On Composed Shapes

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1 Introduction

Most shapes that are useful to render in raytracing are composed of a few primitive functions. Some of these primitives are the sphere, the plane, the cone, the torus, and the cylinder. With creative distortions and combinations of these shapes, many images depicting real-life objects can be rendered. Looking around the room I am writing this article from, I am hard pressed to find objects for which this is not true. There are many shapes which are fundamental, but not considered *primitive.* The box is such a shape, as it is comprised of planes, a more primitive shape. This article aims to prove two important theorems relating composite shapes to their constituents, and provide a framework by which to develop SDFs of the former using those of the latter.

2 Some Definitions

To develop the necessary results, we must first define some useful terms.

Definition 1. A shape can be defined using a *nongeneral signed distance function*, $D : \mathbb{R}^3 \to \mathbb{R}$. The level set $D(\vec{x}) = 0$ represents the surface of the (analogous physical) shape itself, and regions where $D(\vec{x}) < 0$ are analagous to the inside of the shape. We say that $D(\vec{x_{in}}) \leq 0$ is equivalent to $\vec{x_{in}}$ being a *member* of the shape.

Additionally, a shape has a generalized distance function, $\phi : \mathrm{U} \times \mathbb{R}^3 \to \mathbb{R}$, $\mathrm{U} = \{\vec{u} \in \mathbb{R}^3 : \| \vec{u} \| = 1\}$. $\phi(\vec{p}, \hat{v})$ is defined if and only if $\exists s \in \mathbb{R} : D(\vec{p} + s\hat{v}) = 0$. Then, the generalized distance function is defined as the smallest such value, s, that satisfies this condition.

Definition 2. A shape S is *convex* if $\forall \vec{x_1}, \vec{x_2} \in S; \forall 0 \leq k \leq 1; \vec{x_1} + (1-k)\vec{x_2} \in S$. That is, for any two points in the shape, the line segment between them is fully contained in the shape.

Definition 3. A shape S is called *primitive* if there exists a closed-form inclusion function $I : \mathbb{R}^3 \to \mathbb{R}$, and constant $k \in \mathbb{R}$ by which membership of S is defined by $I(\vec{x}) \geq k$, $\forall \vec{x} \in \mathbb{R}^3$. I does not use min or max functions. Furthermore, the nongeneral distance function D must be Lipschitz continuous, and $I(\vec{x}) = k$ is the same as the zero contour of the SDF, $D(\vec{x}) = 0$.

Definition 4. A primitive, S, divides space into two regions, called the *region of inclusion* and *region of exclusion.* They are defined as the regions in which $I(\vec{x}) \geq k$ and $I(\vec{x}) < k$, respectively.

Definition 5. A shape, M is a called a *macroshape* if it can be described using the inclusion functions of one or more primitives. Macroshapes may be composed of other macroshapes or primitives, called *components*. The macroshape has a generalized SDF function, $\phi_M : \mathrm{U} \times \mathbb{R}^3 \to \mathbb{R}$, $\mathrm{U} = \{\vec{u} \in \mathbb{R}^3 :\| \vec{u} \|= 1\}$, that returns the distance from a point $\vec{p} \in \mathbb{R}^3$, along $\hat{v} \in \mathrm{U}$, to the contour $\phi_{d,M}(\vec{x}) = 0$.

Definition 6. We say a macroshape M is a *union macroshape* (union) if $\vec{x} \in M \iff \vec{x} \in S_1 \cup S_2 \cup ... S_n$, and its membership can be described as $I_1(\vec{x}) \geq k_1 \vee I_2(\vec{x}) \geq k_2 \vee ... I_n(\vec{x}) \geq k_n$, or alternatively, $D_1(\vec{x}) \leq 0 \vee D_2(\vec{x}) \leq 0 \vee ... D_n(\vec{x}) \leq 0$.

Definition 7. We say a macroshape M is an *intersection macroshape* (intersection) if $\vec{x} \in M \iff \vec{x} \in S_1 \cap S_2 \cap ... S_n$, and its membership can be described as $I_1(\vec{x}) \geq k_1 \wedge I_2(\vec{x}) \geq k_2 \wedge ... I_n(\vec{x}) \geq k_n$.

3 SDFs of Macroshapes

Now we present two results that allow, in principle, the formulation of an SDF for any macroshape.

