Generalized Signed Distance Functions for Raytracing Algorithms

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Abstract

Currently, many raytracing algorithms use signed distance functions (SDFs) that map points in 3D space to diatances from the objects that are to be imaged. For each pixel, a recursive algorithm using SDF determines the intersection of the light ray passing through the pixel and the object. Optimization of such algorithms is important in making raytracing usable, as it is computationally costly by comparison to other methods of image rendering. We propose to eliminate the necessity for an algorithmic determination of the intersection point, and use a generalized SDF that considers the position and direction of the ray in one evaluation per pixel. The properties of the SDFs that allow for the composition of elaborate multi-object scenes hold for the generalized SDF, making it a potentially faster alternative with few drawbacks.

Standard SDFs and their generalized counterparts

The standard SDF takes the form $\phi(\boldsymbol{p})$, returning the minimum distance from the point $\boldsymbol{p}$ to the object being imaged. It is often better stated in code as many of SDFs are most cleanly expressed in two or three steps, and meant to occur in code anyway.

The generalized SDF takes the form $\phi_G(\hat{\boldsymbol{v}}, \boldsymbol{p})$ and returns the distance from $\boldsymbol{p}$ to the object along the direction of the unit vector, $\hat{\boldsymbol{v}}$. As with the standard SDF, some of them best experessed in multiple lines of code.

All standard SDFs shown are those available on Inigo Quilez's website, `https://iquilezles.org/www/articles/distfunctions/distfunctions.htm`. The standard and generalized versions of the SDFs often sometimes use different descriptions of the same object. This is not a problem, as both descriptions give exactly enough information to define one (and only one) object of that type. The full set of characteristics corresponding to one description implies those of the other. In most cases, I have altered Quilez's SDF in order to match the object definitions and allow for arbitrary translations and rotations.

These SDFs uphold the property that they are negative when $\vec{p}$ is inside the shape. The notion of "inside" depends on the chosen orientation of the shape; on which side of the shape is defined as being "in" the shape. For example, a sphere could be defined with $\|\boldsymbol{p}-\boldsymbol{c}\| \geq r \implies$ inside or $\|\boldsymbol{p}-\boldsymbol{c}\| \leq r \implies$ inside; a plane could be defined as $(\boldsymbol{p}-\boldsymbol{a}) \cdot \hat{\boldsymbol{m}} \geq k$, $(\boldsymbol{p}-\boldsymbol{a}) \cdot \hat{\boldsymbol{m}} \leq k$, or $(\boldsymbol{p}-\boldsymbol{a}) \cdot \hat{\boldsymbol{m}} = k$ defining "inside". Choices of orientation can help us construct interesting "composite shapes" like snowcones or hollowed out spheres.

The plane

Both the generalized and standard SDF use the a point on the plane, $\boldsymbol{s}$, and a unit normal vector, $\hat{\boldsymbol{n}}$ to represent the plane. The SDFs are as follows:

Standard SDF:

$$\boxed{\phi(\boldsymbol{p}) = (\boldsymbol{p}-\boldsymbol{s}) \cdot \boldsymbol{n}}$$

General SDF:

$$\boxed{\phi_G(\hat{\boldsymbol{v}}, \boldsymbol{p}) = \frac{(\boldsymbol{p}-\boldsymbol{s} \cdot \hat{\boldsymbol{n}}}{\hat{v} \cdot \hat{\boldsymbol{n}}}}$$

The sphere

The general and standard both use a center point, $\boldsymbol{c}$, and a radius, r to describe the sphere.

