A joint framework for stochastic correlation and tempered stable process

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Overview

1. Subject:
   - The NTS-OU framework capturing jumps, heavy tails, skewness and stochastic correlation
   - NTS-OU application to quanto option pricing and deriving its closed-form solution
2. Improvement: addressed the empirical phenomena of quanto implied volatility smile and stochastic correlation.
3. Two key processes: Normal Tempered Stable (NTS) process and Ornstein-Uhlenbeck (OU) process
4. Note:
   - NTS-OU can be applied to other examples.
   - Through the NTS-OU, we can infer that the stochastic correlation exists in the risk-neutral world.

Outline

Motivation

What is the best alternative model to address
Implied Volatility Smile and Stochastic Correlation phenomena?

Motivation - *Quanto Option*

- A quanto option is a European option where the payoff at expiry is converted at a prespecified exchange rate to its corresponding currency.

$$F_{fix}(S(T) - K)^{+}$$

- Risk factors: two underlying dynamics, its dependence
- The Black-Scholes model fails to explain: (1) volatility smile and (2) stochastic correlation.

Motivation - *Volatility Smile*

- After Black Monday in 1987, the volatility smile was recognized.
- The volatility smile is the smile pattern shown when implied volatility is plotted against strike prices.
- The empirical distribution of the underlying asset returns exhibits heavy tails, and skewness;

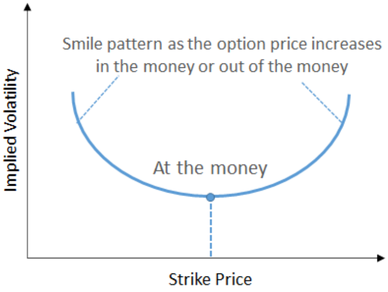

Figure 1: Volatility Smile

Motivation - *Stochastic correlation*

- It is a well-documented fact that the stochastic correlation plays a key role in the pricing of multi-asset financial instruments.

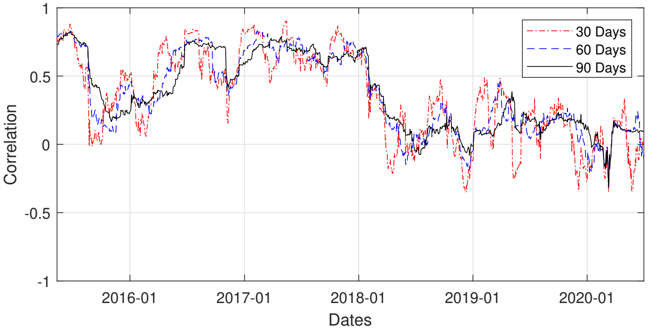

Figure 2: Historical rolling correlation between the S&P 500 and the EUR-USD log-returns. (from January 2015 to June 2020)

Motivation - *Stochastic correlation*

- Two main problems for a constant correlation
  1. Timeframe
  2. Correlation risk; the difference between implied correlation and realized correlation is non-zero.

Preliminaries: Normal Tempered Stable Process

Let $\alpha \in (0,2)$, $\theta, \sigma > 0$, and $\mu, \beta \in \mathbb{R}$. The NTS random variable X with parameters $(\alpha, \theta, \beta, \sigma, \mu)$ is defined as

$$X = \mu - \beta + \beta \mathcal{T} + \sigma \sqrt{\mathcal{T}} W,$$

where $W \sim \mathrm{N}(0,1)$, $\mathcal{T}$ is a positive, non-decreasing random variable called *tempered stable subordinator* with its characteristic function $\phi_{\mathcal{T}}$ being

$$\phi_{\mathcal{T}}(u) = \exp\left(-\frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha}\left((\theta - iu)^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}}\right)\right). \tag{1}$$

$$\phi_{NTS}(u) = \exp\left((\mu - \beta)iu - \frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha}\left(\left(\theta - i\beta u + \frac{\sigma^2 u^2}{2}\right)^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}}\right)\right). \tag{2}$$

Preliminaries: NTS Process Continued

An N-dimensional process $(X(t))_{t\geq 0}$ follows the multivariate NTS if

$$X(t) = \mu t + \beta(\mathcal{T}(t) - t) + \text{diag}(\sigma) R^{\frac{1}{2}} W(\mathcal{T}(t)), \quad t \geq 0$$

where $R = [\rho_{m,n}]_{\{m,n \in 1,2,\cdots,N\}}$ is a dispersion matrix and $R^{\frac{1}{2}}$ given by factorization $R = R^{\frac{1}{2}}(R^{\frac{1}{2}})^T$ such as a Cholesky factorization.

Preliminaries: Ornstein-Uhlenbeck Process

We model the stochastic dependency with the OU process
Let θ be the long-term mean, κ be the reverting speed of the process, and σ be the volatility of the stochastic process. A stochastic process $X(t)$ follows the OU process if the dynamic is as below:

$$dX_{(t)} = -\kappa(X_{(t)} - \theta)dt + \sigma dW(t);$$

where $W(t)$ is a standard Brownian motion. For our study, in particular, we assume the long-term mean θ is zero and the correlation movement in $\mathbb{Q}$-measure is explained by the drift and the Brownian motion part.

Preliminaries: Ornstein-Uhlenbeck Process - Continued

Let's define $(\mathcal{I}(\rho, t))_{t \geq 0}$ as $\mathcal{I}(\rho, t) = \int_0^t \rho(s) ds$. Now, our model of choice for the correlation $(\rho(t))_{t \geq 0}$ over time is the Ornstein-Uhlenbeck (OU) Process as follows:

$$d\rho(t) = -\kappa_{OU}\rho(t)dt + \sigma_{OU}dW(t).$$

The solution of the above OU process is

$$\rho(t) = \rho(0)e^{-\kappa_{OU}t} + \sigma_{OU}\int_0^t e^{-\kappa_{OU}(t-s)}dW(s). \tag{3}$$

