Ve 216: Introduction to Signals and Systems

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Based on Lecture Notes by Prof. Jeffrey A. Fessler

Outline

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Overview

- Primary points
  1. Convolution of discrete-time signals simply becomes multiplication of their z-transforms.
  2. Systematic method for finding the impulse response of LTI systems described by difference equations: partial fraction expansion.
  3. Characterize LTI discrete-time systems in the z-domain
- Secondary points
  1. Characterize discrete-time signals
  2. Characterize LTI discrete-time systems and their response to various input signals

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1. 10. The z-transform
   - The bilateral z-transform (10.1, 10.2)
   - Poles and zeros (10.4)
   - System function and block diagram representations
   - Inversion of the z-transform

The bilateral z-transform

Definition

The **direct z-transform** or **two-sided z-transform** or **bilateral z-transform** or just the **z-transform** of a discrete-time signal $x[n]$is defined as follows.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$X(\cdot) = Z\{x[\cdot]\}$ or shorthand: $x[n] \overset{Z}{\leftrightarrow} X(z)$

- Note capital letter for transform.
- In the math literature, this is called a **power series**.
- It is a mapping from the space of discrete-time signals to the space of functions defined over (some subset of) the complex plane.
- We will also call the complex plane **the z-plane**.

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The **ROC** is the set of values $z \in \mathbb{C}$ for which the sequence $x[n]z^{-n}$ is absolutely summable, *i.e.*,
$\left\{z \in \mathbb{C} : \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < \infty\right\}$

Skill: ***Finding a z-transform completely, including both $X(z)$ and the ROC.***

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z-transform example (1)

Example

$x[n] = \delta[n]$.
$X(z) = 1$ and ROC = $\mathbb{C}$= entire z-plane.

Example

$x[n] = \delta[n-k]$.
$X(z) = z^{-k}$ and

$$\text{ROC} = \begin{cases} \mathbb{C}, & k = 0 \\ \mathbb{C} - \{0\}, & k > 0 \\ \mathbb{C} - \{\infty\}, & k < 0. \end{cases}$$

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z-transform example (2)

Example

$x[n] = \{4, \underline{3}, 0, \pi\}$.
$X(z) = 4z + 3 + \pi z^{-2}$, $\mathrm{ROC} = \mathbb{C} - \{0\} - \{\infty\}$

For a **finite-duration signal**, the ROC is the entire z-plane, possibly excepting $z = 0$ and $z = \infty$.

Question

Why?

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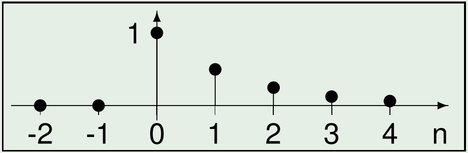
Why?
Because for $k > 0$: z^k is infinite for $z = \infty$ and z^{-k} is infinite for $z = 0$; elsewhere, polynomials in z and z^{-1} are finite.

z-transform example (3)

Skill: *Combining terms to express as geometric series.*

Example

Find z-transform of $x[n] = p^n u[n]$.



z-transform example (4)

Solution

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} p^n z^{-n} = \sum_{n=0}^{\infty} (pz^{-1})^n$$

$$= 1 + \left(\frac{p}{z}\right) + \left(\frac{p}{z}\right)^2 + \left(\frac{p}{z}\right)^3 + \cdots = \frac{1}{1 - pz^{-1}}.$$

The series converges iff $\left|pz^{-1}\right| < 1$*, i.e., if* $\{|z| > |p|\}$*.*

$$\boxed{p^n u[n] \overset{Z}{\leftrightarrow} \frac{1}{1 - pz^{-1}}, \text{ for } |z| > |p|}$$

Smaller $|p|$ *means faster decay means larger ROC.*

z-transform example (5)

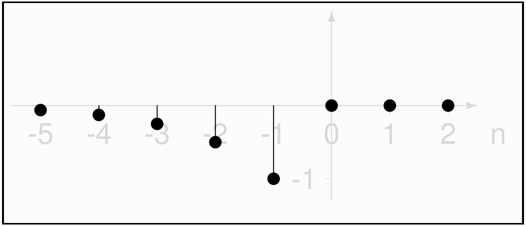
Example

Important special case: $p = 1$ leaves just the unit step function.

$$\boxed{u[n] \overset{Z}{\leftrightarrow} U(z) = \frac{1}{1 - z^{-1}}, \ |z| > 1}$$

Example

$x[n] = -p^n u[-n-1]$ for $p \neq 0$. (An anti-causal signal.)



