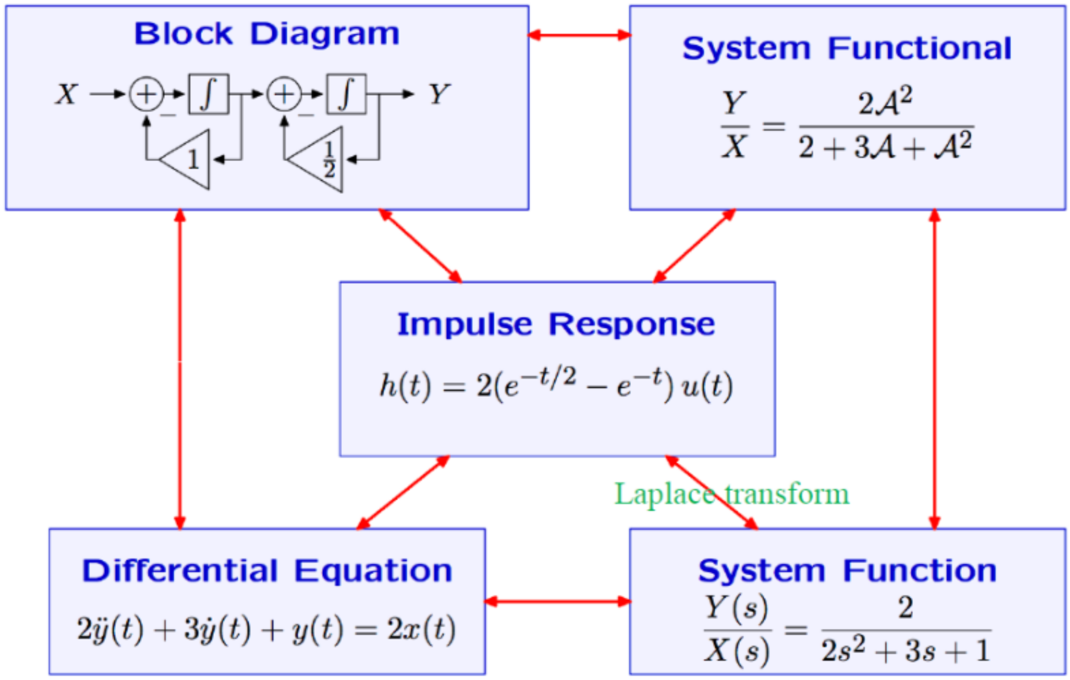
# VE216 Lecture 6

> Laplace Transform

## Concept Map




## Laplace Transform Definition

$X(s) = \int x(t)e^{-st}dt$

### Two versions

- Unilateral: $X(s) = \int_0^\infty x(t)e^{-st}dt$
- Bilateral: $X(s) = \int_{-\infty}^\infty x(t)e^{-st}dt$

(We focus on bilateral version.)

# Region of Convergence (ROC)

This is based on the $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$, we can see from the examples.

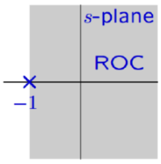
# Example

## Exercise 1

$x_1(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$ Laplace transform.

$X_1(s) = L[x_1(t)] = \int_0^{\infty} e^{-t}e^{-st}dt = -\frac{1}{1+s}\int_0^{\infty} -(1+s)e^{-(1+s)t}dt = -\frac{1}{1+s}e^{-(1+s)t}\Big|_0^{\infty} = -\frac{1}{1+s}$

Since $e^{-(1+s)t}$ is convergent when $1+s<0$, or $Re(s) > -1$, so ROC shows

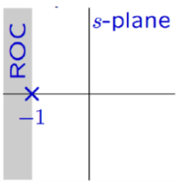



## Exercise 2

$x_2(t) = \begin{cases} -e^{-t} & t \leq 0 \\ 0 & t > 0 \end{cases}$ Laplace transform.

$X_2(s) = \int_{-\infty}^{0} -e^{-t}e^{-st}dt = \frac{1}{1+s}\int_{-\infty}^{0} -(1+s)e^{-(1+s)t}dt = \frac{1}{1+s}$

Since $e^{-(1+s)t}$ is convergent for $x \in (-\infty, 0)$, so $-(1+s) > 0$ or $Re(s) < -1$ with ROC shows

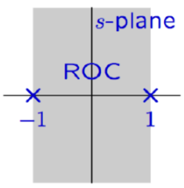



## Exercise 3

$x_3(t) = e^{-|t|}$ Laplace transform.

