# Ve 216: Introduction to Signals and Systems

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Based on Lecture Notes by Prof. Jeffrey A. Fessler

# Outline


1. 2. CT LTI Systems
   - Overview
   - Introduction
   - Techniques for the analysis of linear systems
   - Impulse response: mathematical and physical introduction
   - Impulse representation of CT signals (2.2.1)
   - Convolution for CT LTI systems (2.2.2)
   - Properties of convolution and LTI systems (2.3)
   - LTI system properties via impulse response (2.3.4-7)
   - Step response (2.3.8)
   - CT systems described by differential equation models (2.4)
   - Summary

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     - A-3 Invertibility of LTI systems (2.3.5)
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# Overview

- impulse response
- convolution (graphical, properties, LTI interconnection)
- differential equation (diffeq) systems (important class of LTI systems)
- Skip: 2.1, 2.4.2, 2.5.3 (the doublets part, but do read about unit ramp signal)

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# Linear time-invariant (LTI) systems

Our primary focus hereafter will be on CT **linear time-invariant (LTI)** systems. This chapter is about analyzing such systems.

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Why analysis? So far we only have input-output relationships. For a given system could compute $y(t)$ for a given $x(t)$, but it would be very difficult to design filters etc. by such trial-and-error.

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# Linear and time-invariant system

- Recall, for a **linear** system:

$$a_1 x_1(t) + a_2 x_2(t) \xrightarrow{\mathcal{T}} y(t) = a_1 y_1(t) + a_2 y_2(t)$$

  where $x_1(t) \xrightarrow{\mathcal{T}} y_1(t)$ and $x_2(t) \xrightarrow{\mathcal{T}} y_2(t)$.

- Recall, for a **time-invariant** system,

$$\text{if} \quad x(t) \xrightarrow{\mathcal{T}} y(t), \quad \text{then} \quad x(t-t_0) \xrightarrow{\mathcal{T}} y(t-t_0).$$

These two properties greatly simplify analysis of systems!

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# LTI overview (1)

Overview:

$$x(t) \to \boxed{\text{LTI with impulse response } h(t)} \to y(t) = x(t) * h(t),$$

where $\delta(t) \xrightarrow{\mathcal{T}} h(t)$.

$$\boxed{\text{Input-output relationship: } y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau.}$$

Remarkably, the input-output relationship for any LTI system is given by the above **convolution integral**.

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- An LTI system is completely described by its impulse response $h(t)$.
- We can determine the response $y(t)$ due to any input signal $x(t)$ by performing the convolution of the input signal with the impulse response.

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# Techniques for the analysis of linear systems

General strategy:

1. Decompose input signal $x(t)$ into a weighted sum of elementary functions $x_k(t)$, *i.e.* $x(t) = \sum_k c_k x_k(t)$
   Sometimes this decomposition is itself of interest in terms of studying the signal properties (*e.g.* Fourier analysis).
2. Determine response of system to each elementary function (this should be easy from input-output relationship):
   $$x_k(t) \xrightarrow{\mathcal{T}} y_k(t)$$
3. Apply superposition property:
   $$x(t) = \sum_k c_k x_k(t) \xrightarrow{\mathcal{T}} y(t) = \sum_k c_k y_k(t).$$

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# Elementary functions

Two particularly good choices for the elementary functions $x_k(t)$:

1. impulse functions $\delta(t-\tau)$
2. complex exponentials $e^{j\omega t}$ (later).

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- Time-invariance then further simplifies.

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# Impulse response

## Definition

**Impulse response** $h(t)$ of an LTI system is defined as the response of the system to an input signal that is unit impulse.

$$\delta(t) \xrightarrow{\mathcal{T}} h(t).$$

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# Impulse response: example

**Example**

Find the impulse response of the moving average system $y(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)\,d\tau$. (Verify that it is an LTI system.)

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# A toy car example (1)

## Example

A toy car on a carpet. The input signal $x(t)$ is the applied force, and the output signal $y(t)$ is the velocity of the car.

$$\underset{\text{force}}{x(t)} \rightarrow \boxed{\text{car system}} \rightarrow \underset{\text{velocity}}{y(t)}$$

- If at time $t = 0$ someone hits the (previously stationary) car with a hard stick, then the car begins moving with some velocity.
- But friction gradually decreases the velocity towards zero. So the impulse response is something like:

$$h(t) = ae^{-t/b}u(t), \quad \textbf{\textit{(Picture)}}.$$

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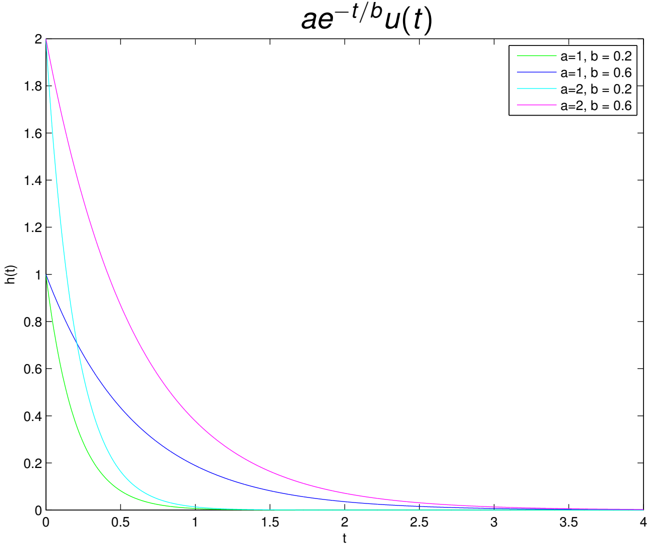

# A toy car example (3)

$$h(t) = ae^{-t/b}u(t), .$$

**Question**

1. *What is the constant a related to?*
2. *What is the constant b related to?*

# A toy car example (3)

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**Question**

1. *What is the constant a related to?* *a is the reciprocal of the mass of the car.*
2. *What is the constant b related to?* *Friction.*

# A toy car example (4)

This toy car system is time-invariant. If we hit the car with the stick at time $t = t_0$ instead, then the velocity would look like

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**Question**

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*What if we hit the car twice with the same force, i.e.* $x(t) = \delta(t) + \delta(t-3)$*?*

*The response will be* $y(t) = h(t) + h(t-3)$*.*
*This is the linearity and time-invariant properties in action.*

# A toy car example (5)

The key question that is answered in this chapter is the following.

