

High-precision bootstrap of multi-matrix quantum mechanics

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Abstract

We consider matrix quantum mechanics with multiple bosonic matrices, including those obtained from dimensional reduction of Yang-Mills theories. Using the matrix bootstrap, we study simple observables like $\langle \text{tr } X^2 \rangle$ in the confining phase of the theory in the infinite N limit. By leveraging the symmetries of these models and using non-linear relaxation, we consider constraints up to level 14, e.g., constraints from traces of words of length ≤ 14 . Our results are more precise than large N , continuum extrapolations of lattice Monte Carlo simulations, including an estimate of certain simple observables up to 8 significant digits.

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1 Introduction

Solving large N gauge theories has been a long-held dream of modern physics. An important tool in this pursuit has been Euclidean Monte Carlo lattice computations. However, this approach requires

extrapolation to large N where computation becomes increasingly expensive, despite the fact that physics is simplifying in the large N limit.

Another approach that has comparatively been less developed is the matrix bootstrap [1–4] which works directly in the infinite N , 't Hooft limit by imposing large N factorization¹.

In this work, we continue developing this approach by performing a high-precision bootstrap analysis of large N bosonic matrix quantum mechanics. These include models arising from dimensional reduction of Yang-Mills theories in $D + 1$ dimensions. Such 0+1 dimensional models share some qualitative aspects with their higher-dimensional counterparts on compact spaces [6], including a similar phase diagram. More precisely, it is believed that at large N these quantum mechanical theories exhibit two continuous phase transitions. For low temperatures $T < T_{c,1}$, the theory is in a confining phase characterized by a $\sim O(N^0)$ free energy. At the first critical temperature $T_{c,1}$, the theory undergoes a second-order Hagedorn transition into a deconfined phase, where the free energy scales as $O(N^2)$. A subsequent third-order Gross-Witten-Wadia transition [7,8] occurs at a temperature $T_{c,2} > T_{c,1}$. Both transitions occur at temperatures of order one in units of the 't Hooft coupling [9–11].

A key improvement in our work is the use of non-linear relaxation techniques to deal with the non-linear system of equations that arise after imposing large N factorization [4]. So far this method has been implemented in the path integral context [4,12]; in this work we implement it in the Hamiltonian formalism. We also apply the ground state positivity condition (7), which boosts the precision of the bootstrap dramatically.

This work may also be viewed as a natural continuation of [13]. The bosonic models considered here are somewhat simpler to bootstrap (in particular, the group theory is somewhat easier) than the supersymmetric BFSS model that we focused on in [13]. The computational methods developed here should inform future efforts, including our planned continuation into the high-precision bootstrap of the BFSS matrix quantum mechanics [14].

Our motivation to push the precision frontier is not merely precision for precision's sake. In the supersymmetric BFSS case, a high-precision measurement of the thermodynamics and/or correlation functions would allow one to “experimentally” measure higher derivative corrections to Type IIA supergravity [15–24]. These corrections are currently unknown using worldsheet methods. So, the high-precision bootstrap could teach us about quantum gravity. In a similar spirit, lattice Monte Carlo studies for large N Yang-Mills theory (see [25] for a review and e.g. [26,27] for recent progress) can teach us about the effective string. We view our work as a starting point for a bootstrap study for higher dimensional Yang-Mills theory in the Hamiltonian lattice gauge theory [28] formalism².

This paper is organized as follows: in section 2 we explain the bootstrap algorithm; in section 3 we present our results and compare to Monte Carlo [10,22,31,32] as well as the large D expansion [11,33] and the numerical master field optimization method [34]. We discuss the implications of our findings and give some future directions in 4. The main text is complemented by several appendices. In Appendices A and B, we provide additional details on the numerical implementation and relaxation procedure. Appendix C describes a particular method for performing the irreducible decomposition of operators. Appendix D summarizes the details of the large D calculation. Finally, in Appendix F, we present a pedagogical example illustrating our bootstrap approach.

¹Alternatively, at finite N one may impose trace relations [5].

²See [1,5,12,29,30] for some bootstrap approaches to the Euclidean path integral approach to pure Yang-Mills theory.

2 Bootstrap ingredients

2.1 The models

We consider systems that consist of D bosonic matrices, satisfying the commutation relations $[(X_I)_{ij}, (P_J)_{kl}] = i\delta_{il}\delta_{jk}\delta_{IJ}$. Here I, J run from 1 to D . These matrices are traceless and Hermitian and transform in the adjoint of $SU(N)$. The Hamiltonian and $SU(N)$ gauge generators are

$$H = \frac{1}{2} \sum_{I=1}^D (\text{Tr } P_I P_I + M^2 \text{Tr } X_I X_I) - \frac{g_{\text{YM}}^2}{4} \sum_{I,J=1}^D \text{Tr}[X_I, X_J]^2 \quad (1)$$

$$C = \sum_{I=1}^D (-i[X_I, P_I] - N\mathbf{1}) \quad (2)$$

The ground state of this model is gauge invariant $C|\Omega\rangle = 0$. The model has an $O(D)$ global symmetry, under which X_I and P_I transform in the fundamental. The case with $M^2 = 0$ is of particular interest, as it arises from the dimensional reduction of pure Yang-Mills theory in $D+1$ dimensions. (For certain values of D , one can also view the massless theory as arising from reducing supersymmetric Yang-Mills theories on tori T^d with anti-periodic boundary conditions for the fermions, see e.g. [9].) For the massless model, we may set $g_{\text{YM}}^2 = 1$ without loss of generality by performing a canonical transformation and rescaling the Hamiltonian; for the massive model there is a dimensionless parameter $\lambda_{\text{eff}} = g_{\text{YM}}^2 N/M^3$.

For the $D = 2$ version, it is convenient to view $SO(2)$ as $U(1)$ and work with complex operators

$$Z = \frac{1}{\sqrt{2}}(X_1 + iX_2), \quad \bar{Z} = \frac{1}{\sqrt{2}}(X_1 - iX_2), \quad (3)$$

and similarly $P = \frac{1}{\sqrt{2}}(P_1 + iP_2)$, $\bar{P} = \frac{1}{\sqrt{2}}(P_1 - iP_2)$. Then the $U(1)$ charge of an operator is simply the difference between (number of Z 's and P 's) – (number of \bar{Z} 's and \bar{P} 's). The additional $O(2)$ reflection symmetry sends $Z \leftrightarrow \bar{Z}$. With this notation, $H = \text{tr } P\bar{P} + M^2 \text{tr } Z\bar{Z} + \frac{1}{2} \text{tr}[Z, \bar{Z}]^2$ and $C = -i[Z, \bar{P}] - i[\bar{Z}, P] - 2$.

Due to large- N factorization, we only need to consider single-trace operators. The operators are “words” composed of “indexed letters” X_I and P_J . Since $\text{Tr } X_I P_I = \frac{D}{2}iN^2$, we perform the following shift³ so that expectation values of single-trace operators are $O(1)$ in the large N limit:

$$\text{Tr} \rightarrow \text{tr} = \frac{1}{N} \text{Tr}, \quad X_I \rightarrow \frac{1}{\sqrt{N}}X_I, \quad P_I \rightarrow \frac{1}{\sqrt{N}}P_I, \quad (4)$$

For the $O(2)$ model, we forgo the indexed letters, and consider words made of the letters $\{Z, \bar{Z}, P, \bar{P}\}$. For practical implementation, it is convenient to define $\Pi = -iP$ so that all correlators are real.

2.2 Constraints

The dynamical constraints include the equations of motion for a stationary state:

$$\langle [H, \mathcal{O}] \rangle = 0. \quad \forall \mathcal{O} \text{ single-trace operator.} \quad (5)$$

We impose positivity of the inner product:

$$\mathcal{M}_{ij} = \langle \text{tr } \bar{\mathcal{O}}_i \mathcal{O}_j \rangle, \quad \mathcal{M} \succeq 0. \quad (6)$$

³This is equivalent to introducing a formal $\hbar = 1/N$ so that $[X_{ij}, P_{kl}] \propto i\hbar$.

level	$D = 2$		$D = 9$	
	free variables	all variables	free variables	all variables
4	3	14	3	7
6	8	94	10	52
8	22	614	43	487
10	77	4086	289	5737
12	326	27830	2859	81442
14	1569	192374	—	1348057

Table 1: The number of free variables in the semi-definite programming problem (after quotienting by the kinematic and dynamical constraints), for the $D = 2$ and $D = 9$ models.

Furthermore, we impose ground state positivity:

$$\mathcal{N}_{ij} = \langle \text{tr } \bar{\mathcal{O}}_i [H, \mathcal{O}_j] \rangle, \quad \mathcal{N} \succeq 0 \quad (7)$$

This inequality immediately follows from demanding that the state $\mathcal{O} |\Omega\rangle$ has an average energy \geq the ground state energy $E_0 = \langle \Omega | H | \Omega \rangle$. This can be viewed as the zero-temperature limit of the finite temperature bootstrap [35–37]. Note that here we consider *adjoint* operators $\mathcal{O}_i, \mathcal{O}_j$ with two open matrix indices. This is justified since the ground state of the ungauged model is a gauge-singlet.

We refer to constraints that do not involve the explicit form of the Hamiltonian as “kinematic constraints.” As in [13], these include cyclicity of the trace and gauge invariance. As in [13] we decompose operators into irreps of $O(D)$ and consider positivity for each irrep separately. For $D = 2$ this is particularly convenient, since the irreps are labeled by charge; the letters Z, P carry charge 1 and \bar{Z}, \bar{P} carry charge -1 .

In addition to the continuous symmetries, we have the additional reflection symmetry that is absent if we only had $SO(D)$ symmetry. For odd D , this means that we only need to consider words with an even number of letters. The ground state of these models are time reversal symmetric, which together with Hermiticity of the matrices implies that $\langle \text{tr } O_1 \cdots O_n \rangle = \pm \langle \text{tr } O_n \cdots O_1 \rangle$ where we choose $+(-)$ if there are an even (odd) number of P ’s in the correlator⁴.

