# GAP AMPLIFICATION AND PRESERVATION

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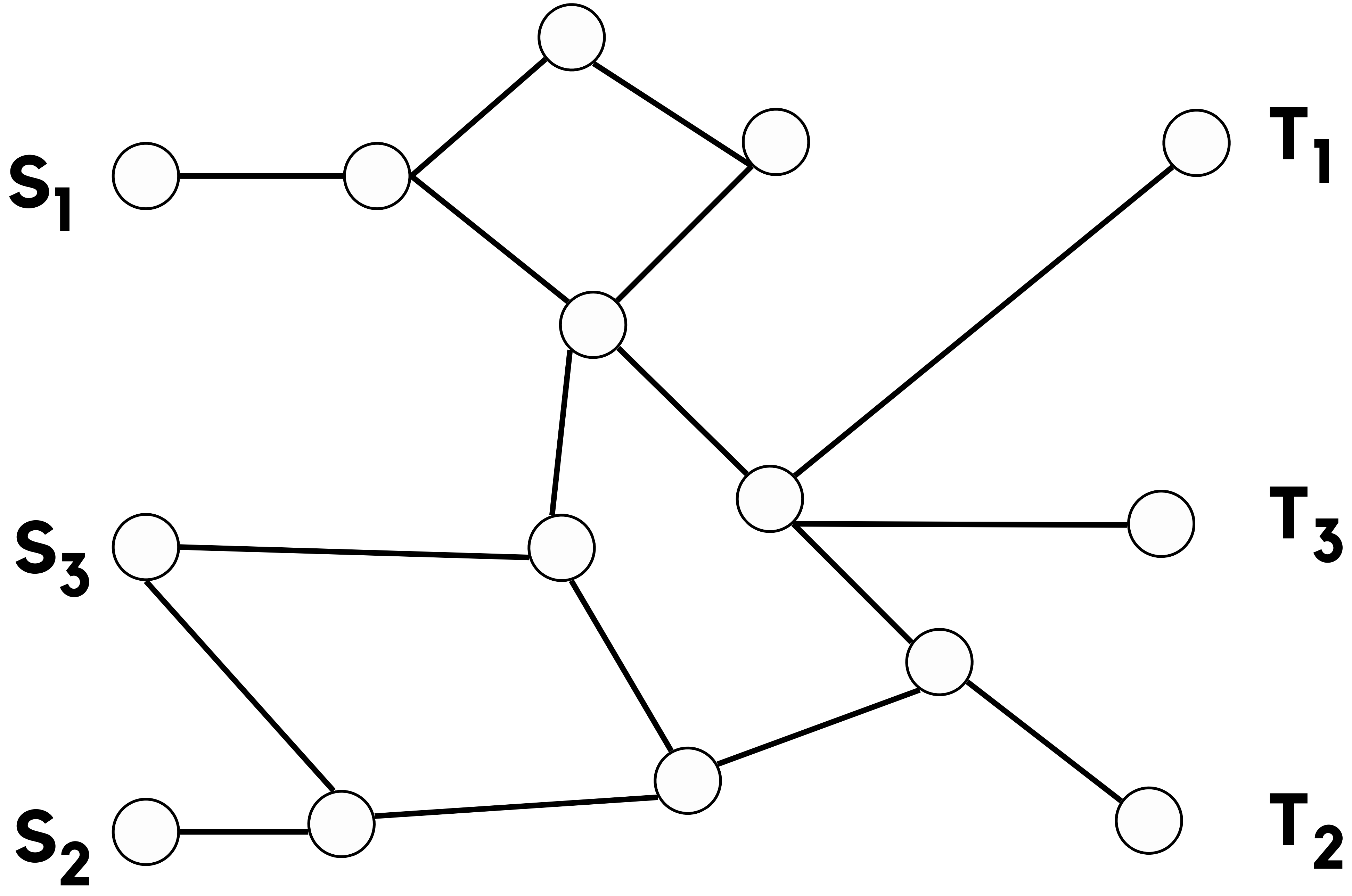
**Theorem 10.6** *The problem* EDP *has no polynomial-time* $(m^{0.5-\varepsilon})$*-approximation for any* $0 < \varepsilon < 1/4$ *unless* $\mathbf{P} = \mathbf{NP}$, *where* $m$ *is the number of edges in the input graph.*

# EDGE DISJOINT PATH

Given a graph $G = (V, E)$

a list $L = ((s_1, t_1), (s_2, t_2), (s_3, t_3))$ of 3 pairs of vertices



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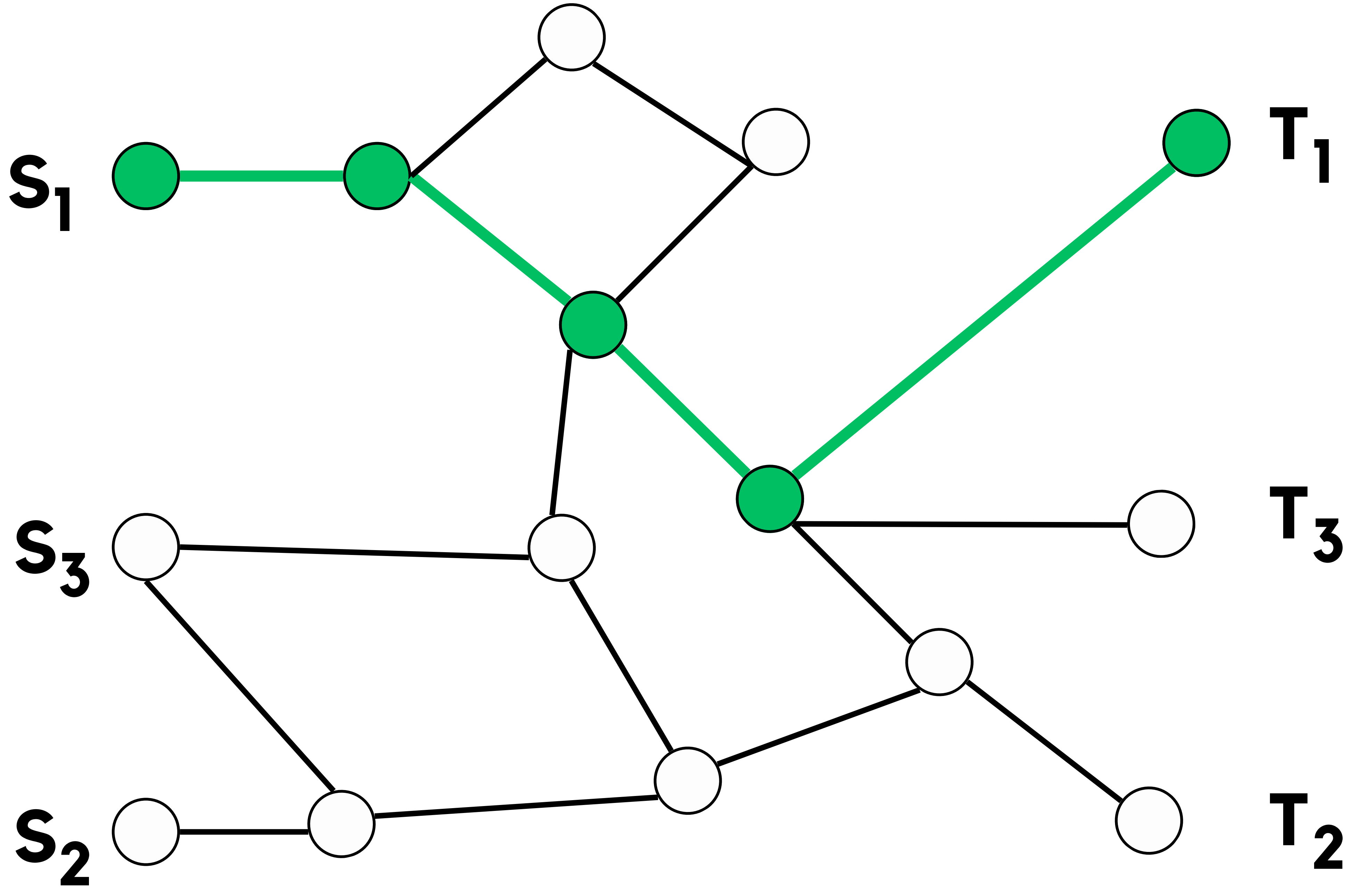
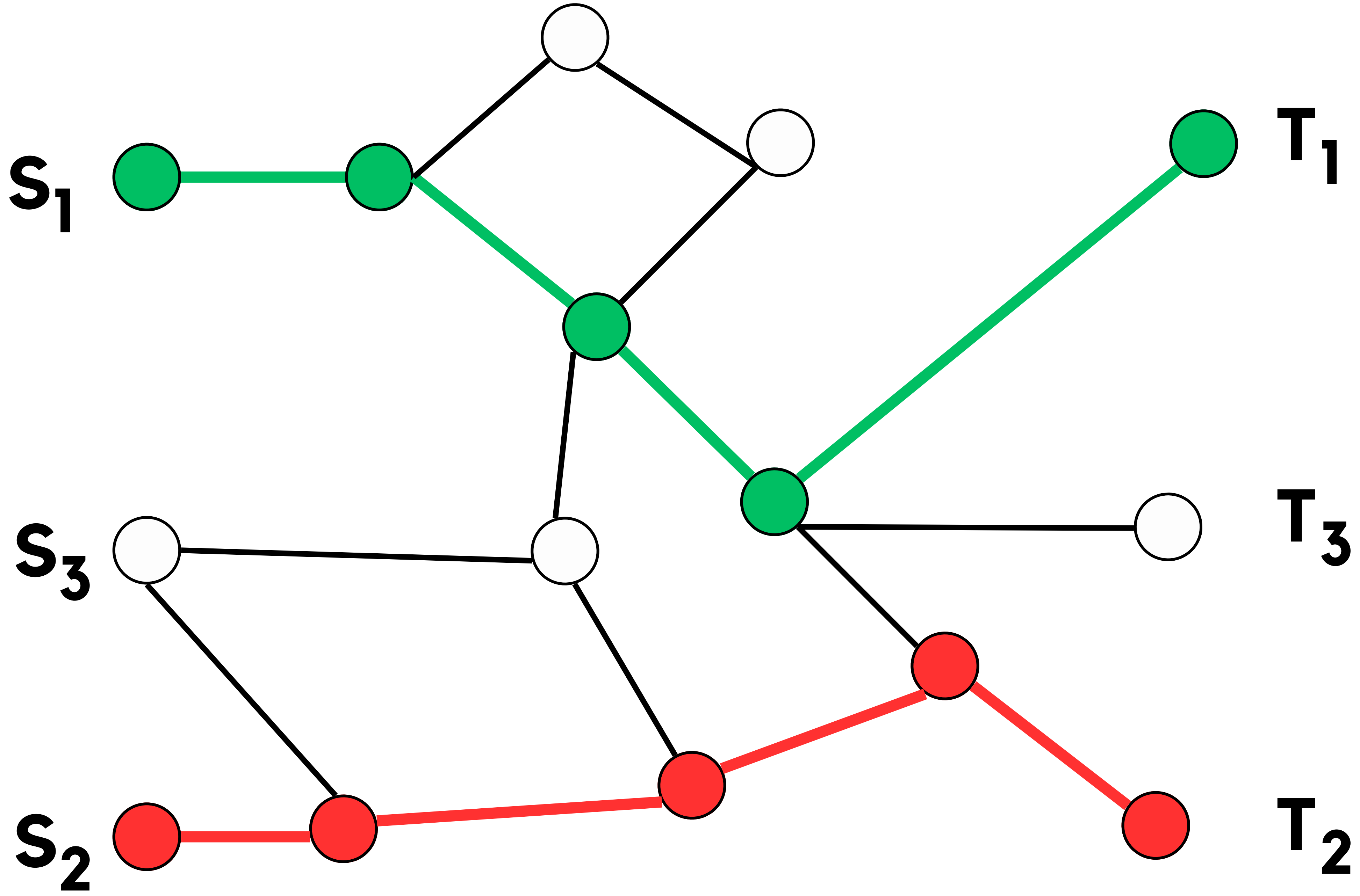