- Suppose that, instead of hitting the car (like an impulse), we give it a gentle push for a couple of seconds. What does the output signal (velocity vs time) look like?
- Now we are considering $x(t) = \text{rect}\left(\frac{t-1}{2}\right)$ for example, and we want to find $y(t)$.

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We know the response of the system to any impulse. We want to find the response of the system to a rect signal (or a general input $x(t)$ for that matter). If only we could somehow express $x(t)$ as a "sum of delta functions", then we could use the LTI properties to say that the output is the "sum of the responses" due to all those delta functions.

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# Superposition of impulse function

**Question**

*How to represent $x(t)$ as a "superposition" of impulse functions?*

One way is to use the sifting property of the unit impulse function:

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau.}$$

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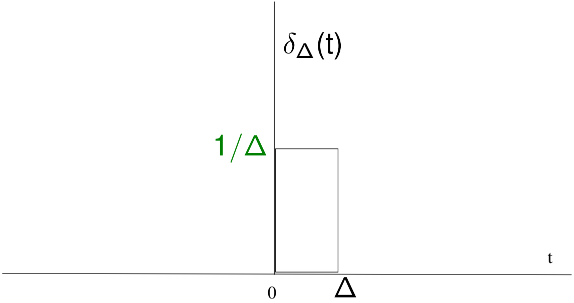
$$\boxed{\text{sifting property } \int_{-\infty}^{\infty} x(t)\delta(t-t_0)\,dt = x(t_0).}$$

# Superposition of impulse function

The following representation may be more intuitive. Practical impulse function, defined for any $\Delta > 0$:

$$\delta_\Delta(t) \triangleq \begin{cases} 1/\Delta, & 0 < t < \Delta \\ 0, & \text{otherwise.} \end{cases} = \frac{1}{\Delta}\,\text{rect}\left(\frac{t}{\Delta} - \frac{1}{2}\right)$$



# Impulse representation of CT signals (2)

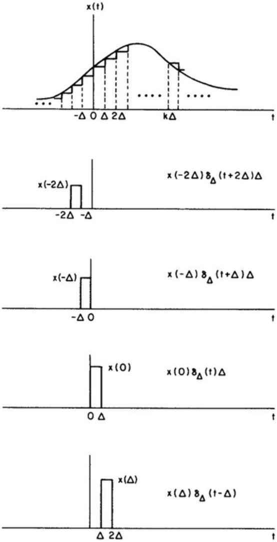
We can approximate any signal $x(t)$ by a stairstep signal formed from these pulse functions as follows

$$x(t) \approx \hat{x}(t) \triangleq \sum_{k=-\infty}^{\infty} x(k\Delta)\, \Delta\delta_{\Delta}(t - k\Delta).$$

Formally: $x(t) = \lim_{\Delta \to 0} \hat{x}(t)$.

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau.}$$

# Impulse representation of CT signals (3)




(MIT Lecture Notes 4-5)

# Outline

1. 2. CT LTI Systems
- Overview
- Introduction
- Techniques for the analysis of linear systems
- Impulse response: mathematical and physical introduction
- Impulse representation of CT signals (2.2.1)
- **Convolution for CT LTI systems (2.2.2)**
- Properties of convolution and LTI systems (2.3)
- LTI system properties via impulse response (2.3.4-7)
  - T-1 Causal LTI systems (2.3.6)
  - T-2 Memory (2.3.4)
  - A-2 Stability of LTI systems (2.3.7)
  - A-3 Invertibility of LTI systems (2.3.5)
- Step response (2.3.8)
- CT systems described by differential equation models (2.4)
  - Solution of linear constant-coefficient diffeqs (2.4.1)
- Summary

# Response of practical impulse function

## Question

*Let $h_\Delta(t)$ denote the response of the system to the practical impulse function $\delta_\Delta(t)$, i.e.*

$$\delta_\Delta(t) \xrightarrow{\mathcal{T}} h_\Delta(t).$$

- *What happens to $\delta_\Delta$ as $\Delta \to 0$?*
- *What happens to $h_\Delta$ as $\Delta \to 0$?*

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- *What happens to $\delta_\Delta$ as $\Delta \to 0$?* $\delta_\Delta(t) \to \delta(t)$.
- *What happens to $h_\Delta$ as $\Delta \to 0$?* $h(t) = \lim_{\Delta \to 0} h_\Delta(t)$.

# Convolution for CT LTI systems (1)

If the system $\mathcal{T}$ is LTI, then shifting the input signal causes a shifted output signal

$$\delta_\Delta(t - k\Delta) \xrightarrow{\mathcal{T}} h_\Delta(t - k\Delta).$$

superposition property of a LTI system

$$\hat{x}(t) \triangleq \sum_{k=-\infty}^{\infty} x(k\Delta)\,\Delta\delta_\Delta(t - k\Delta) \xrightarrow{\mathcal{T}} \hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta h_\Delta(t - k\Delta).$$

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# Convolution for CT LTI systems (2)

On the input side, when we take the limit

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

On the output side, when we take the limit get:

$$\begin{aligned} y(t) &= \lim_{\Delta\to 0} \hat{y}(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\Delta h_{\Delta}(t-k\Delta) \\ &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \end{aligned}$$

**convolution integral:** $x(t) \xrightarrow{\mathcal{T}} y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau.$

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- Every value of the input signal that enters the system causes the system to respond.
- For a linear system, the overall response is the superposition of the contributions due to each input signal value.

