UM-SJTU Joint Institute

Physics Laboratory

(Vp241)

Laboratory Report

Exercise 5

RC, RL, AND RLC CIRCUITS

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Date: 23 November 2019

1. Objective

The objective of this exercise is to understand the physics of alternating-current circuits, in particular the processes of charging/discharging of capacitors, the phenomenon of electromagnetic induction in inductive elements, and other dynamic processes in RC, RL, and RLC series circuits. Moreover, methods for measuring the amplitude-frequency and the phase-frequency characteristics of RC, RL, and RLC series circuits will be studied. The resonance frequency of an RLC circuit as well as the quality factor of the circuit will be found from the amplitude-frequency curve.

2. Theoretical Background

Electric circuits consist of basic elements like resistors, capacitors, and inductors. Depending on a particular arrangement of these elements, RC, RL, RLC alternating-current (AC) circuits may display various features, including transient, steady state, and resonant behavior.

2.1 Transient Processes in RC, RL, RLC Series Circuits

2.1.1 RC Series Circuits

In an RC circuit, the process of charging or discharging of the capacitor is an example of a transient process. Figure 1 shows a RC series circuit with the source signal of a square-wave.

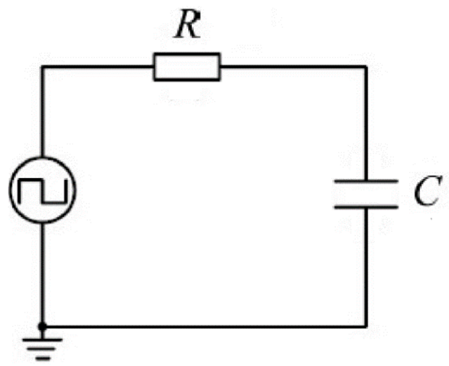

Figure 1 RC series circuit

In the first half of the cycle, the square-wave voltage is U(t) =ε and it charges the capacitor. In the second half-cycle, the square-wave voltage is zero, and the capacitor discharges through the resistor. The loop equation (Kirchhoff's loop rule) for the charging process is

$$RC\frac{dU_C}{dt} + U_C = \varepsilon \qquad (1)$$

With the initial condition U_C(t = 0) = 0, the solution of Eq. (1) can be found as

$$U_C = \varepsilon\left(1 - e^{-\frac{t}{RC}}\right) \; and \; U_R = iR = \varepsilon e^{-\frac{t}{RC}}$$

Hence the voltage across the capacitor U_C increases exponentially with time t, whereas the voltage on the resistor U_R decreases exponentially with time. The curves U(t), U_C(t), and U_R(t) are shown in Figure 2.

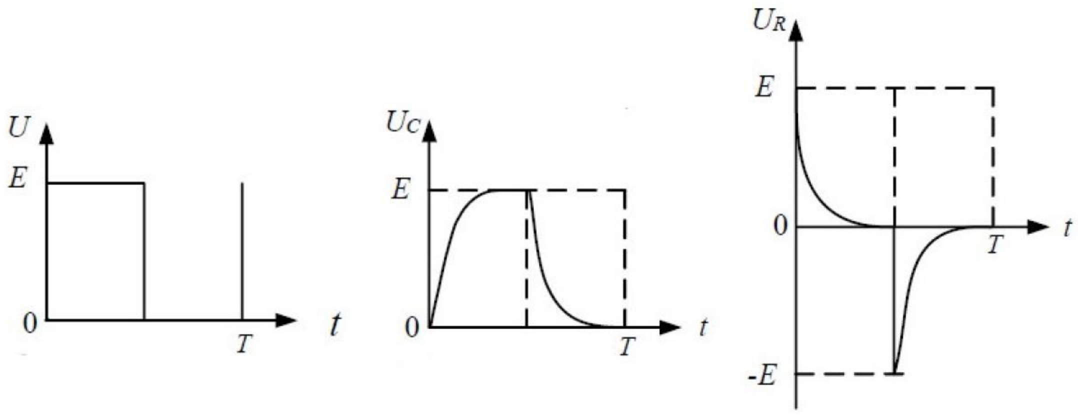

Figure 2 Charging/discharging curves for a RC series circuit

For the discharging process, the loop rule gives

$$RC\frac{dU_C}{dt} + U_C = 0 \qquad (2)$$

The solution of Eq. (2), with the initial condition U_C(t = 0) = ε, is

$$U_C = \varepsilon e^{-\frac{t}{RC}}$$

and, consequently, $U_R = iR = -\varepsilon e^{-\frac{t}{RC}}$

where the magnitudes of both U_C and U_R decrease exponentially with time. Since RC=τ has the units of time, it is called the time constant of the circuit, and characterizes the dynamics of the transient process. There is another characteristic related to the time constant, easier to measure in experiments, which is called the half-life period $T_{1/2}$. The half-life period is the time needed for U_C to decrease to a half of the initial value (or increase to a half of the terminal value), and may be also used to characterize the dynamics of the transient process. Both quantities, in the process with exponential dynamics discussed above, are related by the equation $T_{1/2}$ = τln 2 =0.693 τ.

2.1.2 RL Series Circuit

A similar analysis can be carried out for a RL series circuit. In this case,

$$\tau = \frac{L}{R} \text{ and } T_{1/2} = \frac{L}{R} ln2$$

2.1.3 RLC Series Circuit

First, let us discuss the situation when a power source is suddenly plugged into an RLC circuit. Then the voltage across the capacitor satisfies the differential equation

$$LC\frac{d^2U_C}{dt^2} + RC\frac{dU_C}{dt} + U_C = \varepsilon \qquad (3)$$

following again from the loop rule. Dividing both sides of the equation by LC and introducing the symbols

$$\beta = \frac{R}{2L} \text{ and } \omega_0 = \frac{1}{\sqrt{LC}}, \qquad (4)$$

Eq. (3) can be rewritten as

$$\frac{d^2U_C}{dt^2} + 2\beta\frac{dU_C}{dt} + {\omega_0}^2 U_C = {\omega_0}^2\varepsilon \qquad (5)$$

Note that Eq. (5) is an inhomogeneous differential equation and it is mathematically equivalent to the equation of motion of a damped harmonic oscillator with a constant driving force. Therefore, the complementary homogeneous equation is fully analogous to the equation of motion of a damped harmonic oscillator, with β being the damping coefficient, and ω_0—the natural angular frequency. Moreover, after a specific solution to the inhomogeneous equation is found, a unique solution to the initial value problem consisting of Eq. (5) and the initial conditions

$$U_C(t=0) = 0$$

$$\left.\frac{dU_C}{dt}\right|_{t=0} = 0 \qquad (6)$$

can be found.

Exactly as for mechanical oscillations, depending on the relation between β and ω_0, there are three regimes, as implied by the solution of the complementary homogeneous equation:

- If $\beta^2 - {\omega_0}^2 < 0$(weak damping), the system is in the underdamped regime and the solution to the initial value problem is of the form

$$U_C = \varepsilon - \varepsilon e^{-\beta t}(\cos\omega t + \frac{\beta}{\omega}\sin\omega t)$$

Where $\omega = \sqrt{{\omega_0}^2 - \beta^2}$

- If $\beta^2 - {\omega_0}^2 > 0$(strong damping), the system is in the overdamped regime with the solution of the form

$$U_C = \varepsilon - \frac{\varepsilon}{2\gamma}e^{-\beta t}[(\beta+\gamma)e^{\gamma t} - (\beta-\gamma)e^{-\gamma t}]$$

- Finally, if $\beta^2 - {\omega_0}^2 = 0$,the system is said to be critically damped, and

$$U_C = \varepsilon - \varepsilon(1+\beta t)e^{-\beta t}$$

When the circuit reaches steady state, the power source is suddenly removed (ε= 0). The differential equation for the discharging process is similar to that of the charging process, and there are also three regimes of the process.

