Ch 10: Recursion in Python

- Concept of Recursion
- Recursion Practices
- Divide and Conquer
- Measuring Program Complexity

Definition of Recursion Functions

- A recursive function is one that calls itself.

```
def i_am_recursive(x) :
    maybe do some work
    if there is more work to do :
        i_am_recursive(next(x))
    return the desired result
```

- Infinite loop? Not necessarily, not if `next(x)` needs less work than x.

Recursive Definition [1/2]

- A description of something that refers to itself is called a *recursive* definition

$$n! = n(n-1)(n-2)...(1)$$

$$n! = n(n-1)!$$

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n(n-1)! & \text{otherwise} \end{cases}$$

base case

recursive case

- A recursive definitions should have two key characteristics:
  - There are one or more base cases for which no recursion is applied
  - All chains of recursion eventually end up at one of the base cases

Recursive Definition [2/2]

- Every recursive function definition includes two parts:
  - Base case(s) (non-recursive)
    One or more simple cases that can be done right away
  - Recursive case(s)
    One or more cases that require solving "simpler" version(s) of the original problem.
    - By "simpler", we mean "smaller" or "shorter" or "closer to the base case".

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Recursive Computation Example: Factorial

- $n! = n \times (n-1) \times (n-2) \times \cdots \times 1$

  $2! = \quad 2 \times 1$

  $3! = \quad 3 \times 2 \times 1$

  $4! = \quad 4 \times 3 \times 2 \times 1$

- alternatively:

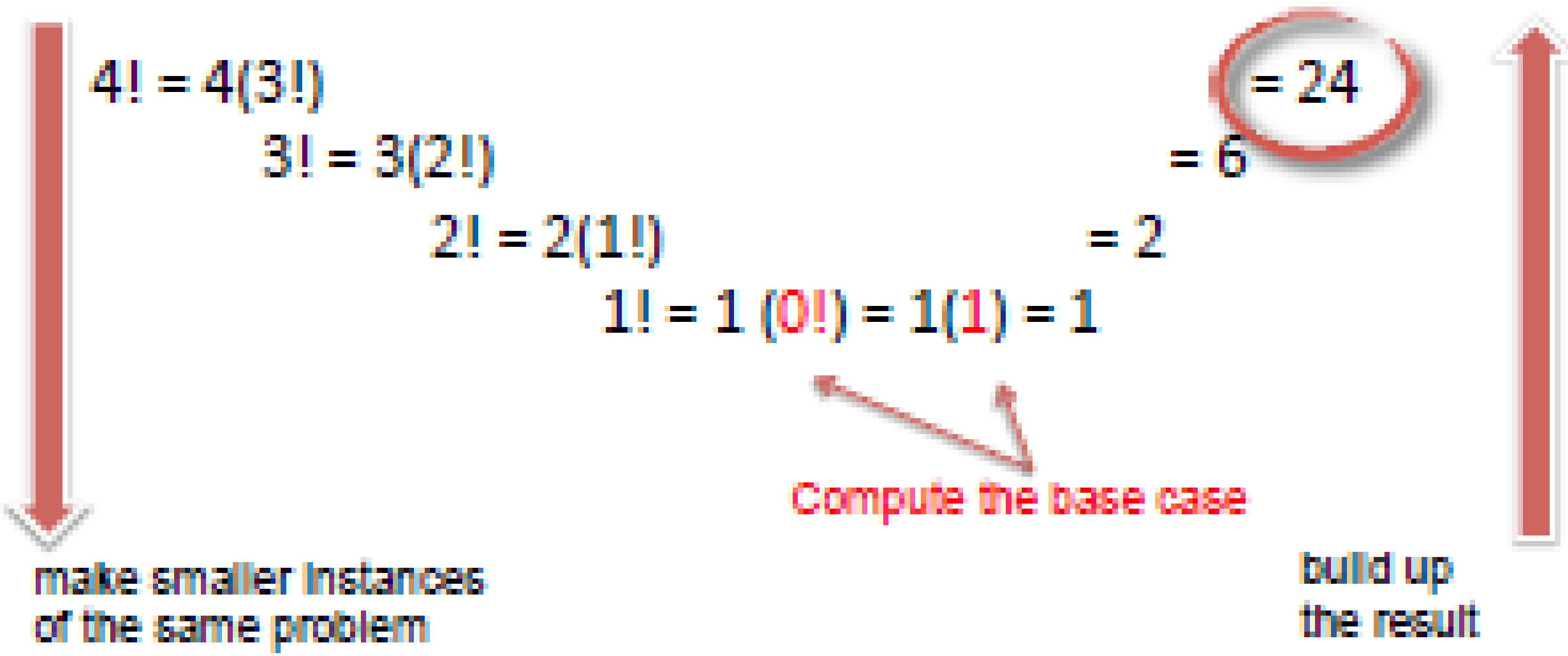
  $0! = 1$ (Base case)

  $n! = n \times (n-1)!$ (Recursive case)

  So $4! = 4 \times 3!$

  And $3! = 3 \times 2!$, $2! = 2 \times 1!$, $1! = 1 \times 0!$

Conceptual Understanding of Recursion



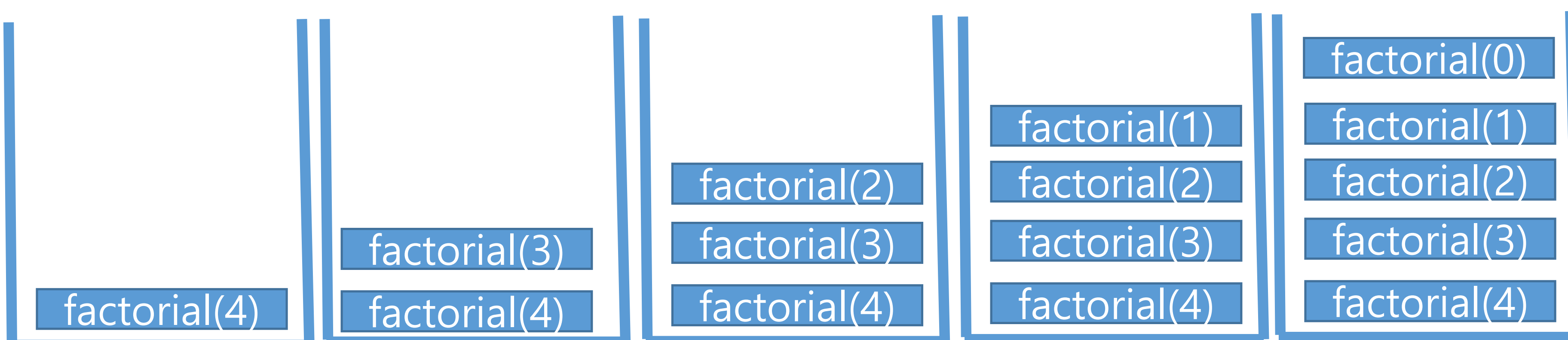
Recursive Factorial Function in Python

```
# 0! = 1 (Base case)
# n! = n × (n-1)! (Recursive case)
def factorial(n):
    if n == 0:      # base case
        return 1
    else:           # recursive case
        return n * factorial(n-1)
```

Inside Python Recursion Processing

S T A C K

```
n=4    factorial(4)? = 4 * factorial(3)
n=3    factorial(3)? = 3 * factorial(2)
n=2    factorial(2)? = 2 * factorial(1)
n=1    factorial(1)? = 1 * factorial(0)
n=0    factorial(0) = 1
```



Recursive Solution vs. Iterative Solution

- For every recursive function, there is an equivalent iterative solution.
- For every iterative function, there is an equivalent recursive solution.
- But some problems are easier to solve one way than the other way.
- And be aware that most recursive programs need space for the stack, behind the scenes

Recursion in Python

- Concept of Recursion
- Recursion Practices
- Divide and Conquer
- Measuring Program Complexity

Iterative Version vs. Recursive Version

Factorial Function in Python (Iterative)

```
def factorial(n):
    result = 1    # initialize accumulator var
    for i in range(1, n+1):
        result = result * i
    return result
```

Factorial Function in Python (Recursive)

```
def factorial(n):
    if n == 0:     # base case
        return 1
    else:          # recursive case
        return n * factorial(n-1)
```

Recursion on Lists: Sum of a List [1/2]

- First we need a way of getting a smaller input from a larger one:
  - Forming a sub-list of a list:

```
>>> a = [1, 11, 111, 1111, 11111, 111111]
>>> a[1:]
[11, 111, 1111, 11111, 111111]
>>> a[2:]
[111, 1111, 11111, 111111]
>>> a[3:]
[1111, 11111, 111111]
>>> a[3:5]
[1111, 11111]
>>>
```

the "tail" of list a

Recursive Sum of a List

```
def sumlist(items):
    if items == []:
        return 0
    else:
        return items[0] + sumlist(items[1:])
```

What if we already know the sum of the list's tail? We can just add the list's first element!