A path found between $s_1, t_1$

Given a graph $G = (V, E)$

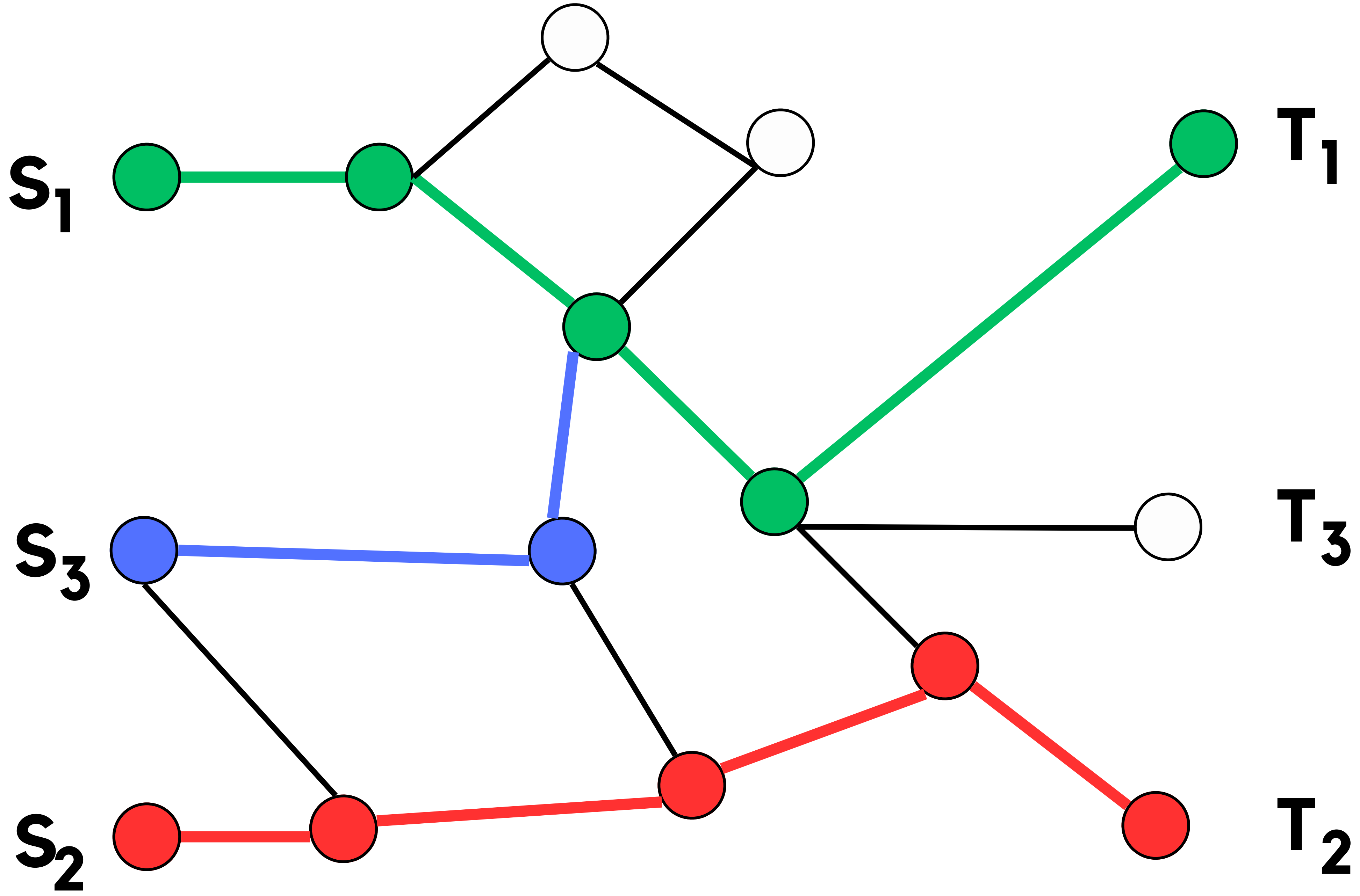
a list $L = ((s_1, t_1), (s_2, t_2), (s_3, t_3))$ of 3 pairs of vertices