Theorem. *M is a union macroshape comprised of n convex primitives; $S_1, S_2, ...S_n$. Then, the generalized SDF, $\phi_M(\vec{p}, \hat{v}) : \mathrm{U} \times \mathbb{R}^3 \to \mathbb{R}$, $\mathrm{U} = \{\vec{u} \in \mathbb{R}^3 :\| \vec{u} \|= 1\}$, satisfies $\phi_M = \min(\phi_1, \phi_2, ...\phi_n)$ for $\vec{p} \notin M$, and $\phi_M = \max(\phi_1, \phi_2, ...\phi_n)$ for $\vec{p} \in M$, where ϕ_k is the general SDF of S_k.*

Proof. We consider this idea in the context of the ray intersection problem, with focal point $\vec{p}$ and ray direction $\hat{v}$. We denote the minimum primitive SDF as $\phi_i = \min(\phi_1, \phi_2, ...\phi_n)$, with ϕ_k being interchangable with $\phi_k(\vec{p}, \hat{v})$.

We evaluate the cases $\vec{p} \notin M$ and $\vec{p} \in M$ separately.

To evaluate the first case, $\vec{p} \notin M$, we consider the point $\vec{p} + \phi_i \hat{v}$.

$s = \phi_k$ is the smallest value for which $D_k(\vec{p} + s\hat{v}) = 0; k \in \{1, 2, ...n\}$.

M, being a union macroshape, is defined with $\vec{x_{in}} \in S_k \subset M \implies \vec{x_{in}} \in M$.

$\phi_i \leq \phi_k; \forall k \leq n$, so $s = \phi_i$ is the smallest value for which $D_k(\vec{p} + s\hat{v}) = 0; k \in \{1, 2, ...n\}$.

$D_i(\vec{p} + \phi_i \hat{v}) = 0 < D_j(\vec{p} + \phi_i \hat{v}) = \phi_j; \forall j \neq i \leq n$.

Thus, we conclude that the shortest distance along $\vec{p} + s\hat{v}$ at which the ray is a member of M is ϕ_i, and therefore $\phi_M = \phi_i$.

In the case $\vec{p} \in M$, we consider that $\vec{p} \in S_1 \wedge \vec{p} \in S_2 \wedge ... \vec{p} \in S_n$. We denote $\max(\phi_1, \phi_2 ... \phi_n) = \phi_a$

The shortest distance, ϕ, at which $D_k(\vec{p} + \phi\hat{v}) = 0; \forall k \leq n$ is ϕ_M, the value of the general SDF.

With $\vec{p} \in M$, for each primitive, S_k there is only one set along $\hat{v}$, $\vec{p} + d\hat{v}; d \in [-\alpha_k, \phi_k]$, where $D(\vec{p} + d\hat{v}) \leq 0$ and $\vec{p} + d\hat{v} \in S_k$. If there was another such region, we could easily see that this would violate convexity, as a line joining the two regions would contain points that are not members of S_k,.

Thus, $D_k(\vec{p}+d\hat{v}); d > \phi_k \vee d < -\alpha_k \implies \vec{p}+d\hat{v} \notin M$.

We search for the condition at which $D_k(\vec{p}+d\hat{v}); d > \phi_k \vee d < -\alpha_k; \forall k \leq n$, or since we only regard forward (positive) solutions, $D_k(\vec{p}+\phi_k\hat{v}); \forall k \leq n$.

$d = \phi_a \geq \phi_k; \forall k \leq n$. ϕ_a is defined as the smallest distance at which $D_a(\vec{p}+d\hat{v}) = 0$, so there exists no smaller value for which $D_M(\vec{p}+d\hat{v}) = 0$.

Thus $\phi_M = \phi_{max}$ for a union macroshape of convex primitives. □

This tells us that we can easily compute the SDF of a union macroshape, given we know the SDFs of the constituents. We also know that it can be done for the computational cost of computing the constituent SDFs. The next theorem uses a similar argument on intersection macroshapes.

Theorem. *M is an intersection macroshape comprised of n convex primitives; $S_1, S_2, ...S_n$. Then, the SDF, $\phi_M : \mathrm{U} \times \mathbb{R}^3 \to \mathbb{R}$, $\mathrm{U} = \{\vec{u} \in \mathbb{R}^3 : \parallel \vec{u} \parallel = 1\}$, satisfies $\phi_M(\vec{p},\hat{v}) = \max(\phi_1,\phi_2,...\phi_n)$ for $\vec{p} \notin M$ and $\phi_M(\vec{p},\hat{v}) = \min(\phi_1,\phi_2,...\phi_n)$, if and only if $\exists \vec{x_{in}} = \vec{p}+s\hat{v} \in M$.*

Proof. Consider the following cases: $\vec{p} \in M$ and $\vec{p} \notin M$.