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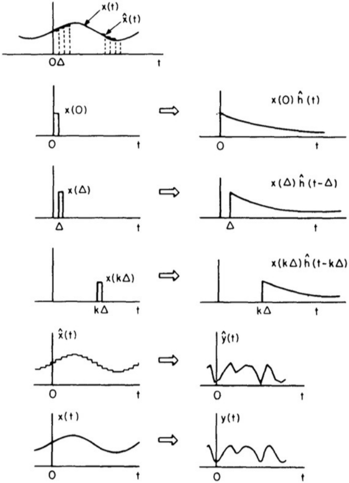


(MIT, Lecture 4.7)

# Convolution for CT LTI systems (5)

The response of an LTI system with impulse response $h(t)$ to an arbitrary input signal $x(t)$ is given by the convolution integral, which we also denote

$$\boxed{y(t) = x(t) * h(t) = (x * h)(t).}$$

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# Convolution for CT LTI systems (6)

Let us now check that the term "impulse response" is appropriately named.

## Example

Suppose the input signal is $\delta(t-5)$. Find the output signal in terms of the impulse response $h(t)$.

By the sifting property of $\delta(t)$.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} \delta(\tau-5)h(t-\tau)\,d\tau = h(t-5)$$

Thus, the convolution integral confirms that the response of the system to a unit impulse at time $t=5$ is the impulse response delayed to "start" at $t=5$.

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# Computing convolution (1)

**Skill: *convolving*.**

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau.}$$

Recipe:

1. Fold: fold $h(\tau)$ about $\tau = 0$ to get $h(-\tau)$
2. Shift: shift $h(-\tau)$ by $t$ to get $h(t-\tau)$
3. Multiply: $x(\tau)$ by $h(t-\tau)$ for every $\tau$
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Repeat for all possible $t$; generally breaks in to a few intervals.

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Mathematically, replace $t$ with $t-\tau$ to complete step 1 and 2.

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# Computing convolution (2)

**Question**

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*Can we fold and shift $x(t)$ instead of $h(t)$ ?*

*We can fold and shift either $h(t)$ or $x(t)$ (commutative property of convolution).*
*More often we use $h(t)$ since we will focus on causal systems, so folding and shifting $h(t)$ is particularly natural.*

# Toy can example (1)

**Example**

Continue the toy car problem, but simplify by letting $b = 1$ so that $h(t) = e^{-at}u(t)$.
Find the response of the input $x(t) = u(t)$ (a steady push beginning at time 0).

# Solution

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} u(\tau)e^{-a(t-\tau)}u(t-\tau)\, d\tau$$

$$= u(t)\int_{0}^{t} e^{-a(t-\tau)}\, d\tau \quad (\tau \geq 0, t-\tau \geq 0 \Longrightarrow 0 \leq \tau \leq t)$$

$$= e^{-at}\left(\frac{1}{a}e^{a\tau}\Big|_{0}^{t}\right)u(t)$$

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***(Picture)****(Example 2.6, p. 98) Video (MIT, Lecture 4, 32.33min)*

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# Toy car example (2)

**Example**

Now return to the problem determining response of the toy car to a short push: $x(t) = \text{rect}\left(\frac{t-1}{2}\right)$ and $h(t) = e^{-t}u(t)$.

# Solution (1)

## Solution

1. *Fold*: $h(-\tau) = e^{-(-\tau)}u(-\tau) = e^{\tau}u(-\tau)$
2. *Shift*: $h(t-\tau) = e^{-(t-\tau)}u(t-\tau)$
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*The product is nonzero only if $t \geq 0$, and even then the "shape" depends whether $t \in [0, 2]$ or $t > 2$.*
*Three cases to consider (we are making a braces function!).*

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# Solution (2)

- *For $t < 0$, the integral $y(t) = 0$.*
- *For $0 < t < 2$ the integral is (using $\nu = -(t-\tau)$):*

$$\int_0^t 1 e^{-(t-\tau)}\, d\tau = \int_{-t}^0 e^{\nu}\, d\nu = 1 - e^{-t}.$$

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*Video (MIT, Lecture 4, 35min)*

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# Toy car example (3)

## Example

A toy car on a carpet. The input signal $x(t)$ is the applied force, and the output signal $y(t)$ is the velocity of the car.

$$\underset{\text{force}}{x(t)} \rightarrow \boxed{\text{car system}} \rightarrow \underset{\text{velocity}}{y(t)}$$

- A gentle push for a couple of seconds. $x(t) = \text{rect}(\frac{t-1}{2})$.
- Impulse response $h(t) = e^{-t}u(t)$.
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Video (MIT, Lecture 4, 36.38min)

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  *The input signal (how hard we push and for how long).* ***forced response***
- *What affects the decay portion?*
  *The system properties (decay due to friction losses).* ***natural response***

# Outline


1. 2. CT LTI Systems
   - Overview
   - Introduction
   - Techniques for the analysis of linear systems
   - Impulse response: mathematical and physical introduction
   - Impulse representation of CT signals (2.2.1)
   - Convolution for CT LTI systems (2.2.2)
   - Properties of convolution and LTI systems (2.3)
   - LTI system properties via impulse response (2.3.4-7)
     - T-1 Causal LTI systems (2.3.6)
     - T-2 Memory (2.3.4)
     - A-2 Stability of LTI systems (2.3.7)
     - A-3 Invertibility of LTI systems (2.3.5)
   - Step response (2.3.8)
   - CT systems described by differential equation models (2.4)
     - Solution of linear constant-coefficient diffeqs (2.4.1)
   - Summary

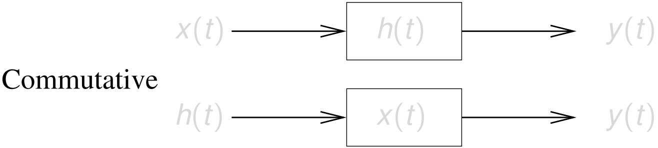
# Commutative property (2.3.1)

**Skill:** ***Use properties to simplify LTI systems.***

**Property**

***commutative property***

$$x(t) * h(t) = h(t) * x(t).$$



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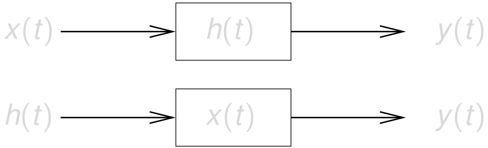

