Vp250 Problem Set 10

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Problem 1

(a) Assume the value of the capacitance is C, the value of the inductance is L. Then the impedance of the capacitor is $-j\dfrac{1}{\omega C}$, the impedance of the inductor is $j\omega L$. So the output voltage can be evaluated as

$$\tilde{V}_{hi} = V_s\angle 0^\circ \frac{R + j\omega L}{R + j(\omega L - \frac{1}{\omega C})} = V_s \frac{\sqrt{(R^2+\omega^2L^2-\frac{L}{C})^2+\frac{R^2}{\omega^2C^2}}}{R^2+(\omega L-\frac{1}{\omega C})^2}\angle \arctan\frac{\frac{R}{\omega C}}{R^2+\omega^2L^2-\frac{L}{C}}$$

So the ratio is

$$V_{hi}/V_s = \frac{\sqrt{(R^2+\omega^2L^2-\frac{L}{C})^2+\frac{R^2}{\omega^2C^2}}}{R^2+(\omega L-\frac{1}{\omega C})^2}$$

When $\omega \to 0$,

$$\begin{aligned}
V_{hi}/V_s &= \frac{\sqrt{(R^2+\omega^2L^2-\frac{L}{C})^2+\frac{R^2}{\omega^2C^2}}}{R^2+(\omega L-\frac{1}{\omega C})^2} \\
&= \frac{\omega C\sqrt{\omega^2C^2(R^2-\frac{L}{C})^2+R^2}}{R^2\omega^2C^2+(\omega^2CL-1)^2} \\
&\quad \text{Take all } \omega^2 \text{ as } 0 \\
&= \omega CR
\end{aligned}$$

So we can see that the ratio is proportional to the ω.
When $\omega \to \infty$,

$$\begin{aligned}
V_{hi}/V_s &= \frac{\sqrt{(R^2+\omega^2L^2-\frac{L}{C})^2+\frac{R^2}{\omega^2C^2}}}{R^2+(\omega L-\frac{1}{\omega C})^2} \\
&= \frac{\sqrt{\frac{1}{\omega^4L^4}(R^2+\omega^2L^2-\frac{L}{C})^2+\frac{R^2}{\omega^6C^2L^4}}}{\frac{R^2}{\omega^2L^2}+(1-\frac{1}{\omega^2CL})^2} \\
&= \frac{\sqrt{(\frac{R^2}{\omega^2L^2}+1-\frac{1}{\omega^2CL})^2+\frac{R^2}{\omega^6C^2L^4}}}{\frac{R^2}{\omega^2L^2}+(1-\frac{1}{\omega^2CL})^2} \\
&\quad \text{Take all } \frac{1}{\omega} \text{ as } 0 \\
&= 1
\end{aligned}$$

(b)

$$\tilde{V}_{lo} = V_s\angle 0^\circ \frac{-j\frac{1}{\omega C}}{R + j(\omega L - \frac{1}{\omega C})} = V_s \frac{\sqrt{(\frac{1}{\omega^2C^2}-\frac{L}{C})^2+\frac{R^2}{\omega^2C^2}}}{R^2+(\omega L-\frac{1}{\omega C})^2}\angle \arctan\frac{-\frac{R}{\omega C}}{\frac{1}{\omega^2C^2}-\frac{L}{C}}$$

So the ratio is

$$V_{lo}/V_s = \frac{\sqrt{(\frac{1}{\omega^2C^2}-\frac{L}{C})^2+\frac{R^2}{\omega^2C^2}}}{R^2+(\omega L-\frac{1}{\omega C})^2}$$

When $\omega \to \infty$,

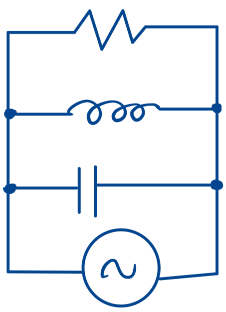
$$\begin{aligned}
V_{lo}/V_s &= \frac{\sqrt{(\frac{1}{\omega^2C^2} - \frac{L}{C})^2 + \frac{R^2}{\omega^2C^2}}}{R^2 + (\omega L - \frac{1}{\omega C})^2} \\
&= \frac{\sqrt{\frac{1}{\omega^4L^4}(\frac{1}{\omega^2L^2} - \frac{L}{C})^2 + \frac{R^2}{\omega^6C^2L^4}}}{\frac{R^2}{\omega^2L^2} + (1 - \frac{1}{\omega^2CL})^2} \\
&\text{Take all } \frac{1}{\omega^2} \text{ as } 0 \\
&= \frac{1}{\omega^2LC}
\end{aligned}$$

So it is proportional to ω^{-2}.
When $\omega \to 0$,

$$\begin{aligned}
V_{lo}/V_s &= \frac{\sqrt{(\frac{1}{\omega^2C^2} - \frac{L}{C})^2 + \frac{R^2}{\omega^2C^2}}}{R^2 + (\omega L - \frac{1}{\omega C})^2} \\
&= \frac{\sqrt{(1 - \omega^2CL)^2 + R^2\omega^2C^2}}{R^2\omega^2C^2 + (\omega^2LC - 1)^2} \\
&\text{Take all } \omega^2 \text{ as } 0 \\
&= 1
\end{aligned}$$

