UM-SJTU Joint Institute, Physics Laboratory I
Measurement Uncertainty Analysis Worksheet*
Exercise 5

WS-1 Natural Angular Frequency

The uncertainty for ten periods is found first. Then the result for the natural frequency is given along with its uncertainty.

The type-B uncertainty for T is $\Delta_{TB} = 0.001$ s. To find the type-A uncertainty, we first find the standard deviation

$$s_T = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(T_i - \overline{T})^2} = \underline{\quad 0 \quad} [\underline{\ s\ }].$$

We have $n = \underline{\quad 4 \quad}$, so the type-A uncertainty Δ_{TA} is calculated as

$$\Delta_{TA} = \frac{t_{0.95}}{\sqrt{n}} s_T = \underline{\ 1.59\ } \times \underline{\quad 0 \quad} = \underline{\quad 0 \quad} [\underline{\ s\ }].$$

Hence the uncertainty for T is given by

$$u_T = \sqrt{\Delta_{TA}^2 + \Delta_{TB}^2} = \underline{\quad 0.001 \quad} [\underline{\ s\ }].$$

Hence the period is given by

$$\boxed{T = \underline{\ 1.556\ } \pm \underline{\ 0.001\ } [\underline{\ s\ }]},$$

*Created by Peng Wenhao, edited by Fan Yixing, Ye Haojie, Li Tianyi, MateuszKrzyzosiak[rev. 1.5] with relative uncertainty

$$\boxed{u_{rT}} = \frac{u_T}{T} \times 100\% = \boxed{\underline{0.06}}\,\%$$

The natural angular frequency ω_0 is found from the formula $\omega_0 = 2\pi/T$, so by the uncertainty propagation formula and the fact that

$$\frac{\partial \omega_0}{\partial T} = -\frac{2\pi}{T^2},$$

we obtain

$$\boxed{u_{\omega_0}} = \left| \frac{\partial \omega_0}{\partial T} u_T \right| = \boxed{\underline{0.003} \quad [\underline{s^{-1}}]}$$

with the relative uncertainty

$$\boxed{u_{r,\omega_0}} = \frac{u_{\omega_0}}{\omega_0} \times 100\% = \boxed{\underline{0.07}\,\%}$$

WS-2 Damping Coefficient

The damping coefficient is found indirectly form measurements of the period T and the amplitude θ as $\beta = \frac{1}{5T}\ln(\theta_i/\theta_{i+5})$.

The uncertainty each single measurement of the amplitude is $u_\theta = \underline{1}\,^\circ$, so the uncertainty of the logarithm of the quotient of them $q_i = \ln(\theta_i/\theta_{i+5})$ is found from the uncertainty propagation formula

$$\Delta_{q_i,B} = \sqrt{\left(\frac{\partial(\ln(\theta_i/\theta_{i+5}))}{\partial \theta_i}\right)^2 u_\theta^2 + \left(\frac{\partial(\ln(\theta_i/\theta_{i+5}))}{\partial \theta_{i+5}}\right)^2 u_\theta^2} = \sqrt{\left(\frac{u_\theta}{\theta_{i+5}}\right)^2 + \left(\frac{u_\theta}{\theta_i}\right)^2}$$

For example, for $i = 1$,

$$\Delta_{q_1,B} = \sqrt{\underline{\left(\frac{1}{\theta_6}\right)^2 + \left(\frac{1}{\theta_1}\right)^2}} = \underline{\cancel{0.02}\ 0.02}.$$

The results for all five sequences of measurements are given in Table WS-1.

i	$\Delta_{q_i,B}$
0	~~0.021~~ 0.02
1	~~0.024~~ 0.02
2	~~0.026~~ 0.03
3	~~0.028~~ 0.03
4	0.03

Table WS-2: Type-B uncertainties for q_i.