Recursion on Lists: Sum of a List [2/2]

Tracing sumlist

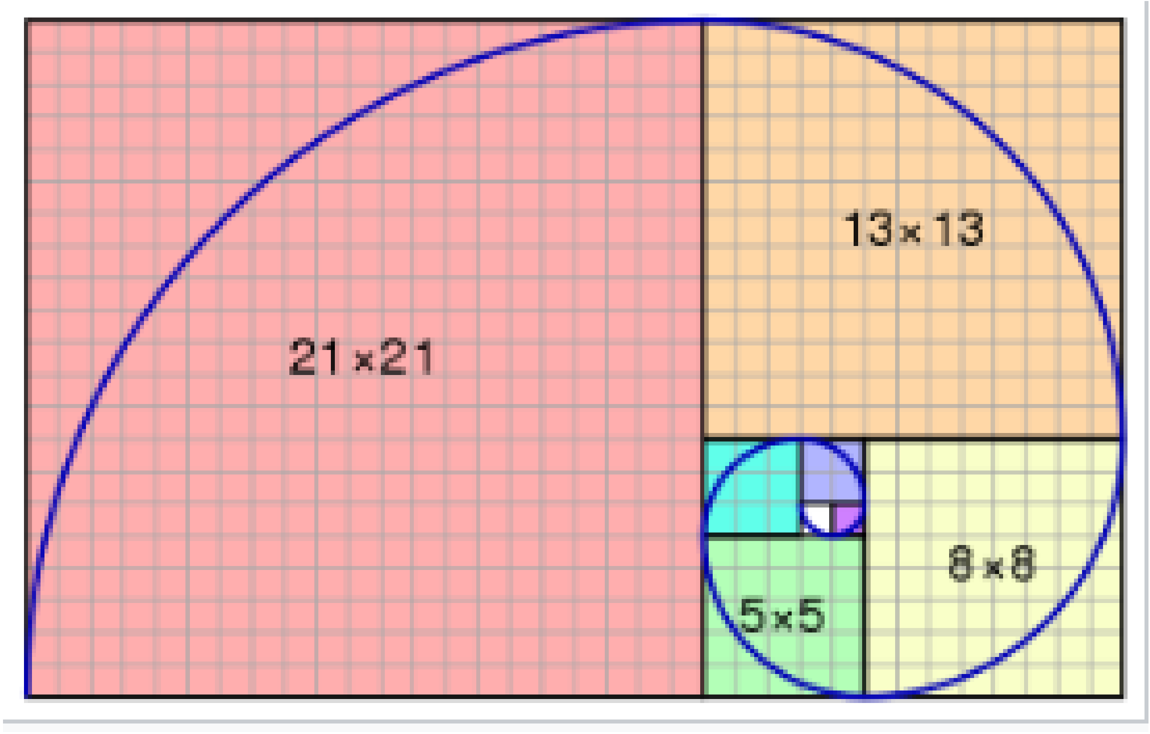
```
def sumlist(items):
    if items == []:
        return 0
    else:
        return items[0] + sumlist(items[1:])
```

```
>>> sumlist([2,5,7])
sumlist([2,5,7]) = 2 + sumlist([5,7])
                       5 + sumlist([7])
                           7 + sumlist([])
                                   0
```

After reaching the base case, the final result is built up by the computer by adding 0+7+5+2.

Fibonacci Number

- Fibonacci (known as Leonardo of Pisa)
- Fibonacci introduced the following pattern in his 1202 Book



The Fibonacci spiral: an approximation of the golden spiral created by drawing circular arcs connecting the opposite corners of squares in the Fibonacci tiling; this one uses squares of sizes 1, 1, 2, 3, 5, 8, 13 and 21.

Multiple Recursive Calls: Fibonacci Numbers

$$fib(n) = fib(n-1) + fib(n-2), \qquad n > 1$$

- A sequence of numbers:

  0
  1
  1
  2
  3
  5
  8
  13
  ...

Recursive Definition of Fibonacci Numbers

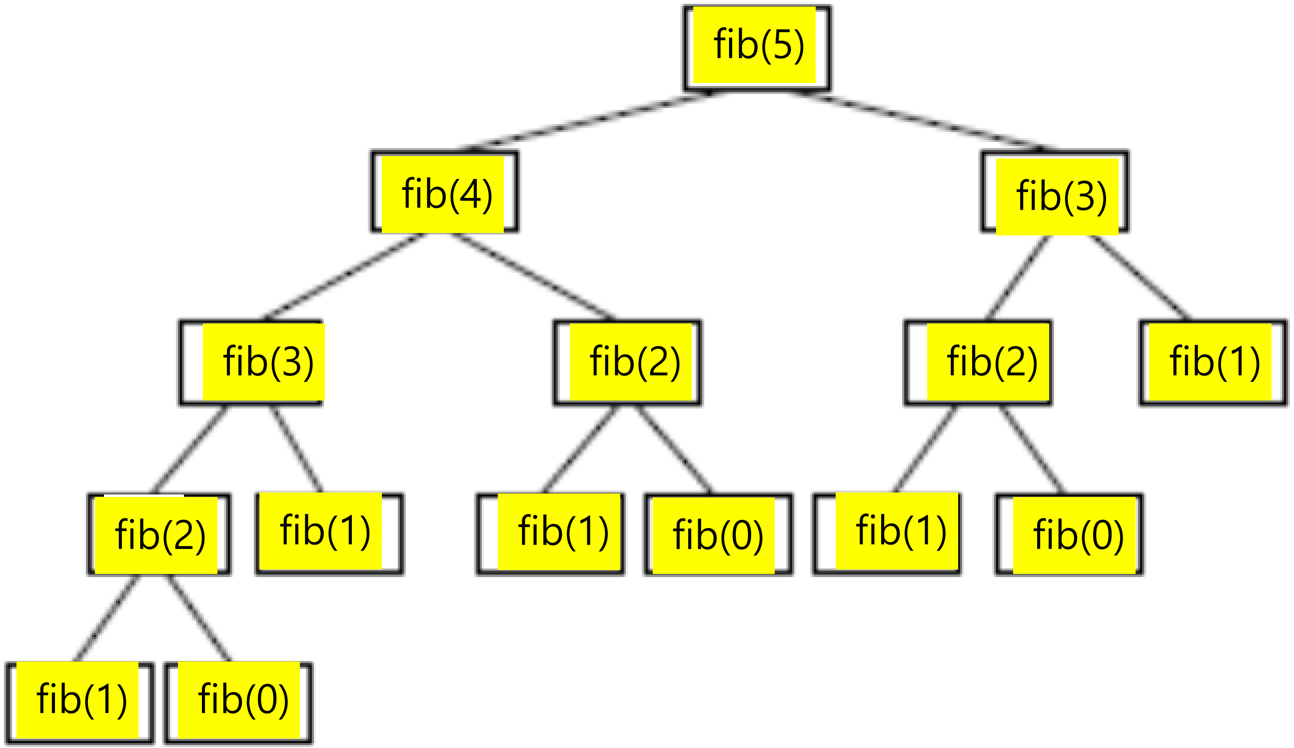
- Let fib(n) = the nth Fibonacci number, $n \geq 0$
  - fib(0) = 0 (base case)
  - fib(1) = 1 (base case)
  - fib(n) = fib(n-1) + fib(n-2), $n > 1$

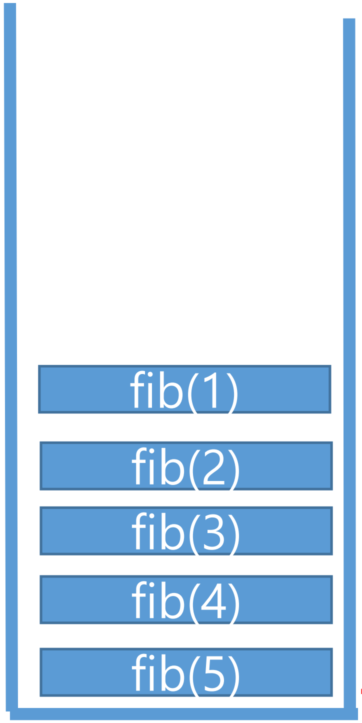
```
def fib(n):
    if n == 0 or n == 1:
        return n
    else:
        return fib(n-1) + fib(n-2)
```

Two recursive calls!

