UM-SJTU Joint Institute
Physics Laboratory
(Vp141)

Laboratory Report

Exercise 5
Damped and Driven Oscillations. Mechanical Resonance

Name: Zhang Yifei ID:519370910103 Group:10

Date: September 17, 2020

1 Introduction

1.1 Objectives

- Study damped and driven oscillations in mechanical systems using the Pohl resonator
- Observe and quantify the mechanical resonance phenomenon for driven oscillations

1.2 Theoretical Background

Forced(driven) oscillation is a kind of motion that a periodically varying external force is applied to a damped harmonic oscillator. Assume the driven force is of the form:

$$F = F_0 \cos \omega t$$

with the amplitude F_0 and angular frequency ω. The resulting steady-state forced oscillations will be simple harmonic with the angular frequency equal to that of the driving force. The amplitude depends on the angular frequency of the driving force, the natural angular frequency, and the damping coefficient. A mechanical resonance happens when the amplitude reaches maximum, and at that time the phase lag is -$\frac{\pi}{2}$.

In this experiment, we'll study motion of a balance wheel acted by a periodic driving torque $\tau_{dr} = \tau_0 \cos \omega t$, a damping torque $\tau_f = -b\frac{d\theta}{dt}$ and a restoring torque $\tau = -k\theta$, and we can get the equation:

$$I\frac{d^2\theta}{dt^2} = -k\theta - b\frac{d\theta}{dt} + \tau_0 \cos \omega t \tag{1}$$

where I is the moment of inertia of the balance wheel, $\tau_0$is the amplitude of the driving torque, and ω is angular frequency of the driving torque, and replace (1) with:

$$\omega_0^2 = \frac{k}{I}, \qquad 2\beta = \frac{b}{I}, \qquad \mu = \frac{\tau_0}{I},$$

And we can get:

$$\frac{d^2\theta}{dt^2} + 2\beta\frac{d\theta}{dt} + \omega_0^2\theta = \mu \cos \omega t \quad \Rightarrow \quad \theta(t) = \theta_{tr}(t) + \theta_{st} \cos(\omega t + \phi)$$

where the former θ_{tr} denotes the transient solution, that depends on the initial condition and will vanish to 0 as t approaches to infinity. θ_{st} represents the amplitude of steady-state oscillation.

$$\theta_{st} = \frac{\mu}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

Take $\omega = \frac{2\pi}{T}$ and then we can calculate β by:

$$\ln \frac{\theta_i}{\theta_j} = \ln \frac{\theta_0 e^{-\beta(iT)}}{\theta_0 e^{-\beta(jT)}} = (j - i)\beta T \tag{2}$$

The phase shift ϕ can be found as $\tan \phi = \frac{2\beta\omega}{\omega^2 - \omega_0^2}$ where $-\pi \leq \phi < 0$, from this we can get to know that ϕ is independent of initial conditions.

And finally we can get resonance frequency $\omega = \omega_{res} = \sqrt{\omega_0^2 - 2\beta^2}$, and correspondingly $\theta_{res} = \theta_{st}(\omega_{res}) = \frac{\mu}{2\beta\sqrt{\omega_0^2 - \beta^2}}$.

2 Experimental setup

2.1 Apparatus

In this experiment, a BG-2 Pohl resonator is used. The BG-2 Pohl resonator consists of two main parts: a vibrometer and a control box. The setups of the vibrometer and the control box are shown respectively in figure 4 and 5 in appendix 1.

The BG-2 Pohl resonator can measure amplitude, time period, and phase lag.

The amplitude of oscillations is measured by counting the notches on the wheel, and this measurement is performed by a photoelectric detector with the result displayed on the electronic control box. After pressing the button, the timer on the control box will start to count the time. The measurement of time has an uncertainty of 0.001s and the uncertainty of the amplitude is 1°.

The phase shift can be measured using the glass turntable with an angle scale and a strobe light. The strobe is controlled by the photoelectric detector above the wheel. When the deep notch passes the equilibrium position, the detector sends a signal and the strobe flashes. In a steady state, a line on the angle scale will be highlighted by the flash of the strobe and the phase difference can be read from the angle scale directly. The phase lag has an uncertainty of 1°. The collection of uncertaintes of each device is given in Table 1.

