Homework 4

VE216 - Introduction to Signal and Systems, Qiao Heng, Spring 2021

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HW Notes:

- Problems where the number of points are followed by an exclamation are basic skill problems and will be graded without partial credit.
- Box your final answer. You will be graded on both the final answer and the steps leading to it. Correct intermediate steps will help earn partial credit. For full credit, ~~cross out~~ any incorrect intermediate step.
- If you need to make any additional assumptions, state them clearly.
- Legible writing will help when it comes to partial credit.
- Simplify your result when possible.

Problems:

1. [20!] Use the table of FT pairs and the table of properties to find the FT of each of the following signals (DO NOT USE INTEGRATION):

(a) [5!] $x(t) = rect\left(\frac{t-1}{2}\right)$

(b) [5!] $x(t) = e^{-3t} rect\left(\frac{t-1}{2}\right)$

(c) [5!] $x(t) = t rect\left(\frac{t-1}{2}\right)$

(d) [5!] $x(t) = \cos(2\pi t) rect\left(\frac{t-1}{2}\right)$

Answer:

(a) $f(t) = rect\left(\frac{t}{T}\right)$ so $F(\omega) = Tsinc\left(T\frac{\omega}{2\pi}\right)$,

suppose $h(t) = rect\left(\frac{t}{2}\right)$ $\quad H(\omega) = 2sinc\left(\frac{\omega}{\pi}\right)$

$x(t) = h(t-1), X(\omega) = H(\omega)e^{-j\omega} = \boxed{2e^{-j\omega}sinc\left(\frac{\omega}{\pi}\right)}$

(b) $x(t) = e^{-3t}u(t) - e^{-3t}u(t-2)$

$X(\omega) = \boxed{\frac{1}{j\omega+3} + \frac{e^{-6-j\omega 2}}{j\omega+3}}$

(c) $h(t) = rect\left(\frac{t-1}{2}\right)$ $\quad H(\omega) = 2e^{-j\omega}4sinc\left(\frac{\omega}{\pi}\right)$

$x(t) = th(t) = j(-jt)h(t) = j\frac{d}{d\omega}$ $\quad F(\omega) = \boxed{\frac{(1+2j\omega)e^{-2j\omega}-1}{\omega^2}}$

(d) $X(\omega) = \frac{H(\omega-2\pi)+H(\omega+2\pi)}{2} = \boxed{e^{-j\omega}\sin\omega\frac{2\omega}{\omega^2-4\pi^2}}$

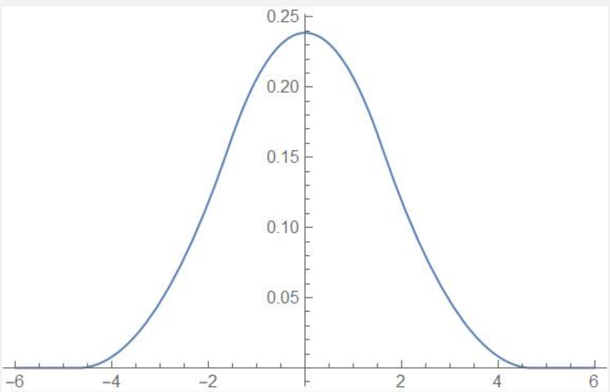
2. [10] Find a mathematical expression and sketch or plot the inverse FT of $F(\omega) = sinc^3(\omega/2)$. Hint: the inverse FT formula would probably be a hard way to do it.

Answer:

when $h(t) = \frac{1}{\pi}rect(\frac{t}{\pi})$, $H(\omega) = sinc(\frac{\omega}{2})$

So if $F(\omega) = H(\omega)^3$, then $f(t) = h(t) * h(t) * h(t) = \frac{1}{\pi^2}tri(\frac{t}{\pi}) * rect\frac{t}{\pi}$

$$\boxed{\text{f(t)}} = \begin{cases} 0 & t \leq -\frac{3}{2}\pi \\ \frac{1}{2\pi^3}t^2 + \frac{3}{2\pi^2}t + \frac{9}{8\pi} & -\frac{3}{2}\pi \leq t \leq -\frac{\pi}{2} \\ \frac{3}{4\pi} - \frac{t^2}{\pi^3} & -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \\ \frac{1}{2\pi^3}t^2 - \frac{3}{2\pi^2}t + \frac{9}{8\pi} & \frac{\pi}{2} \leq t \leq \frac{3\pi}{2} \\ 0 & t \geq \frac{3}{2}\pi \end{cases}$$



3. [10] Find the FT of $t^2 e^{-(t/4)^2}$. Hint: see table of FT pairs.

Answer:
$x(t) = -(-jt)^2 e^(- \frac{t}{4})^2$
so $X(\omega) = -\frac{d^2}{d\omega^2}(4\sqrt{\pi}e^{-4\omega^2}) = \boxed{32\sqrt{\pi}e^{-4\omega^2}(1-8\omega^2)}$

4. [10] Show that if $f(t)$ is real and odd, then $F(\omega)$ is purely imaginary and odd.