Another disjoint path found between $s_2, t_2$

Given a graph $G = (V, E)$

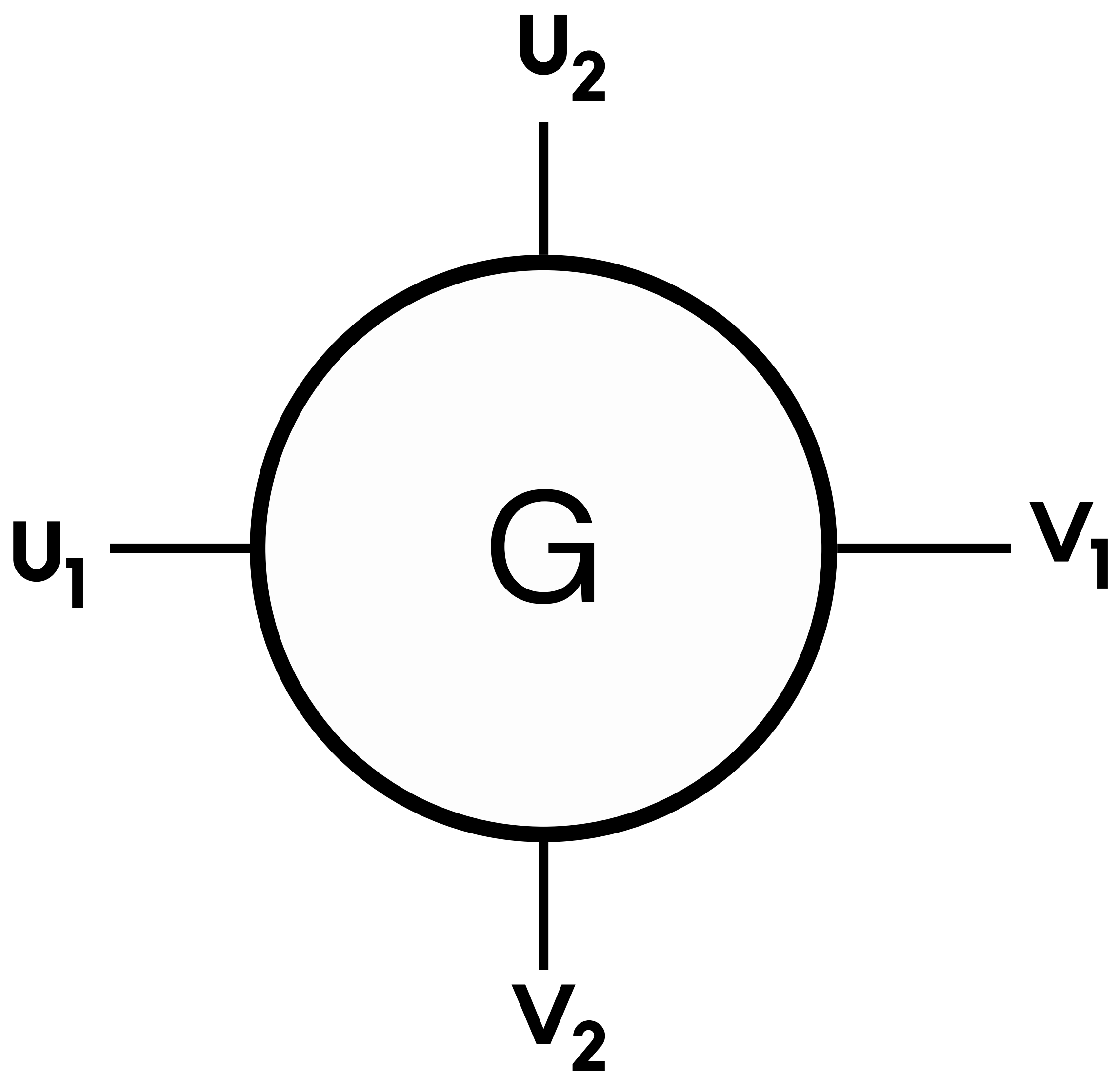
a list $L = ((s_1, t_1), (s_2, t_2), (s_3, t_3))$ of 3 pairs of vertices




No disjoint path found between $s_3, t_3$

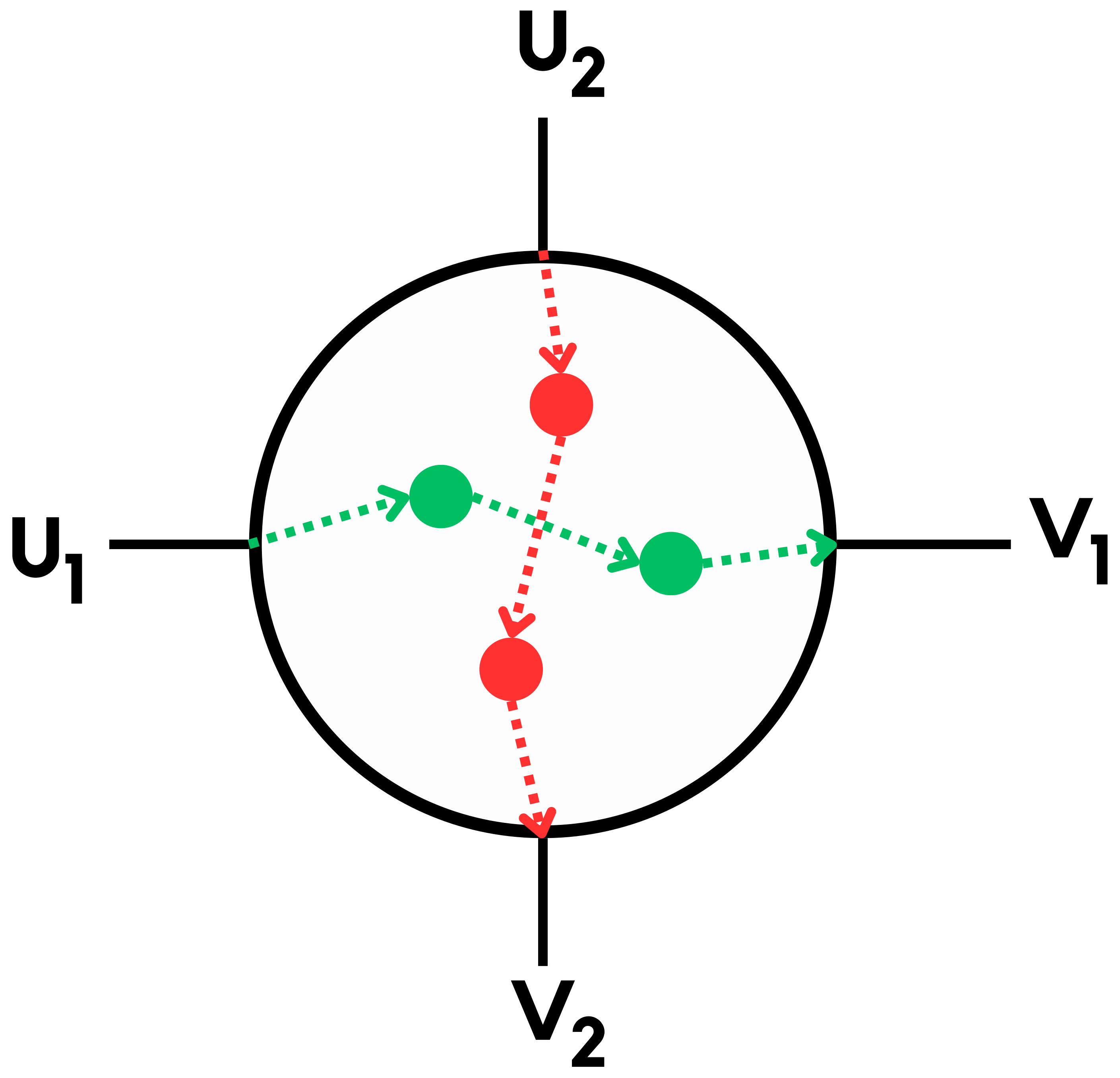
# NP-HARD GAP FOR EDP-2

Given a graph $G = (V, E)$ and
a list $L = ((u_1, v_1), (u_2, v_2))$ of 2 pairs of vertices,