2.3 Hierarchy

Following the approach developed in [5, 13], we introduce a hierarchy among the set of all variables, and impose bootstrap constraints up to some level of the hierarchy. The hierarchy is defined by sorting operators into *levels*: we assign the basic fields $\ell(X_I) = 1$, $\ell(P_I) = 2$. This assignment has the feature that $\ell([H, O]) = \ell(O) + 1$. We enumerate all variables up to a given level and quotient this space by the dynamical constraints. The resulting “search space” can be parameterized by a choice of “free variables” which are the actual variables that enter the semi-definite program. Table 1 displays the number of free variables for each level.

2.4 Non-linear relaxation

The cyclicity of the trace relates single-trace correlators to double-trace correlators. Since we wish to work at large N , we impose large N factorization. This means that the single-trace correlation functions that enter the matrices \mathcal{M}, \mathcal{N} are related by a set of quadratic equations. Hence the matrices \mathcal{M}, \mathcal{N} are in general non-linear functions of the search space parameters. If the number of non-linear parameters is $\lesssim 3$, one can simply scan over them. However, at higher levels, the number of non-linear parameters grows and one should use the technique of non-linear relaxation, first applied to the matrix bootstrap in [4, 12]⁵. The

⁴Here O_i are single letters, e.g., an X_I or a P_J .

⁵In principle, one can also consider other methods like the navigator function [38].

basic idea is to introduce a new variable for each double-trace that appears, e.g., $y = \langle \text{tr } P_I P_I \rangle^2$. Then we may replace the equation $y = \langle \text{tr } P_J P_J \rangle^2$ with the weaker (but rigorous) inequality $y \geq (\langle \text{tr } P_J P_J \rangle)^2$. This can be encoded in a positive semi-definite matrix: $\begin{pmatrix} 1 & x \\ x & y \end{pmatrix} \succeq 0$. In addition, we introduce some new positivity relations (beyond what was done in [4, 12]) by imposing $y \leq \langle \text{tr}(P_I P_I)(P_J P_J) \rangle$ since positivity requires $\langle \text{tr}(P_I P_I)^2 \rangle \geq \langle \text{tr } P_I P_I \rangle^2$. We refer the reader to Appendix B for the detailed implementation⁶. After performing this relaxation, we are left with a standard (convex) semi-definite programming problem with an expanded set of variables.

Although in principle we could relax all quadratic variables, we found it best to scan over $\langle \text{tr } X_I X_I \rangle$ while relaxing the remaining quadratic variables.

3 Bootstrap results

3.1 Analytic results

For low levels, it is possible to analyze the bootstrap constraints by hand and determine which inequalities are saturated at the boundary of the allowed region. At level⁷ 5, we may derive a simple analytic bound on the normalized ground state energy \mathcal{E} and $\langle \text{tr } X^2 \rangle = \frac{1}{D} \langle \text{tr } X_I X_I \rangle$ (see [39] and [13] for similar bounds in the BFSS case.). Let us denote the kinetic, mass, and potential (commutator²) energies by K, M, V . The virial theorem $-2K + 2M + 4V = 0$ and $K + M + V = E_0$, so:

$$\mathcal{E} \equiv \frac{E_0}{N^2} (g_{\text{YM}}^2 N)^{-1/3} = \frac{3}{4} \langle \text{tr } P_I P_I \rangle + \frac{1}{4} M^2 \langle \text{tr } X_I X_I \rangle. \quad (8)$$

Inner product positivity (6) applied to $\{X_I, -iP_I\}$ and also the anti-symmetric operator $X_I X_J - X_J X_I$ gives:

$$\begin{pmatrix} \langle \text{tr } X_I X_I \rangle & D/2 \\ D/2 & \langle \text{tr } P_I P_I \rangle \end{pmatrix} \succeq 0, \quad \langle \text{tr } P_I P_I \rangle \geq M^2 \langle \text{tr } X_I X_I \rangle. \quad (9)$$

Ground state positivity (7) applied to the operators $\{X_I, P_I\}$ yields:

$$\begin{pmatrix} \frac{D}{2} & \langle \text{tr } P_I P_I \rangle \\ \langle \text{tr } P_I P_I \rangle & (D-1) \langle \text{tr } X_I X_I \rangle + \frac{D}{2} M^2 \end{pmatrix} \succeq 0. \quad (10)$$

Then combining with (8) we obtain the bounds

$$\frac{3}{16 \langle \text{tr } X^2 \rangle} + \frac{DM^2 \langle \text{tr } X^2 \rangle}{8} \leq \frac{\mathcal{E}}{D} \leq \frac{1}{8} \left[2M^2 \langle \text{tr } X^2 \rangle + 3(2(D-1) \langle \text{tr } X^2 \rangle + M^2)^{1/2} \right] \quad (11)$$

$$\mathcal{E}/D \geq M^2 \langle \text{tr } X^2 \rangle \quad (12)$$

For the massless case, this yields $\frac{1}{(8D-8)^{1/3}} \leq \langle \text{tr } X^2 \rangle$. Surprisingly, for $M^2 = 0$ adding the bootstrap constraints up to level 8 did not lead to any improvement for these observables. In other words, (11) is the optimal bound up to level 8. The peninsula bound derived here is quite analogous to the level 5

⁶In principle, multi-trace correlators in the Heisenberg formalism are non-commutative, e.g., $\langle \text{tr } O_1 \text{tr } O_2 \rangle \neq \langle \text{tr } O_2 \text{tr } O_1 \rangle$ but to leading order in $1/N$ we may ignore their commutator.

⁷On general grounds, the ground state positivity matrix where O_i has level $\leq \ell_{\max}$ operators yields a matrix whose elements are level $\leq 2\ell_{\max} - 1$. However, the diagonal elements of the ground state matrix have level $\leq 2\ell_{\max} - 3$ since $[H, \text{tr } \bar{O}_i O_i] = 0$. In our analytic bound, since we only included a single level 2 operator P_I , the bounds only relate level 4 operators.

peninsula in the BFSS case. (If we remove (10), we lose the upper bound on \mathcal{E} in (11) and recover a level 4 bound that is also qualitatively similar to the BFSS case.)

The behavior of these bounds is markedly different for the massive case $M^2 > 0$. For the massive models (12) is non-trivial and together with (11), these inequalities describe a compact allowed region (an ‘‘island’’) already at level 5; the island shrinks at higher levels.

3.2 Numerical results

Model	\mathcal{E}	$\langle \text{tr } X_I X_I \rangle$	level 4 operator
$M^2 = 0, D = 2$			$\langle \text{tr } Z^2 \bar{Z}^2 \rangle$
Bootstrap	[0.707832, 0.707868]	[1.15420, 1.15460]	[0.37055, 0.37085]
Monte Carlo [31, 32]	0.7039(11)	—	—
Loop truncation [34]	0.7056(2)	1.172(1)	0.383(2)
Large D expansion [11]	0.756	0.985	0.177
$M^2 = 1, D = 2$			$\langle \text{tr } Z^2 \bar{Z}^2 \rangle$
Bootstrap	[1.172098376, 1.172098408]	[0.77800898, 0.77800934]	[0.15850588, 0.15850607]
Monte Carlo	1.1654(11)	—	—
Loop truncation [34]	1.17198	0.7784	0.1588
$M^2 = 0, D = 9$			$\langle \text{tr } X_I X_I X_J X_J \rangle$
Bootstrap	[6.69946, 6.69968]	[2.29195, 2.29218]	[5.7787, 5.7804]
Monte Carlo [10]	6.695(5)	2.291(1)	—
Large D expansion [11]	6.713	2.279	5.646

Table 2: Ground state energy \mathcal{E} bounds and expectation values of $\langle \text{tr } X_I X_I \rangle$ from bootstrap and Monte Carlo studies for various masses M^2 and dimensions D . Results in blue are level 14 bootstrap bounds; the $D = 9$ bootstrap results are level 11.

We considered the bootstrap for both the massless model ($M = 0$) and the massive model with $M^2 = 1$ (equivalently, $\lambda_{\text{eff}} = 1$). For the case $D = 2$, we performed a bootstrap analysis up to level 14, while for $D = 9$ the analysis was carried out up to level 11. The results are presented in Figure 1 and Figure 2.

For the $D = 2$ model, [31, 32] performed a Euclidean lattice computation using the Hybrid Monte Carlo algorithm. For $D = 2$, a large N and continuum extrapolation yielded the numbers that we display in Table 2. For $D = 9$, one may compare to the $N = 32$ Monte Carlo lattice simulation of [10]. We also compare⁸ to the interesting numerical approach of [34]. Their approach is based on a truncation of the loop equations [34, 40–43].

We also bootstrapped the level 4 operator $\langle \text{tr } Z^2 \bar{Z}^2 \rangle = \frac{1}{4} \sum_{I,J} \langle \text{tr } X_I X_J X_I X_J \rangle$. Together with the constraints, this fully determines the expectation value of all operators at level 4. At level 8, this operator is bounded from both sides for the massive $M^2 = 1$ model (and is unbounded from above at level 6 and below). To find a two-sided bound for the massless model, we had to go to level 10.

3.3 Large D comparison

One can also compare to the large D approximation of this model, see [11, 33, 44]. At large D , one can view the model as a variant of an $O(D)$ vector model and resum bubble diagrams, see Appendix D. In

⁸Our $\lambda_{\text{eff}} = 1$ becomes $m = 2, g = 2$ in the conventions of [34], after a canonical transformation.

