

# Honors Mathematics IV

## Ordinary Differential Equations

With Elements of Linear Algebra and Complex Analysis

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University of Michigan - Shanghai Jiaotong University  
Joint Institute

Fall Term 2020

# Office Hours, Email, TAs

- ▶ Please read the Course Description, which has been uploaded to the Files section on the Canvas course site.
- ▶ My office is Room 441c in the Longbin Building.
- ▶ My email is [horst@sjtu.edu.cn](mailto:horst@sjtu.edu.cn) and I'll try to answer email queries within 24 hours.
- ▶ Office hours will be announced on Canvas.
- ▶ Please also make use of the Discussions page on Canvas for asking questions, making comments or giving feedback on the course.
- ▶ The recitation class schedule and the TA office hours and contact details will be announced on Canvas.

# Coursework

- ▶ There will be weekly coursework (assignments) throughout the term, except in the first week.
- ▶ You will be randomly assigned into **assignment groups** of three students; you are expected to collaborate within each group and hand in a single, common solution paper to each coursework.
- ▶ Each student must achieve **60%** of the total coursework points by the end of the term in order to obtain a passing grade for the course. However, the assignment points have **no effect on the course grade**.
- ▶ Each member of an assignment group will receive the same number of points for each submission. However, there will be an opportunity for team members to anonymously evaluate each others' contributions to the assignments. In cases where one or more group members consistently do not contribute a commensurate share of the work, a TA will investigate the situation and individual group members may lose some or all of their marks.

# Coursework

- ▶ Please hand in your coursework on time, by the date given on each set of course work. Late work will not be accepted unless you come to me personally and I find your explanation for the lateness acceptable.
- ▶ You can be deducted up to **10% of the awarded marks for an assignment** if you fail to write neatly and legibly.
- ▶ You are encouraged to compose your coursework solutions in  $\text{\LaTeX}$ . While this is optional, there will be a **10% bonus to the awarded marks** for those assignment handed in as typed  $\text{\LaTeX}$  manuscripts.

$\text{\LaTeX}$  is open-source software for mathematical typesetting, and there are various implementations available. I suggest that you use Baidu or Google to find a suitable implementation for your computer and OS.  $\text{\LaTeX}$  is widely used for writing theses and scientific papers, so it may be quite useful for you to learn it.

- ▶ Further details can be found in the course description.

## Use of Wikipedia and Other Sources; Honor Code Policy

- ▶ The correct way of using outside sources is to understand the contents of your source and then to write in your own words and without referring back to the source the solution of the problem. Your solution should differ in style significantly from the published solution. **If you are not sure whether you are incorporating too much material from your source in your solutions, then you must cite the source that you used.**
- ▶ You may and are required to collaborate freely with other students in your assignment group. However, you may not communicate at all about concrete coursework with students from other groups. However, discussing general questions regarding the lecture contents with any other student is of course fine and encouraged.

**Do not show or explain your solutions to any student outside your assignment group.**

## Use of Wikipedia and Other Sources; Honor Code Policy

In this course, the following actions are examples of violations of the Honor Code (“another student” means a student outside your assignment group):

- ▶ Showing another student your written solution to a problem.
- ▶ Sending a screenshot of your solution via QQ, email or other means to another student.
- ▶ Showing another student the written solution of a third student; distributing some student’s solution to other students.
- ▶ Viewing another student’s written solution.
- ▶ Copying your solution in electronic form ( $\text{\LaTeX}$  source, PDF, JPG image etc.) to the computer hardware (flash drive, hard disk etc.) of another student. Having another student’s solution in electronic form on your computer hardware.

If you have any questions regarding the application of the Honor Code, please contact me or any of the TAs.

## Grading Policy

- ▶ The grade will be composed of the course work and the exams as follows:
  - ▶ Course Outcome Quizzes: 10 points
  - ▶ First midterm exam: 30 points
  - ▶ Second midterm exam: 30 points
  - ▶ Final exam: 30 points
- ▶ The actual grading scale will **usually** be based on the top approximately 6-12% of students receiving a grade of A+, with the following grades determined by (mostly) fixed point increments.
- ▶ Apart from this normalization, the grade distribution is up to you! If (for example) all students obtain many points in the exams, I am happy to see everyone receive a grade of A. Students are primarily evaluated with respect to a fixed point scale, not with respect to each other.

## Course Grade Example – Vv286 in Fall 2017

Points	Grade	No. of students	% of students	Percentile
0	F	0	0.0%	0.0%
35	D	6	6.5%	0.0%
40	C–	9	9.7%	6.5%
45	C	2	2.2%	16.1%
50	C+	7	7.5%	18.3%
55	B–	6	6.5%	25.8%
65	B	11	11.8%	32.3%
70	B+	10	10.8%	44.1%
75	A–	6	6.5%	54.8%
80	A	27	29.0%	61.3%
90	A+	9	9.7%	90.3%

# Piazza

- ▶ In addition to office hours, I will be answering course-related questions on Piazza. Please also create an account such that your name in pinyin is visible.
- ▶ It is possible to send private messages on Piazza, but most messages should be public so that everyone can see them and the responses or respond themselves. Feel free to answer other students' questions!
- ▶ Please do not post anonymously unless you have a good reason. Don't be shy!

## Mathematica

JI has obtained an unlimited student license for a computer algebra software called **Mathematica**, developed by Wolfram Research. You will be required to make use of the software in your homework assignments, so you should obtain a copy as follows:

- (i) Visit

<https://user.wolfram.com/portal/registration.html>

and create a Wolfram ID. You must use an @sjtu.edu.cn email address and give your first and last names in pinyin (example: Xu Baishen enters last name: Xu and first name: Baishen).

- (ii) Next, visit

<https://user.wolfram.com/portal/requestAK/c51e79e5334a3600a4f740a2b3720961216dbc17>

and request an Activation Number.

# Mathematica

Make a note of the activation number. You will be directed to a page where you can download the installation binaries for the most current version of Mathematica. (You must select whether you want those for Windows, Linux or OS X.) The software binaries are several GB in size; it may perhaps be possible to share them amongst yourselves to save download time. Try it out and let me know.

- (iii) After downloading, you can install the software. You will be asked to enter the Activation Number you noted above and you will need internet access. Mathematica will then run on a temporary two-week license. Your name will be checked against a list, and if successful, the license will automatically be extended for one year. Therefore, it is very important that you enter your name properly when you request the Wolfram ID.

## Class Attendance and Absence for Medical Reasons

I do not formally require that you attend every class. However, if you are unable to attend a significant number of lectures, you should notify me.

The following rules have been laid down by the Academic Office:

- ▶ A student who has been absent from studies for more than one week because of illness or other emergency should consult the program advisor. **[and also talk to me!]**
- ▶ Absence for illness should be supported by a hospital/doctor's certificate. A note that a student visited a medical facility is **not sufficient** excuse for missing an assignment or an exam. The note must specifically indicate that the student was incapable of completing an assignment or taking the exam due to medical problems.

# Class Attendance and Absence for Medical Reasons

- ▶ **Late** medical excuses must satisfy the following criteria to be valid:
  - (i) The problem must be confirmed by the doctor to be so severe that the student could not participate in the exam.
  - (ii) The problem must have occurred so suddenly that it was impractical to contact me in advance.
  - (iii) The student must be in contact with me immediately after the exam with the required documentation.

# Introduction

# Differential Equations

This course is about differential equations. These are equations for an unknown function  $f$ , where the equation involves one or more derivatives of  $f$ . Such equations occur frequently and naturally in physics.

Perhaps the most familiar example involves Newton's law:  $F = m \cdot a$ , force equals mass times acceleration. Since acceleration is simply the second derivative of the position  $x$  as a function of time  $t$ , we can write this as

$$F = m \cdot \ddot{x},$$

using Newton's notation for the derivatives.

Depending on the force  $F$ , this can lead to a variety of differential equations.

## Differential Equations

In the case of a 1D harmonic oscillator satisfying Hooke's law,  $F = -k \cdot x$  and the position is a scalar function of time. Then Newton's law yields

$$m\ddot{x} + kx = 0, \quad x = x(t), \quad t \in \mathbb{R}.$$

This is a **second-order** differential equation (since it involves second derivatives of the unknown function) and also an **ordinary** differential equation since only ordinary (not partial) derivatives are involved.

Newton's law is the main reason why second-order differential equations are the most common type of differential equations encountered in applications.

# Partial Differential Equations

An example of a ***partial differential equation*** is

$$\Delta u = 0, \quad u: \Omega \rightarrow \mathbb{R}, \quad \Omega \subset \mathbb{R}^n,$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

The above equation occurs in electrostatics as well as in the study of ideal fluid flow. We will discuss partial differential equations in the last part of our course.

For the most part of our course we will focus on ordinary differential equations and we start with the simplest kind, involving only derivatives of first order.

# Vector Fields

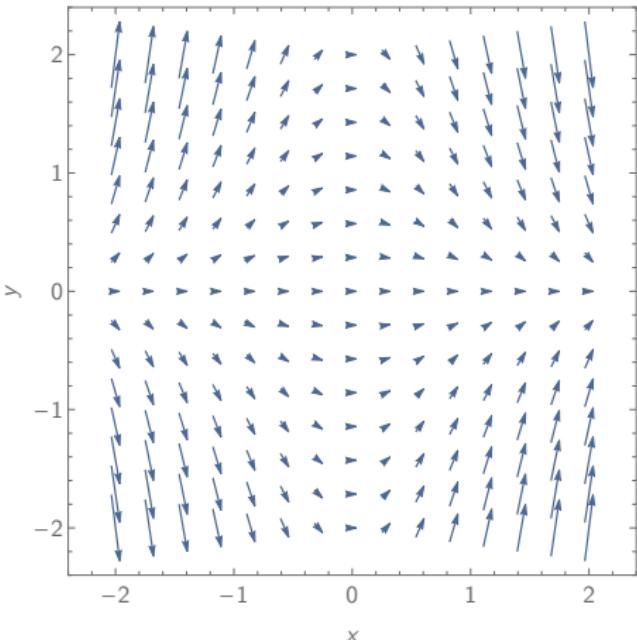
Recall from Vv285 that a vector field on  $\mathbb{R}^2$  is a map

$$F: \mathbb{R}^2 \rightarrow \mathbb{R}^2,$$

$$F(x, y) = \begin{pmatrix} F_1(x, y) \\ F_2(x, y) \end{pmatrix}.$$

A vector field can typically be represented by drawing “direction arrows” at a given point  $(x, y) \in \mathbb{R}^2$ , where the arrow indicates the vector  $F(x, y)$ . The length of the arrow corresponds to the magnitude of  $F$ .

Examples include force fields in mechanics, fields in electrostatics and velocity fields in fluid dynamics.



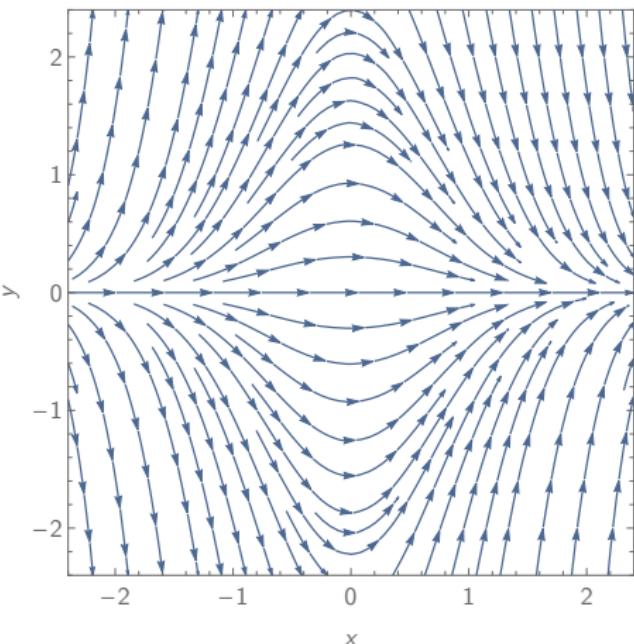
# Trajectories

In the case of a velocity field, we would like to find the **trajectory**

$$\gamma: \mathbb{R} \rightarrow \mathbb{R}^2, \quad \gamma(t) = \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

of a particle in the fluid, i.e., a curve so that at every point the velocity vector field is the tangent to the curve. This involves solving  $\gamma'(t) = F(x(t), y(t))$ , the **system of differential equations**

$$\begin{aligned} \frac{dx}{dt} &= F_1(x, y), \\ \frac{dy}{dt} &= F_2(x, y). \end{aligned} \tag{I.1}$$



## Direction Fields

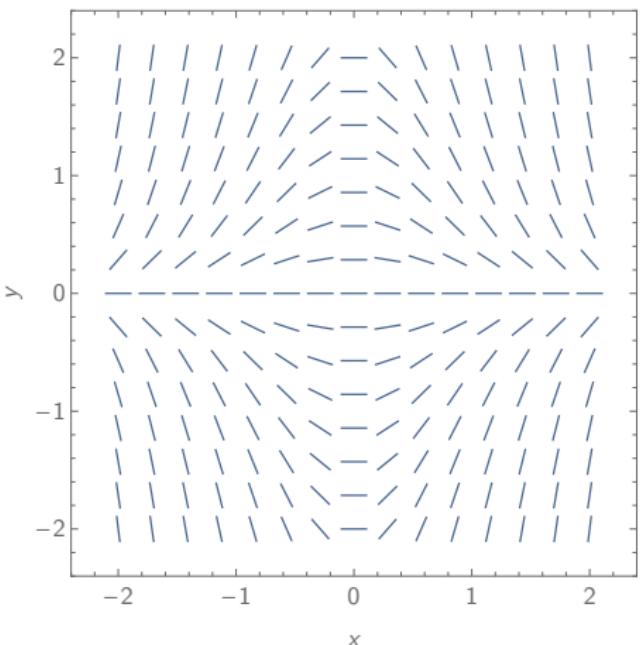
Solving such a system is clearly a very difficult task! So we might set our sights a little lower and instead try to find the **path** of a particle in a velocity field, i.e., simply the range of the trajectory that it travels through.

In that case, neither the length nor the orientation of the velocity field vectors matter, only the slope. We therefore plot the so-called **direction field**.

The direction field consists of **line elements**

$$(x, y, p)$$

where  $p \in \mathbb{R}$  is the slope of the line element at  $(x, y) \in \mathbb{R}^2$ .



## Direction Fields

A trajectory for the direction field will give us the path travelled by the corresponding trajectory of the original vector field. We will denote the direction field by  $G$ ; it is essentially a normalized version of the original vector field  $F$ .

Let us consider a vector field defined on a connected, open set  $\Omega \subset \mathbb{R}^2$ ,  $F: \Omega \rightarrow \mathbb{R}^2$ , and make a simplifying assumption: we suppose that for all  $(x, y) \in \Omega$ , the first component  $F_1(x, y)$  is non-zero. (This means that there are no “vertical” vectors in the vector field.)

In that case, the direction field can be written as

$$G: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad G(x, y) = \begin{pmatrix} 1 \\ f(x, y) \end{pmatrix}$$

where  $p = f(x, y)$  is the slope of the line element at  $(x, y) \in \mathbb{R}^2$ .

## A Differential Equation for the Path

Under our current assumption, the direction field has no vertical vectors. Therefore, the trajectories have no vertical tangent vectors and by the Implicit Function Theorem (see Vv285) they can be parametrized using the  $x$ -variable:

$$\gamma: I \rightarrow \Omega, \quad \gamma(x) = \begin{pmatrix} x \\ y(x) \end{pmatrix},$$

for a suitable interval  $I \subset \mathbb{R}$ .

Then the tangent vector is given by

$$\gamma'(x) = \begin{pmatrix} 1 \\ y'(x) \end{pmatrix} = \begin{pmatrix} 1 \\ f(x, y) \end{pmatrix} = G(x, y)$$

and we can find the trajectories by solving the differential equation

$$y'(x) = f(x, y). \quad (\text{I.2})$$

# A Differential Equation for the Path

Obviously, solving (I.2),  $y' = f(x, y)$ , is a much easier task than solving the system (I.1). But of course (I.2) allows us only to find the path of a trajectory, not the (time-dependent) trajectory itself.

I.1. Example. The previous slides show the vector field

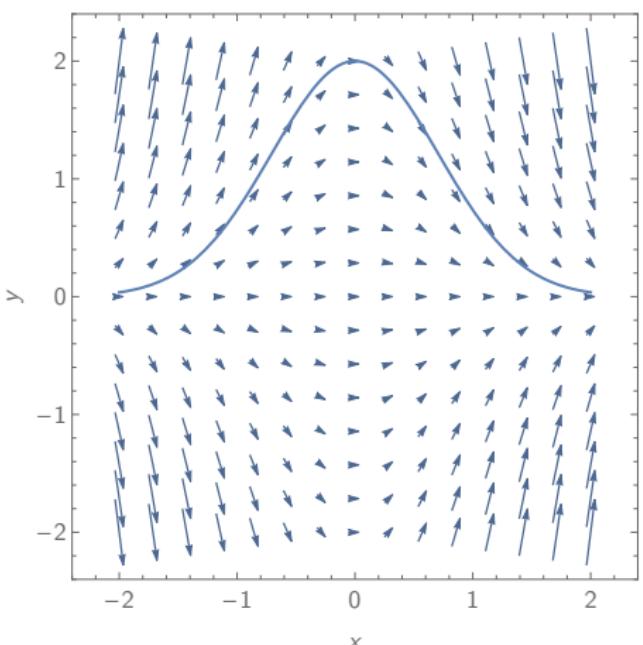
$$F(x, y) = \begin{pmatrix} 1 \\ -2xy \end{pmatrix}$$

and paths are found from solving

$$y' = -2x \cdot y.$$

Solutions are given by

$$y(x) = c \cdot e^{-x^2}, \quad c \in \mathbb{R}.$$



## Terminology: Types of Differential Equations

Both equations (I.1) and (I.2) are ***ordinary differential equations*** (ODEs), since they involve derivatives of an unknown function of a single variable.

The ***order*** of a differential equation is simply the order of the highest derivative occurring in the equation. Both (I.1) and (I.2) are ***first-order*** equations.

One may regard (I.1) as being a single equation for a vector-valued function  $(x(t), y(t))$  or as a ***system of equations*** for the two single variable functions  $x(t)$  and  $y(t)$ .

A system is said to be ***coupled*** if it is impossible to solve the individual equations successively, but we must solve them all at once. Otherwise, it is said to be ***decoupled***.

Solving systems of equations is difficult. We will return to systems in a later section.

## Terminology: Types of Differential Equations

Both (I.1) and (I.2) are examples of ***explicit ODEs***, since they involve a derivative of a (scalar or vector-valued) function on the left-hand side of an equation and an expression involving the function on the right-hand side.

More complicated equations will involve functions of derivatives (for example,  $(y')^2$ ) and are called ***implicit equations***.

If a differential equation involves a function of several variables and corresponding partial derivatives, it is called a ***partial differential equation*** (PDE). We will give a brief introduction to some classical PDEs in the last part of the course.

## Terminology: Solutions of Differential Equations

A **solution** for an explicit, first-order differential equation on an open interval  $I \subset \mathbb{R}$ , given by

$$y'(x) = f(x, y), \quad x \in I \quad (I.3)$$

for a continuous function  $f: I \rightarrow \mathbb{R}^n$ , is a **continuously differentiable function**  $y: I \rightarrow \mathbb{R}^n$  such that (I.3) holds. If  $f$  is not continuous, the concept of a solution is a little more subtle.

