8.4 多元复合函数的求导法则

8.4.1 多元复合函数的求导法则

复合函数的中间变量均为一元函数的情形

定理 4.1 (**链式求导法则**). *若函数$u = \varphi(t), v = \psi(t)$都在点$t$可导，函数$z = f(u,v)$在对应点$(u,v) = (\varphi(t), \psi(t))$可微，则复合函数$z = f(\varphi(t), \psi(t))$ 在点t可导，且有*

$$\frac{dz}{dt} = \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt}, \tag{8.4.1}$$

公式(8.4.1)中的导数称为全导数.

例 4.1. 设$z = uv^2 + \arctan w$，$u = \sin t$，$v = \ln t$，$w = e^t$，求全导数$\frac{dz}{dt}$.

解:

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial u} \cdot \frac{du}{dt} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dt} + \frac{\partial z}{\partial w} \cdot \frac{dw}{dt} \\ &= v^2 \cdot \cos t + 2uv \cdot \frac{1}{t} + \frac{1}{1+w^2} \cdot e^t \\ &= \cos t \ln^2 t + \frac{2}{t} \sin t \ln t + \frac{e^t}{1+e^{2t}}.\end{aligned}$$

例 4.2. 设$z = (\sin t)^{\cos t}$, 求$\frac{dz}{dt}$.

解: 令$u = \sin t$, $v = \cos t$, 则$z = u^v$. 由求导公式得

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial u}\cdot\frac{du}{dt} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dt} = vu^{v-1}\cdot\cos t + u^v\ln u\cdot(-\sin t)\\ &= \cos^2 t(\sin t)^{\cos t-1} - (\sin t)^{\cos t+1}\ln\sin t.\end{aligned}$$

中间变量多于两个的情形

若函数$z = f(x_1, x_2, \cdots, x_n)$在点$(x_1, x_2, \cdots, x_n)$处可微, 而$x_k = \varphi_k(t)$在点$t$处可导$(k = 1, 2, \cdots, n)$, 则复合函数$z = f[\varphi_1(t), \varphi_2(t), \cdots, \varphi_n(t)]$ 在点t处可导, 且

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x_1}\cdot\frac{\mathrm{d}x_1}{\mathrm{d}t} + \frac{\partial z}{\partial x_2}\cdot\frac{\mathrm{d}x_2}{\mathrm{d}t} + \cdots + \frac{\partial z}{\partial x_n}\cdot\frac{\mathrm{d}x_n}{\mathrm{d}t}.$$

中间变量均为二元函数的情形

定理 4.2 (**链式求导法则**). *若函数$u = \varphi(x, y)$, $v = \psi(x, y)$在点(x, y)处偏导数存在, 函数$z = f(u, v)$在对应点$(u, v) = (\varphi(x, y), \psi(x, y))$处可微, 则复合函数*

$$z = f(\varphi(x, y), \psi(x, y))$$

在点(x, y)处偏导数也存在, 且有

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

中间变量既有一元函数又有多元函数的情形

定理 4.3 (**链式求导法则**). *设三元函数$z = f(u, x, y)$在点(u, x, y)处可微, $u = \varphi(x, y)$在点(s, t)处偏导数存在, 则$z = f(\varphi(x, y), x, y)$有*

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial x},$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial y}.$$

注

这里$\frac{\partial z}{\partial x}$和$\frac{\partial f}{\partial x}$是不同的:

- $\frac{\partial z}{\partial x}$是把复合函数$z = f(\varphi(x, y), x, y)$中的$y$看作不变而对$x$求偏导数;

- $\frac{\partial f}{\partial x}$是把函数$z=f(u,x,y)$中的$u$及$y$看作不变而对$x$求偏导数.

例 4.3. 设$z=e^u\sin v$, $u=xy$, $v=\frac{x}{y}$, 求$\frac{\partial z}{\partial x}$和$\frac{\partial z}{\partial y}$.

解:

$$\frac{\partial z}{\partial u}=e^u\sin v,\quad \frac{\partial z}{\partial v}=e^u\cos v,$$

$$\frac{\partial u}{\partial x}=y,\ \frac{\partial u}{\partial y}=x,\ \frac{\partial v}{\partial x}=\frac{1}{y},\ \frac{\partial v}{\partial y}=\frac{-x}{y^2}.$$

根据链式法则可得

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=ye^u\sin v+\frac{1}{y}e^u\cos v=ye^{xy}\sin\frac{x}{y}+\frac{1}{y}e^{xy}\cos\frac{x}{y},$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=xe^u\sin v-\frac{x}{y^2}e^u\cos v=xe^{xy}\sin\frac{x}{y}-\frac{x}{y^2}e^{xy}\cos\frac{x}{y}.$$

例 4.4. 设$z=f(x,y)$有连续偏导数, 且$x=r\cos\theta$, $y=r\sin\theta$, 证明:

$$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2.$$

证明: 因为

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial r}=\frac{\partial z}{\partial x}\cos\theta+\frac{\partial z}{\partial y}\sin\theta,$$

$$\frac{\partial z}{\partial \theta}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}=\frac{\partial z}{\partial x}(-r\sin\theta)+\frac{\partial z}{\partial y}(r\cos\theta),$$

所以

$$\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2=\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2.$$

例 4.5. 设$z=(3x+2y)^{xy^2}$, 求z_x, z_y.

