Homework 2

VE216 - Introduction to Signal and Systems, Qiao Heng, Spring 2021

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HW Notes:

- Problems where the number of points are followed by an exclamation are basic skill problems and will be graded without partial credit.
- $\boxed{\text{Box}}$ your final answer. You will be graded on both the final answer and the steps leading to it. Correct intermediate steps will help earn partial credit. For full credit, ~~cross out~~ any incorrect intermediate step.
- If you need to make any additional assumptions, state them clearly.
- Legible writing will help when it comes to partial credit.
- Simplify your result when possible.

Problems:

1. One of following two statements is correct, and the other is incorrect. The symbol $\boxed{*}$ denotes *convolution*. a is a constant.

- If $y(t) = h(t) * x(t)$ then $y(t-a) = h(t-a) * x(t-a)$;
- Or if $y(t) = h(t) * x(t)$ then $y(t-a) = h(t) * x(t-a)$.

(a) [4!] Give a simple proof of the correct statement.
(b) [3!] Give a simple counterexample for the incorrect statement.
(c) [3!] Repeat (a) and (b) for the following two statements. The symbol $\boxed{\cdot}$ denotes *multiplication*.

- if $y(t) = h(t) \cdot x(t)$ then $y(t-a) = h(t-a) \cdot x(t-a)$;
- Or if $y(t) = h(t) \cdot x(t)$ then $y(t-a) = h(t) \cdot x(t-a)$.

Be careful with the notation $h(t) * x(t)$. More precise notation is $(h * x)(t)$, which makes it clear that convolution is an operation on two signals, not a point-wise operation like multiplication. For question (a), can you utilize the property of delay property of convolution?

Answer:

(a) The $\boxed{\text{second}}$ one is right

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$\to y(t-a) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau+a)d(\tau) = h(t) * x(t-a)$$

(b) let x(t)=$\delta(t)$ and h(t)=u(t) then

$$y(t) = h(t) * x(t) = u(t) \to y(t-a) = u(t-a)$$

$$h(t-a) * x(t-a) = u(t-a) * \delta(t-a) = u(t-2a) \neq y(t-a)$$

(c) The $\boxed{\text{first}}$ one is right

$$\begin{aligned} y(t-a) &= y(t) * \delta(t-a) = (h(t)x(t)) * \delta(t-a) \\ &= h(t) * \delta(t-a) \cdot x(t) * \delta(t-a) = h(t-a)x(t-a) \end{aligned}$$

let x(t)=$\delta(t)$ and h(t)=u(t) and a<0 then

$$y(t) = h(t)x(t) = \delta(t) \to y(t-a) = \delta(t-a)$$

$$h(t)x(t-a) = 0 \neq y(t-a)$$

2. Let $y(t) = (x * h)(t)$. Show the following properties of convolution.

(a) [5!] $\int_{-\infty}^{\infty} y(t)dt = \left[\int_{-\infty}^{\infty} x(t)dt\right] \left[\int_{-\infty}^{\infty} h(t)dt\right]$;

(b) [5!] $\frac{d^n}{dt^n} y(t) = \left[\frac{d^n}{dt^n} x(t)\right] * h(t) = x(t) * \left[\frac{d^n}{dt^n} h(t)\right]$;

Answer:
(a)

$$\begin{aligned}\int_{-\infty}^{\infty} y(t)dt &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(\tau-a)d\tau \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\tau h(t-\tau)dtd\tau \\ &= \int_{-\infty}^{\infty} x(\tau)\int_{-\infty}^{\infty} h(t-\tau)d(t-\tau)d\tau \\ &= [\int_{-\infty}^{\infty} x(\tau)d\tau][\int_{-\infty}^{\infty} h(m)dm] \\ &= \boxed{\int_{-\infty}^{\infty} x(t)dt][\int_{-\infty}^{\infty} h(t)dt]}\end{aligned}$$

