Chapter 6 – DC Circuits

UM-SJTU Joint Institute
Physics II (Fall 2020)
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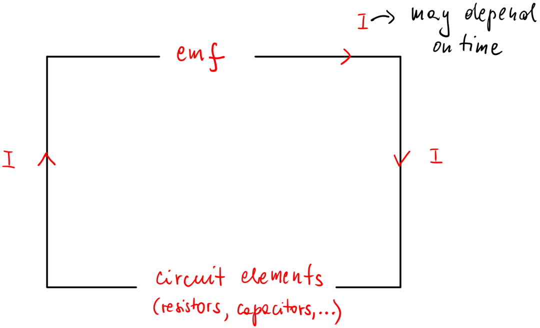
Agenda

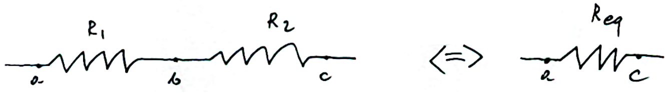
Introduction

In a DC (direct-current) circuit, the electric current flows without a change in the direction, but it's magnitude may be time-dependent.



Systems of Resistors

Series Connection



$$V_{ab} \quad + \quad V_{bc} \quad = \quad V_{ac}$$

which follows from the fact that $V = \int \overline{E}\, d\overline{r}$. The current through each resistor is I, and, from Ohm's law ($V = IR$),

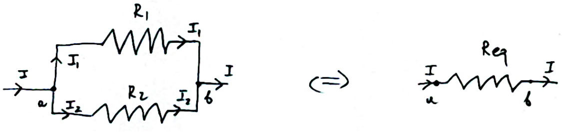
$$IR_1 \quad + \quad IR_2 \quad = \quad IR_{\text{eq}}.$$

Hence, the equivalent resistance of two resistors connected in series is

$$\boxed{R_{\text{eq}} = R_1 + R_2.}$$

Note. This result can be easily generalized to a system of n resistors connected in series, with $R_{\text{eq}} = \sum_{i=1}^{n} R_i$.

Parallel Connection



$$I_1 \quad + \quad I_2 \quad = \quad I,$$

which follows from the conservation of charge (flowing into/from the junction). From the fact that the potential drop on both resistors is equal (V_{ab}), and Ohm's law ($I = V/R$), we have

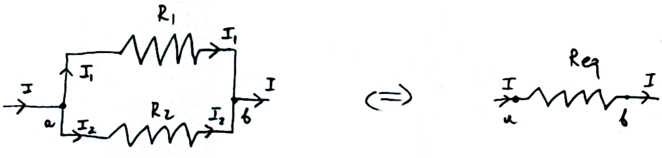
$$\frac{V_{ab}}{R_1} \quad + \quad \frac{V_{ab}}{R_2} \quad = \quad \frac{V_{ab}}{R_{\text{eq}}}.$$

Hence, the inverse of the equivalent resistance of two resistors connected in parallel is

$$\boxed{\frac{1}{R_{\text{eq}}} = \frac{1}{R_1} + \frac{1}{R_2}.}$$

Note. This result can be easily generalized to a system of n resistors connected in series, with $R_{\text{eq}}^{-1} = \sum_{i=1}^{n} R_i^{-1}$.

Resistors in Parallel. Comment



Since

$$V_{ab} = I_1 R_1,$$
$$V_{ab} = I_2 R_2,$$

hence

$$\frac{I_2}{I_1} = \frac{R_1}{R_2}.$$

That is, most of the current flows through the resistor of the least resistance.

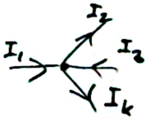
Kirchhoff's Rules

Kirchhoff's Rules (I). Junction Rule

Junction Rule

The algebraic sum of the currents into any junction is zero,

$$\sum_k I_k = 0.$$



Justification

Follows directly form the conservation of the electric charge: the algebraic sum of electric charges flowing through the n wires in and out of a junction at any time interval is constant. Hence,

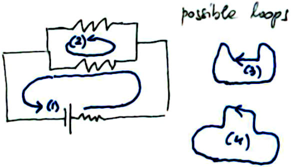
$$\sum_{k=1}^{n} Q_k = \text{const} \qquad \Longrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=1}^{n} Q_k = \sum_{k=1}^{n}\frac{\mathrm{d}Q_k}{\mathrm{d}t} = \sum_{k=1}^{n} I_k = 0$$

Kirchhoff's Rules (II). Loop Rule

Loop Rule

The algebraic sum of the potential differences in any loop (e.g., potential differences across emfs and resistors) is zero,

$$\sum_k V_k = 0.$$



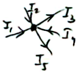
Justification

Recall that the electric force is conservative, hence $\int_{A\to B} \overline{E}\,\mathrm{d}\overline{r}$ does not depend on the path, or, alternatively, $\oint \overline{E}\,\mathrm{d}\overline{r} = 0$.
Consequently,