Standard SDF:

$$\boxed{\phi(\boldsymbol{p}) = \|\boldsymbol{p}-\boldsymbol{c}\| - r}$$

General SDF:

$$\boxed{\phi_G(\hat{v}, \boldsymbol{p}) = -\hat{\boldsymbol{v}} \cdot (\boldsymbol{p}-\boldsymbol{c}) \pm \sqrt{r^2 + (\hat{\boldsymbol{v}} \cdot (\boldsymbol{p}-\boldsymbol{c}))^2 - \|(\boldsymbol{p}-\boldsymbol{c})\|^2}}$$

The infinite cylinder

The cylinder is described by a vector $\hat{\boldsymbol{a}}$ representing the axial direction, a radius r, and a point $\boldsymbol{c}$ which lies on the axis. We first compute the unit vector that is perpendicular to the axial direction, which points from $\boldsymbol{p}$ towards the axis.

$$\hat{\boldsymbol{v}_\perp} = \text{unit}(\hat{\boldsymbol{v}} - (\hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{v}})\hat{\boldsymbol{a}}),$$

Standard SDF:

We think of our solution as being the sphere SDF evaluated along the $\hat{v_\perp}$ direction, as the closest distance is along the direction perpendicular to the axis.

$$\boxed{\phi(\boldsymbol{p}) = \|(\boldsymbol{p}-\boldsymbol{c}) \cdot \hat{\boldsymbol{v}_\perp}\| - r}$$

General SDF:
The solution is effectively a sphere (better conceptualized as a circle here) SDF evaluated along a plane perpendicular to $\hat{\boldsymbol{a}}$, then scaled by $\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{v}_\perp}$ to account for distance in the axial direction.

$$\boxed{\phi_G(\hat{\boldsymbol{v}}, \boldsymbol{p}) = \frac{-\hat{\boldsymbol{v}_\perp} \cdot (\boldsymbol{p} - \boldsymbol{c}) \pm \sqrt{r^2 + (\hat{\boldsymbol{v}_\perp} \cdot (\boldsymbol{p} - \boldsymbol{c}))^2 - \|(\boldsymbol{p} - \boldsymbol{c})\|^2}}{\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{v}_\perp}}}$$

The infinite cone

The cone is defined by a vertex point,$\boldsymbol{b}$, an axial direction unit vector, $\hat{\boldsymbol{a}}$, and an angle θ, representing the angle between the axis and the surface of cone.
Standard SDF:
With the substitutions

$$\boldsymbol{t} = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

$$\boldsymbol{q} = \begin{pmatrix} p_\perp \\ p_a \end{pmatrix},$$

where $p_\perp$ is the length of $\boldsymbol{p}$ projected onto the plane orthogonal to $\hat{\boldsymbol{a}}$, we have

$$\boxed{\phi(\boldsymbol{p}) = \|\boldsymbol{q} - \boldsymbol{t} \max(\boldsymbol{q} \cdot \boldsymbol{t}, 0)\|}.$$

General SDF:

$$\boxed{\phi_G(\hat{\boldsymbol{v}}, \boldsymbol{p}) = \frac{k^2\alpha - \beta\gamma \pm 2k\sqrt{k^2(\alpha^2 - d^2) + \beta^2 + d^2\gamma^2 - 2\alpha\beta\gamma}}{\gamma^2 - k^2}},$$

where

$$\alpha = (\boldsymbol{p} - \boldsymbol{b}) \cdot \hat{\boldsymbol{v}}$$

$$\beta = (\boldsymbol{p} - \boldsymbol{b}) \cdot \hat{\boldsymbol{a}}$$

$$\gamma = \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{a}}$$

$$k = \cos\theta.$$

We accept only positive values, as negative values correspond to distances that must be traveled backward along the ray to reach the shape.

In the context of generalized SDFs, the next few shapes are different than the ones shown up to this point. Geometrically they are the combinations of shapes whose SDFs we have already determined, so their SDFs are determined by the minimum or maximum of the SDFs of their constituents. If such a shape is the intersection of its constituents, we take the maximum of the constituents' SDFs, and if it is the union of its constituents, we take the minimum of its constituents.

There is a similar notion of unions of shapes corresponding to minimums of their SDFs for the standard SDF, but the same does not necessarily hold for intersections and maximums.

The box

The box is defined by three unit vectors, $\hat{\boldsymbol{n_1}}, \hat{\boldsymbol{n_2}}, \hat{\boldsymbol{n_3}}$, which are normal to the three faces, d_1, d_2, d_3, the length of the box in each of those directions, and $\boldsymbol{k}$ the corner of the box closest to $\boldsymbol{p}$.