The overall type-B uncertainty for the quotient can be estimated as the maximum of uncertainties listed in in Table WS-2

$$\Delta_{q,B} = \underline{\ 0.03\ }\ .$$

To estimate the type-A uncertainty of q, the standard deviation of q is calculated as

$$s_q = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(q_i - \overline{q})^2} = \underline{\ \cancel{0.0}\ 0.008\ }$$

Hence the type-A uncertainty for $n = 5$ is calculated as

$$\Delta_{q,A} = \frac{t_{0.95}}{\sqrt{n}} s_q = \underline{\ 1.204\ } \times \underline{\ 0.008\ } = \underline{\ 0.009\ },$$

and the combined uncertainty

$$u_q = \sqrt{\Delta_{q,B}^2 + \Delta_{q,A}^2} = \sqrt{\underline{\ 0.03^2 + 0.009^2\ }} = \underline{\ 0.03\ }.$$

A single measurement for ten periods is recorded as $T_{10} = \underline{\ 15.550\ } \pm \underline{\ 0.001\ }$ [s]. Hence $T = \underline{\ 1.5550\ } \pm \underline{\ 0.0001\ }$ [s].

Then the uncertainty propagation equation is used to calculate the uncertainty for the damping coefficient $\beta = \frac{1}{5T}q$ as

$$\boxed{u_\beta} = \sqrt{\left(\frac{\partial\beta}{\partial T}\right)^2 u_T^2 + \left(\frac{\partial\beta}{\partial q}\right)^2 u_q^2} = \sqrt{\left(-\frac{q}{5T^2}\right)^2 u_T^2 + \left(\frac{1}{5T}\right)^2 u_q^2}$$

$$= \sqrt{\left(-\frac{0.462}{5\times1.5550}\right)^2 \times 0.0001^2 + \left(\frac{1}{5\times1.5550}\right)^2 \times 0.03^2} = \boxed{\ 0.004\ [\ s^{-1}\ ]}$$

with relative uncertainty

$$\boxed{u_{r,\beta}} = \frac{u_\beta}{\beta} \times 100\% = \boxed{\ 6.8\ \%}$$

WS-3 The θ_{st}-ω and φ-ω Characteristics of Forced Oscillations

On the graphs included in the report, the uncertainty is shown in the form of error bars.[1] In both the φ vs. (ω/ω_0) graph and the θ_{st} vs. (ω/ω_0) graph, the

[1] Please follow this part to find the uncertainties and mark them on the graphs of the phase shift φ vs. (ω/ω_0) graph and the amplitude of steady-state oscillations θ_{st} vs. (ω/ω_0).

measurements of φ and θ_{st} are single measurements with uncertainty 1 °, determined by the resolution of our equipment. However, to find the uncertainty of (ω/ω_0) we need to derive it from the uncertainty propagation formula. Let us introduce symbols $Q = \frac{\omega}{\omega_0}$, $T_{\text{natural}} = N$ and $T_{\text{driven}} = D$, where the uncertainty of D is again the minimum scale (resolution) of the equipment used. Since these are single measurements, we have

$$Q = \frac{\omega}{\omega_0} = \frac{T_{\text{natural}}}{T_{\text{driven}}} = \frac{N}{D}$$

and the uncertainty of the ratio Q, found from the uncertainty propagation formula, is

$$u_Q = \sqrt{\left(\frac{\partial Q}{\partial N} u_N\right)^2 + \left(\frac{\partial Q}{\partial D} u_D\right)^2} = \sqrt{\left(\frac{u_N}{D}\right)^2 + \left(\frac{N u_D}{D^2}\right)^2}$$

In particular, with $N =$ 1.556 [s], $u_N =$ 0.001 [s], and $u_D =$ 0.001 [s], so with every set of N and D a unique uncertainty is generated. For instance,[2] for $D =$ 1.476 [s], we can calculate Q as

$$Q = \frac{N}{D} = \frac{1.556}{1.476} = 1.054$$

with uncertainty u_Q calculated as

$$\boxed{u_Q} = \sqrt{\left(\frac{0.001}{1.476}\right)^2 + \left(\frac{1.556}{1.476} \times \cancel{0.0001}\ 0.001\right)^2} = \boxed{0.0012},$$

and

$$\boxed{u_\varphi = 1^\circ = 0.017 \text{ rad}} \qquad \boxed{u_{\theta_{st}} = 1^\circ = 0.017 \text{ rad}}.$$

[2]Here, based on your measurement data, give one sample calculation for a chosen value of ω/ω_0. All values of the calculated uncertainties u_Q that you have used to plot error bars, should be given in the *Results* section, where tables with the data for the plots φ vs. (ω/ω_0) and θ_{st} vs. (ω/ω_0) is included.