The set of **general solution** of (I.3) is a family (set) of functions such that each solution of (I.3) is included in the set of general solutions and each element of the set of general solutions is a solution to (I.3).

## Terminology: Integral Curves and Initial Value Problems

I.2. **Remark.** The problem of finding the path of a trajectory of a vector field leads to that of solving an explicit, first-order ordinary differential equation  $y' = f(x, y)$ . Conversely, given such an ODE, we can plot a corresponding direction field. Solutions to the ODE correspond to trajectories of that direction field.

The trajectories of the direction field are also called **integral curves** of the differential equation  $y' = f(x, y)$ .

We may be led to expect that for each point  $(x, y) \in \mathbb{R}^2$  there is a single trajectory through this point. Whether this is actually the case remains to be investigated. Nevertheless, we say that (I.3) together with an **initial condition**,

$$y' = f(x, y), \quad y(\xi) = \eta$$

for  $\xi \in \bar{I}$  and  $\eta \in \mathbb{R}^n$  is an **initial value problem**.

## Part I

# Differential Equations of First Order

1. Separable Equations
2. Linear Equations
3. Transformable Equations
4. General Integral Curves of First Order ODEs
5. Implicit Equations
6. Systems of Equations
7. The Eigenvalue Problem
8. The Spectral Theorem for Self-Adjoint Matrices
9. Non-Diagonalizable Matrices
10. Solutions to Inhomogeneous, Linear Systems
11. Linear Second-Order Equations and Vibrations

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## Natural Growth and Decay

Of course, first-order differential equations also appear in contexts which initially appear to have nothing to do with the problem of finding trajectories of vector fields. One of the most common examples is the following:

In many physical situations, the rate of growth or decay of a quantity  $x$  as a function of time  $t$  is proportional to its size, i.e.,

$$x'(t) = \beta \cdot x(t) \quad \text{for some } \beta \in \mathbb{R}. \quad (1.1.1)$$

The number  $\beta$  is called the **growth constant** and can be positive or negative, depending on whether  $x$  increases or decreases with time.

Examples include

- ▶ Unlimited growth of a biological population;
- ▶ Cooling or warming of an object to ambient temperature;
- ▶ Weakening of emissions of a radioactive substance.

## Natural Growth and Decay

Let us make a very naive attempt at solving this differential equation:

$$\begin{aligned}\frac{dx}{dt} = \beta \cdot x &\Rightarrow \frac{1}{x} dx = \beta dt \\ &\Rightarrow \int \frac{1}{x} dx = \int \beta dt \\ &\Rightarrow \ln(x) = \beta t + C \\ &\Rightarrow x(t) = e^{\beta t + C} \\ &\Rightarrow x(t) = x(0) \cdot e^{\beta t}\end{aligned}$$

(The integration constant  $C \in \mathbb{R}$  has been eliminated in favor of the constant  $x(0)$ .)

The steps take above seem ridiculous, but the solution obtained does satisfy the differential equation (check this!). In fact, we have found the general solution, as we now show.

# Solution of $y' = \beta y$

1.1.1. Proposition. The unique solution to the initial value problem

$$y' = \beta y, \quad y(0) = y_0 \quad (1.1.2)$$

on  $\mathbb{R}$  is given by

$$y(x) = y_0 e^{\beta x}. \quad (1.1.3)$$

Proof.

It is easy to see that (1.1.3) solves (1.1.2). To see that this solution is unique, suppose that  $\phi$  is an arbitrary solution of (1.1.2). Then

$$(\phi e^{-\beta x})' = (\phi' - \beta \phi) e^{-\beta x} = 0,$$

so  $\phi(x) = c \cdot e^{\beta x}$ . Since  $c = \phi(0) = y_0$ , we see that  $\phi(x) = y_0 e^{\beta x}$ . □

# Solution of $y' = \beta y$

## 1.1.2. Remarks.

- ▶ Depending on the context, we may write  $x = x(t)$  or  $y = y(x)$  for a function of a variable. The former is often used when the variable has the interpretation of a time-like quantity.
- ▶ Since every solution to the differential equation will have some real value at  $x = 0$ , by showing that for each such value  $y(0)$  there exists exactly one solution, we have established that the family of functions of the form

$$y(x; c) = c \cdot e^{\beta x}, \quad c \in \mathbb{R},$$

is the general solution of the ODE.

- ▶ While the “unorthodox” (to say the least) calculation worked in our case, we still need to establish whether we were simply lucky or whether it actually can be used as a general method.

## Separation of Variables - Informally

Suppose that an equation of the form  $y' = f(x) \cdot g(y)$  is given, for suitable functions  $f$  and  $g$ . Then the method of **separation of variables** proceeds as follows: from

$$\frac{dy}{dx} = f(x)g(y) \quad \text{we obtain} \quad \frac{dy}{g(y)} = f(x) dx.$$

Integrating both sides, we have

$$\int \frac{dy}{g(y)} = \int f(x) dx$$

which can be solved for  $y$  to obtain a solution.

To satisfy an initial condition  $y(\xi) = \eta$  for given  $\xi, \eta \in \mathbb{R}$ , we insert the appropriate limits in the integrals,

$$\int_{\eta}^y \frac{ds}{g(s)} = \int_{\xi}^x f(t) dt.$$

## Separation of Variables - Formal Statement

The following theorem gives conditions under which this procedure is allowed. We impose a general hypothesis,

$f$  is continuous in an interval  $I_x \subset \mathbb{R}$ ;

$g$  is continuous in an interval  $I_y \subset \mathbb{R}$ ; (Hyp)

$\xi \in I_x, \eta \in I_y$

**1.1.3. Theorem.** Let  $\eta$  be an interior point of  $I_y$  such that  $g(\eta) \neq 0$  and let (Hyp) hold. Then there exists a neighbourhood of  $\xi$  in  $I_x$  in which the IVP

$$y' = f(x)g(y), \quad y(\xi) = \eta \quad (1.1.4)$$

has a unique solution  $y(x)$ . It can be obtained from

$$\int_{\eta}^y \frac{ds}{g(s)} = \int_{\xi}^x f(t) dt \quad (1.1.5)$$

by solving for  $y$ .

## Separation of Variables - Proof

Proof.

We write

$$G(y) := \int_{\eta}^y \frac{ds}{g(s)}, \quad F(x) := \int_{\xi}^x f(t) dt,$$

so that (1.1.5) becomes  $G(y) = F(x)$ . From the continuity of  $g$  we know that  $g(y) \neq 0$  in some neighborhood of  $\eta$ . Thus  $G(y)$  exists in this neighborhood.

Since  $G' = 1/g \neq 0$  in this neighborhood there exists an inverse function  $H = G^{-1}$ . Then

$$y(x) = H(G(y(x))) = H(F(x)) \tag{1.1.6}$$

We will first show that (1.1.6) solves (1.1.4), i.e., the solution is obtained from (1.1.5).

## Separation of Variables - Proof

Proof (continued).

Since  $H$  and  $F$  are differentiable, the chain rule gives

$$y' = H'(F(x))F'(x) = \frac{1}{G'(H(F(x)))}f(x) = \frac{1}{G'(y)}f(x) = g(y)f(x).$$

Thus (1.1.5) solves the differential equation. Furthermore,

$$y(\xi) = H(F(\xi)) = H(0) = G^{-1}(0) = \eta,$$

so the initial value is also satisfied.

## Separation of Variables - Proof

Proof (continued).

It now remains to check that the solution is unique. Suppose that  $z(x)$  is another solution. Then in a neighborhood of  $\xi$  we have

$$\frac{z'(x)}{g(z(x))} = f(x).$$

Thus, for  $x$  close to  $\xi$ , we have

$$F(x) = \int_{\xi}^x f(t) dt = \int_{\xi}^x \frac{z'(t)}{g(z(t))} dt = \int_{z(\xi)}^{z(x)} \frac{ds}{g(s)} = G(z(x)).$$

This gives  $y(x) = H(F(x)) = z(x)$ .



## Separation of Variables - The case $g(\eta) = 0$

Before we look at examples, we ask what happens when  $g(\eta) = 0$ . In that case, it is clear that a solution to

$$y' = f(x)g(y), \quad y(\xi) = \eta \quad (1.1.7)$$

can be found simply by setting

$$y(x) = \eta \quad (\text{constant}).$$

Whether other solutions exist, found through

$$\int_{\eta}^y \frac{ds}{g(s)} = \int_{\xi}^x f(t) dt$$

effectively depends on whether the integral on the left-hand side exists for  $y$  in a small neighborhood of  $\eta$ . The short answer: if it does not exist,  $y(x) = \eta$  is the unique solution. Otherwise, it is **possible** that there are more solutions.

## Application: Radioactive Decay

Suppose we have a number  $x(t)$  of atoms of radioactive material. We suppose that there exists a decay rate  $\beta > 0$  so that in a sufficiently small time interval  $\Delta t$  a proportion  $\beta \cdot \Delta t$  of the atoms will have decayed. More precisely,

$$x(t + \Delta t) = (1 - \beta \cdot \Delta t)x(t) + o(\Delta t) \quad \text{as } \Delta t \rightarrow 0.$$

This implies

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} = -\beta x(t) + \frac{o(\Delta t)}{\Delta t}$$

and letting  $\Delta t \rightarrow 0$ ,

$$x'(t) = \lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = -\beta x(t). \quad (1.1.8)$$

This is just the equation for natural decay. We have already seen that

$$x(t) = x(0) \cdot e^{-\beta t}.$$

## Decay Rate and Half-Life

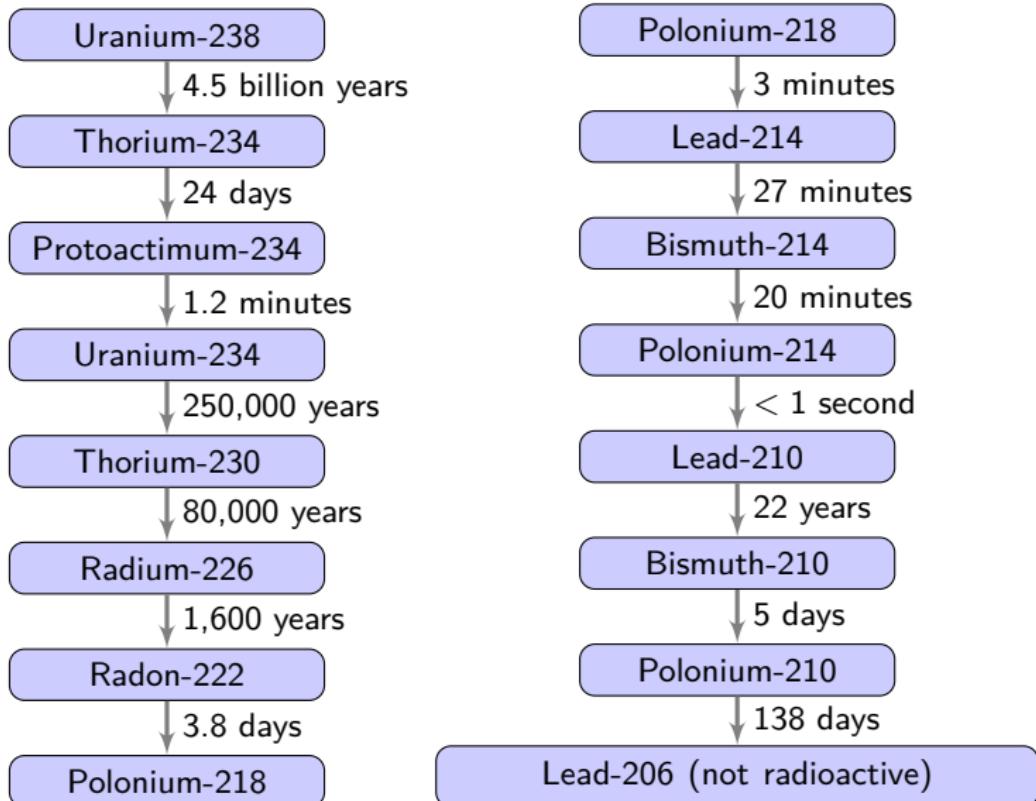
The number  $\beta$  is called the **decay constant** or **decay rate** of the substance.

We define the **half-life**  $T_h$  of a radioactive substance as the time after which half of the atoms have decayed:

$$e^{-\beta T_h} = \frac{1}{2}, \quad \text{i.e.,} \quad T_h = \frac{\ln 2}{\beta}.$$

In general, a radioactive element will not decay into an inert element, but rather into a different radioactive material. As an example, the next slide shows the **Uranium Series** of radioactive decay. The figures on the arrows give the half-life of each decay step and the numbers following each element are the number of protons and neutrons in the nucleus.

# The Uranium Series



## Radioactive Environments

Hence, a radioactive substance usually does not exist in isolation. Suppose that a given Substance 1 decays with rate  $\beta$  but is itself created continuously from another Substance 2 with rate  $\alpha \cdot \beta$ . Supposing that Substance 2 is present essentially in infinite amounts, so that the creation rate does not depend on anything else, the amount of Substance 1 is given by the differential equation

$$x'(t) = \alpha\beta - \beta x(t).$$

We solve this equation by separating variables:

$$\begin{aligned} \frac{dx}{dt} = \alpha\beta - \beta x &\Leftrightarrow \int \frac{dx}{\alpha - x} = \int \beta dt \\ &\Leftrightarrow -\ln(\alpha - x) = \beta x + C \\ &\Leftrightarrow x(t) = \alpha - (\alpha - x(0))e^{-\beta t} \end{aligned} \quad (1.1.9)$$

In our application, we are assuming  $\alpha, \beta > 0$ .

# Equilibrium, Steady-State, Transient Solutions

We will use the solution

$$x(t) = \alpha - (\alpha - x(0))e^{-\beta t}$$

to illustrate some basic ODE concepts. Given a solution  $x(t)$ , we define

- ▶ the **equilibrium solution**  $x_{\text{equi}}$  by

$$x_{\text{equi}}(t) = \text{constant},$$

describing a system that does not change over time. In our example,  $x_{\text{equi}}(t) = \alpha$  is the unique equilibrium solution.

- ▶ the **steady-state** solution  $x_{\text{ss}}$  by

$$x_{\text{ss}} = \lim_{t \rightarrow \infty} x(t),$$

which may depend on the initial conditions. In our case,  $x_{\text{ss}} = \alpha = x_{\text{equi}}$ . It is a deep result that the steady-state solution is often (but not always) equal to the equilibrium solution.

- ▶ the **transient** component by  $x(t) - x_{\text{ss}}$ .

# Radioactive Equilibrium

We see that after a sufficiently long timeframe, the amount of an isotope contained in a substance in contact with its environment will have become very close to its equilibrium amount  $\alpha$ . We then say that the substance is in **radioactive equilibrium** with its environment.

**1.1.4. Example.** In 1949, Willard Libby discovered a method for dating archaeological finds by analyzing the presence of the radioactive carbon isotope C-14, commonly called **radiocarbon**. (He received the nobel prize for Chemistry in 1960 for this work.)

Radiocarbon is created in the atmosphere from nitrogen, which combines with neutrons created from the bombardment with cosmic rays to give C-14. Radiocarbon becomes part of carbon dioxide, which is absorbed by plants and subsequently becomes part of the food chain.



Willard F. Libby (1908 – 1980)  
Nobelprize.org, Nobel Media AB 2014.  
Web. 24 Jul 2016.

## Radiocarbon Dating

Basically anything on earth that absorbs carbon from the atmosphere thus gains a certain proportion of C-14 isotopes. This is, in particular, the case for plants, herbivores that eat these plants, and humans/carnivores that eat the herbivores. While the rates of absorption ( $\beta$ ) will of course differ, all these entities will attain an equilibrium proportion ( $\alpha$ ) of radiocarbon equal to that in the atmosphere.

Radiocarbon dating is based on the fundamental assumption that the rate of bombardment of Earth's atmosphere by cosmic rays has always been constant. Thus the proportion of C-14 in the atmosphere has also been constant for all of known history. When a living object dies, it ceases absorbing C-14 from the Earth's atmosphere, and the equilibrium quantity of C-14 starts to decay without being replaced. By comparing the amount of C-14 left in archaeological finds with that present in current samples, it is possible to determine the age of the find.

# Rate of Disintegration

One measures the ***specific rate of disintegration***

$$\rho(t) = \frac{x'(t)}{m} = \beta \cdot \frac{x(t)}{m},$$

where  $x$  denotes the amount of radiocarbon in a given sample,  $\beta = \ln 2 / T_h$  is the decay constant and  $m$  is the mass of the sample. Living material will be in equilibrium, so the amount of C-14 will be constant, yielding a constant specific rate  $\rho_{\text{equi}}$ . For example, living wood produces 6.68 disintegrations per minute per gram.

The C-14 in dead matter will start to decay without being replenished, so that the disintegration rate of a sample of age  $t$  is

$$\rho_{\text{sample}} = e^{-\beta t} \rho_{\text{equi}} \quad (1.1.10)$$

# Radiocarbon Dating

From (1.1.10), we find the time the sample has been dead as

$$t = \frac{1}{\beta} \ln \frac{\rho_{\text{equi}}}{\rho_{\text{sample}}}.$$

assuming that the equilibrium amount of C-14 has not changed since the sample died.

**1.1.5. Example.** The half-life of C-14 is  $T_h = 5568$  years. Charcoal from the occupation level of the famous Lascaux cave in France gave an average count in 1950 of 0.97 disintegrations per gram per minute.

Thus the date of occupation is approximately

$$\begin{aligned} t &= 1950 \text{ AD} - \frac{1}{\beta} \ln \frac{6.68}{0.97} \\ &= 1950 \text{ AD} - \frac{5568}{\ln 2} \ln 6.887 \\ &\approx 13550 \text{ B.C.} \end{aligned}$$



A painting of the Giant Deer from Lascaux.

Wikimedia Commons. Wikimedia Foundation.

Web. 30 August 2015

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# Linear Equations

A general **linear, first-order** ordinary differential equation on an open interval  $I \subset \mathbb{R}$  has the form

$$a_1(x)y' + a_0(x)y = f(x), \quad x \in I, \quad (1.2.1)$$

where we allow  $a_0, a_1, f$  to be continuous, real-valued functions on  $I$ . The equation (1.2.1) is said to be **homogeneous** if  $f(x) = 0$  for all  $x \in I$ , otherwise it is called **inhomogeneous**.

We define the **differential operator**

$$L = a_1 \frac{d}{dx} + a_0$$

and then express (1.2.1) in the form

$$Ly = f.$$

Similarities to the algebraic systems of equation  $Ax = b$  that we treated in the last semester are not coincidental!

## Structure of Solutions

The equation

$$Ly = a_1(x)y' + a_0(x)y = 0, \quad x \in I, \quad (1.2.2)$$

is called the homogeneous equation associated to (1.2.1).

Suppose that  $y_1$  and  $y_2$  are two solutions of (1.2.2). Then

$$y(x) = c_1y_1(x) + c_2y_2(x), \quad \text{for any } c_1, c_2 \in \mathbb{R}$$

is also a solution of (1.2.2). This is sometimes known as the ***superposition principle***.

