Ch 6: Prime Number Generator with Python

(** With help of CMU 15-110 Class Material)

Prime Numbers

- An integer is "prime" if it is not divisible by any smaller integers except 1.
- 10 is **not** prime because 10 = 2 × 5
- 11 **is** prime
- 12 is **not** prime because 12 = 2 × 6 = 2 × 2 × 3
- 13 **is** prime
- 15 is **not** prime because 15 = 3 × 5

Testing Divisibility in Python

- x is "divisible by" y if the remainder is 0 when we divide x by y
- 15 is divisible by 3 and 5, but not by 2:

```
>>> 15 % 3

0
>>> 15 % 5

0
>> 15 % 2

1
```

IsPrime(): dumb version

```
def IsPrime_dumb(n):
    if (n < 2):
        return False
    for factor in range(2, n):
        if (n % factor == 0):      # 모든숫자 n에 대해서 n번의 modulo 계산필요
            return False
    return True


for i in range(1, 100):
     if IsPrime_dumb(i):
        print(i)
```

A 2000 year old algorithm (procedure) for generating a table of prime numbers.

2, 3, 5, 7, 11, 13, 17, 23, 29, 31, ...

What Is a "Sieve" or "Sifter"?

Separates stuff you want from stuff you don't:

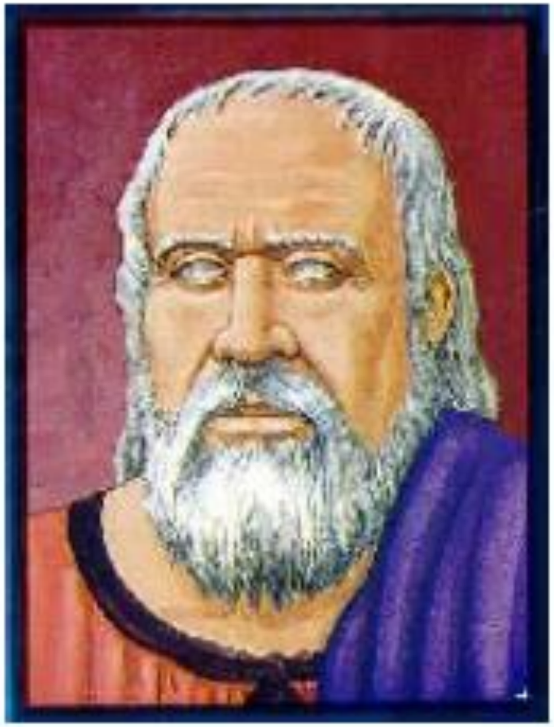
We want to separate prime numbers.

The Sieve of Eratosthenes

Start with a table of integers from 2 to N.

Cross out all the entries that are divisible by the primes known so far.

The first value remaining is the *next* prime.



Finding Primes Between 2 and 50

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

2 is the first prime

Finding Primes Between 2 and 50

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Filter out everything divisible by 2.

Now we see that 3 is the next prime.

Sift!

Finding Primes Between 2 and 50

	2	**3**	4	**5**	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Filter out everything divisible by 5.

Now we see that 7 is the next prime.

Sift!

Finding Primes Between 2 and 50

	2	**3**	4	**5**	6	**7**	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Filter out everything divisible by 7.

Now we see that 11 is the next prime.

Sift!

Finding Primes Between 2 and 50

	2	**3**	4	**5**	6	**7**	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50

Since 11 x 11 > 50, all remaining numbers must be primes. Why?

Sift!

An Algorithm for Sieve of Eratosthenes

Input: A number *n*:

1. Create a list *numlist* with every integer from 2 to n, in order. (Assume $n > 1$.)

2. Create an empty list *primes*.

3. For each element in *numlist*

   a. If element is not marked, copy it to the end of *primes*.

   b. Mark every number that is a multiple of the most recently discovered prime number. Sift!

Output: The list of all prime numbers less than or equal to *n*

Automating the Sieve

numlist	primes
2 3 4 5 6 7 8 9 10 11 12 13 ...	

Use *two* lists: candidates, and confirmed primes.