In the case $\vec{p} \notin M$, $\phi_i = \max(\phi_1,\phi_2,...\phi_n) > 0$.

For each primitive, we define a length, l_k, over which $D_k(\vec{p}+(\phi_k+w)\hat{v}) < 0; \forall 0 \leq w \leq l$.

By the definition of convexity, for each primitive component of M, S_k, $\vec{x_{in}} \in S_k \implies \vec{x_{in}} = \vec{p}+s\hat{v}; \phi_k \leq s \leq \phi_k + l_k$. If there were another separate region along $\hat{v}$ over which solutions were defined, we could see quickly that any line segment joining the two regions would not be fully contained in M.

Given that $\exists \vec{x_{in}} = \vec{p}+s\hat{v} \in M$ (that a solution exists), and that $D_k(\vec{x_{in}}) \leq D_a(\vec{x_{in}}) = 0; \forall k \leq n$ is a necessary condition for inclusion, $\vec{x_{in}} \in M \implies \vec{x_{in}} = \vec{p}+s\hat{v}; \phi_k \leq s \leq \phi_k + l_k; \forall k \leq n$.

Because M is an intersection macroshape, we can specify the previous inequality in terms of the maximum ϕ_k, which is ϕ_a, and the minimum $\phi_k + l_k$, we will denote this as b.

Then, $\vec{x_{in}} \in M \Rightarrow \vec{x_{in}} = \vec{p}+s\hat{v}; \phi_a \leq s \leq b; \forall k \leq n$. We can say that the minimum distance s for which $\vec{p}+s\hat{v} \in M$ is $s = \phi_M = \phi_a$.

Thus, in the case $\vec{p} \notin M$, $\phi_M = \phi_a$

To develop the case $\vec{p} \in M$ we use a similar logic from the $\vec{p} \notin M$ case of the union macroshape. $\phi_i = \min(\phi_1,\phi_2,...\phi_n) > 0$

$s = \phi_k$ is the smallest value for which $D_k(\vec{p}+s\hat{v}) = 0; k \in \{1,2,...n\}$.

M, being an intersection macroshape, satisfies $\vec{x_{in}} \notin S_k \subset M \implies \vec{x_{in}} \notin M$.

$\phi_i \leq \phi_k; \forall k \leq n$, so $s = \phi_i$ is the smallest value for which $D_k(\vec{p}+s\hat{v}) = 0; k \in \{1,2,...n\}$.

$D_i(\vec{p}+\phi_i\hat{v}) = 0 < D_j(\vec{p}+\phi_i\hat{v}) = \phi_j; \forall j \neq i \leq n$.

Thus, we conclude that the shortest distance along $\vec{p}+s\hat{v}$ at which the ray is a member of M is ϕ_i, and therefore $\phi_M = \phi_i$. □

4 Implications

These results tell us that the SDF of any shape comprised of the intersections or unions of convex shapes can be computed at the cost of the constituents. Any shape with both intersections and unions can be subjected to this method, as any two constituents can be represented as a union or intersection, and subsequently combined recursively to obtain the macroshape. In the context of raytracing, this tells us that the only computational constraint for rendering a single pixel is the number of shapes and the cost of their SDFs. We do not need to worry about the "ricochet" effect that happens near the edges of objects in traditional raymarching or uncertainty in the number of iterations needed per pixel.

5 A Brief Almanac of Primitive Shapes

5.1 The Plane

The plane has inclusion function $I(\vec{x}) = (\vec{x} - \vec{a}) \cdot \hat{n} \geq 0$, where $\vec{a}$ rests somewhere on the plane, and $\hat{n}$ is the normal vector. Notice membership is defined as being above the plane.

5.2 The Sphere

We consider the sphere a primitive, with inclusion function $I(\vec{x}) = \| \vec{x} - \vec{c} \| \leq r^2$, where we think of $\vec{c}$ as being the center and r as the radius.

5.3 The Cone

$I(\vec{x}) = \mathrm{unit}(\vec{x} - \vec{b}) \cdot \hat{a} \geq \cos\theta$, where $\vec{b}$ is the corner of the cone, $\hat{a}$ is the axial direction, and θ is the angle between the cone's surface and the axial direction.

5.4 The Cylinder

$I(\vec{x}) = ||(\vec{x} - \vec{b})||^2 - (\vec{x} \cdot \hat{a})^2 \leq r^2$, where $\vec{b}$ is some point on the axis, $\vec{a}$ is the axial direction, and $r$is the radius of the cylinder.