Chapter 2 – Gauss's Law for the Electric Field

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Agenda

1. Flux of Electric Field (Electric Flux)
   - Motivation and Intuitions
   - Definition and Properties
   - Example
2. Gauss's Law
3. Examples and Applications
   - Finding Electric Field Using Gauss's Law
   - Properties of Conductors (I)

Flux of Electric Field (Electric Flux)

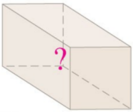
Motivation and Intuitions

Motivation

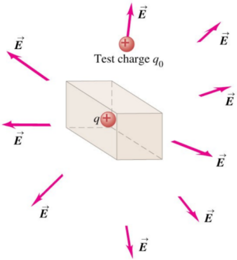
charge distribution $\xrightarrow[\text{\& superposition principle}]{\text{Coulomb's law}}$ electric field

electric field $\xrightarrow{\;?\;}$ charge distribution

(a) A box containing an unknown amount of charge

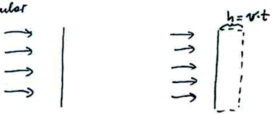


(b) Using a test charge outside the box to probe the amount of charge inside the box



Intuition. Water Flux

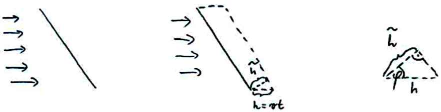
perpendicular



A – cross section

$$\text{volume} = h\cdot A = v\cdot t\cdot A$$

tilted

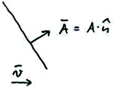


$\tilde{h} = h\cdot\cos\varphi$

$$\text{volume} = \tilde{h}\cdot A = h\cos\varphi\, A = vt\cos\varphi\cdot A$$

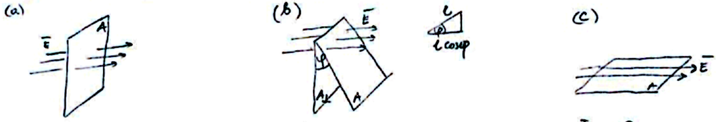
Flow rate

$$\boxed{\frac{\text{volume}}{\text{time}} = vA\cos\varphi = \overline{v}\circ\overline{A}}$$



$\hat{n}$ – normal to the surface

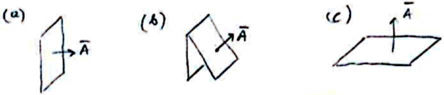
Flux of a Uniform Electric Field Through a Plane



(a) $\Phi_E = EA$ (b) $\Phi_E = EA_\perp = EA\cos\varphi$ (c) $\Phi_E = 0$

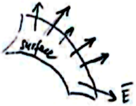
Vector area $\overline{A} = A\hat{n}$

$\hat{n}$ for closed surfaces
↳ points outward



$$\boxed{\Phi_E = \overline{A} \cdot \overline{E}}$$ dot product

Definition and Properties



curved surface
$\bar E$ non uniform

$d\bar A = \hat n\, dA$

$\bar E$ can be treated as uniform over dA

$$d\Phi_E = \bar E\, d\bar A$$

Adding all contributions, the total flux of the electric field through the *surface*

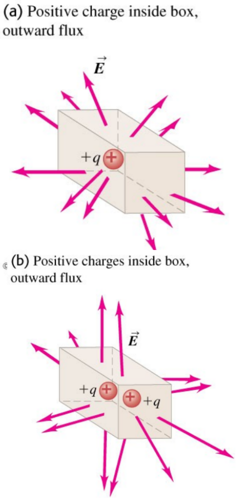
$$\boxed{\Phi_E = \int_{\text{surface}} \overline{E} \circ \mathrm{d}\overline{A}} = \int_{\text{surface}} \overline{E} \circ \hat{n}\, \mathrm{d}A = \int_{\text{surface}} E_\perp\, \mathrm{d}A$$

