smaller, and it has been in use for a shorter period; while the internal wear value of Stairs 1 is larger, and it has been used for a longer period. Therefore, it is easy to conclude that Stairs 1 has experienced wear due to long-term use by a few people, while Stairs 2 has been worn due to short-term but heavy use by many people.

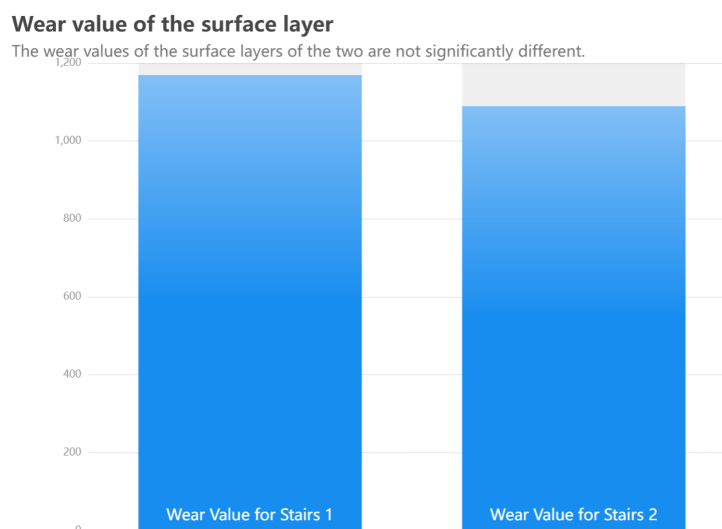


Figure 8: Internal Wear of Two Staircases with Time

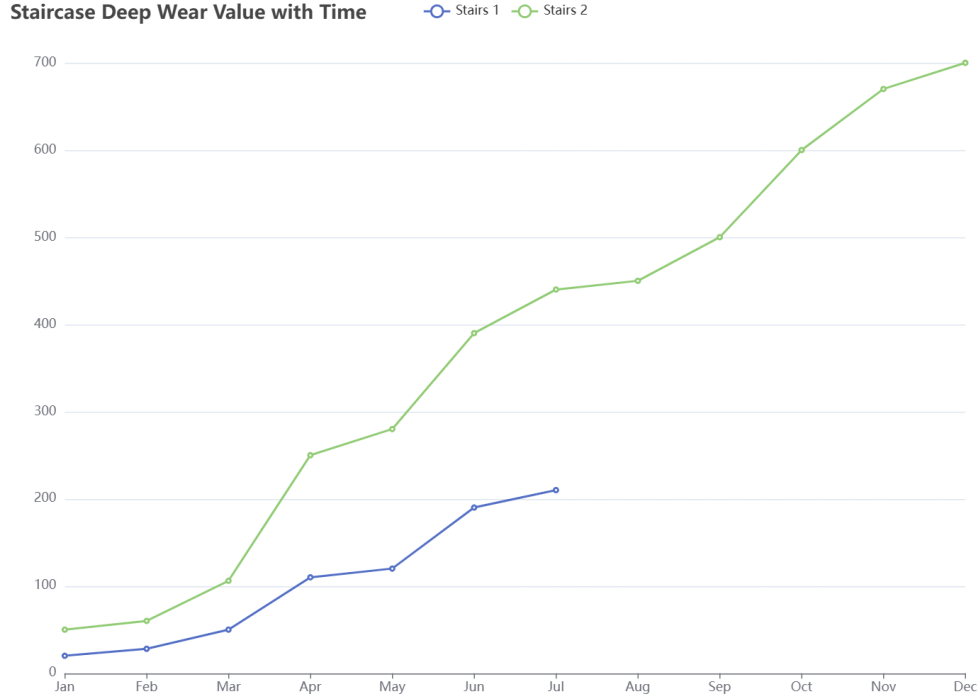


Figure 9: Internal Wear of Two Staircases with Time

#### Problem 4:

To answer this question, we need to determine what constitutes the information available. We assume that it refers to the factual information  $I_{\text{true}}$ , which includes the historical events such as wars, major festivals, repairs and damages, and material composition of the stairs, as inferred by archaeologists from historical records and professional knowledge.