Recursive Call Tree of Fibonacci Number

- Let fib(n) = the nth Fibonacci number, $n \geq 0$
  - fib(0) = 0 (base case)
  - fib(1) = 1 (base case)
  - fib(n) = fib(n-1) + fib(n-2), $n > 1$

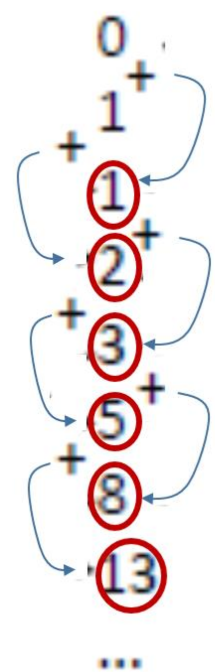




Iterative Fibonacci Python Function

```python
def fib(n):
    x = 0
    next_x = 1
    for i in range(1,n+1):
        x, next_x = next_x, x + next_x
    return x
```

Simultaneous Assignment



Recursion on String: String Reversal [1]

- Write a function to reverse a given string
  - Divide it up into a first character and “all the rest”
  - Reverse the “rest” and append the first character to the end

```
>>> def reverse(s):
      return reverse(s[1:]) + s[0]

>>> reverse("Hello")

Traceback (most recent call last):
File "<pyshell#6>", line 1, in -toplevel-
      reverse("Hello")
File "C:/Program Files/Python 2.3.3/z.py", line 8, in reverse
      return reverse(s[1:]) + s[0]
File "C:/Program Files/Python 2.3.3/z.py", line 8, in reverse
      return reverse(s[1:]) + s[0]
 …
File "C:/Program Files/Python 2.3.3/z.py", line 8, in reverse
      return reverse(s[1:]) + s[0]
RuntimeError: maximum recursion depth exceeded
```

- What happened? There were 1000 lines of errors!

Recursion on String: String Reversal [2]

```
>>> def reverse(s):
     if s == "":
        return s
     else:
        return reverse(s[1:]) + s[0]


>>> reverse("Hello")
'olleH'
```

- Python stops it at 1000 calls, the default "maximum recursion depth."
  - Each time a function is called it takes some memory.

Recursion with Stack Trace: Factorial Function

Iterative Solution

```
def factorial(n):
    factorial = 1
    for i in range(2,n+1):
        factorial *= i
    return factorial

print( factorial(5))
```

Recursive Solution

```
def factorial(n):
    if (n < 2):
        return 1
    else:
        return n*factorial(n-1)

print( factorial(5))
```

Recursive Solution with Stack Trace

```
def factorial(n, depth=0):
    print(“  “ *depth, ”factorial(”,  n,  ”):” )
    if (n < 2):
        result = 1
    else:
        result = n*factorial(n-1,depth+1)
    print(“  “*depth,”→”,  result)
    return result

print( factorial(5))
```

Recursion call 할때 마다 depth가 깊어지는것을 보여주는 print문

" " * depth ➔ blank space

Recursion with Stack Trace: Reverse Function

Iterative Solution

```
>>> def reverse(s):
        reverse = ""
        for ch in s:
                reverse = ch + reverse
        return reverse

>>>
>>> print(reverse("abcd"))
dcba
```

Recursive Solution

```
>>> def reverse(s):
        if (s == ""):
                return ""
        else:
                return reverse(s[1:]) + s[0]


>>>
>>> print(reverse("abcd"))
```

Recursive Solution with Stack Trace

```
>>> def reverse(s, depth = 0):
        print(" "*depth, "reverse(", s, ")")
        if (s == ""):
                return ""
        else:
                return reverse(s[1:], depth + 1) + s[0]
        print(" "*depth, "-->", result)
        return result

>>>
>>> print(reverse("abcd"))
 reverse( abcd )
  reverse( bcd )
   reverse( cd )
    reverse( d )
     reverse(  )
dcba
```

Recursion call 할때 마다 depth가 깊어지는것을 보여주는 print문

Recursion with Stack Trace: Greatest Common Denominator (GCD) Function

Iterative Solution

```
def gcd(x,y):
    while (y > 0):
        oldX = x
        x = y
        y = oldX % y
    return x

print(gcd(500,420))      # 20
```

Recursive Solution

```
def gcd(x,y):
    if (y == 0):
        return x
    else:
        return gcd(y,x%y)

print(gcd(500,420))      # 20
```

Recursive Solution with Stack Trace

```
def gcd(x,y,depth=0):
    print("  "*depth, "gcd(", x, ",",  y, "):")
    if (y == 0):
        result = x
    else:
        result = gcd(y,x%y,depth+1)
    print("  " *depth, "→",  result)
    return result

print gcd(500, 420) # 20
```

Recursion call 할때 마다 depth가 깊어지는것을 보여주는 print문

π Computation in Python [1/4]

π : Mathematical Constant
Many Many Approximations by scholars in Math society

Bailey–Borwein–Plouffe formula

$$\pi = \sum_{i=0}^{\infty} \left[\frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right) \right].$$

Bellard's formula

$$\pi = \frac{1}{2^6} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10n}} \left(-\frac{2^5}{4n+1} - \frac{1}{4n+3} + \frac{2^8}{10n+1} - \frac{2^6}{10n+3} - \frac{2^2}{10n+5} - \frac{2^2}{10n+7} + \frac{1}{10n+9} \right)$$

and

Chudnovsky algorithm

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! (13591409 + 545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}}.$$

π Computation in Python [2/4]

The Algebraic Genius of Euler (1707, Switzerland)

· The Basel Problem

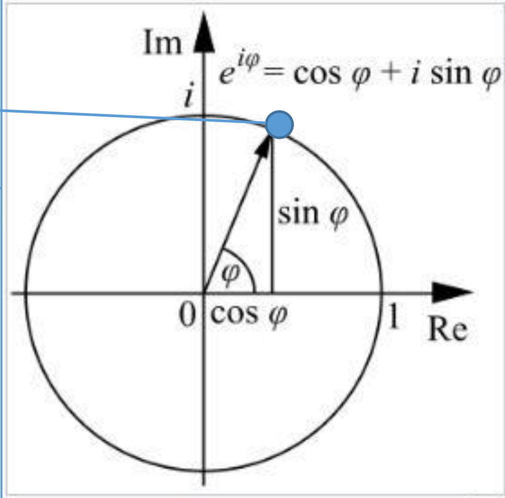
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{n\to\infty} \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \right) = \frac{\pi^2}{6}$$

· Euler's formula

($\cos\varphi$, $\sin\varphi$)

He also defined the exponential function for complex numbers, and discovered its relation to the trigonometric functions. For any real number φ (taken to be radians), Euler's formula states that the complex exponential function satisfies

$$e^{i\varphi} = \cos\varphi + i\sin\varphi.$$



A geometric interpretation of Euler's formula

π Computation in Python [3/4]

Iterative Version of π Computation

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots +$$

$$\pi = \sqrt{6 * \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \right)}$$

```
def pi_series_iter(n) :
    result = 0
    for i in range(1, n+1) :
        result = result + 1/(i**2)
    return result

def pi_approx_iter(n) :
    x = pi_series_iter(n)
    return (6*x)**(.5)
```

π Computation in Python [4/4]

Recursive Version of π Computation

```
def pi_series_r(i) :
    assert(i >= 0)
    # base case
    if i == 0:
        return 0
    # recursive case
    else:
        return pi_series_r(i-1) + 1 / i**2

def pi_approx_r(n) :
    x = pi_series_r(n)
    return (6*x)**(.5)
```

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots +$$

$$\pi = \sqrt{6 * \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \right)}$$

```
def test_pi_approx() :
    # Python's default stack depth limit is 1000, so we can't compute pi_approx_r(1000)
    for i in range(996) :
        assert(pi_approx_r(i) == pi_approx_iter(i))
    print("Done testing pi approximations")
```

Recursion in Python

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Family of Algorithms

- Greedy Methods
- Divide and Conquer
- Dynamic Programming
- Branch and Bound
- Back Tracking

전통적인 Computer Science Algorithms

- Machine Learning Algorithm
- Genetic Algorithm
- Randomized Algorithm

Approximation과 Prediction을 하는 Algorithms

- Mathematical Programming
  - Integer Programming
  - Linear Programming
  - Non-Linear Programming
  - Unconstrained Extrema
  - Constrained Extrema