Preliminaries: Ornstein-Uhlenbeck Process - Continued

$\mathcal{I}(\rho, t)$ follows the normal distribution with the mean

$$E\left(\int_0^t \rho(s)ds\right) = \rho(0)D(t),$$

and the variance

$$\text{var}\left(\int_0^t \rho(s)ds\right) = \frac{\sigma_{OU}^2}{\kappa_{OU}^2}\left(t - D(t) - \frac{\kappa_{OU}}{2}(D(t))^2\right),$$

where

$$D(t) = \frac{1 - e^{-\kappa_{OU} t}}{\kappa_{OU}}.$$

Preliminaries: Ornstein-Uhlenbeck Process - Continued

Finally, the characteristic function of $\mathcal{I}(\rho, t)$ is equal to

$$\begin{aligned}
\phi_{\mathcal{I}(\rho,t)}(u) &:= E\left[\exp\left(iu\int_0^t \rho(s)ds\right)\right] \\
&= \exp\left(iuE\left(\int_0^t \rho(s)ds\right) - \frac{u^2}{2}\mathrm{var}\left(\int_0^t \rho(s)ds\right)\right) \\
&= \exp\left(iu\rho(0)D(t) - \frac{u^2\sigma_{OU}^2}{2\kappa_{OU}^2}\left(t - D(t) - \frac{\kappa_{OU}}{2}(D(t))^2\right)\right).
\end{aligned} \tag{4}$$

We later use this result when calculating the characteristic function of the underlying process $S(t)$.

NTS with Stochastic Correlation: NTS-OU

We extend the NTS framework with the stochastic correlation. Let R be a time-dependent process, $R = (R(t))_{t \geq 0}$. Let process $(\tau(t))_{t\geq 0}$ satisfy $\mathcal{T}(t) = \int_0^t \tau(u)du$, for all $t \geq 0$. Then the N dimensional process is said to follow NTS-OU process,

$$X(t) = \mu t + \beta \int_0^t (\tau(u) - 1)du + \text{diag}(\sigma) \int_0^t R^{1/2}(\mathcal{T}(u))\sqrt{\tau(u)}dW(u) \quad (5)$$

and denoted by $X \sim NTS_{OU}(\alpha, \theta, \mu, \beta, \sigma, R)$. The expectation of this is given as $E[X_n(t)] = \mu_n t$ and the covariance is

$$\text{cov}(X_m(t), X_n(t)) = \sigma_m \sigma_n E\left[\int_0^{\mathcal{T}(t)} \rho_{m,n}(s)ds\right] + \beta_m \beta_n t \left(\frac{2-\alpha}{2\theta}\right).$$

Weighted Sum of NTS-OU Processes

Proposition

Let $w = (w_1, w_2, ..., w_N)^\intercal \in \mathbb{R}^N$ and N-dimensional processes $X \sim NTS_{OU}(\alpha, \theta, \mu, \beta, \sigma, R)$.

$$w^T X(t) = \bar{\mu} t + \bar{\beta}(\mathcal{T}(t) - t) + \int_0^{\mathcal{T}(t)} \bar{\sigma}(s) dW(s), \quad t \geq 0$$

Then $w^T X \sim NTS_{OU}(\alpha, \theta, \bar{\mu}, \bar{\beta}, \bar{\sigma}, R)$

$$\bar{\mu} = \sum_{n=1}^{N} w_n \mu_n, \quad \bar{\beta} = \sum_{n=1}^{N} w_n \beta_n, \quad \bar{\sigma}(t) = \sqrt{\sum_{m=1}^{N} \sum_{n=1}^{N} w_m w_n \sigma_m \sigma_n \rho_{m,n}(t)},$$

and $(W(t))_{t \geq 0}$ is a Brownian motion.

Girsanov Theorem and Risk Neutral Pricing

- Girsanov theorem is important for derivative pricing in the sense that it implicates we can change from physical measure to risk neutral measure to price an instrument.
- Price of the instrument will be the discounted expectation of the future under the risk neutral measure Q. i.e. $P_{derivative} = D(0, T) \cdot E_Q(Payoff(S_T))$
- Girsanov theorem implies that we can shift probability measure to transform an Ito process with a given drift to an Ito process with "arbitrary" drift.
- One important thing is that the market has to be complete to have a unique risk neutral measure for pricing.

Girsanov Theorem - Mathematical Formulation

Theorem (Girsanov)

Let $B(t)$*,* $0 \le t \le T$*, be a Brownian motion on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, and let* $\mathcal{F}(t)$*,* $0 \le t \le T$ *be a filtration for this Brownian motion. Let* $H(t)$*,* $0 \le t \le T$ *be an adopted process. Define*

$$Z(t) = \exp\left(-\int_0^t H(u)dW(u) - \frac{1}{2}\int_0^t H^2(u)du\right),$$

$$W(t) = B(t) + \int_0^t H(u)du,$$

$$\mathbb{E}\left[\int_0^T H^2(u)Z^2(u)du\right] < \infty.$$

Set $Z = Z(t)$*, then* $E(Z) = 1$ *and under the probability measure* $\mathbb{Q}$*:*

$$\mathbb{Q}(A) = \int_A Z(\omega)dP(\omega) \text{ for all } A \in \mathcal{F},$$

the process $W(t)$*,* $0 \le t \le T$*, is a Brownian motion.*

Change of Measure for NTS-OU

Let's apply the theorem 1 to the NTS-OU. We define $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N)$ and $\hat{\beta} = (\hat{\beta}_1, \hat{\beta}_2, \cdots, \hat{\beta}_N)$ that satisfies $\mu - \beta = \lambda - \hat{\beta}$. An N-dimensional process $H(t) = (H_1(t), H_2(t), ..., H_N(t))$ that satisfies the following,

$$\text{diag}(\sigma) R^{1/2}(\mathcal{T}(t)) H(t) = (\beta - \hat{\beta})\sqrt{\tau(t)}. \tag{6}$$

Then equation 5 becomes the following,

$$X(t) = \lambda t + \hat{\beta} \int_0^t (\tau(u) - 1) du + \text{ diag}(\sigma) \left(\int_0^t R^{1/2}(\mathcal{T}(u)) \sqrt{\tau(u)} H(u) du \right.$$
$$\left. + \int_0^t R^{1/2}(\mathcal{T}(u)) \sqrt{\tau(u)} dB(u) \right). \tag{7}$$

With the *Radon-Nikodym derivative*:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{\Xi(T) - \frac{1}{2}[\Xi, \Xi](T)}, \qquad \text{for} \quad \Xi(t) = -\sum_{n=1}^{N} \int_0^t H_n(s) dB_n(s),$$

