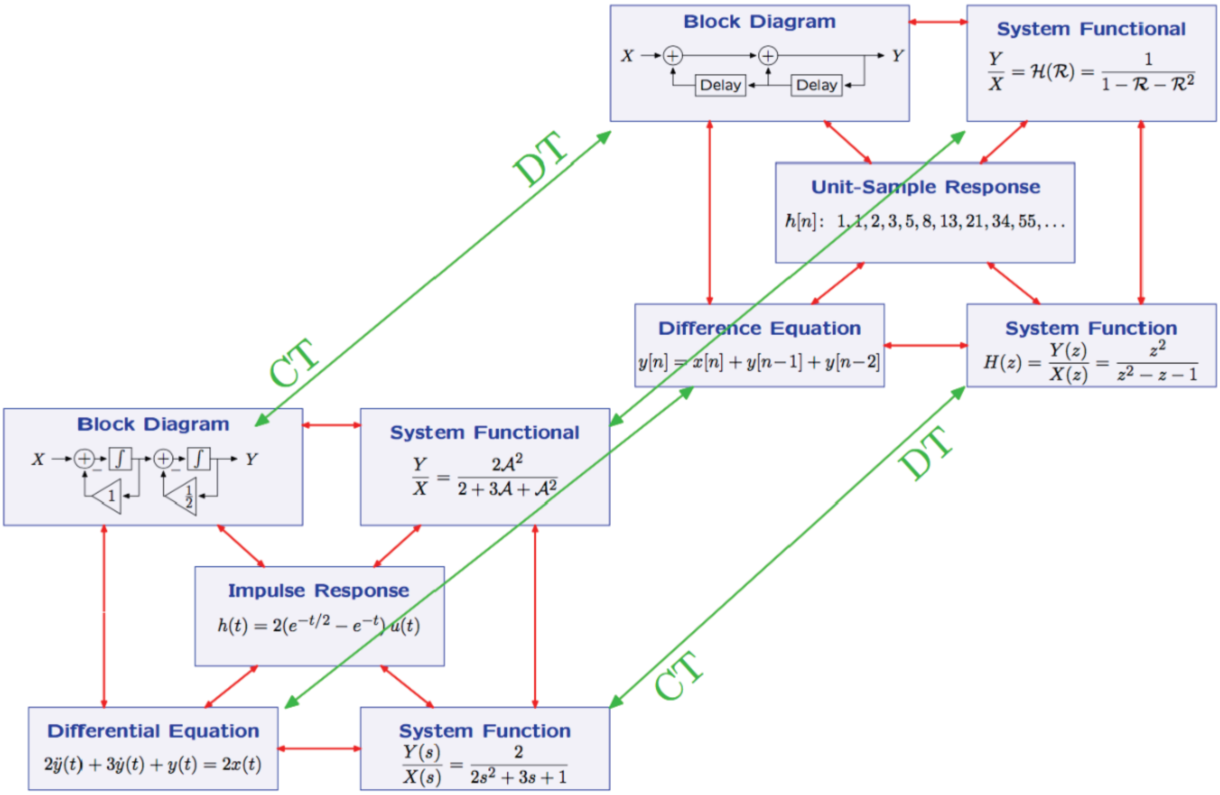
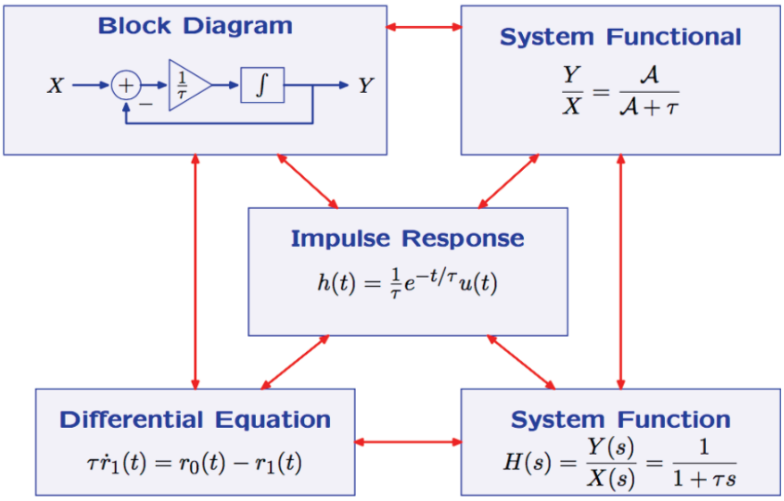
VE216 Lecture 7

