Vp250 Problem Set 9

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Problem 1

Because after the wire is connected, it can form a loop with the top bulb as well as the bottom bulb. The inducted current going through the bottom bulb will cancel each other since they are in opposite direction due to different loops. So the bottom bulb no longer glows.
And as the equivalent resistance of this circuit is smaller (since one bulb is disconnected), the current going through the top bulb becomes greater, which means that it is brighter.

Problem 2

(a) Suppose the current through the small conductor has the positive direction.
Integrate the circle with radius r.

$$\oint \bar{B}\,\mathrm{d}\bar{l} = \mu_0 I \Rightarrow B\cdot 2\pi r = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

Direction: tangent to the circle with radius r.

(b)

$$\mathrm{d}\Phi = Bl\,\mathrm{d}r = \frac{\mu_0 Il}{2\pi r}\,\mathrm{d}r$$

(c) The flux:

$$\Phi = \int_a^b \frac{\mu_0 Il}{2\pi r}\,\mathrm{d}r = \frac{\mu_0 Il}{2\pi}\ln\frac{b}{a}$$

(d) The inductance:

$$L = \frac{\Phi}{I} = \frac{\mu_0 l}{2\pi}\ln\frac{b}{a}$$

(e) The magnetic field density at a distance r from the axis is $\dfrac{B^2(r)}{2\mu_0} = \dfrac{\mu_0 I^2}{8\pi^2 r^2}$. The total energy:

$$U = \int_a^b \frac{\mu_0 I^2}{8\pi^2 r^2} 2\pi r l\,\mathrm{d}r = \frac{\mu_0 I^2 l}{4\pi}\ln\frac{b}{a}$$

Problem 3

Suppose the voltage across the two wrapped inductors is U. Then

$$\begin{cases} U = L_1 I_1' + M I_2' \\ U = L_2 I_2' + M I_1' \\ U = LI = L(I_1' + I_2') \end{cases} \Rightarrow \begin{cases} I_1' = U\dfrac{L_2 - M}{L_1 L_2 - M^2} \\ I_2' = U\dfrac{L_1 - M}{L_1 L_2 - M^2} \end{cases} \rightarrow L = \frac{U}{I_1' + I_2'} = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$$

Problem 4

(a) Suppose the current through the inductor is i and the voltage across the capacitor is v_c. According to the diagram in this question we have:

$$\begin{cases} 40 = 50I + V_c \\ I = C\dfrac{\mathrm{d}V_c}{\mathrm{d}t} + i + \dfrac{V_c}{50} \\ 100i + L\dfrac{\mathrm{d}i}{\mathrm{d}t} = v_c \\ v_c(0) = 0 \\ i(0) = 0 \end{cases} \Rightarrow \begin{cases} V_1(0) = 50I(0) = 40 - v_c(0) = 40V \\ V_2(0) = L\dfrac{\mathrm{d}i}{\mathrm{d}t}\Big|_{t=0} = 0 \\ V_3(0) = 100i(0) = 0 \\ V_4(0) = V_5(0) = v_c(0) = 0 \end{cases} \text{and} \begin{cases} A_1(0) = \dfrac{V_1(0)}{50} = 0.8A \\ A_2(0) = i(0) = 0 \\ A_3(0) = \dfrac{V_4(0)}{50} = 0 \\ A_4(0) = C\dfrac{\mathrm{d}v_c}{\mathrm{d}t}\Big|_{t=0} = 0.8A \end{cases}$$

(b)

$$\begin{cases} 40 = 50I + V_c \\ I = C\dfrac{\mathrm{d}V_c}{\mathrm{d}t} + i + \dfrac{V_c}{50} \\ 100i + L\dfrac{\mathrm{d}i}{\mathrm{d}t} = v_c \end{cases} \Rightarrow 3 \times 10^{-6} i'' + 0.07i' + 250i = 40$$

With $i(0) = 0$, $v_c(0) = 0$, we have $i'(0) = 0$. So $i(t) = 0.05e^{-18931.5t} - 0.21e^{-4401.84t} + 0.16[A]$. Then $v_c(t) = 100i + Li' = 16 + 0.27e^{-18931.5t} - 16.38e^{-4401.84t}$.
So the readings are

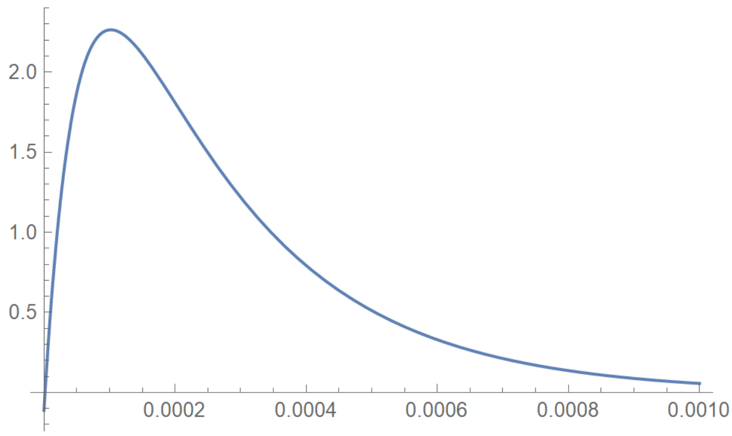
$$\begin{cases} V_1(\infty) = 40 - v_c(\infty) = 24V \\ V_2(\infty) = L\dfrac{\mathrm{d}i}{\mathrm{d}t}\Big|_{t=\infty} = 0 \\ V_3(\infty) = 100i(\infty) = 16V \\ V_4(\infty) = V_5(\infty) = v_c(\infty) = 16V \end{cases} \text{ and } \begin{cases} A_1(\infty) = \dfrac{V_1(\infty)}{50} = 0.48A \\ A_2(\infty) = i(\infty) = 0.16A \\ A_3(\infty) = \dfrac{V_4(\infty)}{50} = 0.32A \\ A_4(\infty) = C\dfrac{\mathrm{d}v_c}{\mathrm{d}t}\Big|_{t=\infty} = 0 \end{cases}$$

(c) First find the maximum voltage across the capacitor:

$$v_c' = 5057.08e^{-18931.5t} + 72093.6e^{-4401.84t} = 0 \Rightarrow t = -0.0002s < 0$$

So the maximum voltage is just when the capacitor is fully charged. Then $v_{c,max} = v_c(\infty) = 16V$. Hence, the maximum charge is $Q_{max} = Cv_{c,max} = 12 \times 10^{-6} \times 16 = 1.92 \times 10^{-4}C$.