The above discussion is valid for an ideal circuit and a step-signal source with zero internal resistance. In the experiment, the ideal system is replaced by a square-wave source with a small internal resistance. The period of the square-signal must be much greater than the time constant of the circuit. Note that, according to the above equations, the voltage across the capacitor U_C will finally reach ε regardless of the regime (Figure 3)

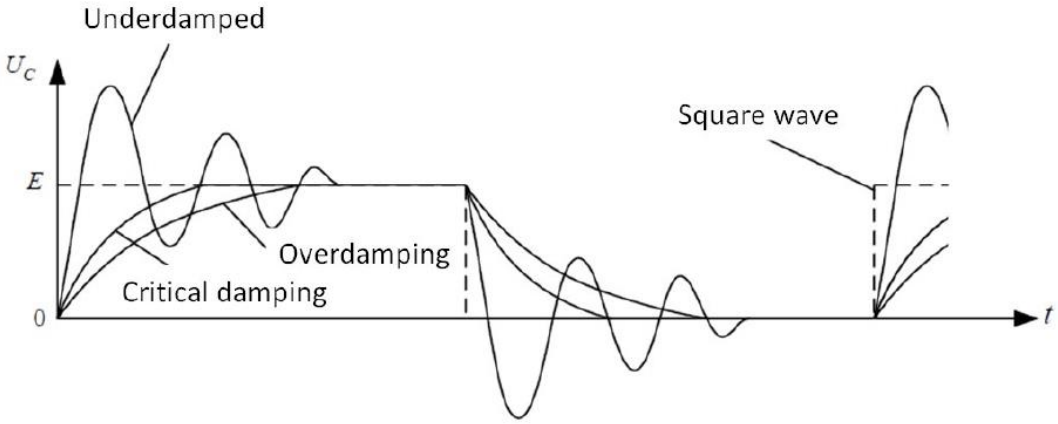

Figure 3 Three different regimes of transient processes in an RLC series circuit

2.2 RC, RL Steady-State Circuits

When a sinusoidal alternating input voltage is provided to a RC (or RL) series circuit, the amplitude and the phase of the voltage across the capacitor and the resistor will change with the frequency of the input voltage. Then the amplitude vs. frequency relation and the phase vs. frequency relation can be obtained by measuring the voltage across the elements in the circuit for different input signal frequencies

$$\phi = tan^{-1}(\frac{U_L}{U_R}) = tan^{-1}(\frac{\omega L}{R})$$

$$\phi = tan^{-1}(-\frac{U_C}{U_R}) = tan^{-1}(-\frac{1}{\omega RC})$$

2.3 RLC Resonant Circuit

2.3.1 RLC Series Circuit

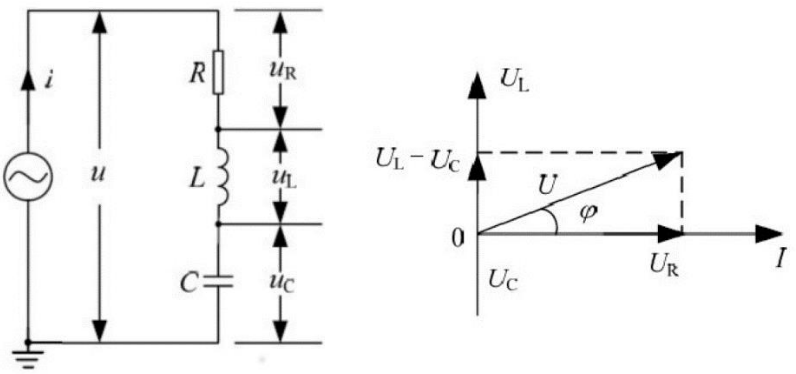

Figure 4 RLC series circuit

A generic RLC series circuit is shown in Figure 4. The impedance and the phase difference in the RLC circuit can be calculated, e.g., by using the phasors technique. Representing the current I by a vector along the horizontal axis, the phase differences between the current and the voltages across the resistor, coil, and capacitor are

$$\varphi_R = 0, \qquad \varphi_L = \frac{\pi}{2}, \qquad \varphi_C = -\frac{\pi}{2},$$

respectively. The corresponding voltage amplitudes across the elements are

$$U_R = IZ = IR, \qquad U_L = IZ_L = I\omega L, \qquad U_C = IZ_C = \frac{I}{\omega C}.$$

Hence, the voltage amplitude

$$U = \sqrt{U_R^2 + (U_L - U_C)^2} \qquad \text{or} \qquad U = I\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2},$$

and the total impedance

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2},$$

with the phase difference between the current and the voltage in the circuit

$$\varphi = \tan^{-1}\left(\frac{U_L - U_C}{U_R}\right) = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right).$$

2.3.2 Resonance

If the frequency of the input signal provided by the source satisfies the condition

$$\omega_0 L = \frac{1}{\omega_0 C}, \qquad \text{or, equivalently,} \qquad \omega_0 = \frac{1}{\sqrt{LC}},$$

the total impedance will reach a minimum, $Z_0 = R$. Note that the resistance R in a real circuit includes the internal resistance and all kinds of alternating-current power losses, so its actual value will be greater than the theoretical one.

When the current reaches its maximum, $I_m = U=R$, the circuit is said to be at resonance. The frequency

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}},$$

at which the resonance phenomenon occurs, is called the resonance frequency.

The total impedance Z, the current I, and the phase difference $\varphi = \varphi_u - \varphi_i$ all depend on the frequency, with generic shapes of the three curves shown in Figure 5. According to Eqs. (8) and (9), when the frequency is low ($f < f_0$, i.e. $1/\omega C > \omega L$), then $\varphi < 0$.In this situation the total voltage lags behind the current and the circuit is said to be capacitive.

When the circuit is resonant ($f = f_0$, i.e. $1/\omega C = \omega L$), then $\varphi = 0$ and the voltages

across the capacitor and the inductor should be equal. The circuit is said to be resistive.
Finally, when the frequency is high ($f > f_0$, i.e. $1/\omega C < \omega L$), then $\varphi > 0$. In this situation the total voltage leads the current, and the circuit is said to be inductive.

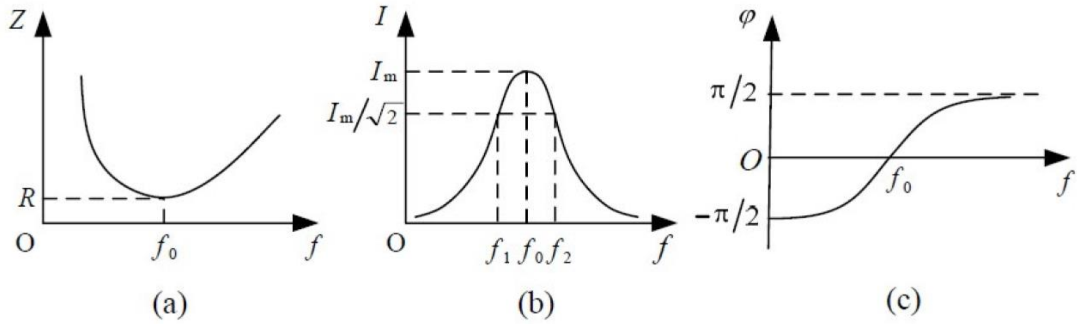

Figure 5 The impedance, the current and the phase difference as functions of the frequency for an RLC series circuit (generic sketches)

2.3.3 Quality Factor in Resonant Circuits

Since $I_m = U/R$, the voltages across the resistor, the inductor, and the capacitor are $U_R = I_m R = U$;

$$\begin{aligned} U_L &= I_{\mathrm{m}} Z_L = \frac{U}{R}\omega L, \\ U_C &= I_{\mathrm{m}} Z_C = \frac{U}{R}\frac{1}{\omega_0 C} = U_L, \end{aligned}$$

respectively. For a circuit driven at the resonance frequency, the ratio of U_L (or U_C) to U is called the quality factor Q of a resonant circuit

$$Q = \frac{U_L}{U} = \frac{\omega_0 L}{R} \qquad \text{or} \qquad Q = \frac{U_C}{U} = \frac{1}{\omega_0 RC}.$$

When the total voltage is fixed, the greater Q is, the greater U_L and U_C are. The value of Q can be used to quantify the efficiency of resonant circuits.
The quality factor can also be found as

$$Q = \frac{f_0}{f_2 - f_1},$$

where f_1 and f_2 are two frequencies such that $I(f_1) = I(f_2) = I_m/\sqrt{2}$ (see Figure 5b).