> Discrete Approximation of Continuous Time Systems

Concept Map



Discrete Approximation of CT Systems



The step response (given $u(t)$ as the input) of the system:

$$\delta(t) \rightarrow \frac{A}{A+\tau} \rightarrow h(t) = \frac{1}{\tau}e^{-\frac{t}{\tau}}u(t)$$

$$u(t) \rightarrow \frac{A}{A+\tau} \rightarrow s(t) = ?$$

$$\delta(t) \rightarrow A \rightarrow u(t) \rightarrow \frac{A}{A+\tau} \rightarrow s(t) = ?$$

$$\delta(t) \rightarrow \frac{A}{A+\tau} \rightarrow h(t) \rightarrow A \rightarrow s(t) = \int_{-\infty}^{t} h(t')dt'$$

$$s(t) = \int_{-\infty}^{t} \frac{1}{\tau}e^{-\frac{t'}{\tau}}u(t')dt' = \int_{0}^{t} \frac{1}{\tau}e^{-\frac{t'}{\tau}}dt' = (1 - e^{-\frac{t}{\tau}})u(t)$$

Forward Euler Approximation

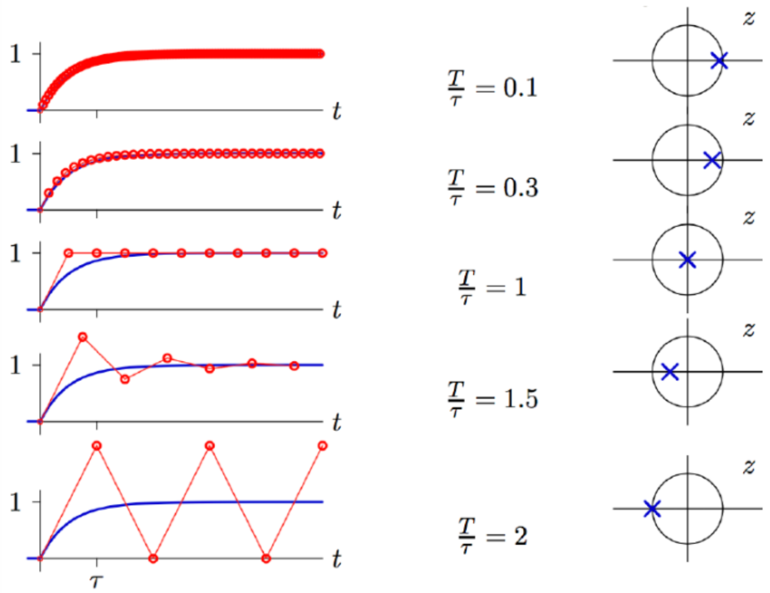
$x_d[n] = x_c(nT)$ and $y_d[n] = y_c(nT)$

$\dot{y}_c(nT) = \frac{1}{T}(y_d[n+1] - y_d[n])$

T is a sampling interval.

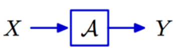
Thus we obtain $\frac{\tau}{T}(y_d[n+1] - y_d[n]) = x_d[n] - y_d[n]$, or $y_d[n+1] - (1 - \frac{T}{\tau})y_d[n] = \frac{T}{\tau}x_d[n]$

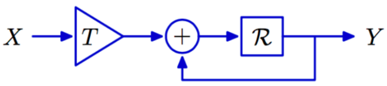
The pole can be achieved by z transform: $zY_d(z) - (1 - \frac{T}{\tau})Y_d(z) = \frac{T}{\tau}X_d(z)$, $z = 1 - \frac{T}{\tau}$.



Dependence of DT pole on Stepsize

> Forward Euler Method

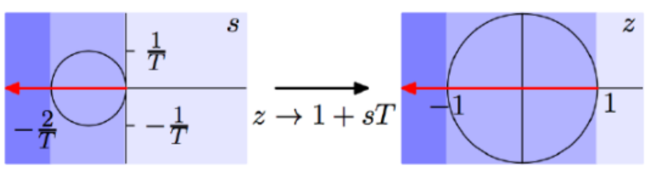
CT: $\dot{y}(t) = x(t)$, 

DT: $\frac{y[n+1]-y[n]}{T} = x[n]$, 

So the change: $A = \frac{1}{s} \to \frac{TR}{1-R} = \frac{T}{z-1}$

Or $s \to \frac{z-1}{T}$, $z = 1 + sT = 1 - \frac{T}{\tau}$

Forward Euler: Mapping CT Poles to DT Poles



CT pole must be inside the circle radius $\frac{1}{T}$ at $s = -\frac{1}{T}$. $-\frac{2}{T} < -\frac{1}{\tau} < 0 \to \frac{T}{\tau} < 2$