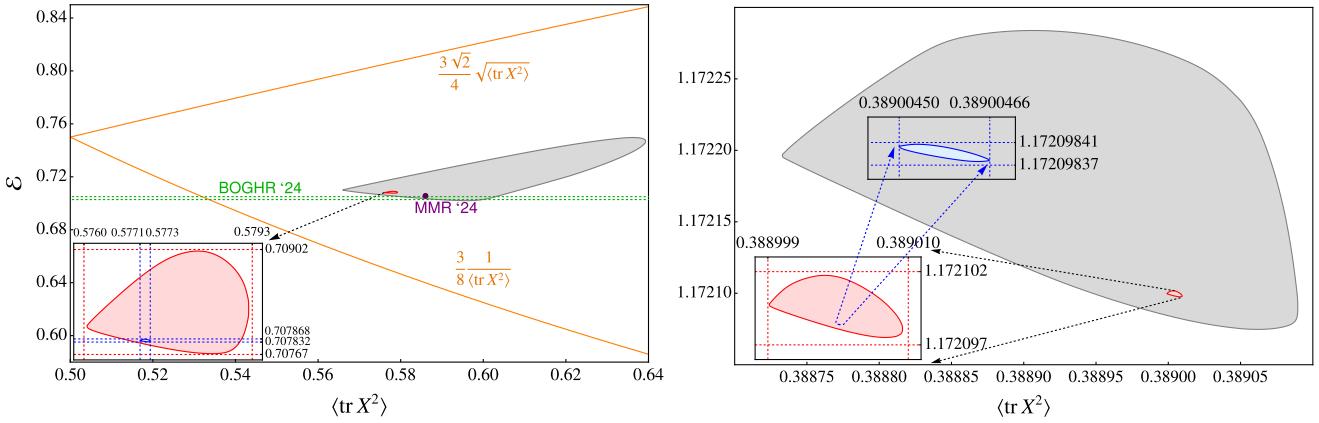


Figure 1: *Left:* Bootstrap constraints on the ground state/confining phase energy $\mathcal{E}/N^2 = E_0/(g_{\text{YM}}^2 N)^{1/3}$ and the simplest observable $\langle \text{tr } X^2 \rangle = \frac{1}{2} \langle \text{tr } X_I X_I \rangle = \langle \text{tr } Z \bar{Z} \rangle$ for the $D = 2$ massless matrix model. The level 6 constraint (11) for the massless $D = 2$ model has the shape of a peninsula. The island first appears at level 10. The island shrinks rapidly as we go to higher levels, so we show them more clearly in the inset panel. The small level 14 island is not visible to the naked eye in the main figure. Comparison to previous numerical results in the literature [32] (green) and [34] (purple) are given. *Right:* the level 10, 12 and 14 constraints for the massive $D = 2$ model with $M^2 = 1$. The level 10 island already excludes the Monte Carlo result [32].

our conventions, this gives the estimates

$$\langle \text{tr } X_I X_I \rangle = \frac{1}{2} D^{2/3} \left[1 + \frac{2}{D} \left(\frac{7\sqrt{5}}{30} - \frac{9}{32} \right) + \dots \right] \approx 2.279 \quad (13)$$

$$\mathcal{E} = D^{4/3} \left[\frac{3}{8} + \frac{1}{D} \left(\frac{\sqrt{5}}{2} - \frac{81}{64} \right) + \dots \right] \approx 6.713 \quad (14)$$

$$\langle \text{tr } X_I X_I X_J X_J \rangle = D^{4/3} \left[\frac{1}{4} + \frac{1}{D} \left(\frac{\sqrt{5}}{3} - \frac{9}{32} \right) + \dots \right] \approx 5.646 \quad (15)$$

(13), (14) were computed in [11]; (15) has not appeared in the literature to our knowledge and is derived in Appendix D. According to the Virial theorem, the potential energy $2V = \langle \text{tr } X_I^2 X_J^2 \rangle - \langle \text{tr } X_I X_J X_I X_J \rangle = 2\mathcal{E}/3$. We observe that to leading order in D , $\langle \text{tr } X_I^2 X_J^2 \rangle = 2\mathcal{E}/3$. This implies that the ‘‘alternating correlator’’ $\langle \text{tr } X_I X_J X_I X_J \rangle$ at leading order in N is sub-leading in $1/D$ compared to $\langle \text{tr } X_I^2 X_J^2 \rangle$.

We have reported the numbers for $D = 9$ in (13)-(15). Although it is a bit extreme, one can also try $D = 2$; one obtains estimates which are unsurprisingly less accurate than for $D = 9$, see Table 2. It would be interesting to repeat the bootstrap at various values of D in order to better test the large D scaling. Some results for $D = 3$ using the loop truncation method have been already obtained [45].

3.4 The adjoint sector

Let us comment on gauge non-singlet states in matrix quantum mechanics. A basic quantity is the energy gap to the adjoint sector ΔE_{adj} . This quantity is related to the adjoint Polyakov loop at low temperatures $P \sim e^{-\beta \Delta E_{\text{adj}}}$. Here we present a simple bootstrap bound:

$$\Delta \mathcal{E}_{\text{adj}} = \Delta E_{\text{adj}} (g_{\text{YM}}^2 N)^{-1/3} \leq \frac{\langle \text{tr } X_I [H, X_I] \rangle}{\langle \text{tr } X_I X_I \rangle} = \frac{1}{2 \langle \text{tr } X^2 \rangle} \quad (16)$$

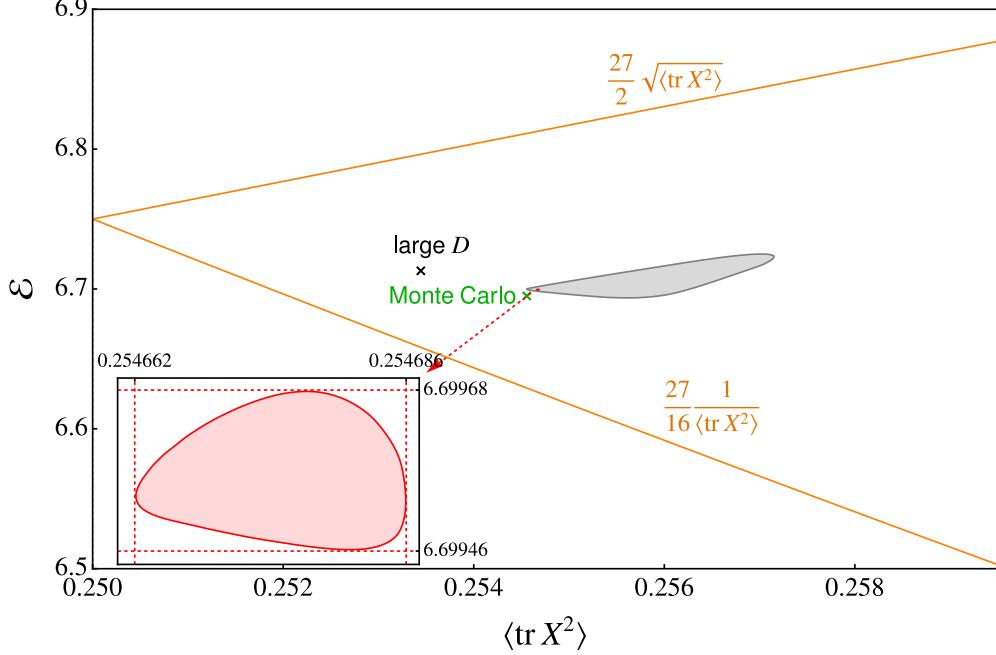


Figure 2: Here we display the level 10 constraints for the massless $D = 9$ model, sometimes referred to as ‘‘bosonic BFSS’’. The cross \times indicates the estimate from the $1/D$ expansion [11] and the green \times is from the $N = 32$ Monte Carlo simulation in [10], see also [19]. The orange curves are the analytic bounds (11). The level 10 island is displayed in gray; the red island (barely visible) near the western tip is the level 11 island; it is displayed more clearly in the inset panel.

where we have used the ground state positivity matrix (10). (We also give a different derivation of this bound in [46], by bootstrapping time-dependent correlators.)

Combined with our $D = 9, M = 0$ level 11 lower bound on $\langle \text{tr } X^2 \rangle$,

$$\Delta E_{\text{adj}} \leq 1.96339, \quad \Delta E_{\text{adj}}^{\text{Monte Carlo}} \approx 2.043(76), \quad D = 9, \quad (17)$$

In (17) we compare our results with a Monte Carlo estimate [22] obtained from comparing the gauged and ungauged thermodynamics. They also report an estimate based on measuring $\langle \text{tr } X^2 \rangle$ in the ungauged model; this gives $\Delta E_{\text{adj}}^{\text{Monte Carlo}} \approx 1.936(71)$. It might seem surprising that the relatively simple bound (16) essentially reproduces the Monte Carlo result; this is because the bound (16) is saturated in the large D limit ⁹.

In the supersymmetric BFSS context [22, 47], our rigorous lower bounds¹⁰ give $\Delta E_{\text{adj}} \leq 1.41(g_{\text{YM}}^2 N)^{1/3}$. The Monte Carlo result [22] reports $\Delta E_{\text{adj}} \leq .92(11)(g_{\text{YM}}^2 N)^{1/3}$. The Monte Carlo results [22, 23] suggest that, unlike the bosonic $D = 9$ model, the supersymmetric model is not too close to saturating (16), which reflects the non-trivial physics of the low-energy limit of the supersymmetric model.

In fact, the simple bound (16) can be improved by simply considering operators other than X_I . For example, one can consider both X_I and P_I and use the full ground state positivity matrix (10); this

⁹At infinite D , the state $X_{ij} |\Omega\rangle$ is an energy eigenstate. This can be verified by considering the expressions for the propagator in appendix D, which take the form of a massive harmonic oscillator.

¹⁰If we take the Monte Carlo result [23] for $\langle \text{tr } X^2 \rangle \approx 0.378$ which is slightly above our level 9 lower bound [13], we get the bound $\Delta E_{\text{adj}} \lesssim 1.3$.

leads to an improved bound¹¹. A method to systematically bound the gap will be reported in upcoming work [48], we hope to apply this method to BFSS in the future [14].