3. Experimental Setup

3.1 Apparatus

The measurement setup consists of the following main elements: a signal generator, an oscilloscope, a digital multimeter, a wiring board, a fixed resistor 100 (2W), a variable resistor 2k (2W), two capacitors 0.47 μF and 0.1 μF, as well as two inductors (10 mH and 33 mH).

3.2 Device Information

The information of each measurement device is shown in Table 1.

apparatus	uncertainty
Signal generator	$\pm0.001\text{Hz}/\pm0.001\text{Vpp}$
Cursor(time)	$\pm0.001/0.01\mu\text{s}$
Cursor(amplitude)	$\pm0.02/0.002\text{Vpp}$
Multimeter(resistance)	$\pm0.01\Omega$
Multimeter(capacitance)	$\pm0.1\text{nF}$

Table 1 Information of Each Measurement Device

4. Measurement Procedure

4.1 RC, RL Series Circuit

1. Choose a capacitor and an inductor to assemble a circuit with the fixed-resistance resistor. Adjust the output frequency of the square-wave signal provided by the signal generator. Observe the change of the waveform when the time constant is smaller or greater than the period of the square-wave. Choose the frequency that allows the capacitor to fully charge/discharge. Use the PRINT function of the oscilloscope to store the waveforms.
2. Adjust display parameters of the oscilloscope and measure $T_{1/2}$ for the studied circuits. Then, calculate the time constant and compare it with the theoretical value. In order to find the time constant accurately, only one period should be displayed on the oscilloscope screen.

4.2 RLC Series Circuit

1. Choose a capacitor and an inductor to assemble an RLC series circuit with the variable resistor. Observe the waveform of the capacitor voltage in the underdamped, critically damped, and overdamped regimes. Use the PRINT function of the oscilloscope to store the waveforms.
2. Adjust the variable resistor to the critically damped regime. According to the definition of the half-life period $T_{1/2}$, we have $\beta T_{1/2} = 1.68$. By finding the value of $T_{1/2}$, the time constant can be found as $\tau = 1/\beta = T_{1/2}/1.68$. Compare your result with the theoretical value.

4.3 RLC Resonant Circuit

Apply a sinusoidal input voltage U_i to the RLC series circuit, change the frequency, then observe the change of the voltage U_R for a fixed resistor R, as well as the phase difference between U_R and U_i. Measure how U_R changes with U_i and calculate the phase difference according to Figure 4. Plot the graphs $I=I_m$ vs. $f=f_0$ and φ vs. $f=f_0$.

Estimate the resonance frequency and calculate the quality factor Q.

4.4 Cautions

- Read manuals carefully before operating the instruments.
- The circuit should be grounded to the same point as the instruments used in the measurements.
- Power supply should be turned on after the circuit is completed.

5. Results

5.1 RC series circuit

$$\mathrm{R} = 99.96[\Omega] \pm 0.01[\Omega]$$
$$f = 700.000[Hz] \pm 0.001[Hz]$$
$$\varepsilon = 4.000[Vpp] \pm 0.001[Vpp]$$
$$C = 460.5[nF] \pm 0.1[nF]$$

$$T_{1/2} = 36.00[\mu s] \pm 0.01[\mu s]$$

We then calculate the time constant τ_{exp}:

$$\tau_{exp} = \frac{T_{1/2}}{ln\,2} = \frac{36}{ln2} = 51.94 \pm 0.01[\mu s]$$

Theoretically, the time constant τ_{theo} should be:

$$\tau_{theo} = RC = 99.96 \times 460.5 \times 10^{-9} = 46003.95[\mathrm{s}] = 46.00 \pm 0.01[\mu s]$$

The relative error is:

$$\delta = \frac{\tau_{exp} - \tau_{theo}}{\tau_{theo}} = \frac{51.94 - 46.00}{46.00} = 12.9\%$$

The error is relatively large, which will be discussed in the error analysis part.

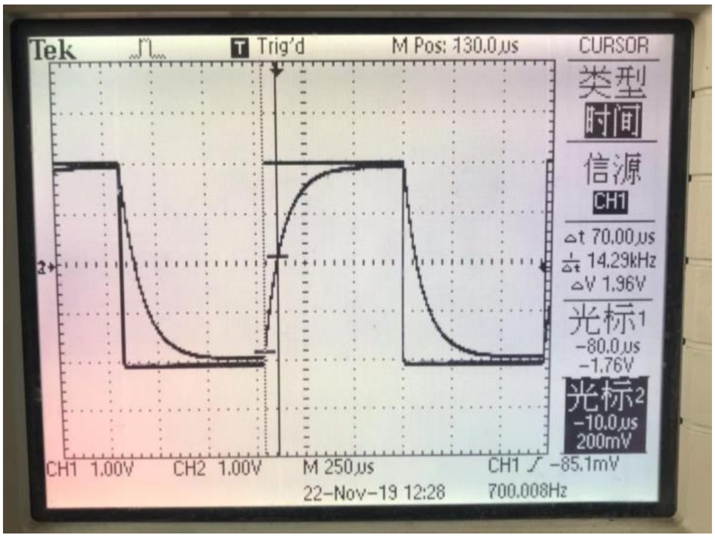

Figure 6 Waveform of RC circuit

5.2 RL series circuit

$$R = 99.96[\Omega] \pm 0.01[\Omega]$$
$$f = 1000.000[Hz] \pm 0.001[Hz]$$
$$\varepsilon = 4.000[Vpp] \pm 0.001[Vpp]$$
$$L = 0.01[H] \pm 0[H]$$

$$T_{1/2} = 70.00[\mu s] \pm 0.01[\mu s]$$

We then calculate the time constant τ_{exp}:

$$\tau_{exp} = \frac{T_{1/2}}{ln\,2} = \frac{70}{ln2} = 100.99 \pm 0.01[\mu s]$$

Theoretically, the time constant τ_{theo} should be:

$$\tau_{the} = \frac{L}{R} = \frac{0.01}{99.96} = 10^{-4}[\mathrm{s}] = 100.00 \pm 0.01[\mu s]$$

The relative error is:

$$\delta = \frac{\tau_{exp} - \tau_{theo}}{\tau_{theo}} = \frac{100.99 - 100.00}{100.00} = 0.99\%$$

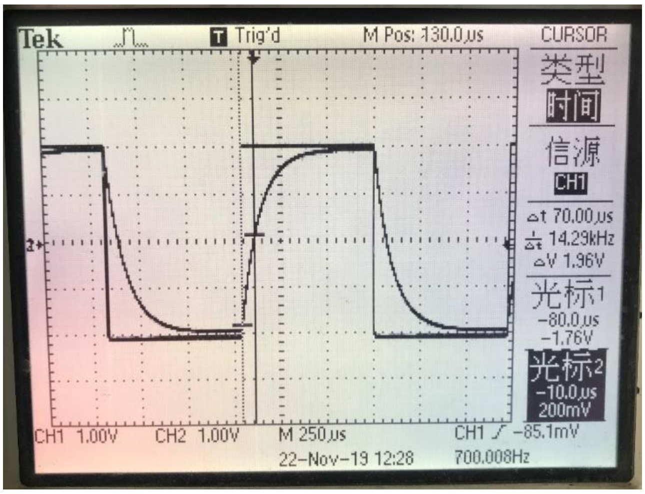

Figure 7 Waveform of RL circuit

5.3 RLC series circuit

When critically damped,

$$f = 400.000[Hz] \pm 0.001[Hz]$$
$$\varepsilon = 4.000[Vpp] \pm 0.001[Vpp]$$
$$L = 0.01[H] \pm 0[H]$$
$$C = 460.5[nF] \pm 0.1[nF]$$
$$\beta t = 1.68$$

$$T_{1/2} = 140.0[\mu S] \pm 0.1[\mu S]$$

We then calculate the time constant τ_{exp}:

$$\tau_{exp} = \frac{T_{1/2}}{\beta t} = \frac{140.0}{1.68} = 83.33 \pm 0.06[\mu s]$$

