Chapter 3 – Electric Potential

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Agenda

Quick Review: Potential Vector Fields

Quick Review: Potential Vector Fields

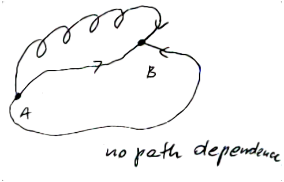
For a potential (conservative) field force field $\overline{F} = \overline{F}(\overline{r})$, the elementary work done by the force on a particle

$$\delta W = -\mathrm{d}U,$$

where U is the potential energy corresponding to the force field.

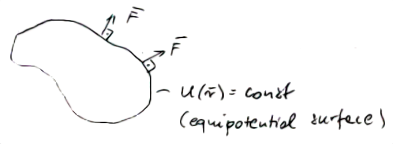
Implications

1. $W_{A\to B} = \int_{\Gamma_{AB}} \overline{F} \circ \mathrm{d}\overline{r} = U(A) - U(B)$



2. Follows from (1): $\oint_\Gamma \overline{F} \circ \mathrm{d}\overline{r} = 0$ for any closed path (loop) Γ.

3. $$\left.\begin{aligned}\delta W &= -\mathrm{d}U = -\nabla U \circ \mathrm{d}\bar{r}\\ \delta W &= \overline{F} \circ \mathrm{d}\bar{r}\end{aligned}\right\} \Longrightarrow \boxed{\overline{F} = -\nabla U = -\mathrm{grad}\, U}$$



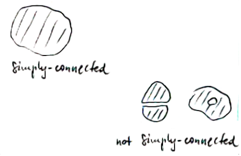
4. Conservation of energy

$$\left.\begin{aligned}\delta W &= -\mathrm{d}U\\ \delta W &= \mathrm{d}K\\ &\text{(work-k.e. thm)}\end{aligned}\right\} \Longrightarrow \mathrm{d}(U+K) = 0 \Longrightarrow \boxed{U + K = \text{const}}$$

Is a Given Force Field Conservative? Useful Criterion

Criterion

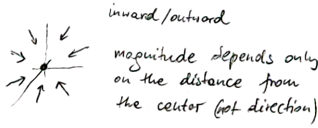
$\overline{F}$ is conservative in a simply connected region $\Omega \iff \operatorname{rot} \overline{F} \equiv 0$ throughout Ω.



Example. *Central force.*

A force $\overline{F}$ is called *central*, if it is of the form

$$\boxed{\overline{F}(\overline{r}) = f(r)\overline{r}.}$$



Examples

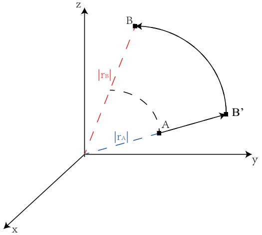
- gravitational force $f(r) = -G\frac{Mm}{r^3}$,
- Coulomb force $f(r) = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^3}$.

To show that a central force is conservative, use the rot $\overline{F}$ criterion (exercise).

Potential Energy for Coulomb Force

Potential Energy for Central Force

Since central forces are conservative, their work does not depend on path. So we will choose the simplest one to deal with in order to find the potential energy function.



Split the entire path up into two parts

- $\Gamma_{AB'}$ (along the radial direction). Here $\overline{F} \parallel \mathrm{d}\overline{r}$,
- $\Gamma_{B'B}$ (along the arc of a circle). Here $\overline{F} \perp \mathrm{d}\overline{r}$.

$$\begin{aligned} W_{A\to B} &= U(A) - U(B) = \int\limits_{\Gamma_{AB'}} \overline{F} \circ \mathrm{d}\overline{r} + \int\limits_{\Gamma_{B'B}} \overline{F} \circ \mathrm{d}\overline{r} \\ &= \int\limits_{\Gamma_{AB'}} f(r)\, r\, \mathrm{d}r \;+\; 0 = \int\limits_{|\overline{r}_A|}^{|\overline{r}_B|} f(r)\, r\, \mathrm{d}r. \end{aligned}$$

Potential Energy for Coulomb Force

In particular, for a point charge q_0 in the force field of another point charge q. we have $f(r) = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^3}$ and

$$U(A) - U(B) = \int_{r_A}^{r_B} \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^2}\,\mathrm{d}r = \frac{qq_0}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right).$$

Only the difference $U(A) - U(B)$ is measurable (*i.e.* physical). The potential energy function itself

$$\boxed{U(r) = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r} + C}$$

is determined up to an additive constant (the *gauge of the potential energy* may be chosen arbitrarily).