# Commutative property (2.3.1)

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## Property

***commutative property***

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$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Commutative

$$h(t) \longrightarrow \boxed{x(t)} \longrightarrow y(t)$$

# Commutative property: proof

**Question**

*Show the commutative property?*

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\begin{aligned}
x(t) * h(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \\
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&= h(t) * x(t)
\end{aligned}
$$

# Associative property (2.3.3)

**Property**

***Associative property***

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$$

This property holds in general for any number of systems connected in series. So the following notation is acceptable:

$$h(t) = h_1(t) * h_2(t) * \cdots * h_k(t).$$

In particular:

$$\begin{aligned}(x * h_1) * h_2 &= x * (h_1 * h_2)\\ &= x * (h_2 * h_1)\\ &= (x * h_2) * h_1\end{aligned}$$

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# Commutative and associative property

**The order of serial connection of LTI systems does not affect the overall impulse response.**

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Commutative

$$h(t) \longrightarrow \boxed{x(t)} \longrightarrow y(t)$$

$$x(t) \longrightarrow \boxed{h_1(t)} \longrightarrow \boxed{h_2(t)} \longrightarrow y(t)$$

Associative

$$x(t) \longrightarrow \boxed{h(t) = h_1(t) * h_2(t)} \longrightarrow y(t)$$

Commutative

$$x(t) \longrightarrow \boxed{h(t) = h_2(t) * h_1(t)} \longrightarrow y(t)$$

Associative

$$x(t) \longrightarrow \boxed{h_2(t)} \longrightarrow \boxed{h_1(t)} \longrightarrow y(t)$$

# Associative property: proof (1)

**Question**

*Show the Associative property.*

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$$

# Associative property: proof (1)

**Question**

*Show the Associative property.*

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$$

*let* $y_1(t) = [x(t) * h_1(t)] * h_2(t)$ *and* $y_2(t) = x(t) * [h_1(t) * h_2(t)]$.
*We must show* $y_1(t) = y_2(t)$.

# Associative property: proof (2)

$$y_1(t) = \int_{-\infty}^{\infty} [x * h_1](\tau)\, h_2(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} x(s) h_1(\tau - s)\, ds \right] h_2(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(s) \left[ \int_{-\infty}^{\infty} h_1(\tau - s) h_2(t-\tau)\, d\tau \right] ds$$

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$$= \int_{-\infty}^{\infty} x(s) [h_1 * h_2](t-s)\, ds$$

$$= (x * [h_1 * h_2])(t) = y_2(t)$$

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\begin{aligned}
y_1(t) &= \int_{-\infty}^{\infty} [x * h_1](\tau)\, h_2(t-\tau)\, d\tau \\
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&= \int_{-\infty}^{\infty} x(s)\, [h_1 * h_2](t-s)\, ds \\
&= (x * [h_1 * h_2])(t) = y_2(t)
\end{aligned}
$$

# Associative property: proof (2)

$$y_1(t) = \int_{-\infty}^{\infty} [x * h_1](\tau)\, h_2(t-\tau)\, d\tau$$

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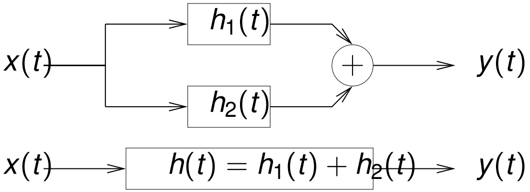
# Distributive property

## Property

***Distributive property***

$$x(t) * [h_1(t) + h_2(t)] = [x(t) * h_1(t)] + [x(t) * h_2(t)]$$

Distributive

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# Distributive property

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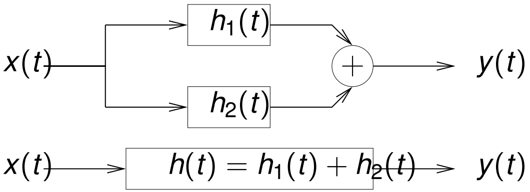


## Question

*Show the Distributive property.*

# Distributive property: proof

$$x(t) * [h_1(t) + h_2(t)] = \int_{-\infty}^{\infty} x(t-\tau)[h_1(\tau) + h_2(\tau)]\, d\tau$$

$$= \int_{-\infty}^{\infty} x(t-\tau)h_1(\tau)\, d\tau + \int_{-\infty}^{\infty} x(t-\tau)h_2(\tau)\, d\tau$$

$$= x(t) * h_1(t) + x(t) * h_2(t)$$

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# Example

## Example

Given $u(t) * h(t) = (1 - e^{-t})u(t)$, find an easier approach to previous problem of finding $y(t) = x(t) * h(t)$ where $x(t) = \text{rect}\left(\frac{t-1}{2}\right)$ and $h(t) = e^{-t}u(t)$.

1. $x(t) = \text{rect}\left(\frac{t-1}{2}\right) = u(t) - u(t-2)$
2. $u(t) - u(t-2) \xrightarrow{\mathcal{T}} z(t) - z(t-2)$ *where* $u(t) \xrightarrow{\mathcal{T}} z(t) = h(t) * u(t) = (1 - e^{-t})u(t)$
3. *Thus* $y(t) = z(t) - z(t-2) = (1 - e^{-t})u(t) - (1 - e^{-(t-2)})u(t-2)$.

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# Properties of convolution and impulse functions (1)

**Property**

$x(t) * \delta(t) = x(t)$

impulse representation of a CT signal

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

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# Properties of convolution and impulse functions (2)

**Property**

*Delay property:* $x(t) * \delta(t - t_0) = x(t - t_0)$

**Question**

*Prove the above property.*

# Properties of convolution and impulse functions (2)

**Property**

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# Solution

## Solution

*Proof.*

$$x(t) * \delta(t - t_0) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau - t_0)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\delta(t - t_0 - \tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\delta(t' - \tau)\, d\tau \quad (t' = t - t_0)$$

$$= x(t') = x(t - t_0)$$

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# Properties of convolution and impulse functions (3)

**Property**

$\delta(t-t_0) * \delta(t-t_1) = \delta(t-t_0-t_1)$

**Property**

*If* $y(t) = x(t) * h(t)$, *then* $x(t-t_0) * h(t-t_1) = y(t-t_0-t_1)$.
*(Due to time invariance of system.)*