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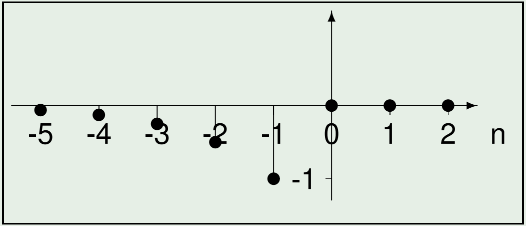

z-transform example (6)

Solution

$$X(z) = \sum_{n=-\infty}^{-1} -p^n z^{-n} = -\sum_{k=1}^{\infty}(p^{-1}z)^k, \quad k = -n$$

$$= -(p^{-1}z)\sum_{k=0}^{\infty}(p^{-1}z)^k = -p^{-1}z\frac{1}{1-p^{-1}z} = \frac{1}{1-pz^{-1}}.$$

The series converges iff $\left|p^{-1}z\right| < 1$, *i.e., if* $|z| < |p|$.

$$\boxed{-p^n u[-n-1] \overset{Z}{\leftrightarrow} \frac{1}{1-pz^{-1}}, \ \textit{for } |z| < |p|}$$

Laplace transform vs. z-transform

$$p^n\, u[n] \overset{Z}{\leftrightarrow} \frac{1}{1 - pz^{-1}}, \text{ for } |z| > |p|$$

$$-p^n\, u[-n-1] \overset{Z}{\leftrightarrow} \frac{1}{1 - pz^{-1}}, \text{ for } |z| < |p|$$

These two signals have the same formula for $X(z)$. The ROC is essential for resolving this ambiguity!

Laplace analogy

$$\mathrm{e}^{at}\, u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s-a}, \qquad \text{real}\{s\} > \text{real}\{a\}$$

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General shape of ROC (1)

Question

In the preceding two examples, we have seen ROC's that are the interior and exterior of circles. ***(Picture MIT Lecture 22.2-3)*** *What is the general shape?*

The ROC is always an annulus, *i.e.*, $\{r_2 < |z| < r_1\}$.

Note that r_2 can be zero (possibly with $\leq$) and r_1 can be ∞ (possibly with $\leq$).

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General shape of ROC (2)

Property

1. *The ROC of a causal signal is the exterior of a circle of some radius r_2.*
2. *The ROC of an anti-causal signal is the interior of a circle of some radius r_1.*

- For a general signal $x[n]$, the ROC will be the intersection of the ROC of its causal and noncausal parts, which is an annulus.
- If $r_2 < r_1$, then that intersection is a (nonempty) annulus. Otherwise the z-transform is undefined (does not exist).

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General shape of ROC: example

Example

Simple example of a signal which has empty ROC?
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Recall

$$u[n] \overset{Z}{\leftrightarrow} X(z) = \frac{1}{1 - z^{-1}} \text{ for } \{|z| > 1\}$$

ROC for the causal part is $\{|z| > 1\}$,
ROC for the anti-causal part is $\{|z| < 1\}$.

z-transform pairs and properties

- Some common z-transform pairs
  TABLE 10.2 (textbook, p.776)
- Properties of the z-transform
  TABLE 10.1 (textbook, p.775)

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   - The bilateral *z*-transform (10.1, 10.2)
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Rational z-transforms

Rational z-transforms is a very important class (*i.e.*, for LTI systems described by difference equations).

$$X(z) = \frac{B(z)}{A(z)} = g\frac{\prod_k (z - z_k)}{\prod_k (z - p_k)}.$$

- The **zeros** of a z-transform $X(z)$ are the values of z where $X(z) = 0$.
- The **poles** of a z-transform $X(z)$ are the values of z where $X(z) = \infty$.
- g is the gain.

Laplace analogy

$$X(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0} = G\frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}$$

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ROC of Rational z-transforms

The ROC will not contain any poles.

Example

- What are possible ROC's in following case?
- Find the corresponding z-transforms.