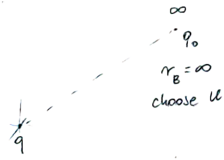
Comments. Interpretation and Choice of Gauge

Interpretation (two possible viewpoints)

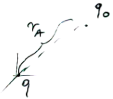
1. $$U_A - U_B = W_{A\to B} = \int_{A\to B} \overline{F} \circ \mathrm{d}\overline{r}$$
   Work done by the Coulomb force on a charge moving from A to B.

2. $$U_A - U_B = \int_{B\to A} \overline{F}_{\mathrm{ext}} \circ \mathrm{d}\overline{r} = \int_{B\to A} (-\overline{F}) \circ \mathrm{d}\overline{r} = \int_{A\to B} \overline{F} \circ \mathrm{d}\overline{r}$$
   Work done by an external force on a charge moving from B to A.

Choice of the gauge (one of infinitely many possible)



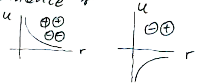
choose $U(r_B) = 0$



$U_A = \frac{qq_0}{4\pi\varepsilon_0}\frac{1}{r_A}$

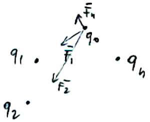
$$\boxed{U = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r}}$$

electric potential energy of two point charges at a distance r



Comments. Generalization to Several Point Charges

($q_1, \dots, q_n$ - static charge configuration)



Superposition principle

$$\bar{F} = \bar{F}_1 + \bar{F}_2 + \dots + \bar{F}_n$$

$$U_A - U_B = \int_{A \to B} \bar{F} \cdot d\bar{r} = \int_{A \to B} \bar{F}_1 \cdot d\bar{r} + \dots + \int_{A \to B} \bar{F}_n \cdot d\bar{r}$$

Hence U - additive

$$\boxed{U = \frac{q_0}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_{0i}}}$$

r_{i0} - distance between q_i and q_0

Potential Energy of Charge Configuration (Self-Energy)

Potential Energy of Charge Configuration (Self-Energy)

Question: *What is the potential energy of the configuration of n static charges itself?*

q_1, q_2, q_3, q_i, q_n

start with all charges far apart and bring them together

$U_1 = 0$

$U_2 = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{12}}$

$U_3 = \frac{1}{4\pi\varepsilon_0} \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right)$

$\ldots$

$U_{conf} = \frac{1}{4\pi\varepsilon_0} \sum_{\substack{i,j=1 \\ i<j}}^{n} \frac{q_i q_j}{r_{ij}}$

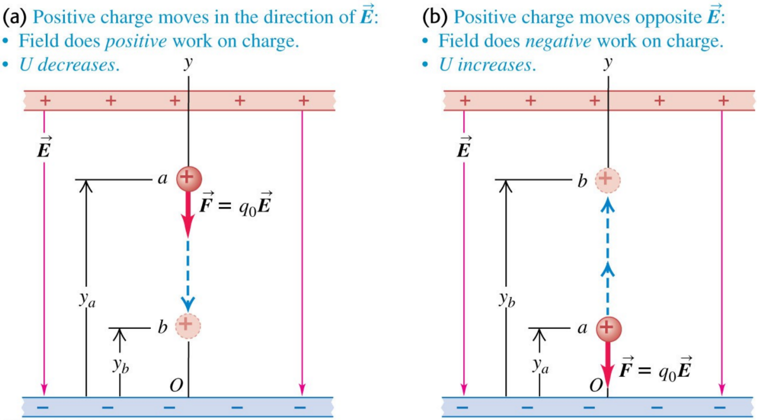
(all pairs; count once)