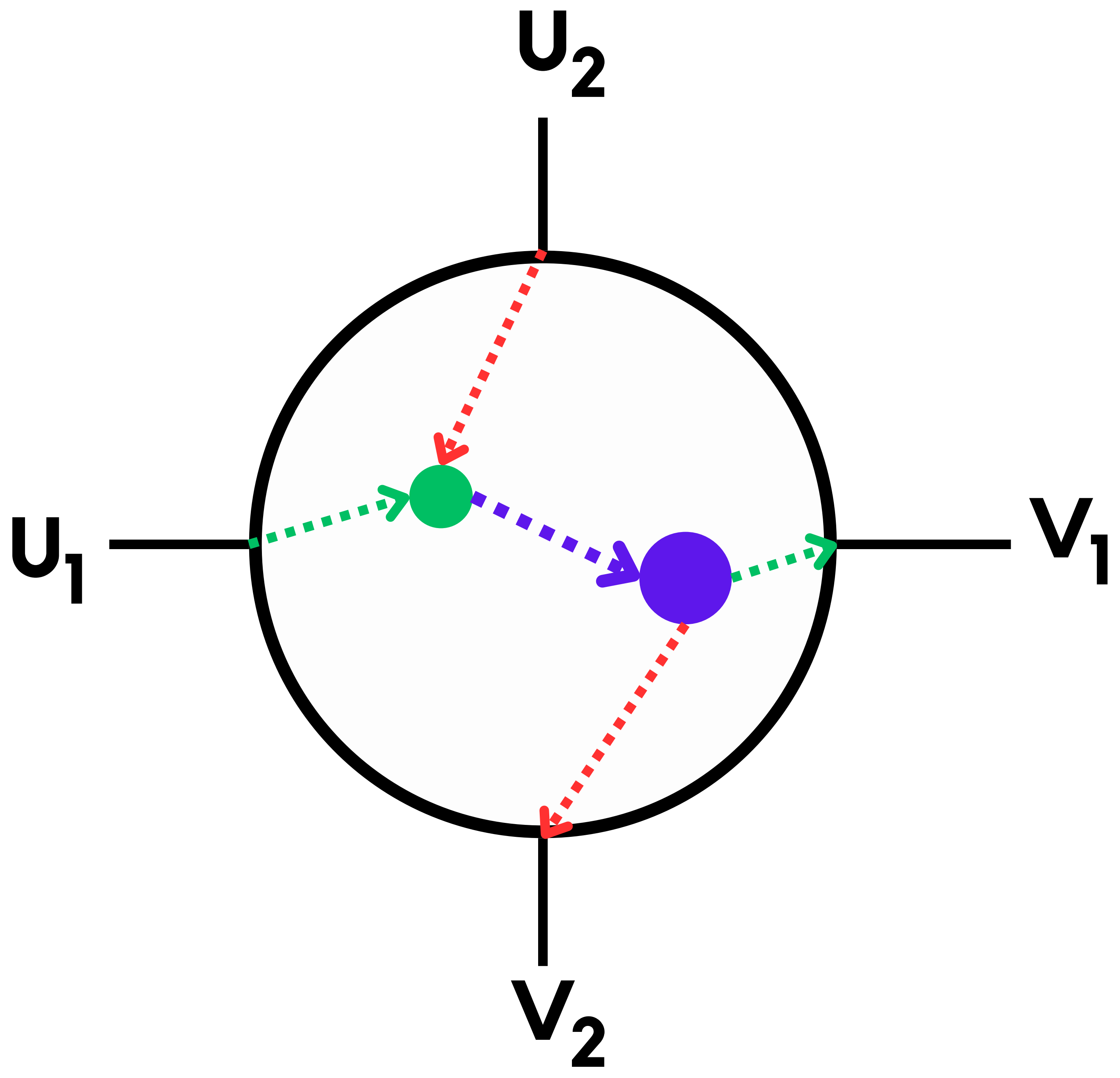

There are two cases

## CASE-1




Two disjoint paths exist. So, OPT $= 2$

**CASE - 2**



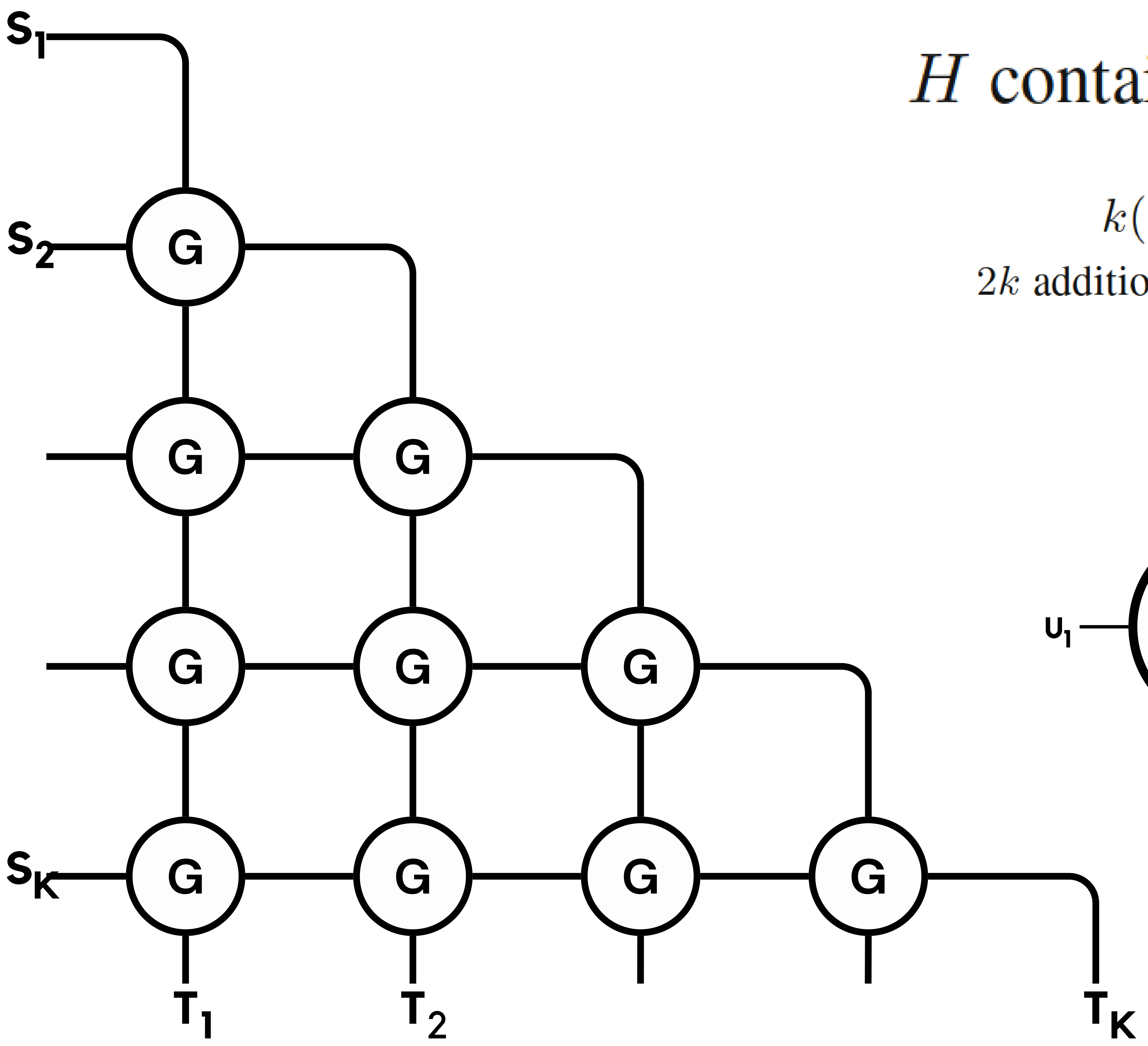

One disjoint path exists. So, OPT = 1 or $OPT < 1 + \epsilon$

**NP**-hard gap $[1+\varepsilon, 2]$

# EDP-2 TO EDP-K REDUCTION

$$[1+\varepsilon, 2] \longrightarrow [1+\varepsilon, k]$$

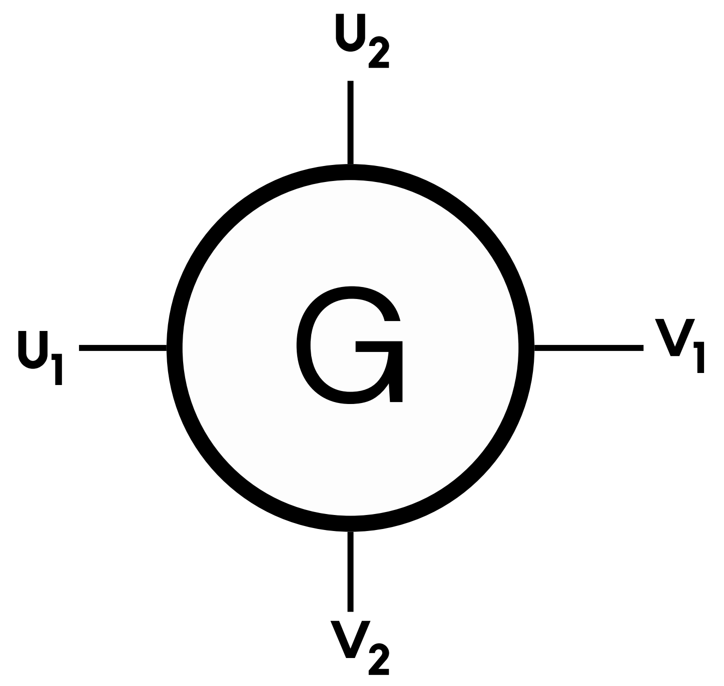
**amplification**




# $H$ contains

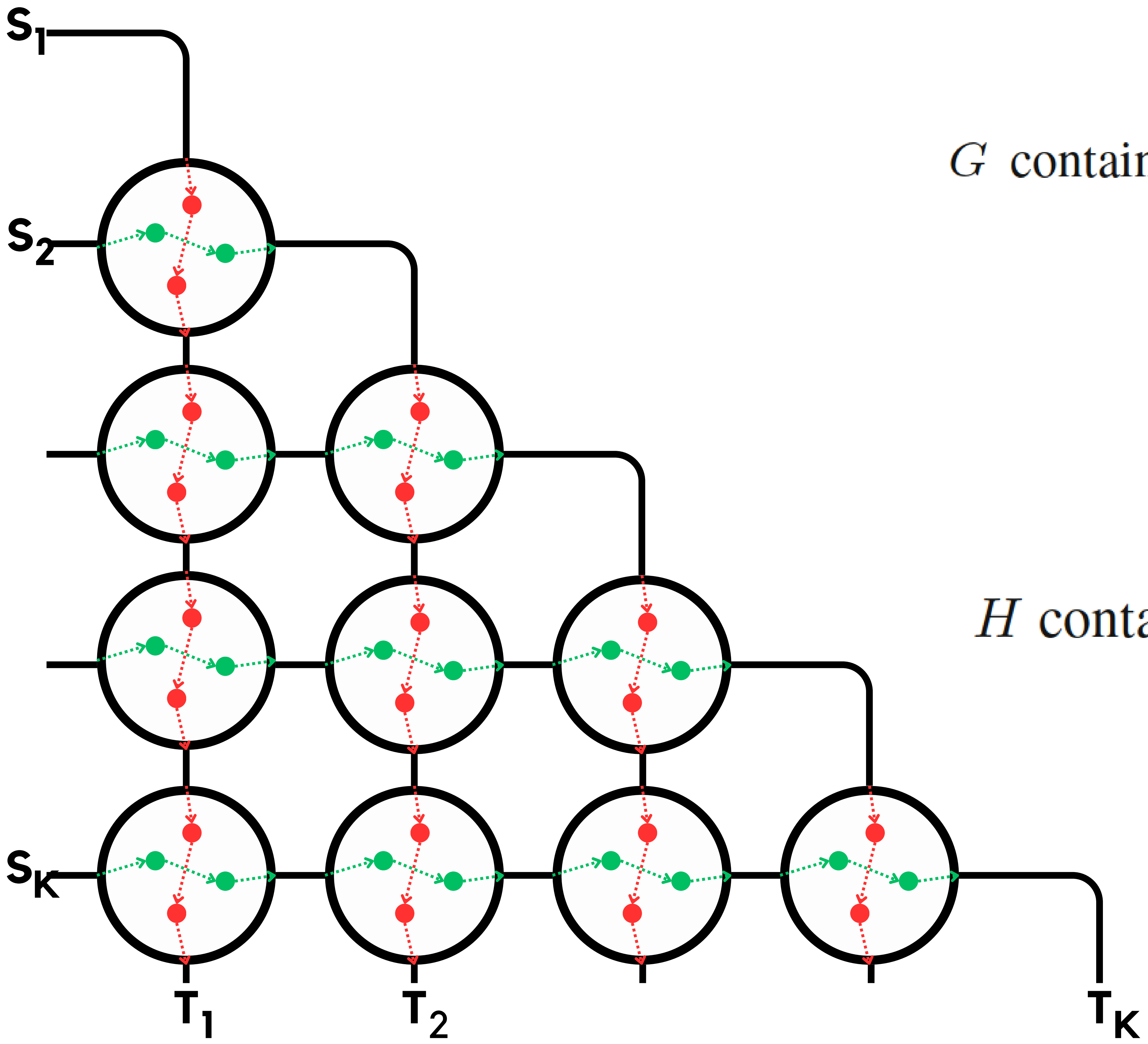
$k(k-1)/2$ copies of $G$

$2k$ additional vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$



## CASE-1

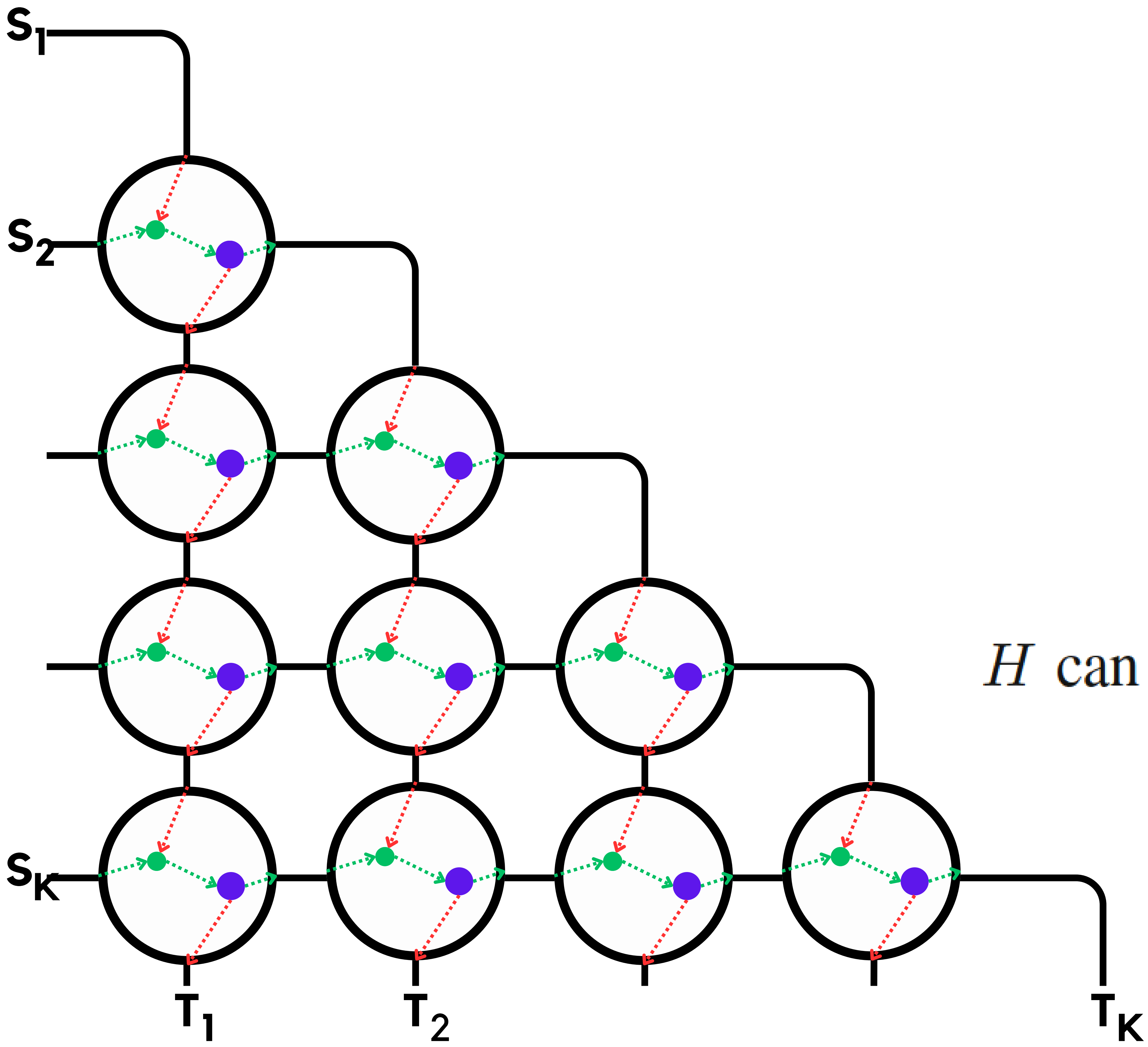
$G$ contains two edge-disjoint paths




$H$ contains $k$ edge-disjoint paths

# CASE - 2

$G$ does not contain two edge-disjoint paths




$H$ can have at most one path

## Number of edges in $H$

$$
\begin{aligned}
m &= \frac{k(k-1)}{2}|E| + k^2 \\
&= \frac{k^2}{2}|E| - \frac{k}{2}|E| + k^2 \\
&\leq \left(\frac{|E|}{2} + 1\right) k^2
\end{aligned}
$$

## Number of edges in $H$

$$m = \frac{k(k-1)}{2}|E| + k^2$$