(b)

$$\begin{aligned}\frac{d^n}{dt^n}y(t) &= \frac{d^n}{dt^n}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} \frac{d^n}{dt^n}x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} [\frac{d^n}{dt^n}x(\tau)]h(t-\tau)d\tau \\ &= \boxed{[\tfrac{d^n}{dt^n}x(t)] * h[t]}\end{aligned}$$

similarly, we can get $\boxed{\frac{d^n}{dt^n}y(t) = x(t)\frac{d^n}{dt^n}h(t)]}$

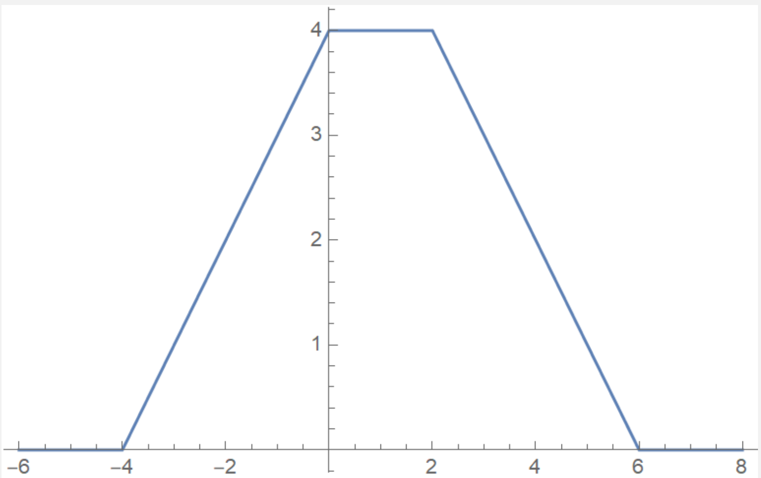
3. Often we will be convolving two signals that are zero everywhere except over some finite range (called **finite support** signals). Suppose $x_1(t)$ is non-zero over the range $a \leq t \leq b$ and that $x_2(t)$ is non-zero over the range $c \leq t \leq d$. Suppose $y(t) = x_1(t) * x_2(t)$.
(a) [5!] Find the range of values of t for which y(t) is possibly non-zero.
(b) [10!] Compute $rect((t-2)/4) * rect((t+1)/6)$ (express answer with braces and carefully sketch). Check your result with part (a).

Answer:
(a) $y(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)dt$ we need to get
$a \le \tau \le b$ and $c \le t - \tau \le d$, we can get $a + c \le t \le b + d$ (b)

$$y(t) = \int_{-\infty}^{\infty} rect(\frac{\tau - 2}{4})rect(\frac{t - \tau + 1}{6})$$

for x_1, $0 \le \tau \le 4$, for x_2, $t - 2 \le \tau \le t + 4$

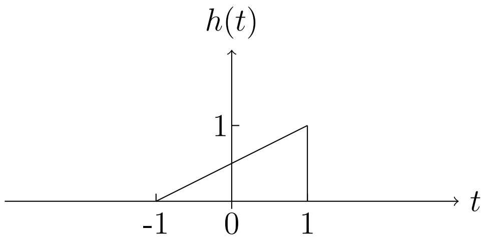
$$\boxed{\text{g(t)}} = \begin{cases} 4 + t & -4 < t \le 0 \\ 4 & 0 < t \le 2 \\ 6 - t & 2 < t \le 6 \\ 0 & t \le -4 \text{ and } 6 \le t \end{cases}$$



4. [15!] Consider a LTI system. Let its impulse response h(t) be the triangular pulse shown below, and x(t) be the *impulse train*

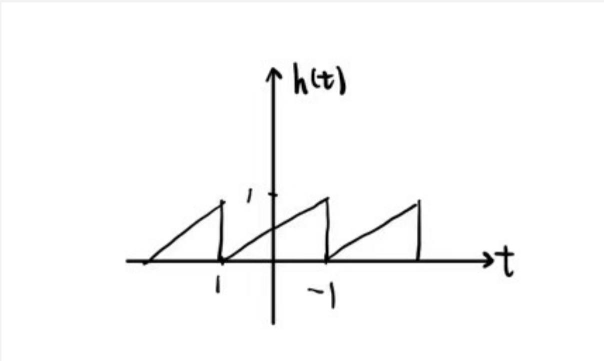
$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

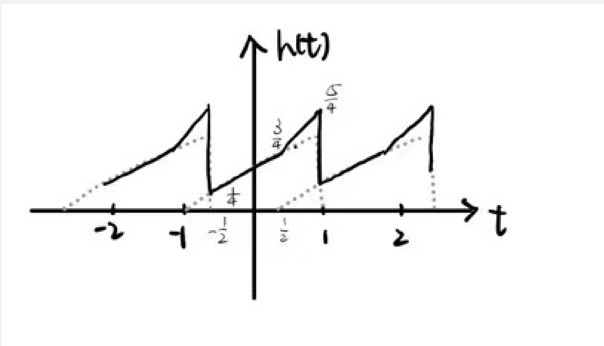
SKETCH $y(t) = x(t) * h(t)$ for T=2, 1.5 and 1. (No formulae are needed though you still want to label your graphs clearly.)