Hence, the potential energy of the configuration of n static charges is

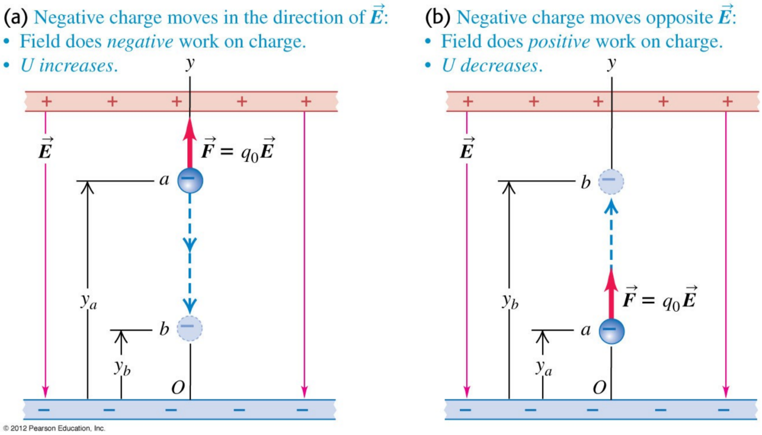
$$\boxed{U_{\mathrm{conf}} = \frac{1}{4\pi\varepsilon_0} \sum_{\substack{i,j=1 \\ i<j}}^{n} \frac{q_i q_j}{r_{ij}} = \frac{1}{8\pi\varepsilon_0} \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \frac{q_i q_j}{r_{ij}}}$$

Interpretation: U_{conf} equals the work needed to be done by an external force to form the system, starting with charges $q_1, q_2, \ldots, q_n$ initially placed far (∞) apart and at rest.

Potential Energy in Uniform Electric Field



Potential Energy in Uniform Electric Field



Electric Potential

Definition and Evaluation Methods

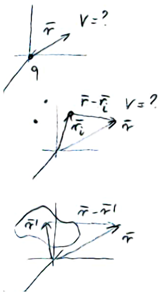
Electric potential (potential energy per unit charge)

$$\boxed{V \stackrel{\text{def}}{=} \frac{U}{q_0}}$$

[SI unit: Volt (V) = Joule (J)/ Coulomb (C)]

How to find?

1st method (the superposition principle)



$V(\bar{r})$	source
$\frac{1}{4\pi\varepsilon_0}\frac{q}{r}$	single point charge
$\frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{\lvert\bar{r}-\bar{r}_i\rvert}$	discrete distribution of charges
$\frac{1}{4\pi\varepsilon_0}\int_{\text{charged object}} \frac{dq}{\lvert\bar{r}-\bar{r}'\rvert}$	continuous distribution of charges: $dq=\rho\, dV$ (3D), $dq=\sigma\, dA$ (2D), $dq=\lambda\, dl$ (1D)

2^{nd} method (relationship between electric potential and electric field)

$$\bar{F} = -\text{grad}\, U \quad /:q_0 \qquad \Longleftrightarrow \qquad U_A - U_B = \int_{A \to B} \bar{F} \cdot d\bar{r} \quad /:q_0$$

$$\bar{E} = -\text{grad}\, V \qquad\qquad V_A - V_B = \int_{A \to B} \bar{E} \cdot d\bar{r}$$

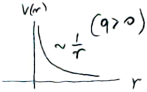
(known $V \Rightarrow$ find $\bar{E}$)

$\hookrightarrow$ any path

(known $\bar{E} \Rightarrow$ find V)

choose $V_B = 0$ (B- reference point)

(a) electric potential due to a point charge
[2^{nd} method; choose $V(\infty) = 0$; reference point — infinity]



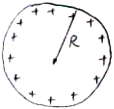
$$\bar{E}(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\frac{\bar{r}}{r}$$

$$V(\bar{r}) = \int_r^\infty \bar{E}(\bar{r})\,d\bar{r} \overset{\bar{E}\parallel d\bar{r}}{=} \int_r^\infty \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\frac{\cancel{r}\,dr}{\cancel{r}} =$$

$$= \int_r^\infty \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,dr = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

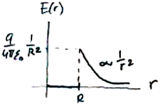
$$\boxed{V(\bar{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}}$$

(b) electric potential due to a uniformly charged conducting sphere
[2^{nd} method; choose $V(\infty) = 0$; reference point — infinity]



Previous lecture:

$$\bar{E}(\bar{r}) = \begin{cases} \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \frac{\bar{r}}{r} & \text{if } r > R \\ 0 & \text{if } r < R \end{cases}$$

