VE216 Lecture 4

> Continuous-time system

Multiple Representations of CT Systems

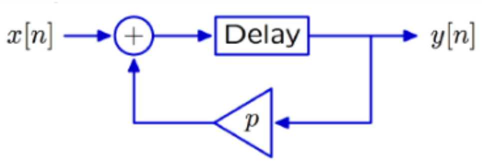
> the same as DT Systems

Differential Equations

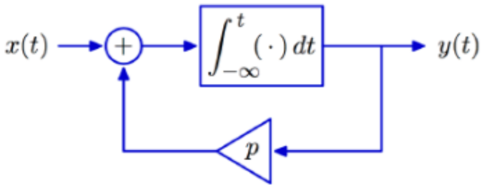
New methods based on **block diagrams** and **operators**, provide new ways to think about system's behaviors.

Block Diagrams

> Key difference is the **delays in DT** are replaced by **integrators in CT**.

- **DT**: 

  adders, scalers, delays

- **CT**: 

  adders, scalers, integrators

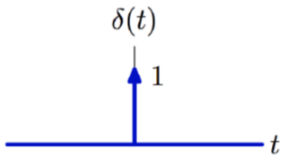
Operator Representation

- use the $\mathcal{A}$ operator.
- $\mathcal{A}$ in CT signal generates a new signal that equals to **the integral of the input signal at all points**.
- $Y = \mathcal{A}X$ is equal to $y(t) = \int_{-\infty}^{t} x(\tau) d\tau$ for all the time t.

Unit Impulse Signal

Properties

- Nonzero only at $t = 0$.
- Integral $(-\infty, +\infty)$ is 1.



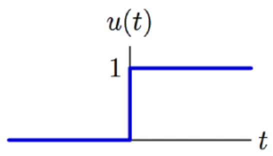
Represented by an arrow with number 1, representing the **area** or **weight**.

Unit Impulse and Unit Step

Unit Step Definition

Indefinite integral of **unit impulse** is **unit step**.

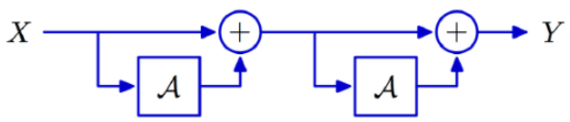
$$u(t) = \int_{-\infty}^{t} \delta(\lambda) d\lambda = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



Then we can see the block diagram: $\delta(t) \longrightarrow \boxed{\mathcal{A}} \longrightarrow u(t)$

Impulse Response of Acyclic CT System

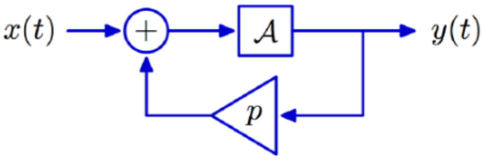
If the block diagram of CT system has no feedback (acyclic, no circle), then the corresponding expression is **imperative**.



$Y = (1 + \mathcal{A})^2 X$

if $x(t) = \delta(t)$, then $y(y) = \delta(t) + 2u(t) + t \cdot u(t)$

CT Feedback Methods



Method 1: Differential Equation Method

$\dot{y}(t) = x(t) + py(t)$ this is a linear, first order equation with constant coefficients.

try $y(t) = Ce^{\alpha t}u(t)$, then we get:

$\dot{y}(t) = \alpha Ce^{\alpha t}u(t) + Ce^{\alpha t}\delta(t) = \alpha Ce^{\alpha t}u(t) + C\delta(t) = \delta(t) + pCe^{\alpha t}u(t) = x(t) + py(t)$

if $y(t) = e^{pt}u(t)$, the equation is satisfied ($\delta(t) = e^{pt}\delta(t)$).