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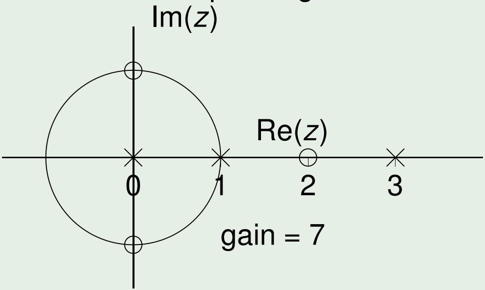

Solution

1. $\{0 < |z| < 1\}$
2. $\{1 < |z| < 3\}$,
3. $\{3 < |z|\}$.

$$X(z) = 7\frac{(z-j)(z+j)(z-2)}{(z-0)(z-1)(z-3)}$$
$$= 7\frac{(1-jz^{-1})(1+jz^{-1})(1-2z^{-1})}{(1-z^{-1})(1-3z^{-1})}$$

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The system function of an LTI system (1)

$$x[n] \rightarrow \boxed{\text{LTI } h[n]} \rightarrow y[n] = x[n] * h[n] \overset{Z}{\leftrightarrow} \boxed{Y(z) = H(z)\, X(z)\,.}$$

- Forward direction: transform $h[n]$ and $x[n]$, multiply, then inverse transform.
- Reverse engineering: put in known signal $x[n]$ with transform $X(z)$; observe output $y[n]$; compute transform $Y(z)$. Divide the two to get the **system function** or **transfer function**

$$\boxed{H(z) = Y(z)\,/\,X(z)\,.}$$

- The third rearrangement

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LCCDE (1)

Now apply these ideas to the analysis of LTI systems that are described by general **linear constant-coefficient difference equations (LCCDE)** (or just **diffeq** systems):

$$y[n] = -\sum_{k=1}^{N} a_k\, y[n-k] + \sum_{k=0}^{M} b_k x[n-k].$$

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LCCDE (2)

Applying linearity and shift properties taking z-transform of both sides of the above:

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k}\, Y(z) + \sum_{k=0}^{M} b_k z^{-k}\, X(z)$$

so

$$\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] Y(z) = \left[\sum_{k=0}^{M} b_k z^{-k}\right] X(z)$$

so, defining $a_0 \triangleq 1$,

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}},}$$

LCCDE (3)

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What is the name for this type of system function?

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What is the name for this type of system function?

It is a rational system function. (Ratio of polynomials in z.) We can also see why studying rational z-transforms is very important.

The system function for a LCCDE system is rational.

Block diagram representation (1)

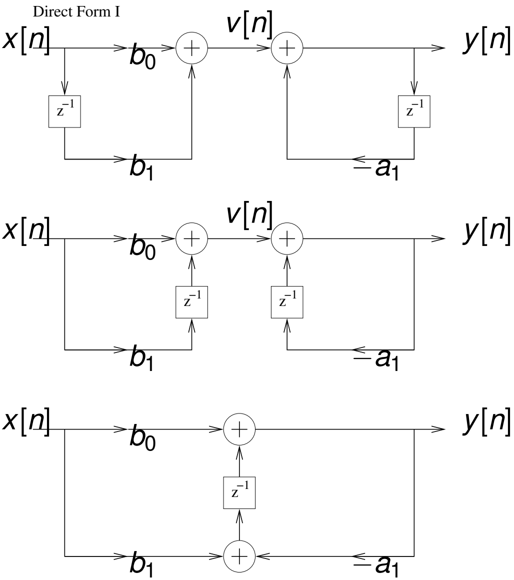
Example

$$y[n] = -a_1\, y[n-1] + b_0 x[n] + b_1 x[n-1]$$

Approach 1:

$$\begin{aligned} y[n] &= -a_1\, y[n-1] + v[n] \\ v[n] &= b_0 x[n] + b_1 x[n-1] \end{aligned}$$

Block diagram representation (2)



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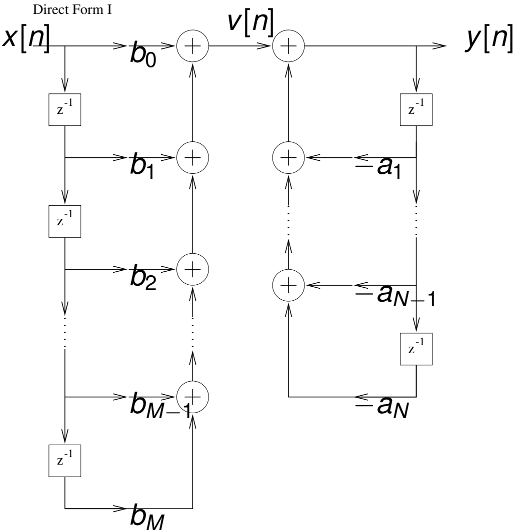
Approach 1:

$$\begin{aligned} v[n] &= \sum_{k=0}^{M} b_k x[n-k] \\ y[n] &= -\sum_{k=1}^{N} a_k\, y[n-k] + v[n] \end{aligned}$$

The first system $v[n] = \dots$ is nonrecursive, where as the second system is recursive.

