Vp250 Problem Set 3

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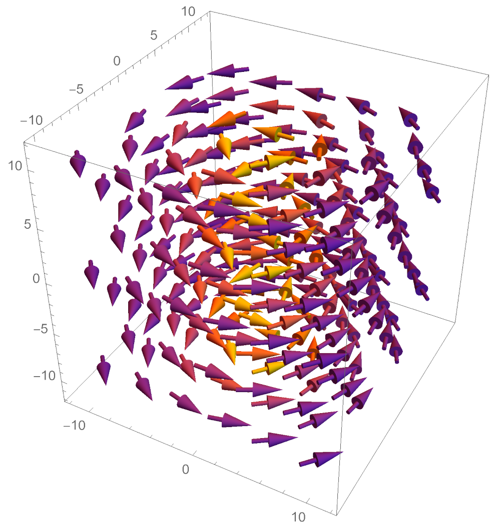
Problem 1

(a) The plot of F is shown below (where the value of A is 1)



(b)

$$rotF = \begin{vmatrix} \hat{n}_x & \hat{n}_y & \hat{n}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{Ay}{x^2+y^2} & \frac{Ax}{x^2+y^2} & 0 \end{vmatrix} = \left(\frac{A(x^2+y^2) - Ax \cdot 2x}{(x^2+y^2)^2} + \frac{A(x^2+y^2) - Ay \cdot 2y}{(x^2+y^2)^2} \right) \hat{n}_z = 0$$

where x and y cannot be 0 at the same time. So the curl is 0 at every point in the space except the z-axis.

(c) Since $x^2 + y^2 = 1$, $\bar{F} = (-Ay, Ax, 0)$

$$W = \int_\Gamma \bar{F}(r)\mathrm{d}\bar{r} = \int_0^{2\pi} < \begin{pmatrix} -A\sin\theta \\ A\cos\theta \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} > \mathrm{d}\theta = \int_0^{2\pi} A\mathrm{d}\theta = 2\pi A$$

(d) It does not contradict. Because the region is not a simply connected region. So even though $rotF$ is 0, F is not a conservative force.

Problem 2

Suppose the charge of alpha particle is $+2e$ and the charge of the electron is $-e$.
The potential energy of the alpha particle when it is at the center is

$$U_i = 4 \cdot \frac{1}{4\pi\varepsilon_0} \cdot \frac{-2e^2}{r_1} = -\frac{2e^2}{\pi\varepsilon_0 r_1}$$

where $r_1 = \dfrac{10 \cdot 10^{-9}}{\sqrt{2}} m$ The potential energy of the alpha particle when it is at the center of one side is

$$U_f = 2 \cdot \frac{1}{4\pi\varepsilon_0} \cdot \frac{-2e^2}{r_2} + 2 \cdot \frac{1}{4\pi\varepsilon_0} \cdot \frac{-2e^2}{r_3} = \frac{-e^2}{\pi\varepsilon_0} \cdot \left(\frac{1}{r_2} + \frac{1}{r_3} \right)$$

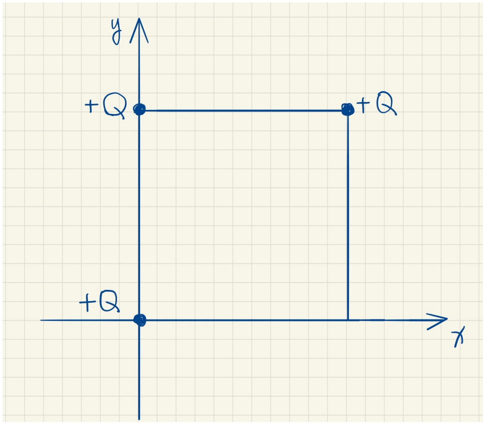
where $r_2 = \dfrac{1}{2} \cdot 10 \cdot 10^{-9}$ and $r_3 = \sqrt{\left(10 \cdot 10^{-9}\right)^2 + \left(\frac{1}{2} \cdot 10 \cdot 10^{-9}\right)^2}$ The work done to move the particle is

$$W = U_f - U_i = -\frac{-e^2}{\pi\varepsilon_0} \cdot \left(\frac{1}{r_2} + \frac{1}{r_3} \right) + \frac{2e^2}{\pi\varepsilon_0 r_1} = -0.038eV$$

Problem 3

(a) The magnitude of the the electric field induced by one of the charge to the corner is

$$E_1 = \frac{1}{4\pi\varepsilon_0}\frac{Q}{a^2} \quad E_2 = \frac{1}{4\pi\varepsilon_0}\frac{Q}{2a^2} = \frac{Q}{8\pi\varepsilon_0 a^2}$$



So the total electric field is

$$\bar{E} = (\frac{Q}{4\pi\varepsilon_0 a^2} + \frac{\sqrt{2}}{2}\frac{Q}{8\pi\varepsilon_0 a^2})\hat{n}_x + (-\frac{Q}{4\pi\varepsilon_0 a^2} - \frac{\sqrt{2}}{2}\frac{Q}{8\pi\varepsilon_0 a^2})\hat{n}_y = \frac{4+\sqrt{2}}{16}\frac{Q}{\pi\varepsilon_0 a^2}\hat{n}_x - \frac{4+\sqrt{2}}{16}\frac{Q}{\pi\varepsilon_0 a^2}\hat{n}_y$$

(b) The electric potential at that corner is

$$V = \frac{1}{4\pi\varepsilon_0}\left(2\frac{Q}{a} + \frac{Q}{\sqrt{2}a}\right) = \frac{(4+\sqrt{2})Q}{8\pi\varepsilon_0 a}$$