Given a solution  $y_{\text{part}}$  of  $Ly = f$  and any solution  $y_{\text{hom}}$  of  $Ly = 0$ , we find that

$$y_{\text{inhom}} := y_{\text{part}} + y_{\text{hom}}$$

solves  $Ly = f$ .

## Initial Value Problems for Linear Equations

Let  $I \subset \mathbb{R}$  be an open interval and  $\xi \in \bar{I}$  (i.e.,  $\xi$  may be a boundary point of  $I$ ). Then an initial value problem on  $I$  has the form

$$Ly = f \quad \text{on } I, \qquad y(\xi) = \eta \quad (1.2.3)$$

where  $\eta \in \mathbb{R}$  is called the **initial condition** for  $y$ . If  $\eta = 0$ , we say that (1.2.3) has **homogeneous initial conditions**. The pair

$$\{f, \eta\}$$

is called the **data** for (1.2.3).

An important observation is the following: If  $y_{\text{part}}$  solves (1.2.3) for data  $\{f, 0\}$  and  $y_{\text{hom}}$  solves for data  $\{0, \eta\}$ , then  $y_{\text{inhom}}$  solves the initial value problem for data  $\{f, \eta\}$ .

## Solving the Homogeneous Equation

The equation with initial value is

$$a_1(x)y' + a_0(x)y = 0.$$

Let us (for the moment) suppose that  $a_1(x) \neq 0$  for  $x \in I$  so that the equation is equivalent to an explicit equation for  $y$ . Then we have simply

$$y' = -\frac{a_0(x)}{a_1(x)}y$$

and the method of separation of variables (Theorem 1.1.3) can be applied to the homogeneous equation with initial value  $y(\xi) = \eta$  to yield the ***unique solution***

$$y_{\text{hom}}(x) = \eta \cdot e^{-\int_{\xi}^x \frac{a_0(t)}{a_1(t)} dt}. \quad (1.2.4)$$

## Solving the Inhomogeneous Equation

The inhomogeneous equation

$$a_1(x)y' + a_0(x)y = f(x)$$

is not separable. In order to derive a solution approach, let us consider the simpler equation we know from radioactive decay as derived on Slide 41. Suppose that the number of atoms  $x(t)$  is replenished at a rate  $f(t)$  (i.e., between time  $t$  and  $t + \Delta t$ ,  $\int_t^{t+\Delta t} f(\tau) d\tau$  atoms are added).

The equation (1.1.8) describing  $x(t)$  then becomes

$$x'(t) + \beta x(t) = f(t).$$

We additionally suppose that there are no atoms present at time  $t = 0$ , i.e.,

$$x(0) = 0.$$

## Solving the Inhomogeneous Equation

In order to approach the problem

$$x'(t) + \beta x(t) = f(t), \quad x(0) = 0 \quad (1.2.5)$$

we can take a different point of view: Suppose that at the instant  $t_1$ , there are exactly  $f(t_2)$  atoms present. These will decay according to the homogeneous equation  $x' - \beta x = 0$ . Now at instant  $t_2$ , an additional  $f(t_2)$  atoms are present. These will also start to decay. We might expect that the total number of atoms present at time  $T$  can be found by adding (integrating) all these contributions from time  $t = 0$  to  $t = T$ .

We would consider the initial value problem

$$x'(t) + \beta x(t) = 0, \quad x(\tau) = 1 \quad (1.2.6)$$

and denote the solution by  $x_\tau$ . Then

$$x(t) = \int_0^t f(\tau) x_\tau(t) d\tau$$

might be expected to solve (1.2.5).

## Duhamel's Principle

This approach of relating the solution of an inhomogeneous equation with homogeneous initial value to that of a homogeneous equation with inhomogeneous initial value is known as ***Duhamel's principle*** and quite general (it applies also to higher-order ODEs and to time-dependent PDEs). We will formulate a simple version here and introduce a more general one in the assignments.



**Jean-Marie Duhamel (1797 – 1872)**  
Wikimedia Commons. Wikimedia Foundation. Web. 31 Jul 2016.

## Duhamel's Principle

1.2.1. **Duhamel's Principle.** Let  $I \subset \mathbb{R}$  be an open interval,  $x_0 \in \bar{I}$ , and  $a_0, a_1, f$  continuous, real-valued functions on  $\bar{I}$ , where  $a_1(x) \neq 0$  for all  $x \in \bar{I}$ . Let  $y_\xi$  solve the initial value problem

$$a_1(x)y'_\xi + a_0(x)y_\xi = 0, \quad y_\xi(\xi) = \frac{1}{a_1(\xi)}$$

for  $x \in \bar{I}$ . Then

$$y(x) = \int_{x_0}^x f(\xi)y_\xi(x) d\xi.$$

solves

$$a_1(x)y' + a_0(x)y = f(x), \quad y(x_0) = 0.$$

## Duhamel's Principle

Proof.

From  $y(x) = \int_{x_0}^x y_\xi(\xi) f(\xi) d\xi$  it is clear that  $y(x_0) = 0$ , so  $y$  satisfies the initial condition. Furthermore, using  $y_\xi(\xi) = 1/a_1(\xi)$  for all  $\xi \in \bar{I}$ , we have

$$\begin{aligned}y'(x) &= \frac{d}{dx} \int_{x_0}^x y_\xi(\xi) f(\xi) d\xi \\&= y_{x_0}(x)f(x) + \int_{x_0}^x y'_\xi(x)f(\xi) d\xi \\&= \frac{f(x)}{a_1(x)} + \int_{x_0}^x y'_\xi(x)f(\xi) d\xi\end{aligned}$$

from which we immediately see that

$$a_1(x)y' + a_0(x)y = f(x) + \underbrace{\left(a_1(x)y'_\xi(x) + a_0(x)y_\xi(x)\right)}_{=0} f(\xi) d\xi. \quad \square$$

## Solution to the Inhomogeneous Equation

From (1.2.4) we know that the solution to problem with data  $\{\eta, 0\}$  is given by

$$y_{\text{hom}}(x) = \eta \cdot e^{-G(x)}, \quad G(x) := \int_{\xi}^x \frac{a_0(t)}{a_1(t)} dt.$$

The function  $y_s(x)$ ,  $s \in \bar{I}$ , is then given by (check this!)

$$y_s(x) = \frac{e^{G(s)}}{a_1(s)} e^{-G(x)}$$

and a solution to the problem with data  $\{0, f\}$  is then

$$y_{\text{part}}(x) = \int_{\xi}^x f(s) y_s(x) ds = e^{-G(x)} \int_{\xi}^x \frac{f(s)}{a_1(s)} e^{G(s)} ds. \quad (1.2.7)$$

## Solution to the Initial Value Problem

From the superposition principle, we see that the solution to

$$a_1(x)y' + a_0(x)y = f(x) \quad \text{on } I, \qquad y(\xi) = \eta \quad (1.2.8)$$

is given by

$$y(x) = \eta \cdot e^{-G(x)} + e^{-G(x)} \int_{\xi}^x \frac{f(s)}{a_1(s)} e^{G(s)} ds$$

for  $x$  in a neighborhood of  $\xi$ .

### 1.2.2. Remarks.

- ▶ The solution to (1.2.8) is unique, as will be shown in the assignments.
- ▶ We will treat the case where  $a_1(x) = 0$  for some  $x$  in a later section.

## General Solution to an Inhomogeneous, Linear Equation

1.2.3. Example. We want to find the general solution to the inhomogeneous linear equation

$$y' + y \sin(x) = \sin^3(x).$$

First, note that the associated homogeneous equation

$$y' + y \sin(x) = 0$$

has the solution  $y_{\text{hom}}(x) = c \cdot e^{\cos(x)}$ ,  $c \in \mathbb{R}$ . We can find a particular solution of the inhomogeneous equation from our formula, yielding

$$\begin{aligned} y_{\text{part}}(x) &= e^{\cos(x)} \int_0^x \sin^3(t) e^{-\cos(t)} dt = e^{\cos(x)} \int_1^{\cos(x)} (s^2 - 1) e^{-s} ds \\ &= \sin^2(x) - 2 \cos(x) - 2 + 4e^{\cos(x)-1}. \end{aligned}$$

Thus the general solution is

$$y_{\text{inhom}}(x; c) = \sin^2(x) - 2 \cos(2) - 2 + c \cdot e^{\cos(x)}, \quad c \in \mathbb{R}.$$

## Problems at Singular Points

1.2.4. Definition. Let  $L = a_1(x) \frac{d}{dx} + a_0(x)$  for suitable functions  $a_0, a_1$ . If  $a_1(x_0) = 0$ , we say that  $x_0$  is a **singular point** for  $L$ .

1.2.5. Example. Consider the equation

$$xy' - y = 0.$$

For  $x > 0$  and  $x < 0$  solutions are given by

$$y(x) = \begin{cases} c_1x, & x < 0, \\ c_2x, & x > 0, \end{cases} \quad c_1, c_2 \in \mathbb{R}.$$

Of course, defining the solution on all of  $\mathbb{R}$  is easy simply by choosing  $c_1 = c_2$  and defining  $y(0) = 0$ . The singular point at  $x = 0$  does not pose any serious problems here.

## Problems at Singular Points

1.2.6. Example. Now consider the very similar-looking equation

$$xy' + y = 0.$$

In this case, solutions are given by

$$y(x) = \begin{cases} \frac{c_1}{x}, & x < 0, \\ \frac{c_2}{x}, & x > 0, \end{cases} \quad c_1, c_2 \in \mathbb{R}.$$

(Check this!) Since both solutions diverge at  $x = 0$ , it is entirely unclear how to “glue together” the solutions on the intervals  $(-\infty, 0)$  and  $(0, \infty)$  to obtain a single solution on  $\mathbb{R}$ .

We will develop techniques to deal with singular points when we learn about differential equations in the complex plane, since then we can “go around” the singularity at  $x_0$  to glue together two separate solutions.

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## Transforming Equations

We now understand the solutions of two basic types of equation: separable equations and linear equations. Of course, there are many other types of first-order equations. The most important technique for solving these is to ***transform them into an equation whose solution is known.***

What this actually means is that one introduces a new function that is determined by a simpler differential equation and, once found by solving this simpler equation, immediately yields the solution to the original problem. The tools for doing this are nothing more than the standard methods of differential calculus.

$$y' = f(ax + by + c); \quad b \neq 0$$

Consider a differential equation of the form

$$y' = f(ax + by + c), \quad a, b, c \in \mathbb{R}, \quad (1.3.1)$$

where  $f$  is some continuous function and we may assume  $b \neq 0$ . (The case  $b = 0$  is easily solved as-is and does not need to be transformed.) Since (1.3.1) is not generally separable, we need to transform it into a more amenable form.

We define

$$u(x) := ax + by(x) + c,$$

Now suppose that  $y$  solves (1.3.1). Then  $u$  satisfies

$$u' = a + by' = a + bf(u).$$

This is a much simpler differential equation and can easily be solved for  $u$ .

$$y' = f(ax + by + c); \quad b \neq 0$$

On the other hand, if we find a function  $u$  such that

$$u' = a + bf(u), \quad (1.3.2)$$

then

$$y(x) = \frac{u(x) - ax - c}{b}$$

solves (1.3.1):

$$y' = \frac{1}{b}(a + bf(u) - a) = f(ax + by + c).$$

It follows that any solution  $y$  of (1.3.1) yields a solution  $u$  of (1.3.2) by transforming  $y$  to  $u$  (and vice-versa).

$$y' = f(ax + by + c)$$

1.3.1. Example. We want to solve  $y' = (x + y)^2$ .

Here  $f(u) = u^2$ ,  $a = 1$ ,  $b = 1$  and  $c = 0$ . We hence first solve

$$u' = a + bf(u) = 1 + u^2.$$

Using the method of separation of variables,

$$\begin{aligned} \frac{du}{dx} = 1 + u^2 &\Leftrightarrow \frac{du}{1 + u^2} = dx \\ &\Leftrightarrow \int \frac{du}{1 + u^2} = \int dx \\ &\Leftrightarrow \arctan u = x + C \\ &\Leftrightarrow u = \tan(x + C). \end{aligned}$$

Then  $y(x; C) = (u(x) - ax - c)/b = \tan(x + C) - x$  is the general solution.

$$y' = f(y/x)$$

A first-order differential equation that can be written in the form

$$y' = f\left(\frac{y}{x}\right)$$

is called **homogeneous**. To solve such an ODE, we use the transformation

$$u(x) = \frac{y(x)}{x}, \quad x \neq 0.$$

We can again show that

$$u' = \frac{f(u) - u}{x} \tag{1.3.3}$$

if and only if  $y' = f(y/x)$ . (Check this yourselves as an exercise!) The ODE (1.3.3) is separable and can be solved for  $u$  using the appropriate methods, whence we obtain  $y$  from

$$y(x) = x \cdot u(x).$$

$$y' = f(y/x)$$

1.3.2. Example. We want to solve  $y' = \frac{y}{x} - \frac{x^2}{y^2}$ ,  $y(1) = 1$ .

Here  $f(z) = z - 1/z^2$ . We hence first solve

$$u' = \frac{f(u) - u}{x} = \frac{-1}{xu^2}, \quad u(1) = \frac{y(1)}{1} = 1.$$

Using the method of separation of variables,

$$\begin{aligned} \frac{du}{dx} = \frac{-1}{xu^2} &\Leftrightarrow \int_1^u z^2 dz = - \int_1^x \frac{dt}{t} \\ &\Leftrightarrow \frac{u^3 - 1}{3} = -\ln x \\ &\Leftrightarrow u = \sqrt[3]{1 - 3\ln x}. \end{aligned}$$

Then  $y(x) = x \cdot u(x) = x\sqrt[3]{1 - 3\ln x}$  solves the initial value problem in the interval  $I = (0, \sqrt[3]{e})$ .

$$y' + gy + hy^\alpha = 0, \quad \alpha \neq 1 \quad (\text{Bernoulli's equation})$$

This non-linear equation can be transformed into a linear equation. We assume that  $g$  and  $h$  are continuous on some interval  $I$ . For general  $\alpha \in \mathbb{R}$ , we need to assume that  $y(x) > 0$  for all  $x \in I$ , so we first discuss this case.

After multiplying with  $(1 - \alpha)y^{-\alpha}$  the equation becomes

$$(y^{1-\alpha})' + (1 - \alpha)g(x)y^{1-\alpha} + (1 - \alpha)h(x) = 0$$

Substituting  $u(x) = y^{1-\alpha}(x)$ , we obtain

$$u' + (1 - \alpha)g(x)u + (1 - \alpha)h(x) = 0 \quad (1.3.4)$$

which is a linear equation for  $u$ . Any strictly positive solution of (1.3.4) can be transformed via  $y(x) = u^{1/(1-\alpha)}(x)$  to yield a strictly positive solution of Bernoulli's equation.

$$y' + gy + hy^\alpha = 0, \quad \alpha \neq 1 \quad (\text{Bernoulli's equation})$$

An initial value  $y(\xi) = \eta > 0$  becomes  $u(\xi) = \eta^{1-\alpha} > 0$ . There exists a unique solution to (1.3.4) with this initial value and this solution will be strictly positive in some neighborhood of  $\xi$ , yielding a (unique) solution of the initial value problem for Bernoulli's equation.

If  $\alpha > 0$ , we may allow  $y(x) = 0$  and in fact  $y = 0$  is a trivial solution of Bernoulli's equation.

If  $\alpha \in \mathbb{Z}$  is a (positive or negative) integer, the function  $y$  may take on arbitrary values. We distinguish two cases:

- ▶  $\alpha$  is odd. Then

$$(-y)' + g(-y) + h(-y)^\alpha = -(y' + gy + hy^\alpha)$$

so every positive solution  $y_+$  induces a negative solution  $y_- = -y_+$  and vice-versa. Solutions with initial conditions  $y(\xi) = \eta < 0$  can be accommodated by taking the negative of the solution with initial condition  $y(\xi) = -\eta > 0$ .

$$y' + gy + hy^\alpha = 0, \quad \alpha \neq 1 \quad (\text{Bernoulli's equation})$$

- $\alpha$  is even. Then  $1 - \alpha$  is odd and  $y(x) < 0$  implies  $u(x) = y^{1-\alpha}(x) < 0$ . We solve (1.3.4) and set

$$y(x) = -|u(x)|^{\frac{1}{1-\alpha}}. \quad (1.3.5)$$

Given an initial value  $y(\xi) = \eta < 0$ , the solution of (1.3.4) with  $u(\xi) = \eta^{1-\alpha} < 0$  will be negative in a neighbourhood of  $\xi$  and (1.3.5) gives a negative solution with  $y(\xi) = \eta$ .

$$y' + gy + hy^\alpha = 0, \quad \alpha \neq 1 \quad (\text{Bernoulli's equation})$$

1.3.3. Example. We want to solve the equation

$$y' + \frac{y}{1+x} + (1+x)y^4 = 0.$$

Since  $\alpha = 4$ , negative as well as positive solutions are permitted. The function  $y = 0$  is a solution, too. We set

$$z = y^{1-\alpha} = \frac{1}{y^3}$$

and by (1.3.4) the transformed equation is

$$z' - \frac{3}{1+x}z - 3(1+x) = 0. \quad (1.3.6)$$

This is an inhomogeneous linear equation. The general solution of the associated homogeneous equation is (check!)

$$z_{\text{hom}}(x) = C \cdot (1+x)^3, \quad C \in \mathbb{R}.$$

$$y' + gy + hy^\alpha = 0, \quad \alpha \neq 1 \quad (\text{Bernoulli's equation})$$

In order to find a particular solution, we may use Duhamel's principle:  
finding first the homogeneous solution with  $z(s) = 1$ , we have

$$z_s(x) = \frac{(1+x)^3}{(1+s)^3}.$$

Then (the lower limit of the integral is irrelevant for us)

$$\begin{aligned} z_{\text{part}}(x) &= \int_1^x z_s(x) \cdot 3(1+s) \, ds = 3 \int_1^x \frac{(1+x)^3}{(1+s)^2} \, ds \\ &= -3(1+x)^2 + \frac{3}{2}(1+x)^3 \end{aligned}$$

Thus the general solution of (1.3.6) is

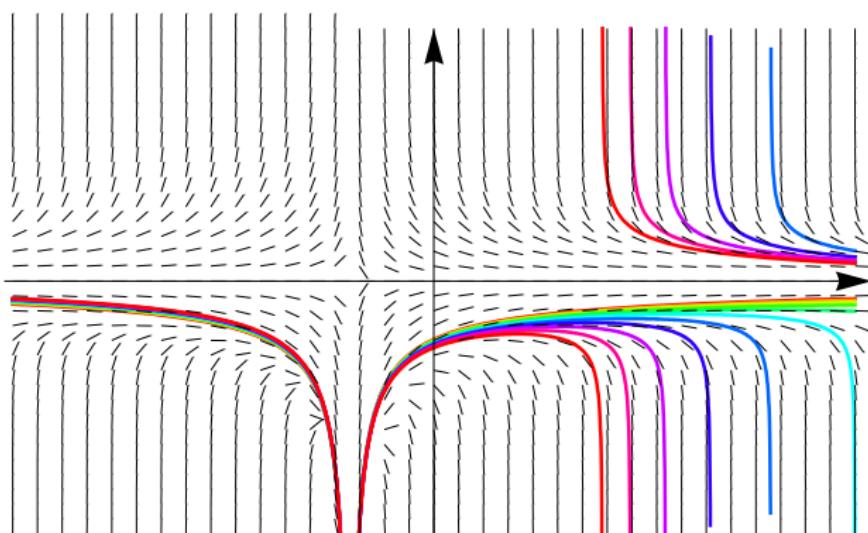
$$z_{\text{inhom}}(x; C) = C(1+x)^3 - 3(1+x)^2 = (1+x)^2(C + Cx - 3).$$

$$y' + gy + hy^\alpha = 0, \quad \alpha \neq 1 \quad (\text{Bernoulli's equation})$$

By (1.3.5),

$$y = \begin{cases} \frac{1}{\sqrt[3]{(1+x)^2(C+Cx-3)}} & \text{if } C + Cx - 3 \geq 0, \\ \frac{-1}{\sqrt[3]{(1+x)^2|C+Cx-3|}} & \text{if } C + Cx - 3 < 0. \end{cases} = \frac{\operatorname{sgn}(C + Cx - 3)}{\sqrt[3]{(1+x)^2|C + Cx - 3|}}$$

Some solution curves for  $C \in [0, 1]$  are plotted below:



$$y' + gy + hy^2 = k \text{ (Riccati's equation)}$$

A Riccati equation is essentially an “inhomogeneous” Bernoulli equation with  $\alpha = 2$ , i.e.,

$$y' + gy + hy^2 = k$$

where  $g$ ,  $h$  and  $k$  are assumed continuous on some interval  $I$ .