4 Discussion

At level $L < 10$, our results for the massless $M^2 = 0$ cases are qualitatively similar to the plots that were obtained in [13] for the maximally supersymmetric Yang-Mills matrix model (the D0 brane or BFSS matrix model). To make the comparison more concrete, note that in the SUSY case, there are three terms in the Hamiltonian $E = K + V + F$ whereas in the bosonic models there are only two terms $E = K + V$. These terms are related by the virial theorem, $-2K + 4V + F = 0$ in the SUSY case and $-2K + 4V = 0$ in the bosonic (massless) case. However, SUSY gives an additional constraint $E = 0$ for the ground state of BFSS, so there is really only one independent variable “energy”-like variable \mathcal{E} in both the SUSY and non-SUSY cases. In addition to this level 4 variable (which can be viewed as the kinetic or potential energy), there is the level 2 variable $\langle \text{tr } X^2 \rangle$ and one other level 4 variable $\langle \text{tr } X_I X_I X_J X_J \rangle$. In all models that we studied, a peninsula in the $\langle \text{tr } X^2 \rangle - \mathcal{E}$ appears exactly at level 5: there is an unconditional lower bound on $\langle \text{tr } X^2 \rangle$ at level 5, but no upper bound. Furthermore, up to level 10, there is a “vertical” peninsula in the $\langle \text{tr } X^2 \rangle - \langle \text{tr } X_I X_I X_J X_J \rangle$ plot that resembles figure 3 in [13].

These features are special to the $M^2 = 0$ models. When we added a mass term, we saw that the bootstrap behaved quite differently; an island appears already at level 4. It seems that the “classically” flat directions present in these potentials make it more difficult to upper bound the size of the ground state wavefunction $\langle \text{tr } X^2 \rangle$.

At level 10, the bootstrap region becomes an island in the massless $D = 2$ and $D = 9$ cases we studied in this paper. We are therefore optimistic that an island will appear at level 10 (or perhaps within a few levels beyond) in the supersymmetric model. Note that in the supersymmetric setting, ground state positivity (7) was implicitly imposed by the supercharge equations of motion [14].

An extension of our work to the future is to study the models at finite temperature, using the approach outlined in [37]. One can also study observables in a microcanonical window of energies E/N^2 above the ground state. Another interesting problem is to compute the Polyakov loop using the bootstrap formalism. This could be done by modifying the gauge constraint. Combining these two future directions, one could confirm the existence of the Gross-Wadia-Witten transition at finite D [9–11]. One could also study time-dependent correlators [46] with the eventual goal of computing the quasinormal modes [49, 50].

Finally, it would be interesting to study supersymmetric models with $\text{SO}(D)$ symmetry, see [31, 51]. These models have fewer supersymmetries than the BFSS model; their gravity duals are much less understood compared to BFSS [52, 53]. In particular, the $\mathcal{N} = 2$ SUSY model [54, 55] would be an interesting bootstrap target. It would be interesting to test whether there are metastable bound states in the planar ’t Hooft limit of these models, which would be akin to the D0 brane black hole in the BFSS model.

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¹¹Defining $p = \langle \text{tr } P_I P_I \rangle / D$, $x = \langle \text{tr } X_I X_I \rangle / D$, this gives the improved bound:

$$\mathcal{E} \leq -\frac{\sqrt{(p - 2(d-1)x^2)^2 - (4px - 1)(2(d-1)x - 4p^2)} - 2(d-1)x^2 + p}{4px - 1}. \quad (18)$$

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A Numerical implementation

To solve the semidefinite programming (SDP) constructed, we used **SDPA-GMP**, an arbitrary-precision arithmetic SDP solver [56]¹². We observed that our SDP problem exhibited significant numerical instability, which motivated our choice of an arbitrary-precision solver. These numerical instabilities can be traced back to the lack of strict feasibility in the primal form of the SDP. For further technical discussion of this type of numerical instability, and possible (though not systematic) strategies for mitigating it, we refer the reader to, for example, [57].

Our [Github repository](#) includes a **Mathematica** notebook as an example for generating the final SDP. Users are free to export the output to other SDP solvers to obtain bounds on specific variables. The built-in solver in **Mathematica** may suffice for lower bootstrap levels, but for more ambitious computations, a higher-precision solver is necessary.

B Relaxation

In this appendix, we provide additional details on the implementation of the relaxation procedure. Let us denote the set of singlet¹³ operators that appear up to some level by $\{\mathcal{O}_i\}$, and define the corresponding column vector of expectations \vec{x} , such that:

$$x_i = \langle \text{tr } \mathcal{O}_i \rangle. \quad (19)$$

The quadratic variables are then given by the elements of the following matrix Q , with entries:

$$Q_{ij} = x_i x_j, \quad (20)$$

or, more compactly,

$$Q = \vec{x} \vec{x}^T. \quad (21)$$

Whenever quadratic terms appear in our constraints, we replace them with the corresponding elements of the matrix Q . In this way, we isolate the non-convexity of our optimization problem in (20). In the following, we study the relaxation of (20).

We observe that the matrix Q satisfies the following two convex conditions. First, the matrix

$$\begin{pmatrix} 1 & \vec{x}^T \\ \vec{x} & Q \end{pmatrix} \succeq 0 \quad (22)$$

is positive semidefinite, since it is the outer product of the vector $(1, \vec{x}^T)^T$ with itself.

Additionally, the following matrix is also positive semidefinite, as implied by (6):

$$\left\langle \text{tr} (\mathcal{O}_i - \langle \mathcal{O}_i \rangle)^\dagger (\mathcal{O}_j - \langle \mathcal{O}_j \rangle) \right\rangle = \mathcal{M}_{ij} - Q_{ij} \succeq 0. \quad (23)$$

In the relaxation, we introduce a new matrix \mathcal{Q} , which, instead of satisfying (20), is required to satisfy the following two conditions:

$$\begin{pmatrix} 1 & \vec{x}^T \\ \vec{x} & \mathcal{Q} \end{pmatrix} \succeq 0, \quad (24)$$

$$\mathcal{M} \succeq \mathcal{Q}. \quad (25)$$

We then replace every occurrence of an element of Q in other constraints with the corresponding element of \mathcal{Q} .

The relaxation implementation described here, namely (24) and (25), is stronger than the previous approach in [4, 12], where only (24) was imposed.

¹²We used the version available at <https://github.com/nakatamaho/sdpa-gmp>.

¹³Here, \mathcal{O}_i should be a singlet under the global symmetry; otherwise, its expectation value vanishes.

C Positivity and O(9) blocks

Consider a collection of operators $O_{1,I_1,\dots,I_n}, O_{2,I_1,\dots,I_n}, \dots, O_{|\alpha|,I_1,\dots,I_n}$ with vector indices. (Here we should think of $|\alpha|$ as the number of different “words” that have a level \leq to some cutoff.) We denote their $O(D)$ indices by the shorthand $\mathbf{i} = \{I_1, I_2, \dots, I_n\}$. Positivity of the inner product yields

$$\mathcal{M}_{\{\alpha,\mathbf{i}\},\{\beta,\mathbf{j}\}} = \langle \text{tr } \bar{O}_{\alpha,\mathbf{i}} O_{\beta,\mathbf{j}} \rangle, \quad \mathcal{M} \succeq 0. \quad (26)$$

If we write out this matrix explicitly, we will have a $|\alpha|D^n \times |\alpha|D^n$ which will quickly become intractable for even modest n . Our goal instead is to use group theory to boil down all the positivity constraints of this explicit matrix into a much smaller set of positivity constraints on the $O(D)$ singlet operators. Since the group theory does not touch the α index, we will henceforth focus on a particular operator, say $\mathcal{O}_i = \mathcal{O}_{1,i}$.

To this end, we decompose the operators into irreps, e.g.,

$$O_i = \sum_R \sum_{r=1}^{\dim R} (C_R)_i^r (O_R)_r \quad (27)$$

An irrep appears in the sum multiple times if the decomposition has multiplicity. Then $SO(9)$ invariance of the state implies that

$$\langle \text{tr} (\bar{O}_{\bar{R}})_{\bar{r}} (O_R)_r \rangle = \delta_{\bar{R},R} \delta_{\bar{r},r} a_{\bar{R},R}, \quad (28)$$

$$\mathcal{M}_{ij} = \sum_{R,\bar{R},r} a_{\bar{R},R} (C_{\bar{R}})_i^r (C_R)_j^r. \quad (29)$$

Here the symbol $\delta_{R,R'} = 1$ if R and R' are equivalent representations of $SO(9)$, or else $\delta_{R,R'} = 0$. Thus we have parameterized a large matrix \mathcal{M} in terms of a smaller number of coefficients $a_{\bar{R},R}$. These coefficients are precisely just the $O(9)$ singlet operators, e.g.,

$$a_{R,\bar{R}} = \langle \text{tr} (\bar{O}_{\bar{R}})_r (O_R)_r \rangle. \quad (30)$$

Furthermore, we can simplify the positivity requirement $\mathcal{M} \succeq 0$ by evaluating the requirement on a nice basis of vectors $\{e_A\}$. A particularly nice basis is the following. First we view (C_R) as a projector from the vector space indexed by \mathcal{I} to the irrep R (a smaller vector space indexed by r). Then we can define the basis e_A to be a collection of the transposed projectors, where $A = (R_A, r_A)$. This spans the bigger vector space of dimension $|\mathcal{I}|$ and satisfies

$$\sum_j e_A^j (C_R)_j^r = \delta_{R,R_A} \delta_{r,r_A}^r. \quad (31)$$

Then the positivity requirement can be expressed as

$$\mathcal{M}_{AB} = \bar{e}_A^i \mathcal{M}_{ij} e_B^j = \sum_{R,\bar{R}} a_{\bar{R}_A,R_B} \delta_{\bar{R}_A,R_B} \delta_{r_A,r_B} \succeq 0. \quad (32)$$

The conclusion is that we only need to impose

$$a_{\bar{R},R} \succeq 0, \quad \bar{R} \sim R. \quad (33)$$

Note that even if the decomposition of \mathbf{i} contains irrep R with unit multiplicity, the generalization to multiple operators will typically lead to a non-trivial matrix a .