Change of Measure for NTS-OU - Continued

by Girsanov's theorem, process

$$W(t) = B(t) + \int_0^t H(u)du,$$

is a $\mathbb{Q}$-Brownian motion, and we have

$$\begin{aligned} X(t) &= \lambda t + \hat{\beta} \int_0^t (\tau(u) - 1)du + \text{diag}(\sigma) \int_0^t R^{1/2}(\mathcal{T}(u))\sqrt{\tau(u)}dW(u) \\ &= \lambda t + \hat{\beta}(\mathcal{T}(t) - t) + \text{diag}(\sigma) \int_0^t R^{1/2}(\mathcal{T}(u))dW(\mathcal{T}(u)) \\ &= \lambda t + \hat{\beta}(\mathcal{T}(t) - t) + \text{diag}(\sigma) \int_0^{\mathcal{T}(t)} R^{1/2}(u)dW(u). \end{aligned} \tag{8}$$

$X \sim \text{NTS}_N(\alpha, \theta, \lambda, \hat{\beta}, \sigma, R)$ is, therefore, an NTS-OU-process under measure $\mathbb{Q}$.

Bivariate Case of the NTS-OU Process

Let two-dimensional stochastic process follow the bivariate NTS-OU as, $X = (X_1, X_2)^\intercal \sim \text{NTS}_{OU}(\alpha, \theta, \mu, \beta, \sigma, R)$ with parameters to be $\alpha \in (0, 2)$, $\theta > 0$, $\mu = (\mu_1, \mu_2)^\intercal$, $\beta = (\beta_1, \beta_2)^\intercal$, $\sigma = (\sigma_1, \sigma_2)^\intercal$, $R = (R(t))_{t \geq 0}$ and

$$R(t) = \begin{pmatrix} 1 & \rho(t) \\ \rho(t) & 1 \end{pmatrix},$$

where correlation $\rho = (\rho(t))_{t \geq 0}$ is a time-dependent stochastic process bounded in $[-1, 1]$. Then we have $X(t) = (X_1(t), X_2(t))^\intercal$ with

$$\begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} t + \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} (\mathcal{T}(t) - t) + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \int_0^{\mathcal{T}(t)} \begin{pmatrix} 1 & 0 \\ \rho(t) & \sqrt{1 - \rho(t)^2} \end{pmatrix} \begin{pmatrix} dB_1(t) \\ dB_2(t) \end{pmatrix}$$

where $\mathcal{T}$ is the tempered stable subordinator, $B = (B(t))_{t \geq 0}$ is the independent two-dimensional Brownian motion.

Bivariate Case of the NTS-OU Process- Continued

Let $Z(t) = w_1 X_1(t) + w_2 X_2(t)$ then $E[Z(t)] = w_1\mu_1 + w_2\mu_2$ We have

$$Z(t) = \bar{\mu}t + \bar{\beta}(\mathcal{T}(t) - t) + \int_0^{\mathcal{T}(t)} \bar{\sigma}(s)dW(s),$$

where $\bar{\mu} = w_1\mu_1 + w_2\mu_2$, $\bar{\beta} = w_1\beta_1 + w_2\beta_2$, and

$$\bar{\sigma}(s) = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho(s)}.$$

Bivariate Case of the NTS-OU Process- Continued

Now we calculate the characteristic function for $Z(t)$,

$$\begin{aligned}\phi_{Z(t)}(u) &= E\left[E\left[E[\exp(iuZ(t))|\mathcal{F}_\rho(t)]|\mathcal{T}(t)\right]\right] \\ &= \exp(iu(\bar{\mu}-\bar{\beta})t)\phi_{\tau(t)}\left(u\bar{\beta}+\frac{iu^2}{2}\left(w_1^2\sigma_1^2+w_2^2\sigma_2^2\right)\right) \\ &\quad\times E\left[\exp\left(-u^2 w_1 w_2\sigma_1\sigma_2\int_0^{\mathcal{T}(t)}\rho(s)ds\right)\right]. \qquad (9)\end{aligned}$$

For convenience, we define a process $(\mathcal{I}(\rho,t))_{t\geq 0}$ as $\mathcal{I}(\rho,t)=\int_0^t \rho(s)ds$ and let $\phi_{\mathcal{I}(\rho,t)}(u)$ be the characteristic function of $\mathcal{I}(\rho,t)$, then

$$\begin{aligned}\phi_{Z(t)}(u) &= \exp(iu(\bar{\mu}-\bar{\beta})t)\phi_{\tau(t)}\left(u\bar{\beta}+\frac{iu^2}{2}\left(w_1^2\sigma_1^2+w_2^2\sigma_2^2\right)\right) \\ &\quad\times E\left[\phi_{\mathcal{I}(\rho,\mathcal{T}(t))}(iu^2 w_1 w_2\sigma_1\sigma_2)\right]. \qquad (10)\end{aligned}$$

Notations and Settings

Here we describe the settings and the notations for pricing.

- Let $r_d \geq 0$ and $r_f \geq 0$ be the instantaneous interest rate for the domestic and foreign currency respectively.
- Let $V(t)$ be the price process of the underlying asset in domestic currency.
- Let $S(t)$ be the price process of the underlying asset in foreign currency.
- Let $F(t)$ be the exchange rate process where for one unit of foreign currency, we get $F(t)$ amount of domestic currency. i.e.