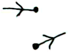
$$\begin{aligned} 0 = \oint \overline{E}\,\mathrm{d}\overline{r} &= \int\limits_{A_1\to A_2} \overline{E}\,\mathrm{d}\overline{r} + \int\limits_{A_2\to A_3} \overline{E}\,\mathrm{d}\overline{r} + \cdots + \int\limits_{A_n\to A_1} \overline{E}\,\mathrm{d}\overline{r} = \\ &= V_{A_1A_2} + V_{A_2A_3} + \cdots + V_{A_nA_1} \end{aligned}$$

Kirchhoff's Rules. Sign Convention

1. Junction Rule



$I_1 + I_2 - I_3 + I_4 - I_5 = 0$

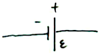


"plus"

"minus"

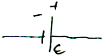
2. Loop Rule

emfs



loop → direction of the loop

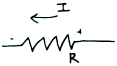
$+\varepsilon$



← loop

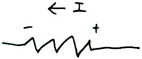
$-\varepsilon$

resistors



loop → opposite to current

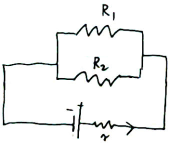
$+IR$



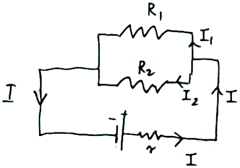
← loop in current direction

$-IR$

Write down equations implied by the junction and the loop rules for the circuit shown in the figure.

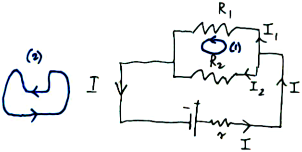


- junction rule



$$I - I_1 - I_2 = 0$$

- loop rule



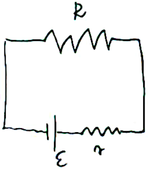
loop (1): $-I_1 R_1 + I_2 R_2 = 0$

loop (2): $-Ir - I_2 R_2 + \mathcal{E} = 0$

The number of (independent) equations is equal to the number of unknowns, hence the system of equations can be solved for I_1, I_2, I.

Example 1

Find the value of the resistance R, such that the power dissipated on the resistor R is maximum. The emf $\mathcal{E}$ and the internal resistance r are given.



From the loop rule

$$-Ir - IR + \mathcal{E} = 0 \qquad \Longrightarrow \qquad I = \frac{\mathcal{E}}{R+r}.$$

Hence the power dissipated on the external resistance

$$P = I^2 R = \frac{\mathcal{E}^2}{(R+r)^2} R.$$

Now, find the maximum of $P = P(R)$.

$$\frac{\mathrm{d}P}{\mathrm{d}R} = \mathcal{E}^2 \left[\frac{1}{(R+r)^2} - \frac{2R}{(R+r)^3} \right] = 0.$$

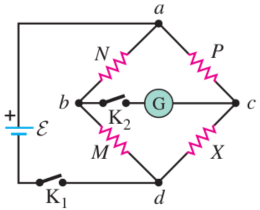
Hence

$$2R = R + r \qquad \Longrightarrow \qquad \boxed{R = r}.$$

Formally, need to check that it is indeed a maximum; or look at the graph $P(R)$.

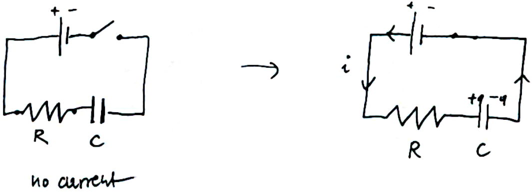
Example 2

Wheatstone bridge (see problem set)



RC Direct-Current Circuit

RC Circuit



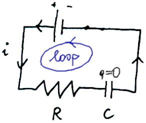
The current is dependent on time, but still flows in one direction. We will assume an ideal emf ($\mathcal{E} = \text{const}, \; r = 0$).

Note. Symbols in lowercase will be used to indicate time dependence, e.g. the current $i = i(t)$, the electric charge on the capacitor $q = q(t)$, and the voltage across the capacitor $v_{ab} = v_{ab}(t)$.

(1) Charging the Capacitor

Initial state: At $t = 0$, we have $q(0) = 0$ and $i(0) = I_0$.