**Question**

*Prove the above property.*

# Properties of convolution and impulse functions (3)

**Property**

$\delta(t - t_0) * \delta(t - t_1) = \delta(t - t_0 - t_1)$

**Property**

*If* $y(t) = x(t) * h(t)$, *then* $x(t - t_0) * h(t - t_1) = y(t - t_0 - t_1)$.
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# Properties of convolution and impulse functions (3)

**Property**

$\delta(t-t_0) * \delta(t-t_1) = \delta(t-t_0-t_1)$

**Property**

*If $y(t) = x(t) * h(t)$, then $x(t-t_0) * h(t-t_1) = y(t-t_0-t_1)$. (Due to time invariance of system.)*

**Question**

*Prove the above property.*

# Proof

$$x(t-t_0) * h(t-t_1) = \int_{-\infty}^{\infty} x(\tau - t_0) h(t - \tau - t_1)\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau') h(t - t_1 - (\tau' + t_0))\, d\tau' \quad (\tau' = \tau - t_0)$$

$$= \int_{-\infty}^{\infty} x(\tau') h(t' - \tau')\, d\tau' \quad (t' = t - t_0 - t_1)$$

$$= y(t') = y(t - t_0 - t_1)$$

# Proof

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# LTI system properties via impulse response

Since an LTI system is completely characterized by its impulse response, we should be able to express the remaining four properties in terms of $h(t)$.

- T-1 causality
- T-2 memory
- A-2 stability
- A-3 invertibility

# Outline

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# T-1 Causal LTI systems (1)

Recall a system is **causal** iff output $y(t)$ depends only on present and past values of input.

---

For an LTI system with impulse response $h(t)$:

$$
\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau \\
&= \int_{0}^{\infty} h(\tau)x(t-\tau)\,d\tau + \int_{-\infty}^{0^-} h(\tau)x(t-\tau)\,d\tau.
\end{aligned}
$$

1. The first term depends on present and past input values $x(t), x(t-\tau)$ for $\tau \geq 0$.
2. The second term depends on future input values $x(t-\tau)$ for $\tau < 0$.

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1. The first term depends on present and past input values $x(t), x(t-\tau)$ for $\tau \geq 0$.
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# T-1 Causal LTI systems (1)

Recall a system is **causal** iff output $y(t)$ depends only on present and past values of input.

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# T-1 Causal LTI systems (2)

Thus the system is causal iff the impulse response terms corresponding to the second term are zero. These terms are $h(\tau)$ for $\tau < 0$.

**Definition**

An LTI system is **causal** iff its impulse response $h(t) = 0$ for all $t < 0$.

In the causal case the convolution integral simplifies slightly since we can drop the right term above:

$$y(t) = \int_0^{\infty} h(\tau) x(t-\tau)\, d\tau = \int_{-\infty}^{t} x(\tau') h(t-\tau')\, d\tau' \text{ (using } \tau' = t - \tau\text{)}.$$

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**Example**

Is the LTI system with $h(t) = u(t - t_0 - 5)$ causal?

# T-1 Causal LTI systems (3)

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Is the LTI system with $h(t) = u(t - t_0 - 5)$ causal?

**Solution**

*It is causal only if* $h(t) = 0$ *for all* $t < 0$. $\Rightarrow t_0 + 5 \geq 0$.

# T-1 Causal LTI systems (4)

**Definition**

A **causal signal** is a signal $x(t)$ which is zero for all $t < 0$.

If the input to a causal LTI system is a causal signal, then the output is simply

$$y(t) = \begin{cases} 0, & t < 0 \\ \int_0^t h(\tau)x(t-\tau)\,d\tau = \int_0^t x(\tau)h(t-\tau)\,d\tau, & t \geq 0. \end{cases}$$

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MATLAB's `conv` function computes a discrete-time version of the above integral, for finite-duration $x(t)$ and $h(t)$.

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Recall a system is **static** or **memoryless** if the output $y(t)$ depends only on the current input $x(t)$, not on previous or future values of the input signal.

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**Definition**

An LTI system is **memoryless** iff its impulse response is $h(t) = a\delta(t)$. Otherwise the system is **dynamic** (has memory).

In this case the response is
$y(t) = x(t) * h(t) = x(t) * a\delta(t) = ax(t)$.

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There are two classes of dynamic systems.

**Definition**

A **finite impulse response** or **FIR** system has $h(t)$ that is nonzero only over some finite interval $t_1 < t < t_2$.

**Example**

$h(t) = \mathrm{rect}(t)$

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# A-2 Stability of LTI systems (1)

Recall $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau$. Suppose $x(t)$ is a bounded input signal, *i.e.* $|x(t)| \leq M_x < \infty\ \forall t$. Under what conditions on $h(t)$ is the response $y(t)$ a bounded signal?

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\begin{aligned}
|y(t)| &= \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\, d\tau \right| \\
&\leq \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)|\, d\tau \quad \text{(triangle inequality)} \\
&= \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)|\, d\tau \leq M_x \int_{-\infty}^{\infty} |h(\tau)|\, d\tau.
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An LTI system is **BIBO stable** iff its impulse response is absolutely integrable, *i.e.* $\int_{-\infty}^{\infty} |h(t)| \, dt < \infty$.

**Example**

Integrator: $y(t) = \int_{-\infty}^{t} x(\tau) \, d\tau$.
What is impulse response? Is it stable?

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Integrator: $y(t) = \int_{-\infty}^{t} x(\tau) \, d\tau$.
What is impulse response? Is it stable?

# Solution (1)

## Solution

*Approach 1:*

*let $x(t) = \delta(t)$, then perform the running integral, i.e.,*

$$h(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$$

# Solution (2)

## Solution

*Approach 2:*

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau = \int_{-\infty}^{\infty} u(t-\tau) x(\tau)\, d\tau.$$

*Since this has the exact form of a convolution integral with $u(t-\tau)$ in the place where $h(t-\tau)$ goes, we can conclude that $h(t-\tau) = u(t-\tau)$, or equivalently simply that $h(t) = u(t)$.*

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# Solution (3)

**Solution**

$\int_{-\infty}^{\infty} |u(t)|\, dt = \int_{0}^{\infty} 1\, dt = \infty$, so *unstable*.

