VE216 Lecture 14

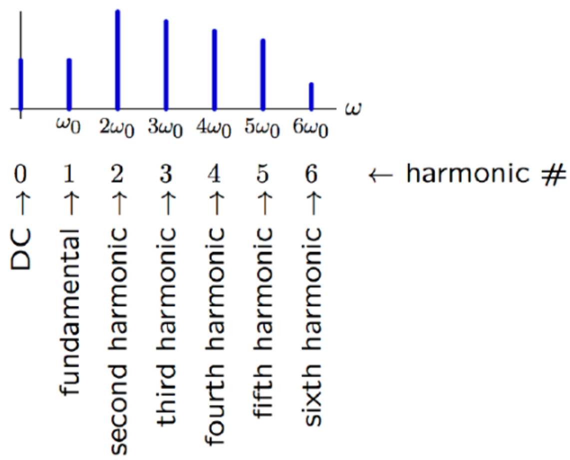
> Fourier Representations

Fourier Representation

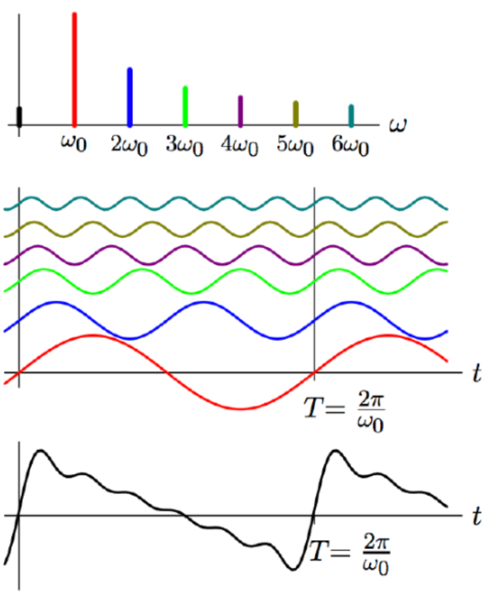
- Represent signals in terms of sinusoids.
- New representation for **systems** as **filters**.

Fourier Series

The harmonic components:



Harmonic Representations



- The sum of **harmonic components** can only be **periodic components**.
- All harmonic of ω_0 are periodic in $T = 2\pi/\omega_0$.
- all the periodic signals with harmonics can be represented as Fourier representation.

Harmonics Properties

- Multiplying **two harmonics** produces a **new harmonics** with same **fundamental frequency**: $e^{j\omega_0 kt} \times e^{j\omega_0 lt} = e^{j\omega_0 (l+k)t}$
- The integral of a **harmonic** over **any time interval** with length equal to a period T is **zero** unless the harmonic is at **DC**: $$\int_{t_0}^{t_0+T} e^{jk\omega_0 t} dt = \int_T e^{jk\omega_0 t} dt = \begin{cases} 0, k \neq 0 \\ T, k = 0 \end{cases} = T\delta[k]$$

Separating Harmonic Components

$x(t) = x(t+T) = \sum_{-\infty}^{\infty} a_k e^{j\omega_0 kt}$

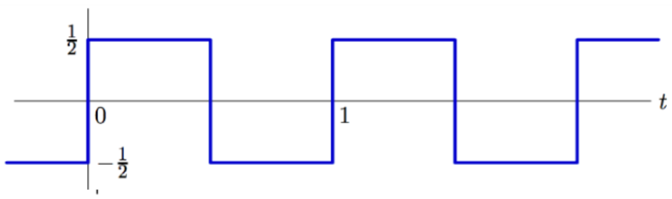
Then we see $\displaystyle\int_T x(t) e^{-jl\omega_0 t} = \int_T \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 (k-l)t} = \sum_{k=-\infty}^{\infty} a_k T\delta[k-l] = Ta_l$

So $\displaystyle a_k = \frac{1}{T} \int_T x(t) e^{-j\omega_0 kt} dt = \frac{1}{T} \int_T x(t) e^{-j\frac{2\pi}{T}kt} dt$

Fourier Series

- analysis equation: $\displaystyle a_k = \frac{1}{T} \int_T x(t) e^{-j\omega_0 kt} dt = \frac{1}{T} \int_T x(t) e^{-j\frac{2\pi}{T}kt} dt$
- synthesis equation: $\displaystyle x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$

Example

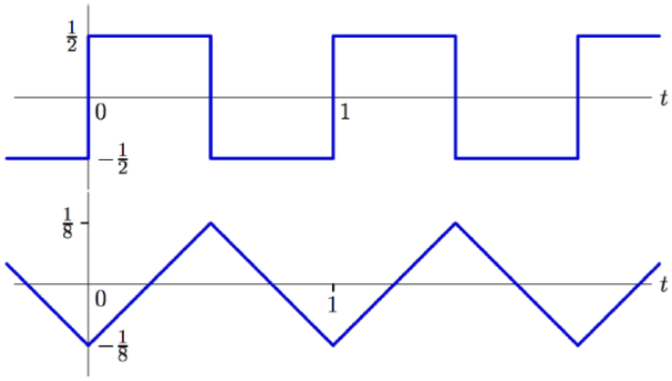


$$\begin{aligned} a_k &= \frac{1}{T} \int_T x(t) e^{-j\frac{2\pi}{T}kt} dt = -\frac{1}{2} \int_{-\frac{1}{2}}^{0} e^{-j2\pi kt} dt + \frac{1}{2} \int_0^{\frac{1}{2}} e^{-j2\pi kt} dt \\ &= \frac{1}{j4\pi k} (2 - e^{j\pi k} - e^{-j\pi k}) \\ &= \begin{cases} \frac{1}{j\pi k}, mod(k,2) = 1 \\ 0, otherwise \end{cases} \end{aligned}$$

Fourier Series Properties

If a signal is differentiated in time t, its Fourier coefficients are multiplied by $j\frac{2\pi}{T}k$.

Example

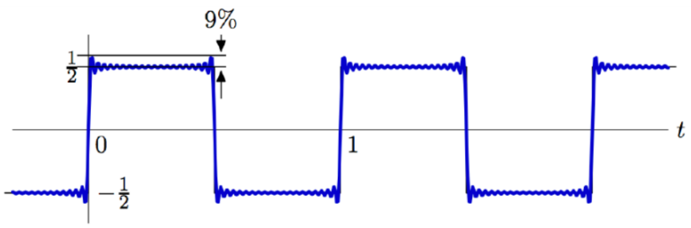


Since the triangle wave is the integral of square wave, then each of triangle wave's Fourier coefficient is multiplied by $\frac{T}{j2\pi k} = \frac{1}{j2\pi k}$ here.

Convergence of Fourier Series

Fourier series representations of functions with discontinuous slopes **converge toward** functions with discontinuous slopes.

Remark: Gibb's Phenomenon



Partial sums of discontinuous functions' Fourier series **"ring" around discontinuities**.

The reason of "9%" difference on triangle wave is: its Fourier coefficients only decreases at $\frac{1}{k}$.

So the triangle wave decreases at $\frac{1}{k^2}$.

Decrease or eliminating the ringing by **decreasing the magnitudes of Fourier coefficients at higher frequency**.