Theoretically, the time constant τ_{theo} should be:

$$\tau_{theo} = \sqrt{LC} = \sqrt{0.01 \times 460.5 \times 10^{-9}} = 6.786 \times 10^{-5}[s]$$
$$= 67.86 \pm 0.007[\mu s]$$

The relative error is:

$$\delta = \frac{\tau_{exp} - \tau_{theo}}{\tau_{theo}} = \frac{83.33 - 67.86}{67.86} = 22.8\%$$

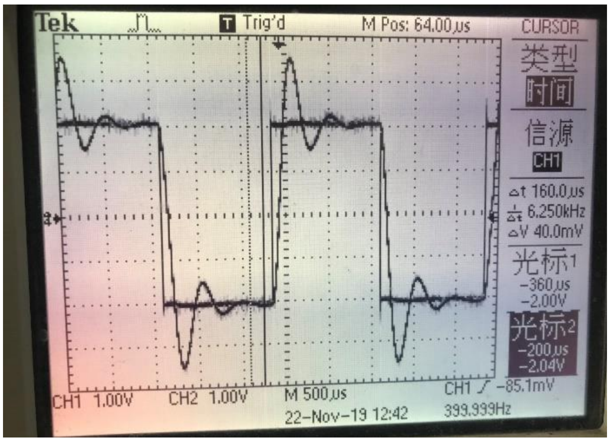

Figure8 Waveform of Under-damped regime of RLC circuit

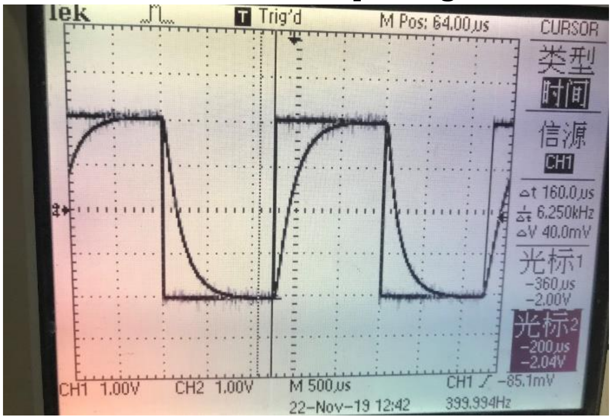

Figure9 Waveform of critically-damped regime of RLC circuit

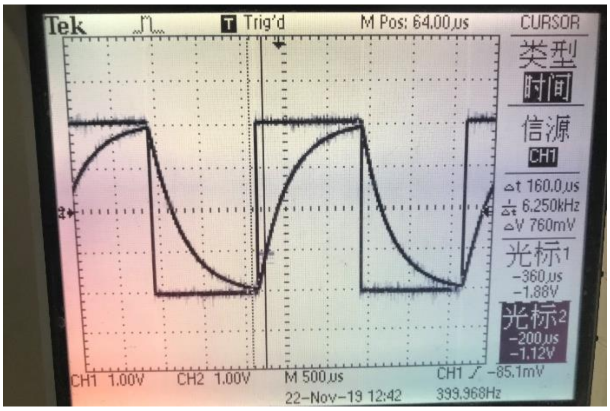

Figure10 Waveform of over-damped regime of RLC circuit

5.4 RLC resonant circuit

$$\mathrm{R} = 99.96[\Omega] \pm 0.01[\Omega]$$
$$\mathrm{L} = 0.01[\mathrm{H}] \pm 0[\mathrm{H}]$$
$$\mathrm{C} = 460.5[\mathrm{nF}] \pm 0.1[\mathrm{nF}]$$
$$\mathrm{f0} = 2300.000[\mathrm{Hz}] \pm 0.001[\mathrm{Hz}]$$
$$\varepsilon = 4.000[\mathrm{Vpp}] \pm 0.001[\mathrm{Vpp}]$$

	$U_R[Vpp]$	$u_{U_R}[Vpp]$	$\mathrm{f[Hz]}\pm 0.001[Hz]$
1	0.520	0.002	400.000
2	0.920	0.002	700.000
3	1.40	0.02	1000.000
4	1.96	0.02	1300.000
5	2.20	0.02	1400.000
6	2.88	0.02	1700.000
7	3.12	0.02	1800.000
8	3.36	0.02	1900.000
9	3.56	0.02	2000.000
10	3.72	0.02	2100.000
11	3.84	0.02	2200.000
12	3.92	0.02	2300.000
13	3.88	0.02	2400.000
14	3.80	0.02	2500.000
15	3.72	0.02	2600.000
16	3.48	0.02	2800.000
17	3.20	0.02	3000.000
18	2.76	0.02	3300.000
19	2.20	0.02	3900.000
20	1.88	0.02	4300.000
21	1.32	0.02	5600.000

Table 2 Measurement of RLC Resonant Circuit

5.4.1 Relationship between $\frac{I}{I_m}$ and $\frac{f}{f_0}$

We want to find the relation between I/I_m and f/f0. Thus, we calculate the value of I/I_m and f/f0.

$$\frac{I}{I_m} = \frac{\frac{U_R}{R}}{\frac{U_R}{R}} = \frac{U_R}{U_m}$$

$$U_m = 3.92 \pm 0.02[Vpp]$$

$$f_0 = 2300.000 \pm 0.001[Hz]$$

Take the first row of data for example:

$$\frac{I}{I_m} = \frac{U_R}{U_m} = \frac{0.520}{3.92} = 0.133 \pm 0.0008$$

$$\frac{f}{f_0} = \frac{400.000}{2300.000} = 0.174 \pm 4 \times 10^{-7}$$

We show the result of all data and their uncertainties in Table 3:

	$\frac{I}{I_m}$	$u_{\frac{I}{I_m}}$	$\frac{f}{f_0}$	$u_{\frac{f}{f_0}}$
1	0.133	0.0008	0.174	4×10^{-7}
2	0.235	0.001	0.304	5×10^{-7}
3	0.357	0.005	0.435	5×10^{-7}
4	0.500	0.006	0.565	5×10^{-7}
5	0.561	0.006	0.609	5×10^{-7}
6	0.735	0.006	0.739	5×10^{-7}
7	0.796	0.007	0.783	6×10^{-7}
8	0.857	0.007	0.826	6×10^{-7}
9	0.908	0.007	0.870	6×10^{-7}
10	0.949	0.007	0.913	6×10^{-7}
11	0.980	0.007	0.957	6×10^{-7}
12	1.000	0.007	1.000	6×10^{-7}
13	0.990	0.007	1.043	6×10^{-7}
14	0.969	0.007	1.087	6×10^{-7}
15	0.949	0.007	1.130	7×10^{-7}
16	0.888	0.007	1.217	7×10^{-7}
17	0.816	0.007	1.304	7×10^{-7}
18	0.704	0.006	1.435	8×10^{-7}
19	0.561	0.006	1.696	9×10^{-7}
20	0.480	0.006	1.870	9×10^{-7}
21	0.337	0.005	2.435	1×10^{-6}

Table 3 relation between I/I_m and f/f0

Using Originlab, we plot the figure of I/Im vs f/f0, shown in Figure11:

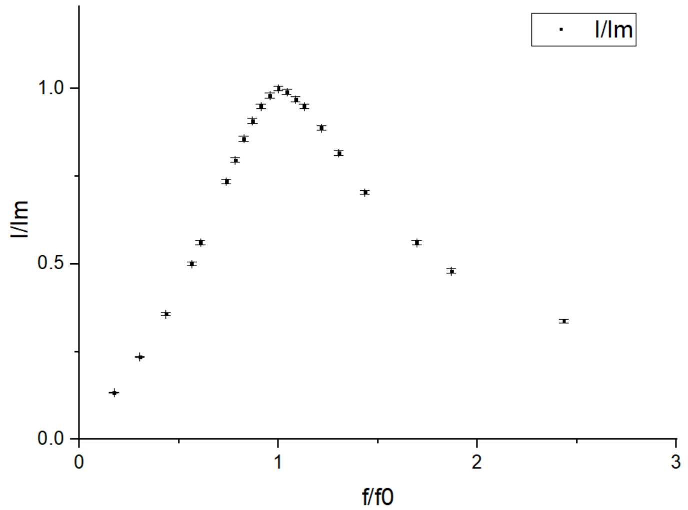

Figure 11 relation between I/I_m and f/f0

5.4.2 Phase Shift

$$\varphi_{theo} = tan^{-1}(\frac{2\pi f L - \frac{1}{2\pi f C}}{R})$$

$$\varphi_{exp} = cos^{-1}(\frac{U_R}{U_m})$$

Take the first row of data for example:

$$\varphi_{theo} = tan^{-1}(\frac{2\pi f L - \frac{1}{2\pi f C}}{R}) = tan^{-1}(\frac{2\pi \times 400 \times 0.01 - \frac{1}{2\pi \times 400 \times 460.5 \times 10^{-9}}}{99.96})$$

$$= -1.45 \pm 3 \times 10^{-5}[rad]$$

$$\varphi_{exp} = cos^{-1}(\frac{U_R}{U_m}) = cos^{-1}(\frac{0.520}{3.92}) = 1.44 \pm 0.0009[\text{rad}]$$

Then we calculate the result of all the data, arranged in table 4:

Caution: for data 1-11, we add a negative sign in front of φ_{exp}.

	$\frac{f}{f_0}$	$u_{\frac{f}{f_0}}$	φ_{theo}	$u_{\varphi_{theo}}$	φ_{exp}	$u_{\varphi_{exp}}$
1	0.174	4×10^{-7}	-1.45	3×10^{-5}	-1.44	0.0009
2	0.304	5×10^{-7}	-1.35	5×10^{-5}	-1.33	0.001
3	0.435	5×10^{-7}	-1.23	9×10^{-5}	-1.21	0.006
4	0.565	5×10^{-7}	-1.07	1×10^{-4}	-1.05	0.007
5	0.609	5×10^{-7}	-1.01	2×10^{-4}	-0.97	0.007
6	0.739	5×10^{-7}	-0.77	2×10^{-4}	-0.75	0.009
7	0.783	6×10^{-7}	-0.67	3×10^{-4}	-0.65	0.01
8	0.826	6×10^{-7}	-0.56	3×10^{-4}	-0.54	0.01
9	0.870	6×10^{-7}	-0.44	3×10^{-4}	-0.43	0.02
10	0.913	6×10^{-7}	-0.32	3×10^{-4}	-0.32	0.02
11	0.957	6×10^{-7}	-0.19	3×10^{-4}	-0.20	0.04
12	1.000	6×10^{-7}	-0.06	3×10^{-4}	0.00	/
13	1.043	6×10^{-7}	0.07	3×10^{-4}	0.14	0.05
14	1.087	6×10^{-7}	0.19	3×10^{-4}	0.25	0.03
15	1.130	7×10^{-7}	0.30	3×10^{-4}	0.32	0.02
16	1.217	7×10^{-7}	0.48	2×10^{-4}	0.48	0.01
17	1.304	7×10^{-7}	0.63	2×10^{-4}	0.62	0.01
18	1.435	8×10^{-7}	0.80	1×10^{-4}	0.79	0.009
19	1.696	9×10^{-7}	1.00	7×10^{-5}	0.97	0.007
20	1.870	9×10^{-7}	1.09	6×10^{-5}	1.07	0.006
21	2.435	1×10^{-6}	1.24	3×10^{-5}	1.23	0.006

Table4 Calculation result of phase shift

Then we use origin lab to plot the figures of $\varphi_{theo}\ vs\frac{f}{f0}and\ \varphi_{exp}\ vs\ f/f0$

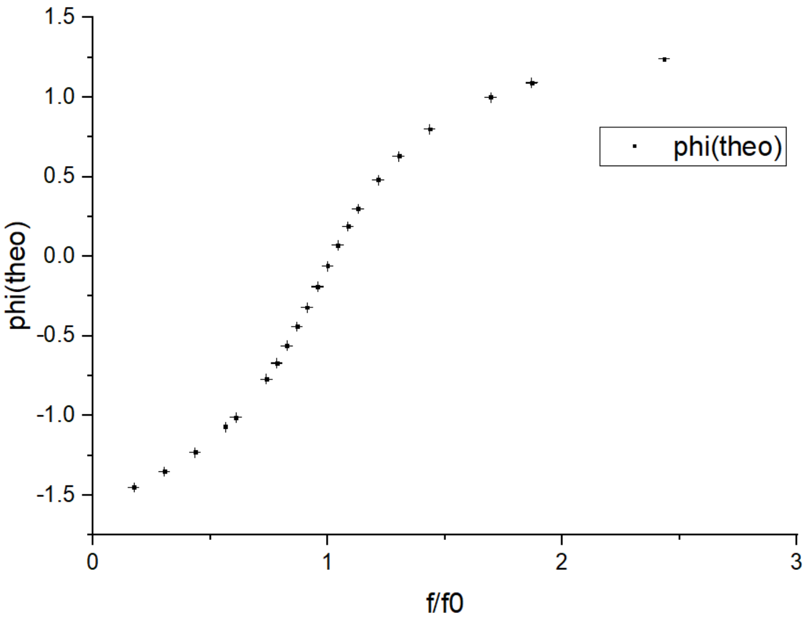

Figure 12 $\varphi_{theo}\ vs\frac{f}{f0}$

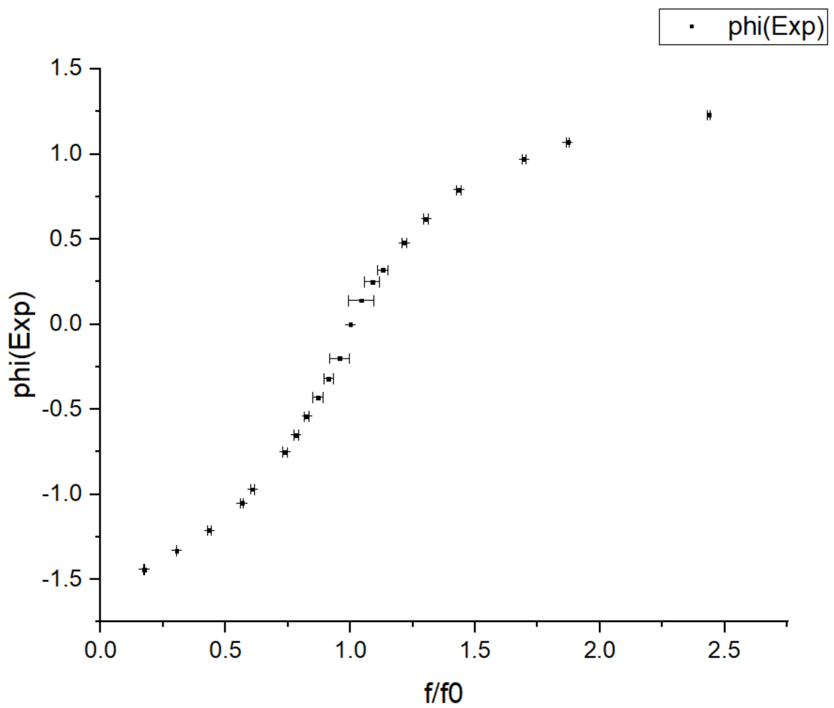

Figure 13 $\varphi_{exp}\ vs\ f/f0$

The theoretical resonate frequency

$$f_{0,theo}=\frac{1}{2\pi\sqrt{LC}}=\frac{1}{2\pi\sqrt{0.01\times 460.5\times 10^{-9}}}=2345.3[Hz]\pm 0.3[Hz]$$