Block diagram representation (4)



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$$Y(z) = -a_1 z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \frac{W(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}}$$

Derivation

$$\frac{Y(z)}{W(z)} = b_0 + b_1 z^{-1}$$

$$\Longrightarrow Y(z) = b_0 W(z) + b_1 z^{-1} W(z)$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1}}$$

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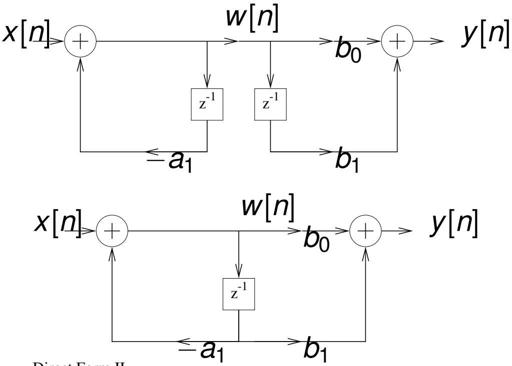
$$\Longrightarrow W(z) = -a_1 z^{-1} W(z) + X(z)$$

Block diagram representation (6)



Direct Form II

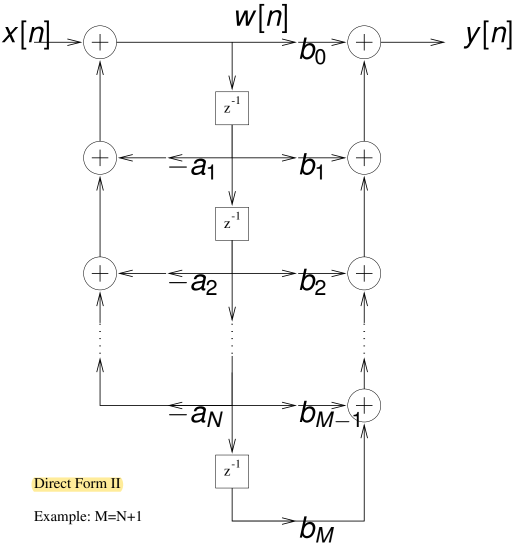
Block diagram representation (7)

$$y[n] = -\sum_{k=1}^{N} a_k\, y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

Approach 2:

$$\begin{aligned} w[n] &= -\sum_{k=1}^{N} a_k\, w[n-k] + x[n] \\ y[n] &= \sum_{k=0}^{M} b_k\, w[n-k] \end{aligned}$$

Block diagram representation (8)



Direct Form II

Example: M=N+1

Outline

1. 10. The z-transform
   - The bilateral z-transform (10.1, 10.2)
   - Poles and zeros (10.4)
   - System function and block diagram representations
   - Inversion of the z-transform

Inversion of the z-transform

Skill: ***Choosing and performing simplest approach to inverting a z-transform.***

Methods for inverse z-transform

- Table lookup, using properties
- Contour integration
- Series expansion into powers of z and z^{-1}
- Partial-fraction expansion and table lookup

The inverse z-transform by contour integration

$$\boxed{x[n] = \frac{1}{2\pi j} \oint X(z)\, z^{n-1}\, \mathrm{d}z}$$

The integral is a **contour integral** over a closed path C that must

- enclose the origin,
- lie in the ROC of $X(z)$.

Typically C is just a circle centered at the origin and within the ROC.

The inverse z-transform by power series expansion

The inverse z-transform by **power series expansion**, aka "**coefficient matching**"

If we can expand the z-transform into a power series (considering its ROC), then "by the uniqueness of the z-transform:"

$$\boxed{\text{if } \; X(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n} \text{ then } \; x[n] = c_n,}$$

i.e., the signal sample values in the time-domain are the corresponding coefficients of the power series expansion.

Example

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Find impulse response $h[n]$ for system described by $y[n] = 2\,y[n-3] + x[n]$.

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Question

Do we want an expansion in terms of powers of z or z^{-1}?

We want z^{-1}.

Solution

Using geometric series:

$$H(z) = \frac{1}{1 - 2z^{-3}} = \sum_{k=0}^{\infty} (2z^{-3})^k = \sum_{k=0}^{\infty} 2^k z^{-3k}$$

$$= 1 + 2z^{-3} + 2^2 z^{-6} + \dots$$

Thus

$$h[n] = \{\underline{1}, 0, 0, 2, 0, 0, 4, \dots\} = \sum_{k=0}^{\infty} 2^k \, \delta[n - 3k]$$

This case was easy since the power series was just the familiar geometric series.
In general one must use tedious ***long division*** *if the power series is not easy to find.*

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