# A-2 Stability of LTI systems (3)

**Example**

Is the moving average $h(t) = \frac{1}{T}\,\mathrm{rect}\left(t/T - \frac{1}{2}\right)$ stable?

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Is the moving average $h(t) = \frac{1}{T}\,\mathrm{rect}\left(t/T - \frac{1}{2}\right)$ stable?

$\int_{-\infty}^{\infty} |h(t)|\, dt = \frac{1}{T}\int_0^T 1\, dt = 1 < \infty$. *Thus stable.*

# A-2 Stability of LTI systems (4)

**Example**

Show the stability of the decaying sinusoid system with impulse response of $h(t) = e^{-t}(\cos t)u(t)$.

# Solution

*We can avoid a messy integral by thinking:*

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\, dt &= \int_{-\infty}^{\infty} |e^{-t}(\cos t)u(t)|\, dt \\
&= \int_{0}^{\infty} e^{-t} |\cos t|\, dt \\
&\leq \int_{0}^{\infty} e^{-t} 1\, dt \\
&= 1.
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# Outline


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# A-3 Invertibility of LTI systems (1)

Recall if a system $\mathcal{T}$ is **invertible**, then there exists a system $\mathcal{T}^{-1}$ such that

$$x(t) \to \boxed{\mathcal{T}} \to y(t) \to \boxed{\mathcal{T}^{-1}} \to z(t) = x(t).$$

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Fact: if a system is LTI, then if it is also invertible then the inverse system is also LTI (Problem 2.50, textbook). So

$$x(t) \to \boxed{\text{LTI } h(t)} \to y(t) \to \boxed{\text{LTI } h_i(t)} \to z(t) = x(t),$$

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- The cascade of two LTI systems is also LTI, so can be characterized by its impulse response.
- If the input is $x(t) = \delta(t)$, then the output is

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**Definition**

An LTI system is **invertible** iff there exist an inverse system whose impulse response $h_i(t)$ satisfies the following relationship with $h(t)$

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**Example**

Consider the delay system with $h(t) = 2\delta(t-5)$. Is it invertible? If yes, find the impulse response of its inverse system.

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## Solution

*It is invertible. The inverse system has impulse response* $h_i(t) = \frac{1}{2}\delta(t+5)$.

*Later, using Fourier transforms, we will see how to find* $h_i(t)$ *more generally, and how to determine when a given* $h(t)$ *does and does not correspond to an invertible system.*

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# A-3 Invertibility of LTI systems (3)

**Example**

The impulse response of an invertible LTI system is $h(t) = \delta(t) - e^{-t}u(t)$. Show a system with the impulse response of $h_i(t) = \delta(t) + u(t)$ is its inverse system.

# Solution (1)

*It suffices to show that* $h(t) * h_i(t) = \delta(t)$.

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\begin{aligned}
& h(t) * h_i(t) \\
= \;& [\delta(t) - e^{-t}u(t)] * [\delta(t) + u(t)] \quad (\textit{distributive property}) \\
= \;& \delta(t) * \delta(t) + \delta(t) * u(t) - (e^{-t}u(t) * \delta(t)) - (e^{-t}u(t) * u(t)) \\
= \;& \delta(t) + u(t) - e^{-t}u(t) - \int_{-\infty}^{\infty} e^{-(t-\tau)}u(t-\tau)u(\tau)\, d\tau \\
= \;& \delta(t) + \left(1 - e^{-t}\right) u(t) - \left(\int_0^t 1 e^{-(t-\tau)}\, d\tau\right) u(t) \\
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# Outline

1 2. CT LTI Systems
- Overview
- Introduction
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- Impulse response: mathematical and physical introduction
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# Impulse response of an LTI system

**Question**

*How can we find the impulse response $h(t)$ of an LTI system in practice?*

1. One way is to put a "practical impulse" through the system and observe output. This approach can be difficult since a large input spike could drive some systems into nonlinear regime.
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# Step response and impulse response

**Question**

*Find the* ***step response****, and then differentiate to find the* ***impulse response*** $h(t)$*. Why does this work?*

$$u(t) \to \boxed{d/dt} \to \delta(t) \to \boxed{\text{LTI}} \to h(t)$$

by associative/commutative property:

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# Step response

**Definition**

The **(unit) step response** of an LTI system is the running integral of its impulse response, or

$$s(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)\,d\tau = \int_{-\infty}^{t} h(\tau)\,d\tau$$

$\delta(t) \xrightarrow{\mathcal{T}} h(t)$ called **impulse response**

$u(t) \xrightarrow{\mathcal{T}} s(t)$ called **step response**

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## Definition

The **(unit) step response** of an LTI system is the running integral of its impulse response, or

$$s(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)\,d\tau = \int_{-\infty}^{t} h(\tau)\,d\tau$$

$\delta(t) \xrightarrow{\mathcal{T}} h(t)$ called **impulse response**

$u(t) \xrightarrow{\mathcal{T}} s(t)$ called **step response**

# Step response

If the system is causal, then $h(t) = 0$ for $t < 0$ so

$$\boxed{s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau = \left[\int_{0}^{t} h(\tau)\, d\tau\right] u(t)}$$

Relations:

$$\begin{aligned} s(t) &= \left[\int_{0}^{t} h(\tau)\, d\tau\right] u(t) \\ h(t) &= \frac{\mathrm{d}}{\mathrm{d}t} s(t) \end{aligned}$$

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# Example

**Example**

Earlier for car problem we showed that step response was $s(t) = (1 - e^{-t})u(t)$. Find the impulse response.