Let us consider a tensor of rank n . We are interested in the projectors

$$(\pi_{\mathsf{T}})_{b_1 b_2 \dots b_r}^{a_1 a_2 \dots a_r}. \quad (34)$$

These are defined to be orthonormal projectors:

$$(\pi_{\mathsf{S}})_{b_1 b_2 \dots b_r}^{a_1 a_2 \dots a_r} (\pi_{\mathsf{T}})_{c_1 c_2 \dots c_r}^{b_1 b_2 \dots b_r} = \delta_{\mathsf{ST}} \cdot (\pi_{\mathsf{T}})_{c_1 c_2 \dots c_r}^{a_1 a_2 \dots a_r}. \quad (35)$$

These projectors are also traceless, meaning

$$(\pi_{\mathsf{T}})_{b_1 b_2 \dots b_r}^{a_1 a_2 \dots a_r} \delta_{b_i b_j} = 0. \quad (36)$$

A subtlety with this procedure occurs when a given irrep appears more than once in the above decomposition. In this case, the orthogonality of the projectors is potentially misleading. Since each degenerate irrep is isomorphic, there exists a linear map

$$\iota_{m,n} : R_m \rightarrow R_n \quad (37)$$

Using this map, we can consider the contraction

$$(\pi_{R_m})_{b_1 \dots b_r}^{a_1 \dots a_r} (\iota_{m,n})_{c_1 \dots c_r}^{b_1 \dots b_r} (\pi_{R_n})_{a_1 \dots a_r}^{c_1 \dots c_r} (\iota_{1,m}) \quad (38)$$

It will be convenient to simply consider the

$$\iota_{m,n} = \iota_{m,1} \cdot \iota_{1,n} \quad (39)$$

Then, (28) becomes

$$\mathcal{M}_{ij} = \mathcal{M}_{i_1 \dots i_r}^{j_1 \dots j_r} = \sum_R \sum_{m,n} a_{m,n}^R (\pi_{R_m})_{i_1 \dots i_m}^{b_1 \dots b_r} (\iota_{m,n})_{b_1 \dots b_r}^{c_1 \dots c_r} (\pi_{R_n})_{c_1 \dots c_r}^{j_1 \dots j_r}. \quad (40)$$

$$\mathcal{M}_{AB} = (\pi_{R_m})_{c_1 \dots c_r}^{i_1 \dots i_m} \mathcal{M}_{i_1 \dots i_m}^{j_1 \dots j_m} (\pi_{R_n})_{j_1 \dots j_m}^{b_1 \dots b_r} (\iota_{n,m})_{b_1 \dots b_r}^{c_1 \dots c_r} \quad (41)$$

The conclusion is that we can dress

$$\pi_m \rightarrow \pi_m \cdot \iota_{m,1} \quad (42)$$

C.1 constructing the projectors

To construct the $O(D)$ projectors, we do so in a two step process, by first constructing the $GL(D)$ projectors, and then removing the traces to satisfy (36).

For a given diagram, one can define the $GL(D)$ projector by first symmetrizing over the rows and then anti-symmetrizing over the columns of the given diagram:

$$\tilde{\pi}_{\text{Young Diagram}} = \prod_{\text{columns}} \text{antisymmetrizer} \cdot \prod_{\text{rows}} \text{symmetrizer} \quad (43)$$

To illustrate this procedure concretely, let us consider the mixed-symmetry **231** irreps. This appears in the rank 3 tensor decomposition:

$$\mathbf{9} \times \mathbf{9} \times \mathbf{9} = 3(\mathbf{9}) + \mathbf{84} + \mathbf{156} + 2(\mathbf{231}). \quad (44)$$

There are two isomorphic **231** irreps, corresponding to the two Young diagrams

$$\mathbf{231}_+ = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} = \text{Diagram with white box and black box}, \quad (45)$$

$$\mathbf{231}_- = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} = \text{Diagram with black box and white box}, \quad (46)$$

Here we are using ‘‘birdtracks’’ notation; we have indicated the symmetrizer using a white box and an anti-symmetrizer using a black box [58]. This defines some projectors $\tilde{\pi}_{b_1 \dots b_r}^{a_1 \dots a_r}$ which satisfy

$$\tilde{\pi}_{b_1 \dots b_r}^{a_1 \dots a_r} - \tilde{\pi}_{b_1 \dots b_r}^{\tau(a_1 \dots a_r)} = 0, \quad \tau \in \{\text{row transpositions}\} \quad (47)$$

$$\tilde{\pi}_{b_1 \dots b_r}^{a_1 \dots a_r} + \tilde{\pi}_{\tau(b_1 \dots b_r)}^{a_1 \dots a_r} = 0, \quad \tau \in \{\text{column transpositions}\} \quad (48)$$

Here row (column) transpositions generate the subgroup of the permutation group that preserve the row (column) structure of the Young tableau. Now to obtain the $O(9)$ projectors, we also need to subtract off any lower irreps (in this case **9**). To do so we construct various Wick contractions, e.g.,

$$\delta_{a_1 a_2} \delta^{b_1 b_2} \delta_{a_3}^{b_3} = \text{Diagram with two loops and one line}, \quad \delta_{a_2 a_3} \delta^{b_2 b_3} \delta_{a_1}^{b_1} = \text{Diagram with two loops and one line}, \quad (49)$$

$$\delta_{a_1 a_3} \delta^{b_1 b_3} \delta_{a_2}^{b_2} = \text{Diagram with two loops and one line}, \quad (50)$$

Then for each irrep/Young diagram Y , we sandwich these Wick contractions according to $\tilde{\Delta}_Y = \tilde{\pi} \cdot \Delta \cdot \tilde{\pi}$. This results in a tensor with the same symmetry properties as (47) and (48). For example, taking the first tensor in (49) and using the $\mathbf{231}_+$ projector gives

$$\tilde{\Delta}_{\mathbf{231}_+} = \text{Diagram with two loops and two black boxes}, \quad (51)$$

Sandwiching the second tensor in (49) gives zero, and sandwiching (50) produces a tensor proportional to the above. We introduce the $O(D)$ inner product on tensors

$$\langle S, T \rangle = \delta_{i_1 k_1} \dots \delta_{i_r k_r} \delta^{j_1 l_1} \dots \delta^{j_s l_s} S_{j_1 \dots j_s}^{i_1 \dots i_r} T_{l_1 \dots l_s}^{k_1 \dots k_r} \quad (52)$$

$$= S_{k_1 \dots k_r}^{l_1 \dots l_s} T_{l_1 \dots l_s}^{k_1 \dots k_r}. \quad (53)$$

In general, we should carry out a Gram-Schmidt procedure using this inner product on the resulting set $\{\tilde{\Delta}_Y\} \xrightarrow{\text{GS}} \{\Delta_Y\}$; in this example this produces only one tensor. We may choose the normalization of Δ_Y so that $(\Delta_Y)_i \cdot (\Delta_Y)_j = \delta_{ij} (\Delta_Y)_i$. We then demand that the final projector π_Y is orthogonal to $\{\Delta_Y\}$:

$$\langle \pi_Y, \Delta_Y \rangle = 0 \quad (54)$$

$$\pi_{\mathbf{231}_+} = \text{Diagram with two loops and two black boxes} - \text{Diagram with two loops and two black boxes} \quad (55)$$

As another example, consider the mixed symmetry irrep

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} = \text{Diagram with two loops and two black boxes} \quad (56)$$

The orthonormalization procedure on the Wick contractions produces 2 vectors $\{\Delta_Y\}$ and we find $\pi_Y = \tilde{\pi}_Y - (\Delta_Y)_1 - (\Delta_Y)_2$.

D The Large D expansion

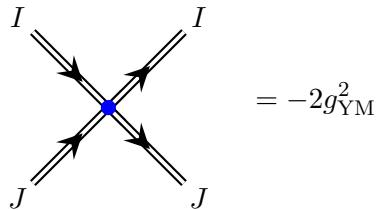
In this appendix, we review the solution to the large N and large D Yang-Mills quantum mechanics and compute the 4-pt function. In this limit, we take $N \rightarrow \infty, D \rightarrow \infty$ holding fixed $\tilde{\lambda} = g_{\text{YM}}^2 ND$. (More precisely, we hold fixed the dimensionless combination $\tilde{\lambda}\tau^3$ where τ is some timescale in the problem.)

We will use a diagrammatic approach, see [11, 33, 44]. The bare propagator

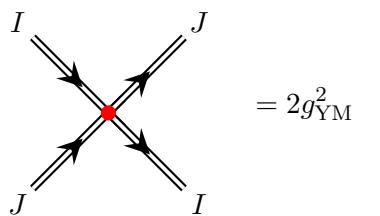
$$\langle(X_I)_{ij}(X_J)_{kl}\rangle_{\text{bare}} = \text{---} \Rightarrow \frac{1}{(\omega - (\alpha_j - \alpha_i))^2} \delta_{il} \delta_{jk} \delta_{IJ} \quad (57)$$

Here α_i are the eigenvalues of the gauge field A_0 . We also have the two vertices from the potential:

$$V = \frac{g_{\text{YM}}^2}{4} \sum_{I,J} 2 \text{Tr } X_I^2 X_J^2 - 2 \text{Tr}(X_I X_J)^2 \quad (58)$$



$$= -2g_{\text{YM}}^2 \quad (59)$$



$$= 2g_{\text{YM}}^2 \quad (60)$$

At large N and large D , the diagrams that contribute are the planar bubble diagrams. The most important vertex is the blue vertex; at large N bubble diagrams that are made of red vertices are either non-planar or are suppressed by $1/D$ (due to the index contraction pattern).

We can sum bubble diagrams using the Schwinger-Dyson equation:

$$I = \text{---} \circ I = \text{---} + I \text{---} \circ K \text{---} I \quad (61)$$

$$G(\omega) = \frac{1}{(\omega - (\alpha_j - \alpha_i))^2} - \frac{1}{\omega^2} \Delta^2 G(\omega), \quad \Delta^2 = 2g_{\text{YM}}^2 DN \int \frac{d\omega'}{2\pi} G(\omega') \quad (62)$$

$$G(\omega) = \frac{1}{(\omega - (\alpha_j - \alpha_i))^2 + \Delta^2}, \quad \Delta^2 = \frac{g_{\text{YM}}^2 DN}{\Delta} \Rightarrow \Delta = \tilde{\lambda}^{1/3} \quad (63)$$

Here we have adopted the shorthand for the dressed propagator $G(\omega) = G_{ijji}(\omega)$ and $G(\tau)$ is the Fourier transform:

$$G_{ijkl}(\tau) = \frac{1}{2\Delta_0} e^{i(\alpha_j - \alpha_i)|\tau| - \Delta|\tau|} \delta_{il} \delta_{jk} \Rightarrow \langle \text{tr } X_I X_I \rangle = \frac{D}{2\tilde{\lambda}^{1/3}} \quad (64)$$

This agrees with the leading answer in $1/D$ quoted in (13) and derived in [11]. Note that in the final answer, the eigenvalues of the gauge field drop out. In the confining phase, we can essentially neglect the eigenvalues. Since the ground state is an $SU(N)$ singlet in the ungauged model, we should get identical correlators at zero temperature if we study the ungauged model where there is no gauge field.