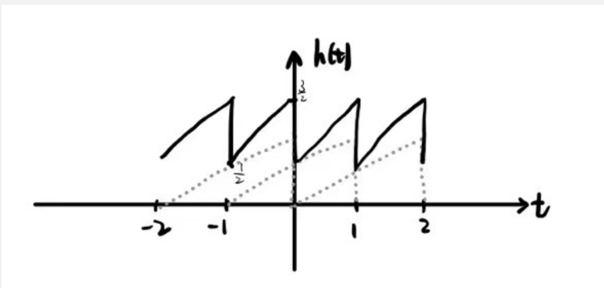


Answer:

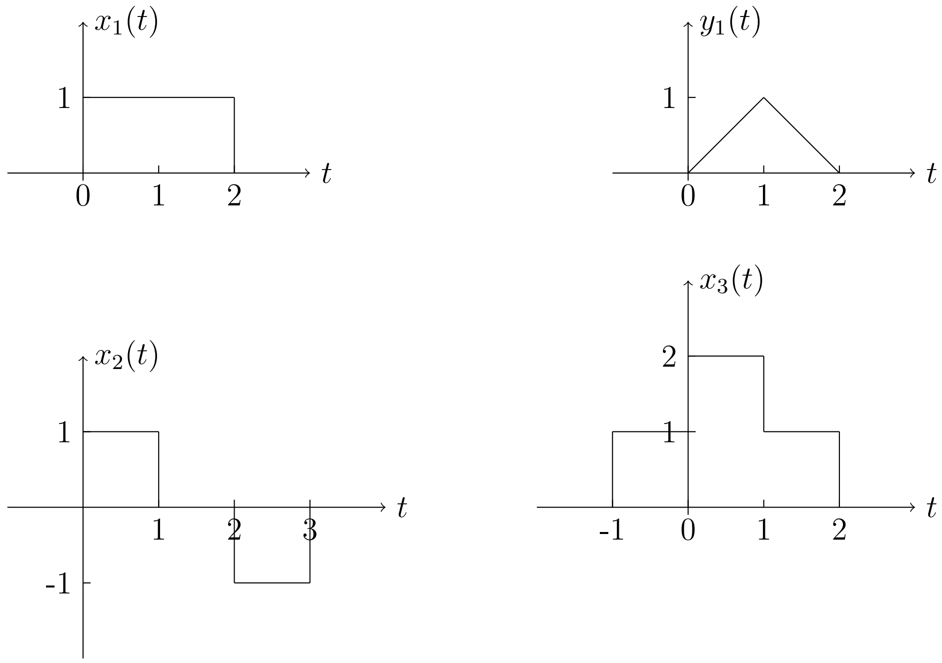
$$y(t) = h(t) * x(t) = \sum_{n=-\infty}^{\infty} h(t - nT)$$



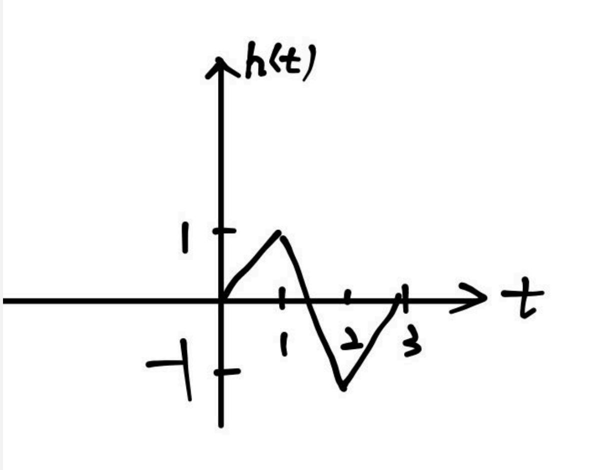


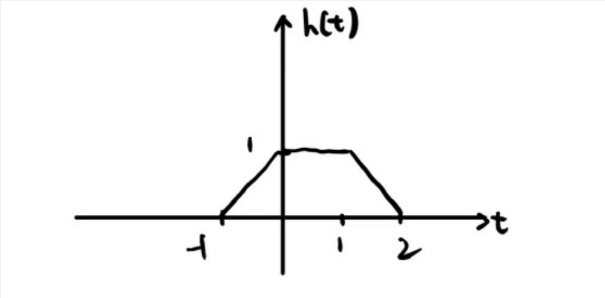


5. [2×5!] Consider an LTI system whose response to the signal $x_1(t)$ is the signal $y_1(t)$ which are illustrated below.
(a) Determine and sketch carefully the response of the system to the input $x_2(t)$ depicted below.
(b) Determine and sketch carefully the response of the system to the input $x_3(t)$ depicted below.