# Solution

## Solution

*The impulse response is*

$$
\begin{aligned}
h(t) &= \frac{\mathrm{d}}{\mathrm{d}t}s(t) = \frac{\mathrm{d}}{\mathrm{d}t}[(1-e^{-t})u(t)] \\
&= [\frac{\mathrm{d}}{\mathrm{d}t}(1-e^{-t})]u(t) + (1-e^{-t})[\frac{\mathrm{d}}{\mathrm{d}t}u(t)] \\
&= e^{-t}u(t) + (1-e^{-t})\delta(t) \\
&= e^{-t}u(t) + (1-e^{-0})\delta(t) \quad \textit{sampling property of } \delta(t) \\
&= e^{-t}u(t),
\end{aligned}
$$

*which indeed is the impulse response we had started with originally in that example.*

# Outline


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## CT systems described by differential equation models (1)

- The convolution integral $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$ looks fine on paper, but what about hardware (*e.g.* circuit) implementation?
- We might use mathematical analysis to design the "perfect" impulse response $h(t)$, but if there is no physical system that has that impulse response, then our design is of limited use.

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Let us focus on RLC circuits for the moment, and see what types of input-output relationships (and hence what types of impulse response functions) can be realized.

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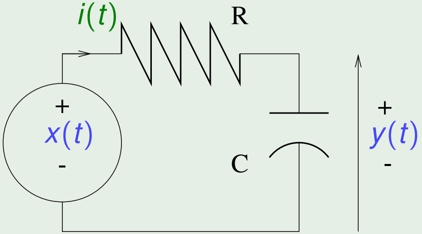
# CT systems described by differential equation models (2)

## Example

The input signal $x(t)$ is the voltage source, and the output signal $y(t)$ is the voltage across the capacitor.




The current is $i(t) = (x(t) - y(t))/R$, and the current is also $C$ times the derivative of the voltage across the capacitor: $i(t) = C\frac{\mathrm{d}}{\mathrm{d}t}y(t)$. Thus

$$\boxed{\frac{x(t) - y(t)}{R} = C\frac{\mathrm{d}}{\mathrm{d}t}y(t) \quad \text{or} \quad y(t) + RC\frac{\mathrm{d}}{\mathrm{d}t}y(t) = x(t)}$$

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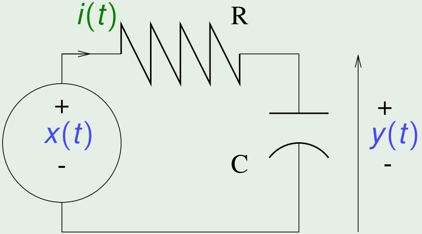


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# CT systems described by differential equation models (3)

Had we used a larger number of circuit elements, we would have found a differential equation (diffeq) with more terms.

**Definition**

The input-output relationship for an RLC circuit (with a finite number of components) is a diffeq of the following form, called a **linear constant-coefficient differential equation**:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

Fortunately this class of systems is sufficiently large to be interesting and useful!

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$$\boxed{\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t).}$$

- This is an implicit input-output relationship.
- We assume hereafter that that $a_k$'s and $b_k$'s are real.
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**Example**

What is order of above RC circuit?

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**Example**

What is order of above RC circuit?

*It is called first order, since* $N = 1$.

# CT systems described by differential equation models (4)

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In general there is not a simple time-domain method for finding the impulse response of such systems. We will do it systematically later using frequency response and the partial fraction expansion (PFE) method.

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# Solution of linear constant-coefficient diffeqs (1)

The general solution to a diffeq can be decomposed into the sum of two parts:

$$y(t) = y_h(t) + y_p(t).$$

- $y_h(t)$ is called the **homogeneous solution** or **zero-input solution** in the diffeq world, or the **natural response** in the systems world.
- The general form of $y_h(t)$ only depends on the system itself, although the coefficients in the form depend on the input signal.

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# Method of undetermined coefficients (1)

The **natural response** is found by solving the zero-input equation:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = 0. \tag{1}$$

e,;

It was discovered long ago that the solutions to the above equation have components of the form

$$y(t) = Ce^{st}$$

for values $C \neq 0$ and $s$ to be determined. Plugging into (1) yields

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Clearly this is only zero for all $t$ if $s$ satisfies

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*i.e.* if $s$ is a root of the above **characteristic equation** or **characteristic polynomial**.

The characteristic equation has $N$ roots, let us label them $s_1, \ldots, s_N$. (They may be distinct or not, real or complex.)

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- A solution associated with one root

$$y_{hl}(t) = C_l e^{s_l t}$$

satisfies the zero-input equation (1).

- Because the differential equation is linear, the sum of these solutions is also a soultion.
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For the general case of an $r$-th order root $s_i$ in the characteristic equation, the (almost) general natural response is

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**Question**

*How to determine the coefficients $C_1, C_2, \ldots, C_N$?*

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**si这个根有r个**

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*Need N auxiliary conditions. e.g.,*

$$y(t), \frac{\mathrm{d}}{\mathrm{d}t}y(t), \ldots, \frac{d^{N-1}}{dt^{N-1}}y(t) \quad at \quad t = t_0$$

# Method of undetermined coefficients (5)

- Different choices for these auxiliary conditions result in different input-output relationships. (Textbook, Problem 2.34)
- Depending on how the auxiliary information is stated or whether the auxiliary information is available, the system may or may not correspond to a linear system, may or may not correspond to a linear time-invariant (LTI) system, may or may not correspond to a causal and LTI system.
- We focus on differential equations used to describe systems that are LTI and causal.

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- Recall zero input signal implies zero output signal.

$$\boxed{\begin{gathered}\text{linear system} \iff \text{auxiliary conditions} = 0 \\ y(t_0) = \frac{\mathrm{d}}{\mathrm{d}t}y(t_0) = \ldots = \frac{d^{N-1}}{dt^{N-1}}y(t_0) = 0\end{gathered}}$$

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# Method of undetermined coefficients (6)

**Definition**

**Initial rest** states that if $x(t) = 0$ for $t \leq t_0$, then $y(t) = 0$ for $t \leq t_0$.

In words the output must be zero up until the time when the input becomes nonzero.

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# Initial rest (1)

- It is relatively straightforward to see

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intuition as to why initial rest $\Longrightarrow$ time-invariant system.

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If we perform identical experiments on two successive days, where the circuit starts from initial rest at noon on each day, then we would expect to see identical responses, *i.e.*, responses that are simply time-shifted by one day with respect to each other.