D.1 4pt function

We now compute the 4-pt function at equal times. It is easier to compute the alternating correlator. This is given by dressing the bare interaction vertex:

$$\langle \text{tr } X_I X_J X_I X_J \rangle_c = \begin{array}{c} \text{Diagram: Four external lines (two horizontal, two vertical) meeting at a central red dot} \end{array} = 2D^2 \frac{\tilde{g}^2 N}{D} \int_{-\infty}^{\infty} d\tau G(\tau)^4 = \frac{D \tilde{\lambda}}{16 \Delta^5} = \frac{D}{16 \tilde{\lambda}^{2/3}} \quad (65)$$

Here we have used

$$\int_{-\infty}^{\infty} d\tau G(\tau)^4 = \frac{2}{64 \Delta_0^5}. \quad (66)$$

Since the $O(D)$ indices are not aligned, the disconnected component is down by a factor of $1/D$, but contributes at the same order as the connected diagram:

$$\langle \text{tr } X_I X_J X_I X_J \rangle = \begin{array}{c} \text{Diagram: Four external lines (two horizontal, two vertical) meeting at a central red dot} \end{array} + \begin{array}{c} \text{Diagram: Two vertical lines (I up, J down) and two horizontal lines (J left, I right)} \end{array} + \begin{array}{c} \text{Diagram: Two horizontal lines (I left, J right) and two vertical lines (I up, J down)} \end{array} \quad (67)$$

$$= \frac{D}{4 \tilde{\lambda}^{2/3}} + \frac{D}{4 \tilde{\lambda}^{2/3}} + \frac{D}{16 \tilde{\lambda}^{2/3}} = \frac{9 D^{1/3}}{16 \tilde{\lambda}^{2/3}} \quad (68)$$

One can then use the Virial theorem $-2K + 4V = 0$ together with $K + V = \mathcal{E}$ to convert this to give the other level 4 correlator

$$\frac{2\mathcal{E}}{3} = \frac{\tilde{\lambda}^{2/3}}{D^{2/3}} \sum_{I,J} \langle \text{tr } X_I^2 X_J^2 \rangle - \langle \text{tr } X_I X_J X_I X_J \rangle. \quad (69)$$

Then combining with (14), we obtain the result cited in the main text (15):

$$\langle \text{tr } X_I^2 X_J^2 \rangle = \frac{D^2}{\tilde{\lambda}^{2/3}} \left[\frac{1}{4} + \frac{1}{D} \left(\frac{\sqrt{5}}{3} - \frac{9}{32} \right) + \dots \right]. \quad (70)$$

D.2 Non-alternating correlator

As a check of the above computation, we can also compute the non-alternating correlator directly. In the $O(D)$ vector model context, this correlator is a “double trace” in the $O(D)$ indices; the leading term is given by the factorized answer. To compute the first non-trivial correction, we consider the connected Feynman diagrams:

$$\langle \text{tr } X_I X_I X_J X_J \rangle_c = \begin{array}{c} \text{Diagram: Four external lines (two horizontal, two vertical) meeting at a central blue dot} \end{array} + \begin{array}{c} \text{Diagram: Four external lines (two horizontal, two vertical) with a central loop involving indices K and J} \end{array} + \begin{array}{c} \text{Diagram: Four external lines (two horizontal, two vertical) with a central loop involving indices K, L, and J} \end{array} + \dots \quad (71)$$

$$= -\tilde{\lambda} D \frac{1}{\omega_1^2 + \Delta^2} \frac{1}{\omega_2^2 + \Delta^2} \frac{1}{\omega_3^2 + \Delta^2} \frac{1}{\omega_4^2 + \Delta^2} (1 - \tilde{\lambda} B(\omega_{12})) \quad (72)$$

Note the symmetry factors of 1/2. Then one can sum over such bubble diagrams [33] which produces the effective propagator (in the large D vector model context, this is the σ propagator):

$$B(\omega) = \text{---} \rightarrow \text{---} = \frac{1}{\tilde{\lambda}^{1/3}} \frac{1}{\omega^2 + 5\tilde{\lambda}^{2/3}}, \quad B(\tau) = \frac{1}{2\sqrt{5}\tilde{\lambda}^{1/3}\tilde{\lambda}^{1/3}} e^{-\sqrt{5}\tilde{\lambda}^{1/3}|\tau|} \quad (73)$$

So to compute the bubble diagram correction we have

$$\int d\tau_1 d\tau_2 G^2(\tau_1)B(\tau_2 - \tau_1)G^2(-\tau_2) = \frac{1}{2^5 \tilde{\lambda}^{6/3}} \int \frac{d\tilde{\tau}_1 d\tilde{\tau}_2}{\sqrt{5}\tilde{\lambda}^{2/3}} \exp \left[-(2|\tilde{\tau}_1| + 2|\tilde{\tau}_2| + \sqrt{5}|\tilde{\tau}_2 - \tilde{\tau}_1|) \right] \quad (74)$$

$$= \frac{16 - 7\sqrt{5}}{32\sqrt{5}\tilde{\lambda}^{8/3}} \quad (75)$$

We also need the 1-loop corrected disconnected contribution

$$\langle \text{tr } X_I X_I \rangle = \frac{D}{2\tilde{\lambda}^{1/3}} \left[1 + \frac{2}{D} \left(\frac{7\sqrt{5}}{30} - \frac{9}{32} \right) + \dots \right] \quad (76)$$

Putting together these ingredients,

$$\langle \text{tr } X_I X_I X_J X_J \rangle = \begin{array}{c} \text{Diagram 1: Two vertical lines } I \text{ and } J \text{ meeting at two nodes.} \\ \text{Diagram 2: Two vertical lines } I \text{ and } J \text{ meeting at two nodes, with a dashed line connecting them.} \\ \text{Diagram 3: A single vertical line } I \text{ with a node.} \\ \text{Diagram 4: A single horizontal line } I = J \end{array} + \quad (77)$$

$$= -\frac{1}{2} \frac{D}{16\tilde{\lambda}^{2/3}} + \tilde{\lambda}^2 \frac{16 - 7\sqrt{5}}{32\sqrt{5}\tilde{\lambda}^{8/3}} + \frac{D^2}{4\tilde{\lambda}^{2/3}} \left[1 + \frac{4}{D} \left(\frac{7\sqrt{5}}{30} - \frac{9}{32} \right) \right] + \frac{D}{4\tilde{\lambda}^{2/3}} + \dots \quad (78)$$

$$= \frac{D^2}{\tilde{\lambda}^{2/3}} \left[\frac{1}{4} + \frac{1}{D} \left(\frac{\sqrt{5}}{3} - \frac{9}{32} \right) + \dots \right] \quad (79)$$

We again recover the answer quoted in the main text (15).

E Supersymmetric model

$$Q = 2i \text{Tr } \bar{\psi} P + i \text{Tr} [Z, \bar{Z}] \psi \quad (80)$$

$$Q = -2i \text{Tr } \psi \bar{P} - i \text{Tr} [Z, \bar{Z}] \bar{\psi} \quad (81)$$

$$\{Q, \bar{Q}\} = 2H = \text{Tr } P \bar{P} - \frac{1}{4}[Z, \bar{Z}]^2 + i \text{Tr } \bar{\psi} [Z, \bar{\psi}] + i \text{Tr } \psi [\bar{Z}, \psi] \quad (82)$$

$$Q^2 = i \text{Tr } ZG = 0 \quad (83)$$

$$\bar{Q}^2 = -i \text{Tr } \bar{Z}G = 0 \quad (84)$$

$$[J, Q] = \frac{1}{2}Q, \quad [J, \bar{Q}] = -\frac{1}{2}\bar{Q} \quad (85)$$

$$[J, Z] = Z, \quad [J, \bar{Z}] = -\bar{Z} \quad (86)$$

F A worked example: level 5

In this section, we present a detailed worked example of the Hamiltonian discussed in the main text:

$$H = \frac{1}{2} \sum_{I=1}^D (-\text{Tr} \Pi_I \Pi_I + M^2 \text{Tr} X_I X_I) - \frac{1}{4} \sum_{I,J=1}^D \text{Tr} [X_I, X_J]^2 \quad (87)$$

$$C = \sum_{I=1}^D ([X_I, \Pi_I] - \mathbf{1}) \quad (88)$$

Here, we have introduced $\Pi_I = -iP_I$. The advantage of this replacement is that the correlators of words composed of the letters Π_I and X_I are real numbers.

F.1 Equations

Up to level 5, we have the following 19 variables:

$$\begin{aligned} & \langle \text{tr} \Pi_I \Pi_I \rangle, \langle \text{tr} \Pi_I X_I \rangle, \langle \text{tr} X_I \Pi_I \rangle, \langle \text{tr} X_I X_I \rangle, \langle \text{tr} \Pi_I X_I X_J X_J \rangle, \langle \text{tr} \Pi_I X_J X_I X_J \rangle, \\ & \langle \text{tr} \Pi_I X_J X_J X_I \rangle, \langle \text{tr} X_I \Pi_I X_J X_J \rangle, \langle \text{tr} X_I \Pi_J X_I X_J \rangle, \langle \text{tr} X_I \Pi_J X_J X_I \rangle, \langle \text{tr} X_I X_I \Pi_J X_J \rangle, \\ & \langle \text{tr} X_I X_I X_J \Pi_J \rangle, \langle \text{tr} X_I X_I X_J X_J \rangle, \langle \text{tr} X_I X_J \Pi_I X_J \rangle, \langle \text{tr} X_I X_J \Pi_J X_I \rangle, \langle \text{tr} X_I X_J X_I \Pi_J \rangle, \\ & \langle \text{tr} X_I X_J X_I X_J \rangle, \langle \text{tr} X_I X_J X_J \Pi_I \rangle, \langle \text{tr} X_I X_J X_J X_I \rangle \end{aligned} \quad (89)$$

The notation for the variables used here follows that introduced in [13], where repeated indices I, J, K, \dots are implicitly summed over.