Measurements	**Uncertainty**
Time	0.001s
Amplitude (θ)	1°
Phase lag (ϕ)	1°

Table 1: Device Uncertainty

3 Measurement Procedure

3.1 Natural Angular Frequency

1. After selecting the mode, rotate the balance wheel to the initial angular position $\theta_0 \approx 90°$ and record the period on the panel.

2. Repeat for four times and calculate the natural angular frequency ω_0.

3.2 Damping Coefficient

1. Select damping mode 2, rotate the balance wheel like section 1. Record the period for ten periods and the amplitdue of each period with the help of "recall".

2. Calculate the damping coefficient β by recalling function(2) in the theoretical backgroud part.

$$\ln \frac{\theta_i}{\theta_j} = \ln \frac{\theta_0 e^{-\beta(iT)}}{\theta_0 e^{-\beta(jT)}} = (j-i)\beta T \qquad \Rightarrow \qquad \beta = \frac{1}{5T} \ln \frac{\theta_i}{\theta_{i+5}}$$

3.3 θ_{st} vs. ω and ϕ vs. ω Characteristics of Forced Oscillations

1. Keep the damping selection at 2 and set the speed of the motor. Record the amplitdue θ_{st} ,the period T and the phase shift ϕ after reaching steady-state.
2. Change the speed of the motor and repeat step 1 about 15 times.
3. Choose damping selection 1 or 3, repeat the above steps.
4. Plot the $\theta_{st}(\omega)$ and $\phi(\omega)$ characteristics with ω/ω_0 on the horizontal axis.

4 Results

4.1 Measurement of Natural Angular Frequency

To get natural angular frequency, we measure the time of period. The values are calculated based on table 2.

	T[s]±0.001[s]
1	1.557
2	1.557
3	1.557
4	1.557

Table 2

That gives the value of the period to be $T = 1.557 \pm 0.001[s]$ (Detailed calculation for part 4 is put in appendix)

And the relative uncertainty is 0.06%.

Hence the natural frequency can be calculated by $\omega_0 = 2\pi/T = 4.030 \pm 0.003[s^{-1}]$ with the relative uncertainty of 0.06%

4.2 Damping Coefficient

The measurements of damping coefficient is shown in 3.2. The values are calculated based on Table 3.

Amplitude[°]± 1[°]		Amplitude[°]± 1[°]		$\ln(\theta_i/\theta_{i+5})$
θ_0	80	θ_5	47	0.532
θ_1	72	θ_6	42	0.539
θ_2	65	θ_7	38	0.537
θ_3	58	θ_8	34	0.534
θ_4	51	θ_9	30	0.531
The average value of				0.535

Table 3

We need to measure the time of 10 periods and the amplitude of each of them. The result is shown as follow and the uncertainty of q ($\ln(\theta_i/\theta_{i+5})$) is 0.006.

Since the measurement for 10 periods is $T_{10} = 15.595 \pm 0.001s$, then T=1.5595 $\pm$ 0.0001s.

Then we can calculated β by

$$\beta = \frac{1}{5T} \ln \frac{\theta_i}{\theta_{i+5}} = 0.069 \pm 0.005 s^{-1}$$

and the relative uncertainty is 7.3%

4.3 The $\theta_{st} - \omega$ and $\phi - \omega$ Characteristics of Forced Oscillations

In this section, we calculate the value of ω/ω_0 and use this as x-axis.

Below are the calculated data used for plotting and the figure. (Raw data, uncertainty and calculations are placed in appendix)

Damping Selection 2		Damping Selection 3	
ω/ω_0	$\theta_{st}\pm1$ [°]	ω/ω_0	$\theta_{st}\pm1$ [°]
1.057	36	1.057	36
1.051	40	1.045	44
1.045	44	1.034	56
1.039	50	1.022	76
1.033	57	1.011	108
1.022	78	1.002	130
1.011	113	1.001	131
1.000	140	1.000	132
1.014	139	0.999	132
1.001	140	0.998	130
0.999	140	0.995	126
0.998	138	0.989	112
0.996	135	0.984	96
0.991	123	0.979	82
0.981	89	0.973	71
0.973	72	0.968	62
0.968	63	0.963	55
0.963	56		