Standard SDF:

To easily notate the basis of $\hat{\boldsymbol{n_1}}, \hat{\boldsymbol{n_2}}, \hat{\boldsymbol{n_3}}$, we say

$$x_k \equiv \boldsymbol{x} \cdot \hat{\boldsymbol{n_k}}.$$

We then substitute

$$\boldsymbol{d} = \boldsymbol{p} - \boldsymbol{k}$$

$$\boldsymbol{m} = \begin{pmatrix} \max(0, d_1) \\ \max(0, d_2) \\ \max(0, d_3) \end{pmatrix}$$

$$\boxed{\phi(\boldsymbol{p}) = \|\boldsymbol{m}\| + \min(\max(d_1, d_2, d_3), 0)}$$

General SDF:

We call an object defined by

$$(\boldsymbol{x} - \boldsymbol{a}) \cdot \hat{m} \geq k$$

a plane inequality. The box, being the intersection of three plane inequalities, we simply take the maximum of the three planes:

$$\boxed{\phi_G(\hat{\boldsymbol{v}}, \boldsymbol{p}) = \max\left(\phi_{p1}, \phi_{p2}, \phi_{p3}\right)},$$

where ϕ_{pk} is the SDF of the plane normal to $\hat{\boldsymbol{n_k}}$, passing through $\boldsymbol{k}$.

The finite cylinder

The finite cylinder is determined by an infinite cylinder, and two planes (to cap the cylinder on either end). The planes are oriented to be positive in the direction away form eachother (if you flipped the orientation of the planes, you would have an infinite cylinder with a finite gap between the two planes). In addition to the characteristics defining the two constituent planes and cylinder, we introduce $\boldsymbol{b}$, the center of the circular face, and the height, h, of the cylinder. We substitute $\boldsymbol{d} = \boldsymbol{p} - \boldsymbol{b}$.

Standard SDF:
We substitute

$$\boldsymbol{d} = \boldsymbol{p} - \boldsymbol{b}$$

and

$$\boldsymbol{d} \cdot \hat{\boldsymbol{a}} = d_a$$

$$\boxed{\phi(\boldsymbol{p}) = \sqrt{\text{clamp}(h, d_a, 0)^2 + (\|(\boldsymbol{d})_\perp\| - r)^2}}.$$

General SDF:
The finite cylinder is composed of two parallel plane inequalities with opposite orientations (one plane has inclusion above and the other has inclusion below) and the infinite cylinder.

$$\boxed{\phi_G(\hat{\boldsymbol{v}}, \boldsymbol{p}) \max\left(\phi_{p1}, \phi_{p2}, \phi_c\right)},$$

where ϕ_c is the SDF of the cylinder, and $\phi_{p1,2}$ are the SDFs of the planes.

The finite cone

To define the finite cone we simply use the characteristics associated with the corresponding infinite cone and the plane containing the flat face. Choice of plane orientation determines whether the conic piece contains the vertex or the ever-expanding base of the cone. One could also use two planes to take a finite piece of the cone not containing the vertex, or render a cone with a missing chunk.

Standard SDF:

$$\boldsymbol{q} = h \begin{pmatrix} \tan\theta \\ -1 \end{pmatrix}$$

$$\boldsymbol{w} = \begin{pmatrix} \|\boldsymbol{p} - (\boldsymbol{p} \cdot \hat{\boldsymbol{a}})\| \\ \\ \boldsymbol{p} \cdot \hat{\boldsymbol{a}} \end{pmatrix}$$

$$\boldsymbol{a} = \boldsymbol{w} - \boldsymbol{q}\text{clamp}(\frac{\boldsymbol{w} \cdot \boldsymbol{q}}{\|\boldsymbol{q}\|^2}, 0, 1)$$

$$\boldsymbol{b} = \boldsymbol{w} - \boldsymbol{q} \begin{pmatrix} \text{clamp}(\frac{\|\boldsymbol{p} - (\boldsymbol{p} \cdot \hat{\boldsymbol{a}})\|}{\tan\theta}, 0, 1) \\ \\ 1 \end{pmatrix}$$