$$= \frac{k^2}{2}|E| - \frac{k}{2}|E| + k^2$$

$$\leq \left(\frac{|E|}{2} + 1\right)k^2$$

$$\leq k^{\frac{4\epsilon}{(1-2\epsilon)}} \cdot k^2$$

smaller epsilon results in larger amplification

$$\boxed{k = \left\lceil \left(\frac{|E|}{2} + 1\right)^{(1-2\varepsilon)/4\varepsilon} \right\rceil.}$$

## Number of edges in $H$

$$\leq k^{\frac{4\epsilon}{(1-2\epsilon)}} \cdot k^2$$

$$= k^{\frac{4\epsilon}{1-2\epsilon}+2}$$

$$= k^{\frac{4\epsilon+2-4\epsilon}{1-2\epsilon}}$$

$$= k^{\frac{2}{1-2\epsilon}}$$

$$\therefore m \leq k^{\frac{2}{1-2\epsilon}}$$

## Number of edges in $H$

$$k \geq m^{\frac{(1-2\epsilon)}{2}}$$

$$k \geq m^{\frac{1}{2}-\epsilon}$$

**Theorem 10.7** *The problem* MAX-3SAT *does not have a polynomial-time* $(8/7 - \varepsilon)$*-approximation for any* $\varepsilon > 0$ *unless* $\mathbf{P} = \mathbf{NP}$.

A finite field having 2 values

MAXIMUM 3-LINEAR EQUATIONS (MAX-3LIN): Given a system of linear equations over $GF(2)$, where each equation contains exactly three variables, find an assignment to variables that satisfies the maximum number of equations.