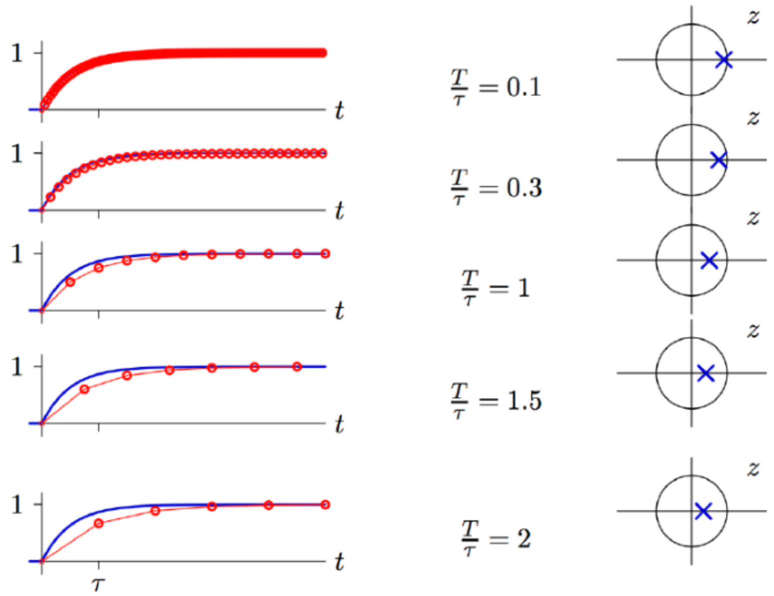
Backward Euler Approximation

$\dot{y}_c(nT) = \frac{1}{T}(y_d[n] - y_d[n-1])$

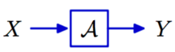
In the example, $\frac{\tau}{T}(y_d[n] - y_d[n-1]) = x_d[n] - y_d[n]$ or $(1 + \frac{\tau}{T})y_d[n] - y_d[n-1] = \frac{T}{\tau}x_d[n]$

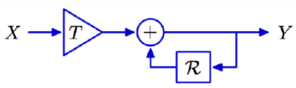
After z transform, $(1 + \frac{\tau}{T})Y_d(z) - z^{-1}Y_d(z) = \frac{T}{\tau}X_d(z)$, thus $H(z) = \frac{\frac{T}{\tau}z}{(1+\frac{T}{\tau})z-1}$

So the pole is $z = \frac{1}{1+\frac{T}{\tau}}$



Dependence of DT pole on Stepsize

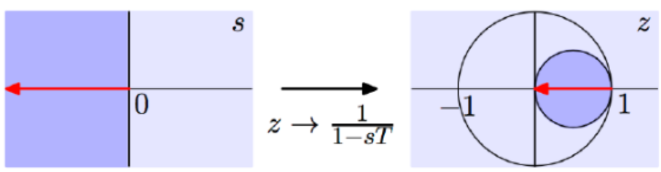
CT: $\dot{y}(t) = x(t)$, 

DT: $\frac{y[n]-y[n-1]}{T} = x[n]$, 

Since $A = \frac{1}{s} \rightarrow \frac{T}{1-\mathcal{R}} = \frac{T}{1-\frac{1}{z}}$

Thus $z \rightarrow \frac{1}{1-sT} = \frac{1}{1+\frac{T}{\tau}}$

Backward Euler: Mapping CT Poles to DT Poles



If CT is stable, then DT is stable.

Trapezoidal Rule

$$\frac{1}{T}(y[n]-y[n-1])=\frac{1}{2}(x[n]+x[n-1])$$

After z transform, $H(z)=\frac{T}{2}(\frac{1+z^{-1}}{1-z^{-1}})=\frac{T}{2}(\frac{z+1}{z-1})$

$A=\frac{1}{s}\rightarrow\frac{T}{2}(\frac{z+1}{z-1})$

$$z\rightarrow\frac{1+\frac{sT}{2}}{1-\frac{sT}{2}}$$

s	$\rightarrow$	$z=\frac{1+\frac{sT}{2}}{1-\frac{sT}{2}}$
0		1
$-\frac{1}{T}$		$\frac{1}{3}$
$-\frac{2}{T}$		0
$-\infty$		-1
$j\omega$		$\frac{2+j\omega T}{2-j\omega T}$