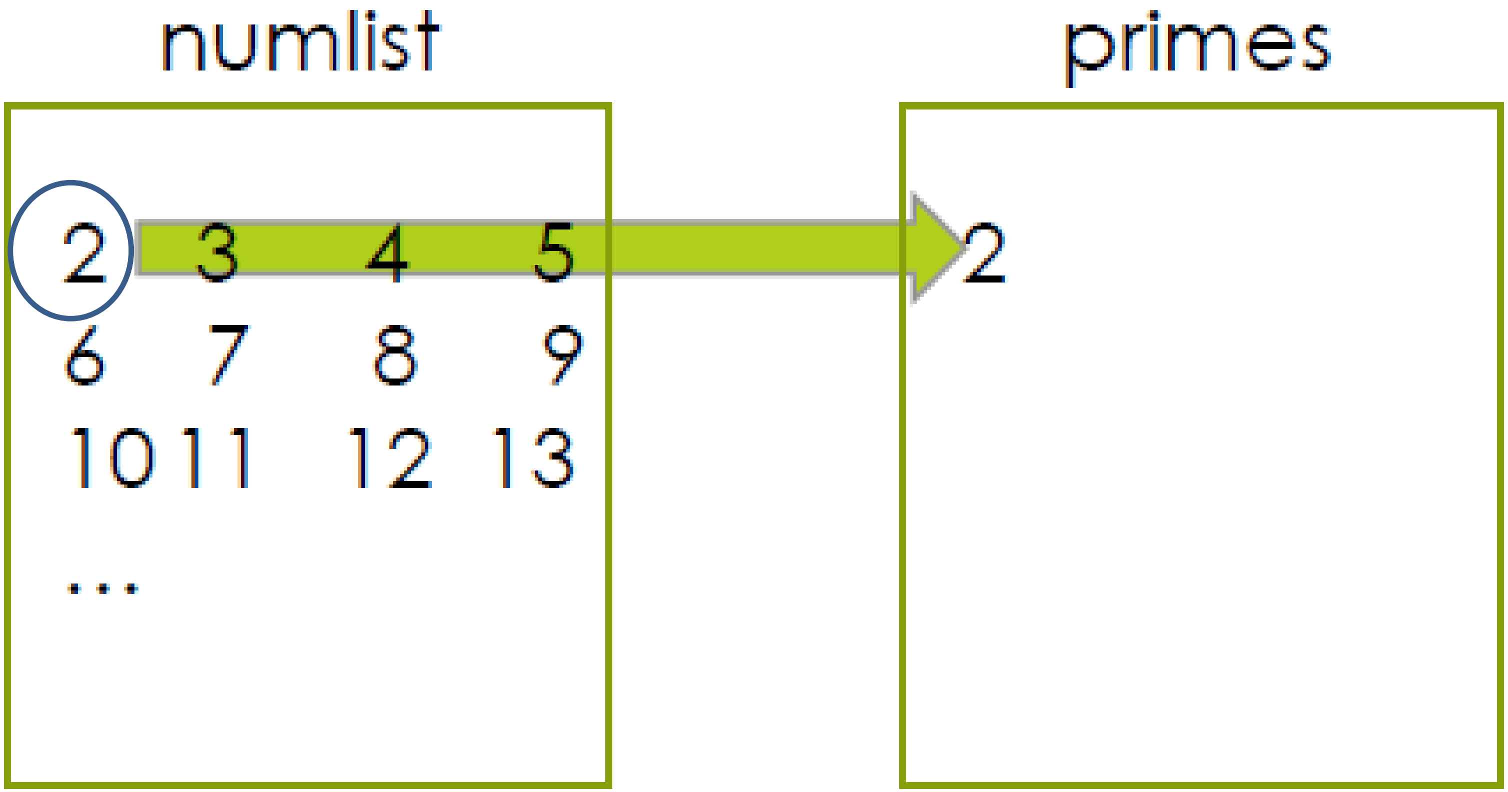
Steps 1 and 2

numlist

2	3	4	5
6	7	8	9
10	11	12	13
...			

primes

Step 3a



Append the current number in numlist to the end of primes.