$$S(t) = \frac{V(t)}{F(t)}$$

Bivariate NTS-OU and Quanto Option

To obtain the NTS–based quanto Option value with stochastic dependence, we assume that processes $(V(t))_{t\geq 0}$ and $(F(t))_{t\geq 0}$ are given by

$$V(t) = V(0)\exp\left(\mu_X t + X(t)\right) \quad \text{and} \quad F(t) = F(0)\exp\left(\mu_Y t + Y(t)\right), \qquad (11)$$

where $\mu_X, \mu_Y \in \mathbb{R}$ and

$$(X, Y) \sim \mathrm{NTS}_{OU_2}\left(\alpha, \theta, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \beta_X \\ \beta_Y \end{pmatrix}, \begin{pmatrix} \sigma_X \\ \sigma_Y \end{pmatrix}, \begin{pmatrix} 1 & \rho(t) \\ \rho(t) & 1 \end{pmatrix}_{t\geq 0}\right)$$

for a bounded stochastic process $\rho = (\rho(t))_{t\geq 0}$ under the physical (or market) measure $\mathbb{P}$. With $\lambda = (\lambda_X, \lambda_Y)^\mathsf{T}$, then, we can find equivalent measure $\mathbb{Q}_\lambda$, under which

$$(X, Y) \sim \mathrm{NTS}_{OU_2}\left(\alpha, \theta, \begin{pmatrix} \lambda_X^* \\ \lambda_Y^* \end{pmatrix}, \begin{pmatrix} \beta_X + \lambda_X^* \\ \beta_Y + \lambda_Y^* \end{pmatrix}, \begin{pmatrix} \sigma_X \\ \sigma_Y \end{pmatrix}, \begin{pmatrix} 1 & \rho(t) \\ \rho(t) & 1 \end{pmatrix}_{t\geq 0}\right)$$

Equivalent Martingale Measure

We find an equivalent measure (or risk-neutral measure), $\mathbb{Q}_{\lambda^*}$, with $\lambda^* = (\lambda^*_X, \lambda^*_Y)^\intercal$. This means

$$E_{\mathbb{Q}_{\lambda^*}}\left[\tilde{V}(t)\right] = V(0) \text{ and } E_{\mathbb{Q}_{\lambda^*}}\left[\tilde{F}(t)\right] = F(0)$$

The discounted price processes are $(\tilde{V}(t))_{t\geq 0}$ and $(\tilde{F}(t))_{t\geq 0}$ defined as follows with

$$\tilde{V}(t) = e^{-r_d t} V(t) \text{ and } \tilde{F}(t) = e^{(-r_d + r_f)t} F(t)$$

This is equivalent to

$$E_{\mathbb{Q}_{\lambda^*}}\left[e^{X(t)}\right] = e^{-(\mu_X - r_d)t}, \quad and \quad E_{\mathbb{Q}_{\lambda^*}}\left[e^{Y(t)}\right] = e^{-(\mu_Y - r_d + r_f)t}$$

.

Equivalent Martingale Measure - continued

This implies λ^* has to satisfy the following two conditions:

Condition 1: $\lambda_X^* < \theta - \beta_X - \frac{\sigma_X^2}{2}$ and $\lambda_Y^* < \theta - \beta_Y - \frac{\sigma_Y^2}{2}$ so that $E_{\mathbb{Q}_{\lambda^*}}\left[e^{X(t)}\right]$ and $E_{\mathbb{Q}_{\lambda^*}}\left[e^{Y(t)}\right]$ exist.

Condition 2: $\mu_X - r_d + w(\lambda_X^*) = 0$ and $\mu_Y - r_d + r_f + w(\lambda_Y^*) = 0$, where

$$w(\lambda_X^*) = \log E_{\mathbb{Q}_{\lambda^*}}\left[e^{X(1)}\right] = -\beta_X - \frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha}\left(\left(\theta - \beta_X - \lambda_X^* - \frac{\sigma_X^2}{2}\right)^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}}\right),$$

$$w(\lambda_Y^*) = \log E_{\mathbb{Q}_{\lambda^*}}\left[e^{Y(1)}\right] = -\beta_Y - \frac{2\theta^{1-\frac{\alpha}{2}}}{\alpha}\left(\left(\theta - \beta_Y - \lambda_Y^* - \frac{\sigma_Y^2}{2}\right)^{\frac{\alpha}{2}} - \theta^{\frac{\alpha}{2}}\right).$$

Price Processes under $\mathbb{Q}$ measure

Since, we have $\mu_X = r_d - w(\lambda_X^*)$ and $\mu_Y = r_d - r_f - w(\lambda_Y^*)$, we also have

$$
\begin{aligned}
V(t) &= V(0)\exp\left((r_d - w(\lambda_X^*))t + X(t)\right), \\
F(t) &= F(0)\exp\left((r_d - r_f - w(\lambda_Y^*))t + Y(t)\right).
\end{aligned}
$$

Thus, the asset price in foreign currency, $S(t)$, is obtained by

$$
S(t) = \frac{V(t)}{F(t)} = S(0)\exp\left(\left(r_f - w(\lambda_X^*) + w(\lambda_Y^*)\right) t + Z(t)\right) \tag{12}
$$

where $Z(t) = X(t) - Y(t)$, under the risk-neutral measure $\mathbb{Q}_{\lambda^*}$

Price Processes under $\mathbb{Q}$ measure - Continued

With $w = (1, -1)^\intercal$, $Z = (Z(t))_{t \geq 0}$ now follows

$$Z(t) = \lambda_Z t + \beta_Z(\mathcal{T}(t) - t) + \int_0^{\mathcal{T}(t)} \sigma_Z(s) dW(s), \quad t \geq 0,$$

where $\lambda_Z = \lambda_X^* - \lambda_Y^*$, $\beta_Z = \beta_X + \lambda_X^* - \beta_Y - \lambda_Y^*$, $\sigma_Z(t) = \sqrt{\sigma_X^2 + \sigma_Y^2 - 2\sigma_X \sigma_Y \rho(t)}$ and $(W(t))_{t \geq 0}$ is an 1-dimensional Brownian motion independent of ρ and $\mathcal{T}$.

Price Processes under $\mathbb{Q}$ measure - Continued

Hence, by equation (10), we can quickly check the characteristic function of $Z(t)$ becoming

$$\phi_{Z(t)}(u) = \exp(iu(\lambda_Z - \beta_Z)t)\phi_{\tau(t)}\left(u\beta_Z + \frac{iu^2}{2}\left(\sigma_X^2 + \sigma_Y^2\right)\right) \times E\left[\phi_{\mathcal{I}(\rho,\mathcal{T}(t))}(-iu^2\sigma_X\sigma_Y)\right] \tag{13}$$

Note that the characteristic function for the correlation i.e. $\phi_{\mathcal{I}(\rho,\mathcal{T}(t))}$ is given in equation 4.

General European Option Pricing Formula

Now we can leverage the general European option pricing formula to price the Quanto option with 1 dimensional NTS-OU process.

