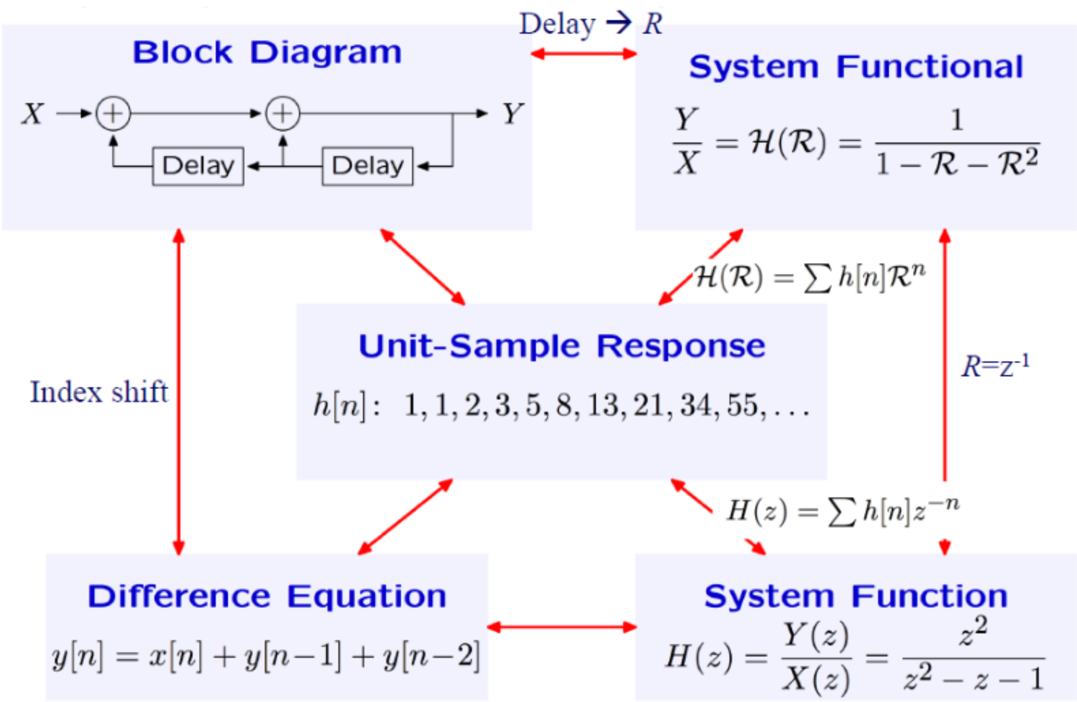
VE216 Lecture 5

> Z Transform

Concept Map



System Function in terms of Unit-sample Response

$\frac{Y}{X} = H(R) = h[0] + h[1]R + h[2]R^2 + \cdots = \sum_n h[n]R^n$

Z Transform

$H(z) = \sum h[n]z^{-n} \to X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$

Z Transform Pairs

example: $x[n] = (\frac{7}{8})^n u[n] \leftrightarrow X(z) = \frac{1}{1-\frac{7}{8}z^{-1}}$ with $\left|\frac{7}{8}z^{-1}\right| < 1$, i.e., $|z| > \frac{7}{8}$.

Region of Convergence (ROC)

For the example above, the `ROC` is $|z| > \frac{7}{8}$

Z Transform Properties

Linearity Property

Predicate

- $x_1[n] \leftrightarrow X_1(z)$ for z in ROC_1
- $x_2[n] \leftrightarrow X_2(z)$ for z in ROC_2

Conclusion

- $x_1[n] + x_2[n] \leftrightarrow X_1(z) + X_2(z)$ for z in $(ROC_1 \cap ROC_2)$

Delay Property

Predicate

$x[n] \leftrightarrow X(z)$ for z in ROC

Conclusion

$x[n-1] \leftrightarrow z^{-1}X(z)$ for z in ROC

Generality

- $\delta[n] \leftrightarrow 1$
- $\delta[n-1] \leftrightarrow z^{-1}$
- $X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$
  Let $y[n] = x[n-1]$, then $Y(z) = \sum_{-\infty}^{\infty} x[n-1]z^{-n}$
  So $z^{-1}X(z) = Y(z)$

Rational Polynomials

A system can be described in **linear difference equation** with **constant coefficients** can also be described by a **Z transform** that is a **ratio of polynomials in** z.

$\sum_{i=0}^{k} b_i y[n-i] = \sum_{i=0}^{k} a_i x[n-i]$

$\sum_{i=0}^{k} b_i z^{-i} Y(z) = \sum_{i=0}^{k} a_i z^{-i} X(z)$

$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{k} a_i z^{-i}}{\sum_{i=0}^{k} b_i z^{-i}}$

$(x[n-p] = X(z)z^{-p})$

Poles and Holes

$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{k} a_i z^{-i}}{\sum_{i=0}^{k} b_i z^{-i}} = \frac{\sum_{i=0}^{k} a_i z^{k-i}}{\sum_{i=0}^{k} b_i z^{k-i}} = \frac{\Pi_{i=0}^{k}(z-z_i)}{\Pi_{i=0}^{k}(z-p_i)}$

So the roots for numerator are **holes**.