(c)

$$W = -Q(V_c - V_\infty) = -\frac{(4+\sqrt{2})Q^2}{8\pi\varepsilon_0 a}$$

(d)

$$U_{conf} = \frac{1}{4\pi\varepsilon_0}\left(2\frac{Q^2}{a} + \frac{Q^2}{\sqrt{2}a}\right) = \frac{(4+\sqrt{2})Q^2}{8\pi\varepsilon_0 a}$$

Problem 4

When $0 < r < a$: $\bar{E}(r) = 0$. When $a < r < b$: $\bar{E}(r) = \dfrac{k(r^2-a^2)}{2r^2\varepsilon_0}\hat{n}_r$ ($\hat{n}_r$ is the unit vector in the direction of r). When $r > b$: $\bar{E}(r) = \dfrac{k(b^2-a^2)}{2\varepsilon_0 r^2}\hat{n}_r$ ($\hat{n}_r$ is the unit vector in the direction of r).
Reference point: Infinity.

(i) $0 < r < a$:

$$V(\bar{r}) - V(\infty) = \int_{\bar{r}\to\infty} \bar{E}\mathrm{d}\bar{r} = \int_r^a E(r)\mathrm{d}r + \int_a^b E(r)\mathrm{d}r + \int_b^\infty E(r)\mathrm{d}r = 0 + \int_a^b \frac{k(r^2-a^2)}{2r^2\varepsilon_0}\mathrm{d}r + \int_b^\infty \frac{k(b^2-a^2)}{2\varepsilon_0 r^2}\mathrm{d}r$$

$$\Rightarrow V(\bar{r}) = \frac{k(b-a)}{\varepsilon_0}$$

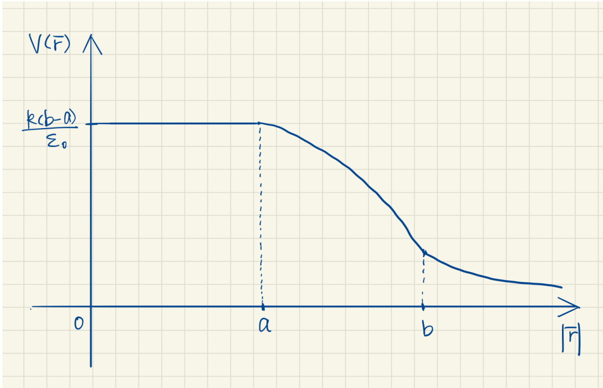
(ii) $a < r < b$:

$$V(\bar{r}) - V(\infty) = \int_{\bar{r}\to\infty} \bar{E}\mathrm{d}\bar{r} = \int_r^b E(r)\mathrm{d}r + \int_b^\infty E(r)\mathrm{d}r = \int_r^b \frac{k(r^2-a^2)}{2\varepsilon_0 r^2}\mathrm{d}r + \int_b^\infty \frac{k(b^2-a^2)}{2\varepsilon_0 r^2}\mathrm{d}r$$

$$\Rightarrow V(\bar{r}) = \frac{k}{\varepsilon_0}\left(b - \frac{r}{2} - \frac{a^2}{2r}\right)$$

(iii) $r > b$:

$$V(\bar{r}) - V(\infty) = \int_r^\infty \frac{k(b^2-a^2)}{2\varepsilon_0 r^2}\mathrm{d}r = \frac{k(b^2-a^2)}{2\varepsilon_0 r} \Rightarrow V(\bar{r}) = \frac{k(b^2-a^2)}{2\varepsilon_0 r}$$



Problem 5

Reference point: a point with distance from the wire s_0.
Gauss surface: a cylinder with side-wall parallel with the wire and height is dl and radius is s. Using Gauss's law:

$$\frac{\lambda \mathrm{d}l}{\varepsilon_0} = \int_\Sigma \bar{E}(\bar{s})\mathrm{d}\bar{A} = E(\bar{s})2\pi s\mathrm{d}l \Rightarrow \bar{E}(\bar{s}) = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{n}_s$$

(where $\hat{n}_s$ is the unit vector in the direction of $\bar{s}$)

$$V(\bar{s}) - V(s_0) = \int_{\bar{s}\to s_0} \bar{E}(\bar{r})\mathrm{d}\bar{r} = \int_{\bar{s}\to s_0} \frac{\lambda}{2\pi\varepsilon_0 s}\mathrm{d}r \Rightarrow V(\bar{s}) = \frac{\lambda}{2\pi\varepsilon_0} ln\frac{s_0}{s}$$

$$-\bigtriangledown V = \frac{\lambda}{2\pi\varepsilon_0 s} = E(\bar{s})$$

Problem 6

The charge density of the solid sphere is

$$\rho = \frac{q}{\frac{4}{3}\pi R^3} = \frac{3q}{4\pi R^3}$$

Set the reference point as infinity.
The work needed to move all the charges that can form a thin sphere with radius r from infinity is

$$U(r) - U(\infty) = -\int_r^\infty F(l)\mathrm{d}l = -\int_r^\infty \frac{1}{4\pi\varepsilon_0}\frac{\frac{4\pi r^3}{3}\rho\cdot 4\pi r^2\mathrm{d}r\rho}{l^2}\mathrm{d}l$$

(where l is the distance between the thin sphere and the origin) So $U(r) = -\frac{4\pi\rho^2}{3\varepsilon_0}r^4\mathrm{d}r$ For the whole solid sphere:

$$U_{conf} = \int_0^R U(r) = \int_0^R -\frac{4\pi\rho^2}{3\varepsilon_0}r^4\mathrm{d}r = -\frac{4\pi\rho^2 R^5}{15\varepsilon_0} = \frac{3q^2}{20\pi\varepsilon_0 R}$$