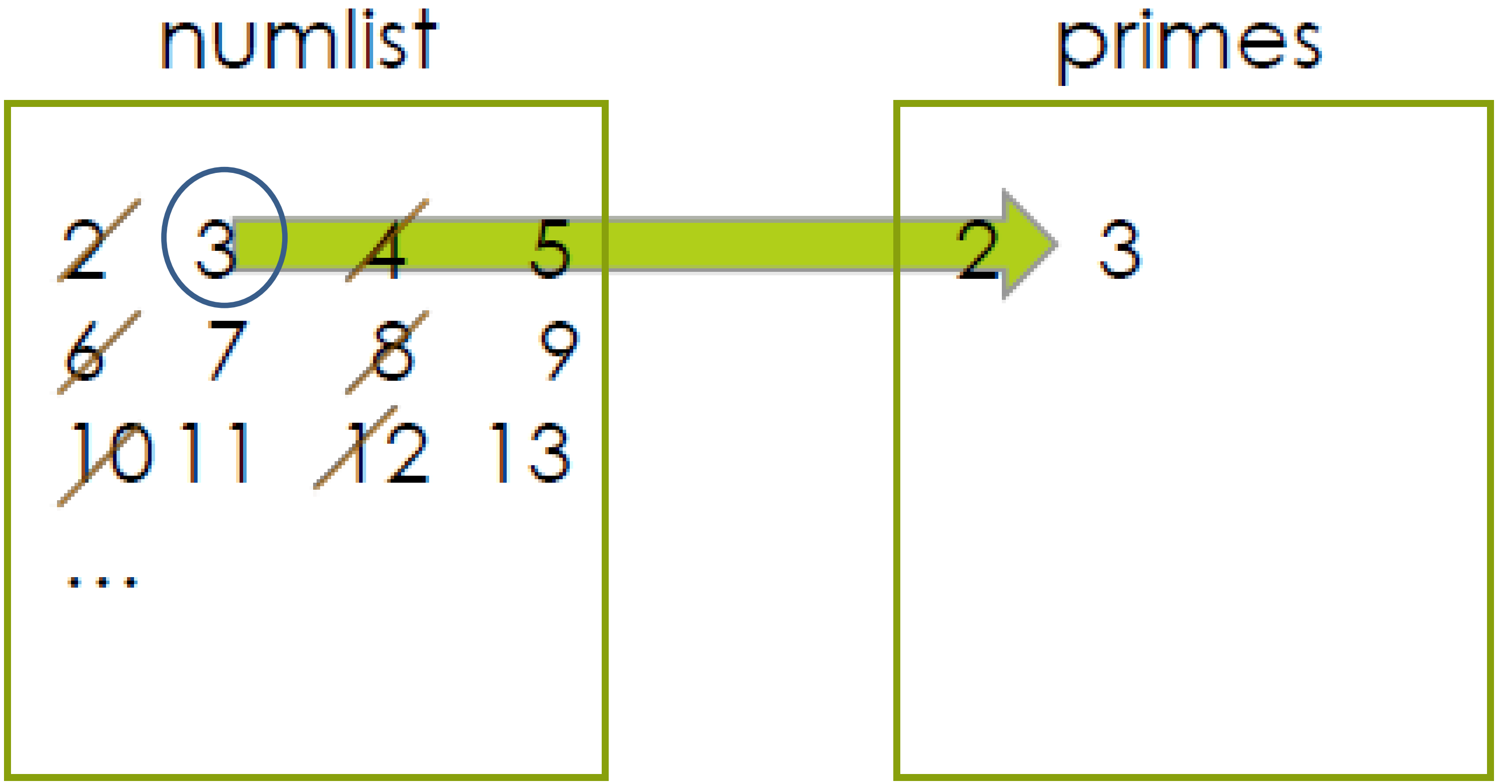
Step 3b

numlist			
~~2~~	3	~~4~~	5
~~6~~	7	~~8~~	9
~~10~~	11	~~12~~	13
…			

primes
2

Cross out all the multiples of the last number in primes.

Iterations



Append the current number in numlist to the end of primes.