Applied Mathematics or Industrial Engineering에서 하는 Algorithms

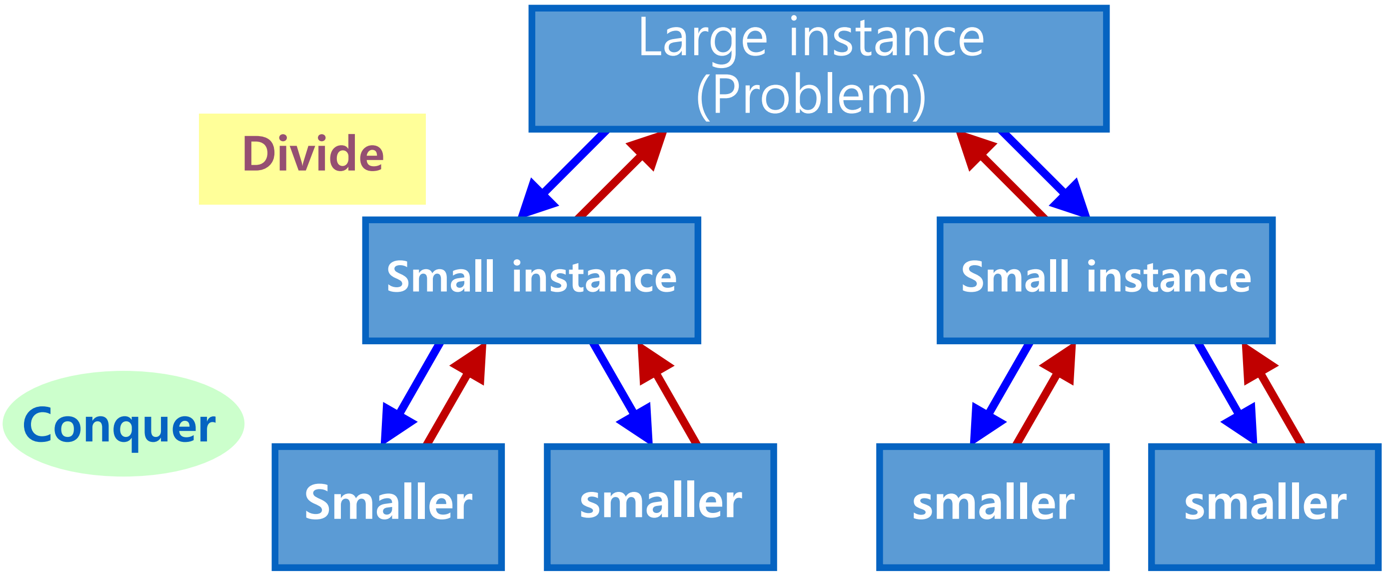
Divide and Conquer

- In computation:
  - Divide the problem into "simpler" versions of itself.
  - Conquer each problem using the same process (usually recursively).
  - Combine the results of the "simpler" versions to form your final solution.
- Problems that fit well with Divide & Conquer recursion style
  - Tower of Hanoi
  - Fractals
  - Binary Search
  - Merge Sort
  - Quicksort
  - Many many more .....

Divide and Conquer style programming은 recursion이 자연스럽다!

Divide and Conquer Style Algorithm

- Distinguish between small and large instances
- Small instances solved differently from large ones
- All instances are non-overlapping



Recursion Example: Fast Exponentiation [1]

- One way to compute a^n : multiply a by itself n times

```
def loopPower(a, n):
    ans = 1
    for i in range(n):
        ans = ans * a
    return ans
```

- Another way to compute a^n : divide and conquer!
  - $a^n = a^{n//2}(a^{n//2})$?

$$a^n = \begin{cases} a^{n//2}(a^{n//2}) & \text{if } n \text{ is even} \\ a^{n//2}(a^{n//2})(a) & \text{if } n \text{ is odd} \end{cases}$$

  - $2^8 = 2^4(2^4)$
  - $2^9 = 2^4(2^4)2$

Recursion Example: Fast Exponentiation [2]

```
def recursivePower(a, n) :
    # raises a to the int power n
    if n ==  0:
        return 1
    else:
        factor = recursivePower(a, n//2)
        if n%2 == 0:                           # n is even
            return factor * factor
        else:                                  # n is odd
            return factor * factor * a
```

- The temporary variable *factor* is used so that we don't need to calculate $a^{n//2}$ more than once

Sorting Algorithms

- The sorting problem
  - take a list of *n* elements
  - and rearrange it so that the values are in increasing (or decreasing) order.

- *Selection sort*
  - For *n* elements, we find the smallest value and put it in the 0^{th} position.
  - Then we find the smallest remaining value from position 1 to (*n*-1) and put it into position 1.
  - The smallest value from position 2 to (*n*-1) goes in position 2.
  - …

Naive Sorting: Selection Sort [1/2]

6	4	1	2	10	11	12	7	4
0	1	2	3	4	5	6	7	8

bottom

1	4	6	2	10	11	12	7	4
0	1	2	3	4	5	6	7	8

bottom

1	2	6	4	10	11	12	7	4
0	1	2	3	4	5	6	7	8

bottom

1	2	4	6	10	11	12	7	4
0	1	2	3	4	5	6	7	8

bottom

Naive Sorting: Selection Sort [2/2]

```
def selSort(nums):
   # sort nums into ascending order

   n = len(nums)

   # For each position in the list (except the very last)

   for bottom in range(n-1):
       # find the smallest item in nums[bottom]...nums[n-1]

       mp = bottom                      # bottom is smallest initially
       for i in range(bottom+1, n):         # look at each position
           if nums[i] < nums[mp]:           # for loop 이 끝날때 가장 작은값의 Index는 i
               mp = i                       # i를 mp에 저장

       # swap the smallest item to the bottom
       nums[bottom], nums[mp] = nums[mp], nums[bottom]
```

bottom mp

nums [29, 64, 73, 34, 20,]