To determine whether this information aligns with the information inferred from our wear value model, we simply need to calculate and derive the model information  $I_{\text{model}}$  based on the solutions to Problems 1, 2, 3, 5, 6, 7, and 8, along with the relevant models and approaches, and check whether there is an inclusion relationship between the two:

$$I_{\text{true}} \subseteq I_{\text{model}}$$

If the above relationship holds, then we can conclude: The wear is consistent with the information available.

## 5 Sensitivity Analysis

### 5.1 The Sensitivity Analysis of the Wear Value Model

The equation we derived for the wear value  $W$  is:

$$W = rV_{\text{total}} \cdot \left( \frac{\rho_0}{\rho} \right)^{\frac{1}{n}}$$

Now, we can compute the derivative of  $W$  with respect to both  $V_{\text{total}}$  and  $\frac{\rho}{\rho_0}$  to understand how  $W$  changes as both variables vary:

$$\frac{\partial W}{\partial V_{\text{total}}} = r \left( \frac{\rho_0}{\rho} \right)^{\frac{1}{n}}$$

$$\frac{\partial W}{\partial \left( \frac{\rho}{\rho_0} \right)} = -r V_{\text{total}} \cdot \frac{1}{n} \cdot \left( \frac{\rho_0}{\rho} \right)^{\frac{1}{n}} \cdot \frac{\rho_0}{\rho}$$

The first derivative indicates that  $W$  increases as  $V_{\text{total}}$  increases. The second derivative indicates that  $W$  decreases as  $\frac{\rho}{\rho_0}$  increases. The rate of change depends on the exponent  $n$ , the reference resistivity  $\rho_0$ , and the total volume  $V_{\text{total}}$ .

### 5.1.1 Sensitivity Factors

- If  $V_{\text{total}}$  is large,  $W$  will be more sensitive to changes in  $V_{\text{total}}$ , as the total volume increases the wear value directly.
- The combination of both  $V_{\text{total}}$  and  $\frac{\rho}{\rho_0}$  leads to a more complex sensitivity relationship, where both variables affect  $W$  in different ways.

To visualize this relationship, we can plot  $W$  as a function of both  $V_{\text{total}}$  and  $\frac{\rho}{\rho_0}$  for different values of  $n$ .