Method 2: Operator Method

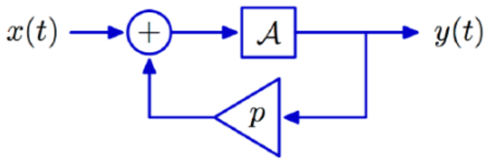
$Y = A(X + pY) \leftrightarrow \frac{Y}{X} = \frac{A}{1-pA} = A(1 + pA + p^2A^2 + \cdots)$

Let $X = x(t) = \delta(t)$ then:

$$
\begin{aligned}
& y(t) = A(1 + pA + p^2A^2 + \ldots)\delta(t) \\
& = (1 + pA + p^2A^2 + \ldots)u(t) \\
& = (1 + pt + \tfrac{1}{2}p^2t^2 + \tfrac{1}{6}p^3t^3 + \cdots)u(t) = e^{pt}u(t)
\end{aligned}
$$

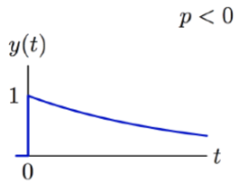
Convergent and Divergent Poles

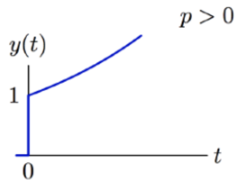
check the **CT system**:

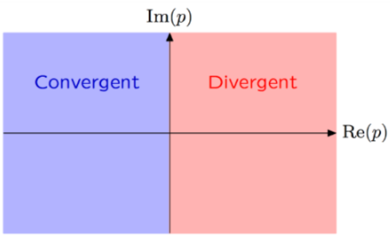


then we get a $y(t) = e^{pt}u(t)$.

We decide the convergence by this method:





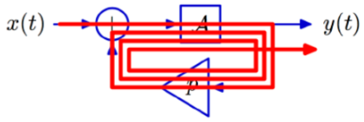


if $Re(p) < 0$ then it is convergent, $Re(p) > 0$ then it is divergent.

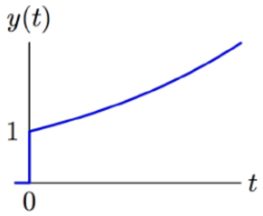
Feedback Comparison (CT, DT)

CT Feedback

In CT, each cycle adds new integration.



$$y(t) = A(1 + pA + p^2A^2 + \ldots)\delta(t)$$
$$= (1 + pA + p^2A^2 + \ldots)u(t)$$
$$= (1 + pt + \tfrac{1}{2}p^2t^2 + \tfrac{1}{6}p^3t^3 + \cdots)u(t) = e^{pt}u(t)$$



PT Feedback

In DT, each cycle creates another sample in output:

$$y[n] = (1 + pR + p^2R^2 + p^3R^3 + \cdots)\delta[n]$$
$$= \delta[n] + p\delta[n-1] + p^2\delta[n-2] + p^3\delta[n-3] + \cdots$$

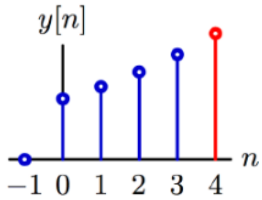
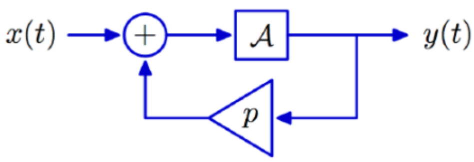
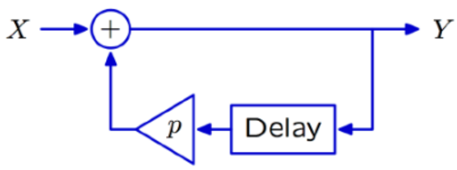
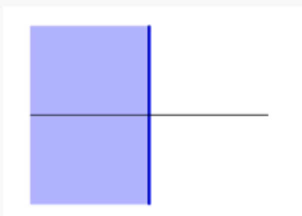
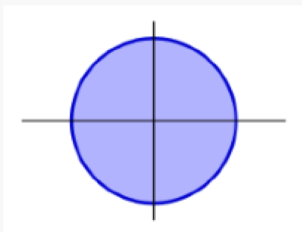


Diagram Comparison

Equation:	$\dot{y}(t) = x(t) + py(t)$	$y[n] = x[n] + py[n-1]$
Block diagram:		
Operator:	$\frac{A}{1-pA}$	$\frac{1}{1-pR}$
Solution	$e^{pt}u(t)$	$p^nu[n]$
Convergence:		

Convergent? (Exercise)

- $\frac{1}{1-\frac{1}{4}R^2}$

  $Y = \frac{1}{1-\frac{1}{4}R^2}X$

  $= k_1\frac{1}{1-\frac{1}{2}R}X + k_2\frac{1}{1+\frac{1}{2}R}X$

  $= k_1(1+\frac{1}{2}R+\frac{1}{4}R^2+\cdots)X + k_2(1+(-\frac{1}{2}R)+(-\frac{1}{2}R)^2+\cdots)X$