The roots for denominator are **poles**.

Region of Convergence (ROC)

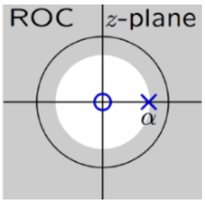
Regions of convergence for Z transform are delimited by circles in Z-plane.

Edges of circles are at the poles.

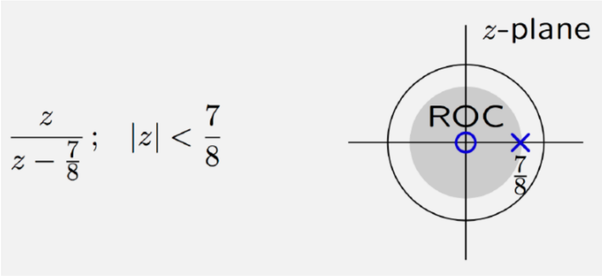
Example

$x[n] = \alpha^n u[n]$, so $X(z) = \sum_{k=0}^{\infty} \alpha^k z^{-k} = \frac{1}{1-\alpha z^{-1}}$.

We need $|\alpha z^{-1}| < 1$, so $|z| > |\alpha|$. So ROC is in outer region.



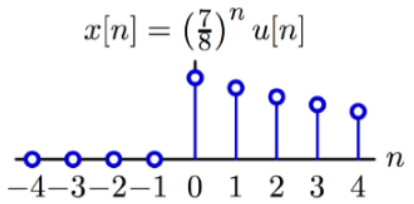
Question: DT signal with ROC in Inner Region?



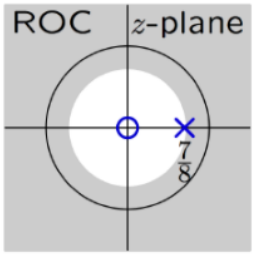
We get the same difference equation for the system: $y[n+1] - \frac{7}{8}y[n] = x[n+1]$

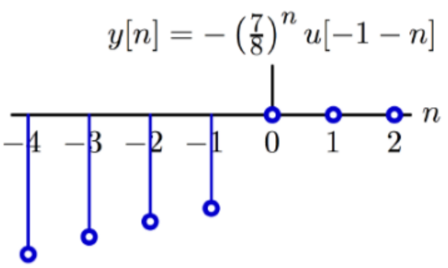
So corresponding to $y[n] = \alpha^n u[n]$ with $\alpha = \frac{7}{8}$ here and ROC outer region, the ROC inner region shows the **result** $y[n] = -\alpha^n u[-(n+1)]$.

ROC Comparison

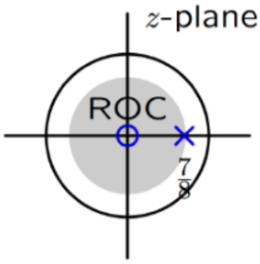


$$\frac{z}{z-\frac{7}{8}}$$





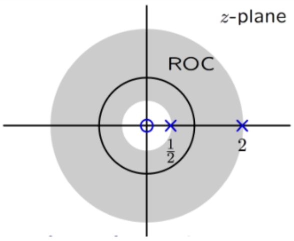
$$\frac{z}{z-\frac{7}{8}}$$



Example Exercise

$X(z) = \frac{-3z}{2z^2-5z+2}$ inverse transform, ROC has the unit circle on z-plane.

$X(z) = \frac{z}{z-\frac{1}{2}} - \frac{z}{z-2}$.



So $\frac{1}{2}$ pole is outer region and $\mathbf{2}$ pole is inner region.

So $x[n] = (\frac{1}{2})^n u[n] - (-2^n u[-n-1]) = (\frac{1}{2})^n u[n] + 2^n u[-n-1]$

Z Transform Properties

Property	$x[n]$	$X(z)$	ROC
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	$\supset (R_1 \cap R_2)$
Delay	$x[n-1]$	$z^{-1}X(z)$	R
Multiply by n	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R
Convolve in n	$\sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]$	$X_1(z)X_2(z)$	$\supset (R_1 \cap R_2)$

Example

$Y(z) = (\frac{z}{z-1})^2$ inverse transform with $|z| > 1$ (assume $x[n] = \delta[n]$)

$Y(z) = (\frac{z}{z-1})^2$, so $\frac{Y}{X} = (\frac{1}{1-R})^2 = (1 + R + R^2 + \cdots)^2 = \sum_{n=0}^{\infty}(n+1)R^n = (n+1)u[n]$

$h[n] = \frac{Y}{X} = (n+1)u[n] = y[n]$

Another approach

$$\frac{z}{z-1} \leftrightarrow u[n]$$

$$-z\frac{d}{dz}(\frac{z}{z-1}) = z(\frac{1}{z-1})^2 \leftrightarrow nu[n]$$

$$z \times (-z\frac{d}{dz}(\frac{z}{z-1})) = (\frac{z^2}{z-1})^2 \leftrightarrow (n+1)u[n]$$