20, 64, 73, 34, 29,

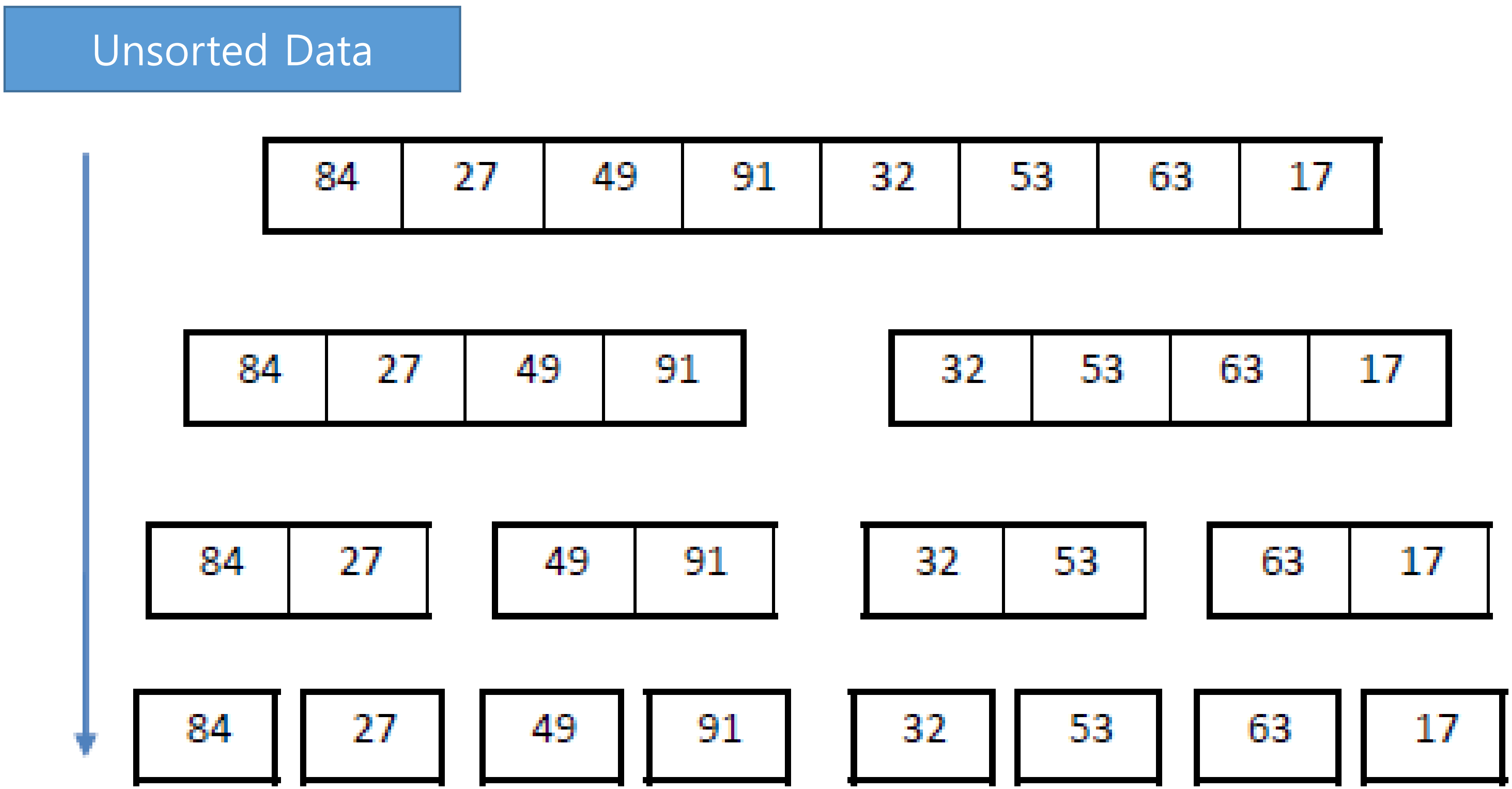
20, 29, 73, 34, 64

20, 29, 34, 73, 64

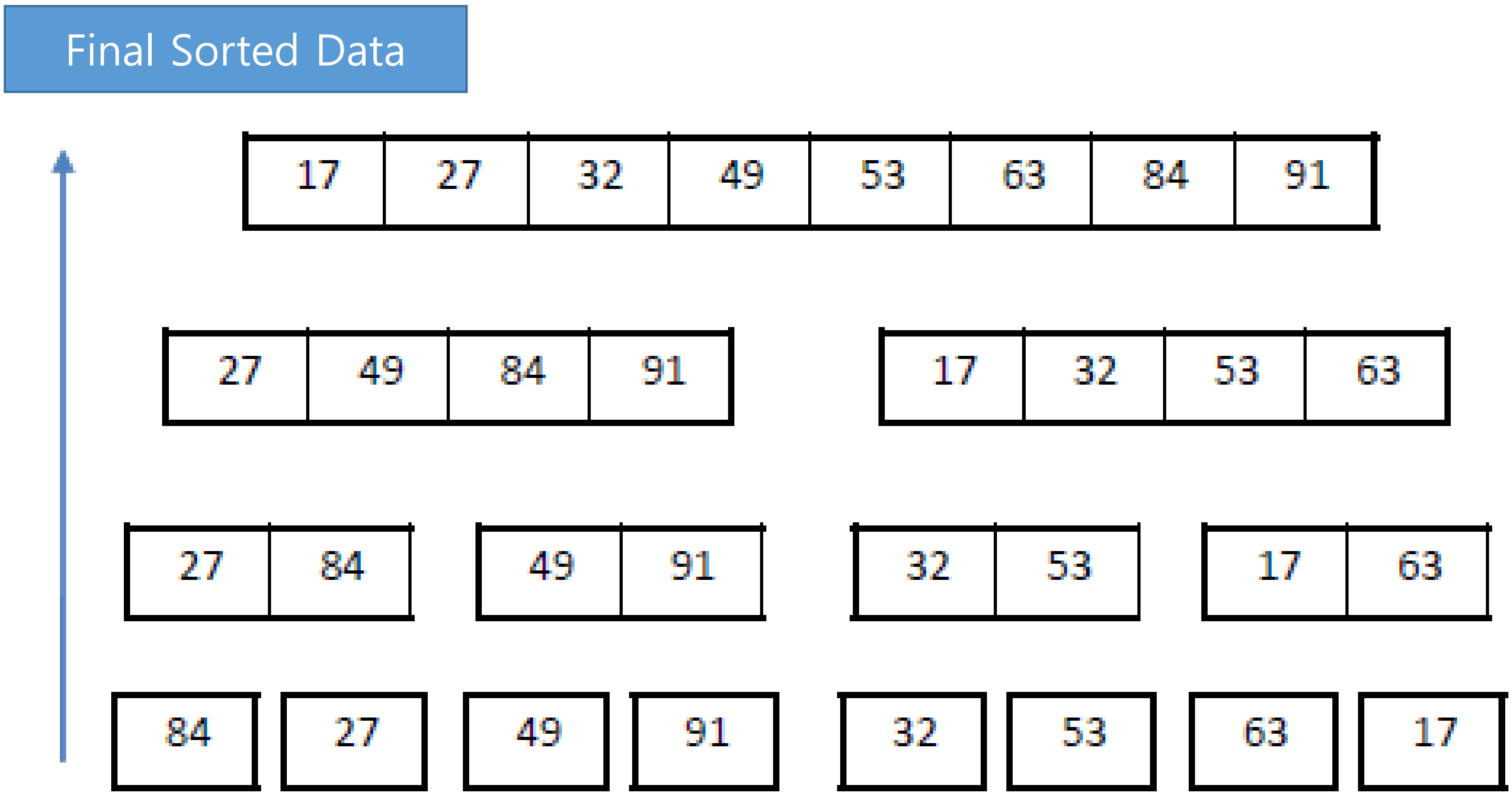
20, 29, 34, 64, 73

가장 작은값을 찾아서 첫번째 자리에 있는 값과 교체

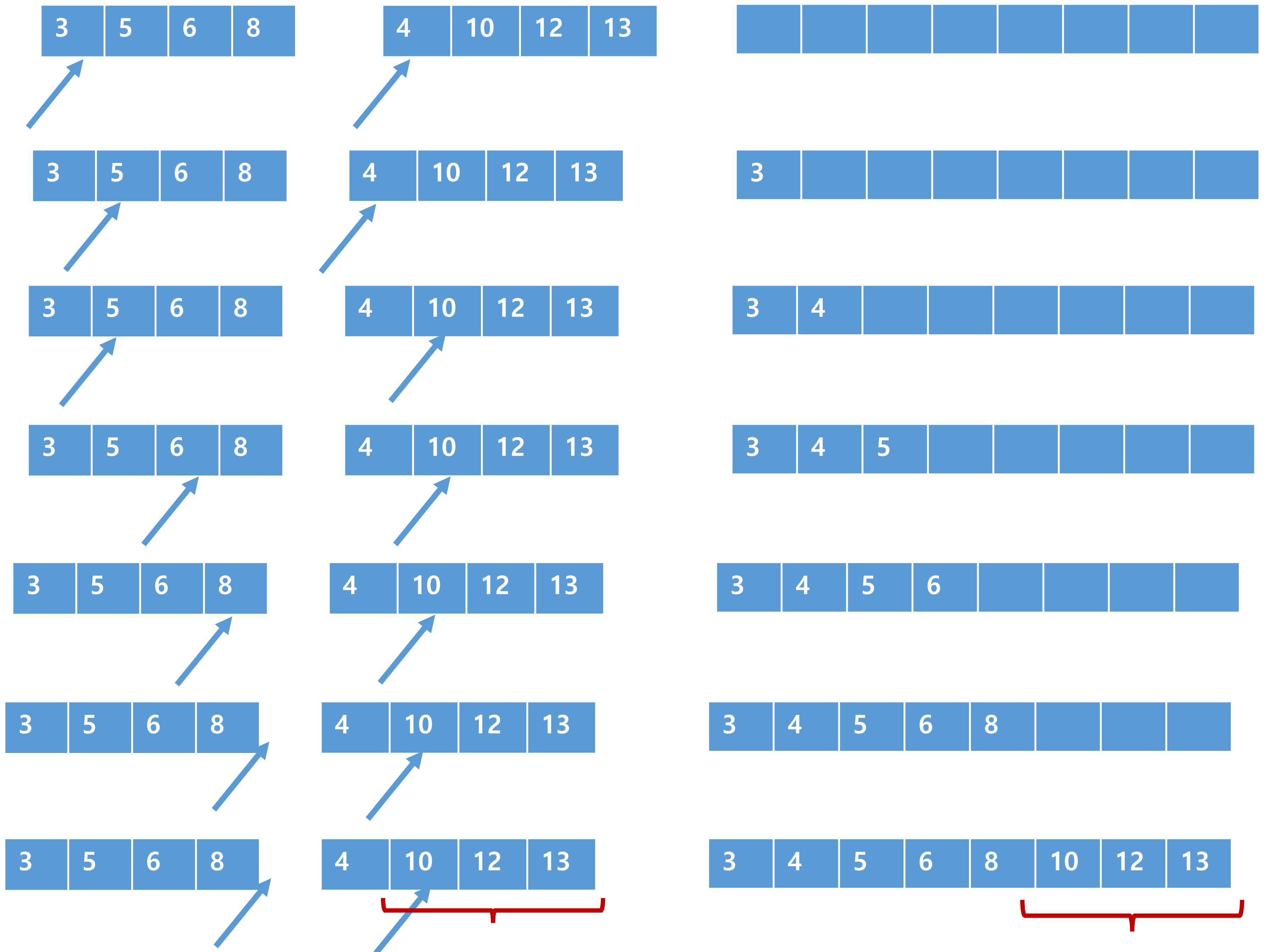
Merge-Sort: Divide Step (Split)