The solutions of this equation cannot be given explicitly even in simple particular cases. However, if one solution is known, all others can be calculated.

Let  $\phi$  be a known solution of the Riccati equation. Let  $y$  be any other solution. Then the difference between the two solutions,  $u = y - \phi$ , satisfies

$$u' + gu + h(y^2 - \phi^2) = 0.$$

With  $y^2 - \phi^2 = (y - \phi)(y + \phi) = u(u + 2\phi)$ , we have

$$u' + (g + 2\phi h)u + hu^2 = 0, \quad (1.3.7)$$

which is a Bernoulli equation with  $\alpha = 2$ .

$$y' + gy + hy^2 = k \text{ (Riccati's equation)}$$

The Bernoulli equation (1.3.7) can be transformed into a linear equation by setting  $z = u^{-1}$ , giving

$$z' - (g + 2\phi h)z = h. \quad (1.3.8)$$

Thus  $y = \varphi + 1/z$ , where  $z$  is the solution to (1.3.8). An example of this procedure is given in **Walter**, Ch. 1, §2, IV and you are encouraged to follow the steps in the calculations there.

1. Separable Equations
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## Vector Fields

We recall some notions that we first encountered in the introduction. We will study vector fields in  $\mathbb{R}^2$ , i.e., functions of the form.

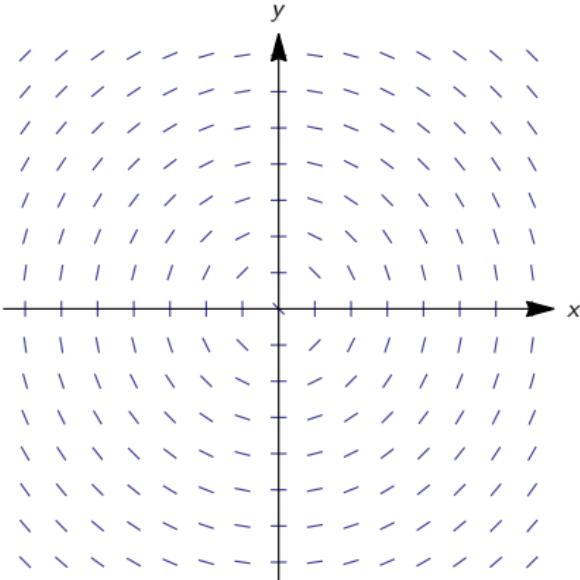
$$F: \mathbb{R}^2 \rightarrow \mathbb{R}^2,$$

$$F(x, y) = \begin{pmatrix} F_1(x, y) \\ F_2(x, y) \end{pmatrix}.$$

1.4.1. Example. The field given by

$$F(x, y) = \begin{pmatrix} -y \\ x \end{pmatrix}$$

is shown at right.



## Integral Curves of Vector Fields

1.4.2. Definition. An **trajectory** of a vector field  $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a curve (called an **integral curve**)  $\mathcal{C} \subset \mathbb{R}^2$  together with a parametrization

$$\gamma: I \rightarrow \mathcal{C}, \quad \gamma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad I \subset \mathbb{R}.$$

such that

$$\gamma'(t) = F \circ \gamma(t), \quad t \in I.$$

Hence, at every point  $(x, y) \in \mathcal{C}$ , the vector  $F(x, y)$  is tangent to  $\mathcal{C}$ .

Physically speaking, the parametrization  $\gamma$  represents the trajectory of a particle moving along a path given by  $\mathcal{C}$ . Of course, several different trajectories may share the same path, moving at different velocities.

## Finding the Integral Curves

The vector field  $F$  is tangent to  $\mathcal{C}$  at some point  $(x, y) \in \mathcal{C}$  if

$$\dot{x}(t) = c \cdot F_1(x(t), y(t)), \quad \dot{y}(t) = c \cdot F_2(x(t), y(t)). \quad (1.4.1)$$

for some  $c \in \mathbb{R}$ . This is a coupled system of first-order equations, which can be quite difficult to solve.

However, it turns out that we can use a trick to find the integral curves: we define the orthogonal vector field

$$F^\perp: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad F^\perp(x, y) = \begin{pmatrix} F_2(x, y) \\ -F_1(x, y) \end{pmatrix}.$$

Let us suppose that  $F^\perp$  is a potential field, i.e., there exists a function  $U^\perp: \mathbb{R}^2 \rightarrow \mathbb{R}$  such that

$$F^\perp(x, y) = \nabla U^\perp(x, y).$$

## Finding the Integral Curves

1.4.3. Lemma. A curve  $\mathcal{C} \subset \mathbb{R}^2$  with parametrization  $\gamma$  is an integral curve of  $F$  if and only if  $U^\perp$  is constant along  $\mathcal{C}$ , i.e.,

$$U^\perp \circ \gamma(t) = \text{constant}.$$

Proof.

Applying the chain rule,

$$\begin{aligned}\frac{d}{dt} U^\perp(x(t), y(t)) &= U_x^\perp(x(t), y(t))\dot{x}(t) + U_y^\perp(x(t), y(t))\dot{y}(t) \\ &= [F_2 \circ \gamma(t)] \cdot \dot{x}(t) - [F_1 \circ \gamma(t)] \cdot \dot{y}(t) \\ &= 0\end{aligned}$$

if and only if (1.4.1) holds. □

# Finding the Integral Curves

1.4.4. Example. Consider the vector field

$$F: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad F(x, y) = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

Then

$$F^\perp(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}$$

and a potential function is

$$U^\perp(x, y) = \frac{1}{2}(x^2 + y^2).$$

The integral curves are given precisely by

$$x^2 + y^2 = \text{constant},$$

i.e., they are circles about the origin.

## Connection to Differential Equations

We now apply this method of finding integral curves to differential equations. Suppose we have an equation of the form

$$h(x, y)y' + g(x, y) = 0, \quad x \in I \subset \mathbb{R},$$

for suitable, continuous functions  $g$  and  $h$ . If  $h(x, y) \neq 0$  for all  $x \in I$ , it may be transformed into the explicit equation

$$y' = -\frac{g(x, y)}{h(x, y)} \tag{1.4.2}$$

We have seen in (I.2) that the solutions of (1.4.2) are precisely the integral curves of the vector field

$$G(x, y) = \begin{pmatrix} 1 \\ -\frac{g(x, y)}{h(x, y)} \end{pmatrix} \tag{1.4.3}$$

## Generalized Concept of a Solution to an ODE

Since  $h(x, y) \neq 0$ , we may instead consider the integral curves to the vector field

$$F(x, y) = c \cdot \begin{pmatrix} -h(x, y) \\ g(x, y) \end{pmatrix} \quad (1.4.4)$$

with an arbitrary scaling constant  $c \in \mathbb{R}$ . While the trajectories of  $G$  and  $F$  will differ, their paths (integral curves) are the same. The trajectories merely differ in the velocities.

Considering again

$$h(x, y)y' + g(x, y) = 0 \quad (1.4.5)$$

we may ask, fundamentally, what the meaning of this equation is when  $h(x, y) = 0$ . The answer is now clear: **solutions to (1.4.5) are integral curves of (1.4.4)**.

## Generalized Concept of a Solution to an ODE

Note that we thereby significantly generalize the concept of a solution to a differential equation: solutions are no longer functions, but rather curves.

1.4.5. Example. The “solution” to the differential equation

$$yy' + x = 0 \quad (1.4.6)$$

are the integral curves of the vector field  $F(x, y) = \begin{pmatrix} -y \\ x \end{pmatrix}$ . As we have seen, these integral curves are given by

$$y^2 + x^2 = c, \quad c \in \mathbb{R}.$$

This is the **general solution of** (1.4.6).

A problem that may arise is that the vector field  $F^\perp$  associated to (1.4.4) may not have a potential function, making it impossible to find the integral curves as described before.

## Variable Velocities

The basic approach of finding integral curves of (1.4.4) relies on solving the equations (1.4.1),

$$\dot{x}(t) = -c \cdot h(x(t), y(t)), \quad \dot{y}(t) = c \cdot g(x(t), y(t)).$$

The constant  $c$  represents merely a factor influencing the velocity. But there is no need for  $c$  to be constant; the velocity of the trajectory may change at every instant without affecting the integral curve.

Hence, we may introduce a function  $M(x, y)$  ("variable velocity") and solve instead

$$\begin{aligned}\dot{x}(t) &= -M(x(t), y(t))h(x(t), y(t)), \\ \dot{y}(t) &= M(x(t), y(t))g(x(t), y(t)).\end{aligned}\tag{1.4.7}$$

Both systems of equations yield the same integral curve. By choosing  $M$  appropriately, the vector field associated to (1.4.7) is a potential field, and the integral curves can be determined.

# Integrating factors (Euler Multipliers)

1.4.6. Example. Consider the differential equation

$$2xy' = -y \quad \text{with} \quad F = \begin{pmatrix} -2x \\ y \end{pmatrix}, \quad F^\perp = \begin{pmatrix} y \\ 2x \end{pmatrix}.$$

Clearly,  $\partial_y F_1^\perp = 1 \neq 2 = \partial_x F_2^\perp$  so  $F^\perp$  does not have a potential.

However, for  $x > 0$  we can multiply the ODE (equivalently: scale the vector field  $F^\perp$ , or change the velocity of the integral curve) by  $1/\sqrt{x}$ , yielding

$$F^\perp(x, y) = \begin{pmatrix} y/\sqrt{x} \\ 2\sqrt{x} \end{pmatrix}.$$

This field has potential function  $U^\perp(x, y) = 2y\sqrt{x}$  and we see that integral curves are given by

$$y\sqrt{x} = c \quad \Leftrightarrow \quad y = \frac{c}{\sqrt{x}}, \quad x > 0, \quad c \in \mathbb{R}.$$

## Integrating factors (Euler Multipliers)

1.4.7. Definition. Let  $g, h$  be continuous functions on an open set  $D \subset \mathbb{R}^2$ . A function  $M$  with  $M(x, y) \neq 0$  defined on  $D$  is said to be an **integrating factor** or **Euler multiplier** for the differential equation

$$h(x, y)y' + g(x, y) = 0 \quad (1.4.8)$$

if the vector field

$$F^\perp(x, y) = \begin{pmatrix} M(x, y)g(x, y) \\ M(x, y)h(x, y) \end{pmatrix}$$

has a potential function.

Of course, the main difficulty now is **finding the correct integrating factor**  $M(x, y)$ .

## Finding Integrating factors

1.4.8. Remark. If  $D$  is open, simply connected and  $g, h, M \in C^1(D)$ , then

$$(Mg)_y = (Mh)_x, \quad \text{i.e.,} \quad M_y g + M g_y = M_x h + M h_x, \quad (1.4.9)$$

is necessary and sufficient for  $M$  to be a multiplier.

In general, it is quite difficult to find a multiplier  $M$ , since (1.4.9) is a first-order, partial differential equation.

However, if one can assume that  $M$  depends only on  $x$  or only on  $y$ , the task becomes manageable. For example, consider the case of a multiplier depending on  $x$  only. Then (1.4.9) becomes

$$\frac{g_y - h_x}{h} = \frac{M'}{M} = (\ln M)',$$

which is solvable.

# Finding Integrating factors

1.4.9. Example. Consider the differential equation

$$y' = -\frac{(2x^2 + 2xy^2 + 1)y}{3y^2 + x}.$$

Here

$$F^\perp(x, y) = \begin{pmatrix} (2x^2 + 2xy^2 + 1)y \\ 3y^2 + x \end{pmatrix}$$

but this is not a potential field. However,

$$\frac{g_y - h_x}{h} = \frac{2x^2 + 2xy^2 + 1 + 4xy^2 - 1}{3y^2 + x} = 2x$$

so  $M$  depends only on  $x$  and  $M(x) = e^{x^2}$  is the desired Euler multiplier.

# Finding Integrating factors

It is easy to see that

$$\begin{pmatrix} (2x^2 + 2xy^2 + 1)ye^{x^2} \\ (3y^2 + x)e^{x^2} \end{pmatrix}$$

is a potential field (check this!) and has the potential function

$$U(x, y) = ye^{x^2}(x + y^2),$$

so all integral curves are given by  $ye^{x^2}(x + y^2) = c$ ,  $c \in \mathbb{R}$ .

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## Regular and Singular Line Elements

An implicit first-order differential equation has the form

$$F(y, y'; x) = 0 \quad (1.5.1)$$

for a continuous function  $F$ . Just as with explicit equations, (1.5.1) defines a direction field of line elements  $(x, y, p)$  given by

$$F(y, p; x) = 0. \quad (1.5.2)$$

However, now  $p$  is not necessarily unique for every  $(x, y) \in \mathbb{R}^2$  any more!

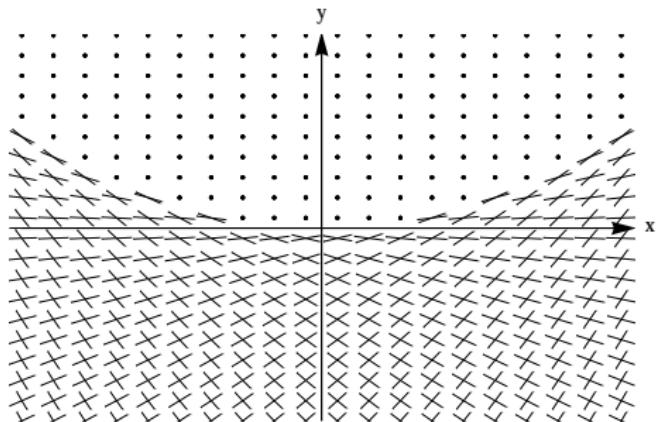
**1.5.1. Definition.** If  $F(y_0, p_0; x_0) = 0$  and (1.5.2) can be solved for  $p$  as a function of  $x$  and  $y$  in some neighborhood  $U$  of  $(x_0, y_0, p_0)$ , then  $(x_0, y_0, p_0)$  is said to be a ***regular line element***, otherwise a ***singular line element***.

## Regular and Singular Line Elements

1.5.2. Example. Consider the implicit differential equation

$$(y')^2 - xy' + y = 0.$$

The line elements are obtained from  $p^2 + xp - y = 0$  and yield  $p = \frac{1}{2}(x \pm \sqrt{x^2 - 4y})$ . Thus the direction field does not exist when  $y > x^2/4$ , the line elements are regular for  $y = x^2/4$  and singular elsewhere.



# Regular Solutions and Singular Points

## 1.5.3. Definition.

- (i) A solution  $y$  of the implicit ODE  $F(y, y'; x) = 0$  is said to be **regular** on an interval  $I \subset \mathbb{R}$  if for all  $x \in I$  the line elements  $(x, y(x), y'(x))$  are regular.
- (ii) A point  $(x, y)$  is said to be a **singular point** of the ODE if there exists a singular line element  $(x, y, p)$ .

The following result follows the Implicit Function Theorem of Vv285:

**1.5.4. Theorem.** If  $F$  and  $\partial_p F$  are continuous in a neighborhood of  $(x_0, y_0, p_0)$  and if

$$F(y_0, p_0; x_0) = 0, \quad \frac{\partial F}{\partial p}(y_0, p_0; x_0) \neq 0,$$

then  $(x_0, y_0, p_0)$  is a regular line element.

Note that even if  $F(y_0, p_0; x_0) = F_p(y_0, p_0; x_0) = 0$  the line element is not necessarily singular; take, e.g.,  $F(y, p; x) = (p - f(x, y))^2$ .

## Slope parametrization

Recall the direction field of Example 1.5.2. We would expect an integral curve to follow along the slopes of the line elements in a “smooth” fashion; even though a line element might be singular, we would not want a solution to suddenly “bend” into a different direction. In other words, the integral curve should be continuously differentiable and smooth.

If a given solution  $y$  has the property that  $y''$  exists and  $y'' \neq 0$ , then its slope  $y'$  is a monotonic (hence bijective) function of  $x$  and we can use the slope to parametrize the solution curve. This use of a geometric curve property as a parametrization is similar in spirit to the curve length parametrization we introduced in Vv285.

## The Slope of a Plane Curve

Let  $\mathcal{C}$  be a smooth curve in  $\mathbb{R}^2$  (called a **plane curve**), parametrized by a differentiable function

$$\gamma: I \rightarrow \mathbb{R}^2, \quad \gamma(t) = \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix}$$

with non-zero derivative everywhere. Then we define the **slope** of  $\mathcal{C}$  at  $\gamma(t)$  by

$$p = \frac{\dot{\gamma}_1(t)}{\dot{\gamma}_2(t)}$$

provided that  $\dot{\gamma}_2(t) \neq 0$ . If  $\dot{\gamma}_2(t) = 0$ , we say that the slope is undefined or infinite.

(To prevent confusion, we will use the “dot notation” for the derivative when we differentiate a curve parametrization with respect to its parameter.)

## Slope Parametrization of Curves

By the implicit function theorem, whenever the slope of a curve exists, the curve may be represented as the graph of a function  $f: J \rightarrow \mathbb{R}$  for some suitable interval  $J$ . In that case, the slope at  $\gamma(p)$  is equal to the derivative of  $f$  at  $\gamma_1(p)$ , i.e.,

$$p = f'(\gamma_1(p))$$

We say that a curve (segment) is imbued with **slope parametrization** if the parametrization  $\gamma: I \rightarrow \mathcal{C}$  satisfies

$$\dot{\gamma}_1(p) = p \cdot \dot{\gamma}_2(p),$$

i.e., the slope of the curve at  $\gamma(p)$  is equal to the parameter  $p$ .

## Slope Parametrization of Curves

Along curve segments where the slope is not constant but instead changes monotonically, it is possible to choose the slope value as a parametrization of the curve. This will clearly only be possible if every point on the curve segment has a different slope, so that such a point can be identified by the value of the slope. A **sufficient condition** for this is that

$$f''(x) \neq 0$$

on that curve segment. For example, straight lines may not be parametrized by their slope.