解: 令$u=3x+2y$, $v=xy^2$, 则$z=u^v$. 于是

$$\begin{aligned}z_x=&z_uu_x+z_vv_x=vu^{v-1}\cdot 3+u^v\ln u\cdot y^2\\=&3xy^2(3x+2y)^{xy^2-1}+y^2(3x+2y)^{xy^2}\ln(3x+2y),\end{aligned}$$

$$\begin{aligned}z_y=&z_uu_y+z_vv_y=vu^{v-1}\cdot 2+u^v\ln u\cdot 2xy\\=&2xy^2(3x+2y)^{xy^2-1}+2xy(3x+2y)^{xy^2}\ln(3x+2y).\end{aligned}$$

例 4.6. 设$z = f(x^2 - y^2, y^2 - x^2)$, f有对各个变量的连续偏导数, 证明
$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0.$$
解: 令$u = x^2 - y^2$, $v = y^2 - x^2$, 则$z = f(u, v)$. 于是
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = 2x\frac{\partial z}{\partial u} - 2x\frac{\partial z}{\partial v},$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = -2y\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v}.$$
所以
$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = y\left(2x\frac{\partial z}{\partial u} - 2x\frac{\partial z}{\partial v}\right) + x\left(-2y\frac{\partial z}{\partial u} + 2y\frac{\partial z}{\partial v}\right) = 0.$$

例 4.7. 设$z = f(x + y, xy)$, 其中f有二阶连续偏导数, 求$\frac{\partial z}{\partial x}$, $\frac{\partial^2 z}{\partial x \partial y}$.

解: 令$u = x + y$, $v = xy$, 则$z = f(u, v)$. 于是
$$\frac{\partial z}{\partial x} = f_u u_x + f_v v_x = f_u + y f_v,$$
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(f_u + y f_v) = \frac{\partial f_u}{\partial y} + f_v + y\frac{\partial f_v}{\partial y}.$$
又$f_u = f_u(x + y, xy)$, $f_v = f_v(x + y, xy)$, 故
$$\frac{\partial f_u}{\partial y} = f_{uu} u_y + f_{uv} v_y = f_{uu} + x f_{uv},$$
$$\frac{\partial f_v}{\partial y} = f_{vu} u_y + f_{vv} v_y = f_{vu} + x f_{vv}.$$
因此
$$\frac{\partial^2 z}{\partial x \partial y} = f_{uu} + x f_{uv} + f_v + y(f_{vu} + x f_{vv}).$$

由于f有二阶连续偏导数, 有$f_{uv} = f_{vu}$, 所以
$$\frac{\partial^2 z}{\partial x \partial y} = f_{uu} + (x + y) f_{uv} + xy f_{vv} + f_v.$$

在本例中复合函数$f(x + y, xy)$的中间变量没有明显写出, 为了简便起见, 通常可用f_i'表示f对第i个中间变量的偏导数, 用f_{ij}''表示f先对第i个中间变量后对第j个中间变量的二阶偏导数. 这样, 上述f_u, f_v, f_{uv}, f_{vu}, f_{uu}, f_{vv} 就可以写成f_1', f_2', f_{12}'', f_{21}'', f_{11}'', f_{22}''. 于是本例的结果可改写成
$$\frac{\partial z}{\partial x} = f_1' + y f_2',$$
$$\frac{\partial^2 z}{\partial x \partial y} = f_{11}'' + (x + y) f_{12}'' + xy f_{22}'' + f_2'.$$

例 4.8. 设$z = e^{x^2+y^2+u^2}$, $u = x\sin y$, 求$\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

解: 令$z = f(u,x,y) = e^{x^2+y^2+u^2}$, 则有

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial x} = e^{x^2+y^2+u^2}2u\cdot\sin y + e^{x^2+y^2+u^2}\cdot 2x\\ &= 2x(\sin^2 y+1)e^{x^2+y^2+x^2\sin^2 y},\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial y} = e^{x^2+y^2+u^2}2u\cdot x\cos y + e^{x^2+y^2+u^2}\cdot 2y\\ &= 2(x^2\sin y\cos y + y)e^{x^2+y^2+x^2\sin^2 y}.\end{aligned}$$

例 4.9. 设$z = uv + \sin t$, $u = e^t$, $v = \cos t$, 求全导数$\frac{\mathrm{d}z}{\mathrm{d}t}$.