Table 4: $\theta_{st} - \omega$ values

Damping Selection 2		Damping Selection 3	
ω/ω_0	$\phi\pm1$ [°]	ω/ω_0	$\phi\pm1$ [°]
1.057	-167	1.057	-164
1.051	-165	1.045	-160
1.045	-164	1.034	-155
1.039	-161	1.022	-145
1.033	-157	1.011	-126
1.022	-147	1.002	-99
1.011	-127	1.001	-95
1.000	-90	1.000	-91
1.014	-97	0.999	-87
1.001	-94	0.998	-83
0.999	-86	0.995	-73
0.998	-82	0.989	-57
0.996	-75	0.984	-47
0.991	-62	0.979	-39
0.981	-40	0.973	-33
0.973	-31	0.968	-28
0.968	-26	0.963	-25
0.963	-23		

Table 5: $\phi - \omega$ values

Use ω/ω_0 as x-axis, θ_{st} and ϕ as y-axis respectively, and we can get the graphs (with error bars) as shown below (Figure 1 and Figure 2).

From the graph of θ_{st} vs (ω/ω_0) we can see when (ω/ω_0) is close to 1, the θ_{st} (amplitude of the oscillation) reaches maximum, which means the wheel is at mechanical resonance. Also we can find that the line of damping selection 2 is higher than selection 3. Recall that

$$\theta_{st} = \frac{\mu}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

we can deduce that $\beta_2 < \beta_3$

We can see whatever the damping is, when ϕ is close to -90 °, the slope reaches the maximum, where ω/ω_0 is close to 1, at this time the wheel reaches mechanical resonance.

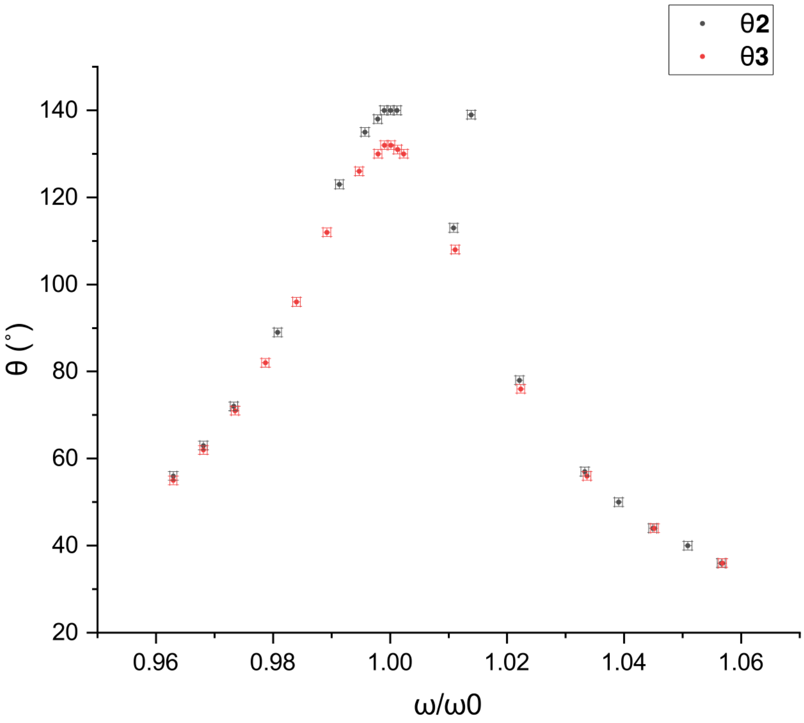

Figure 1: graphs of θ_{st} vs. (ω/ω_0)