## Slope Parametrization of Curves

1.5.5. Example. Consider the curve  $\mathcal{C} \subset \mathbb{R}^2$  given by  $y = x^3$ ,  $x \in \mathbb{R}$ . The slope parametrization must satisfy

$$p = y'(\gamma_1(p)) = 3\gamma_1(p)^2$$

so that

$$\gamma_1(p) = \pm\sqrt{p/3}.$$

Furthermore,

$$\gamma_2(p) = p \cdot \gamma_1(p) = \pm(p/3)^{3/2}.$$

We note that of course  $\gamma_2(p) = \gamma_1(p)^3$ , as desired.

1.5.6. Remark. From now on, following **Walter**, we will use the notation

$$\gamma(p) = \begin{pmatrix} \gamma_1(p) \\ \gamma_2(p) \end{pmatrix} =: \begin{pmatrix} x(p) \\ y(p) \end{pmatrix}$$

which will hopefully not lead to confusion.

# Implicit Differential Equations

We use the following procedure for solving implicit differential equations:

- ▶ An equation  $F(y, y'; x) = 0$  is given.
- ▶ Assuming that the solution curve admits (at least locally) the slope parametrization  $(x(p), y(p))$ , we set

$$F(y(p), p; x(p)) = 0. \quad (1.5.3)$$

- ▶ Using (1.5.3) and  $\dot{y}(p) = p \cdot \dot{x}(p)$  we solve for  $x(p)$  and  $y(p)$ .

This procedure will find solutions for which  $y''(x) \neq 0$ , so it will not yield solutions that are straight lines.

We can in particular solve differential equations of the forms

- ▶  $x = g(y')$  and
- ▶  $y = g(y')$

using this method. (See **Walter**, §4 III and IV.)

# $y = xy' + g(y')$ (Clairaut's equation)

We suppose that  $g \in C^1(I)$  for some interval  $I$ . Assuming the integral curve admits a slope parametrization, we have



Alexis Claude Clairaut (1713-1765).  
Wikimedia Commons. Wikimedia Foundation. Web. 8 August 2016.

$$y(p) = x(p) \cdot p + g(p).$$

Differentiating,

$$\dot{y}(p) = p \cdot \dot{x}(p) + x(p) + \dot{g}(p).$$

With  $\dot{y} = p\dot{x}$  we obtain  $x + \dot{g} = 0$ , i.e.,

$$x(p) = -\dot{g}(p), \quad y(p) = -p\dot{g}(p) + g(p). \quad (1.5.4)$$

However, there are more solutions: it is easy to see that the straight lines

$$y = cx + g(c), \quad c \in I, \quad (1.5.5)$$

also solve the equation.

$$y = xy' + g(y') \quad (\text{Clairaut's equation})$$

1.5.7. Example. Consider Clairaut's equation with  $g(x) = e^x$ ,

$$y = xy' + e^{y'}.$$

Then (1.5.4) becomes

$$x(p) = -e^p, \quad y(p) = -pe^p + e^p,$$

giving the curve ( $s = -e^p$ )

$$\gamma(s) = \begin{pmatrix} s \\ s(\ln(-s) - 1) \end{pmatrix}, \quad s < 0.$$

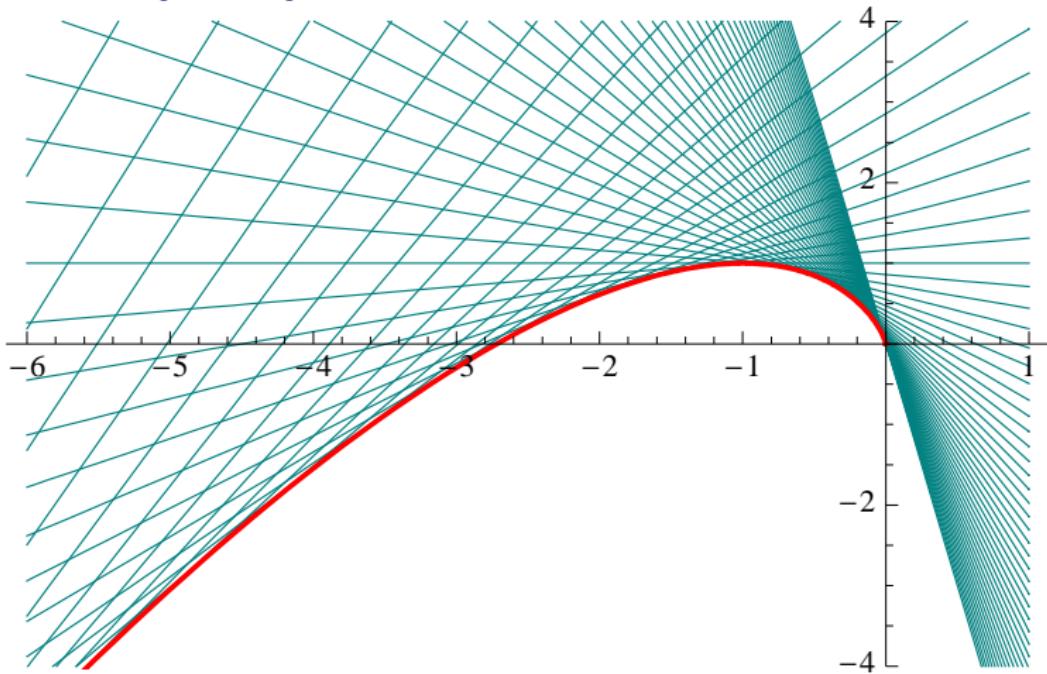
Hence, we obtain the solution

$$y(x) = x(\ln|x| - 1), \quad x < 0. \quad (1.5.6)$$

Furthermore, from (1.5.5), straight-line solutions are

$$y(x; c) = c \cdot x + e^c, \quad c \in \mathbb{R}.$$

# Solutions of $y = xy' + e^{y'}$



The solution (1.5.6) is shown in red - remarkably, it appears to be what is called the **envelope** of the family of straight-line solutions.

## Families of Curves and Envelopes

Let us investigate this phenomenon more closely:

1.5.8. Definition. A **one-parameter family of smooth curves in  $\mathbb{R}^2$**  is a set

$$F = \{\mathcal{C}_s, s \in I\}$$

where  $I \subset \mathbb{R}$  is some interval and each  $\mathcal{C}_s$  is a smooth curve.

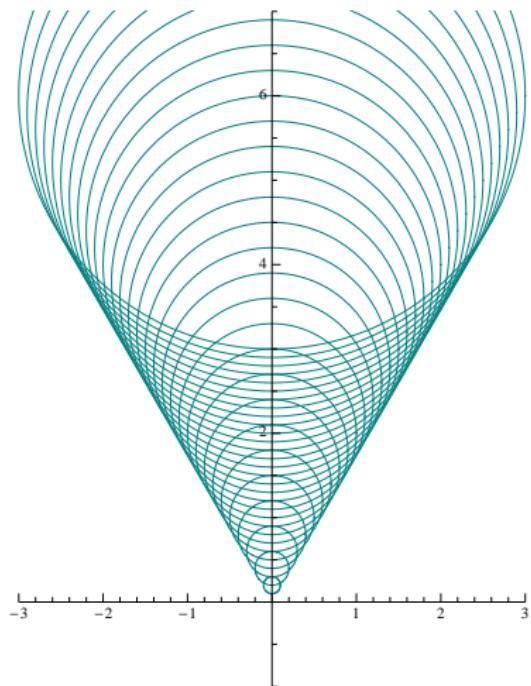
An **envelope** of  $F$  is a curve  $\mathcal{E}$  such that every point of  $\mathcal{E}$  is tangent to a curve in  $F$ .

Of course, not every family of curves has an envelope. However, given a family of smooth curves with each curve  $\mathcal{C}_s$  parametrized by a function

$$\gamma(s, \cdot): J \rightarrow \mathcal{C}_s, \quad t \mapsto \gamma(s, t)$$

with the same interval  $J \subset \mathbb{R}$  used for all  $\mathcal{C}_s$ , we would like to find a parametrization for the envelope curve, if it exists.

# A Family of Circles



1.5.9. Example. We will study the family of circles  $F = \{\mathcal{C}_R : R > 0\}$  with radius  $R$  and center  $(0, 2R)$ , i.e., parametrized by

$$\gamma(R, t) = (R \cos t, R \sin t + 2R).$$

An envelope  $\mathcal{E}$  is clearly visible in the shape of two straight lines. Note that we can use  $R$  to parametrize  $\mathcal{E}$  **locally** but not globally.

# Calculating Envelopes in $\mathbb{R}^2$

We will assume that the map

$$\gamma: I \times J \rightarrow \mathbb{R}^2, \quad (s, t) \mapsto \gamma(s, t)$$

is smooth, in particular, differentiable. Let us also assume that at a point  $p \in \mathcal{E}$  the envelope is tangent to a curve  $\mathcal{C}_s$ . Then the tangent vector of  $\mathcal{E}$  is parallel to the tangent vector of  $\mathcal{C}_s$  at  $p$ .

Now for each curve  $\mathcal{C}_s$ , there exists a parameter value  $t = \psi(s)$  such that  $p \in \mathcal{C}_s \cap \mathcal{E}$  is given by

$$p = \gamma(s, \psi(s)).$$

The hereby defined function  $\psi$  is differentiable because  $\gamma$  is differentiable. We can use  $s$  to parametrize (at least locally) the envelope  $\mathcal{E}$ .

# Calculating Envelopes in $\mathbb{R}^2$

The tangent vector of the envelope at  $p \in \mathcal{C}_s \cap \mathcal{E}$  is given by

$$\frac{\partial}{\partial s} \gamma(s, \psi(s)) = \partial_1 \gamma(s, \psi(s)) + \psi'(s) \cdot \partial_2 \gamma(s, \psi(s))$$

while the tangent vector of  $\mathcal{C}_s$  at  $p$  is simply

$$\left. \frac{\partial}{\partial t} \gamma(s, t) \right|_{t=\psi(s)} = \partial_2 \gamma(s, \psi(s)).$$

Since the two vectors are parallel,

$$\begin{aligned} 0 &= \det \left( \frac{\partial}{\partial s} \gamma(s, \psi(s)), \left. \frac{\partial}{\partial t} \gamma(s, t) \right|_{t=\psi(s)} \right) \\ &= \det (\partial_1 \gamma(s, \psi(s)), \partial_2 \gamma(s, \psi(s))) + \underbrace{\det (\psi'(s) \partial_2 \gamma(s, \psi(s)), \partial_2 \gamma(s, \psi(s)))}_{=0} \\ &= \det \left( \begin{matrix} \frac{\partial \gamma_1}{\partial s} & \frac{\partial \gamma_1}{\partial t} \\ \frac{\partial \gamma_2}{\partial s} & \frac{\partial \gamma_2}{\partial t} \end{matrix} \right) \Big|_{t=\psi(s)} \end{aligned}$$

## The Envelope Equation

We thus obtain the **envelope equation**

$$\frac{\partial \gamma_1}{\partial s} \frac{\partial \gamma_2}{\partial t} = \frac{\partial \gamma_1}{\partial t} \frac{\partial \gamma_2}{\partial s}, \quad t = \psi(s). \quad (1.5.7)$$

This allows us to determine  $\psi$ . Inserting into the parametrization  $\gamma$ , we then obtain the parametrization of  $\mathcal{E}$  as  $\gamma(s, \psi(s))$ ,  $s \in I$ .

**1.5.10. Example.** Consider the family of circles parametrized by

$$\gamma(R, t) = \begin{pmatrix} R \cos t \\ R \sin t + 2R \end{pmatrix}.$$

The relation (1.5.7) then yields

$$\cos t \cdot R \cos t = -R \sin t (\sin t + 2),$$

which is equivalent to  $1 = -2 \sin t$  or  $\sin t = -1/2$ . Hence  $\psi(t) = \arcsin(-1/2)$ , so  $t$  is constant in this example.

## Envelope of a Family of Circles

Thus the envelope  $\mathcal{E}$  is parametrized by

$$\gamma(R, t)|_{\sin t = -1/2} = \begin{pmatrix} \pm R\sqrt{1 - 1/4} \\ 3R/2 \end{pmatrix} = \frac{\sqrt{3}}{2} \begin{pmatrix} \pm R \\ \sqrt{3}R \end{pmatrix}$$

The factor  $\sqrt{3}/2$  is irrelevant for the shape of  $\mathcal{E}$ , and we see that  $\mathcal{E}$  is the graph of  $y = \sqrt{3}|x|$ .

**1.5.11. Remark.** It should not come as a surprise that envelopes play an important role in ***geometric optics***, the theory of light rays (an approximation to the wave theory of light). Envelopes of light rays correspond to ***caustics*** or ***focal points*** and it turns out that the intensity of light has a singularity at these points. An example will be given in the assignments.

## Envelopes and Clairaut's Equation

1.5.12. Remark. It is not difficult to show that the solution for Clairaut's equation arrived at through parametrizing with  $p = y'$  is actually the envelope of the straight-line solutions. The proof is left for the assignments.

Thus, there are two ways to solve Clairaut's Differential Equation:

- (i) After writing down the straight-line solutions, parametrize with  $p = y'$  to find the non-trivial solution; or
- (ii) After writing down the straight-line solutions, calculate their envelope to find the non-trivial solution.

The non-trivial solution is also sometimes called the **singular solution** of Clairaut's equation.

# $y = xf(y') + g(y')$ (d'Alembert's equation)



Jean Le Rond d'Alembert (1717-1783) at  
age 36. Wikimedia Commons. Wikimedia  
Foundation. Web. 28 February 2012.

Let  $f, g \in C^1(I)$  for some interval  $I \subset \mathbb{R}$ . With the slope parameterization ansatz gives

$$y(p) = x(p)f(p) + g(p).$$

Differentiating this,

$$\dot{y} = \dot{x}f + x\dot{f} + \dot{g}.$$

With  $\dot{y} = p\dot{x}$  we obtain

$$\dot{x} = \frac{x\dot{f}(p) + \dot{g}(p)}{p - f(p)}$$

from which  $x(p)$  and then  $y(p)$  can be determined. A line  $y = cx + d$  is a solution if and only if  $f(c) = c$  and  $d = g(c)$ .

$$y = xf(y') + g(y') \text{ (d'Alembert's equation)}$$

1.5.13. Example. Consider the equation

$$y = x \left( y' + \frac{1}{y'} \right) + (y')^4.$$

We differentiate

$$y(p) = x(p)(p + 1/p) + p^4 \quad (1.5.8)$$

to obtain

$$\dot{y} = \dot{x}(p + 1/p) + x(1 - p^{-2}) + 4p^3.$$

Replacing  $\dot{y} = p\dot{x}$ , this gives

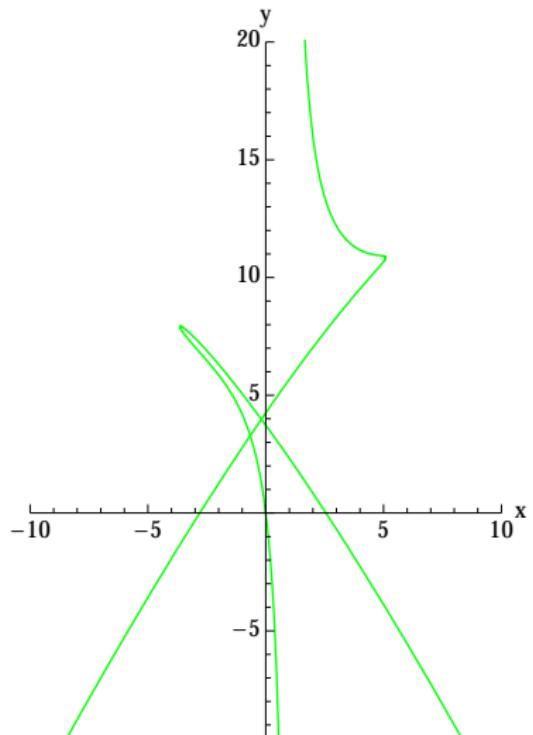
$$\dot{x} = x \left( \frac{1}{p} - p \right) - 4p^4.$$

This is an inhomogeneous linear equation for  $x(p)$ . It is not difficult to find the general solution

$$x(p) = C \cdot pe^{-p^2/2} + 8p - 4p^3, \quad C \in \mathbb{R}.$$

$$y = xf(y') + g(y') \text{ (d'Alembert's equation)}$$

From (1.5.8) we can then find  $y(p)$  and hence obtain a parametrization of the integral curves. The integral curve for  $C = 1$  is plotted below.



## Slope Parametrization for General Implicit Equations

We now consider the slope parametrization approach for general implicit equations of the form

$$F(y, y'; x) = 0.$$

The slope parametrization ansatz gives

$$F(y(p), p; x(p)) = 0.$$

Differentiating, the chain rule yields

$$F_x \dot{x} + F_y \dot{y} + F_p = 0.$$

Since  $\dot{y} = p\dot{x}$  we find

$$\dot{x} = -\frac{F_p}{F_x + pF_y}, \quad \dot{y} = -\frac{pF_p}{F_x + pF_y}. \quad (1.5.9)$$

## Slope Parametrization for General Implicit Equations

The system (1.5.9) is a coupled system of explicit differential equations for  $x$  and  $y$  and in general quite difficult to solve.

However, in special cases the system (1.5.9) decouples, i.e., the first equation does not contain  $y$  or the second does not contain  $x$ . In that case, they can be solved one after the other.

This decoupling occurs, for example if  $F(y, y'; x) = G(x, y') - y$  or  $F(y, y'; x) = H(y, y') - x$ . In these cases we find, respectively,

$$\begin{aligned}\dot{x} &= \frac{G_p(x, p)}{p - G_x(x, p)}, & y(p) &= G(x(p), p), \\ \dot{y} &= \frac{pH_p(y, p)}{1 - pH_y(y, p)}, & x(p) &= H(y(p), p).\end{aligned}$$

An example of such a situation is d'Alembert's equation. (Verify this!)

1. Separable Equations
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## Systems of First-Order ODEs

In our course, we will only study ***explicit systems of n first-order differential equations***. These have the form

$$\dot{x}(t) = F(x, t) \quad (1.6.1)$$

where

$$x: \mathbb{R} \rightarrow \mathbb{R}^n, \quad F: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n.$$

Writing out (1.6.1) in components, we have

$$\dot{x}_1(t) = F_1(x_1, \dots, x_n, t),$$

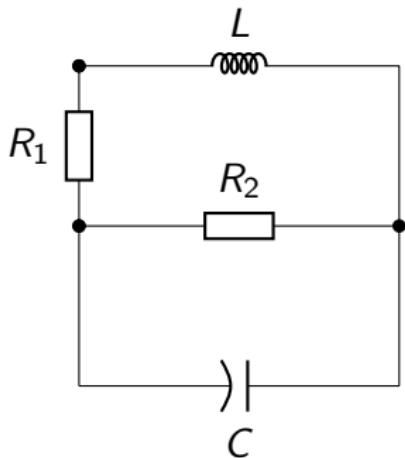
$$\dot{x}_2(t) = F_2(x_1, \dots, x_n, t),$$

$$\vdots$$

$$\dot{x}_n(t) = F_n(x_1, \dots, x_n, t).$$

## Example: An RCL Circuit

1.6.1. Example. Consider the so-called **RCL circuit** pictured below:



Here  $C$  is the capacitance of the capacitor,  $L$  is the inductance of the spool and  $R_1$  and  $R_2$  are resistances. It can be shown that the voltage  $V$  and the current  $I$  satisfy the equations

$$L \frac{dI}{dt} = -R_1 I - V,$$

$$C \frac{dV}{dt} = I - \frac{V}{R_2}.$$

# Systems of ODEs and Higher Order Equations

Given an explicit ODE of order  $n$ ,

$$x^{(n)}(t) = f(x, x', x'', \dots, x^{(n-1)}, t), \quad (1.6.2)$$

we can introduce “new variables” by setting

$$x_1 := x, \quad x_2 := x', \quad x_3 := x'', \quad \dots, \quad x_n := x^{(n-1)}.$$

We then rewrite (1.6.2) as a system of equations

$$\begin{pmatrix} x'_1(t) \\ x'_2(t) \\ x'_3(t) \\ \vdots \\ x'_n(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ x_3(t) \\ x_4(t) \\ \vdots \\ f(x_1, x_2, \dots, x_n, t) \end{pmatrix} \quad (1.6.3)$$

By solving (1.6.3) we can obtain a solution to (1.6.2).