Theorem (Lewis)

Let $h(x)$ be a payoff function of a given European option with $x = \log S(T)$ and $\hat{h}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} h(x) dx$. Suppose $\hat{h}(\xi)$ is defined for all $\xi \in R_h = \{z \in \mathbb{C} \colon \mathrm{Im}(z) \in I_h\}$, for some open interval I_h. The driving process $(U(T))_{t\geq 0}$, with $U(t) = \ln S(t)$, is a Lévy process, such that a characteristic function $\phi_{U(T-t)}(u)$ of $U(T-t)$ is defined for all $\xi \in R_\phi = \{z \in \mathbb{C} \colon \mathrm{Im}(z) \in I_\phi\}$, for some open interval I_ϕ. Then, the European option price $C(t)$ at time t is determined by

$$C(t) = \frac{e^{-r_d(T-t)}}{2\pi} \int_{-\infty}^{\infty} (S(t))^{i(u+i\zeta)} \phi_{U(T-t)}(u+i\zeta) \hat{h}(u+i\zeta) du, \ \zeta \in I_h \cap I_\phi. \tag{14}$$

General European Option Pricing Formula - continued

We layout the components of the above equation (7). The payoff function of the quanto option is $F_{fix}(S(T)-K)^+$, thus we rewrite it to be

$$h(x) = F_{fix}(e^x - K)^+ \text{ and } \hat{h}(\xi) = -F_{fix}K^{1-i\xi}/\xi(\xi+i). \tag{15}$$

$\hat{h}(\xi)$ is well defined for $\xi \in \{z \in \mathbb{C} : \mathrm{Im}(z) \in I_h = (-\infty, -1)\}$.
the characteristic function of $U(T-t)$ becomes

$$\phi_{U(T-t)}(\xi) = e^{i\xi(r_f - w(\lambda_X^*) + w(\lambda_Y^*))(T-t)}\phi_{Z(T-t)}(\xi). \tag{16}$$

General European Option Pricing Formula - continued

Let $T \geq 0$ then by theorem 2 the quanto call option price is

$$\begin{aligned} C_t^{quanto}(K,T) &= exp(-r_d(T-t))E^{\mathbb{Q}}\left[F_{fix}(S(T)-K)^+ | F_t\right] \\ &= \frac{e^{-r_d(T-t)}}{2\pi}\int_{-\infty}^{\infty}(S(t))^{i(u+i\zeta)} \\ &\times \frac{F_{fix}K^{1-i(u+i\zeta)}\phi_{U(T-t)}(u+i\zeta)}{(-1)(u+i\zeta)(u+i(\zeta+1))}du, \end{aligned} \tag{17}$$

By looking at equation (13) and equation (16), we can see that we can directly use the equation (17) to calculate the quanto call option price.

Empirical Results

1. Quanto Option with S&P 500 and EUR-USD
2. Quanto Option with DJIA and BTC-USD

Empirical Results

1. Quanto Option with S&P 500 and EUR-USD
    1. In-sample test
    2. Kolmogorov Smirnov (KS) Test
    3. Stochastic Correlation
    4. Parameter Calibration
    5. Goodness-of-fit-test for the parameter calibration
    6. Term structure
2. Quanto Option with DJIA and BTC-USD

Empirical Results - S&P 500 and EUR-USD

- In-sample test: Summary Statistics

	S&P 500	EUR-USD
Mean	3.1688×10^{-04}	-4.9179×10^{-05}
Standard Deviation	0.0125	0.0053
Skewness	-0.9548	-0.0394
Kurtosis	20.8514	5.6368
$Q_{.01}$	-0.1034	-0.0269
$Q_{.05}$	-0.0482	-0.0173
$Q_{.1}$	-0.0342	-0.0142
$Q_{.5}$	-0.0178	-0.0081
$Q_{.95}$	0.0156	0.0086
$Q_{.99}$	0.0309	0.0139
$Q_{.995}$	0.0470	0.0155
$Q_{.999}$	0.0889	0.0226

Table 1: Summary statistics for daily log-returns of the S&P 500 and the EUR-USD exchange rate from January 2015 to June 2020.

Empirical Results - S&P 500 and EUR-USD

- In-sample test: Density Distributions and the Q-Q plots

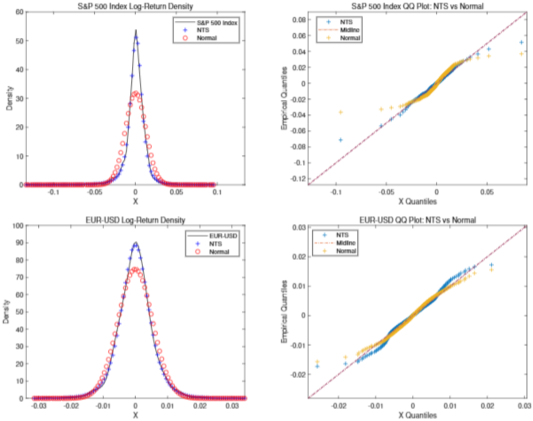

Figure 3: The log-return density distribution (left) and the QQ-plot (right) for both the S&P 500 index and the EUR-USD. The NTS distribution and the nor-mal distribution are fitted.

Empirical Results - S&P 500 and EUR-USD

- In-sample test: Kolmogorov-Smirnov test

$$KS = \sup |\hat{F}(x) - F(x)|$$

	S&P 500		EUR-USD	
Distribution	KS Statistics	*p-value*	KS Statistics	*p-value*
Normal	0.1056	0.0001	0.0469	0.0034
Student's T	0.023	0.422	0.0169	0.7993
NTS	0.0072	0.9997	0.0056	0.9921

Table 2: *p*-values of the KS test for three candidate distribution in the S&P 500 and the EUR-USD at 5% of the significance level. The NTS strongly beats the other candidates the normal and the student's T.