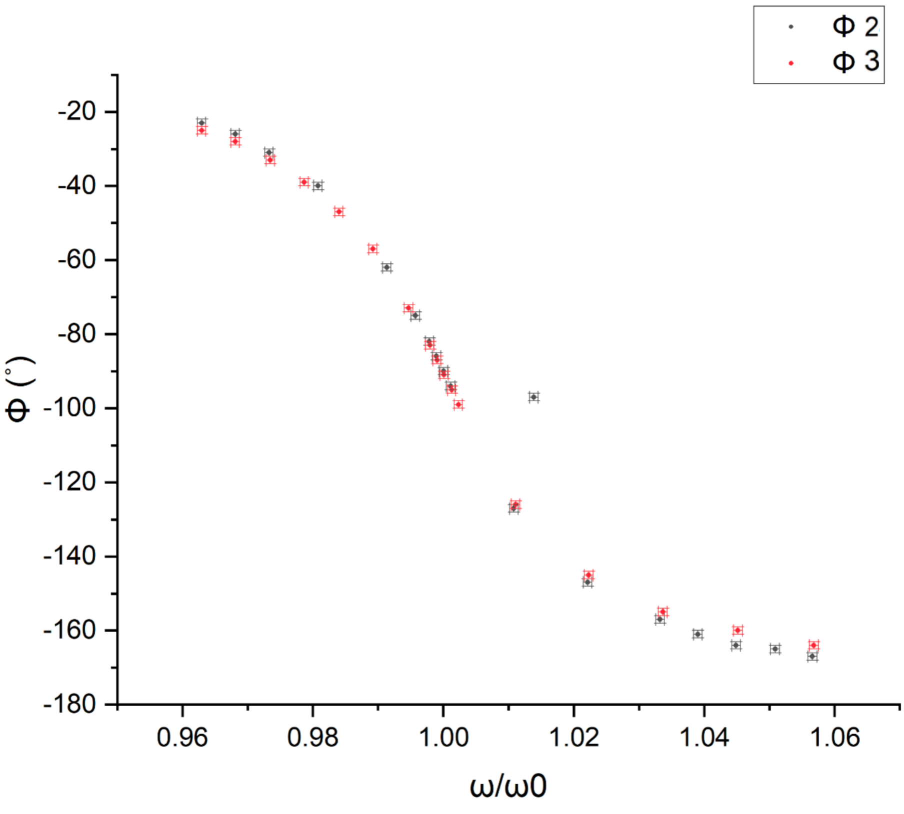

Figure 2: graphs of ϕ vs. (ω/ω_0)

5 Conclusions and Discussion

5.1 Summary

In this experiment, the natural angular frequency we calculated was:

$$\omega_0 = 2\pi/T = 4.030 \pm 0.003[s^{-1}]$$

with relative uncertainty of 0.06%. And the damping coefficient, which is

$$\beta = \frac{1}{5T} \ln \frac{\theta_i}{\theta_{i+5}} = 0.069 \pm 0.005 s^{-1}$$

with relative uncertainty 7.3 %. And we also plotted the θ_{st} vs. ω/ω_0 and ϕ vs. ω/ω_0 graphs to study the characteristics of forced oscillations and get that the value of damping coefficient of selection 2 is smaller than selection 3.

5.2 Discussions

5.2.1 Natural Angular Frequency

In this part, we got 0.06% for the relative uncertainty and in the Table 2 we can find that we get four identical values which means this value is very precise.

However in this part we neglect the air drag, which actually does have some influence on the measurement, so the calculated value is slightly smaller than the actual value.

5.2.2 Damping Coefficient

In this part, we got 7.3% for the relative uncertainty, which is non-negligible. But from calculation we can find that the uncertainty for each element is actually not very large, so I think that this uncertainty can be concluded to due to the complex calculation.

5.2.3 The $\theta_{st} - \omega$ and $\phi - \omega$ Characteristics of Forced Oscillations

In this part, we get two figures. For the $\theta_{st} - \omega$ graph (Figure 1) we can see that the value reaches maximum when ω/ω_0 approaches 1. And generally the line of damping selection 2 is higher than the line of damping selection 3. For the $\phi - \omega$ graph (Figure 2) we can find that both lines declines when ω/ω_0 increases and the slope reaches maximum when $\omega/\omega_0 = 1$.

From figure 4 in appendix which shows the therotical line, we can see that these results generally meets what we expected before the experiment, but quite lower than theoretical line. Recall that:

$$\theta_{st} = \frac{\mu}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

and we can deduce that β, with 7.3% relative uncertainty, may be the main cause of this phenomenon. (β is larger than theoretical value)

It's worth discussing that in both graphs, there is a outlier, which gives by the data that is close to resonance. And I think that is because near resonance, the change is not so obvious and I change too little that the machine can't show the difference at its precision.

In this lab experiment, I think there are two parts that create much uncertainty. First is the above part when I change a little, I can't get precise changes. And the second is the measurement of ϕ, which will show two values on the machine and is hard for us to read. So my suggestion is to add precision and use device to read the value of ϕ.