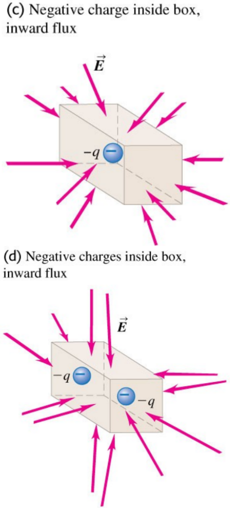
Comment. Electric flux through closed surface

$$\Phi_E = \oint_\Sigma \bar E \cdot d\bar A = \oint_\Sigma \bar E \cdot \hat n\, dA$$

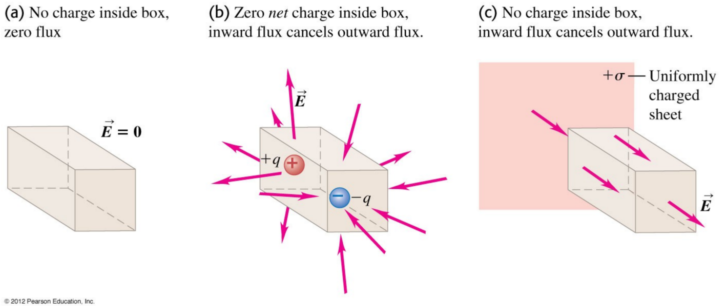
Closed surface

Electric Flux Through Closed Surface. Observations





Electric Flux Through Closed Surface. Zero Flux



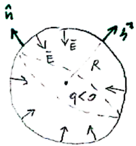
Observations

net charge inside	flux
(+)	outward (positive)
(−)	inward (negative)
0	none (zero)

Charges outside of the closed surface do not contribute to the flux. (The flux depends on the net charge enclosed by the surface.)

Example

Find the electric flux through a sphere with radius R and a point charge $q < 0$ placed at its center.



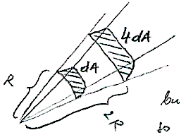
$$\bar{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\hat{n} \qquad \text{electric field on the sphere}$$

$$\Phi_E = \oint_\Sigma \bar{E}\, d\bar{A} = \oint_\Sigma \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\overbrace{\hat{n}\cdot\hat{n}}^{=1}\, dA = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\underbrace{\oint_\Sigma dA}_{=4\pi R^2} = \frac{q}{\varepsilon_0}$$

$$\boxed{\Phi_E = \frac{q}{\varepsilon_0}}$$

Observations

- The flux does not depend on the radius of the sphere.



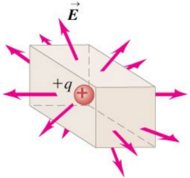
but $E(r) \sim \frac{1}{r^2}$

so $E(2R) = \frac{1}{4}E(R)$

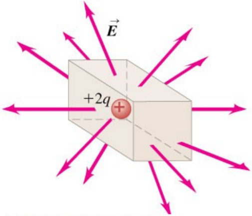
- The flux is proportional to the charge enclosed by the sphere.

Scaling Properties. Valid for Any Surface?

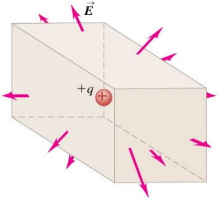
(a) A box containing a charge



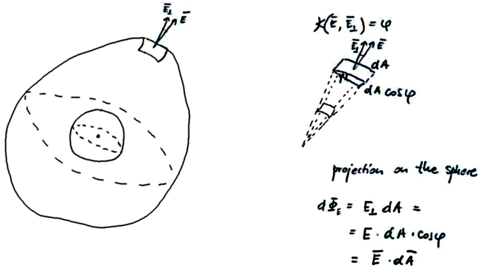
(b) Doubling the enclosed charge doubles the flux.



(c) Doubling the box dimensions *does not change* the flux.