  Poles of $\pm\frac{1}{2}$ for discrete case, so convergent.
- $\frac{1}{1-\frac{1}{4}A^2}$

  $Y = \frac{1}{1-\frac{1}{4}A^2}X$

  $= k_1\frac{1}{1-\frac{1}{2}A}X + k_2\frac{1}{1+\frac{1}{2}A}X$

  $= k_1(1+\frac{1}{2}A+(\frac{1}{2}A)^2+\cdots)X + k_2(1+(-\frac{1}{2}A)+(-\frac{1}{2}A)^2+\cdots)X$

  $= k_1e^{1/2t} + k_2e^{-1/2t}$

  Poles of $\pm\frac{1}{2}$ in continuous case, so divergent ($\frac{1}{2}$).
- $\frac{1}{1+2R+\frac{3}{4}R^2}$

  $Y = \frac{1}{1+2R+\frac{3}{4}R^2}X$

  $= k_1\frac{1}{1+\frac{1}{2}R}X + k_2\frac{1}{1+\frac{3}{2}R}X$

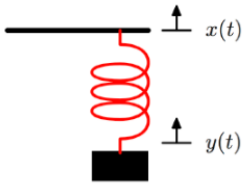
  We have pole for $-\frac{3}{2}$ and $-\frac{1}{2}$ in discrete case, so divergent ($-\frac{3}{2}$).
- $\frac{1}{1+2A+\frac{3}{4}A^2}$

  $Y = \frac{1}{1+2A+\frac{3}{4}A^2}X$

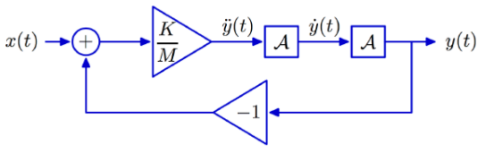
  $= k_1\frac{1}{1+\frac{1}{2}A}X + k_2\frac{1}{1+\frac{3}{2}A}X$

  The poles are $-\frac{1}{2}$ and $-\frac{3}{2}$ in continuous case, so convergent.

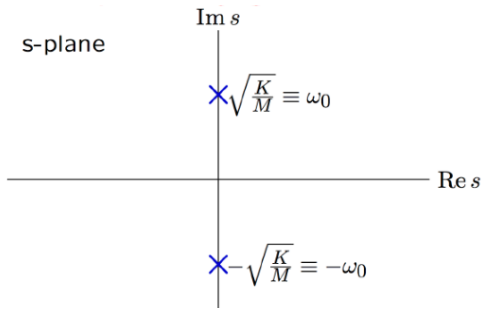
Mass-Spring System (Exercise)



$F = K(x(t) - y(t)) = M\ddot{y}(t)$



$\frac{Y}{X} = \frac{\frac{K}{M}A^2}{1+\frac{K}{M}A^2}$, thus $p = \pm j\sqrt{\frac{K}{M}}$.



$e^{j\omega_0 t} = cos\omega_0 t + jsin\omega_0 t$ and $e^{-j\omega_0 t} = cos\omega_0 t - jsin\omega_0 t$.

$\frac{Y}{X} = \frac{\omega_0}{2j}(\frac{A}{1-j\omega_0 A}) - \frac{\omega_0}{2j}(\frac{A}{1+j\omega_0 A})$, with $\frac{A}{1\pm j\omega_0 A}$ as two modes (check lecture 2 for **fundamental modes**)

Then check lecture 3 for **fundamental mode** and **complex poles**, we get:

$y(t) = \frac{\omega_0}{2j}(e^{j\omega_0 t} - e^{-j\omega_0 t}) = \omega_0 sin\omega_0 t, t > 0$.

An alternative (ugly) approach

$\frac{Y}{X} = \frac{\omega_0^2 A^2}{1+\omega_0^2 A^2} = \omega_0^2 A^2 \sum_{l=0}^{\infty}(-\omega_0^2 A^2)^l$

then if $x(t) = \delta(t)$:

$y(t) = \sum_{l=0}^{\infty} \omega_0^2 (-\omega_0^2)^l A^{2l+2} \delta(t) = \omega_0^2 t - \omega_0^4 \frac{t^3}{3!} + \omega_0^6 \frac{t^5}{5!} \cdots = \omega_0 sin\omega_0 t$