Problem 2

(a) The circuit graph:



For three loops,

$$\begin{cases} v_R - v_L = 0 \\ v_L - v_c = 0 \\ v_c - v = 0 \end{cases} \Rightarrow v_R = v_L = v_C = v$$

$$\begin{cases} v_R = i_R R \\ v_L = i_L Z_L \\ v_C = i_C Z_C \\ v = i(R||Z_L||Z_C) = \end{cases} \Rightarrow i = i_R + i_C + i_L$$

(b) $\tilde{v} = V\angle 0°$.

$$\tilde{i}_R = \frac{\tilde{v}}{R} = \frac{V}{R}\angle 0°$$

$$\tilde{i}_C = \frac{\tilde{v}}{\frac{1}{j\omega C}} = V\omega C\angle 90°$$

$$\tilde{i}_L = \frac{\tilde{v}}{j\omega L} = \frac{V}{\omega L}\angle -90°$$

(c)

$$\tilde{i} = \frac{V}{R}\angle 0^\circ + V\omega C\angle 90^\circ + \frac{V}{\omega L}\angle - 90^\circ = \frac{V}{R} + j\left(V\omega C - \frac{V}{\omega L}\right)$$

So

$$I = \sqrt{\left(\frac{V}{R}\right)^2 + \left(V\omega C - \frac{V}{\omega L}\right)^2} = \sqrt{I_R^2 + (I_C - I_L)^2} = V\sqrt{\left(\frac{1}{R}\right)^2 + \left(\omega C - \frac{1}{\omega L}\right)^2}$$

(d) When $\omega_0 = \dfrac{1}{\sqrt{LC}}$,

$$\tilde{i}_L = V\sqrt{\frac{C}{L}}\angle - 90^\circ \Rightarrow I_L = V\sqrt{\frac{C}{L}}$$

$$\tilde{i}_C = V\sqrt{\frac{C}{L}}\angle 90^\circ \Rightarrow I_C = V\sqrt{\frac{C}{L}}$$

Hence, $I_C = I_L$.
For I,

$$I = \frac{V}{R}$$

, which is the minimum of I because $\left(\omega C - \dfrac{1}{\omega L}\right)^2 \geqslant 0$.

The power of the resistor is $P = \dfrac{V^2}{2R}$, which is a constant. So you can say that it is a minimum when $\omega = \omega_0$.

(e) when it is at the resonance,

$$\tilde{i} = \frac{V}{R} + j\left(V\omega C - \frac{V}{\omega L}\right) = \sqrt{\frac{V^2}{R^2} + \left(V\omega C - \frac{V}{\omega L}\right)^2}\angle \arctan\frac{V\omega C - \frac{V}{\omega L}}{\frac{V}{R}}$$

$$\Rightarrow \tilde{i} = \sqrt{\frac{V^2}{R^2} + \left(V\omega C - \frac{V}{\omega L}\right)^2}\angle \arctan\frac{RV\omega^2 CL - VR}{V\omega L}$$

So the phase angle is

$$\phi = \arctan\frac{RV\omega^2 CL - VR}{V\omega L} = 0$$

The phase lag in RLC series circuit is $\tan\phi = \dfrac{\omega L - \frac{1}{\omega C}}{R}$. When $\omega = \omega_0$, the phase shift is 0, the same as the phase shift as the parallel condition.

Problem 3

For f_1

$$\begin{cases} \dfrac{\partial^2 f_1}{\partial x^2} = 4Ak^2(x-tv)^2 e^{-k(x-tv)^2} - 2Ake^{-k(x-tv)^2} \\ \dfrac{1}{v^2}\cdot\dfrac{\partial^2 f_1}{\partial t^2} = \frac{1}{v^2}(4Ak^2v^2(x-tv)^2 e^{-k(x-tv)^2} - 2Akv^2 e^{-k(x-tv)^2}) \end{cases} \Rightarrow \frac{\partial^2 f_1}{\partial x^2} = \frac{1}{v^2}\cdot\frac{\partial^2 f_1}{\partial t^2}$$

So f_1 satisfies.
For f_2

$$\begin{cases} \dfrac{\partial^2 f_2}{\partial x^2} = -Ak^2\sin(k(x-tv)) \\ \dfrac{1}{v^2}\cdot\dfrac{\partial^2 f_2}{\partial t^2} = \dfrac{1}{v^2}(-Ak^2v^2\sin(k(x-tv))) \end{cases} \Rightarrow \frac{\partial^2 f_2}{\partial x^2} = \frac{1}{v^2}\cdot\frac{\partial^2 f_2}{\partial t^2}$$