$$k = \text{sign}(\boldsymbol{q} \cdot \hat{\boldsymbol{a}})$$

$$d = \min(\boldsymbol{a}^2, \boldsymbol{b}^2)$$

$$s = \max(k\boldsymbol{w} \times \boldsymbol{q}, k(\boldsymbol{p} \cdot \hat{\boldsymbol{a}} + h))$$

$$\boxed{\phi(\boldsymbol{p}) = \text{sign}(s)\sqrt{d}}$$

General SDF:
As the cone is the union of a plane inequality and a cone, the SDF is simply

$$\boxed{\phi_G(\hat{\boldsymbol{v}}, \boldsymbol{p}) = \max(\phi_{cone}, \phi_{plane})}.$$

The capsule/spherical capped cylinder

For the spherical capped cylinder, the standard and general SDFs take the same approach, geometrically, but end up looking rather different. The definition uses two points, $\boldsymbol{A}$ and $\boldsymbol{B}$ and a radius r. The normal vector for the planes is the same as the axial direction, $\text{unit}(\boldsymbol{A} - \boldsymbol{B})$, with sign being fairly arbitrary, as long as all relevant orientations match.

Standard SDF:
This solution is based on the line segment's SDF, ϕ_s, and asks how far away the point $\boldsymbol{p}$ is from the surface $\phi_s = r$ Quilez breaks the solution down into three regions, $(\boldsymbol{p} - \boldsymbol{A}) \cdot \hat{\boldsymbol{a}} < 0$, $0 \leq (\boldsymbol{p} - \boldsymbol{A}) \cdot \hat{\boldsymbol{a}} \leq h$, and $(\boldsymbol{p} - \boldsymbol{A}) \cdot \hat{\boldsymbol{a}} > h$. The solutions are then combined through the clamp function.

$$\boldsymbol{\alpha} = \boldsymbol{p} - \boldsymbol{a}$$

$$\boldsymbol{\beta} = \boldsymbol{b} - \boldsymbol{a}$$

$$h = \text{clamp}(\frac{\boldsymbol{\alpha} \cdot \boldsymbol{\beta}}{\boldsymbol{\beta}^2}, 0, 1)$$

$$\phi(\boldsymbol{p}) = \|\boldsymbol{\alpha} - h\boldsymbol{\beta}\| - r$$

General SDF:
Similarly to the standard SDF, the general solution is derived from asking: when is $\boldsymbol{p} + \phi\hat{\boldsymbol{v}}$ a distance, r from the line segment, S? It turns out that when deriving the solution in each region, we end up with the same strategy as our other shapes, taking mins and maxs of primitives; in this case, spheres and cylinders (after all, spheres and cylinders of radius r are defined as being r units away from a point or line).

$$\boxed{\phi_G(\hat{\boldsymbol{v}}, \boldsymbol{p}) = \min(\max(\phi_{s1}, -\phi_{p1}), \max(\phi_{p1}, \phi_c, -\phi_{p2}), \max(\phi_{s2}, \phi_{p2}))}.$$

The two spheres are centered at $\boldsymbol{A}$ and $\boldsymbol{B}$, with radius r. The two planes pass through $\boldsymbol{A}$ and $\boldsymbol{B}$, normal to the axial direction, $\hat{\boldsymbol{a}} = \text{unit}(\boldsymbol{A} - \boldsymbol{B})$. The cylinder has height $\|\boldsymbol{A} - \boldsymbol{B}\|$. It is important to remember that the signs of ϕ_{p1} and ϕ_{p2} will depend on the chosen orientations of those planes. The spherical cap regions are bounded by a sphere and a plane oriented with inclusion away from the other sphere. The cylindrical region is bounded by the cylinder and the two planes, oriented towards eachother.

As with the capsule, the SDF of any shape that combines the "primitives" (sphere, plane, cone, cylinder), can be composed easily. This means the SDF of the box frame, hexagonal prism, triangular prism, capped cone, solid angle, rounded cone, rhombus, octrahedron, pyramid, triangle, and quad listed on Quilez's website have rather simple solutions. Computing any of these solutions is as costly as the "sum of its parts", i.e. the cost of computing each constituent SDF separately.