## Example: The Simple Harmonic Oscillator

1.6.2. Example. The motion of a simple harmonic oscillator is described by the second order differential equation

$$\ddot{x} = -kx, \quad k > 0. \quad (1.6.4)$$

Setting  $x_1 := x$ ,  $x_2 := \dot{x}$ , we obtain

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -kx_1 \end{pmatrix} \quad (1.6.5)$$

1.6.3. Remark. It is clear that every single, higher-order ODE can be transformed into a system of first-order ODES, but not every system comes from such a single, higher-order ODE. The theory of systems encompasses that of single ODEs and is in fact more general.

## Initial Value Problems and Integral Equations

One approach to solving the ***initial value problem*** for systems,

$$\frac{dx}{dt} = F(x, t), \quad x(t_0) = x_0 \in \mathbb{R}^n \quad (1.6.6)$$

is to re-write it in the form

$$x(t) = x_0 + \int_{t_0}^t F(x(s), s) ds. \quad (1.6.7)$$

This is an ***integral equation*** and it is not difficult to see that a solution of (1.6.6) must also satisfy (1.6.7). Integral equations arise naturally in physics (e.g., from conservation laws) and may be regarded as more fundamental than differential equations. However, to find exact solutions the machinery available for differential equations is more powerful, which is why the focus is usually on (1.6.6) instead of (1.6.7).

**1.6.4. Remark.** The problem (1.6.6) is not quite equivalent to (1.6.7). Why?

# Picard Iteration



**Charles Émile Picard (1856 – 1941)**  
MacTutor History of Mathematics archive. School of Mathematics and Statistics University of St Andrews, Scotland. Web. 21 Aug 2016.

An approximate method for finding a solution to (1.6.7) is known as **Picard iteration**. This works as follows: Guess a function  $x^{(0)}(t)$ , e.g.,  $x^{(0)}(t) = x_0$  (constant). Then set

$$x^{(k+1)}(t) := x_0 + \int_{t_0}^t F(x^{(k)}(s), s) ds, \quad k \in \mathbb{N}.$$

This yields a sequence of functions  $(x^{(k)})$ .

Under suitable conditions on the function  $F$ , the contraction mapping principle (cf. Exercise 4.2 of Vv186) can be used to show that the sequence converges to a (unique) function  $x(t)$ . Of course, this function  $x$  then satisfies

$$x(t) = x_0 + \int_{t_0}^t F(x(s), s) ds$$

and hence solves the integral equation (1.6.7). The conditions on  $F$  ensure that  $x$  is differentiable and satisfies the initial value problem (1.6.6).

# The Fundamental Existence and Uniqueness Theorem

1.6.5. Theorem of Picard-Lindelöf. Let  $x_0 \in \Omega$ , where  $\Omega \subset \mathbb{R}^n$  is open and let  $t_0 \in I$ , where  $I \subset \mathbb{R}$  is an interval.

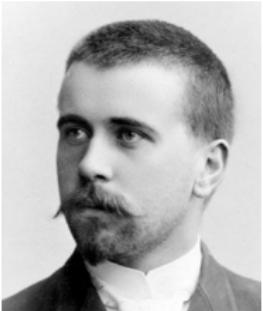
Suppose  $F: \Omega \times I \rightarrow \mathbb{R}^n$  is a continuous function satisfying a **Lipschitz estimate** in  $x$ : there exists an  $L > 0$  such that for all  $x, y \in \Omega$  and all  $t \in I$

$$\|F(x, t) - F(y, t)\| \leq L\|x - y\|. \quad (1.6.8)$$

Then the initial value problem

$$\frac{dx}{dt} = F(x, t), \quad x(t_0) = x_0 \quad (1.6.9)$$

has a unique solution in some  $t$ -interval containing  $t_0$ .



Ernst Lindelöf (1870 – 1946)  
Wikimedia Commons. Wikimedia Foundation. Web. 21 Aug 2016.



Rudolf Lipschitz (1832 – 1903)  
Wikimedia Commons. Wikimedia Foundation. Web. 21 Aug 2016.

## Existence, Uniqueness, Stability

1.6.6. Remark. The Lipschitz estimate (1.6.8) is satisfied, for example, if  $F$  is differentiable with respect to  $x$  and the derivative is bounded on  $\Omega$ .

The proof of the Picard-Lindelöf Theorem 1.6.5 is quite technical; a version can be found in Taylor's book. In fact, the existence of solutions can be obtained by a much less strict condition on  $F$  than the Lipschitz estimate (1.6.8). Instead, we will focus on the uniqueness of solutions, which will be much more important for us.

The next result actually tells us something about the ***stability of solutions***: how much can a small change in initial conditions affect the solution of the differential equation?

## Gronwall's Inequality

1.6.7. **Gronwall's Inequality.** Suppose that all the conditions of Theorem 1.6.5 are satisfied and that  $x$  and  $y$  satisfy the differential equation with initial values  $x_0, y_0 \in \mathbb{R}^n$ , i.e.,

$$\begin{aligned}\frac{dx}{dt} &= F(x, t), & x(t_0) &= x_0, \\ \frac{dy}{dt} &= F(y, t), & y(t_0) &= y_0.\end{aligned}$$

Then

$$\|x(t) - y(t)\| \leq e^{L \cdot |t - t_0|} \|x_0 - y_0\|. \quad (1.6.10)$$



Thomas H. Gronwall (1877 – 1932)  
MacTutor History of Mathematics archive, School of Mathematics and Statistics University of St Andrews, Scotland. Web. 21 Aug 2016.

1.6.8. **Corollary.** The solution of the initial value problem (1.6.9) is unique, since  $x_0 = y_0$  implies  $x(t) = y(t)$  for all  $t$ .

# Gronwall's Inequality

Proof.

We first note that

$$\frac{d}{dt} \|x(t)\| = \lim_{h \rightarrow 0} \frac{\|x(t+h)\| - \|x(t)\|}{h}$$

From the reverse triangle inequality,

$$\frac{\|x(t+h)\| - \|x(t)\|}{h} \leq \frac{\|x(t+h) - x(t)\|}{h},$$

so we have, on taking limits,

$$\frac{d}{dt} \|x(t)\| \leq \left\| \frac{d}{dt} x(t) \right\|.$$

# Gronwall's Inequality

Proof.

Then, the calculation is straightforward:

$$\begin{aligned}\frac{d}{dt} \|x(t) - y(t)\| &\leq \left\| \frac{d}{dt}(x(t) - y(t)) \right\| \\&= \|F(x(t), t) - F(y(t), t)\| \\&\leq L \cdot \|x(t) - y(t)\|\end{aligned}$$

Now comes the trick:

$$\frac{d}{dt} \|x(t) - y(t)\| - L \cdot \|x(t) - y(t)\| = e^{Lt} \frac{d}{dt} \left( e^{-Lt} \|x(t) - y(t)\| \right)$$

so

$$\frac{d}{dt} \left( e^{-Lt} \|x(t) - y(t)\| \right) \leq 0.$$

Integrating from  $t_0$  to  $t$ , we obtain (1.6.10). □

## Linear Systems of ODEs

Solving general systems of equations is quite difficult; Picard iteration gives us a way of finding an approximate solution and of course there are yet more sophisticated numerical methods. From this point onwards, however, we will restrict ourselves to ***linear systems of equations***, i.e., those having the form

$$\frac{dx}{dt} = A(t)x + b(t), \quad t \in I \subset \mathbb{R}, \quad (1.6.11)$$

where  $A: I \rightarrow \text{Mat}(n \times n, \mathbb{R})$  is a matrix-valued function of  $t$  and  $b: I \rightarrow \mathbb{R}^n$ .

- ▶ If  $b = 0$  in (1.6.11), the system is said to be ***homogeneous***, otherwise ***inhomogeneous***.
- ▶ If  $A$  does not depend on  $t$ , i.e.,  $A$  is constant, then the system is said to have ***constant coefficients***, otherwise ***variable coefficients***.

## Linear Systems of ODEs

1.6.9. Example. The harmonic oscillator system (1.6.5) is linear, since we can write

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -kx_1 \end{pmatrix}$$

as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (1.6.12)$$

This system has constant coefficients and is homogeneous.

## Solution Space and Fundamental System of Solutions

The structure of the solution set of homogeneous, linear systems is very similar to that for the systems of algebraic equations  $y = Ax$  that we discussed in Vv285.

1.6.10. Superposition Principle. Let  $x^{(1)}$  and  $x^{(2)}$  be two solutions of

$$\frac{dx}{dt} = A(t)x. \quad (1.6.13)$$

and let  $\lambda, \mu \in \mathbb{R}$ . Then

$$\lambda \cdot x^{(1)} + \mu \cdot x^{(2)}$$

is also a solution of (1.6.13).

1.6.11. Remark. We see that the set of all solutions of (1.6.13) is a vector space, called the **solution space**. A set of functions giving a basis of the solution space is called a **fundamental system of solutions**.

## Construction of Solutions

The immediate question arises: is the solution space finite-dimensional?  
Consider the initial value problem

$$\frac{dx}{dt} = A(t)x, \quad x(t_0) = x_0 \in \mathbb{R}^n. \quad (1.6.14)$$

Now suppose that for some basis  $\mathcal{B} = \{b_1, \dots, b_n\}$  of  $\mathbb{R}^n$  we have functions  $x^{(k)}$ ,  $k = 1, \dots, n$ , that are solutions of

$$\frac{dx^{(k)}}{dt} = A(t)x^{(k)}, \quad x^{(k)}(t_0) = b_k. \quad (1.6.15)$$

Suppose that the initial vector  $x_0$  in (1.6.14) has the basis representation

$$x_0 = \sum_{k=1}^n \lambda_k b_k$$

for numbers  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ .

## Construction of Solutions

Then, by the Superposition Principle (1.6.10) the linear combination

$$x(t) := \sum_{k=1}^n \lambda_k x^{(k)}(t)$$

satisfies the initial value problem (1.6.14). We conclude that at most the  $n$  solutions (1.6.15) are necessary to construct a solution to (1.6.14). In terms of linear algebra, the solution space is a subset of

$$\text{span}\{x^{(1)}, \dots, x^{(n)}\}$$

and hence is at most  $n$ -dimensional. We would like to establish that the  $n$  solutions (1.6.15) are actually independent, yielding a fundamental system and establishing that the solution space is exactly  $n$ -dimensional.

## Independence of Functions

We recall the definition of independence:

1.6.12. Remark. A set of functions  $\{x^{(1)}, \dots, x^{(m)}\}$  mapping  $I \rightarrow \mathbb{R}^n$ ,  $I \subset \mathbb{R}$ , is said to be independent if and only if

$$\sum_{k=1}^m \lambda_k x^{(k)} = 0 \quad \Rightarrow \quad \lambda_1 = \dots = \lambda_m = 0.$$

This linear combination is a combination of functions; expressed in terms of their values, the condition becomes

$$\left( \forall t \in I, \sum_{k=1}^m \lambda_k x^{(k)}(t) = 0 \right) \quad \Rightarrow \quad \lambda_1 = \dots = \lambda_m = 0. \quad (1.6.16)$$

## Independence of Solutions

We see immediately that the solutions constructed on Slide 135 will always be independent:

$$\begin{aligned} \forall t \in I \sum_{k=1}^m \lambda_k x^{(k)}(t) = 0 &\Rightarrow \sum_{k=1}^m \lambda_k x^{(k)}(t_0) = 0 \\ &\Rightarrow \sum_{k=1}^m \lambda_k b_k = 0 \\ &\Rightarrow \lambda_1 = \dots = \lambda_m = 0. \end{aligned}$$

Hence, any solution of the system can be represented **uniquely** in terms of these constructed functions. We now formalize this notion.

## Fundamental System of Solutions

We have now established the following result:

**1.6.13. Proposition.** Let  $\{b_1, \dots, b_n\}$  be a basis of  $\mathbb{R}^n$ ,  $I \subset \mathbb{R}$  an interval and let  $x^{(k)}: I \rightarrow \mathbb{R}^n$ ,  $k = 1, \dots, n$ , satisfy the systems

$$\frac{dx^{(k)}}{dt} = A(t)x^{(k)}, \quad x^{(k)}(t_0) = b_k$$

with initial point  $t_0 \in I$ .

Then  $\{x^{(1)}, \dots, x^{(n)}\}$  is a fundamental system for the equation  $\dot{x} = A(t)x$ ,  $t \in I$ .

The matrix  $X: I \rightarrow \text{Mat}(n \times n, \mathbb{R})$  given by

$$X(t) = (x^{(1)}(t), \dots, x^{(n)}(t))$$

is called a **fundamental matrix** for the initial value problem.

## Systems of Linear ODEs with Constant Coefficients

Let us now consider the most basic case, that of the linear, homogeneous system

$$\frac{dx}{dt} = Ax, \quad x(0) = x_0, \quad (1.6.17)$$

where the matrix  $A$  is constant, i.e., independent of  $t$ . Looking at this simple equation, we are tempted to claim that the (unique!) solution is given by

$$x(t) = e^{At}x_0. \quad (1.6.18)$$

What does the right-hand side actually mean? Well, to start with, we define the exponential function by

$$e^{At} := \mathbb{1} + \sum_{k=1}^{\infty} \frac{A^k t^k}{k!}. \quad (1.6.19)$$

## The Set of $n \times n$ Matrices and the Operator Norm

The series is understood as a series of elements in a vector space, namely in the space of square matrices  $\text{Mat}(n \times n; \mathbb{R})$ . For such a series to make sense, the vector space needs to be endowed with a norm (why?) and we choose the operator norm

$$\|A\| := \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|Ax|}{|x|}$$

where we have used the modulus symbol  $|\cdot|$  for the euclidean norm in  $\mathbb{R}^n$ . This norm is useful, because it satisfies

$$\|AB\| \leq \|A\| \cdot \|B\|, \quad A, B \in \text{Mat}(n \times n; \mathbb{R}).$$

This is not a “standard norm property” but rather a specific property of this particular norm.

## The Matrix Exponential

Then we can prove that the series (1.6.19) converges absolutely, since

$$\begin{aligned}\sum_{k=1}^{\infty} \left\| \frac{A^k t^k}{k!} \right\| &= \sum_{k=1}^{\infty} \frac{\|A^k\| \cdot |t|^k}{k!} \\ &\leq \sum_{k=1}^{\infty} \frac{\|A\|^k \cdot |t|^k}{k!} \\ &= e^{|t|\|A\|} - 1 < \infty\end{aligned}$$

for any  $A$  and  $t$ . Since absolutely convergent series in a vector space also converge, this shows that the series (1.6.19) is well-defined, i.e., equal to some matrix in  $\text{Mat}(n \times n; \mathbb{R})$ .

So, at least the right-hand side of (1.6.18) makes sense. But does it solve the differential equation?

## Justification of the Formal Solution

A formal calculation gives

$$\begin{aligned}\frac{d}{dt} e^{At} &= \frac{d}{dt} \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{d}{dt} \frac{A^k t^k}{k!} \\ &= \sum_{k=1}^{\infty} \frac{A^k t^{k-1}}{(k-1)!} = A \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \\ &= Ae^{At},\end{aligned}\tag{1.6.20}$$

where the exchange of summation and differentiation is justified by the absolute convergence of the exponential series for all  $t$ .

Furthermore,

$$e^{At}|_{t=0} = \mathbb{1},$$

so (1.6.18) actually satisfies (1.6.17).

## The Eigenvalue Problem

However, we have no idea how to actually calculate the exponential series in (1.6.18) in practice!

It turns out that the solution to this “minor difficulty” can be found in the following question:

*Given a linear map, is it possible to find some elements of its domain such that the linear map acts on them simply by multiplication with a number?*

This seemingly harmless question has wide-ranging consequences in all of analysis. It turns out that such elements (which we will call **eigenvectors**) correspond to **states** in quantum mechanics, to **elementary oscillations** in mechanics and also appear in many other contexts.

We will therefore embark on an extended digression into linear algebra.

1. Separable Equations
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# The Eigenvalue Problem

Let  $V$  be a real or complex vector space. We want to investigate if for some  $L \in \mathcal{L}(V, V)$  there exist  $x \in V$ ,  $x \neq 0$  and  $\lambda \in \mathbb{F}$  such that

$$Lx = \lambda x.$$

**1.7.1. Definition.** Let  $V$  be a real or complex vector space and  $L \in \mathcal{L}(V, V)$ . Then a number  $\lambda \in \mathbb{F}$  such that

$$Lx = \lambda x \tag{1.7.1}$$

for some  $x \neq 0$  is called an **eigenvalue of  $L$** . Any  $x$  such that (1.7.1) holds is called an **eigenvector for the eigenvalue  $\lambda$** . The subspace

$$V_\lambda = \{x \in V : Lx = \lambda x\} \tag{1.7.2}$$

is called the **eigenspace for the eigenvalue  $\lambda$** . The dimension  $\dim V_\lambda$  is called the **geometric multiplicity of  $\lambda$** .

## The General Eigenvalue Problem

1.7.2. Remark. The eigenvalue problem is formulated for a linear map  $L: V \rightarrow V$ , where  $V$  is any vector space. If  $V$  is a space whose elements are functions, then the eigenvectors are of course functions and accordingly often called **eigenfunctions**. An example, involving the so-called **Hermite polynomials**, will be discussed in the assignments.

It should be emphasized that, although in this chapter we will primarily consider the case  $V = \mathbb{R}^n$ ,  $A \in \text{Mat}(n \times n, \mathbb{R})$ , the eigenvalue problem is itself much more general.

1.7.3. Example. Consider the map

$$(\cdot)^T: \text{Mat}(n \times n, \mathbb{R}) \rightarrow \text{Mat}(n \times n, \mathbb{R}), \quad A \mapsto A^T,$$

where  $A^T$  denotes the transpose of  $A$ . The eigenvalue problem is

$$A^T = \lambda A$$

for suitable  $A$  and  $\lambda \in \mathbb{R}$ .

## The General Eigenvalue Problem

We first try to find possible eigenvalues. Since  $(A^T)^T = A$ , we have

$$A = \lambda^2 A$$

for any eigenvector ("eigenmatrix")  $A$ . Thus  $\lambda = \pm 1$  are the only possible eigenvalues.