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- A necessary/sufficient condition: the right-hand side of the differential equation contains no impulses or derivatives of impulses.
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$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

# The method of undetermined coefficients

- The **method of undetermined coefficients** assumes that the **forced response** or **particular solution** is the sum of functions of the mathematical form of the excitation $x(t)$ and all (distinct, nonzero) derivatives of $x(t)$ that differ in form from $x(t)$ *i.e.*,

$$\boxed{y_p(t) = P_0 x(t) + P_1 \frac{\mathrm{d}}{\mathrm{d}t} x(t) + \cdots .}$$

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# Forced response examples

## Example

1. $$x(t) = 5e^{-7t}u(t) \Longrightarrow y_p(t) = P_0 e^{-7t}u(t)$$

2. $$x(t) = 170\cos(377t)\,u(t) \Longrightarrow$$
$$y_p(t) = [P_0\cos(377t) + P_1\sin(377t)]\,u(t)$$

3. $$x(t) = 10u(t) \Longrightarrow y_p(t) = P_0 u(t)$$

# Method of undetermined coefficients (7)

Combining the **natural response** and **forced response**, the overall solution has the form

$$\boxed{\begin{aligned} y(t) &= y_h(t) + y_p(t) \\ &= \sum_{l=1}^{N} C_l e^{s_l t} + P_0 x(t) + P_1 \frac{\mathrm{d}}{\mathrm{d}t} x(t) + \cdots \end{aligned}}$$

which we plug into the differential equation and then solve for the undetermined coefficients (the $C_l$'s and the $P_k$'s).

# Impulse response of LTI diffeq systems (1)

- In a diffeq class, the focus is on computing solutions to the diffeq for various input signals.
- In systems analysis, the diffeq indirectly describes the input-output relationship of an LTI system, and we are more interested in finding the impulse response $h(t)$ of that system (from which we can then look at stability and other system properties).

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# Impulse response of LTI diffeq systems (2)

- In principle we could find $h(t)$ by letting $x(t) = \delta(t)$ and solving the diffeq.
- Typically this time-domain approach requires more work and is less insightful than the frequency-domain / PFE approach described later.
- Nevertheless, for completeness we work one example here.
- In this example, we simplify things by first finding the step response, and then differentiating to find the impulse response, as in earlier discussion of step response.

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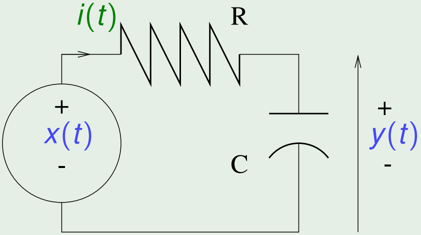
# Example (1)

## Example

Find step response and impulse response of the following RC circuit, which had diffeq

$$y(t) + \frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}y(t) = x(t)$$

for $\alpha = 1/RC$.



# Solution (1)

Find **natural response** $y_h(t) = Ce^{st}$.

$$y_h(t) + \frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}t}y_h(t) = 0$$

$$\Longrightarrow Ce^{st} + \frac{1}{\alpha}sCe^{st} = 0$$

$$\Longrightarrow s + \alpha = 0, \quad (C \neq 0, e^{st} \neq 0)$$

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# Solution (2)

We analyze a unit-step input signal

$$x(t) = u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & otherwise \end{cases}.$$

Since $x(t) = 0$ for $t \leq 0$, the condition of initial rest implies the auxiliary condition $y(t) = 0$ for $t \leq 0$.

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Since $x(t) = 1$, $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = 0$ for $t \geq 0$, the forced response is

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# Solution (3)

Plugging into the diffeq yields:

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\begin{aligned}
& y(t)+\frac{1}{\alpha} \frac{\mathrm{d}}{\mathrm{d} t} y(t)=x(t) \\
\Longrightarrow \quad & \left[C e^{-\alpha t}+P\right]+\frac{1}{\alpha}\left[-\alpha C e^{-\alpha t}\right]=1 \quad(\text{for} \quad t \geq 0) \\
\Longrightarrow \quad & P=1 .
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Furthermore, using the initial condition

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# Solution (4)

$$y_h(t) = -e^{-\alpha t} \text{ and } y_p(t) = 1 \text{ for } t \geq 0.$$

Note that

$$y_h(t) \to 0 \text{ as } t \to \infty, \quad y(t) \to y_p(t) \text{ as } t \to \infty,$$

These effects are called the **transient response** and the **steady-state response**

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Our final solution for the step response is

$$\boxed{y(t) = \begin{cases} [1 - e^{-\alpha t}], & t \geq 0 \\ 0, & t < 0 \end{cases} = [1 - e^{-\alpha t}]u(t).}$$

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- We have the impulse response, we can determine the response of the system to any other input signal simply by performing convolution, rather than re-solving the differential equation!
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# Summary of diffeqs

- Main point is hardware implementation with RLC circuits yields systems described by diffeqs. Thus this class of systems is extremely important.
- Analog filter design is partly about how to make efficient approximations to a desired impulse response! (*n* and *m* limited by cost/complexity)
- Just because the world is going digital does not obviate the need for analog. As we will see later (especially in 451), the first component of a DSP system is the sampler in an A/D converter, which requires a high-quality analog anti-alias filter!

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# Outline


1. 2. CT LTI Systems
   - Overview
   - Introduction
   - Techniques for the analysis of linear systems
   - Impulse response: mathematical and physical introduction
   - Impulse representation of CT signals (2.2.1)
   - Convolution for CT LTI systems (2.2.2)
   - Properties of convolution and LTI systems (2.3)
   - LTI system properties via impulse response (2.3.4-7)
     - T-1 Causal LTI systems (2.3.6)
     - T-2 Memory (2.3.4)
     - A-2 Stability of LTI systems (2.3.7)
     - A-3 Invertibility of LTI systems (2.3.5)
   - Step response (2.3.8)
   - CT systems described by differential equation models (2.4)
     - Solution of linear constant-coefficient diffeqs (2.4.1)
   - Summary

# Summary

- impulse response
- convolution integral for LTI systems
- graphical convolution
- properties of convolution
- convolution and LTI system interconnection
- impulse response vs step response
- LTI system properties characterized by $h(t)$ (causality, memory, stability, invertibility)
- diffeq systems
- solutions of diffeqs