6 Reference

Qin Tian, Wang Yin, Tianyi Li, Mateusz Krzyzosiak, Physics Laboratory VP141 Exercise 5 Damped and Driven Oscillations. Mechanical Resonance

Appendix I

Some Figures

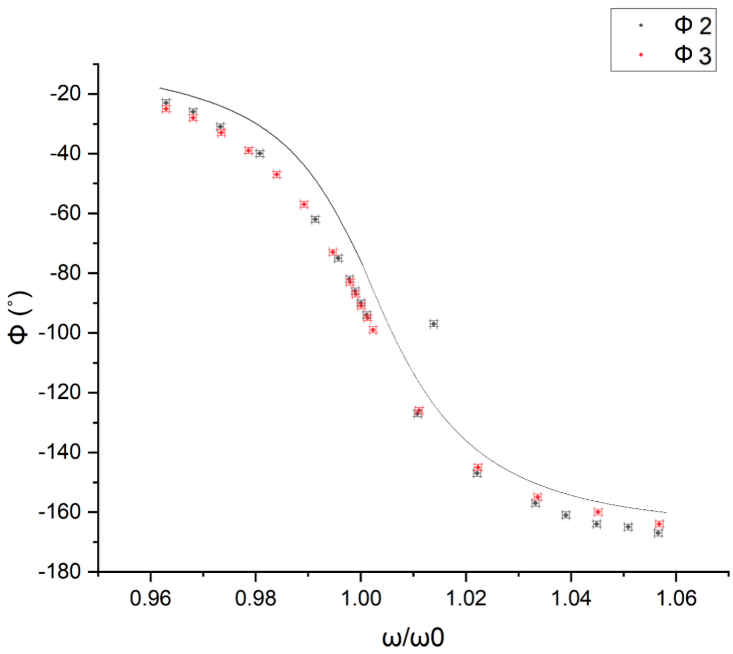

Figure 3: theoretical graph

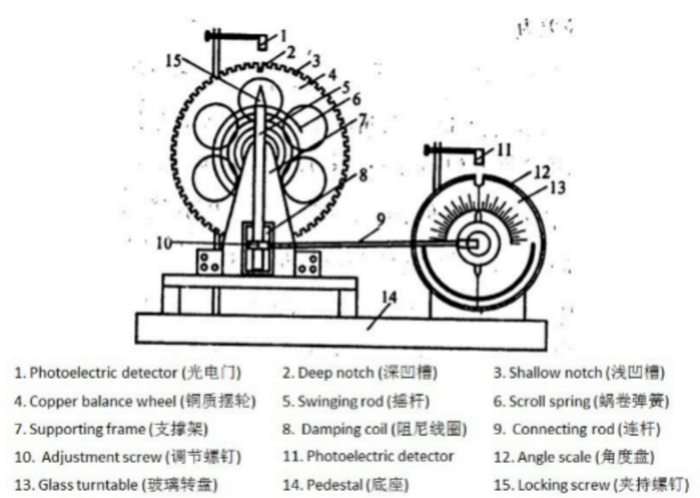

Figure 4: vibrometer

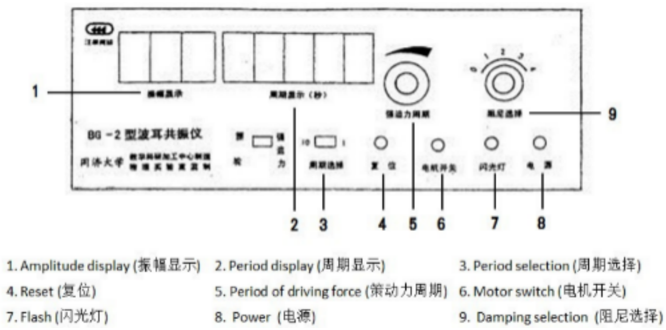

Figure 5: control box

Appendix II

Data and uncertainty analysis

	T_{10}	T	ω/ω_0	uncertainty for ω/ω_0	$\theta_{st}\pm 1$ [°]	$\phi\pm 1$ [°]
1	14.736	1.4736	1.057	0.00068	167	-36
2	14.816	1.4816	1.051	0.00068	165	-40
3	14.901	1.4901	1.045	0.00067	164	-44
4	14.985	1.4985	1.039	0.00067	161	-50
5	15.069	1.5069	1.033	0.00067	157	-57
6	15.233	1.5233	1.022	0.00066	147	-78
7	15.403	1.5403	1.011	0.00065	127	-113
8	15.569	1.5569	1.000	0.00065	90	-140
9	15.357	1.5357	1.014	0.00065	97	-139
10	15.552	1.5552	1.001	0.00065	94	-140
11	15.586	1.5586	0.999	0.00064	86	-140
12	15.603	1.5603	0.998	0.00064	82	-138
13	15.637	1.5637	0.996	0.00064	75	-135
14	15.706	1.5706	0.991	0.00064	62	-123
15	15.875	1.5875	0.981	0.00063	40	-89
16	15.997	1.5997	0.973	0.00063	31	-72
17	16.083	1.6083	0.968	0.00062	26	-63
18	16.169	1.6169	0.963	0.00062	23	-56