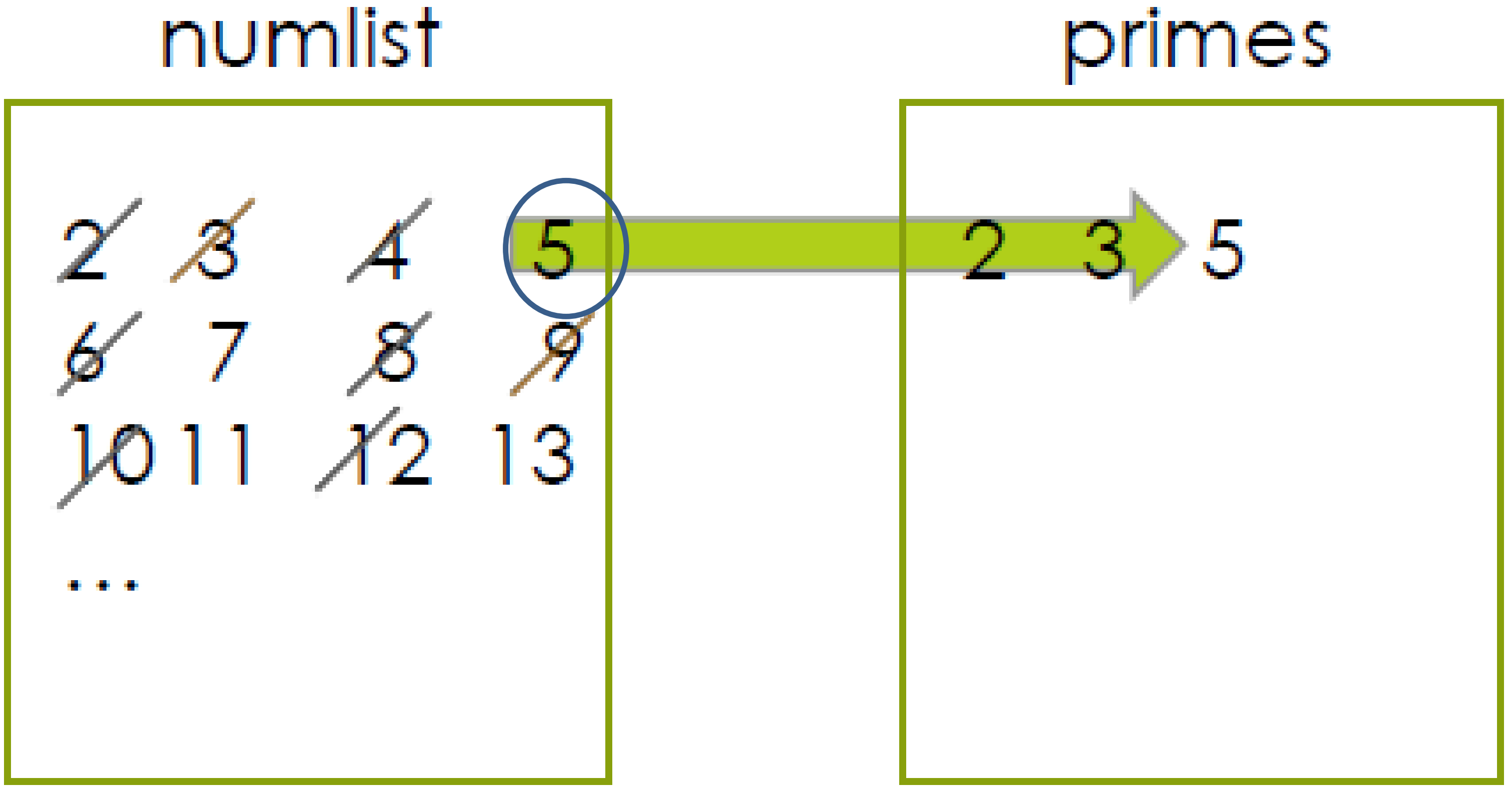
Iterations

numlist	primes
~~2~~ ~~3~~ ~~4~~ 5 ~~6~~ 7 ~~8~~ ~~9~~ ~~10~~ 11 ~~12~~ 13 …	2 3

Cross out all the multiples of the last number in primes.

numlist 에서 marking이 안된 가장 작은값은 prime number !

Iterations



Append the <u>current</u> number in numlist to the <u>end</u> of primes.

numlist 에서 marking이 안된 가장 작은값은 prime number !

Iterations

numlist

~~2~~ ~~3~~ ~~4~~ ~~5~~
~~6~~ 7 ~~8~~ ~~9~~
~~10~~ 11 ~~12~~ 13
...

primes

2 3 5

Cross out all the multiples of the last number in primes.

numlist 에서 marking이 안된 가장 작은값은 prime number !

Why? → sift( ) 에서 자기 보다 작은값에 의해서는 divide 안된다는것이 보장

An Algorithm for Sieve of Eratosthenes

Input: A number *n*:

1. Create a list *numlist* with every integer from 2 to n, in order. (Assume $n > 1$.)

2. Create an empty list *primes*.

3. For each element in *numlist*

   a. If element is not marked, copy it to the end of *primes*.

   b. Mark every number that is a multiple of the most recently discovered prime number. Sift!

Output: The list of all prime numbers less than or equal to *n*

Implementation Decisions

- How to implement *numlist* and *primes?*
  - *For numlist* we will use a list in which crossed out elements are marked with the special value None. For example,

```
[None, 3, None, 5, None, 7, None]
```

- Use a helper function for step 3.b. We will call it sift.