Empirical Results - S&P 500 and EUR-USD

- Historical Correlation

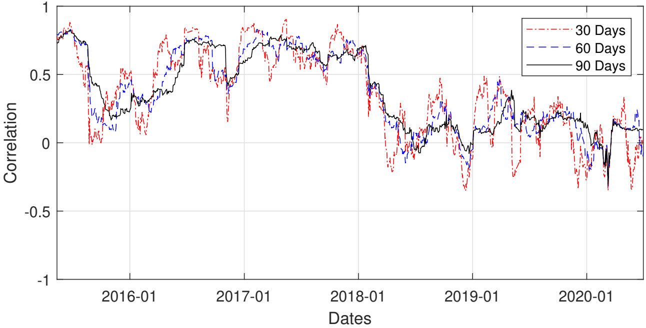

Figure 4: Illustration of the historical rolling correlation between the S&P 500 and the EUR-USD returns over the period of January 2015 and June 2020

- Generating proxy data for parameter calibration

  The full quanto option price is calculated by

$$e^{-r_d(T-t)} F_{fix} E_Q[(S(T)-K)^+ | \mathcal{F}_t]$$

but

$$E_Q[(S(T)-K)^+ | \mathcal{F}_t]$$

part are replaced with the market price of S&P 500 index call and put options.

Empirical Results - S&P 500 and EUR-USD

- Market data fitting : NTS-OU $>$ NTS $>$ BS

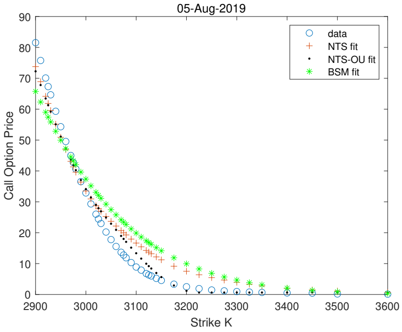

Figure 5: Comparing the estimated prices for the quanto option of the S&P 500 index and EUR-USD on August 5, 2019. The NTS-OU is the best performer, followed by the NTS and then the BS.

Empirical Results - S&P 500 and EUR-USD

- Parameter Calibration - 6 samples

Date	Model	α	θ	λ_Z	β_Z	σ_1	σ_2	μ_Z	ρ_0	σ_{OU}	κ_{OU}
05-Aug-2019	BS					0.7762	0.6258		0.9965		
	NTS	0.0001	1.9557	-0.0052	0.1790	0.1502	0.0906	0.0005			
	NTS-OU	0.2549	2.1592	0.0018	0.3702	0.0735	0.0735	0.0002	0.2416	2.6914	13.4179
08-Aug-2019	BS					0.7434	0.6113		1.0000		
	NTS	1.3372	0.1143	0.0052	0.0810	0.1508	0.0908	-0.0005			
	NTS-OU	1.3198	1.5552	-0.0173	0.3664	0.1268	0.0379	0.0018	0.9999	0.0000	0.6678
13-Aug-2019	BS					0.7313	0.5936		1.0000		
	NTS	0.0001	1.6484	-0.0191	0.1422	0.1186	0.0756	-0.0029			
	NTS-OU	0.5193	1.4461	-0.0104	0.2750	0.0668	0.0669	0.0010	0.0423	3.7562	15.0887
15-Aug-2019	BS					0.7185	0.5601		1.0000		
	NTS	0.0001	2.7481	-0.0070	0.1561	0.1512	0.09309	0.0008			
	NTS-OU	1.3978	3.33E-05	-0.0045	0.5329	0.0010	0.1088	0.0004	-0.2119	-0.2855	2.3407
21-Aug-2019	BS					0.7390	0.6099		0.9993		
	NTS	1.3866	0.0986	0.0054	0.0660	0.1519	0.0905	-0.0005			
	NTS-OU	1.0706	4.6120	-0.0095	0.4499	0.0760	0.0695	0.0010	0.7028	3.3230	9.3295
23-Aug-2019	BS					0.6835	0.5340		0.9984		
	NTS	0.0001	1.9362	-0.0198	0.1310	0.1524	0.0905	0.0015			
	NTS-OU	1.2044	2.4331	-0.0209	0.5734	0.0010	0.0010	0.0016	0.0671	3.2780	14.1898

Table 3: Fig.5. Calibrated risk-neutral parameters under the BS, the NTS and the NTS-OU models for the S&P 500 and the EUR-USD quanto option. Note that the all estimated α are well below 2 which implies the heavy tails and skewness in this quanto option dynamics

Empirical Results - S&P 500 and EUR-USD

- Goodness-of-fit test

Date	Model	RMSE	AAE	APE
05-Aug-2019	BS	9.3650	7.9902	0.2477
	NTS	3.4094	2.5545	0.0792
	NTS-OU	2.7870	2.1422	0.0664
08-Aug-2019	BS	8.6772	7.2288	0.3117
	NTS	3.4306	2.7598	0.1190
	NTS-OU	0.6912	0.5450	0.0235
13-Aug-2019	BS	9.3630	7.9459	0.3466
	NTS	2.0207	1.2874	0.0562
	NTS-OU	1.7385	1.3069	0.0570
15-Aug-2019	BS	18.9876	2.3157	6.3008
	NTS	10.1730	1.5348	4.1765
	NTS-OU	6.4119	1.5083	4.1037
21-Aug-2019	BS	7.2555	6.0643	0.3340
	NTS	3.8922	3.1852	0.1754
	NTS-OU	1.8426	1.4862	0.0819
23-Aug-2019	BS	8.1294	6.9202	0.3103
	NTS	4.1080	2.9506	0.1323
	NTS-OU	2.3988	1.8563	0.0832

Table 4: The goodness-of-fit test result for calibrated parameters shown. This test is performed on the selected ten trading days in August 2019 for the S&P 500 Index and the EUR-USD quanto options. The NTS-OU consistently show the lowest value on all test measures except for a few cases, while the BS model comprehensively underperform.

Empirical Results - S&P 500 and EUR-USD

- Goodness-of-fit metrics definition.