Table 6: calculated data for damping selection 2

	T_{10}	T	ω/ω_0	uncertainty for ω/ω_0	$\theta_{st}\pm1$ [°]	$\phi\pm1$ [°]
1	14.733	1.4733	1.057	0.00068	164	-36
2	14.897	1.4897	1.045	0.00067	160	-44
3	15.063	1.5063	1.034	0.00067	155	-56
4	15.23	1.523	1.022	0.00066	145	-76
5	15.399	1.5399	1.011	0.00065	126	-108
6	15.534	1.5534	1.002	0.00065	99	-130
7	15.55	1.555	1.001	0.00065	95	-131
8	15.568	1.5568	1.000	0.00065	91	-132
9	15.585	1.5585	0.999	0.00064	87	-132
10	15.602	1.5602	0.998	0.00064	83	-130
11	15.653	1.5653	0.995	0.00064	73	-126
12	15.74	1.574	0.989	0.00064	57	-112
13	15.823	1.5823	0.984	0.00064	47	-96
14	15.909	1.5909	0.979	0.00063	39	-82
15	15.994	1.5994	0.973	0.00063	33	-71
16	16.083	1.6083	0.968	0.00062	28	-62
17	16.169	1.6169	0.963	0.00062	25	-55

Table 7: calculated data for damping selection 3

UM-SJTU Joint Institute
Physics Laboratory
Data Sheet (Exercise 5)

Name: 张远飞. Student ID: 519370910103 .

Name: ______ Student ID: ______

Group: 10 Date: 2020/9/12 .

NOTICE. Please remember to show the data sheet to your instructor before leaving the laboratory. The data sheet will not be accepted if the data are recorded with pencil or modified by correction fluid/tape. If a mistake is made in recording a datum item, cancel the wrong value by drawing a fine line through it, record the correct value legibly, and ask your instructor to confirm the correction. Please remember to take a record of the precision of the instruments used.
You are required to hand in the original data with your lab report, so please keep the data sheet properly.

	T [s] ± 0.001 [s]
1	1.557
2	1.557
3	1.557
4	1.557

Table 1. Measurement of the natural frequency.

Damping Selection: 2

Amplitude [°] ± 1 [°]		Amplitude [°] ± 1 [°]		$\ln(\theta_i/\theta_{i+5})$
θ_0	80	θ_5	47	0.532
θ_1	72	θ_6	42	0.539
θ_2	65	θ_7	38	0.537
θ_3	58	θ_8	34	0.534
θ_4	51	θ_9	30 .	0.531 .
The average value of $\ln(\theta_i/\theta_{i+5})$				0.535

$10T =$ 15.595 [s] ± 0.001 [s]

Table 2. Measurement of the damping coefficient.

Instructor's signature: [illegible]

1

Damping Selection: 2

	$10T$ [s] ± 0.001 [s]	φ [°] ± 1 [°]	θ [°] ± 1 [°]
1	~~14.769~~ 14.736	~~166~~ 167	~~35~~ 36
2	~~14.833~~ 14.816	~~145~~ ~~150~~ 165	~~38~~ 40
3	~~14.909~~ 14.901	~~142~~ 164	~~42~~ 44
4	~~14.995~~ 14.985	~~136~~ 167	~~48~~ 50
5	~~15.077~~ 15.069	~~134~~ 157	~~58~~ 57
(3). 6	~~15.239~~ 15.233	~~100~~ 147	~~78~~ 78
7	15.403	127	113
8	15.569	90	140
9	15.537	97	139
10	15.552	94	140
11	15.586	86	140
12	15.603	~~83~~ 82	138
13	15.637	75	135
14	15.706	62	123
15	15.875	40	89
16	15.997	~~34~~ 31	~~72~~ 72
17	16.083	26	63
18	16.169	~~24~~ 23	56
19			
20			
21			
22			
23			
24			
25			
26			
27			
28			
29			
30			

Table 3. θ vs. ω and φ vs. ω characteristics.