MAX-3LIN has an **NP**-hard gap of $[(0.5+\varepsilon)m, (1-\varepsilon)m]$

Out of scope

$$\text{m equations}\left\{\begin{array}{l} X_1 \oplus X_2 \oplus X_3 = 1 \\ X_4 \oplus X_2 \oplus X_3 = 0 \\ X_1 \oplus X_2 \oplus X_5 = 1 \\ \cdots\cdots \\ \cdots\cdots \\ X_I \oplus X_J \oplus X_K = 1 \end{array}\right.$$

Find an assignment that maximizes satisfied equations

# MAX-3-LIN TO MAX-3-SAT REDUCTION

## CASE-1

$$x_i \oplus x_j \oplus x_k = 1 \;\; = \overline{0}$$

| | | | |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

$(\overline{x_i} \wedge \overline{x_j} \wedge \overline{x_k})$

$(\overline{x_i} \wedge x_j \wedge x_k)$

$(x_i \wedge \overline{x_j} \wedge x_k)$

$(x_i \wedge x_j \wedge \overline{x_k})$

$$x_i \oplus x_j \oplus x_k = 1 \;\; = \bar{0}$$

$$= \overline{(\overline{x_i} \wedge \overline{x_j} \wedge \overline{x_k}) \vee (\overline{x_i} \wedge x_j \wedge x_k) \vee (x_i \wedge \overline{x_j} \wedge x_k) \vee (x_i \wedge x_j \wedge \overline{x_k})}$$

$$x_i \oplus x_j \oplus x_k = 1 \;= \overline{0}$$

$$= \overline{(\overline{x_i} \wedge \overline{x_j} \wedge \overline{x_k}) \vee (\overline{x_i} \wedge x_j \wedge x_k) \vee (x_i \wedge \overline{x_j} \wedge x_k) \vee (x_i \wedge x_j \wedge \overline{x_k})}$$

$$= \overline{(\overline{x_i} \wedge \overline{x_j} \wedge \overline{x_k})} \wedge \overline{(\overline{x_i} \wedge x_j \wedge x_k)} \wedge \overline{(x_i \wedge \overline{x_j} \wedge x_k)} \wedge \overline{(x_i \wedge x_j \wedge \overline{x_k})}$$

$$x_i \oplus x_j \oplus x_k = 1 \ = \overline{0}$$

$$= \overline{(\overline{x_i} \wedge \overline{x_j} \wedge \overline{x_k}) \vee (\overline{x_i} \wedge x_j \wedge x_k) \vee (x_i \wedge \overline{x_j} \wedge x_k) \vee (x_i \wedge x_j \wedge \overline{x_k})}$$

$$= \overline{(\overline{x_i} \wedge \overline{x_j} \wedge \overline{x_k})} \wedge \overline{(\overline{x_i} \wedge x_j \wedge x_k)} \wedge \overline{(x_i \wedge \overline{x_j} \wedge x_k)} \wedge \overline{(x_i \wedge x_j \wedge \overline{x_k})}$$

$$= (x_i \vee x_j \vee x_k) \wedge (x_i \vee \overline{x_j} \vee \overline{x_k}) \wedge (\overline{x_i} \vee x_j \vee \overline{x_k}) \wedge (\overline{x_i} \vee \overline{x_j} \vee x_k)$$

## CASE - 2

$$x_i \oplus x_j \oplus x_k = 0 \quad = \overline{1}$$

| | | | | |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | |
| 0 | 0 | 1 | 1 | $(\overline{x_i} \wedge \overline{x_j} \wedge x_k)$ |
| 0 | 1 | 0 | 1 | $(\overline{x_i} \wedge x_j \wedge \overline{x_k})$ |
| 0 | 1 | 1 | 0 | |
| 1 | 0 | 0 | 1 | $(x_i \wedge \overline{x_j} \wedge \overline{x_k})$ |
| 1 | 0 | 1 | 0 | |
| 1 | 1 | 0 | 0 | |
| 1 | 1 | 1 | 1 | $(x_i \wedge x_j \wedge x_k)$ |

$$x_i \oplus x_j \oplus x_k = 0 \;\; = \overline{1}$$

$$= \overline{(\overline{x_i} \wedge \overline{x_j} \wedge x_k) \vee (\overline{x_i} \wedge x_j \wedge \overline{x_k}) \vee (x_i \wedge \overline{x_j} \wedge \overline{x_k}) \vee (x_i \wedge x_j \wedge x_k)}$$

$$x_i \oplus x_j \oplus x_k = 0 \ = \overline{1}$$

$$= \overline{(\overline{x_i} \wedge \overline{x_j} \wedge x_k) \vee (\overline{x_i} \wedge x_j \wedge \overline{x_k}) \vee (x_i \wedge \overline{x_j} \wedge \overline{x_k}) \vee (x_i \wedge x_j \wedge x_k)}$$

$$= \overline{(\overline{x_i} \wedge \overline{x_j} \wedge x_k)} \wedge \overline{(\overline{x_i} \wedge x_j \wedge \overline{x_k})} \wedge \overline{(x_i \wedge \overline{x_j} \wedge \overline{x_k})} \wedge \overline{(x_i \wedge x_j \wedge x_k)}$$

$$x_i \oplus x_j \oplus x_k = 0 \quad = \overline{1}$$

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$$= (x_i \vee x_j \vee \overline{x_k}) \wedge (x_i \vee \overline{x_j} \vee x_k) \wedge (\overline{x_i} \vee x_j \vee x_k) \wedge (\overline{x_i} \vee \overline{x_j} \vee \overline{x_k})$$

(i) If an assignment satisfies $e$ (or, $e'$), then the same assignment satisfies four clauses in $f_e$ (or, respectively, in $f_{e'}$).

(ii) If an assignment does not satisfy $e$ (or, $e'$), then the same assignment satisfies exactly three clauses in $f_e$ (or, respectively, in $f_{e'}$).

# CASE-1

$\boxed{\text{Max-3Lin has an } \mathbf{NP}\text{-hard gap of } [(0.5+\varepsilon)m, (1-\varepsilon)m]}$

$$\text{Satisfied equations} \implies (0.5+\epsilon)m$$

$$\begin{aligned}\text{Unsatisfied equations} \implies &\{1-(0.5+\epsilon)\}m \\ &= (0.5-\epsilon)m\end{aligned}$$

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$$\text{Satisfied equations} \implies (0.5+\epsilon)m$$

$$\begin{aligned}\text{Unsatisfied equations} &\implies \{1-(0.5+\epsilon)\}m \\ &= (0.5-\epsilon)m\end{aligned}$$

**For satisfied equations**

$$\begin{aligned}\text{No. of clauses satisfied} &\implies 4 \times (0.5+\epsilon)m \\ &= (2+4\epsilon)m\end{aligned}$$

**For unsatisfied equations**

$$\begin{aligned}\text{No. of satisfied clauses} &\implies 3 \times (0.5-\epsilon)m \\ &= (1.5-3\epsilon)m\end{aligned}$$

# CASE-1

$\boxed{\text{MAX-3LIN has an } \mathbf{NP}\text{-hard gap of } [(0.5+\varepsilon)m, (1-\varepsilon)m]}$

$$\text{Satisfied equations} \implies (0.5+\epsilon)m$$

$$\begin{aligned}\text{Unsatisfied equations} &\implies \{1-(0.5+\epsilon)\}m \\ &= (0.5-\epsilon)m\end{aligned}$$

## For satisfied equations

$$\begin{aligned}\text{No. of clauses satisfied} &\implies 4\times(0.5+\epsilon)m \\ &= (2+4\epsilon)m\end{aligned}$$

## For unsatisfied equations

$$\begin{aligned}\text{No. of satisfied clauses} &\implies 3\times(0.5-\epsilon)m \\ &= (1.5-3\epsilon)m\end{aligned}$$

$$\therefore \text{Total satisfied clauses} \implies (3.5+\epsilon)m$$

# CASE - 2

$\boxed{\textsc{Max-3Lin} \text{ has an } \mathbf{NP}\text{-hard gap of } [(0.5+\varepsilon)m, (1-\varepsilon)m]}$

$$\text{Satisfied equations} \implies (1-\epsilon)m$$

$$\text{Unsatisfied equations} \implies \{1-(1-\epsilon)\}m$$

**For satisfied equations**

$$\text{No. of clauses satisfied} \implies 4 \times (1-\epsilon)m$$
$$= (4-4\epsilon)m$$

**For unsatisfied equations**

$$\text{No. of satisfied clauses} \implies 3 \times \epsilon m$$
$$= 3\epsilon m$$

$$\therefore \text{Total satisfied clauses} \implies (4-\epsilon)m$$

MAX-3SAT has an **NP**-hard gap of $[(3.5+\varepsilon)m, (4-\varepsilon)m]$

## Approximation Ratio

$$\text{Approximation ratio} \Rightarrow \frac{4-\epsilon}{3.5+\epsilon}$$

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$$\text{Approximation ratio} \Rightarrow \frac{4-\epsilon}{3.5+\epsilon}$$

$$> \frac{4-\epsilon-2.5\epsilon}{3.5+\epsilon-\epsilon}$$

$$\frac{a}{b} > \frac{a-mc}{b-c}$$

$$a(b-c) > b(a-mc)$$

$$ab-ac > ab-bmc$$

$$-ac > -bmc$$

$$a < mb$$

## Approximation Ratio

$$\begin{aligned}
\text{Approximation ratio} \Rightarrow & \frac{4-\epsilon}{3.5+\epsilon} \\
& > \frac{4-\epsilon-2.5\epsilon}{3.5+\epsilon-\epsilon} \\
& = \frac{4-3.5\epsilon}{3.5} \\
& = \frac{4}{3.5}-\epsilon \\
& = \frac{8}{7}-\epsilon
\end{aligned}$$

$$\therefore \text{Ratio} \Rightarrow \frac{8}{7}-\epsilon$$

# Approximation Ratio

$$\frac{4-\varepsilon}{3.5+\varepsilon} \longrightarrow \frac{8}{7},$$

as $\varepsilon \to 0$. This completes the proof of this theorem.