Merge-Sort: Conquer Step (Merge)

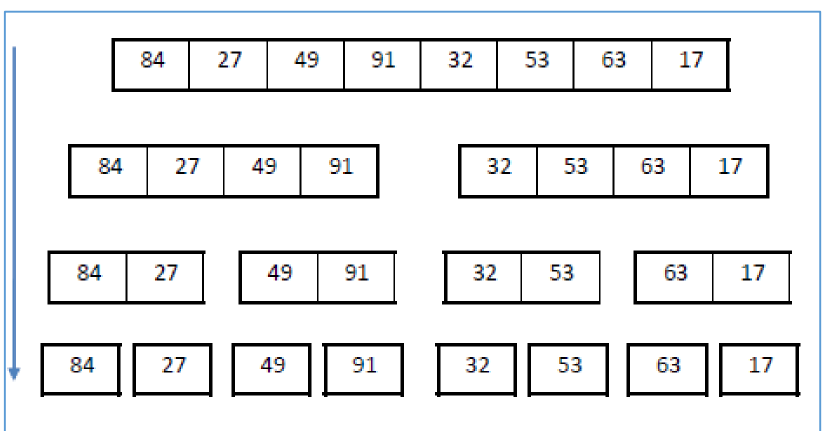


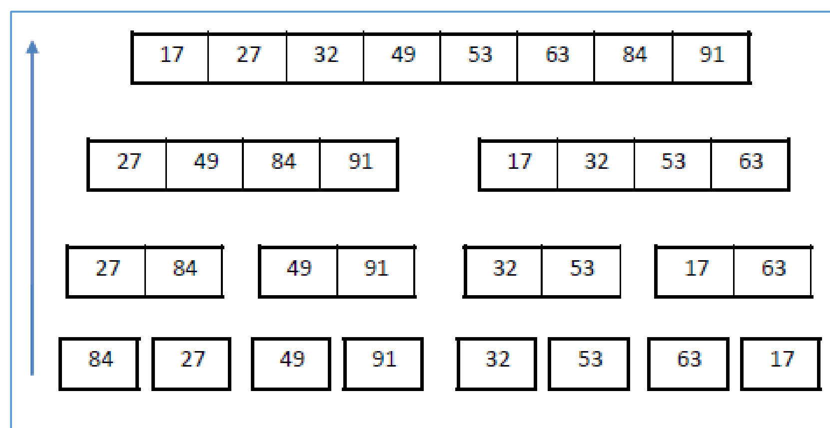
Shots of Merge Step



Outline of msort() - Merge Sort

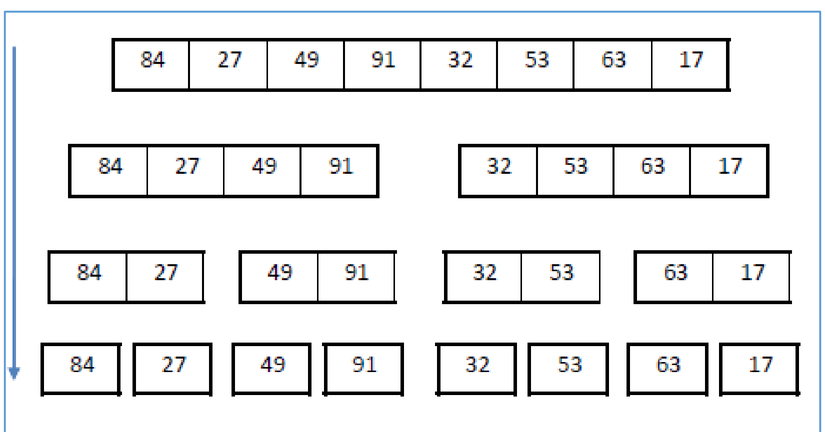
- Input: List *a* of *n* elements.
- Output: Returns **a new list** containing the same elements in sorted order.
- Algorithm:
  1. If less than two elements, return a **copy** of the list **(base case!)**
  2. Sort the first half using merge sort. **(recursive!)**
  3. Sort the second half using merge sort. **(recursive!)**
  4. **Merge** the two sorted halves to obtain the final sorted array.

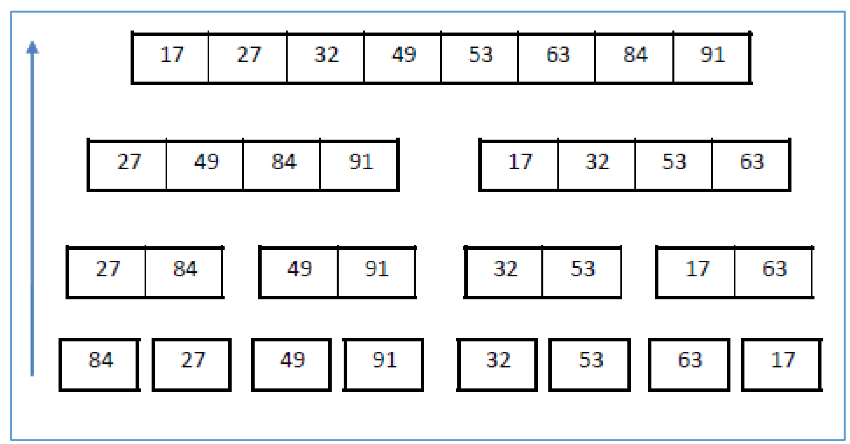




Python Code for msort( )

```python
def msort(list):
    if len(list) == 0 or len(list) == 1: # base case
        return list[:len(list)] # copy the input
    # recursive case
    halfway = len(list) // 2
    list1 = list[0:halfway]
    list2 = list[halfway:len(list)]
    newlist1 = msort(list1) # recursively sort left half
    newlist2 = msort(list2) # recursively sort right half
    newlist = merge(newlist1, newlist2)
    return newlist
```

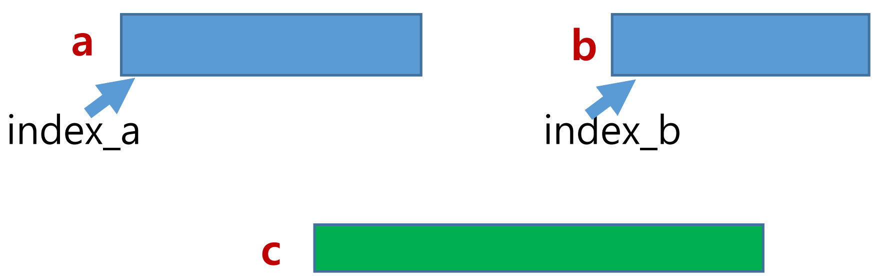




Outline of merge(): Merging Two Sorted Lists

- Input: Two lists *a* and *b*, already sorted
- Output: A new list containing the elements of *a* and *b* merged together in sorted order.
- Algorithm:

1. Create an empty list *c*, set *index_a* and *index_b* to 0
2. While *index_a* < length of *a* and *index_b* < length of b
   a. Add the smaller of *a*[*index_a*] and *b*[*index_b*] to the end of *c*, and increment the index of the list with the smaller element
3. If any elements are left over in *a* or *b*, add them to the end of *c*, in order
4. Return *c*



Python Code for merge( )

```python
def merge(a, b):
    index_a = 0 # the current index in list a
    index_b = 0 # the current index in list b
    c = []
    while index_a < len(a) and index_b < len(b):
        if a[index_a] <= b[index_b]:
            c.append(a[index_a])
            index_a = index_a + 1
        else:
            c.append(b[index_b])
            index_b = index_b + 1
    # when we exit the loop
    # we are at the end of at least one of the lists
    c.extend(a[index_a:])
    c.extend(b[index_b:])
    return c
```