Damping Selection: 3

	$10T$ [s] ± 0.001 [s]	φ [°] ± 1 [°]	θ [°] ± 1 [°]
1	14.733	164	36
2	14.897	160	44
3	15.063	155	56
4	15.230	145	76
5	15.399	126	108
6	15.534	99	130
7	15.550	95	131
8	15.568	91	132
9	15.585	87	132
10	15.602	83	130
11	15.653	73	126
12	15.740	57	112
13	15.823	47	96
14	15.909	39	82
15	15.994	33	71
16	16.083	28	62
17	16.169	25	~~55~~ 55
18			
19			
20			
21			
22			
23			
24			
25			
26			
27			
28			
29			
30			

Table 4. θ vs. ω and φ vs. ω characteristics.

Instructor's signature:

UM-SJTU Joint Institute, Physics Laboratory I
Measurement Uncertainty Analysis Worksheet*
Exercise 5

WS-1 Natural Angular Frequency

The uncertainty for ten periods is found first. Then the result for the natural frequency is given along with its uncertainty.

The type-B uncertainty for T is $\Delta_{TB} = 0.001$ s. To find the type-A uncertainty, we first find the standard deviation

$$s_T = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(T_i - \overline{T})^2} = \underline{\quad 0 \quad} \quad [\underline{\ s\ }].$$

We have $n = \underline{\ 4\ }$, so the type-A uncertainty Δ_{TA} is calculated as

$$\Delta_{TA} = \frac{t_{0.95}}{\sqrt{n}} s_T = \underline{\ 1.59\ } \times \underline{\ 0\ } = \underline{\quad 0 \quad} \quad [\underline{\ s\ }].$$

Hence the uncertainty for T is given by

$$u_T = \sqrt{\Delta_{TA}^2 + \Delta_{TB}^2} = \underline{\ 0.001\ } \quad [\underline{\ s\ }].$$

Hence the period is given by

$$\boxed{T = \underline{\ 1.557\ } \pm \underline{\ 0.001\ } \ [\underline{\ s\ }]},$$

*Created by Peng Wenhao, edited by Fan Yixing, Ye Haojie, Li Tianyi, MateuszKrzyzosiak[rev. 1.5] with relative uncertainty

$$\boxed{u_{rT}} = \frac{u_T}{T} \times 100\% = \boxed{\underline{\ 0.06\ }\%}$$

The natural angular frequency ω_0 is found from the formula $\omega_0 = 2\pi/T$, so by the uncertainty propagation formula and the fact that

$$\frac{\partial \omega_0}{\partial T} = -\frac{2\pi}{T^2},$$

we obtain

$$\boxed{u_{\omega_0}} = \left|\frac{\partial \omega_0}{\partial T} u_T\right| = \boxed{\underline{\ 0.007...\ }\ [\underline{\ s^{-1}\ }]}$$

with the relative uncertainty

$$\boxed{u_{r,\omega_0}} = \frac{u_{\omega_0}}{\omega_0} \times 100\% = \boxed{\underline{\ 0.06\ }\%}$$

WS-2 Damping Coefficient

The damping coefficient is found indirectly form measurements of the period T and the amplitude θ as $\beta = \frac{1}{5T}\ln(\theta_i/\theta_{i+5})$.

The uncertainty each single measurement of the amplitude is $u_\theta = \underline{\ 1\ }°$, so the uncertainty of the logarithm of the quotient of them $q_i = \ln(\theta_i/\theta_{i+5})$ is found from the uncertainty propagation formula

$$\Delta_{q_i,B} = \sqrt{\left(\frac{\partial(\ln(\theta_i/\theta_{i+5}))}{\partial \theta_i}\right)^2 u_\theta^2 + \left(\frac{\partial(\ln(\theta_i/\theta_{i+5}))}{\partial \theta_{i+5}}\right)^2 u_\theta^2} = \sqrt{\left(\frac{u_\theta}{\theta_{i+5}}\right)^2 + \left(\frac{u_\theta}{\theta_i}\right)^2}$$

For example, for $i = 1$,

$$\Delta_{q_1,B} = \sqrt{\underline{\ 1/5184 + 1/1764\ }} = \underline{\ 0.028\ }.$$

The results for all five sequences of measurements are given in Table WS-1.

	i	$\Delta_{q_i,B}$
0	1	0.025
1	2	0.028
2	3	0.030
3	4	0.034
4	5	0.039

Table WS-2: Type-B uncertainties for q_i.