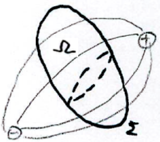
Flux Through a Non-Spherical Closed Surface (Single Charge Inside)



divide the non-spherical surface into small elements $\mathrm{d}A \longrightarrow$ project them onto a spherical surface $\longrightarrow$ calculate the contribution to the flux $\longrightarrow$ add all contributions

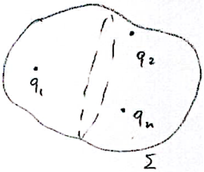
$$\oint_{\text{non-spherical surface}} \overline{E} \circ \mathrm{d}\overline{A} = \frac{q}{\varepsilon_0}.$$

Surface enclosing no charge



$$\oint_{\Sigma} \bar{E}\, d\bar{A} \Rightarrow \Phi_E = 0$$

Surface enclosing multiple point charges



$$\Phi_E = \oint_{\Sigma} \bar{E} \cdot d\bar{A} = \frac{q_1}{\varepsilon_0} + \frac{q_2}{\varepsilon_0} + \ldots + \frac{q_n}{\varepsilon_0} = \frac{q_{encl}}{\varepsilon_0}$$

↳ total field (use superposition principle)

$$\Phi_E = \frac{q_{encl}}{\varepsilon_0}$$

→ net charge enclosed by Σ

Gauss's Law

Gauss's Law (Integral Form)

Gauss's Law for the Electric Field

The flux of the electric field through any closed surface Σ is equal to $q_{\mathrm{encl}}/\varepsilon_0$, where q_{encl} is the net electric charge enclosed by the surface Σ

$$\boxed{\oint_{\Sigma} \overline{E} \circ \mathrm{d}\overline{A} = \frac{q_{\mathrm{encl}}}{\varepsilon_0}.}$$

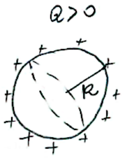
Note. The surface Σ is often referred to as the *Gaussian surface*.
$\overline{E}$ — total electric field.

total electric field $\xrightarrow{\text{Gauss's law}}$ charge distribution

Examples and Applications

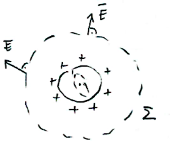
Finding Electric Field Using Gauss's Law

Example 1. Uniformly Charged Sphere



Gaussian surface Σ – sphere of radius r with the center at the center of charged sphere

Case 1. $r > R$



$\bar{E} = E(r)\,\hat{n}$

spherical symmetry

$$\Phi_E = \oint_\Sigma \bar{E}\, d\bar{A} = \oint_\Sigma E(r)\,\hat{n}\cdot\hat{n}\, dA = \oint_\Sigma E(r)\, dA =$$

$$= E(r) \oint_\Sigma dA = E(r)\, 4\pi r^2$$

Gauss's law

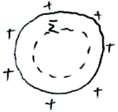
$(Q_{enc} = Q)$ $\qquad E(r)\, 4\pi r^2 = \dfrac{Q}{\varepsilon_0} \quad \Rightarrow \quad E(r) = \dfrac{Q}{4\pi\varepsilon_0 r^2}$

$$\bar{E}(\bar{r}) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}\,\hat{n} = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2}\,\hat{r}$$

$\hat{r} = \hat{n}$ (points outward)

Note. Same result as for a point charge Q placed at the center of the sphere.

Case 2. $r < R$



Gaussian surface Σ – sphere of radius r with the center at the center of charged sphere

$$\Phi_E = \oint_{\Sigma} \bar{E}\, d\bar{A} = 4\pi r^2 E(r)$$

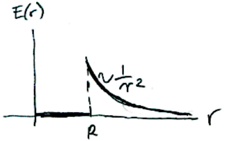
as above $\bar{E}$ has to have spherical symmetry (may be trivial)

$Q_{encl} = 0$

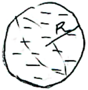
so

$$\oint_{\Sigma} \bar{E}\, d\bar{A} = 0 \iff \boxed{\bar{E} \equiv 0} \quad \text{for } r < R$$

Summary

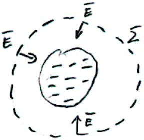


Example 2. Insulating, Uniformly Charged Ball ($Q < 0$)



Gaussian surface Σ – sphere of radius r with the center at the center of charged ball

Case 1. $r > R$



Σ - Gaussian surface

$\bar{E} = E(r)\,\hat{n} \parallel d\bar{A}$ spherical symmetry (here $E(r) < 0$)

$\Phi_E = \oint_\Sigma \bar{E} \cdot d\bar{A} = \ldots$ as in ex. (1) $\ldots = E(r)\, 4\pi r^2$

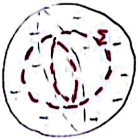
$Q_{encl} = Q$

From the Gauss's Law $E(r) \cdot 4\pi r^2 = \dfrac{Q}{\varepsilon_0} \Rightarrow E(r) = \dfrac{1}{4\pi\varepsilon_0} \dfrac{Q}{r^2}$ (< 0)

$$\boxed{\bar{E}(\bar{r}) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2}\, \hat{r}}$$ for $r > R$ (points inwards)

Note. Same result as for a point charge Q placed at the center of the ball.