The experimental resonate frequency is 2300±0.001[Hz]

The relative error of f0 is:

$$\delta = \frac{f_0 - f_{0,theo}}{f_{0,theo}} = \frac{2300.000 - 2345.3}{2345.3} = -1.93\%$$

5.5 Quality Factor

$$Q_{theo} = \frac{\sqrt{LC}}{RC} = \frac{\sqrt{0.01 \times 460.5 \times 10^{-9}}}{99.96 \times 460.5 \times 10^{-9}} = 1.472 \pm 0.0002$$

when $I(f_1) = I(f_2) = I_m/\sqrt{2}$, $U_R(f_1) = U_R(f_2) = \frac{U_m}{\sqrt{2}} = \frac{3.92}{\sqrt{2}} = 2.77Vpp$

The closet frequency is f_1=1700 f_2 =3300Hz.
Hence,

$$Q_{exp} = \frac{f_0}{f_2 - f_1} = \frac{2300}{3300 - 1700} = 1.438 \pm 1 \times 10^{-6}$$

The relative error of Q is:

$$\delta = \frac{Q_{exp} - Q_{theo}}{Q_{theo}} = \frac{1.438 - 1.472}{1.472} = -2.31\%$$

6. Conclusions and discussion

6.1 Conclusions

In this lab, we have understood the physics of alternating-current circuits, in particular the processes of charging/discharging of capacitors, the phenomenon of electromagnetic induction in inductive elements, and other dynamic processes in RC, RL, and RLC series circuits. Moreover, we have studied methods for measuring the amplitude-frequency and the phase-frequency characteristics of RC, RL, and RLC series circuits. Finally, we found the resonance frequency of an RLC circuit as well as the quality factor of the circuit.

6.1.1 RC series circuit

The time constant is found:

$$\tau_{exp} = 51.94 \pm 0.01[\mu s]$$
$$\tau_{theo} = 46.00 \pm 0.01[\mu s]$$

The relative error is 12.9%

6.1.2 RL series circuit

The time constant is found:

$$\tau_{exp} = 100.99 \pm 0.01[\mu s]$$
$$\tau_{theo} = 100.00 \pm 0.01[\mu s]$$

The relative error is 0.99%

6.1.3 RLC series circuit (critically damped)

The time constant is found:

$$\tau_{exp} = 83.33 \pm 0.06[\mu s]$$

$$\tau_{theo} = 67.86 \pm 0.007[\mu s]$$

The relative error is 22.8%

6.1.4 RLC series resonant circuit

6.1.4.1 Relationship between I and f

From Figure11 we can see that when f=f0, the circuit reaches resonance and I reaches its maximum. Moving further from f0, I becomes smaller, which matches our expectation.

6.1.4.2 Phase shift

Comparing Figure12 and Figure 13, we can see that the phase shift is increasing. When f=f0, the phase shift is 0. When f<f0, the phase shift is negative. When f>f0, the phase shift is positive. When f is very small or very large, the phase shift is close to $\pi/2$.

6.1.4.3 Resonant frequency

The resonant frequency is found at f0=2300.000$\pm$0.001Hz.Compared with the theoretical value: $f_{0,theo}$ =2345.3±0.3Hz, the relative error of f0 is -1.93%.

6.1.4.4 Quality factor

The quality factor is found:

$$Q_{theo} = 1.472 \pm 0.0002$$
$$Q_{exp} = 1.438 \pm 1 \times 10^{-6}$$

The relative error is -2.31%.

6.2 Error Analysis

- In RC and RLC series circuit, the relative error of time constant is quite large. This may due to the inaccuracy in the reading of $T_{1/2}$ because the magnitude of U cannot reach precisely at half of Um. This error might be magnified during following calculation.
- In the RLC series circuit section, it is hard to determine whether the critically-damped regime is reached by eye.
- The oscilloscope may have resistance.
- The resistance of the circuit may change because of rise of temperature.
- When calculating the experimental quality factor, we just take the closest values from $\mathrm{Im}/\sqrt{2}$ as f1 and f2 from our experimental data. This step causes huge error.

6.3 Improvements

- Make the resistance of the changeable resistor visible so that the critically-damped regime can be reached more precisely.
- Improve the accuracy of the equipment.
- Repeat the experiment of RLC resonant series to find f1 and f2 precisely.

7. Reference

[1] M. Krzyzosiak (2019). Exercise 5 - lab manual.pdf Shanghai: UMJI-SJTU.

A. Measurement uncertainty analysis

A.1 Uncertainty in RC Series Circuit

$$u_{\tau,theo} = RC$$

$$u_{\tau,theo} = \sqrt{(\frac{\partial\tau, theo}{\partial R}u_R)^2 + (\frac{\partial\tau, theo}{\partial C}u_C)^2} = \sqrt{(Cu_R)^2 + (Ru_C)^2}$$

$$= \sqrt{(460.5 \times 10^{-9} \times 0.01)^2 + (99.96 \times 0.1 \times 10^{-9})^2} = 1 \times 10^{-8}[s]$$
$$= 0.01[\mu s]$$

$$\tau_{exp} = \frac{T_{1/2}}{ln\,2}$$

$$u_{\tau,exp} = \sqrt{(\frac{\partial\tau, exp}{\partial T_{1/2}}u_{T_{1/2}})^2} = \sqrt{(\frac{1}{ln2} \times 0.01 \times 10^{-6})^2} = 1 \times 10^{-8}[s] = 0.01[[\mu s]$$

A.2 Uncertainty in RL Series Circuit

$$u_{\tau,theo} = \frac{L}{R}$$

$$u_{\tau,theo} = \sqrt{(\frac{\partial\tau, theo}{\partial R}u_R)^2 + (\frac{\partial\tau, theo}{\partial L}u_L)^2} = \sqrt{(\frac{L}{R^2}u_R)^2 + (\frac{1}{R}u_L)^2}$$

$$= \sqrt{\left(\frac{0.01}{99.96^2} \times 0.01\right)^2 + \left(\frac{1}{99.96} \times 0\right)^2} = 1 \times 10^{-8}[s] = 0.01[\mu s]$$

$$\tau_{exp} = \frac{T_{1/2}}{ln\,2}$$

$$u_{\tau,exp} = \sqrt{(\frac{\partial\tau,exp}{\partial T_{\frac{1}{2}}} u_{T_{\frac{1}{2}}})^2} = \sqrt{(\frac{1}{ln2} \times 0.01 \times 10^{-6})^2} = 1 \times 10^{-8}[s] = 0.01[[\mu s]$$

A.3 Uncertainty in RLC Series Circuit

$$\tau_{theo} = \sqrt{LC}$$

$$u_{\tau,theo} = \sqrt{(\frac{\partial\tau,theo}{\partial L} u_L)^2 + (\frac{\partial\tau,theo}{\partial C} u_C)^2} = \sqrt{(\frac{1}{2}\sqrt{\frac{L}{C}} u_C)^2}$$

$$= \sqrt{\left(\frac{1}{2} \times \sqrt{\frac{0.01}{460.5 \times 10^{-9}}} \times 0.1 \times 10^{-9}\right)^2} = 7 \times 10^{-9}[s] = 0.007[\mu s]$$

$$\tau_{exp} = \frac{T_{1/2}}{\beta t} = \frac{T_{1/2}}{1.68}$$

$$u_{\tau,exp} = \sqrt{(\frac{\partial\tau,exp}{\partial T_{\frac{1}{2}}} u_{T_{\frac{1}{2}}})^2} = \sqrt{(\frac{1}{1.68} \times 0.1 \times 10^{-6})^2} = 6 \times 10^{-8}[s] = 0.06[[\mu s]$$