解:

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial z}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{\partial z}{\partial t}\\ &= ve^t - u\sin t + \cos t = e^t(\cos t - \sin t) + \cos t.\end{aligned}$$

多元函数链式法则可以推广到两个以上中间变量的情形.

例 4.10. 设$z = uv\ln w$, $u = x+y$, $v = y-x$, $w = 1+xy$, 求$\frac{\partial z}{\partial x}$.

解:

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x}\\ &= v\ln w\cdot 1 + u\ln w\cdot(-1) + \frac{uv}{w}\cdot y\\ &= -2x\ln(1+xy) + \frac{y(y^2-x^2)}{1+xy}.\end{aligned}$$

例 4.11. 设$F = f(x, xy, xyz)$, 求$\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$.

解: 令$u = x$, $v = xy$, $w = xyz$, 有$F = f(u,v,w)$. 于是

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = f_1' + yf_2' + yzf_3',$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} = xf_2' + xzf_3',$$

$$\frac{\partial F}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} = xyf_3'.$$

8.4.2 一阶全微分形式不变性

设$z=f(u,v)$具有连续偏导数, 则全微分

$$\mathrm{d}z=\frac{\partial z}{\partial u}\,\mathrm{d}u+\frac{\partial z}{\partial v}\,\mathrm{d}v.$$

如果$z=f(u,v)$具有连续偏导数, 而$u=\varphi(x,y)$, $v=\psi(x,y)$也具有连续偏导数, 则全微分

$$\begin{aligned}\mathrm{d}z&=\frac{\partial z}{\partial x}\,\mathrm{d}x+\frac{\partial z}{\partial y}\,\mathrm{d}y\\&=\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)\mathrm{d}x+\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)\mathrm{d}y\\&=\frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}\,\mathrm{d}x+\frac{\partial u}{\partial y}\,\mathrm{d}y\right)+\frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}\,\mathrm{d}x+\frac{\partial v}{\partial y}\,\mathrm{d}y\right)=\frac{\partial z}{\partial u}\,\mathrm{d}u+\frac{\partial z}{\partial v}\,\mathrm{d}v.\end{aligned}$$

无论u,v是自变量还是中间变量, z的全微分形式是一样的. 这个性质叫做一阶全微分形式不变性.

例 4.12. 设$z=e^{xy}\sin(x+y)$, 求$\mathrm{d}z$, 并由此导出$\frac{\partial z}{\partial x}$和$\frac{\partial z}{\partial y}$.

解: 令$u=xy$, $v=x+y$, 则$z=f(u,v)=e^u\sin v$. 于是

$$\begin{aligned}\mathrm{d}z&=\mathrm{d}(e^u\sin v)=e^u\sin v\,\mathrm{d}u+e^u\cos v\,\mathrm{d}v\\&=e^u\sin v\,\mathrm{d}(xy)+e^u\cos v\,\mathrm{d}(x+y)\\&=e^u\sin v(y\,\mathrm{d}x+x\,\mathrm{d}y)+e^u\cos v(\,\mathrm{d}x+\,\mathrm{d}y)\\&=e^u(y\sin v+\cos v)\,\mathrm{d}x+e^u(x\sin v+\cos v)\,\mathrm{d}y\\&=e^{xy}(y\sin(x+y)+\cos(x+y))\,\mathrm{d}x+e^{xy}(x\sin(x+y)+\cos(x+y))\,\mathrm{d}y.\end{aligned}$$

所以

$$\frac{\partial z}{\partial x}=e^{xy}(y\sin(x+y)+\cos(x+y)),\qquad\frac{\partial z}{\partial y}=e^{xy}(x\sin(x+y)+\cos(x+y)).$$

高阶微分没有微分形式不变性这一性质.

设$z=x+y$, $x=u^2v$, $y=u+v$, 则

$$\begin{aligned}\mathrm{d}^2z&=\mathrm{d}\left(\frac{\partial z}{\partial u}\,\mathrm{d}u+\frac{\partial z}{\partial v}\,\mathrm{d}v\right)=\mathrm{d}\left(\frac{\partial z}{\partial u}\right)\mathrm{d}u+\mathrm{d}\left(\frac{\partial z}{\partial v}\right)\mathrm{d}v\\&=\frac{\partial^2z}{\partial u^2}\,\mathrm{d}u^2+2\frac{\partial^2z}{\partial u\partial v}\,\mathrm{d}u\,\mathrm{d}v+\frac{\partial^2z}{\partial v^2}\,\mathrm{d}v^2=2v\,\mathrm{d}u^2+2u\,\mathrm{d}u\,\mathrm{d}v.\end{aligned}$$

以上为x,y是中间变量的结果. 若x,y为自变量, 则

$$\mathrm{d}^2z=\mathrm{d}^2(x+y)=\mathrm{d}^2x+\mathrm{d}^2y=0.$$

8.4.3 思考与练习

练习 8.12. 设$z=f\left(\arctan\frac{y}{x}\right)$, 其中$f(x)$为可微函数, 且$f(x)$是$x^2$的一个原函数, 则$\frac{\partial z}{\partial x}\big|_{(1,1)}=$ ____.