Answer:
(a) $x_2(t) = x_1(t) - x_1(t-1) \rightarrow y_2(t) = y_1(t) - y_1(t-1)$
(b) $x_2(t) = x_1(t) + x_1(t+1) \rightarrow y_2(t) = y_1(t) + y_1(t+1)$





6. [10!] The triangular pulse is defined as $tri(t) = (1 - |t|)rect(t/2)$. Compute $x(t) = tri(t/2) * rect(\frac{t-1}{2})$. Express your answer using braces, and carefully sketch.

Answer:
$x(t) = (1-|\frac{t}{2}|)rect(\frac{t}{4}) * rect(\frac{t-1}{2}) = \int_{-\infty}^{\infty}(1-|\frac{\tau}{2}|)rect(\frac{\tau}{4})rect(\frac{t-\tau-1}{2})d\tau$

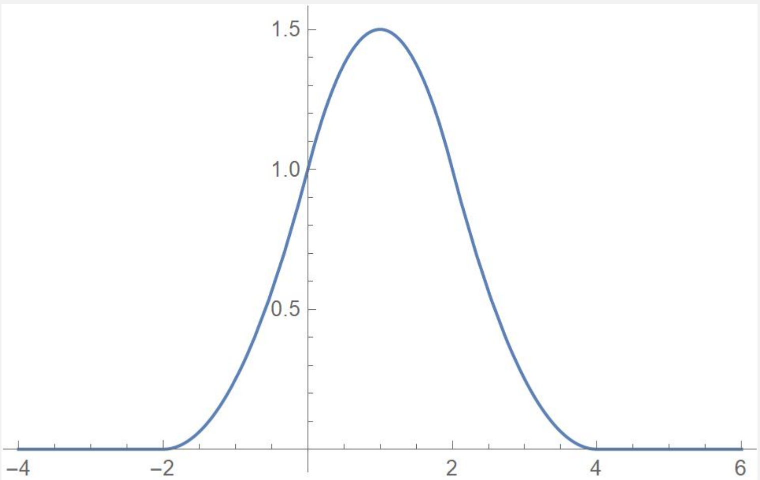
(1)-2<t<0 $x(t) = \int_{-2}^{t}(1+\frac{\tau}{2})d\tau = \frac{t^2}{4}+t+1$

(2)0≤ t<2 $x(t) = \int_{0}^{t}(1-\frac{\tau}{2})d\tau + \int_{t-2}^{0}(1+\frac{\tau}{2})d\tau = -\frac{t^2}{2}+t+1$

(3)2≤ t ≤4 $x(t) = \int_{t-2}^{2}(1-\frac{\tau}{2})d\tau = \frac{t^2}{4}-2t+4$

(4) others 0

$$\boxed{\text{g(t)}} = \begin{cases} \frac{t^2}{4}+t+1 & -2<t<0 \\ -\frac{t^2}{2}+t+1 & 0\le t<2 \\ \frac{t^2}{4}-2t+4 & 2\le t\le 4 \\ 0 & t \le others \end{cases}$$



7. [4×4!] Find the impulse response of the following LTI systems and further determine whether they are causal, stable and static.

(a) $y(t) = \int_{-\infty}^{t} (\tau - t)e^{-2(t-\tau)}x(\tau)d\tau$

(b) $y(t) = \int_{t-2}^{t} e^{-(t-\tau)}x(\tau)d\tau$

(c) $y(t) = \int_{-\infty}^{t} \left[\int_{-\infty}^{s} x(\tau - 5)d\tau\right] ds$

(d) $y(t) = \int_{-3}^{3} \tau^2 x(t - \tau)d\tau + \int_{-\infty}^{t+1} (t - \tau + 3)^{-2}x(\tau)d\tau$

Answer:

(a) $y(t) = \int_{-\infty}^{t}(\tau - t)e^{-2(t-\tau)}x(\tau)d\tau = \int_{-\infty}^{\infty}(\tau - t)e^{-2(t-\tau)}u(t-\tau)x(\tau)d\tau$
$= x(t) * (-te^{-2t}u(t)) \rightarrow h(t) = -te^{-2t}u(t)$

Casual: It is casual because h(t)=0 when t<0

Stable: It is stable because $\int_{-\infty}^{\infty} |-te^{-2t}u(t)|dt = \int_{0}^{\infty} te^{-2t}dt = \frac{1}{4} \neq \infty$

Static: It is not static because h(t)$\neq$kδ(t)

So it is casual, stable and not static.