A.4 Uncertainty in RLC Resonant Circuit

A.4.1 Uncertainty in Analysis of the Relationship between $\frac{I}{I_m}$ and $\frac{f}{f_0}$

$$\frac{I}{I_m} = \frac{U_R}{U_m}$$

$$\frac{f}{f_0}$$

$$u_{\frac{I}{I_m}} = \sqrt{\left(\frac{\partial\left(\frac{U_R}{U_m}\right)}{\partial U_R} \cdot u_{U_R}\right)^2 + \left(\frac{\partial\left(\frac{U_R}{U_m}\right)}{\partial U_m} \cdot u_{U_m}\right)^2} = \sqrt{\left(\frac{1}{U_m} \cdot u_{U_R}\right)^2 + \left(-\frac{U_R}{U_m^2} \cdot u_{U_m}\right)^2}$$

$$u_{\frac{f}{f_0}} = \sqrt{\left(\frac{\partial(\frac{f}{f_0})}{\partial f} \cdot u_f\right)^2 + \left(\frac{\partial(\frac{f}{f_0})}{\partial f_0} \cdot u_{f_0}\right)^2} = \sqrt{\left(\frac{1}{f_0} \cdot u_f\right)^2 + \left(-\frac{f}{f_0^2} \cdot u_{f_0}\right)^2}$$

Take the first row of data for example:

$$u_{\frac{I}{I_m}} = \sqrt{\left(\frac{1}{3.92} \times 0.002\right)^2 + \left(-\frac{0.520}{3.92^2} \cdot 0.02\right)^2} = 8 \times 10^{-4}$$

$$u_{\frac{f}{f_0}} = \sqrt{\left(\frac{1}{2300} \cdot 0.001\right)^2 + \left(-\frac{400}{2300^2} \cdot 0.001\right)^2} = 4 \times 10^{-7}$$

We calculate the result of all data, arranged in Table 5:

	U_R $[Vpp]$	u_{U_R} $[Vpp]$	f[Hz] $\pm 0.001[Hz]$	$\frac{I}{I_m}$	$u_{\frac{I}{I_m}}$	$\frac{f}{f_0}$	$u_{\frac{f}{f_0}}$
1	0.520	0.002	400.000	0.133	0.0008	0.174	4×10^{-7}
2	0.920	0.002	700.000	0.235	0.001	0.304	5×10^{-7}
3	1.40	0.02	1000.000	0.357	0.005	0.435	5×10^{-7}
4	1.96	0.02	1300.000	0.500	0.006	0.565	5×10^{-7}
5	2.20	0.02	1400.000	0.561	0.006	0.609	5×10^{-7}
6	2.88	0.02	1700.000	0.735	0.006	0.739	5×10^{-7}
7	3.12	0.02	1800.000	0.796	0.007	0.783	6×10^{-7}
8	3.36	0.02	1900.000	0.857	0.007	0.826	6×10^{-7}
9	3.56	0.02	2000.000	0.908	0.007	0.870	6×10^{-7}
10	3.72	0.02	2100.000	0.949	0.007	0.913	6×10^{-7}
11	3.84	0.02	2200.000	0.980	0.007	0.957	6×10^{-7}
12	3.92	0.02	2300.000	1.000	0.007	1.000	6×10^{-7}
13	3.88	0.02	2400.000	0.990	0.007	1.043	6×10^{-7}
14	3.80	0.02	2500.000	0.969	0.007	1.087	6×10^{-7}
15	3.72	0.02	2600.000	0.949	0.007	1.130	7×10^{-7}
16	3.48	0.02	2800.000	0.888	0.007	1.217	7×10^{-7}
17	3.20	0.02	3000.000	0.816	0.007	1.304	7×10^{-7}
18	2.76	0.02	3300.000	0.704	0.006	1.435	8×10^{-7}
19	2.20	0.02	3900.000	0.561	0.006	1.696	9×10^{-7}
20	1.88	0.02	4300.000	0.480	0.006	1.870	9×10^{-7}
21	1.32	0.02	5600.000	0.337	0.005	2.435	1×10^{-6}

Table 5 uncertainties of I/Im and f/f0

A.4.2 Uncertainty in Phase Shift

A.4.2.1 Uncertainty of φ_{theo}

$$\varphi_{theo} = tan^{-1}\left(\frac{2\pi fL - \frac{1}{2\pi fC}}{R}\right)$$

$$u_{\varphi_{theo}} = \sqrt{\left(\frac{\partial\varphi_{theo}}{\partial f} \cdot u_f\right)^2 + \left(\frac{\partial\varphi_{theo}}{\partial C} \cdot u_C\right)^2 + \left(\frac{\partial\varphi_{theo}}{\partial R} \cdot u_R\right)^2}$$

$$\frac{\partial\varphi_{theo}}{\partial f} = \frac{R\left(2\pi L + \frac{1}{2\pi f^2 C}\right)}{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2}$$

$$\frac{\partial \varphi_{theo}}{\partial C} = \frac{R}{2\pi f C^2[(2\pi f L - \frac{1}{2\pi f C})^2 + R^2]}$$

$$\frac{\partial \varphi_{theo}}{\partial R} = \frac{\frac{1}{2\pi f C} - 2\pi f L}{(2\pi f L - \frac{1}{2\pi f C})^2 + R^2}$$

Take the first row of data for example:

$$\frac{\partial \varphi_{theo}}{\partial f} == \frac{99.96 \times (2\pi \times 0.01 + \frac{1}{2\pi \times 400^2 \times 460.5 \times 10^{-9}})}{99.96^2 + (2\pi \times 400 \times 0.01 - \frac{1}{2\pi \times 400 \times 460.5 \times 10^{-9}})^2} = 3.11 \times 10^{-4}$$

$$\frac{\partial \varphi_{theo}}{\partial C} = \frac{99.96}{2\pi \times 400 \times (460.5 \times 10^{-9})^2[(2\pi \times 400 \times 0.01 - \frac{1}{2\pi \times 400 \times 460.5 \times 10^{-9}})^2 + 99.96^2]}$$

$$=262774.543$$

$$\frac{\partial \varphi_{theo}}{\partial R} = \frac{\frac{1}{2\pi \times 400 \times 460.5 \times 10^{-9}} - 2\pi \times 400 \times 0.01}{(2\pi \times 400 \times 0.01 - \frac{1}{2\pi \times 400 \times 460.5 \times 10^{-9}})^2 + 99.96^2} = 0.0011753$$

Hence,

$$u_{\varphi_{theo}} = \sqrt{(3.11 \times 10^{-4} \cdot 0.001)^2 + (262774.543 \cdot 10^{-10})^2 + (0.0011753 \cdot 0.01)^2}$$

$$= 3 \times 10^{-5}$$

A.4.2.2 Uncertainty of φ_{exp}

$$\varphi_{exp} = cos^{-1}(\frac{U_R}{U_m})$$

$$u_{\varphi_{exp}} = \sqrt{(\frac{\partial \varphi_{exp}}{\partial U_m} u_{U_m})^2 + (\frac{\partial \varphi_{exp}}{\partial U_R} u_{U_R})^2}$$

$$\frac{\partial \varphi_{exp}}{\partial U_R} = -\frac{1}{\sqrt{U_m^2 - U_R^2}}$$

$$\frac{\partial \varphi_{exp}}{\partial U_m} = \frac{1}{\sqrt{U_m^2 - U_R^2}} \times \frac{U_R}{U_m}$$

Take the first row of data for example:

$$\frac{\partial \varphi_{exp}}{\partial U_R} = -\frac{1}{\sqrt{3.92^2 - 0.520^2}} = -0.2574$$

$$\frac{\partial \varphi_{exp}}{\partial U_m} = \frac{1}{\sqrt{3.92^2 - 0.520^2}} \times \frac{0.520}{3.92} = 0.0341$$

Hence,

$$u_{\varphi_{exp}} = \sqrt{(0.0341 \times 0.02)^2 + (0.2574 \times 0.002)^2} = 0.0009$$