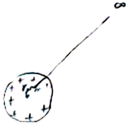


1° $r > R$

$$V(r) - V(\infty) = \int_r^\infty \underset{\bar{E} \parallel d\bar{r}}{\bar{E}(\bar{r})\, d\bar{r}} = \int_r^\infty \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \frac{\bar{r}\, dr}{r} =$$

$$= \underline{\frac{1}{4\pi\varepsilon_0} \frac{q}{r}}$$

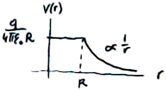
2° $r < R$



$$V(r) - V(\infty) = \int_r^\infty \bar{E}(\bar{r})\, d\bar{r} = \int_r^R \underbrace{\bar{E}(\bar{r})}_{\substack{\equiv 0 \\ (\text{inside})}} d\bar{r} + \int_R^\infty \bar{E}(\bar{r})\, d\bar{r} =$$
$$\qquad\qquad\qquad\qquad\qquad (\text{outside})$$

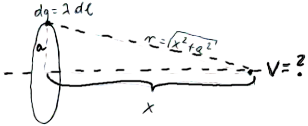
$$= \int_R^\infty \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2}\, dr = \underline{\frac{q}{4\pi\varepsilon_0} \frac{1}{R}}$$

$$V(\bar{r}) = \begin{cases} \frac{1}{4\pi\varepsilon_0} \frac{q}{r} & \text{if } r > R \\ \frac{1}{4\pi\varepsilon_0} \frac{q}{R} & \text{if } r < R \end{cases}$$



(c) electric potential due to a ring of radius a, charged uniformly with linear charge density $\lambda =$ const, on the axis of symmetry perpendicular to the ring's plane

[1^{st} method; choose $V(\infty) = 0$; reference point — infinity]



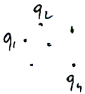
Superposition principle: divide the ring into small elements (can be treated as point charges) add all contributions.

$$V = \frac{1}{4\pi\varepsilon_0} \int_{ring} \frac{dq}{r} = \frac{1}{4\pi\varepsilon_0} \frac{1}{\sqrt{x^2+a^2}} \int_{ring} \lambda \, dl = \frac{1}{4\pi\varepsilon_0} \frac{\overbrace{(2\pi a \lambda)}^{\text{total charge on the ring}}}{\sqrt{x^2+a^2}}$$

Potential Energy of a Continuous Charge Configuration

Potential Energy of Charge Configuration. Discrete Charge Distribution

Recall that for a discrete charge distribution

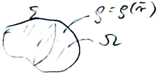


$$U_{conf} = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{q_i q_j}{r_{ij}} = \frac{1}{2}\sum_{i=1}^{n} q_i \underbrace{\left(\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{4\pi\varepsilon_0}\frac{q_j}{r_{ij}}\right)}_{V(\bar{r}_i)}$$

potential due to all charges (except q_i) at the position of q_i

$$\boxed{U_{\text{conf}} = \frac{1}{2}\sum_{i=1}^{n} q_i V(\bar{r}_i)} \qquad (\star)$$

For a continuous charge distribution (3D)



$$U_{conf} = \frac{1}{2}\int_{\Omega} \rho V \, d\tau$$

↳ element of volume

Example (2D)

Find the energy of a uniformly charged spherical shell of radius R and total charge q.

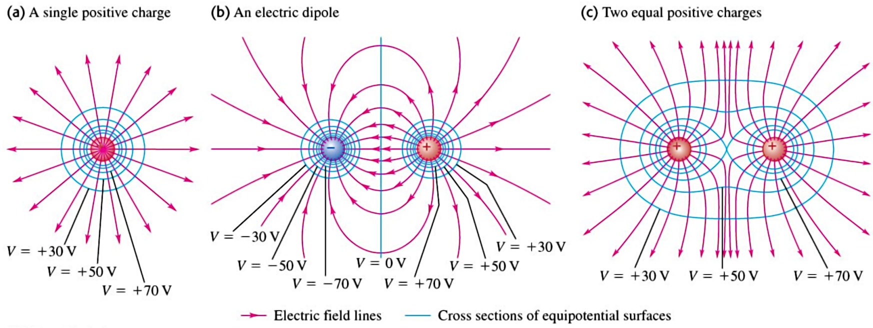
$$U_{conf} = \frac{1}{2} \int_{sphere} \sigma V \, dA = \frac{1}{2} \int_{sphere} \sigma \frac{1}{4\pi\varepsilon_0} \frac{q}{R} \, dA =$$

$$= \frac{1}{8\pi\varepsilon_0} \frac{q}{R} \int_{sphere} \sigma \, dA = \frac{1}{8\pi\varepsilon_0} \frac{q^2}{R}$$

Equipotential Surfaces

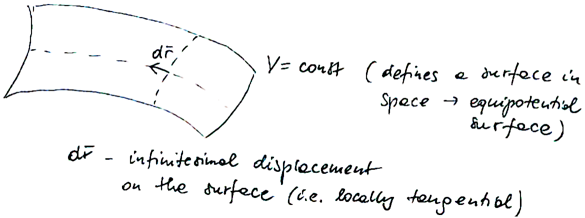
Equipotential Surfaces

An equipotential surface (or a line, in 2D) is a surface with the property that the electric potential is constant on that surface.