1. For  $\lambda_1 = 1$  we have  $A^T = A$ , so  $A$  is **symmetric** and has the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & & \vdots \\ \vdots & & \ddots & a_{(n-1)n} \\ a_{1n} & \cdots & a_{(n-1)n} & a_{nn} \end{pmatrix}$$

Thus  $\dim V_{\lambda_1} = \dim V_1 = n(n+1)/2$ . This is the geometric multiplicity of the eigenvalue  $\lambda_1 = 1$ .

## The General Eigenvalue Problem

2. For  $\lambda_2 = -1$  we have  $A^T = -A$ , so  $A$  is ***anti-symmetric*** and has the form

$$A = \begin{pmatrix} 0 & a_{12} & \dots & a_{1n} \\ -a_{12} & 0 & & \vdots \\ \vdots & & \ddots & a_{(n-1)n} \\ -a_{1n} & \dots & -a_{(n-1)n} & 0 \end{pmatrix}$$

Thus  $\dim V_{\lambda_2} = \dim V_{-1} = n(n-1)/2$ . This is the geometric multiplicity of the eigenvalue  $\lambda_2 = -1$ .

Note that the eigenspaces are disjoint (except for  $0 \in V$ ) and the sum of their dimensions is  $n^2$ , so we can write

$$\text{Mat}(n \times n, \mathbb{R}) = V_1 \oplus V_{-1}$$

In particular, for any  $A \in \text{Mat}(n \times n, \mathbb{R})$  we have

$$A = \underbrace{\frac{1}{2}(A + A^T)}_{\in V_1} + \underbrace{\frac{1}{2}(A - A^T)}_{\in V_{-1}}.$$

## Independence of Eigenvectors

Our observation that eigenspaces to different eigenvalues are “disjoint” (omitting the zero element) can of course be verified in general:

**1.7.4. Lemma.** Let  $V$  be a real or complex vector space,  $A \in \mathcal{L}(V, V)$  with eigenvalues  $\lambda_1, \dots, \lambda_n \in \mathbb{F}$  such that  $\lambda_i \neq \lambda_j$  for  $i \neq j$ . Then the set of associated eigenvectors  $\{v_1, \dots, v_n\}$  is an independent set.

**1.7.5. Corollary.** Let  $V$  be a real or complex vector space with  $\dim V = n$ .

- (i) A linear map  $A \in \mathcal{L}(V, V)$  can have at most  $n$  distinct eigenvalues.
- (ii) If  $A$  has  $n$  distinct eigenvalues  $\lambda_1, \dots, \lambda_n$ , then it has precisely  $n$  independent eigenvectors  $v_1, \dots, v_n$ . Thus  $\mathcal{B} = (v_1, \dots, v_n)$  constitutes a basis of  $V$  and

$$V = \bigoplus_{i=1}^n V_{\lambda_i}.$$

# Independence of Eigenvectors

## Proof.

We proceed by induction in the number of eigenvectors. The first eigenvector  $\{v_1\}$  obviously gives an independent set. Next, assume that we have  $k + 1$  eigenvectors  $v_1, \dots, v_{k+1}$  to distinct eigenvalues  $\lambda_1, \dots, \lambda_{k+1}$  and that  $\{v_1, \dots, v_k\}$  is an independent set. Then we have

$$\sum_{i=1}^{k+1} \alpha_i v_i = 0 \quad \Rightarrow \quad A \left( \sum_{i=1}^{k+1} \alpha_i v_i \right) = \sum_{i=1}^{k+1} \alpha_i \lambda_i v_i = 0 \quad (1.7.3)$$

On the other hand,

$$\sum_{i=1}^{k+1} \alpha_i v_i = 0 \quad \Rightarrow \quad \lambda_{k+1} \sum_{i=1}^{k+1} \alpha_i v_i = \sum_{i=1}^{k+1} \alpha_i \lambda_{k+1} v_i = 0 \quad (1.7.4)$$

# Independence of Eigenvectors

Proof (continued).

Subtracting from (1.7.4) from (1.7.3), we have

$$\sum_{i=1}^k \alpha_i(\lambda_i - \lambda_{k+1})v_i = 0$$

Since we have assumed  $\{v_1, \dots, v_k\}$  to be an independent set, this implies

$$\alpha_i(\lambda_i - \lambda_{k+1}) = 0, \quad i = 1, \dots, k.$$

The eigenvalues are assumed to be distinct, so  $\lambda_i - \lambda_{k+1} \neq 0$  for all  $i$ .

Thus

$$\alpha_1 = \alpha_2 = \dots = \alpha_k = 0.$$

Inserting this into  $\sum_{i=1}^{k+1} \alpha_i v_i = 0$ , we obtain  $\alpha_{k+1} = 0$ . This proves that  $\{v_1, \dots, v_{k+1}\}$  is an independent set. □

## The Eigenvalue Problem for Matrices

We will now discuss how to find the eigenvectors and eigenvalues for the case of  $V = \mathbb{R}^n$  (hence  $A \in \text{Mat}(n, \mathbb{R})$ ): First,

$$Ax = \lambda x \Leftrightarrow Ax - \lambda x = 0 \Leftrightarrow (A - \lambda \mathbb{1})x = 0$$

where  $\mathbb{1}$  is the unit matrix. This is a homogeneous system of linear equations which has a solution  $x \in \mathbb{R}^n$  if and only if

$$p(\lambda) = \det(A - \lambda \mathbb{1}) = 0.$$

here  $p(\lambda)$  is a polynomial of degree  $n$ , called the **characteristic polynomial**.

We thus find possible eigenvalues by finding the zeroes of the characteristic polynomial. After having found all possible eigenvalues, we proceed to solve  $(A - \lambda \mathbb{1})x = 0$  to obtain the eigenvectors  $x$  corresponding to each  $\lambda$ .

# Eigenvalues of a Matrix

1.7.6. Example. Let us consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

We first determine the eigenvalues of  $A$ . The characteristic polynomial is

$$\begin{aligned} p(\lambda) &= \det(A - \lambda \mathbb{1}) = \det \begin{pmatrix} 1 - \lambda & 0 & 0 \\ 0 & 2 - \lambda & 1 \\ 0 & 1 & 2 - \lambda \end{pmatrix} \\ &= (1 - \lambda)[(2 - \lambda)^2 - 1] \end{aligned}$$

and clearly has the zeroes 1 and 3. We obtain the eigenvalues  $\lambda_1 = 3$  and  $\lambda_2 = 1$ .

## Eigenvectors of a Matrix

We can now obtain the eigenvectors corresponding to the eigenvalues  $\lambda_1$  and  $\lambda_2$  by solving the homogeneous systems of equations  $(A - \lambda_i \mathbb{1})v = 0$ .

$\lambda_1$ : We have

$$\begin{aligned}(A - \lambda_1 \mathbb{1})v = 0 \quad &\Leftrightarrow \quad \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ &\Leftrightarrow \quad v_1 = 0 \wedge v_2 = v_3\end{aligned}$$

Hence any vector of the form  $v = \alpha \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ ,  $\alpha \in \mathbb{R} \setminus \{0\}$  is an eigenvector  $\lambda_1 = 3$  and the corresponding eigenspace is

$$V_{\lambda_1} = V_3 = \left\{ v \in \mathbb{R}^3 : v = \alpha \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R} \right\}$$

# Eigenvectors of a Matrix

$\lambda_2$ : We have

$$(A - \lambda_2 \mathbb{1})v = 0 \Leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\Leftrightarrow v_1 \in \mathbb{R} \text{ arbitrary} \wedge v_2 = -v_3$$

We can choose the two independent eigenvectors  $v = \alpha \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$  and  $u = \beta \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ ,  $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ . The eigenspace is

$$V_{\lambda_2} = V_1 = \left\{ v \in \mathbb{R}^3 : v = \alpha \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \alpha, \beta \in \mathbb{R} \right\}.$$

Since  $\dim V_1 = 2$ , the geometric multiplicity of  $\lambda_2$  is also 2.

## Eigenvalues of Matrices

1.7.7. Remark. Not every linear map has eigenvalues! Take, for example, the rotation

$$A: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is easily seen that there are no real eigenvectors or eigenvalues for  $A$ . However, if the matrix  $A$  is considered as a map

$$A: \mathbb{C}^2 \rightarrow \mathbb{C}^2, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we do find complex eigenvalues  $\lambda = \pm i$  and corresponding eigenvectors.

From now on, if we simply say “the eigenvalues of a matrix  $A$ ”, we mean the possibly complex eigenvalues when the matrix is regarded as a map  $\mathbb{C}^n \rightarrow \mathbb{C}^n$ .

## Fundamental Theorem of Algebra

The following result will be proved later in our course:

**1.7.8. Fundamental Theorem of Algebra.** Let  $p(x) = a_0 + a_1x + \cdots + a_nx^n$  be a polynomial of degree  $n$ . Then there exist numbers  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$  such that

$$p(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n). \quad (1.7.5)$$

We say that the  $\lambda_k$  are the **roots** or **zeroes** of  $p$ . The representation (1.7.5) is called the **factorization** of  $p$ .

If exactly  $m$  of the  $\lambda_k$  in (1.7.5) are equal to the same number  $\lambda \in \mathbb{C}$ , then  $m$  is said to be the **multiplicity** of the root  $\lambda$ .

**1.7.9. Remark.** Clearly, the sum of the multiplicities of the zeroes of a polynomial is equal to its degree.

## Algebraic Multiplicity of Eigenvalues

1.7.10. **Definition.** Let  $\lambda$  be an eigenvalue of a matrix  $A \in \text{Mat}(n \times n, \mathbb{R})$ . Then the multiplicity of the zero of the characteristic polynomial  $p(\lambda) = 0$  is called the ***algebraic multiplicity*** of  $\lambda$ .

The geometric multiplicity gives us the number of independent eigenvectors associated to each eigenvalue, while the algebraic multiplicity tells us how often the eigenvalue repeats in the factorization of the characteristic polynomial. We will prove shortly (Lemma 1.7.16) that

$$\text{geometric multiplicity of } \lambda \leq \text{algebraic multiplicity of } \lambda. \quad (1.7.6)$$

## Diagonalizable Matrices

By the fundamental theorem of algebra and (1.7.6), the sum of the geometric multiplicities of a matrix  $A \in \text{Mat}(n \times n, \mathbb{R})$  is at most  $n$ . If (and only if) the sum of geometric multiplicities equals  $n$ , we have

$$\mathbb{R}^n = \bigoplus_{j=1}^k V_{\lambda_j} \quad (1.7.7)$$

where  $\lambda_1, \dots, \lambda_k \in \mathbb{R}$  are  $k \leq n$  distinct eigenvalues of  $A$ . In other words:

*If the sum of the geometric multiplicities is  $n$ , there exists a basis of eigenvectors of  $\mathbb{R}^n$ .*

**1.7.11. Definition.** A matrix  $A \in \text{Mat}(n \times n, \mathbb{R})$  whose eigenvectors form a basis of  $\mathbb{R}^n$  is called **diagonalizable**.

## Diagonalization and Diagonal Form

Suppose that  $A \in \text{Mat}(n \times n, \mathbb{R})$  is diagonalizable with eigenvalues  $\lambda_1, \dots, \lambda_n$  and corresponding independent eigenvectors  $v_1, \dots, v_n$ . Then

$$U = (v_1, \dots, v_n)$$

is invertible. The matrix  $U$  represents the transformation of the standard basis  $(e_1, \dots, e_n)$  into the basis of eigenvectors. It follows that

$$D := U^{-1}AU = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{pmatrix} \quad (1.7.8)$$

is a **diagonal matrix** whose elements are the eigenvalues of  $A$ . We say that  $D$  is the **diagonal form** of  $A$ .

# Diagonalization and Diagonal Form

Proof of (1.7.8).

Let  $e_k$  be the  $k$ th standard basis vector in  $\mathbb{R}^n$ . Then the  $k$ th column of  $D$  is given by  $D e_k$ . We have

$$D e_k = U^{-1} A U e_k = U^{-1} A v_k = U^{-1} \lambda_k v_k = \lambda_k U^{-1} v_k = \lambda_k e_k. \quad \square$$

Since diagonal matrices occur frequently, we often write

$$\text{diag}(\lambda_1, \dots, \lambda_n) := \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

for short.

## Diagonalization of Matrices

1.7.12. Example. Returning to the matrix  $A$  of Example 1.7.6, we have

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \quad U = (v_1, v_2, v_3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

Then

$$U^{-1} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ -2 & 1 & -1 \end{pmatrix}$$

and

$$D = U^{-1}AU = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

# Matrix Powers

The diagonal form of a quadratic matrix  $A$  is very convenient for calculating powers  $A^k$ . Observe that

$$\begin{aligned} D^k &= (U^{-1}AU)^k = \underbrace{(U^{-1}AU)(U^{-1}AU) \dots (U^{-1}AU)}_{k \text{ times}} \\ &= U^{-1}A\underbrace{UU^{-1}}_{=I}AU \dots U^{-1}AU \\ &= U^{-1}A^kU \end{aligned}$$

gives us

$$A^k = UD^kU^{-1}.$$

This is a useful expression, since

$$D^k = \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix}$$

is easy to calculate.

## Functional Calculus

This now allows us to define analytic functions of  $A$  through their power series expansion: if

$$f(x) = \sum_{j=0}^{\infty} c_j x^j,$$

where we assume that the power series has infinite radius of convergence,

$$\begin{aligned} f(A) &:= \sum_{j=0}^{\infty} c_j A^j = \sum_{j=0}^{\infty} c_j U D^j U^{-1} = \lim_{N \rightarrow \infty} \sum_{j=0}^N c_j U D^j U^{-1} \\ &= \lim_{N \rightarrow \infty} U \left( \sum_{j=0}^N c_j D^j \right) U^{-1} = U \left( \lim_{N \rightarrow \infty} \sum_{j=0}^N c_j D^j \right) U^{-1}. \end{aligned}$$

Here we have used the continuity of matrix multiplication.

# Functional Calculus

Now

$$\sum_{j=0}^N c_j D^j = \sum_{j=0}^N c_j \begin{pmatrix} \lambda_1^j & & 0 \\ & \ddots & \\ 0 & & \lambda_n^j \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^N c_j \lambda_1^j & & 0 \\ & \ddots & \\ 0 & & \sum_{j=0}^N c_j \lambda_n^j \end{pmatrix}.$$

It follows that

$$\begin{aligned} f(A) &= U \left( \lim_{N \rightarrow \infty} \sum_{j=0}^N c_j D^j \right) U^{-1} = U \begin{pmatrix} \sum_{j=0}^{\infty} c_j \lambda_1^j & & 0 \\ & \ddots & \\ 0 & & \sum_{j=0}^{\infty} c_j \lambda_n^j \end{pmatrix} U^{-1} \\ &= U \begin{pmatrix} f(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & f(\lambda_n) \end{pmatrix} U^{-1} \end{aligned}$$

# Matrix Exponential

In particular, for a diagonalizable matrix  $A$ ,

$$e^A = U \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix} U^{-1}$$

1.7.13. Example. Referring to Examples 1.7.6 and 1.7.12, we have

$$\begin{aligned} e^A &= U e^D U^{-1} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} e^3 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ -2 & 1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} e & 0 & 0 \\ 0 & e + e^3 & -e + e^3 \\ 0 & -e + e^3 & e + e^3 \end{pmatrix} \end{aligned}$$

## Matrix Exponential

Note that if  $Av = \lambda v$  for some eigenvalue  $\lambda$  and eigenvector  $v$ , then for any  $t \in \mathbb{R}$ ,

$$(At)v = (\lambda t)v,$$

so if  $v$  is an eigenvector for the eigenvalue  $\lambda$ , it is also an eigenvector for the eigenvalue  $\lambda t$ . It follows that if  $A$  is diagonalizable,

$$e^{At} = U \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} U^{-1}$$

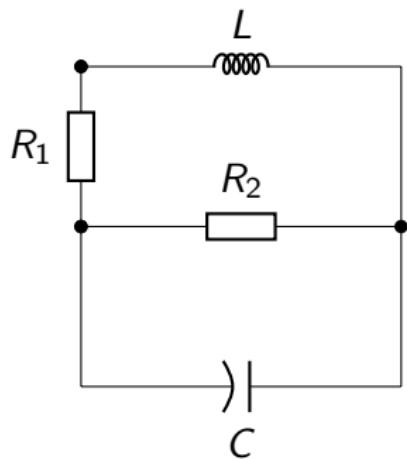
where  $U = (v_1, \dots, v_n)$  is the matrix of eigenvectors. We are now finally able to solve linear systems of ODEs of the form

$$\frac{dx}{dt} = Ax, \quad x(0) = x_0 \in \mathbb{R}^n$$

if  $A$  is a constant, diagonalizable matrix.

# Electric Circuits

1.7.14. Example. Consider the RCL circuit below:



It can be shown that the voltage  $V$  and current  $I$  satisfy the system of ODEs

$$L \frac{dI}{dt} = -R_1 I - V, \quad C \frac{dV}{dt} = I - \frac{V}{R_2}.$$

which we may write as

$$\begin{pmatrix} \dot{I} \\ \dot{V} \end{pmatrix} = \begin{pmatrix} -R_1/L & -1/L \\ 1/C & -1/(CR_2) \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}.$$

Assume that  $R_1 = 1 \Omega$ ,  $R_2 = 3/5 \Omega$ ,  $L = 2 \text{ H}$  and  $C = 2/3 \text{ F}$ . Then

$$\begin{pmatrix} \dot{I} \\ \dot{V} \end{pmatrix} = \begin{pmatrix} -1/2 & -1/2 \\ 3/2 & -5/2 \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}. \quad (1.7.9)$$

## Electric Circuits

We consider the initial value problem with  $I(0) = I_0$  and  $V(0) = V_0$ . Here

$$A = \begin{pmatrix} -1/2 & -1/2 \\ 3/2 & -5/2 \end{pmatrix}$$

has eigenvalues

- ▶  $\lambda_1 = -2$  with eigenvector  $v_1 = \begin{pmatrix} 1/3 \\ 1 \end{pmatrix}$ ,
- ▶  $\lambda_2 = -1$  with eigenvector  $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

We have

$$U = \begin{pmatrix} 1/3 & 1 \\ 1 & 1 \end{pmatrix}, \quad e^{At} = U \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{pmatrix} U^{-1}$$

and calculate

$$e^{At} = \frac{1}{2} \begin{pmatrix} 3e^{-t} - e^{-2t} & e^{-2t} - e^{-t} \\ 3e^{-t} - 3e^{-2t} & 3e^{-2t} - e^{-t} \end{pmatrix}.$$

# Electric Circuits

We obtain

$$\begin{pmatrix} I(t) \\ V(t) \end{pmatrix} = e^{At} \begin{pmatrix} I_0 \\ V_0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3e^{-t} - e^{-2t} \\ 3e^{-t} - 3e^{-2t} \end{pmatrix} I_0 + \frac{1}{2} \begin{pmatrix} e^{-2t} - e^{-t} \\ 3e^{-2t} - e^{-t} \end{pmatrix} V_0.$$

It is easy to verify that

$$I(t) = \frac{1}{2}(3e^{-t} - e^{-2t})I_0 + \frac{1}{2}(e^{-2t} - e^{-t})V_0,$$

$$V(t) = \frac{1}{2}(3e^{-t} - 3e^{-2t})I_0 + \frac{1}{2}(3e^{-2t} - e^{-t})V_0$$

actually satisfy the initial value problem.