Case 2. $r < R$



Σ - Gaussian surface $(r < R)$

again $\bar{E}$ is spherically symmetric

$$\bar{E} = E(r)\,\hat{n} \qquad (\hat{n} = \hat{r})$$

$$\Phi_E = \oint_\Sigma \bar{E}\, d\bar{A} = \ldots = E(r)\, 4\pi r^2$$

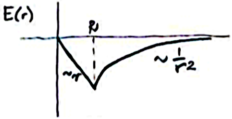
but now Σ encloses only a fraction of charge

$$Q_{encl} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}\, Q = \frac{r^3}{R^3}\, Q$$

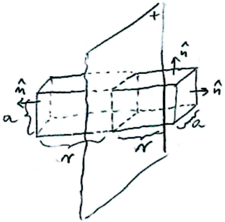
Gauss's law

$$E(r)\, 4\pi r^2 = \frac{Q}{\varepsilon_0} \frac{r^3}{R^3}$$

$$E(r) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{R^3}\, r \quad \Rightarrow \quad \boxed{\bar{E}(r) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{R^3}\, r\, \hat{r}}$$



Example 3. Infinite Plane, Charged Uniformly ($\sigma > 0$)



$\overline{E}$ – perpendicular to the plane (symmetry), pointing away from plane

Gaussian surface

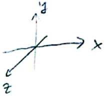
$$\Sigma = \text{left base} \ \cup \ \text{side walls} \ \cup \ \text{right base}$$

$$\Phi_E = \oint_{\Sigma} \bar{E}\, d\bar{A} = \int_{\substack{\text{left}\\ \text{base}}} \bar{E}\, d\bar{A} + \underbrace{\int_{\substack{\text{side}\\ \text{walls}}} \bar{E}\, d\bar{A}}_{=0,\ \bar{E}\perp\hat{n}} + \int_{\substack{\text{right}\\ \text{base}}} \bar{E}\, d\bar{A}$$

$$= \dots = E(r)\int_{\text{l. base}} dA + 0 + E(r)\int_{\text{r. base}} dA = 2\cdot E(r)\, a^2$$

Gauss's Law

$$\Phi_E = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow 2E(r)\,a^2 = \frac{\sigma a^2}{\varepsilon_0} \Rightarrow E(r) = \frac{\sigma}{2\varepsilon_0}$$



$$\boxed{\bar{E}(\bar{r}) = \begin{cases} \dfrac{\sigma}{2\varepsilon_0}\,\hat{n}_x & \text{for } x>0 \\ -\dfrac{\sigma}{2\varepsilon_0}\,\hat{n}_x & \text{for } x<0 \end{cases}}$$

Comments

- Uniform electric field in each of the two regions of space.
- Compare with the result obtained using the superposition principle.

Example 4. Non-uniformly charged object (see problem set)

$\rho = \rho(r)$

$\downarrow$

cylindrical symmetry

e.g. inside

$Q_{encl} = \int_{\Omega} \rho(r)\, dV$

$\Omega \hookrightarrow$ solid enclosed by surface Σ

$\oint_{\Sigma} \bar{E}\, d\bar{A}$ - only the lateral surface contributes

General comment: The symmetry of a Gaussian surface should be compatible with the symmetry of the electric field. Then the flux can be easily found.

Properties of Conductors (I)

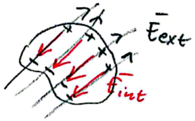
General Fact. Under electrostatic conditions, the electric field at every point inside a conductor is zero.