As discussed in the main text, these variables are subject to both kinematic and dynamical constraints.

Below, we list the linear combinations of variables that vanish as a consequence of these constraints¹⁴:

$$\begin{aligned}
& D + \langle \text{tr } \Pi_I X_I \rangle - \langle \text{tr } X_I \Pi_I \rangle \\
& - \langle \text{tr } \Pi_I X_I \rangle + \langle \text{tr } X_I \Pi_I \rangle - D \\
& \langle \text{tr } \Pi_I X_I \rangle + \langle \text{tr } X_I \Pi_I \rangle \\
& - \langle \text{tr } \Pi_I X_J X_J X_I \rangle + \langle \text{tr } X_I \Pi_I X_J X_J \rangle - D \langle \text{tr } X_I X_I \rangle \\
& - \langle \text{tr } X_I X_I \rangle - \langle \text{tr } \Pi_I X_J X_I X_J \rangle + \langle \text{tr } X_I \Pi_J X_I X_J \rangle \\
& - \langle \text{tr } X_I X_I \rangle - \langle \text{tr } \Pi_I X_I X_J X_J \rangle + \langle \text{tr } X_I \Pi_J X_J X_I \rangle \\
& \langle \text{tr } X_I \Pi_I X_J X_J \rangle + \langle \text{tr } X_I X_I \Pi_J X_J \rangle \\
& \langle \text{tr } X_I X_I \Pi_J X_J \rangle - \langle \text{tr } X_I \Pi_J X_J X_I \rangle \\
& \langle \text{tr } X_I X_I \rangle + \langle \text{tr } \Pi_I X_J X_J X_I \rangle - \langle \text{tr } X_I X_I X_J \Pi_J \rangle + D \langle \text{tr } X_I X_I \rangle \\
& \langle \text{tr } \Pi_I X_I X_J X_J \rangle + \langle \text{tr } X_I X_I X_J \Pi_J \rangle \\
& - \langle \text{tr } X_I X_I \Pi_J X_J \rangle + \langle \text{tr } X_I X_I X_J \Pi_J \rangle - D \langle \text{tr } X_I X_I \rangle \\
& \langle \text{tr } \Pi_I X_I X_J X_J \rangle + \langle \text{tr } X_I \Pi_I X_J X_J \rangle + \langle \text{tr } X_I X_I \Pi_J X_J \rangle + \langle \text{tr } X_I X_I X_J \Pi_J \rangle \\
& \langle \text{tr } X_I X_J \Pi_I X_J \rangle - \langle \text{tr } X_I \Pi_J X_I X_J \rangle \\
& \langle \text{tr } X_I \Pi_J X_I X_J \rangle + \langle \text{tr } X_I X_J \Pi_I X_J \rangle \\
& - \langle \text{tr } X_I X_I \rangle + \langle \text{tr } X_I X_I X_J \Pi_J \rangle - \langle \text{tr } X_I X_J \Pi_J X_I \rangle \\
& \langle \text{tr } X_I X_J \Pi_J X_I \rangle - \langle \text{tr } X_I \Pi_I X_J X_J \rangle \\
& \langle \text{tr } X_I \Pi_J X_J X_I \rangle + \langle \text{tr } X_I X_J \Pi_J X_I \rangle \\
& 2 \langle \text{tr } X_I X_I \rangle + \langle \text{tr } \Pi_I X_J X_I X_J \rangle - \langle \text{tr } X_I X_J X_I \Pi_J \rangle \\
& \langle \text{tr } \Pi_I X_J X_I X_J \rangle + \langle \text{tr } X_I X_J X_I \Pi_J \rangle \\
& - \langle \text{tr } X_I X_I \rangle - \langle \text{tr } X_I X_J \Pi_I X_J \rangle + \langle \text{tr } X_I X_J X_I \Pi_J \rangle \\
& \langle \text{tr } \Pi_I X_J X_I X_J \rangle + \langle \text{tr } X_I \Pi_J X_I X_J \rangle + \langle \text{tr } X_I X_J \Pi_I X_J \rangle + \langle \text{tr } X_I X_J X_I \Pi_J \rangle \\
& \langle \text{tr } X_I X_I \rangle + \langle \text{tr } \Pi_I X_I X_J X_J \rangle - \langle \text{tr } X_I X_J X_J \Pi_I \rangle + D \langle \text{tr } X_I X_I \rangle \\
& \langle \text{tr } \Pi_I X_J X_J X_I \rangle + \langle \text{tr } X_I X_J X_I \Pi_I \rangle \\
& - \langle \text{tr } X_I X_I \Pi_J X_J \rangle + \langle \text{tr } X_I X_J X_J \Pi_I \rangle - D \langle \text{tr } X_I X_I \rangle \\
& \langle \text{tr } \Pi_I X_J X_J X_I \rangle + \langle \text{tr } X_I \Pi_J X_J X_I \rangle + \langle \text{tr } X_I X_J \Pi_J X_I \rangle + \langle \text{tr } X_I X_J X_J \Pi_I \rangle \\
& \langle \text{tr } X_I X_I X_J X_J \rangle - \langle \text{tr } X_I X_J X_J X_I \rangle \\
& \langle \text{tr } X_I X_J X_J X_I \rangle - \langle \text{tr } X_I X_I X_J X_J \rangle \\
& \langle \text{tr } \Pi_I \Pi_I \rangle + \langle \text{tr } X_I X_I X_J X_J \rangle - 2 \langle \text{tr } X_I X_J X_I X_J \rangle + \langle \text{tr } X_I X_J X_J X_I \rangle + M^2 \langle \text{tr } X_I X_I \rangle
\end{aligned} \tag{91}$$

and one longer equation corresponds to the vanishing of $\langle [H, \text{tr } \Pi_I \Pi_I] \rangle$:

$$\begin{aligned}
& \langle \text{tr } \Pi_I X_I X_J X_J \rangle - 2 \langle \text{tr } \Pi_I X_J X_I X_J \rangle + \langle \text{tr } \Pi_I X_J X_J X_I \rangle + \langle \text{tr } X_I X_I X_J \Pi_J \rangle - 2 \langle \text{tr } X_I X_J X_I \Pi_J \rangle \\
& + \langle \text{tr } X_I X_J X_J \Pi_I \rangle + M^2 \langle \text{tr } \Pi_I X_I \rangle + M^2 \langle \text{tr } X_I \Pi_I \rangle = 0.
\end{aligned} \tag{92}$$

Solving these equations, we find that only three variables remain independent: $\langle \text{tr } \Pi_I \Pi_I \rangle$, $\langle \text{tr } X_I X_I \rangle$, and

¹⁴Up to level 5, all the equations listed are linear. Quadratic equations arise from the cyclicity properties of the moments, which introduce double-trace operators. The first quadratic equation appears at level 7, when we cyclically permute, for example, the moment $\langle \text{tr } X_I X_J X_J \Pi_I X_K X_K \rangle$:

$$0 = \langle \text{tr } X_I X_J X_J \Pi_I X_K X_K \rangle - \langle \text{tr } X_I X_I \Pi_J X_K X_K X_J \rangle + D \langle \text{tr } X_I X_I \rangle^2 \tag{90}$$

$\langle \text{tr } X_I X_I X_J X_J \rangle$. The remaining variables are determined by:

$$\begin{aligned}
\langle \text{tr } \Pi_I X_I \rangle &\rightarrow -\frac{1}{2}D \\
\langle \text{tr } X_I \Pi_I \rangle &\rightarrow \frac{1}{2}D \\
\langle \text{tr } X_I X_J X_I X_J \rangle &\rightarrow \langle \text{tr } X_I X_I X_J X_J \rangle + \frac{1}{2}M^2 \langle \text{tr } X_I X_I \rangle + \frac{1}{2} \langle \text{tr } \Pi_I \Pi_I \rangle \\
\langle \text{tr } X_I X_J X_J X_I \rangle &\rightarrow \langle \text{tr } X_I X_I X_J X_J \rangle \\
\langle \text{tr } \Pi_I X_I X_J X_J \rangle &\rightarrow -\frac{1}{2}D \langle \text{tr } X_I X_I \rangle - \frac{1}{2} \langle \text{tr } X_I X_I \rangle \\
\langle \text{tr } \Pi_I X_J X_I X_J \rangle &\rightarrow -\langle \text{tr } X_I X_I \rangle \\
\langle \text{tr } \Pi_I X_J X_J X_I \rangle &\rightarrow -\frac{1}{2}D \langle \text{tr } X_I X_I \rangle - \frac{1}{2} \langle \text{tr } X_I X_I \rangle \\
\langle \text{tr } X_I \Pi_I X_J X_J \rangle &\rightarrow \frac{1}{2}D \langle \text{tr } X_I X_I \rangle - \frac{1}{2} \langle \text{tr } X_I X_I \rangle \\
\langle \text{tr } X_I \Pi_J X_I X_J \rangle &\rightarrow 0 \\
\langle \text{tr } X_I \Pi_J X_J X_I \rangle &\rightarrow -\frac{1}{2}D \langle \text{tr } X_I X_I \rangle + \frac{1}{2} \langle \text{tr } X_I X_I \rangle \\
\langle \text{tr } X_I X_I \Pi_J X_J \rangle &\rightarrow -\frac{1}{2}D \langle \text{tr } X_I X_I \rangle + \frac{1}{2} \langle \text{tr } X_I X_I \rangle \\
\langle \text{tr } X_I X_I X_J \Pi_J \rangle &\rightarrow \frac{1}{2}D \langle \text{tr } X_I X_I \rangle + \frac{1}{2} \langle \text{tr } X_I X_I \rangle \\
\langle \text{tr } X_I X_J \Pi_I X_J \rangle &\rightarrow 0 \\
\langle \text{tr } X_I X_J \Pi_J X_I \rangle &\rightarrow \frac{1}{2}D \langle \text{tr } X_I X_I \rangle - \frac{1}{2} \langle \text{tr } X_I X_I \rangle \\
\langle \text{tr } X_I X_J X_I \Pi_J \rangle &\rightarrow \langle \text{tr } X_I X_I \rangle \\
\langle \text{tr } X_I X_J X_J \Pi_I \rangle &\rightarrow \frac{1}{2}D \langle \text{tr } X_I X_I \rangle + \frac{1}{2} \langle \text{tr } X_I X_I \rangle
\end{aligned} \tag{93}$$