  The error estimators follows:

  $AAE(Average\ Absolute\ Error) = \sum_{j=1}^{N} \frac{|\hat{P}_j - P_j|}{N}$

  $APE(Average\ Prediction\ Error) = \frac{\sum_{j=1}^{N} |\hat{P}_j - P_j|/N}{\sum_{j=1}^{N} |\hat{P}_j|/N}$

  $RMSE(Root\ Mean\ \text{-}\ Square\ Error) = \sqrt{\sum_{j=1}^{N} \frac{(\hat{P}_j - P_j)^2}{N}}$

Empirical Results - S&P 500 and EUR-USD

- Term Structure

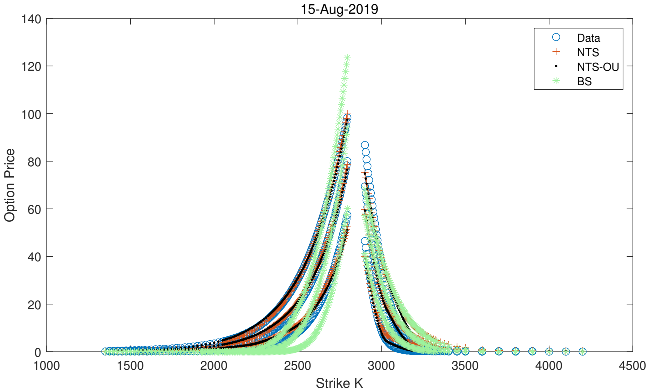

Figure 6: A term structure on the S&P 500 Index with the EUR-USD quanto option. Based on the RMSE, the BS model (18.9876) is clearly underperforms for all strike prices. The NTS model (10.173) is the next whereas the NTS-OU (6.4119) provides the best fitting capability.

Empirical Results

1. Quanto Option with S&P 500 and EUR-USD
2. Quanto Option with DJIA and BTC-USD
    1. In-sample test
    2. Kolmogorov Smirnov (KS) Test
    3. Stochastic Correlation
    4. Parameter Calibration
    5. Goodness-of-fit-test for the parameter calibration
    6. Term structure

Empirical Results - DJIA and BTC-USD

- In-sample test: Summary Statistics

	DJIA	Bitcoin
Mean	2.7389×10^{-04}	0.0022
Standard Deviation	0.0125	0.0428
Skewness	-1.1219	-1.1268
Kurtosis	27.9564	17.2957
$Q_{.01}$	-0.1094	-0.2661
$Q_{.05}$	-0.0469	-0.1688
$Q_{.1}$	-0.0363	-0.1283
$Q_{.5}$	-0.0177	-0.0630
$Q_{.95}$	0.0147	0.0672
$Q_{.99}$	0.0313	0.1184
$Q_{.995}$	0.0483	0.1444
$Q_{.999}$	0.0917	0.2159

Table 5: Summary statistics for daily log-returns of the DJIA and the Bitcoin exchange rate from January 2015 to June 2020. The high kurtosis value far exceeding 3 (the normal distribution case) gives us the confidence to consider the NTS assumption.

Empirical Results - DJIA and BTC-USD

- In-sample test: Density Distributions and the Q-Q plots

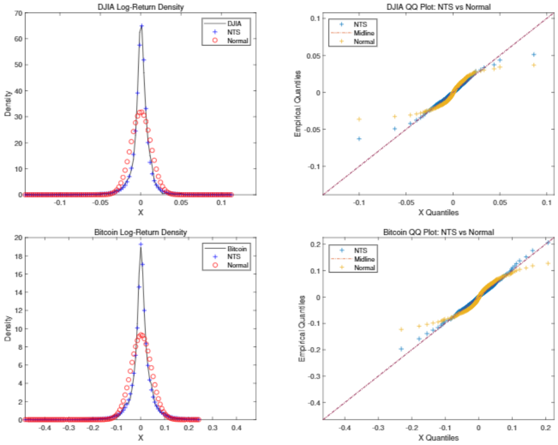

Figure 7: The log-return density distribution (left) and the QQ-plot (right) for both the DJIA index and the Bitcoin-USD. The NTS distribution and the normal distribution are fitted.

Empirical Results - DJIA and BTC-USD

- In-sample test: Kolmogorov-Smirnov test

	DJIA		Bitcoin	
Distribution	KS Statistics	*p-value*	KS Statistics	*p-value*
Normal	0.1407	0.0001	0.1178	0.0001
Student's T	0.0228	0.4367	0.0251	0.3157
NTS	0.0101	0.8938	0.0057	0.9995

Table 6: p-values of the KS test for three candidate distribution in the DJIA and the BTC-USD at 5% of the significance level. The NTS displays the high p-value of 0.8938 and 0.9995 for both the DJIA and the Bitcoin movements whereas the normal assumption is clearly failed to describe the empirical distribution.

Empirical Results - DJIA and BTC-USD

- Historical Correlation

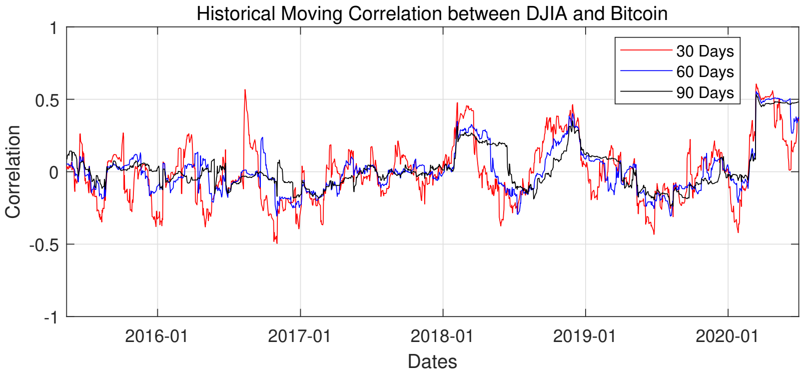

Figure 8: Historical rolling correlation between the DJIA and the BTC-USD returns over the period of January 2015 and June 2020.

Empirical Results - DJIA and BTC-USD

- Market data fitting : NTS-OU > NTS > BS

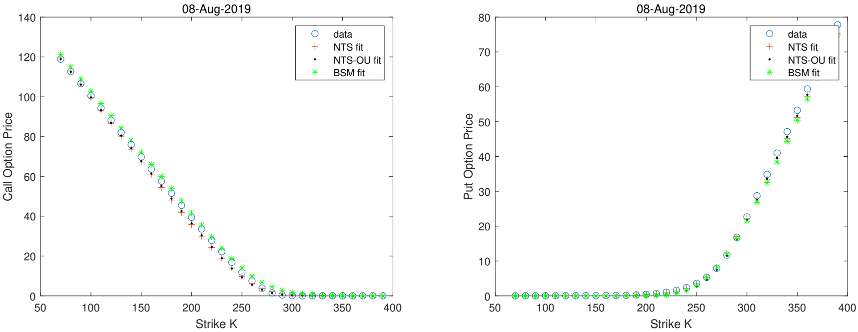

Figure 9: Comparing the estimated prices for the quanto option of the DIA with BTC-USD on August 8, 2019. Due to the limited data points, the difference between the NTS and the NTS-OU model is indistinguishable, but we note that the BS model does not perform well around the out-of-the-money strikes. The lower RMSE of the NTS-OU model (4.9258) than the BS model (5.5368) supports this observation.