**Theorem 10.8** *The problem* MIN-VC *does not have a polynomial-time* $(7/6 - \varepsilon)$*-approximation for any* $\varepsilon > 0$ *unless* $\mathbf{P} = \mathbf{NP}$.

# MAX-3-LIN TO MIN-VC REDUCTION

$$\text{m equations} \begin{cases} X_1 \oplus X_2 \oplus X_3 = 1 \\ X_4 \oplus X_2 \oplus X_3 = 0 \\ X_1 \oplus X_2 \oplus X_5 = 1 \\ \cdots\cdots \\ \cdots\cdots \\ X_I \oplus X_J \oplus X_K = 1 \end{cases}$$

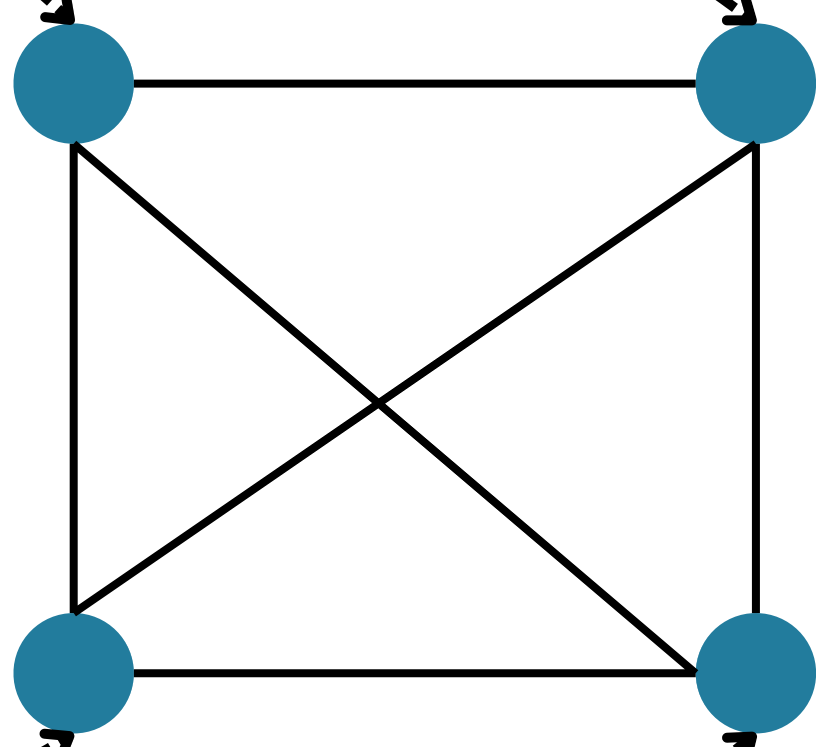
Find an assignment that maximizes satisfied equations