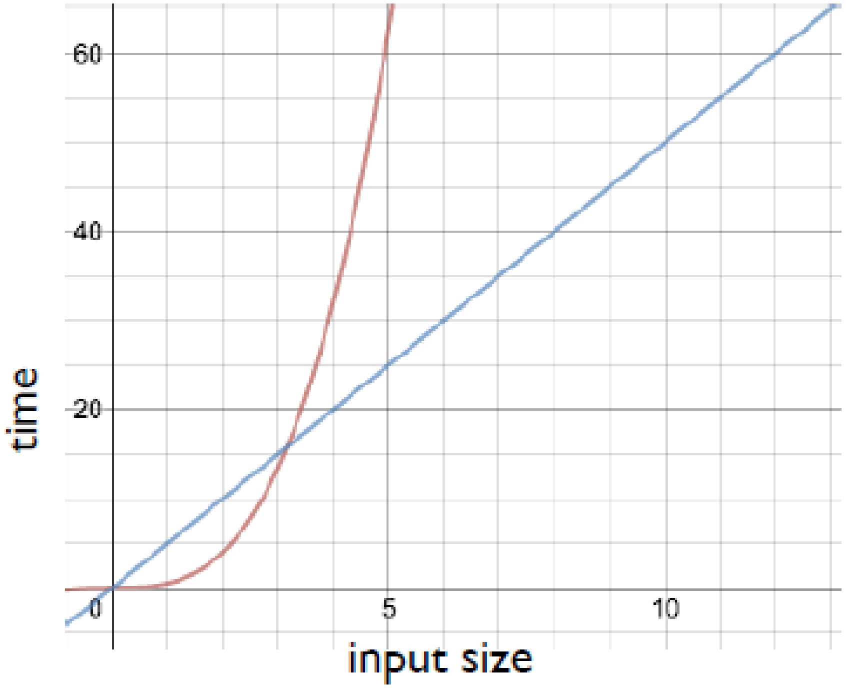
Recursion in Python

- Concept of Recursion
- Recursion Practices
- Divide and Conquer
- Measuring Program Complexity

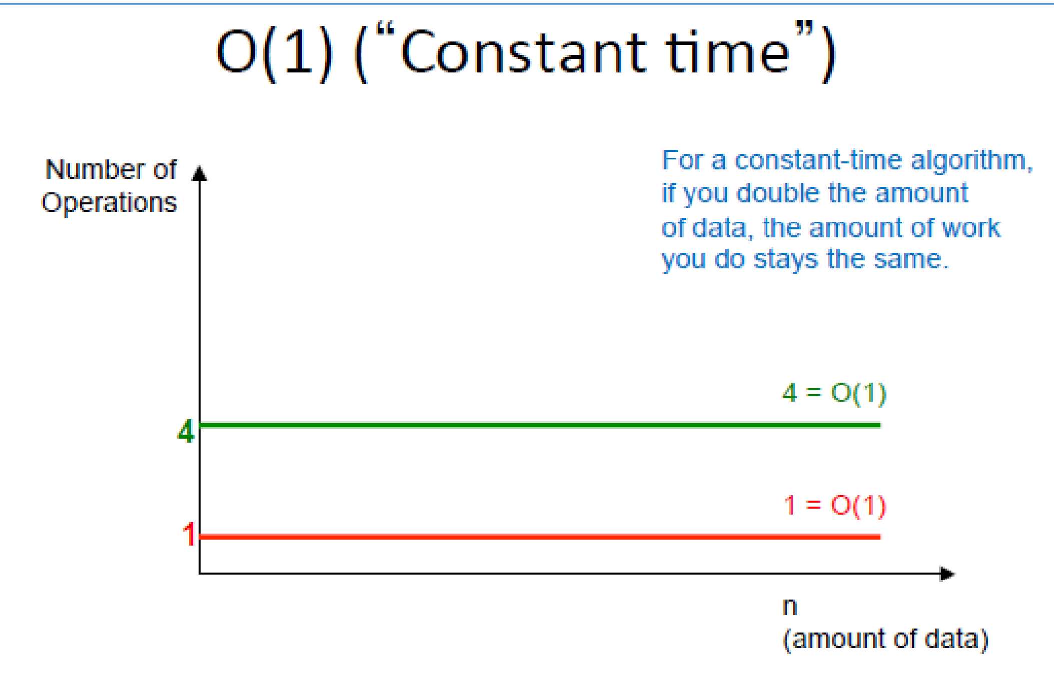
Measuring Algorithm Efficiency

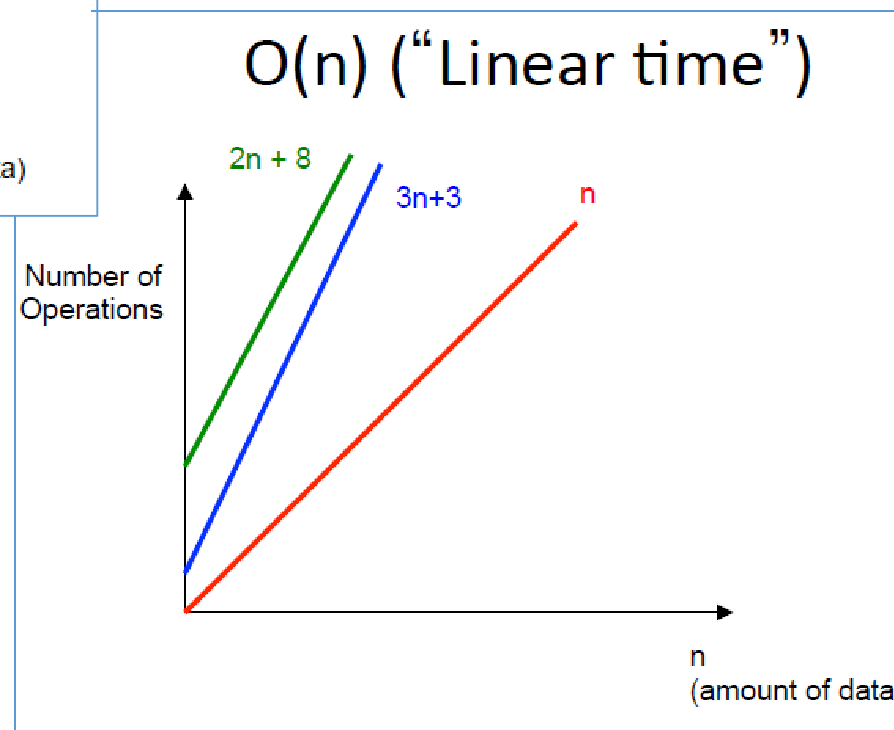
Run time as function of input size

- Graphs of the time complexity of a more efficient and a less efficient algorithm