F.2 Positivity

To express the positivity conditions, we classify all vectors in the $O(D)$ irreducible representations up to level 2:

$$\begin{aligned}
&1, \quad X_I X_I \\
&X_I, \quad \Pi_I \\
&\quad -\frac{1}{D} \delta_{IJ} X_K X_K + \frac{1}{2} X_I X_J + \frac{1}{2} X_J X_I \\
&\quad \frac{1}{2} X_I X_J - \frac{1}{2} X_J X_I
\end{aligned} \tag{94}$$

These correspond, respectively, to the singlet, vector, traceless-symmetric, and antisymmetric representations. When computing the inner product as defined in (6), vectors belonging to different irreducible representations are orthogonal, as required by group theoretical considerations. The vectors in (94) should be regarded as tensors with abstract indices. The inner product is defined by first taking the trace of the product of matrices with their adjoints, followed by summing over the corresponding abstract indices

from 1 to D . As an example, the corresponding positivity condition for the vectors in (94) is given by:

$$\begin{aligned} \begin{pmatrix} 1 & \langle \text{tr } X_I X_I \rangle \\ \langle \text{tr } X_I X_I \rangle & \langle \text{tr } X_I X_I X_J X_J \rangle \end{pmatrix} &\succeq 0, \quad \begin{pmatrix} \langle \text{tr } X_I X_I \rangle & \langle \text{tr } X_I \Pi_I \rangle \\ -\langle \text{tr } \Pi_I X_I \rangle & -\langle \text{tr } \Pi_I \Pi_I \rangle \end{pmatrix} \succeq 0, \\ -\frac{1}{D} \langle \text{tr } X_I X_I X_J X_J \rangle + \frac{1}{2} \langle \text{tr } X_I X_J X_I X_J \rangle + \frac{1}{2} \langle \text{tr } X_I X_J X_J X_I \rangle &\geq 0, \\ \frac{1}{2} \langle \text{tr } X_I X_J X_J X_I \rangle - \frac{1}{2} \langle \text{tr } X_I X_J X_I X_J \rangle &\geq 0, \end{aligned} \tag{95}$$

Substituting the solution from (93), we obtain the following inequality constraints for the three free variables:

$$\begin{aligned} \begin{pmatrix} 1 & \langle \text{tr } X_I X_I \rangle \\ \langle \text{tr } X_I X_I \rangle & \langle \text{tr } X_I X_I X_J X_J \rangle \end{pmatrix} &\succeq 0, \quad \begin{pmatrix} \langle \text{tr } X_I X_I \rangle & \frac{1}{2} D \\ \frac{1}{2} D & -\langle \text{tr } \Pi_I \Pi_I \rangle \end{pmatrix} \succeq 0, \\ \frac{1}{4} \langle \text{tr } \Pi_I \Pi_I \rangle + \frac{D-1}{D} \langle \text{tr } X_I X_I X_J X_J \rangle + \frac{1}{4} M^2 \langle \text{tr } X_I X_I \rangle &\geq 0 \\ -\frac{1}{4} M^2 \langle \text{tr } X_I X_I \rangle - \frac{1}{4} \langle \text{tr } \Pi_I \Pi_I \rangle &\geq 0 \end{aligned} \tag{96}$$

Similarly, for the ground state positivity condition (7), we have:

$$\begin{aligned} \langle \text{tr } X_I X_I \rangle &\geq 0, \\ \begin{pmatrix} \frac{1}{2} D & -\langle \text{tr } \Pi_I \Pi_I \rangle \\ -\langle \text{tr } \Pi_I \Pi_I \rangle & (D-1) \langle \text{tr } X_I X_I \rangle + \frac{1}{2} D M^2 \end{pmatrix} &\succeq 0, \\ \left(\frac{D+1}{2} - \frac{1}{D} \right) \langle \text{tr } X_I X_I \rangle &\geq 0, \\ \frac{D-1}{2} \langle \text{tr } X_I X_I \rangle &\geq 0. \end{aligned} \tag{97}$$

Hence, we reproduce the analytic result presented in Section 3.1.

F.3 Higher levels

The constraints discussed in this appendix so far are universal for any D . However, as we proceed to higher levels in the bootstrap, additional conditions arise that are non-universal and depend explicitly on the value of D . Thus, the role of D in the bootstrap is not merely as a normalization factor, but also reflects deeper structural features of the theory.

One such example is the vanishing of certain δ products at $D = 2$:

$$0 = \varepsilon_{IJK} \varepsilon_{LMN} = \delta_{IL} \delta_{JM} \delta_{KN} - \delta_{IM} \delta_{JL} \delta_{KN} - \delta_{IL} \delta_{JN} \delta_{KM} - \delta_{IN} \delta_{JM} \delta_{KL} + \delta_{IM} \delta_{JN} \delta_{KL} + \delta_{IN} \delta_{JL} \delta_{KM} \tag{98}$$

The right-hand side implies non-trivial kinematic relations among six-letter words, which explains why there are significantly fewer free variables in $D = 2$ compared to $D = 9$, as shown in Table 1. A similar truncation occurs for the positivity conditions when considering the irreducible decomposition of operators. The first such example arises from the fact that there is no $(D+1)$ -indexed antisymmetric irreducible representation in an $O(D)$ symmetric matrix model, as guaranteed by the pigeonhole principle.

As an example of higher levels, we present the level 6 irreducible representation vectors corresponding to the following Young diagram (assuming $D \geq 3$):

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \tag{99}$$

$$\begin{aligned}
& - \frac{1}{2} \frac{1}{D-1} \delta_{IJ} \text{tr } X_L X_L X_K + \frac{1}{4} \frac{1}{D-1} \delta_{IJ} \text{tr } X_K X_L X_L + \frac{1}{4} \frac{1}{D-1} \delta_{IJ} \text{tr } X_L X_K X_L \\
& - \frac{1}{4} \frac{1}{D-1} \delta_{JK} \text{tr } X_I X_L X_L - \frac{1}{4} \frac{1}{D-1} \delta_{JK} \text{tr } X_L X_I X_L + \frac{1}{2} \frac{1}{D-1} \delta_{JK} \text{tr } X_L X_L X_I \\
& + \frac{1}{4} \text{tr } X_I X_J X_K + \frac{1}{4} \text{tr } X_J X_I X_K - \frac{1}{4} \text{tr } X_J X_K X_I - \frac{1}{4} \text{tr } X_K X_J X_I, \\
& - \frac{1}{2} \frac{1}{D-1} \delta_{IJ} \text{tr } X_L X_K X_L + \frac{1}{4} \frac{1}{D-1} \delta_{IJ} \text{tr } X_K X_L X_L + \frac{1}{4} \frac{1}{D-1} \delta_{IJ} \text{tr } X_L X_L X_K \\
& - \frac{1}{4} \frac{1}{D-1} \delta_{JK} \text{tr } X_I X_L X_L - \frac{1}{4} \frac{1}{D-1} \delta_{JK} \text{tr } X_L X_L X_I + \frac{1}{2} \frac{1}{D-1} \delta_{JK} \text{tr } X_L X_I X_L \\
& + \frac{1}{4} \text{tr } X_I X_K X_J - \frac{1}{4} \text{tr } X_J X_I X_K + \frac{1}{4} \text{tr } X_J X_K X_I - \frac{1}{4} \text{tr } X_K X_I X_J,
\end{aligned} \tag{100}$$

For $D = 2$ the general irrep decomposition is in terms of charge. Let us consider an operator \mathcal{O}_k with $U(1)$ charge k . It is related to an operator $\mathcal{O}_{I_1 I_2 \dots I_k}$ in the fully symmetric representation via

$$v^I = (1, i), \quad \delta_{IJ} v^I v^J = 0 \tag{101}$$

$$\mathcal{O}_k = v^{I_1} v^{I_2} \dots v^{I_k} \mathcal{O}_{I_1 \dots I_k}. \tag{102}$$

For example, let \mathcal{O}_1 and \mathcal{O}_2 be invariant under the $O(2)$ transformations. Then the $k = 2$ the operator $\mathcal{O}_1 Z \mathcal{O}_2 Z$ may be written

$$\mathcal{O}_1 Z \mathcal{O}_2 Z = v^I v^I \left[\frac{1}{2} (\mathcal{O}_1 X_I \mathcal{O}_2 X_J + \mathcal{O}_1 X_J \mathcal{O}_2 X_I) - \frac{1}{2} \delta_{IJ} \mathcal{O}_1 X_K \mathcal{O}_2 X_K \right]. \tag{103}$$

Similarly, a charge $-k$ operator \mathcal{O}_{-k} can be obtained in a similar fashion by using the vectors $\bar{v}^I = (1, -i)$. Since the symmetry of the model is actually $O(2)$ and not just $U(1)$, the irreps are actually 2-dimensional for $k \neq 0$, spanned by $\{\mathcal{O}_k, \mathcal{O}_{-k}\}$, with the reflection element $r \in O(2)$ acting by $U_k(r) \mathcal{O}_k = \mathcal{O}_{-k}$. So the irreps of $O(2)$ may be labeled by just a non-negative integer $k \geq 0$.

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