$-\frac{\pi^2}{32}$

练习 8.13. 已知$f(x,y)|_{y=x^2}=1$, $f_1'(x,y)|_{y=x^2}=2x$, 求$f_2'(x,y)|_{y=x^2}$.

解: 由$f(x,x^2)=1$两边对x求导, 得

$$f_1'(x,x^2)+2xf_2'(x,x^2)=0.$$

故$f_2'(x,x^2)=-1$.

练习 8.14. 设函数$z=f(x,y)$在点$(1,1)$处可微, 且

$$f(1,1)=1,\quad \frac{\partial f}{\partial x}|_{(1,1)}=2,\quad \frac{\partial f}{\partial y}|_{(1,1)}=3,$$

$\varphi(x)=f(x,f(x,x))$, 求$\frac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)|_{x=1}$.

解: 由题设$\varphi(1)=f(1,f(1,1))=f(1,1)=1$. 故

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}x}\varphi^3(x)|_{x=1}=3\varphi^2(x)\frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\Big|_{x=1}\\ =&3\left[f_1'(x,f(x,x))+f_2'(x,f(x,x))(f_1'(x,x)+f_2'(x,x))\right]|_{x=1}\\ =&3[2+3(2+3)]=51.\end{aligned}$$

练习 8.15. 设$z=f(x,y)$有连续偏导数, 且$x=r\cos\theta$, $y=r\sin\theta$, 证明:

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial r^2}+\frac{1}{r}\frac{\partial z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2}=\frac{1}{r^2}\left[r\frac{\partial}{\partial r}\left(r\frac{\partial z}{\partial r}\right)+\frac{\partial^2 z}{\partial\theta^2}\right].$$

练习 8.16. 设$w=f(x+y+z,xyz)$, f具有二阶连续偏导数, 求$\frac{\partial w}{\partial x}$, $\frac{\partial^2 w}{\partial x\partial z}$.

$$\begin{aligned}\frac{\partial w}{\partial x}&=f_1'(x+y+z,xyz)+yzf_2'(x+y+z,xyz),\\ \frac{\partial^2 w}{\partial x\partial z}&=f_{11}''+y(x+z)f_{12}''+xy^2zf_{22}''+yf_2'.\end{aligned}$$

练习 8.17. 设$f(u,v)$具有二阶连续偏导数, 且满足$\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}=1$, 设$g(x,y)=f(xy,(x^2-y^2)/2)$, 求$\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}$.

$$\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}=(x^2+y^2)(f''_{11}+f''_{22})=x^2+y^2.$$

练习 8.18. 设$g(y)$在实轴上有二阶连续导数，$\phi(\xi,\eta)$有二阶连续偏导数，$f(x,y)=\phi(x,x+y)+g(xy)$，求$\frac{\partial^2 f(x,y)}{\partial y\partial x}$.

$$\frac{\partial f(x,y)}{\partial x}=\phi'_1+\phi'_2+g'(xy)y,\quad \frac{\partial^2 f(x,y)}{\partial y\partial x}=\phi''_{12}+\phi''_{22}+g'(xy)+g''(xy)xy.$$

练习 8.19. 设$u=f(x,xy,xyz)$，其中f具有连续的二阶偏导数，求$\frac{\partial^2 f}{\partial y\partial z}$.

$xf'_3+x^2yf''_{32}+x^2yzf''_{33}$

练习 8.20. 设$z=y^2f\left(\frac{y}{x},\frac{x}{y^2}\right)$，其中$f(u,v)$具有连续的二阶偏导数，求$\frac{\partial^2 z}{\partial x\partial y}$.

$-\frac{3y^2}{x^2}f'_1-\frac{y^3}{x^3}f''_{11}+\frac{3}{x}f''_{12}-\frac{2x}{y^3}f''_{22}$

练习 8.21. 设函数$z=f(xy,yg(x))$，其中函数f具有二阶连续偏导数，函数$g(x)$可导且在$x=1$处取得极值$g(1)=1$，求$\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{\substack{x=1\\y=1}}$.

由题意知$g'(1)=0$. $\left.\frac{\partial^2 z}{\partial x\partial y}\right|_{\substack{x=1\\y=1}}=f''_{11}(1,1)+f''_{12}(1,1)+f'_1(1,1)$