Relational Operators

- If we want to compare two integers to determine their relationship, we can use these relational operators:

<	less than	<=	less than or equal to
>	greater than	>=	greater than or equal to
==	equal to	!=	not equal to

- We can also write compound expressions using the Boolean operators and and or.

x >= 1 and x <= 1

Sifting: Removing Multiples of a Number

```
def sift(lst,k):
    # marks multiples of k with None
    i = 0
    while i < len(lst):
        if (lst[i] != None) and lst[i] % k == 0:
            lst[i] = None
        i = i + 1
    return lst
```

Filters out the multiples of the number k from list by marking them with the special value None (greyed out ones).

lst 에 있는 data가 k로 나누어지면 None으로 교체!

Sifting: Removing Multiples of a Number (Alternative Vresion)

```
def sift2(lst,k):
    i = 0
    while i < len(lst):
        if lst[i] % k == 0:
            lst.remove(lst[i])
        else:
            i = i + 1
    return lst
```

Filters out the multiples of the number k from list
by modifying the list. Be careful in handling indices.

lst 에 있는 data를 직접 remove 하면 lst 가 계속 변경되므로 곤란!

A Working Sieve

```
def sieve(n):
    numlist = list(range(2,n+1))
    primes = []
    for i in range(0,len(numlist)):
        if numlist[i] != None:
            primes.append(numlist[i])
            sift(numlist,numlist[i])
    return primes
```

Use the first version of sift in this function, which does the filtering using Nones.

We could have used `primes[len(primes)-1]` instead.

Helper function that we defined before

numlist 에서 marking이 안된 가장 작은값은 prime number !

Why? → sift( ) 에서 자기 보다 작은값에 의해서는 divide 안된다는것이 보장

Observation for a Better Sieve

We stopped at 11 because all the remaining entries must be prime since $11 \times 11 > 50$.

2	**3**	4	**5**	6	**7**	8	9	10	
11	12	**13**	14	15	16	**17**	18	**19**	20
21	22	**23**	24	25	26	27	28	**29**	30
31	32	33	34	35	36	**37**	38	39	40
41	42	**43**	44	45	46	**47**	48	49	50

Jonathan Sorenson (1990, Univ Wisconsin)는 sieve algorithm에서 n까지 확인을 할 필요없이 square_root(n)까지만 확인하면 남아있는 숫자들은 전부 prime이라는것을 증명

A Better Sieve

```
def   better_sieve(n):
      numlist = list(range(2, n+1))
      primes = [ ]
      i  = 0   # index 0 contains  number 2
      while   (i + 2)   <=  math.sqrt(n):
              if    numlist[i] != None;
                     primes.append(numlist[i])
                     sift(numlist,  numlist[i])
              i = i + 1
       temp_list = [ ]
       for i in range(0, len(numlist)):
              if numlist[i] != None:
                     temp_list.append(numlist[i])
       numlist = temp_list
       return primes + numlist
```

- Owing to Sorenson [1990]

Algorithm-Inspired Sculpture

The Sieve of Eratosthenes, 1999 sculpture by Mark di Suvero. Displayed at Stanford University.

Prime Number Checker
IsPrime(): dumb version

```
def IsPrime_dumb(n):
    if (n < 2):
        return False
    for factor in range(2, n):
        if (n % factor == 0):       # 모든숫자 n에 대해서 n번의 modulo 계산필요
            return False
    return True


for i in range(1, 100):
     if IsPrime_dumb(i):
        print(i)
```

- 앞에 있었던 sieve(n)은 n까지의 모든 prime number를 생성하는 algorithm
- IsPrime(i)는 i가 prime인지 아닌지만 결정하는 algorithm

Prime Number Checker
IsPrime_better(): better version

```
def IsPrime_better(n):
    if (n < 2):
        return False
    if (n == 2):
        return True
    if (n % 2 == 0):
        return False
    for factor in range(3, n, 2):     #  2로 divide 안되었다면 2의 배수로 divide 안될것이므로
        if (n % factor == 0):         # 2의 배수는 skip해도 OK!
            return False
    return True


for i in range(1, 100):
    if IsPrime_better(i):
        print(i)
```

Prime Number Checker
IsPrime_best(): best version

```
def IsPrime_best(n):
    if (n < 2):
        return False
    if (n == 2):
        return True
    if (n % 2 == 0):
        return False
    maxFactor = round(n**0.5)       # Sorenson방식을 이용한다면
    for factor in range(3, maxFactor+1, 2):
        if (n % factor == 0):            # Modulo 계산횟수가 획기적으로 줄어듬
            return False
    return True

for i in range(1,100):
    if IsPrime_best(i):
        print(i)
```