(d) The graph:



Problem 5

Assume that the flux passing through each turn of the coil is ϕ. Then the total flux through the primary one and secondary one are

$$\Phi_p = N_1\phi \qquad \Phi_s = N_2\phi$$

$$\mathcal{E}_1 = -N_1\frac{\mathrm{d}\phi}{\mathrm{d}t} \qquad \mathcal{E}_2 = -N_2\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

So

$$\frac{\mathcal{E}_1}{\mathcal{E}_2} = \frac{N_1}{N_2}$$

Problem 6

(a) Assume that the flux through each turn is ϕ. And the current in the primary coil is I_1, the current passing through the secondary coil is I_2.
Then M can be represented as

$$M = \frac{N_1\phi}{I_2} = \frac{N_2\phi}{I_1}$$

So

$$M^2 = \frac{N_1\phi}{I_2} \cdot \frac{N_2\phi}{I_1} = \frac{N_1\phi}{I_1} \cdot \frac{N_2\phi}{I_2} = L_1L_2 \tag{1}$$

(b) Let ϕ_1 denotes the flux through each turn in primary coil, ϕ_2 denotes the flux through each turn in secondary coil.
Then

$$L_1\frac{\mathrm{d}I_1}{\mathrm{d}t} + M\frac{\mathrm{d}I_2}{\mathrm{d}t} = N_1\frac{\mathrm{d}\phi_1}{\mathrm{d}t}$$

And as

$$N_1\frac{\mathrm{d}\phi_1}{\mathrm{d}t} = V_{in}$$

So

$$L_1\frac{\mathrm{d}I_1}{\mathrm{d}t} + M\frac{\mathrm{d}I_2}{\mathrm{d}t} = V_1\cos\omega t$$

For the secondary coil, the voltage across the coil is equal to the voltage across the resistor but in different direction. So

$$L_2\frac{\mathrm{d}I_2}{\mathrm{d}t} + M\frac{\mathrm{d}I_1}{\mathrm{d}t} = -I_2R$$

(c)

$$\begin{cases} L_1\frac{\mathrm{d}I_1}{\mathrm{d}t} + M\frac{\mathrm{d}I_2}{\mathrm{d}t} = V_1\cos\omega t \\ L_2\frac{\mathrm{d}I_2}{\mathrm{d}t} + M\frac{\mathrm{d}I_1}{\mathrm{d}t} = -I_2R \end{cases} \Rightarrow \left(L_2 - \frac{M^2}{L_1}\right)I_2' + \frac{M}{L_1}V_1\cos\omega t = -I_2Rs$$

Since $M^2 = L_1L_2$,

$$I_2 = -\frac{M}{L_1R}V_1\cos\omega t$$

Then

$$I_1' = \frac{V_1\cos\omega t - MI_2'}{L_1} = \frac{V_1R\cos\omega t - L_2V_1\omega\sin\omega t}{L_1R}$$

$$\Rightarrow I_1 = \frac{V_1}{\omega L_1}\sin\omega t + \frac{L_2V_1}{L_1R}\cos\omega t$$

(d)

$$V_{out} = I_2R = -L_2\frac{\mathrm{d}I_2}{\mathrm{d}t} - M\frac{\mathrm{d}I_1}{\mathrm{d}t} = N_2\frac{\mathrm{d}\phi_2}{\mathrm{d}t}$$

$$V_{in} = L_1\frac{\mathrm{d}I_1}{\mathrm{d}t} + M\frac{\mathrm{d}I_2}{\mathrm{d}t} = N_1\frac{\mathrm{d}\phi_1}{\mathrm{d}t}$$

Since this is a ideal transformer, $\phi_1 = \phi_2$,

$$\frac{V_{out}}{V_{in}} = \frac{N_2}{N_1}$$

(e)

$$P_{in} = V_{in}I_1 = V_1\cos\omega t \cdot \left(\frac{V_1}{\omega L_1}\sin\omega + \frac{L_2V_1}{L_1R}\cos\omega t\right) = \frac{V_1^2}{\omega L_1}\sin\omega t\cos\omega t + \frac{L_2V_1^2}{L_1R}\cos^2\omega t$$

$$P_{out} = V_{out}I_2 = I_2^2R = \frac{L_2}{L_1R}V_1^2\cos^2\omega t$$

The time needed for a full circle is $\frac{2\pi}{\omega}$. So the average rate of input is

$$\bar{P}_{in} = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{V_1^2}{\omega L_1}\sin\omega t\cos\omega t + \frac{L_2V_1^2}{L_1R}\cos^2\omega t\,\mathrm{d}t = \frac{L_2V_1^2}{2L_1R}$$

The average rate of the output is

$$\bar{P}_{out} = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{L_2}{L_1R}V_1^2\cos^2\omega t = \frac{L_2V_1^2}{2L_1R}$$

We can conclude that

$$\bar{P}_{in} = \bar{P}_{out}$$