Justification ("reductio ad absurdum")

assume $\bar{E} \neq 0$ inside $\xrightarrow[\text{in the conductor}]{\text{mobile charges}}$ moving charges $\Rightarrow$ NO ELECTROSTATICS

× CONTRADICTION

Illustration (neutral, i.e. uncharged, conductor in an external electric field)



$$\bar{E}_{ext} + \bar{E}_{int} = 0$$

Property 1. Charge Distribution On a Solid Conductor

Property 1

Any excess charge, placed on a solid conductor, resides entirely on the conductor's surface.

$\overline{E}$ inside is zero, hence Φ_E through any Σ is zero.

$\Downarrow$ (Gauss's Law)

Charge enclosed by Σ is zero.

$\Downarrow$ (shrink Σ to a point)

Charge at that point is zero.

This can be repeated for any surface Σ enclosing any point inside the conductor.

Property 2a. Charge Distribution on Hollow Conductor

Property 2a (hollow conductor with empty cavity)

For a hollow charged conductor with an empty cavity, there is no net charge on the cavity's surface.

$\overline{E}$ everywhere on the Gaussian surface is zero, hence Φ_E through it is zero.

$\Downarrow$ (Gauss's Law)

Charge enclosed by Σ is zero.

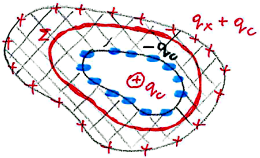
$\Downarrow$ (shrink Σ to fit the cavity's shape)

Net charge on the empty cavity's surface is zero.

Property 2b. Charge Distribution on Hollow Conductor

Property 2b (cavity with a charge inside)

For a hollow charged (q_x) conductor with a cavity and a point charge q_c inside the cavity, the net charge on the cavity's surface is $-q_c$ and the charge on the conductor's surface is $q_x + q_c$.



Again, $\overline{E}$ everywhere on the Gaussian surface is zero, hence Φ_E through it is zero.

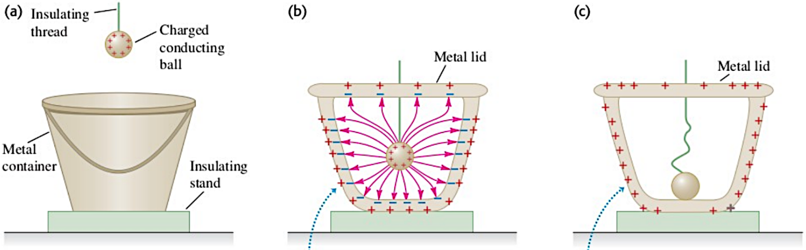
$\Downarrow$ (Gauss's Law)

Charge enclosed by Σ is zero.

$\Downarrow$ (shrink Σ to fit the cavity's shape)

There is charge $-q_c$ appearing on the surface of the cavity, so that the charge on the conductor's outer surface is $q_x + q_c$.

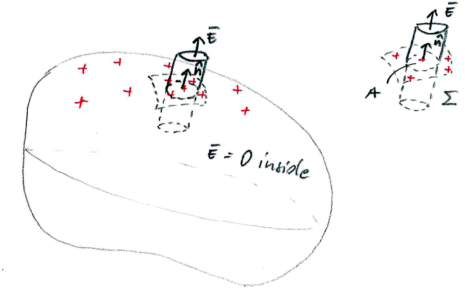
Example. Faraday's Ice-pail Experiment



Charged ball induces charges on the interior and exterior of the container.

Once the ball touches the container, it is part of the interior surface; all the charge moves to the container's exterior.

Property 3. Electric Field at Charged Conductor's Surface



Fact 1. $\overline{E}$ has to be normal to the conductor's surface, otherwise the charges would be moving.

Fact 2. $|\overline{E}|$ at a given point of the surface is directly proportional to the surface charge density at that point.

Use Gauss law with Σ

$$\underbrace{AE}_{\Phi_E} = \underbrace{\frac{\sigma A}{\varepsilon_0}}_{\frac{Q_{encl}}{\varepsilon_0}} \Rightarrow \boxed{\bar{E} = \frac{\sigma}{\varepsilon_0}\hat{n}}$$