(b) $y(t) = \int_{t-2}^{t} e^{-(t-\tau)}x(\tau)d\tau = \int_{-\infty}^{\infty} e^{-(t-\tau)}x(\tau)u(\tau - t + 2)u(t-\tau)d\tau$
$h(t) = e^{-t}u(2-t)u(t)$

Casual: It is casual because h(t)=0 when t<0

Stable: It is stable because $\int_{-\infty}^{\infty} |e^{-t}u(2-t)u(t)|dt = \int_{0}^{2} e^{-t}dt = 1-e^{-2} \neq \infty$

Static: It is not static because h(t)$\neq$kδ(t)

So it is $\boxed{\text{casual, stable and not static}}$.

(c) $y(t) = \int_{-\infty}^{t}[\int_{-\infty}^{s} x(\tau - 5)d\tau]ds = \int_{-\infty}^{t}[\int_{-\infty}^{\infty} x(\tau - 5)u(s-\tau)d\tau]ds$
$= \int_{-\infty}^{t} x(s)u(s-5)ds = \int_{-\infty}^{\infty} x(s)u(s-5)u(t-s)ds$
$\rightarrow h(t) = u(t-5) * u(t) = (t-5)u(t-5)$

Casual: It is casual because h(t)=0 when t<0

Stable: It is not stable because $\int_{-\infty}^{\infty} |h(t)|dt = \infty$

Static: It is not static because h(t)$\neq$kδ(t)

So it is $\boxed{\text{casual, not stable and not static}}$.

(d)

$$y(t) = \int_{-3}^{3} \tau^2 x(t-\tau)d\tau + \int_{-\infty}^{t+1} (t-\tau+3)^{-2}x(\tau)d\tau$$
$$= x(t) * (u(t+3)u(3-t)t^2) + x(t) * (u(t+1)(t+3)^{-2})$$
$$\rightarrow h(t) = u(t+3)u(3-t)t^2 + u(t+1)(t+3)^{-2}$$

Casual: It is not casual because h(t)$\neq$0 when t<0

Stable: It is stable because $\int_{-\infty}^{\infty} |h(t)|dt < \infty$

Static: It is not static because h(t)$\neq$kδ(t)

So it is $\boxed{\text{not casual, stable and not static}}$.

8. [10!] Consider an LTI system S and a signal $x(t) = e^{-5t}u(t-1)$. If $x(t) \rightarrow y(t)$ and
$$\frac{d}{dt}x(t) \rightarrow -5y(t) + e^{-t}u(t)$$
determine the impulse response h(t) of S.

Answer:
$\frac{d}{dt}x(t) = -5e^{-5t}u(t-1) + e^{-5t}\delta(t-1)$
Since it's an LTI system, we cna know that $e^{-5t}\delta(t-1) \to e^{-t}u(t)$
Since $\delta(t-1)$ is only suitable for t=1: $e^{-5}\delta(t-1) \to e^{-t}u(t)$
then $\delta(t-1) \to e^{-t+5}u(t)$
$h(t-1) = e^{-t+5}u(t) \to h(t) = \boxed{e^{-t+4}u(t+1)}$

9. [4!] Find the expression of response of the CT system described by the linear constant-coefficient differential equation.

$$\frac{d}{dt}y(t) + 10y(t) = 2x(t)$$

where

$$y(0) = 1; x(t) = u(t)$$

Answer:
First, we found the solution for $\frac{d}{dt}y(t) + 10y(t) = 0$ and $y_h(t) = Ce^{-10t}$
x(t)=u(t), so $\frac{d}{dt}x(t) = 0$ for x>0, so $y_p = P$
$-10Ce^{-10t} + 10Ce^{-10t} + 10P = 2$, so $P = \frac{1}{5}$
$y(0) = 1 \to C + \frac{1}{5} = 1 \to C = \frac{4}{5}$
$y = y_h + y_p = \frac{4}{5}e^{-10t} + \frac{1}{5}$, but y is 0 when $y \leq 0$, so $y(t) = (\frac{4}{5}e^{-10t} + \frac{1}{5})u(t)$
$h(t) = \frac{d}{dt}y(t) = \boxed{-8e^{-10t}u(t) + \delta(t)}$