Hillary Yang

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Operations on Surreal Numbers

Addition

Definition. *For surreals $x = \langle x^L \mid x^R \rangle$ and $y = \langle y^L \mid y^R \rangle$, we define $x + y = \langle x + y^L \cup x^L + y \mid x + y^R \cup x^R + y \rangle$.*

Motivation: By Noah's aspiration, we want the left set of $x + y$ to be less than $x + y$ itself. Also by Noah's aspiration, we know that every element of y^L is less than y, so $x + y^L < x + y$, making it a good lower bound for $x + y$.

By similar logic, $x^L + y < x + y$. Taking the union of these two sets gives us the left set of $x + y$, which is $x + y^L \cup x^L + y$.

Similarly, for the upper bound, we want $(x + y)^R$ to exceed $x + y$. "Boosting" each of x and y up to their right sets gives us $x + y^R \cup x^R + y$, completing our definition for addition.

Negation

Definition. *For a surreal number x, $-x = \langle -x^R \mid -x^L \rangle$.*

If we take $x + y = 0$, then we want to find $y = -x$.

Using our definition for audition, we have $\langle x + y^L \cup x^L + y \mid x + y^R \cup x^R + y \rangle = 0$. By the Squeeze Theorem, we want $x^L + y < 0$, because it's in the left set of the sum. Similarly, we want $x^R + y > 0$ because it's in the right set.

From these two inequalities, we have $y < -x^L$ and $y > -x^R$, which give us our lower and upper bounds for $-x$. Using Noah's aspiration, we have $-x^R < y < -x^L$ and $y = -x = \langle -x^R \mid -x^L \rangle$.

Multiplication

Definition. *For surreals x and y, $xy = \langle xy^L + x^L y - x^L y^L \cup xy^R + x^R y - x^R y^R \mid xy^R + x^L y - x^L y^R \cup xy - xy^L - x^R y + x^R y^L \rangle$.*

Four statements that we know to be true by Noah's aspiration:

1) $x - x^L > 0$
2) $x - x^R < 0$
3) $y - y^L > 0$
4) $y - y^R < 0$.

Lemma. *If surreals a and b are both positive or both negative, then $ab > 0$. Otherwise, $ab < 0$.*

With this lemma, we know that the result from multiplying 1) and 3) is positive.

$$(x - x^L)(y - y^L) > 0$$

$$xy - xy^L - x^L y + x^L y^L > 0$$

$$xy > xy^L + x^L y - x^L y^L$$

which is a lower bound for the product.

Multiplying 1) and 4) gives us: $xy < xy^R + x^L y - x^L y^R$.
2) and 3): $xy < xy - xy^L - x^R y + x^R y^L$.
2) and 4): $xy > xy^R + x^R y - x^R y^R$.

This gives us two lower bounds and two upper bounds which we union to get the product xy.

Reciprocation

Definition. *For surreal y, we have*

$$\frac{1}{y} = \langle 0 \cup \frac{(\frac{1}{y})^R (y^L - y) + 1}{y^L} \cup \frac{(\frac{1}{y})^L (y^R - y) + 1}{y^R} \mid \frac{(\frac{1}{y})^R (y^R - y) + 1}{y^R} \cup \frac{(\frac{1}{y})^L (y^L - y) + 1}{y^L} \rangle.$$

Reciprocation follows many of the same procedures as multiplication. We have the four statements:

1) $y^L - y < 0$
2) $y^R - y > 0$
3) $(\frac{1}{y})^L - \frac{1}{y} < 0$
4) $(\frac{1}{y})^R - \frac{1}{y} > 0$.

Again, we multiply pairs to obtain:

1) $\dfrac{1}{y} > \dfrac{(\frac{1}{y}^R)(y^L - y) + 1}{y^L}$
2) $\dfrac{1}{y} > \dfrac{(\frac{1}{y}^L)(y^R - y) + 1}{y^R}$
3) $\dfrac{1}{y} < \dfrac{(\frac{1}{y}^L)(y^L - y) + 1}{y^L}$
4) $\dfrac{1}{y} < \dfrac{(\frac{1}{y}^R)(y^R - y) + 1}{y^R}$.

However, if we directly union these as we did for our previous definitions, we find an issue. Included in the definition of $(\frac{1}{y})^L$ is a call for $(\frac{1}{y})^R$, and vice versa. Neither is defined, as both of $\frac{1}{y}$'s sets are initially empty!

The solution is to union the left set with 0, which inserts a 0 into our left set. This then allows an element of $(\frac{1}{y})^R$ to be defined, which defines an element of $(\frac{1}{y})^L$, and so on. This can be thought of as a "bouncing" between the left and right sets, recursively defining elements in each.

Square Roots

(I'm not completely sure about this one, but I did find some results.)

Definition. *For surreal $x > 1$, we have*

$$y = \sqrt{x} = \langle 1 \cup \frac{y^R y^L + y^2}{y^L + y^R} \mid \frac{(y^L)^2 + y^2}{2y^L} \cup \frac{(y^R)^2 + y^2}{2y^R} \rangle.$$

We know the following two statements about y, y^L, and y^R.

1) $y - y^L > 0$
2) $y - y^R < 0$.

Multiplying 1) by 1) gives us

$$(y - y^L)^2 > 0$$
$$y^2 - 2yy^L + (y^L)^2 > 0$$
$$y < \frac{y^2 + (y^L)^2}{2y^L}$$

Similarly, multiplying 2) by 2) yields

$$y < \frac{y^2 + (y^R)^2}{2y^R}$$

which should be in the right set/be the upper bound of y.