Empirical Results - DJIA and BTC-USD

- Parameter Calibration

date	model	α	θ	λ_Z	β_Z	σ_1	σ_2	μ_Z	ρ_0	σ_{OU}	κ_{OU}
05-Aug-2019	BS					0.1283	0.2042		0.6262		
	NTS	0.5920	2.3819	-0.0281	0.2152	0.1512	0.0931	0.0028			
	NTS-OU	0.6959	7.0222	-0.0111	0.2532	0.1485	0.0891	0.0011	-0.5891	2.4830	12.811
08-Aug-2019	BS					0.2238	0.1012		0.7847		
	NTS	0.6312	0.8274	-0.0170	0.0403	0.1508	0.0908	0.0017			
	NTS-OU	1.9012	8.2731	-0.0161	0.8061	0.0707	0.0663	0.0016	-0.3967	3.0990	11.0851
16-Aug-2019	BS					0.2308	0.1035		0.6984		
	NTS	0.7263	0.5895	-0.0148	0.0475	0.1186	0.0757	0.0015			
	NTS-OU	1.9530	8.3134	-0.0153	0.9111	0.0585	0.0554	0.0015	-0.3893	3.0982	11.0872
19-Aug-2019	BS					0.2255	0.1031		0.6977		
	NTS	0.8614	2.3498	-0.0119	0.1604	0.0983	0.0632	0.0012			
	NTS-OU	1.8874	9.5927	-0.0159	0.8433	0.0395	0.0347	0.0016	0.6865	3.3568	11.3882
20-Aug-2019	BS					0.2139	0.0893		0.7314		
	NTS	0.6478	0.8418	-0.0134	0.0430	0.1515	0.0903	0.0013			
	NTS-OU	1.8391	9.1712	-0.0162	0.3068	0.0879	0.0866	0.0016	0.9999	2.8940	11.7439
21-Aug-2019	BS					0.2230	0.0996		0.7858		
	NTS	0.4300	0.5422	-0.0172	0.0109	0.1519	0.0905	0.0017			
	NTS-OU	1.9182	9.1790	-0.0183	0.8845	0.0351	0.0212	0.0018	-0.2364	0.4304	12.3552

Table 7: Calibrated parameter comparison for the DJIA with Bitcoin quanto options between BS, NTS and NTS-OU.

Empirical Results - DJIA and BTC-USD

- Goodness-of-fit test

Date	Model	RMSE	AAE (bp[1])	APE (%)
05-Aug-2019	BS	8.9876	2.3157	6.3008
	NTS	5.0641	1.5348	4.1765
	NTS-OU	4.6683	1.5083	4.1037
08-Aug-2019	BS	5.5368	4.5863	3.3023
	NTS	1.8003	1.4018	1.2406
	NTS-OU	4.9258	3.8356	3.3946
16-Aug-2019	BS	7.0598	2.0326	4.8581
	NTS	4.4422	1.4505	3.4669
	NTS-OU	4.3475	1.4265	3.4094
19-Aug-2019	BS	6.9423	2.0062	5.0348
	NTS	3.2871	1.1909	2.9886
	NTS-OU	3.1332	1.1907	2.9882
20-Aug-2019	BS	6.8902	2.0123	4.9955
	NTS	4.7131	1.4260	3.5401
	NTS-OU	3.5251	1.2756	3.1666
21-Aug-2019	BS	5.6648	1.8138	4.2243
	NTS	5.0603	1.4871	3.4632
	NTS-OU	3.5799	1.2941	3.0137

Table 8: Quanto option price goodness of fit for DJIA with Bitcoin, The unit is in Bitcoin: RMSE, AAE, APE. Note that 05 August 2019 is run with multiple expiries to capture the term structure. The NTS-OU model show the lowest RMSE, AAE, and APE with a few exceptions.

- Term Structure

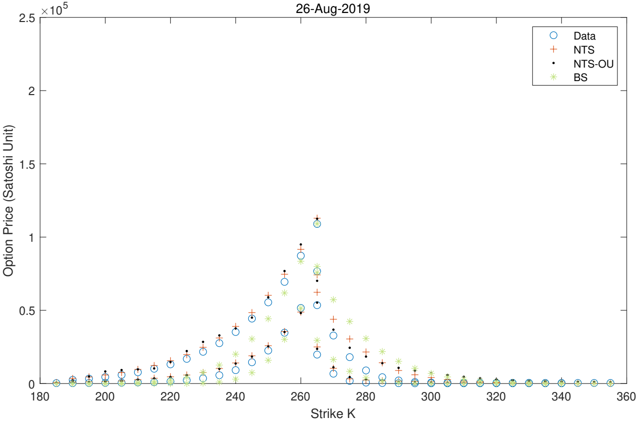

Figure 10: A term structure of the DIA and the BTC-USD quanto option with different expiries. We deleted the price of the deep in-the-money strikes.

Conclusion

- In this study, we propose the NTS-OU pricing model for quanto option by combining the OU process with the previous NTS framework.
- We also derived a closed-form solution under the risk-neutral measure by applying Girsanov's Theorem.
- For the effective numerical calculation, the characteristic function is provided to be directly used for fast Fourier transform.

Conclusion

- In the empirical illustration, we tested two empirical case studies to understand the model performance in different market environments.
- For both case studies, across all selected trading dates, the NTS model consistently displays a superior price estimation than the BS model and the NTS model.
- This conclusion is supported by the statistical summary, multiple goodness-of-fit metrics, and the term structure.

Future Works

- Delta-hedging strategies for the NTS-OU model; we can apply the local risk-minimizing hedging strategy.
- Discuss the tradeoff between flexibility by adding parameters and computational effectiveness.

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Q&A

Questions and Answers