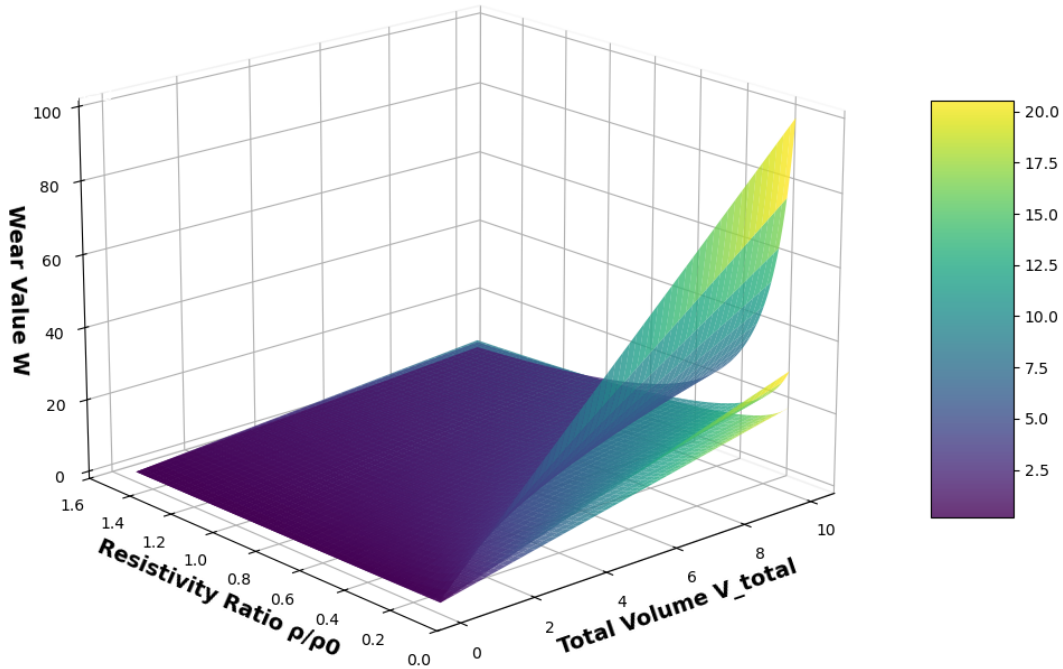


Figure 10: Sensitivity of  $W$  to  $\frac{\rho}{\rho_0}$  and  $V_{\text{total}}$  for Different  $n$  Values

### 5.1.2 Sensitivity of Wear Value $W$ to Volume $V_{\text{total}}$ and Resistivity Ratio $\frac{\rho}{\rho_0}$

The plot above shows the sensitivity of the wear value  $W$  to both the volume  $V_{\text{total}}$  and the resistivity ratio  $\frac{\rho}{\rho_0}$  for different values of the exponent  $n$  in Archie's law.

From the graph, you can observe:

- As  $V_{\text{total}}$  increases, the wear value  $W$  increases directly.
- As  $\frac{\rho}{\rho_0}$  increases (i.e., as the resistivity  $\rho$  increases), the wear value  $W$  decreases.
- The sensitivity of  $W$  increases with larger values of  $n$ , meaning that  $W$  is more responsive to changes in both  $V_{\text{total}}$  and  $\frac{\rho}{\rho_0}$  when  $n$  is large.

This sensitivity analysis provides insights into how changes in both volume and resistivity affect wear, particularly in materials with varying porosities. The combined effect of these two factors is crucial for understanding wear behavior under different conditions.

## 6 Model Evaluation and Future Work

### 6.1 Model Advantages

**Interdisciplinary Integration:** The model integrates theories from physics (friction, Archard's theory), mathematics (probability distributions, equations), materials science (Archie's law), and archaeology, demonstrating the power of interdisciplinary collaboration. This enhances both the theoretical depth and practical applicability of the model.

**High Adaptability:** The model adapts to different scenarios by incorporating various probability models and adjusting parameters. For example, the walking direction preference can be modeled by modifying the correction factor  $\beta$ , while staircase materials can be identified by comparing K/H values. This flexibility enables the model to be applied to a wide range of archaeological problems.

### 6.2 Model Innovations

**Pedestrian Behavior Consideration:** The model accounts for footfall distribution using a probability model, allowing precise wear estimation across different staircase areas. This improves the model's spatial accuracy in studying wear patterns.

**Comprehensive Construction:** Our model considers both surface and internal wear, unifying them to enhance its flexibility, generality, and functionality. This provides more detailed historical insights for archaeologists, helping to reveal the staircase's usage and material characteristics, and is significant for archaeological preservation.

**Multiple Solutions for One Problem:** The model can solve multiple problems with one approach, as demonstrated in Problems 2 and 3, where the footfall model, single-use wear formula, and wear value model are combined. It also offers multiple solutions for a single problem, as seen in Problem 8, with different approaches based on surface and deep-layer wear models.



## 6.3 Limitations

**Simplified Assumptions:** Assumptions such as uniform, linearly additive wear and footfall positions described by probability functions may not align with real-world scenarios, potentially affecting model accuracy.

**Model Complexity:** While the model is versatile, its complexity can make data processing and computation challenging in practical applications.

**Data Dependency:** The model relies on high-quality data, such as footfall probability, material properties, and resistivity measurements. In practice, it may be difficult to obtain detailed and accurate data, impacting the model's accuracy and reliability.

## 6.4 Future Work

We plan to enhance the model by either incorporating more factors for precision or simplifying it by focusing on key elements:

1. **Model Precision Enhancement:**

We could introduce nonlinear factors to improve accuracy. Currently, the model assumes linear additive wear, but real-world factors may require a nonlinear approach.

2. **Model Simplification:**

The model can be simplified by reducing mathematical complexity, refining the footfall probability model, and minimizing adjustable parameters.