Fact. *An electric field line passing through a point is normal to the equipotential surface at that point.*

Justification



Recall that $\overline{E} = -\text{grad}\, V$. Hence $\mathrm{d}V = \text{grad}\, V \circ \mathrm{d}\bar{r} = -\overline{E} \circ \mathrm{d}\bar{r}$.

For $V = \text{const}$, we have $\mathrm{d}V = 0$. But $\mathrm{d}V = -\overline{E} \circ \mathrm{d}\bar{r}$, hence

$$\overline{E} \circ \mathrm{d}\bar{r} = 0 \qquad \Longrightarrow \qquad \boxed{\overline{E} \perp \mathrm{d}\bar{r}.}$$

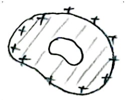
Properties of Conductors (II)

Further Properties of Conductors

1. When all charges are at rest, the surface of a conductor is an equipotential surface.

   Justification. $\overline{E}$ is perpendicular to the conductor's surface (cf. discussion of Gauss's law for conductors in previous lecture). So $q_0\overline{E} \circ \mathrm{d}\overline{r} = \overline{F} \circ \mathrm{d}\overline{r} = 0$ when moving along the conductor's surface. Hence $\overline{F} \circ \mathrm{d}\overline{r} = -\mathrm{d}U = 0$, which implies $U = \text{const}$.

2. Consider a conductor (may be charged, in general) with an empty cavity.

   Under electrostatic conditions, there is no charge anywhere on the surface of the cavity.

   Justification. Idea: Show that $\overline{E} \equiv 0$ inside the cavity. Then the surface charge density on the surface of the cavity is zero ($\sigma \equiv 0$).

A – cavity's surface

Σ – Gaussian surface

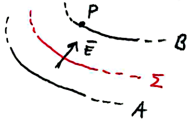
B – surface within Σ; through point P

Step 1. Every point in the empty cavity is at the same potential.

(a) Suppose that P is at a different potential.

(b) Construct an equipotential surface B through point P.

(c) Consider a Gaussian surface Σ between B and A.



$$V_A \neq V_B \quad \text{(from (a))}$$
$$\Downarrow$$
$$\bar{E} \neq 0$$
$$\Downarrow$$
$$\Phi_E \text{ through } \Sigma \text{ is non-zero}$$
$$\Downarrow \text{ Gauss's law}$$
$$\text{there is charge inside cavity}$$

CONTRADICTION! We assumed that the cavity is empty! Hence, every point in the empty cavity must be at the same potential.

Step 2. Because of the statement in *Step 1*, *the electric field must be zero everywhere inside the cavity.*

Step 3. But $|\overline{E}|$ at any point of the conductor's surface is proportional to σ. Hence, *the surface density of charge is zero at every point on the wall of the cavity.*

Example. Faraday's cage.

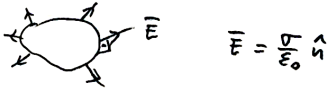
Properties of Conductors. Summary

(i) $\overline{E} = 0$ inside a conductor.

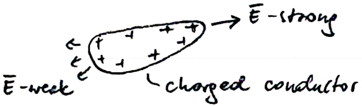
(ii) No net (excess) charge inside a conductor; any net charge resides on the surface.

(iii) A conductor is an equipotential.

(iv) Just outside of the conductor, $\overline{E}$ is perpendicular to the conductor's surface.



Properties of Conductors. Digression



$\bar{E}$-depends on curvature

WHY?