$X_3(s) = \int_{-\infty}^{\infty} e^{-|t|-st}dt = \int_0^{\infty} e^{-(1+s)t}dt + \int_{-\infty}^{0} e^{(1-s)t}dt = \frac{e^{(1-s)t}}{1-s}\Big|_{-\infty}^{0} - \frac{e^{-(1+s)t}}{1+s}\Big|_0^{\infty} = \frac{1}{1-s} + \frac{1}{1+s}$

So $1-s>0$ and $1+s>0$, $1>s>-1$, with ROC shows



## Exercise 4

$\frac{2s}{s^2-4}$ Laplace inverse transform, how many possible solution and ROCs?

$\frac{2s}{s^2-4} = \frac{1}{s+2} + \frac{1}{s-2}$, with $\pm 2$ as poles.

- $\frac{1}{s+2}$ has 2 forms with 2 ROCs.
    - $\frac{1}{s+2} = \int_{-\infty}^{\infty} e^{-st}e^{-2t}u(t)dt = \int_{0}^{\infty} e^{-(2+s)t}dt = -\frac{1}{s+2}e^{-(2+s)t}\Big|_0^{\infty}$, so $2+s>0$ with $Re(s) > -2$.
    - $\frac{1}{s+2} = \int_{-\infty}^{\infty} -e^{-st}e^{-2t}u(-t)dt = \int_{-\infty}^{0} -e^{-(s+2)t} = \frac{1}{s+2}e^{-(s+2)t}\Big|_{-\infty}^{0}$, so $2+s<0$ with $Re(s) < -2$.
- $\frac{1}{s-2}$ has 2 forms with 2 ROCs
    - $\frac{1}{s-2} = \int_{-\infty}^{\infty} e^{-st}e^{2t}u(t)dt = \int_{0}^{\infty} e^{(2-s)t}dt = \frac{1}{2-s}e^{(2-s)t}\Big|_0^{\infty}$, so $2-s<0$ with $Re(s) > 2$.
    - $\frac{1}{s-2} = \int_{-\infty}^{\infty} -e^{-st}e^{2t}u(-t)dt = \int_{-\infty}^{0} -e^{(2-s)t}dt = \frac{1}{s-2}e^{(2-s)t}\Big|_{-\infty}^{0}$, so $2-s>0$ with $Re(s) < 2$.

So there are totally 3 solutions:

- $x(t) = e^{-2t}u(t) + e^{2t}u(t)$ with $Re(s) > 2$
- $x(t) = -e^{-2t}u(-t) - e^{2t}u(-t)$ with $Re(s) < -2$
- $x(t) = e^{-2t}u(-t) - e^{2t}u(-t)$ with $2 > Re(s) > -2$

# Laplace Transform of a Derivative

$X_d(s) = \int_{-\infty}^{\infty} x'(t)e^{-st}dt = x(t)e^{-st}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x(t)(-s)e^{-st}dt$

Since $X(s)$ is convergent, so $x(t)e^{-st}\Big|_{-\infty}^{\infty} = 0$, thus $X_d(s) = sX(s)$.

# Laplace Transform Properties

| Property | $x(t)$ | $X(s)$ | ROC |
|---|---|---|---|
| Linearity | $ax_1(t)+bx_2(t)$ | $aX_1(s)+bX_2(s)$ | $\supset (R_1 \cap R_2)$ |
| Delay by $T$ | $x(t-T)$ | $X(s)e^{-sT}$ | $R$ |
| Multiply by $t$ | $tx(t)$ | $-\frac{dX(s)}{ds}$ | $R$ |
| Multiply by $e^{-\alpha t}$ | $x(t)e^{-\alpha t}$ | $X(s+\alpha)$ | shift $R$ by $-\alpha$ |
| Differentiate in $t$ | $\frac{dx(t)}{dt}$ | $sX(s)$ | $\supset R$ |
| Integrate in $t$ | $\int_{-\infty}^{t} x(\tau)\,d\tau$ | $\frac{X(s)}{s}$ | $\supset \big(R \cap (\mathrm{Re}(s)>0)\big)$ |
| Convolve in $t$ | $\int^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau$ | $X_1(s)X_2(s)$ | $\supset (R_1 \cap R_2)$ |

## Initial Value Theorem

If $x(t) = 0\, \forall t < 0$, $x(t)$ contains no impulses at $t = 0$, then

$x(0^+) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \int_0^\infty x(t) s e^{-st} dt = \lim_{s\to\infty} \int_0^\infty x(t)\delta(t) dt = x(0^+)$

(As $s \to \infty$ $e^{-st}$ shrink to 0 very fast.)

## Final Value Theorem

If $x(t) = 0\, \forall t < 0$, $x(t)$ contains no impulses at $t = 0$, then

$x(\infty) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} \int_0^\infty x(t) s e^{-st} dt$

Since $se^{-st}$ is flattened when $s \to 0$, $se^{-st}$ covers area of 1 whatever $s$.

So $x(\infty) = \frac{1}{\infty} \int_0^\infty x(t) dt = x(\infty)$.