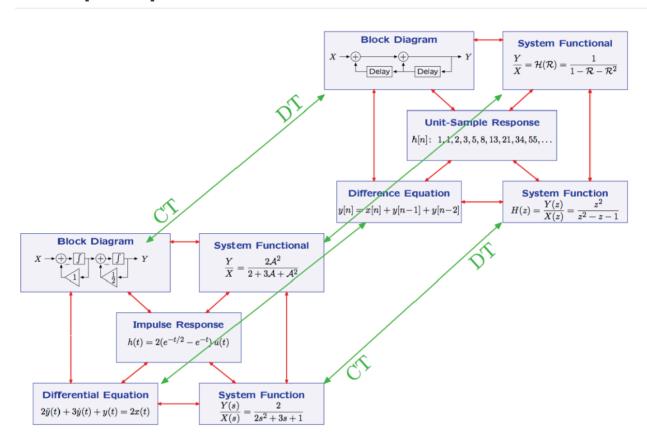
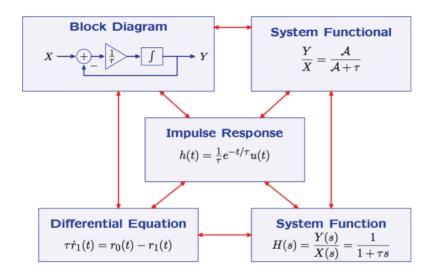
VE216 Lecture 7

Discrete Approximation of Continuous Time Systems

Concept Map



Discrete Approximation of CT Systems



The step response (given u(t) as the input) of the system:

$$egin{aligned} \delta(t)
ightarrow rac{A}{A+ au}
ightarrow h(t) &= rac{1}{ au} e^{-rac{t}{ au}} u(t) \ u(t)
ightarrow rac{A}{A+ au}
ightarrow s(t) = ? \ \delta(t)
ightarrow A
ightarrow u(t)
ightarrow rac{A}{A+ au}
ightarrow s(t) = ? \ \delta(t)
ightarrow rac{A}{A+ au}
ightarrow h(t)
ightarrow A
ightarrow s(t) = \int_{-\infty}^t h(t') dt' \ s(t) &= \int_{-\infty}^t rac{1}{ au} e^{-rac{t'}{ au}} u(t') dt' = \int_0^t rac{1}{ au} e^{-rac{t'}{ au}} dt' = (1-e^{-rac{t}{ au}}) u(t) \end{aligned}$$

Forward Euler Approximation

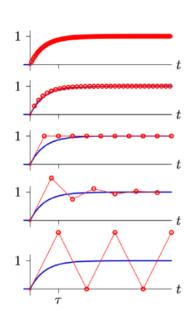
 $x_d[n] = x_c(nT)$ and $y_d[n] = y_c(nT)$

$$\dot{y_c}(nT) = rac{1}{T}(y_d[n+1] - y_d[n])$$

 ${m T}$ is a sampling interval.

Thus we obtain $rac{ au}{T}(y_d[n+1]-y_d[n])=x_d[n]-y_d[n]$, or $y_d[n+1]-(1-rac{T}{ au})y_d[n]=rac{T}{ au}x_d[n]$

The pole can be achieved by z transform: $zY_d(z)-(1-\frac{T}{\tau})Y_d(z)=\frac{T}{\tau}X_d(z)$, $z=1-\frac{T}{\tau}$.



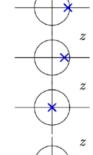
$$rac{T}{ au}=0.1$$

$$\frac{T}{\tau} = 0.3$$

$$\frac{T}{\tau} = 1$$

$$\frac{T}{\tau} = 1.5$$

$$\frac{T}{\tau} = 2$$



Dependence of DT pole on Stepsize

Forward Euler Method

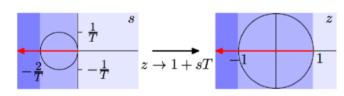
CT:
$$\dot{y}(t) = x(t)$$
, $X \longrightarrow A \longrightarrow Y$

DT:
$$\frac{y[n+1]-y[n]}{T} = x[n], X \longrightarrow T \longrightarrow \mathbb{R}$$

So the change: $A=rac{1}{s}
ightarrowrac{TR}{1-R}=rac{T}{z-1}$

Or
$$s
ightarrow rac{z-1}{T}$$
 , $z=1+sT=1-rac{T}{ au}$

Forward Euler: Mapping CT Poles to DT Poles



CT pole must be inside the circle radius $rac{1}{T}$ at $s=-rac{1}{T}.-rac{2}{T}<-rac{1}{ au}<0
ightarrowrac{T}{ au}<2$

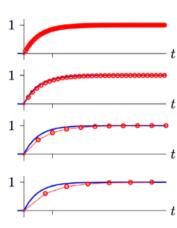
Backward Euler Approximation

$$\dot{y_c}(nT)=rac{1}{T}(y_d[n]-y_d[n-1])$$

In the example,
$$rac{ au}{T}(y_d[n]-y_d[n-1])=x_d[n]-y_d[n]$$
 or $(1+rac{ au}{T})y_d[n]-y_d[n-1]=rac{T}{ au}x_d[n]$

After
$$z$$
 transform, $(1+rac{ au}{T})Y_d(z)-z^{-1}Y_d(z)=rac{ au}{ au}X_d(z)$, thus $H(z)=rac{ frac{ au}{ au}z}{(1+rac{ au}{T})z-1}$

So the pole is $z=\frac{1}{1+\frac{T}{z}}$



$$\frac{T}{\tau} = 0.1$$



$$\frac{1}{\tau} = 0.3$$

$$\frac{T}{\tau} = 1$$

$$\frac{T}{\tau} = 1.5$$

$$rac{T}{ au}=2$$



Dependence of DT pole on Stepsize

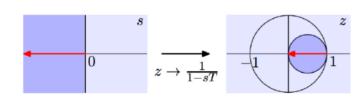
CT:
$$\dot{y}(t) = x(t)$$
, $X \longrightarrow A \longrightarrow Y$

DT:
$$\frac{y[n]-y[n-1]}{T}=x[n]$$
, X

Since
$$A=rac{1}{s}
ightarrowrac{T}{1-R}=rac{T}{1-rac{1}{z}}$$

Thus
$$z
ightarrow rac{1}{1-sT} = rac{1}{1+rac{T}{ au}}$$

Backward Euler: Mapping CT Poles to DT Poles



If CT is stable, then DT is stable.

Trapezoidal Rule

$$\frac{1}{T}(y[n]-y[n-1])=\frac{1}{2}(x[n]+x[n-1])$$

After
$$z$$
 transform, $H(z)=rac{T}{2}(rac{1+z^{-1}}{1-z^{-1}})=rac{T}{2}(rac{z+1}{z-1})$

$$A=rac{1}{s}
ightarrowrac{T}{2}(rac{z+1}{z-1})$$

$$z
ightarrowrac{1+rac{sT}{2}}{1-rac{sT}{2}}$$

$$s \qquad \rightarrow \qquad z = \frac{1 + \frac{sT}{2}}{1 - \frac{sT}{2}}$$

$$-\frac{1}{T}$$

 $j\omega$

$$\frac{1}{3}$$







