VE216 Lecture 18

> DT Fourier Representations

FFT

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{bmatrix} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^0 & W_8^2 & W_8^4 & W_8^6 \\ W_8^0 & W_8^3 & W_8^6 & W_8^1 & W_8^4 & W_8^7 & W_8^2 & W_8^5 \\ W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 \\ W_8^0 & W_8^5 & W_8^2 & W_8^7 & W_8^4 & W_8^1 & W_8^6 & W_8^3 \\ W_8^0 & W_8^6 & W_8^4 & W_8^2 & W_8^0 & W_8^6 & W_8^4 & W_8^2 \\ W_8^0 & W_8^7 & W_8^6 & W_8^5 & W_8^4 & W_8^3 & W_8^2 & W_8^1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}$$

where $W_N = e^{-j\frac{2\pi}{N}}$

This is actually 64 multiplications.

So we can generate the FFT calculation method.

FFT - Even numbered and Odd numbered Part

Even numbered	Odd Numbered
$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^0 & W_4^2 \\ W_4^0 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \\ x[4] \\ x[6] \end{bmatrix}$	$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^0 & W_4^2 \\ W_4^0 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \\ x[5] \\ x[7] \end{bmatrix}$

Total 32 multiplications.

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^0 & W_4^2 \\ W_4^0 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \\ x[4] \\ x[6] \end{bmatrix} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 \\ W_8^0 & W_8^4 & W_8^0 & W_8^4 \\ W_8^0 & W_8^6 & W_8^4 & W_8^2 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \\ x[4] \\ x[6] \end{bmatrix}$$

$$\boxed{\begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 \\ W_8^0 & W_8^4 & W_8^0 & W_8^4 \\ W_8^0 & W_8^6 & W_8^4 & W_8^2 \\ W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 \\ W_8^0 & W_8^4 & W_8^0 & W_8^4 \\ W_8^0 & W_8^6 & W_8^4 & W_8^2 \end{bmatrix} \begin{bmatrix} x[0] \\ x[2] \\ x[4] \\ x[6] \end{bmatrix}$$

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^0 & W_4^2 \\ W_4^0 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \\ x[5] \\ x[7] \end{bmatrix} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 \\ W_8^0 & W_8^4 & W_8^0 & W_8^4 \\ W_8^0 & W_8^6 & W_8^4 & W_8^2 \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \\ x[5] \\ x[7] \end{bmatrix}$$

$$\boxed{\begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{bmatrix} = \begin{bmatrix} W_8^0\, b_0 \\ W_8^1\, b_1 \\ W_8^2\, b_2 \\ W_8^3\, b_3 \\ W_8^4\, b_0 \\ W_8^5\, b_1 \\ W_8^6\, b_2 \\ W_8^7\, b_3 \end{bmatrix}} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^1 & W_8^3 & W_8^5 & W_8^7 \\ W_8^2 & W_8^6 & W_8^2 & W_8^6 \\ W_8^3 & W_8^1 & W_8^7 & W_8^5 \\ W_8^4 & W_8^4 & W_8^4 & W_8^4 \\ W_8^5 & W_8^7 & W_8^1 & W_8^3 \\ W_8^6 & W_8^2 & W_8^6 & W_8^2 \\ W_8^7 & W_8^5 & W_8^3 & W_8^1 \end{bmatrix} \begin{bmatrix} x[1] \\ x[3] \\ x[5] \\ x[7] \end{bmatrix}$$

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{bmatrix} = \begin{bmatrix} d_0 + e_0 \\ d_1 + e_1 \\ d_2 + e_2 \\ d_3 + e_3 \\ d_4 + e_4 \\ d_5 + e_5 \\ d_6 + e_6 \\ d_7 + e_7 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} W_8^0\, b_0 \\ W_8^1\, b_1 \\ W_8^2\, b_2 \\ W_8^3\, b_3 \\ W_8^4\, b_0 \\ W_8^5\, b_1 \\ W_8^6\, b_2 \\ W_8^7\, b_3 \end{bmatrix}$$

Fourier Transform: Aperiodic Signals

$x[n] \to$ aperiodic DT signal.

Then we have the periodic extension: $x_N[n] = \sum_{k=-\infty}^{\infty} x[n+kN]$

So $x[n] = \lim_{N\to\infty} x_N[n]$

- Analysis Equation: $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} = H(z)_{z=e^{j\Omega}}$ with $H(z) = \sum x[n]z^{-n}$
- Synthesis Equation: $x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\Omega})e^{j\Omega n}\,d\Omega$

Property	$x[n]$	$X(z)$	$X(e^{j\Omega})$
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(s) + bX_2(s)$	$aX_1(e^{j\Omega}) + bX_2(e^{j\Omega})$
Time shift	$x[n-n_0]$	$z^{-n_0}X(z)$	$e^{-j\Omega n_0}X(e^{j\Omega})$
Multiply by n	$nx[n]$	$-z\frac{d}{dz}X(z)$	$j\frac{d}{d\Omega}X(e^{j\Omega})$
Convolution	$(x_1 * x_2)[n]$	$X_1(z) \times X_2(z)$	$X_1(e^{j\Omega}) \times X_2(e^{j\Omega})$