## References

- [1] L. Prandtl, Fluid motions with very small friction, Proceedings of the 3rd International Mathematical Congress, Heidelberg: H. Schlichting, 1904, 484-491.
- [2] Hu, Duoxiang. "Research on Improving the Wear - resistance of the Mixing Drum of Concrete Mixer Trucks".Xiangtan university,2022,TU528.53.
- [3] Li, Bohan. "Measurement of Concrete Surface Wear Based on 3D Scanning Technology".China Water Power and Electrification,2022,1673-8241 ( 2022) 03-004-06
- [4] Xia Jin, Zheng Yuhang, Wu Renjie, Chen Keyu, Zhao Yuxi, Jin Weiliang. Research on the Influence of Size Effect on the Measurement of Concrete Resistivity by Wenner Technology [J/OL]. Journal of Building Structures. <https://link.cnki.net/urlid/11.1931.tu.20240719.1534.002>
- [5] Li, Shuang. "Research on the Wear Performance of Silicon Nitride Full - Ceramic Ball Bearings\_Chapter 4\_Determination of the Wear Coefficient of Silicon Nitride Materials".Shenyang Jianzhu University,2022,TH133.3;TH117.1
- [6] "Research on the Dry Friction and Wear Characteristics of 1D Composites Based on the Archard Wear Model".Acta Materiae Compositae Sinica ,2024,1000-3851(2024)11-6206-09 10.13801/j.cnki.fhclxb.20240318.005

- [7] Jiao, Yu. "Modeling of Human Kinetics Based on Habits". SCIENCE CHINA PRESS, 2010.
- [8] Wang, Xuan, et al. "Simulation analysis of cyclic wear of reciprocating mechanism based on improved Archard model." Journal of Naval University of Engineering 36.6 (2024): 46 - 52.
- [9] Pejman Azarsa and Rishi Gupta. "Electrical Resistivity of Concrete for Durability Evaluation: A Review". Advances in Materials Science and Engineering, 2017, Article ID 8453095, 30 pages. DOI: 10.1155/2017/8453095.
- [10] Wang, Sicheng. "Research on the Mechanical Characteristics of Contact between Rough Surfaces of Different - Formed Concrete". Master's Thesis, Chongqing University.

## Appendices

---

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

r = 1.0
rho_0 = 1.0
V_values = np.linspace(0.1, 10, 200)
rho_values = np.linspace(0.1, 1.5, 200)
n_values = [1, 2, 3]
V_grid, rho_grid = np.meshgrid(V_values, rho_values)

fig = plt.figure(figsize=(12, 8))
ax = fig.add_subplot(111, projection='3d')

for n in n_values:
    phi_grid = (rho_0 / rho_grid) ** (1 / n)
    W_grid = r * V_grid * phi_grid
    surf = ax.plot_surface(V_grid, rho_grid, W_grid, cmap='viridis', alpha
        ↳ =0.8, edgecolor='none')
    ax.text2D(0.05, 0.9 - 0.05 * n, f'n={n}', transform=ax.transAxes, fontsize
        ↳ =12, color='black')

ax.set_xlabel('Total Volume V_total', fontsize=14, fontweight='bold')
ax.set_ylabel('Resistivity Ratio / 0 ', fontsize=14, fontweight='bold')
ax.set_zlabel('Wear Value W', fontsize=14, fontweight='bold')
ax.set_title('3D Sensitivity of Wear Value (W) to V_total and Resistivity
Ratio ( / _0 )', fontsize=16, fontweight='bold')
fig.colorbar(surf, ax=ax, shrink=0.5, aspect=5)
fig.patch.set_facecolor('white')