Then we calculate the result of all data, arranged in Table6:

	$\frac{f}{f_0}$	$u_{\frac{f}{f_0}}$	φ_{theo}	$u_{\varphi_{theo}}$	φ_{exp}	$u_{\varphi_{exp}}$
1	0.174	4×10^{-7}	-1.45	3×10^{-5}	1.44	0.0009
2	0.304	5×10^{-7}	-1.35	5×10^{-5}	1.33	0.001
3	0.435	5×10^{-7}	-1.23	9×10^{-5}	1.21	0.006
4	0.565	5×10^{-7}	-1.07	1×10^{-4}	1.05	0.007
5	0.609	5×10^{-7}	-1.01	2×10^{-4}	0.97	0.007
6	0.739	5×10^{-7}	-0.77	2×10^{-4}	0.75	0.009
7	0.783	6×10^{-7}	-0.67	3×10^{-4}	0.65	0.01
8	0.826	6×10^{-7}	-0.56	3×10^{-4}	0.54	0.01
9	0.870	6×10^{-7}	-0.44	3×10^{-4}	0.43	0.02
10	0.913	6×10^{-7}	-0.32	3×10^{-4}	0.32	0.02
11	0.957	6×10^{-7}	-0.19	3×10^{-4}	0.20	0.04
12	1.000	6×10^{-7}	-0.06	3×10^{-4}	0.00	/
13	1.043	6×10^{-7}	0.07	3×10^{-4}	0.14	0.05
14	1.087	6×10^{-7}	0.19	3×10^{-4}	0.25	0.03
15	1.130	7×10^{-7}	0.30	3×10^{-4}	0.32	0.02
16	1.217	7×10^{-7}	0.48	2×10^{-4}	0.48	0.01
17	1.304	7×10^{-7}	0.63	2×10^{-4}	0.62	0.01
18	1.435	8×10^{-7}	0.80	1×10^{-4}	0.79	0.009
19	1.696	9×10^{-7}	1.00	7×10^{-5}	0.97	0.007
20	1.870	9×10^{-7}	1.09	6×10^{-5}	1.07	0.006
21	2.435	1×10^{-6}	1.24	3×10^{-5}	1.23	0.006

Table 6 Uncertainties of φ_{exp} $\boldsymbol{and}$ φ_{theo}

A.4.2.3 Uncertainty of the theoretical f_0

$$f_{0,theo} = \frac{1}{2\pi\sqrt{LC}}$$

$$\frac{\partial f_{0,theo}}{\partial C} = \frac{-C^{-\frac{3}{2}}}{4\pi\sqrt{L}}$$

$$u_{f_{0,theo}} = \sqrt{\left(\frac{\partial f_{0,theo}}{\partial C}u_C\right)^2} = \sqrt{\left(\frac{-(460.5\times 10^{-9})^{-\frac{3}{2}}\times 10^{-10}}{4\pi\sqrt{0.01}}\right)^2} = 0.3[Hz]$$

A.4 Uncertainty of Quality Factor

$$Q_{theo} = \frac{\sqrt{LC}}{RC}$$

$$\frac{\partial Q_{theo}}{\partial R} = \frac{-\sqrt{LC}}{CR^2}$$

$$\frac{\partial Q_{theo}}{\partial C} = \frac{-\sqrt{L}}{2R}C^{-\frac{3}{2}}$$

$$u_{Q_{theo}} = \sqrt{(\frac{\partial Q_{theo}}{\partial R} u_R)^2 + (\frac{\partial Q_{theo}}{\partial C} u_C)^2}$$

$$= \sqrt{\frac{L}{C}(\frac{u_R}{R^2})^2 + \frac{L}{4C^3}(\frac{u_C}{R})^2} = \sqrt{\frac{0.01}{460.5 \times 10^{-9}}(\frac{0.01}{99.96^2})^2 + \frac{0.01}{4(460.5 \times 10^{-9})^3}(\frac{10^{-10}}{99.96})^2} = 0.0002$$

$$Q_{exp} = \frac{f_0}{f_2 - f_1}$$

$$u_{Q_{exp}} = \sqrt{(\frac{\partial Q_{exp}}{\partial f_0} \cdot u_{f_0})^2 + (\frac{\partial Q_{exp}}{\partial f_1} \cdot u_{f_1})^2 + (\frac{\partial Q_{exp}}{\partial f_2} \cdot u_{f_2})^2}$$

$$= \sqrt{(\frac{1}{f_2 - f_1} \cdot u_{f_0})^2 + (\frac{f_0}{(f_2 - f_1)^2} \cdot u_{f_1})^2 + (\frac{f_0}{(f_2 - f_1)^2} \cdot u_{f_2})^2}$$

$$= \sqrt{(\frac{1}{3300 - 1700} \cdot 0.001)^2 + (\frac{2300}{(3300 - 1700)^2} \cdot 0.001)^2 + (\frac{2300}{(3300 - 1700)^2} \cdot 0.001)^2}$$

$$= 1 \times 10^{-6}$$

UM-SJTU PHYSICS LABORATORY VP241

DATA SHEET (EXERCISE 5)

Name: 何雯欣 Student ID: 518370910117

Group: 1 Date: 2019/11/22

NOTICE. Please remember to show the data sheet to your instructor before leaving the laboratory. The data sheet will not be accepted if the data are recorded with pencil or modified by correction fluid/tape. If a mistake is made in recording a datum item, cancel the wrong value by drawing a fine line through it, record the correct value legibly, and ask your instructor to confirm the correction. Please remember to take a record of the precision of the instruments used. You are required to hand in the original data with your lab report, so please keep the data sheet properly.

R 99.96 [Ω] ± 0.01 [Ω], f 1000 [Hz] ± 0.001 [Hz], $\mathcal{E}$ 4.000 [Vpp] ± 0.001 [Vpp]	
C 460.5 [nF] ± 0.1 [nF]	$T_{1/2}$ 36.00 [μs] ± 0.01 [μs]

Table 1. $T_{1/2}$ measurement data for a RC series circuit.

R 99.96 [Ω] ± 0.01 [Ω], f 700 [Hz] ± 0.001 [Hz], $\mathcal{E}$ 4.000 [Vpp] ± 0.001 [Vpp]	
L 0.01 [H] ± 0 [H]	$T_{1/2}$ 70.00 [μs] ± 0.01 [μs]

Table 2. $T_{1/2}$ measurement data for a RL series circuit.

L 0.01 [H] ± 0 [H], C 460.5 [nF] ± 0.1 [nF], $\mathcal{E}$ 4.000 [Vpp] ± 0.001 [Vpp], f 400 [Hz] ± 0.001 [Hz]	
$\beta t = 1.68$	$T_{1/2} =$ 140.0 [μs] ± 0.1 [μs]

Table 3. $T_{1/2}$ measurement data for a critically damped RLC series circuit.

Instructor's signature:

R 99.96 [Ω] ± 0.01 [Ω], L 0.01 [H] ± 0 [H], C 40.5 [nF] ± 0.1 [nF]		
f_0 2300 [Hz] ± 0.001 [Hz], $\mathcal{E}$ 4.000 [Vpp] ± 0.001 [Vpp]		
	U_R [Vpp] ± 0.02 [Vpp]	f [Hz] ± 0.001 [Hz]
1	0.520 ±0.002Vpp	400
2	0.920	700
3	1.40	1000
4	1.96	1300
5	2.20	1400
6	2.88	1700
7	3.12	1800
8	3.36	1900
9	3.56	2000
10	3.72	2100
11	3.84	2200
12	3.92	2300
13	3.88	2400
14	3.80	2500
15	3.72	2600
16	3.48	2800
17	3.20	3000
18	2.76	3300
19	2.20	3900
20	1.88	4300
21	1.32	5600

Table 4. Measurement data for the U_R vs. f dependence for a RLC resonant circuit.

Instructor's signature: 郑

rev. 3