So f_2 satisfies.
For f_3

$$\frac{\partial^2 f_3}{\partial x^2} = 16Ak^4x^2 e^{-k\left(kx^2+tv\right)^2}\left(kx^2+tv\right)^2 - 8Ak^3x^2 e^{-k\left(kx^2+tv\right)^2} - 4Ak^2 e^{-k\left(kx^2+tv\right)^2}\left(kx^2+tv\right)$$

$$\frac{1}{v^2}\cdot\frac{\partial^2 f_3}{\partial t^2}=\frac{1}{v^2}(4Ak^2v^2e^{-k\left(kx^2+tv\right)^2}\left(kx^2+tv\right)^2-2Akv^2e^{-k\left(kx^2+tv\right)^2})$$

these two are not equal, so f_3 does not satisfy.
For f_4

$$\frac{\partial^2 f_4}{\partial x^2}=-Ak^2\sin(kx)\cos^3(ktv)$$

$$\frac{1}{v^2}\cdot\frac{\partial^2 f_4}{\partial t^2}=\frac{1}{v^2}(6Ak^2v^2\sin(kx)\sin^2(ktv)\cos(ktv)-3Ak^2v^2\sin(kx)\cos^3(ktv))$$

these two are not equal, so f_4 does not satisfy.

Problem 4

(a)

$$\begin{cases}\dfrac{\partial^2\xi}{\partial x^2}=-Ak^2\sin(kx)\cos(\omega t)\\ \dfrac{1}{v^2}\cdot\dfrac{\partial^2\xi}{\partial t^2}=\dfrac{k^2}{\omega^2}(-A\omega^2\sin(kx)\cos(\omega t))\end{cases}\Rightarrow\frac{\partial^2\xi}{\partial x^2}=\frac{1}{v^2}\cdot\frac{\partial^2\xi}{\partial t^2}$$

So it satisfies the wave equation.

(b)

$$\xi(x,t)=A\sin(kx)\cos(\omega t)=\frac{1}{2}\sin(kx+\omega t)+\frac{1}{2}\cos(kx-\omega t)$$

Problem 5

(a) Reflective. By the conservation of momentum, when the sail is reflective, the spaceship can obtain a higher speed opposite to the photons.

(b) Suppose the distance between the spaceship and the sun is d, the mass of sun is M, the mass of the spaceship is m, P is the power, c is the speed of the light and I is the intensity.

$$\begin{cases}I=\dfrac{P}{4\pi d^2}\\ p=\dfrac{2I}{c}\\ p\cdot 4\pi d^2=\dfrac{GMm}{d^2}\end{cases}\Rightarrow\begin{aligned}A=4\pi d^2&=\frac{2\pi GMmc}{P}\\&=\frac{2\pi\cdot 6.67\cdot 10^{-11}\cdot 1.989\cdot 10^{30}\cdot 10\cdot 10^3\cdot 3\cdot 10^8}{3.9\cdot 10^{26}}\\&=6.41\cdot 10^6 m^2=6.41km^2\end{aligned}$$

(c) From (b) we can see that the formula for the area of the sail has nothing to do with the distance.

Problem 6

(a) Assume a unit vector $\hat{l}$ which is parallel to the axis of the cylinder, $\hat{t}$, tangent to the circle which is concentric with the cylinder.

$$|B|\cdot 2\pi a=\mu_0 I\Rightarrow|B|=\frac{\mu_0 I}{2\pi a}\Rightarrow\bar{B}=\frac{\mu_0 I}{2\pi a}\hat{t}$$

The resistance:

$$R=\rho\frac{\mathrm{d}l}{\pi a^2}$$

The infinitesimal potential difference between a $\mathrm{d}l$ long distance in the conductor is

$$\mathrm{d}U=|E|\,\mathrm{d}l\Rightarrow|E|=\frac{\mathrm{d}U}{\mathrm{d}l}=\frac{IR}{\mathrm{d}l}=\frac{I\rho\frac{\mathrm{d}l}{\pi a^2}}{\mathrm{d}l}=\frac{I\rho}{\pi a^2}\Rightarrow\bar{E}=\frac{I\rho}{\pi a^2}\hat{l}$$

(b)

$$|S|=\frac{1}{\mu_0}|E|\cdot|B|=\frac{1}{\mu_0}\cdot\frac{\mu_0 I}{2\pi a}\cdot\frac{I\rho}{\pi a^2}=\frac{I^2\rho}{2\pi^2a^3}\Rightarrow\bar{S}=-\frac{I^2\rho}{2\pi^2a^3}\hat{n}$$

where $\hat{n}$ is the unit vector which normal to the surface of the cylinder and points out of the surface.

(c) Denote the total flow rate of the energy as r.

$$r = \int_{\Sigma} \bar{S}\, \mathrm{d}\bar{A} = |S| \cdot 2\pi a l = \frac{I^2 \rho l}{\pi a^2}$$

(d)

$$P = I^2 R = I^2 \rho \frac{l}{\pi a^2} = \frac{I^2 \rho l}{\pi a^2}$$

It is the same as in (c). And since the energy can be generated within this conductor, the direction of S is pointing out the conductor, which is the positive direction as we defined.