ax.set_facecolor('white')
ax.xaxis.pane.fill = False
ax.yaxis.pane.fill = False
ax.zaxis.pane.fill = False
ax.view_init(elev=30, azimuth=45)
plt.tight_layout()
plt.show()
```

---

## Report on Use of AI

### 1. OpenAI ChatGPT(Jan 26, 2025 version, ChatGPT-4)

Query1:For this formula, what aspects can I perform sensitivity analysis on?

$$W = rV_{\text{total}} \cdot \left(\frac{\rho_0}{\rho}\right)^{\frac{1}{n}}$$

Output:For the formula

$$W = rV_{\text{total}} \cdot \left(\frac{\rho_0}{\rho}\right)^{\frac{1}{n}}$$

you can perform sensitivity analysis on several aspects:

- **Total Volume** ( $V_{\text{total}}$ ): Sensitivity analysis can be done to evaluate how changes in the total volume of the material affect the wear value. As the volume increases, the wear value will increase proportionally.
- **Resistivity Ratio** ( $\frac{\rho_0}{\rho}$ ): Sensitivity analysis can be performed to examine how variations in the resistivity ratio (the ratio of reference resistivity to current resistivity) impact the wear value. If the resistivity ratio increases, the wear value decreases.
- **Exponent** ( $n$ ): The sensitivity to changes in the exponent  $n$  in the formula can be analyzed. A larger  $n$  increases the sensitivity of the wear value to changes in resistivity. If  $n$  is large, even small changes in resistivity can cause significant changes in the wear value.
- **Proportional Coefficient** ( $r$ ): Sensitivity analysis can also focus on the proportional coefficient  $r$ , which relates the wear value to other factors in the model. Changes in  $r$  may significantly alter the calculated wear value.

By analyzing these factors, you can determine how each contributes to variations in the wear value, providing deeper insights into the material's wear behavior under different conditions.

### 2. OpenAI ChatGPT(Jan 27, 2025 version, ChatGPT-4)

Query1: How can I neatly place Python code in the appendix of a LaTeX document?

Output: To neatly place Python code in the appendix of a LaTeX document, you can use the `listings` package, which is designed for displaying code in a well-organized and readable format. Here's how you can do it:

1. Include the `listings` package in your LaTeX preamble:

---

```
\usepackage{listings}
\usepackage{xcolor} % Optional, for adding color
```

---

2. Set up the `listings` environment to specify the style of the code (e.g., language, color):

---

```
\lstset{
  language=Python,
  basicstyle=\ttfamily\footnotesize,
  numbers=left,
  numberstyle=\tiny\color{gray},
  stepnumber=1,
  numbersep=5pt,
  backgroundcolor=\color{white},
  showspace=false,
  showstringspaces=false,
  showtabs=false,
  frame=single,
  rulecolor=\color{black},
  captionpos=b,
  breaklines=true,
  breakatwhitespace=false,
  escapeinside={\%*}{*}
}
```

---

3. Then, in your appendix, use the `lstlisting` environment to display your Python code:

---

```
\appendix
\section{Python Code}

\begin{lstlisting}[caption={Python Code Example}, label={lst:python_code}]
import numpy as np
import matplotlib.pyplot as plt

def plot_graph(data):
    plt.plot(data)
    plt.show()

data = np.random.randn(100)
plot_graph(data)
```

---

### 3. OpenAI ChatGPT(Jan 26, 2025 version, ChatGPT-4)

Query1: Where is the best place to find data about stair wear?

Output:To find data about stair wear, the following sources would be helpful:

**1. Scientific Journals and Research Papers:** You can search for published papers on tribology, material science, or structural engineering. Journals such as "Wear," "Tribology International," or "Journal of Materials Science" often publish studies related to material wear, including staircases.

**2. Government and Academic Institutions:** Some government agencies or universities might have data on public infrastructure and building materials. Websites like Google Scholar, ResearchGate, or university libraries can be useful for finding relevant studies.

**3. Construction and Architecture Databases:** Databases such as ASTM (American Society for Testing and Materials), ISO (International Organization for Standardization), and other

construction standards organizations may have research on the wear characteristics of building materials, including stairs.

**4. Engineering Companies and Consulting Firms:** Some companies in the field of civil engineering and material testing may have proprietary data on the wear of various building materials, including stairs. These companies often conduct studies for commercial or industrial purposes.

**5. Historical Data and Archaeological Studies:** For historical staircases, studies related to archaeology, especially those focusing on the preservation and wear of ancient buildings, could provide valuable data. You can look for publications or research related to the physical condition of historical buildings.