Then, for the lower bound, we combine 1) and 2):

$$(y - y^L)(y - y^R) < 0$$
$$y^2 - y^R y - y^L y + y^L y^R < 0$$
$$y > \frac{y^2 + y^L y^R}{y^L + y^R}$$

which gives us a left set.

The reason for the 1 in the union is very similar to that for reciprocation. We can't start recursively defining the square root without any element initially in the left set. Since we know that $y > 1$, (because $x > 1$) we can add a 1 to the left set without any issues.

Hillary Yang

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HCSSiM Workshop Final Writeup

Problem Set 9: Big, Bigger, Biggest #1

Hi

We will use $A \sim B$ to denote a perfect matching between A and B.

To compare the sizes of $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, we can relate $\mathbb{Z}$ to each of the other two sets. Furthermore, since $\mathbb{Z} \sim \mathbb{N}$, (shown below) we can use $\mathbb{N}$ as an "intermediary" for $\mathbb{Z}$: if we show $\mathbb{N} \sim X$, then $\mathbb{Z} \sim X$, where X is some set. In other words, since $|\mathbb{Z}| = |\mathbb{N}|$, if we show that $|\mathbb{N}| = |X|$, then $|\mathbb{Z}| = |X|$ and there exists a perfect matching between these sets.

Mapping $\mathbb{N}$ to $\mathbb{Z}$

$\mathbb{N}$	7	5	3	1	2	4	6
$\mathbb{Z}$	-3	-2	-1	0	1	2	3

Mapping $\mathbb{N}$ to $\mathbb{Q}$

First consider $\mathbb{Q}^+$. We can represent the elements of this set in the first quadrant, starting from the origin and extending infinitely up and right. The value at grid location (x, y) is given by $\frac{y}{x}$, ensuring that every element of $\mathbb{Q}^+$ is represented. A diagram illustrating this is shown below.

$\vdots$	$\vdots$	$\vdots$	
$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\cdots$
$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\cdots$
$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\cdots$

Notice that we can map $\mathbb{Q}^+$ to $\mathbb{N}^2$ using this layout. Along each of the rows, we have the denominators corresponding exactly to $\mathbb{N}$, and along the columns with the numerators. Therefore, $\mathbb{Q}^+ \sim \mathbb{N}^2$.

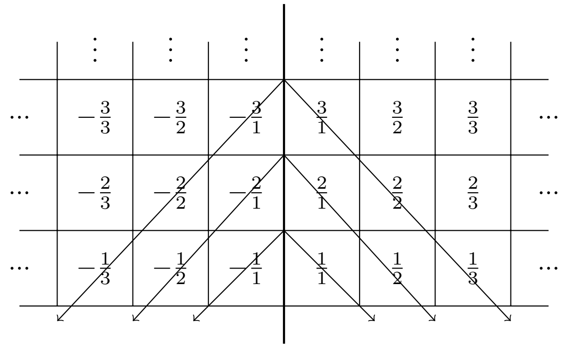
Extending this to $\mathbb{Q}$, we must now consider 0 and the negative rationals. Zero is negligible, since with any matching, we can always shift it right to accommodate any finite amount of elements.

Considering how we mapped $\mathbb{Z}$ to $\mathbb{N}$ with the alternating positive/negative technique (shown above), we can apply the same idea here by representing $\mathbb{Q}$ with the first two quadrants, then alternating to map them entirely to $\mathbb{N}$.

	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\cdots$	$-\frac{3}{3}$	$-\frac{3}{2}$	$-\frac{3}{1}$	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\cdots$
$\cdots$	$-\frac{2}{3}$	$-\frac{2}{2}$	$-\frac{2}{1}$	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\cdots$
$\cdots$	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{1}{1}$	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\cdots$

Our mapping procedure should start along the bottom row, with $\frac{1}{1}$ mapping to 1, $-\frac{1}{1}$ to 2, $\frac{1}{2}$ to 3, and so on. However, we run into a problem. Each row has infinite length, so we'll never finish mapping even the first row. Therefore, this mapping does not work.

Instead, we can travel along the diagonals while alternating between quadrants I and II, similar to what we did with the integers. Such a mapping is shown below.



Because of their finite lengths, we can always reach the end of each diagonal. Therefore, this mapping can be completed and is a successful perfect matching between $\mathbb{N}$ and $\mathbb{Q}$.

Mapping $\mathbb{N}$ to $\mathbb{R}$

There are infinitely many reals in the set $[0,1]$, which we will assume can be mapped to $\mathbb{N}$. Considering their decimal representations, we have:

$$r_1 = 0.a_1a_2a_3a_4...$$
$$r_2 = 0.b_1b_2b_3b_4...$$
$$r_3 = 0.c_1c_2c_3c_4...$$
$$...$$

where $a, b, c...$ are digits from $0-9$ and the table above extends infinitely down and to the right. However, we can show that there cannot exist a mapping from $\mathbb{N}$ to this list, by the fact that we can always find an element x of $\mathbb{R}$ in $[0,1]$ which has not been previously mapped. This is done by taking the first digit of r_1, a_1, and appending to x a digit that does not equal a_1, then for the second digit of r_2, adding some digit that does not equal b_2, and so on.

The resulting x is, by virtue of its construction, a decimal that differs from every element in our existing list by at least one digit. Therefore, there is always a new element in $[0,1]$ that has not been mapped. This implies that $\mathbb{N} \nsim \mathbb{R}$.

Therefore, we have shown that $\mathbb{N} \sim \mathbb{Q}$ and $\mathbb{N} \nsim \mathbb{R}$. Therefore, $\mathbb{Z} \sim \mathbb{Q}$ and $\mathbb{Z} \nsim \mathbb{R}$, implying that $|\mathbb{Z}| = |\mathbb{Q}|$.