Explanation

$$V_A = V_B \Rightarrow \frac{Q_A}{R_A} = \frac{Q_B}{R_B} \Rightarrow \overset{\sigma_A}{\frac{Q_A}{4\pi R_A^2}} R_A = \overset{\sigma_B}{\frac{Q_B}{4\pi R_B^2}} R_B$$

Hence $\dfrac{\sigma_A}{\sigma_B} = \dfrac{R_B}{R_A} \xRightarrow{R_B < R_A} \sigma_A < \sigma_B$ and $|\bar{E}| \sim \sigma$

Conclusion. The magnitude of the electric field can get very large at sharp edges of a charged conductor.

Poisson's Equation and Laplace's Equation*

Gauss's Law (Differential Form)

$$\oint_\Sigma \overline{E} \circ \mathrm{d}\overline{A} = \frac{q_{\mathrm{encl}}}{\varepsilon_0}$$

Transform the lhs using Gauss–Ostrogradsky theorem (divergence theorem)

$$\oint_\Sigma \bar{E}\, d\bar{A} = \int_\Omega \operatorname{div}\bar{E}\, dV$$

For the rhs

$$\frac{q_{encl}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_\Omega \rho(\bar{r})\, dV$$

↳ bulk density of charge

Hence, comparing the right-hand sides (valid for any region Ω), we obtain Gauss's law in the differential (local) form

$$\boxed{\operatorname{div}\bar{E}(\bar{r}) = \frac{\rho(\bar{r})}{\varepsilon_0}}$$

Poisson's Equation and Laplace's Equation

From the differential form of the Gauss's law $\operatorname{div} \overline{E} = \frac{\varrho}{\varepsilon_0}$. On the other hand, $\overline{E} = -\operatorname{grad} V$, and

$$\operatorname{div} \overline{E} = \operatorname{div}(-\operatorname{grad} V) = \nabla \circ (-\nabla V) = -\nabla^2 V,$$

where ∇^2 is the *Laplace's operator* (*Laplacian*). Hence

$$\boxed{\nabla^2 V = -\frac{\varrho}{\varepsilon_0}} \qquad \text{Poisson's equation}$$

Note. Given ϱ (charge distribution), can find V (electric potential) by solving a PDE. It is a 3^{rd} method of finding the electric potential.

Comments

(i) If $\varrho \equiv 0$ is some region, then in that region

$$\boxed{\nabla^2 V = 0} \qquad \text{Laplaces's equation}$$

(ii) Poisson's/Laplace's equations are PDEs.
(Recall in Cartesian coordinates $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.)

(iii) Complete problem

boundary of Ω

equation in a region Ω + the value of V on Σ

$\Vert$

Boundary Value Problem

Final Remarks

Final Remarks

Mutual relations between the charge density, the electric field, and the electric potential.

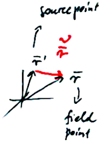
$$V(\bar{r}) = \int_{\substack{\text{charge}\\ \text{distribution}}} \frac{1}{4\pi\varepsilon_0}\frac{\rho(\bar{r}')\,d\tau}{|\bar{\mathscr{r}}|} \quad (*)$$

(reference point at ∞)

$$-\frac{\rho}{\varepsilon_0} = \nabla^2 V$$

$$\frac{\rho}{\varepsilon_0} = \operatorname{div}\bar{E}$$

$$\bar{E}(\bar{r}) = \int_{\substack{\text{charge}\\ \text{distribution}}} \frac{1}{4\pi\varepsilon_0}\frac{\bar{\mathscr{r}}}{|\bar{\mathscr{r}}|}\frac{\rho(\bar{r}')}{|\bar{\mathscr{r}}|^2}\,d\tau \quad (*)$$



$$\bar{\mathscr{r}} = \bar{r} - \bar{r}'$$

$$\rho \quad V \quad \bar{E}$$

$$\bar{E} = -\operatorname{grad} V$$

$$V(\bar{r}) - V(\bar{r}_0) = -\int_{\bar{r}_0 \to \bar{r}} \bar{E}(\bar{r})\,d\bar{r}$$

↳ reference point (usually $V(\bar{r}_0) = 0$)

→ any path

(*) - superposition principle

Final Remarks

Note. In many problems, in order to find $\overline{E}$, we usually first find V (it is a scalar; easier to deal with when e.g. using the superposition principle), and then find $\overline{E} = -\text{grad}\, V$.

SI Units

ϱ	[C/m^3 or C/m^2 or C/m] for 3D, 2D, 1D, respectively
V	[V] Volt
$\overline{E}$	[V/m or N/C]