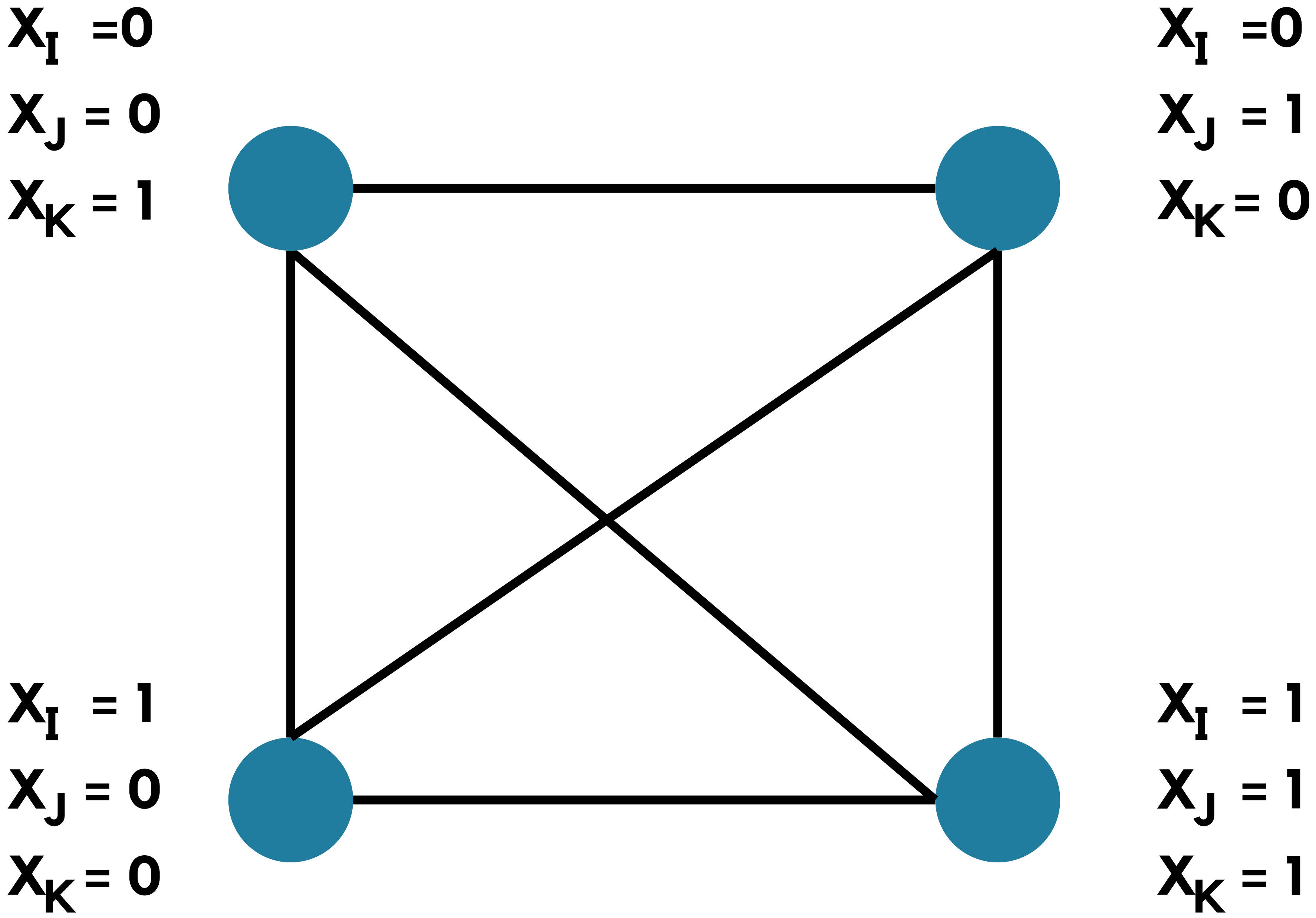
## CASE-1

$$X_I \oplus X_J \oplus X_K = 1$$

| $X_I$ | $X_J$ | $X_K$ | |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

4-NODE COMPLETE GRAPH



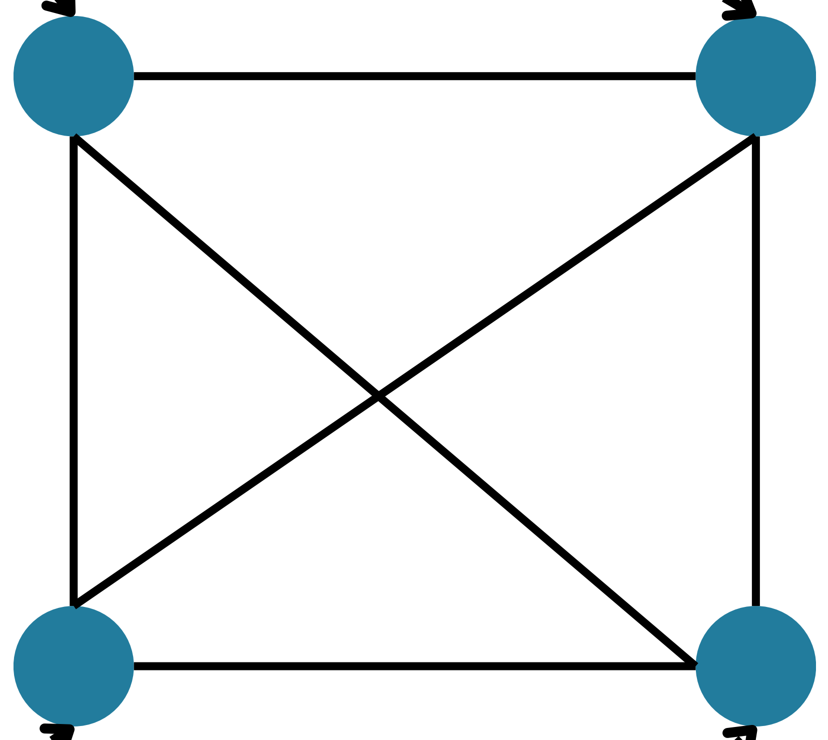


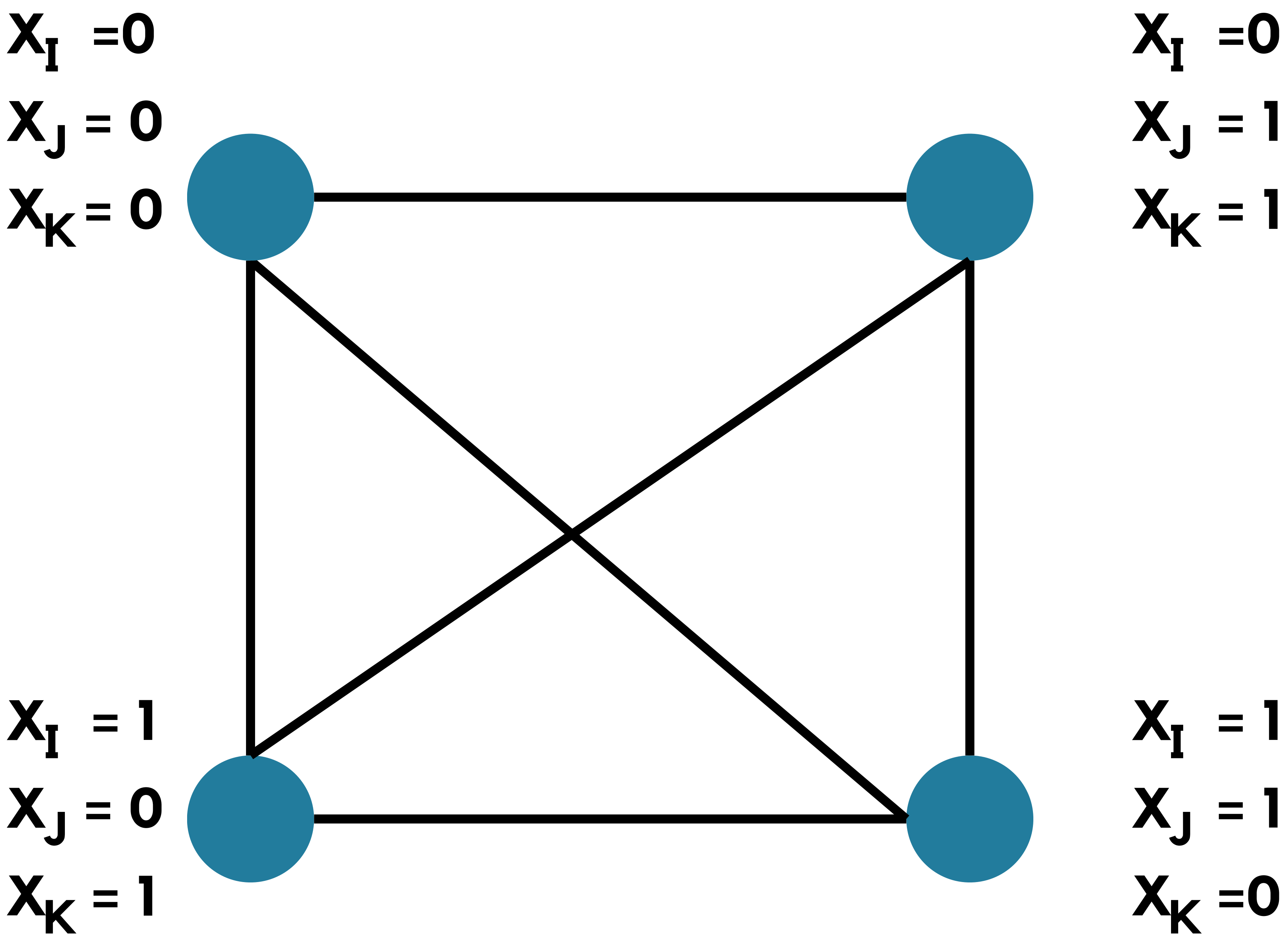
$X_I$ =0
$X_J$ = 0
$X_K$ = 1
$X_I$ =0
$X_J$ = 1
$X_K$ = 0
$X_I$ = 1
$X_J$ = 0
$X_K$ = 0
$X_I$ = 1
$X_J$ = 1
$X_K$ = 1

## CASE - 2

$$X_I \oplus X_J \oplus X_K = 0$$

| $X_I$ | $X_J$ | $X_K$ | |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |




$X_I = 0$
$X_J = 0$
$X_K = 0$
$X_I = 0$
$X_J = 1$
$X_K = 1$
$X_I = 1$
$X_J = 0$
$X_K = 1$
$X_I = 1$
$X_J = 1$
$X_K = 0$

# CASE-1

$\boxed{\text{MAX-3LIN has an } \mathbf{NP}\text{-hard gap of } [(0.5+\varepsilon)m, (1-\varepsilon)m]}$

At least $(1-\epsilon)m$ equations are satisfied.

$\implies$ At least $(1-\epsilon)m$ labels of vertices are satisfied [one vertex per graph].

## CASE-1

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At least $(1-\epsilon)m$ equations are satisfied.

$\implies$ At least $(1-\epsilon)m$ labels of vertices are satisfied [one vertex per graph].

$\therefore$ Independent set size $\geq (1-\epsilon)m$

Total vertices $= 4m$

# CASE-1

$\boxed{\textsc{Max-3Lin} \text{ has an } \textbf{NP}\text{-hard gap of } [(0.5+\varepsilon)m, (1-\varepsilon)m]}$

At least $(1-\epsilon)m$ equations are satisfied.

$\implies$ At least $(1-\epsilon)m$ labels of vertices are satisfied [one vertex per graph].

$\therefore$ Independent set size $\geq (1-\epsilon)m$

Total vertices $= 4m$

$\therefore$ Vertex cover size $\leq 4m - (1-\epsilon)m = (3+\epsilon)m$

## CASE - 2

$\boxed{\text{MAX-3LIN has an \textbf{NP}-hard gap of } [(0.5+\varepsilon)m, (1-\varepsilon)m]}$

fewer than $(0.5 + \epsilon)m$ equations are satisfied

$\Rightarrow$ fewer than $(0.5 + \epsilon)m$ labels of vertices are satisfied [one vertex per graph].

$\therefore$ Independent set size $< (0.5 + \epsilon)m$

Total vertices $= 4m$

$\therefore$ Vertex cover size $> 4m - (0.5 + \epsilon)m = (3.5 - \epsilon)m$

MIN-VC has an **NP**-hard gap of $[(3+\varepsilon)m, (3.5-\varepsilon)m]$

## Approximation Ratio

$$\text{Approximation ratio} \Rightarrow \frac{3.5-\epsilon}{3+\epsilon}$$

# Approximation Ratio

$$\text{Approximation ratio} \Rightarrow \frac{3.5-\epsilon}{3+\epsilon}$$

$$> \frac{3.5-\epsilon-2\epsilon}{3+\epsilon-\epsilon}$$

$$\frac{a}{b} > \frac{a-mc}{b-c}$$

$$a(b-c) > b(a-mc)$$

$$ab-ac > ab-bmc$$

$$-ac > -bmc$$

$$a < mb$$

## Approximation Ratio

$$\text{Approximation ratio} \Rightarrow \frac{3.5-\epsilon}{3+\epsilon}$$

$$> \frac{3.5-\epsilon-2\epsilon}{3+\epsilon-\epsilon}$$

$$= \frac{3.5-3\epsilon}{3}$$

$$= \frac{3.5}{3}-\epsilon$$

$$= \frac{7}{6}-\epsilon$$

$$\therefore \text{Ratio} \Rightarrow \frac{7}{6}-\epsilon$$

## Approximation Ratio

$$\frac{3.5 - \varepsilon}{3 + \varepsilon} \longrightarrow \frac{7}{6}$$

as $\varepsilon \to 0$.

THANK YOU....