Answer:
f(t) is odd, so $f(t) + f(-t) = 0$, then $f(t) + f(-t) \xleftrightarrow{\mathcal{F}} F(\omega) + F(-\omega) = 0$, so $F(\omega)$ is odd
f(t) is real, so $f(t) - f*(t) = 0$, then $f(t) - f*(t) \xleftrightarrow{\mathcal{F}} F(\omega) - F*(\omega) = 0$, so $F(\omega)$ is real

5. [10] Consider a real signal $f(t)$ and let

$$f(t) \overset{\mathcal{F}}{\leftrightarrow} F(\omega), \quad F(\omega) = real\{F(\omega)\} + jimag\{F(\omega)\}$$

and

$$f(t) = f_e(t) + f_o(t)$$

where $f_e(t)$ and $f_o(t)$ are the even and odd component of $f(t)$ respectively. Show that

$$f_e(t) \overset{\mathcal{F}}{\leftrightarrow} real\{F(\omega)\} \quad f_o(t) \xleftrightarrow{\mathcal{F}} jimag\{F(\omega)\}$$

Answer:
$f(t) \xleftrightarrow{\mathcal{F}} F(\omega)$, so $f_e(t) + f_o(t) \xleftrightarrow{\mathcal{F}} F_e(\omega) + F_o(\omega)$
As in problem 4, $f_o(t)$ is odd and real, so $F_o(\omega)$ is odd and real
Also, f(t) is even, so $f(t) - f(-t) = 0$, then $f(t) - f(-t) \xleftrightarrow{\mathcal{F}} F(\omega) - F(-\omega) = 0$, so $F(\omega)$ is even
f(t) is real, so $f(t) - f*(t) = 0$, then $f(t) - f*(t) \xleftrightarrow{\mathcal{F}} F(\omega) - F*(\omega) = 0$, so $F(\omega)$ is real
So $f_e(t)$ is even and real, so $F_e(\omega)$ is even and real
So, $f_e(t) \overset{\mathcal{F}}{\leftrightarrow} real\{F(\omega)\} \quad f_o(t) \xleftrightarrow{\mathcal{F}} jimag\{F(\omega)\}$

6. [10!] Find the energy of the signal $x(t) = tsinc^2(t)$ by Fourier methods.

Answer:
$sinc^2(t) \xleftrightarrow{\mathcal{F}} tri(\frac{\omega}{2\pi})$

$$\begin{aligned} t\sin c^2(t) \xleftrightarrow{\mathcal{F}} & j\frac{d}{d\omega}tri\left(\frac{\omega}{2\pi}\right) \\ &= \frac{j}{2\pi}\frac{d}{d\omega}rect\left(\frac{\omega}{2\pi}\right) * rect\left(\frac{\omega}{2\pi}\right) \\ &= \frac{j}{2\pi}(\delta(\omega+\pi) - \delta(\omega-\pi)) * rect\left(\frac{\omega}{2\pi}\right) \\ &= \frac{j}{2\pi}\left(rect\left(\frac{\omega+\pi}{2\pi}\right) - rect\left(\frac{\omega-\pi}{2\pi}\right)\right) \\ |X(\omega)| &= \frac{1}{2\pi}rect\left(\frac{\omega}{4\pi}\right) \end{aligned}$$

$E = \frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega = \boxed{\frac{1}{2\pi^2}}$

7. [10] What percentage of the total energy in the energy signal $f(t) = e^{-t}u(t)$ is contained in the frequency band $-8\text{rad/s} \leq \omega \leq 8\text{rad/s}$

Answer:
$X(\omega) = \frac{1}{j\omega - 1}$, so $E_{part} = \frac{1}{2\pi}\int_{-8}^{8}|F(\omega)|^2 d\omega = \frac{arctan(8)}{\pi}$

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2 d\omega = \frac{1}{2}$$

$$P = \frac{E_{part}}{E} = \boxed{92\%}$$

8. [20] A LTI system has the following frequency response:

$$H(j\omega) = \frac{-\omega^2 + j\omega + 1}{(-\omega^2 + 6j\omega + 25)(j\omega + 2)}$$

(a) [10] Find the impulse response of the LTI system. Hint: first find the partial differential equation.
(b) [10] Find the differential equation corresponding to the LTI system. Hint: write $H(\omega) = Y(\omega)/X(\omega)$ and cross multiply.

Answer:
(a) $H(s) = \frac{s^2+s+1}{(s^2+6s+25)(s+2)}$
$\frac{b}{s+3-4j} + \frac{c}{s+3+4j} + \frac{a}{s+2} = \frac{s^2+s+1}{(s^2+6s+25)(s+2)} \Rightarrow a = \frac{3}{17} \quad b = \frac{7}{17} + \frac{71}{136}j \quad c = \frac{7}{17} - \frac{71}{136}j$
h(t)= $\boxed{\left(\frac{7}{17} + \frac{71}{136}j\right) e^{(4j-3)t}u(t) + \left(\frac{7}{17} - \frac{71}{136}j\right) e^{(-4j-3)t}u(t) + \frac{3}{17}e^{-2t}u(t)}$
(b)

$$\frac{Y(\omega)}{X(\omega)} = \frac{-\omega^2 + j\omega + 1}{(-\omega^2 + 6j\omega + 25)(j\omega + 2)}$$

$$\left(s^3 + 8s^2 + 37s + 50\right) Y(\omega) = \left(s^2 + s + 1\right) X(\omega)$$

$$\boxed{y'''(t) + 8y''(t) + 37y'(t) + 50y(t) = x''(t) + x'(t) + x(t)}$$