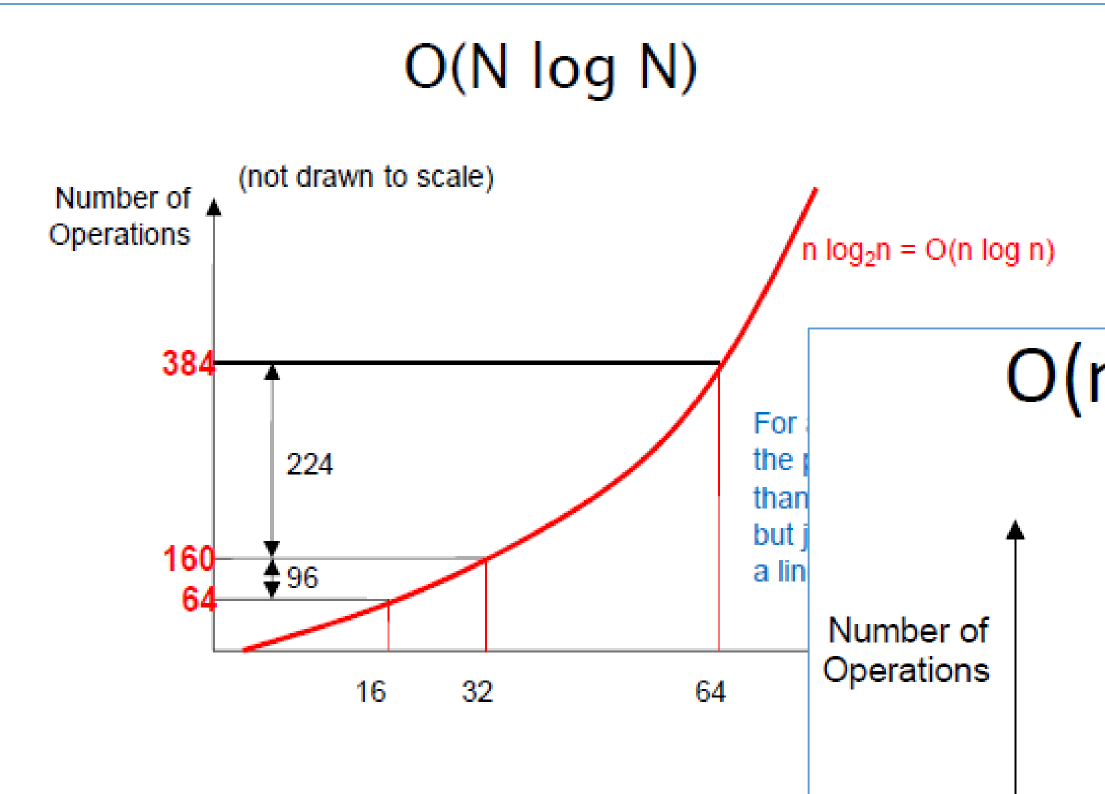


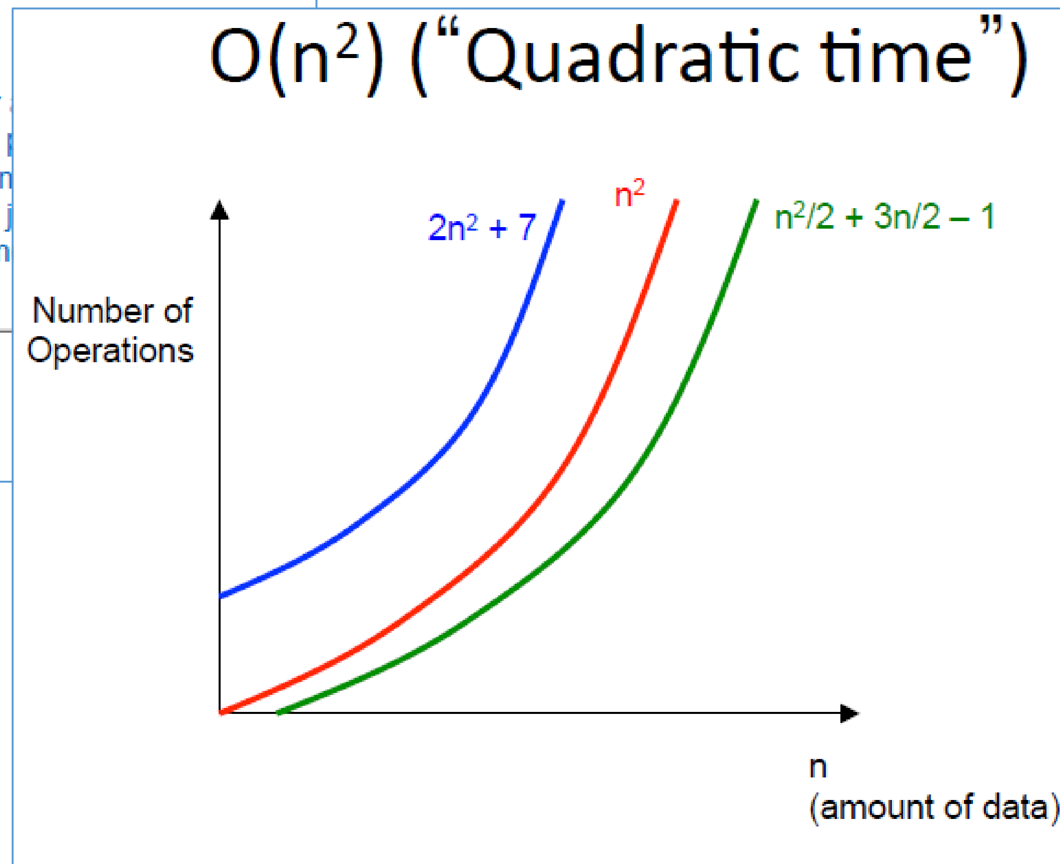
Big-Oh Notation [1/2]



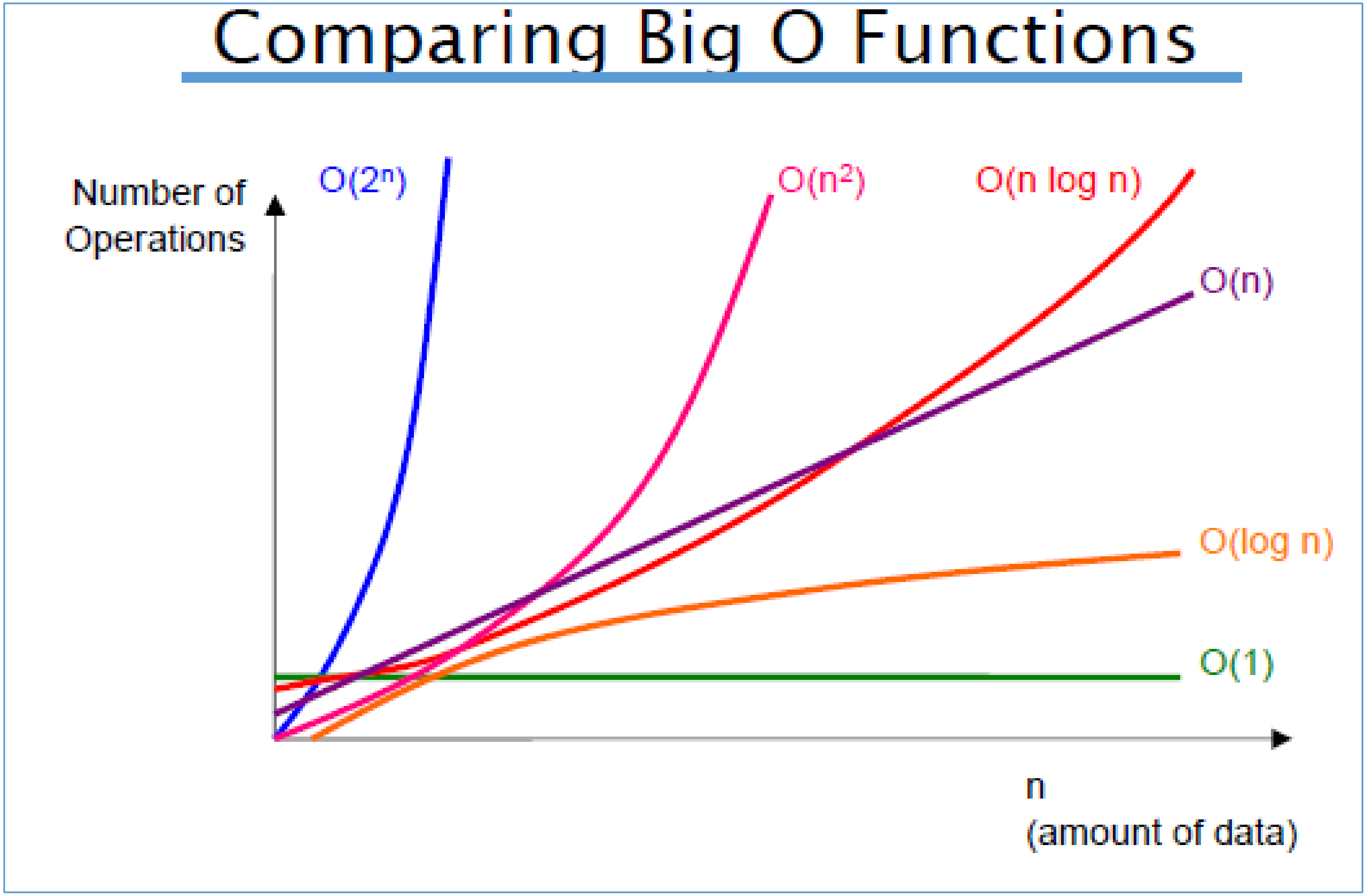


Big-Oh Notation [2/2]





Comparing Algorithms



See you in Data Structure class, Algorithms class….!!!

Comparing Sorts using Time Complexity

- **Selection Sort** (➔ $O(n^2)$ algorithm)
- For a list of size *n*
  - To find the smallest element, the algorithm inspects all *n* items
  - The next time through the loop, it inspects the remaining *n*-1 items
- The total number of comparisons in iterations is:

$$n + (n-1) + (n-2) + (n-3) + \ldots + 1 \quad = \quad \frac{n(n+1)}{2}$$

- contains an n^2 term: the number of steps in the algorithm is proportional to the square of the size of the list

- **Merge Sort** (➔ O(n*log(n)) algorithm)
- For a list of size *n*
- The number of levels: $\log_2 n$
- The number of comparisons in merge step of each level: a little bit less than n
  - M개 원소의 2개 list를 merge: best case ➔ (M/2)번 비교, worse case ➔ (M-1)번 비교
  - 각 level에서 merge를 할때에 가장 worst한 경우에는 거의 n개의 비교를 해야할수 있다

=> total work required to sort *n* items: $n*\log_2 n$