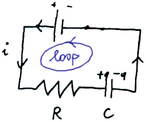
Kirchhoffs loop rule



$$-\underbrace{I_0 R}_{v_{ab}(0)} + \mathcal{E} = 0 \implies \boxed{I_0 = \frac{\mathcal{E}}{R}}$$

For $t > 0$

Kirchhoffs loop rule



$$-iR - \frac{q}{C} + \mathcal{E} = 0 \implies i = \frac{\mathcal{E}}{R} - \frac{q}{RC}$$

But $i = dq/dt$, and

$$\frac{dq}{dt} = \frac{\mathcal{E}}{R} - \frac{q}{RC}$$

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\mathcal{E}}{R} - \frac{q}{RC}$$

It is an example of a 1^{st} order ordinary differential equation (ODE) with separable variables. Solution strategy: separate variables and integrate

$$\int_0^{q(t)} \frac{\mathrm{d}q}{q - C\mathcal{E}} = -\int_0^t \frac{\mathrm{d}t}{RC}$$

$$\ln\left|\frac{q(t) - C\mathcal{E}}{-C\mathcal{E}}\right| = -\frac{t}{RC}$$

But $q(t) < C\mathcal{E} = Q_{\mathrm{max}}$, hence

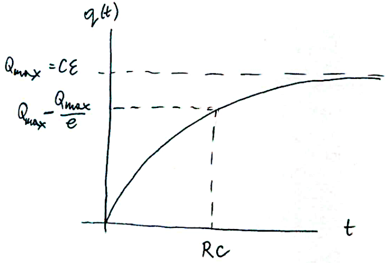
$$\ln\frac{C\mathcal{E} - q(t)}{C\mathcal{E}} = -\frac{t}{RC}.$$

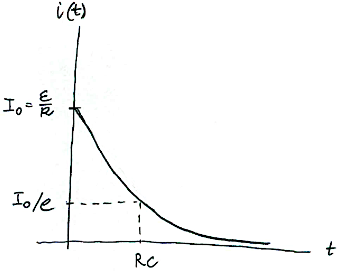
Solving for $q(t)$ yields

$$\boxed{q(t) = C\mathcal{E}\left(1 - e^{-t/RC}\right).}$$

The corresponding current

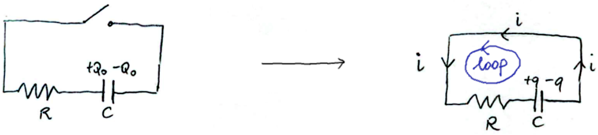
$$\boxed{i(t) = \frac{\mathrm{d}q(t)}{\mathrm{d}t} = \frac{\mathcal{E}}{R} e^{-t/RC} = I_0 e^{-t/RC}.}$$





$RC = \tau$ is the *time constant* for the RC circuit (units [s]).

(2) Discharging the Capacitor



Kirchhoffs loop rule yields $-iR - \frac{q}{C} = 0$, hence $i = -\frac{q}{RC}$.

Initial state: At $t = 0$, we have $q(0) = Q_0$ and $i(0) = -\frac{Q_0}{RC} = I_0$.

For $t > 0$

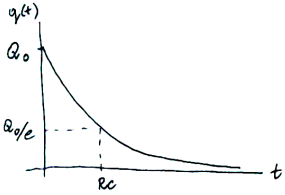
$$i = -\frac{q}{RC} \quad \Longrightarrow \quad \frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{q}{RC} \quad \Longrightarrow \quad \int_{Q_0}^{q(t)} \frac{\mathrm{d}q}{q} = -\int_0^t \frac{\mathrm{d}t}{RC}$$

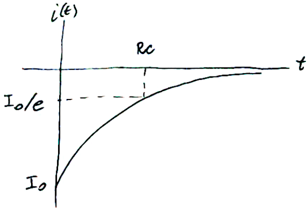
Hence

$$\ln \frac{q(t)}{Q_0} = -\frac{t}{RC} \quad \Longrightarrow \quad \boxed{q(t) = Q_0 e^{-t/RC}}$$

The corresponding current

$$\boxed{i(t) = -\frac{q}{RC} = -\frac{Q_0}{RC} e^{-t/RC} = I_0 e^{-t/RC}.}$$





Comment. Energy in the Charging Process

$$-iR - \frac{q}{C} + \mathcal{E} = 0 \quad / \cdot i$$

$$-i^2R - \frac{iq}{C} + \mathcal{E}i = 0$$

power dissipated in resistor — part stored in capacitor — power supplied by battery

Total energy

- supplied by the battery

$$E_{batt} = \int_0^\infty \mathcal{E}\, i\, dt = I_0 \mathcal{E} \int_0^\infty e^{-t/RC}\, dt = I_0 RC\mathcal{E} \overset{I_0 = \frac{\mathcal{E}}{R}}{=} C\mathcal{E}^2$$

- dissipated in the resistor

$$E_{res} = \int_0^\infty i^2 R\, dt = I_0^2 R \int_0^\infty e^{-2t/RC}\, dt = \frac{C\mathcal{E}^2}{2}$$

- stored in the capacitor

$$E_{cap} = \int_0^\infty \frac{iq}{C} = I_0 C\mathcal{E} \int_0^\infty \left(e^{-t/RC} - e^{-2t/RC}\right) = \frac{C\mathcal{E}^2}{2}$$

Note. The energy supplied splits half–half between the energy stored in the capacitor and the energy dissipated in the resistor.