Page WS-2

The overall type-B uncertainty for the quotient can be estimated as the maximum of uncertainties listed in in Table WS-2

$$\Delta_{q,B} = \underline{0.039}.$$

To estimate the type-A uncertainty of q, the standard deviation of q is calculated as

$$s_q = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(q_i - \bar{q})^2} = \underline{0.003}$$

Hence the type-A uncertainty for $n = 5$ is calculated as

$$\Delta_{q,A} = \frac{t_{0.95}}{\sqrt{n}} s_q = \underline{1.204} \times \underline{0.003} = \underline{0.004},$$

and the combined uncertainty

$$u_q = \sqrt{\Delta_{q,B}^2 + \Delta_{q,A}^2} = \sqrt{\underline{0.039^2 + 0.004^2}} = \underline{0.04}.$$

A single measurement for ten periods is recorded as $T_{10} = \underline{15.595} \pm \underline{0.001}$ [$\underline{s}$]. Hence $T = \underline{1.5595} \pm \underline{0.0001}$ [$\underline{s}$].

Then the uncertainty propagation equation is used to calculate the uncertainty for the damping coefficient $\beta = \frac{1}{5T}q$ as

$$\boxed{u_\beta} = \sqrt{\left(\frac{\partial \beta}{\partial T}\right)^2 u_T^2 + \left(\frac{\partial \beta}{\partial q}\right)^2 u_q^2} = \sqrt{\left(-\frac{q}{5T^2}\right)^2 u_T^2 + \left(\frac{1}{5T}\right)^2 u_q^2}$$

$$= \boxed{\sqrt{\left(-\frac{0.535}{5\times 1.5595^2}\right)^2 \times 0.0001^2 + \left(\frac{1}{5\times 1.5595}\right)^2 \times 0.04^2} \quad \underline{0.005} \; [\, s^{-1} \,]}$$

with relative uncertainty

$$\boxed{u_{r,\beta}} = \frac{u_\beta}{\beta} \times 100\% = \boxed{\underline{7.3}\ \%}$$

WS-3 The θ_{st}-ω and φ-ω Characteristics of Forced Oscillations

On the graphs included in the report, the uncertainty is shown in the form of error bars.[1] In both the φ vs. (ω/ω_0) graph and the θ_{st} vs. (ω/ω_0) graph, the

[1] Please follow this part to find the uncertainties and mark them on the graphs of the phase shift φ vs. (ω/ω_0) graph and the amplitude of steady-state oscillations θ_{st} vs. (ω/ω_0).

measurements of φ and θ_{st} are single measurements with uncertainty 1 °, determined by the resolution of our equipment. However, to find the uncertainty of (ω/ω_0) we need to derive it from the uncertainty propagation formula. Let us introduce symbols $Q = \frac{\omega}{\omega_0}$, $T_{\text{natural}} = N$ and $T_{\text{driven}} = D$, where the uncertainty of D is again the minimum scale (resolution) of the equipment used. Since these are single measurements, we have

$$Q = \frac{\omega}{\omega_0} = \frac{T_{\text{natural}}}{T_{\text{driven}}} = \frac{N}{D}$$

and the uncertainty of the ratio Q, found from the uncertainty propagation formula, is

$$u_Q = \sqrt{\left(\frac{\partial Q}{\partial N} u_N\right)^2 + \left(\frac{\partial Q}{\partial D} u_D\right)^2} = \sqrt{\left(\frac{u_N}{D}\right)^2 + \left(\frac{N u_D}{D^2}\right)^2}$$

In particular, with $N =$ 1.557 [s], $u_N =$ 0.001 [s], and $u_D =$ 0.0001 [s], so with every set of N and D a unique uncertainty is generated. For instance,[2] for $D =$ 1.4733 [s], we can calculate Q as

$$Q = \frac{N}{D} = \frac{1.557}{1.4733} = 1.057$$

with uncertainty u_Q calculated as

$$\boxed{u_Q} = \sqrt{\left(\frac{0.001}{1.4733}\right)^2 + \left(\frac{1.557}{1.4733^2} \times 0.0001\right)^2} = \boxed{6. \quad \times 10^{-4}},$$

and

$$\boxed{u_\varphi = 1° = 0.017 \text{ rad}} \qquad \boxed{u_{\theta_{st}} = 1° = 0.017 \text{ rad}}.$$