**1.7.15. Remark.** Regardless of the initial values, the current and voltage will drop off to zero as  $t \rightarrow \infty$ . It is clear that the reason for this lies in the negativity of the eigenvalues. We will investigate this in more detail later.

## Geometric and Algebraic Multiplicity

1.7.16. Lemma. The geometric multiplicity of an eigenvalue is never greater than its algebraic multiplicity.

### Proof.

Suppose  $\lambda_0$  is an eigenvalue of  $A$  with geometric multiplicity  $m \leq n$ . Then there exist  $m$  independent eigenvectors  $v_1, \dots, v_m$  to  $\lambda_0$ . Adding suitable independent vectors, we obtain a basis  $\mathcal{B} = \{v_1, \dots, v_m, b_1, \dots, b_{n-m}\}$  of  $\mathbb{R}^n$ . If  $U = (v_1, \dots, v_m, b_1, \dots, b_{n-m})$ , then

$$U^{-1}AU = \begin{pmatrix} \lambda_0 \mathbb{1} & C \\ 0 & B \end{pmatrix}$$

in block matrix notation, where  $\mathbb{1}$  is an  $m \times m$  unit matrix and  $B \in \text{Mat}((n-m) \times (n-m); \mathbb{R})$ ,  $C \in \text{Mat}(n \times m; \mathbb{R})$  are arbitrary matrices.

# Geometric and Algebraic Multiplicity

Proof (continued).

Noting that  $\det(AB) = \det A \det B$  for  $n \times n$  matrices  $A$  and  $B$ ,

$$1 = \det(\mathbb{1}) = \det(U^{-1}U) = \det(U^{-1})\det(U).$$

We hence see

$$\begin{aligned} p(\lambda) &= \det(A - \lambda\mathbb{1}) = \det(U^{-1})\det(A - \lambda\mathbb{1})\det U \\ &= \det(U^{-1}(A - \lambda\mathbb{1})U) \\ &= \det(U^{-1}AU - \lambda\mathbb{1}) \\ &= (\lambda - \lambda_0)^m \det(B - \lambda\mathbb{1}), \end{aligned}$$

so  $\lambda_0$  is a root of multiplicity at least  $m$  of  $p(\lambda)$ . By definition, the algebraic multiplicity of  $\lambda$  is at least  $m$ .



1. Separable Equations
2. Linear Equations
3. Transformable Equations
4. General Integral Curves of First Order ODEs
5. Implicit Equations
6. Systems of Equations
7. The Eigenvalue Problem
8. The Spectral Theorem for Self-Adjoint Matrices
9. Non-Diagonalizable Matrices
10. Solutions to Inhomogeneous, Linear Systems
11. Linear Second-Order Equations and Vibrations

## Self-Adjoint Matrices

It turns out that not every matrix is diagonalizable; in particular, a given matrix may have an eigenvalue where the algebraic and geometric multiplicities differ. We will show that this never happens for a certain class of matrices:

Recall that if  $(V, \langle \cdot, \cdot \rangle)$  is an inner product space and  $A: V \rightarrow V$ , the adjoint of  $A$  is defined through the relation

$$\langle x, Ay \rangle = \langle A^*x, y \rangle \quad \text{for all } x, y \in V.$$

If  $A = A^*$ , the map  $A$  is said to be **self-adjoint**. In the case of matrices, if  $A \in \text{Mat}(n \times n; \mathbb{C})$ ,

$$A = A^* \quad \Leftrightarrow \quad A = \overline{A}^T,$$

where  $A^T$  denotes the transpose of  $A$  and  $\overline{A}$  is the complex conjugate of  $A$ , taken in each component.

# Eigenvalues of Self-Adjoint Matrices

1.8.1. Lemma. Let  $A = A^*$  be a self-adjoint matrix. Then all eigenvalues of  $A$  are real.

## Proof.

Let  $A = A^*$  and suppose that  $\lambda$  is an eigenvalue of  $A$ , i.e.,  $p(\lambda) = 0$ , where  $p$  is the characteristic polynomial. Then the system of equations  $(A - \lambda \mathbb{1})x = 0$  has a non-trivial solution, i.e., there exists an  $x \neq 0$  in  $\mathbb{C}^n$  such that  $Ax = \lambda x$  and

$$\lambda \|x\|^2 = \lambda \langle x, x \rangle = \langle x, Ax \rangle = \langle Ax, x \rangle = \langle \lambda x, x \rangle = \bar{\lambda} \langle x, x \rangle = \bar{\lambda} \|x\|^2$$

from which we deduce  $\lambda = \bar{\lambda}$ . □

## Existence of Eigenvalues of Self-Adjoint Matrices

Every matrix has at least one eigenvalue, as the characteristic polynomial will have at least one (in general, complex) root. The previous lemma allows us to deduce a little more:

1.8.2. Corollary. A linear map  $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $A = A^T$  has at least one eigenvalue.

### Proof.

We can regard  $A$  as a matrix with real coefficients, so  $A = A^T$  means that  $A$  is self-adjoint. The characteristic polynomial will have at least one complex zero. By Lemma 1.8.1 this zero is real and hence an eigenvalue of  $A$ . □

# Orthogonal Complement and Invariant Subspaces

Recall the following definition from last term:

1.8.3. Definition. If  $U \subset \mathbb{R}^n$  is a subspace, the set

$$U^\perp := \left\{ y \in \mathbb{R}^n : \forall_{x \in U} \langle x, y \rangle = 0 \right\}$$

is called the **orthogonal complement** of  $U$ .

We have shown in the last term that if  $U$  is a subspace,  $U^\perp$  is also a subspace.

1.8.4. Definition. Let  $U \subset \mathbb{R}^n$  be a subspace and  $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$  a linear map. We say that  $U$  is **invariant under  $A$**  if

$$x \in U \quad \Rightarrow \quad Ax \in U.$$

## Orthogonal Complement and Invariant Subspaces

1.8.5. Lemma. Let  $A = A^* \in \text{Mat}(n \times n; \mathbb{R})$  be self-adjoint and  $U \subset \mathbb{R}^n$  invariant under  $A$ . Then  $U^\perp$  is invariant under  $A$ .

Proof.

Let  $y \in U^\perp$ . Then for all  $x \in U$  we have  $0 = \langle y, Ax \rangle = \langle Ay, x \rangle$ , so  $Ay \perp x$ , i.e.,  $Ay \in U^\perp$ . □

# The Spectral Theorem

Reference **Jänich**, Section 10.3 (“Principal Axis Transformation”)

**1.8.6. Spectral Theorem.** Let  $A = A^* \in \text{Mat}(n \times n; \mathbb{R})$  be a self-adjoint matrix. Then there exists an orthonormal basis of  $\mathbb{R}^n$  consisting of eigenvectors of  $A$ .

**1.8.7. Corollary.** Every self-adjoint matrix  $A$  is diagonalizable. Furthermore, if  $(v_1, \dots, v_n)$  is an orthonormal basis of eigenvectors and  $U = (v_1, \dots, v_n)$ , then  $U^{-1} = U^*$ . Hence, if  $A$  is self-adjoint, there exists an orthogonal matrix  $U$  such that

$$D = U^*AU$$

is the diagonalization of  $A$ .

# The Spectral Theorem

## Proof of the Spectral Theorem.

We prove the theorem by induction. Consider the case  $n = 1$ . By Corollary 1.8.2  $A$  has an eigenvalue  $\lambda \in \mathbb{R}$  and hence an associated eigenvector, which corresponds to a basis in  $\mathbb{R}$ .

Now assume that for any self-adjoint map  $\mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$  there exists an orthonormal basis of eigenvectors. Let  $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be self-adjoint. Then  $A$  has at least one eigenvalue  $\lambda_1 \in \mathbb{R}$  and a normed eigenvector  $v_1$ . Let  $V_1 = \text{span}\{v_1\}$  be the associated eigenspace. This eigenspace is of course invariant under  $A$ ,

$$A|_{V_1}: V_1 \rightarrow V_1,$$

so by Lemma 1.8.5 the same is true of  $V_1^\perp$ . Now consider  $A$  as a map

$$A|_{V_1^\perp}: V_1^\perp \rightarrow V_1^\perp.$$

# The Spectral Theorem

Proof of the Spectral Theorem (continued).

After a change of basis,  $A|_{V_1^\perp}$  may be regarded as a self-adjoint map  $\mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ . By the induction hypothesis, there exists an orthonormal basis of  $V_1^\perp$  consisting of eigenvectors of  $A|_{V_1^\perp}$ . Since these basis vectors are orthogonal to  $v_1$ , we obtain an orthonormal basis of  $\mathbb{R}^n$  after adding  $v_1$  to these basis vectors. □

Recall that a matrix  $A \in \text{Mat}(n \times n, \mathbb{R})$  is said to be **positive definite** if

$$\langle x, Ax \rangle > 0 \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}.$$

$A$  is said to be **negative definite** if  $-A$  is positive definite, i.e., if

$$\langle x, Ax \rangle < 0 \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}.$$

## Remark on Positive Definite Linear Maps

1.8.8. Lemma. Let  $A \in \text{Mat}(n \times n, \mathbb{R})$  be self-adjoint. Then  $A$  is positive definite if and only if all eigenvalues of  $A$  are strictly greater than zero.

Proof.

( $\Rightarrow$ ) We show the contraposition. Suppose  $A$  has one eigenvalue  $\lambda$  that is less than or equal to zero. Let  $v$  be the associated eigenvector. Then

$$\langle v, Av \rangle = \lambda \langle v, v \rangle = \lambda \|v\|^2 \leq 0,$$

so  $A$  is not positive definite.

( $\Leftarrow$ ) Suppose that all eigenvalues  $\lambda_1, \dots, \lambda_n$  are strictly positive. By Corollary 1.8.7 there exists an orthogonal matrix  $U$  such that  $D = UAU^* = \text{diag}(\lambda_1, \dots, \lambda_n)$ . Let  $x \in \mathbb{R}^n$  and  $y := Ux$ . Then

$$\langle x, Ax \rangle = \langle x, U^*DUx \rangle = \langle Ux, DUx \rangle = \sum_i \lambda_i y_i^2 > 0.$$

□

1. Separable Equations
2. Linear Equations
3. Transformable Equations
4. General Integral Curves of First Order ODEs
5. Implicit Equations
6. Systems of Equations
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9. Non-Diagonalizable Matrices
10. Solutions to Inhomogeneous, Linear Systems
11. Linear Second-Order Equations and Vibrations

# Extending Diagonalization to Arbitrary Matrices

Reference **Jänich**, Section 11.3

We already know that any  $n \times n$  matrix will have  $n$  complex eigenvalues (counting with algebraic multiplicities) and that the eigenvalues may have differing algebraic and geometric multiplicities. For a diagonalizable matrix, the diagonal form consists of the eigenvalues along the diagonal. If a matrix is not diagonalizable, we might hope to find a non-diagonal form which nevertheless has the (complex) eigenvalues along the diagonal, but perhaps also certain other non-zero entries.

For a diagonalizable matrix  $A$ , the matrix  $U$  that transforms  $A$  into diagonal form consists of eigenvectors of  $A$ . If  $A$  is not diagonalizable, we essentially have too few eigenvectors, i.e., they do not span the whole space. A solution to this problem is to add certain **generalized eigenvectors** to  $U$ .

## Extending the Eigenspace

Let  $\lambda$  be an eigenvalue of a matrix  $A$  and denote by  $a_\lambda$  the algebraic multiplicity of  $\lambda$ . Suppose that the algebraic multiplicity is greater than its geometric multiplicity, i.e.,

$$\dim V_\lambda < a_\lambda.$$

Our problem is that  $V_\lambda$  is too “small.” We now try to find a suitable extension of  $V_\lambda$ .

We set

$$E_1 := V_\lambda = \ker(A - \lambda \mathbb{1})$$

and define

$$E_2 := \ker(A - \lambda \mathbb{1})^2 = \{v \in V : (A - \lambda \mathbb{1})^2 v = 0\}.$$

It is obvious that  $E_1 \subset E_2$ , so  $E_2$  is an extension of  $V_\lambda$ . The elements of  $E_2$  are called **generalized eigenvectors** of  $\lambda$ .

## Extending the Eigenspace

What are the properties of the vectors that we have added to  $V_\lambda$ ? By construction,

$$E_2 \setminus E_1 = \{v \in V : (A - \lambda \mathbb{1})^2 v = 0 \text{ and } (A - \lambda \mathbb{1})v \neq 0\}.$$

Now if  $\dim E_2 = a_\lambda$ , we have found an extension that is “large enough”. If not, we further extend  $E_2$  by setting

$$E_3 := \ker(A - \lambda \mathbb{1})^3 = \{v \in V : (A - \lambda \mathbb{1})^3 v = 0\}$$

where again

$$E_3 \setminus E_2 = \{v \in V : (A - \lambda \mathbb{1})^3 v = 0 \text{ and } (A - \lambda \mathbb{1})^2 v \neq 0\}.$$

This process can be continued until  $\dim E_k = a_\lambda$  for some  $k \in \mathbb{N}$ . We will now discuss these extensions in more detail, show how to construct the generalized eigenvectors and how they enable us to “almost-diagonalize” any matrix.

## Generalized Eigenvectors

1.9.1. Definition. Let  $\lambda$  be an eigenvalue of a matrix  $A$ . Then  $v$  is a **generalized eigenvector of rank  $r$** ,  $r \in \mathbb{N} \setminus \{0\}$ , if

$$(A - \lambda \mathbb{1})^r v = 0 \quad \text{and} \quad (A - \lambda \mathbb{1})^{r-1} v \neq 0.$$

### 1.9.2. Remark.

1. An “ordinary” eigenvector is hence a generalized eigenvector of rank 1.
2. In the notation of the preceding slide, a generalized eigenvector of rank  $r$  is an element of  $E_r \setminus E_{r-1}$ .

## Finding Generalized Eigenvectors “Bottom-up”

So how do we find generalized eigenvectors? There are two main ways of constructing them, one is “from the bottom up”, the other is “from the top down”.

**Bottom-up Method:** Suppose that  $E_1$  is given, i.e., the eigenspace has been found. Suppose that  $w \in E_2 \setminus E_1$ . Then

$$(A - \lambda \mathbb{1})^2 w = (A - \lambda \mathbb{1})(A - \lambda \mathbb{1})w = 0,$$

so  $(A - \lambda \mathbb{1})w \in E_1$  and  $(A - \lambda \mathbb{1})w$  is non-zero. Hence, there exists an eigenvector  $v \in E_1$  such that

$$(A - \lambda \mathbb{1})w = v. \tag{1.9.1}$$

We can now reverse this argument: choosing a suitable  $v^{(1)} \in E_1$ , we solve

$$(A - \lambda \mathbb{1})v^{(2)} = v^{(1)} \tag{1.9.2}$$

to find one  $v^{(2)} \in E_2 \setminus E_1$ .

## Finding Generalized Eigenvectors

Having found such a  $v^{(2)}$ ,

$$E_1 \oplus \text{span}\{v^{(2)}\} \subset E_2$$

and  $\dim E_1 \oplus \text{span}\{v^{(2)}\} = \dim E_1 + 1$ . If the dimension is still smaller than  $a_\lambda$ , we can repeat the procedure, i.e., find some vector  $v^{(3)}$  such that

$$(A - \lambda \mathbb{1})v^{(3)} = v^{(2)}$$

and then consider

$$E_1 \oplus \text{span}\{v^{(2)}, v^{(3)}\} \subset E_3.$$

This process is repeated until

$$(A - \lambda \mathbb{1})v^{(m+1)} = v^{(m)}$$

does not yield a solution. If the dimension is still not large enough, one chooses another vector  $\tilde{v}^{(1)} \in E_1$  (independent of  $v^{(1)}$ ) and starts the process again by solving (1.9.2). Of course, the success of the process is contingent on choosing the initial  $v^{(1)} \in \text{ran}(A - \lambda \mathbb{1})$ .

# Generalized Eigenvectors with Mathematica

1.9.3. Example. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 3 & -1 \\ 4 & -1 & 3 \end{pmatrix}.$$

In this example, we will use Mathematica code to perform the calculations. First, we find the characteristic polynomial and the eigenvalues. We set up  $A - \lambda \mathbb{1}$ :

```
A = {{0, 1, -1}, {0, 3, -1}, {4, -1, 3}};  
MatrixForm[A - λ IdentityMatrix[3]]
```

$$\begin{pmatrix} -\lambda & 1 & -1 \\ 0 & 3 - \lambda & -1 \\ 4 & -1 & 3 - \lambda \end{pmatrix}$$

## Characteristic Polynomial and Eigenvalues

The characteristic polynomial and the eigenvalues are:

```
p[λ_] := Det[A - λ IdentityMatrix[3]]; p[λ]
```

$$8 - 12\lambda + 6\lambda^2 - \lambda^3$$

```
Solve[p[λ] == 0, λ]
```

$$\{\{\lambda \rightarrow 2\}, \{\lambda \rightarrow 2\}, \{\lambda \rightarrow 2\}\}$$

Of course, there is a built-in function in to give the eigenvalues directly:

```
Eigenvalues[A]
```

$$\{2, 2, 2\}$$

Thus  $A$  has a single eigenvalue  $\lambda = 2$  with algebraic multiplicity 3.

# Eigenvectors

We set up the equations for finding the eigenvectors

```
eqns = Thread[(A - λ IdentityMatrix[3]).{x, y, z} == {0, 0, 0}] /. λ → 2
{-2 x + y - z == 0, y - z == 0, 4 x - y + z == 0}
```

and then solve them:

```
Solve[eqns, {x, y, z}]
```

Solve::svars : Equations may not give solutions for all "solve" variables. >

```
{ {x → 0, z → y} }
```

Mathematica gives a warning message because there are more unknowns than equations. We hence have

$$V_\lambda = \left\{ v \in \mathbb{R}^3 : v = \alpha \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R} \right\}.$$

## Eigenvectors

There is only a single independent eigenvector,  $\dim V_\lambda = 1$ , and so  $\lambda$  has geometric multiplicity 1. The matrix  $A$  is not diagonalizable. The built-in command for eigenvectors in Mathematica gives

```
Eigenvectors[A]
```

```
{ {0, 1, 1}, {0, 0, 0}, {0, 0, 0} }
```

Mathematica returns zero vectors in place of the “missing” two eigenvectors.

We will now construct generalized eigenvectors. We choose the eigenvector

$$v^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

and solve  $(A - 2\mathbb{1})v = v^{(1)}$  to obtain a solution  $v^{(2)}$ .

# First Generalized Eigenvector

```
Solve[
  Thread[(A - 2 IdentityMatrix[3]).{x, y, z} == {0, 1, 1}],
  {x, y, z}]
```

Solve::svars: Equations may not give solutions for all "solve" variables. >>

$$\left\{ \left\{ x \rightarrow \frac{1}{2}, y \rightarrow 1 + z \right\} \right\}$$

We choose the solution

$$v^{(2)} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

which is a generalized eigenvector of rank 2. Since the algebraic multiplicity of  $\lambda$  is 3, we need a further generalized eigenvector, so we solve  $(A - 2\mathbb{1})v = v^{(2)}$ .

## Second Generalized Eigenvector

```
Solve[
  Thread[(A - 2 IdentityMatrix[3]).{x, y, z} ==
    {1/2, 1/2, -1/2}], {x, y, z}]
```

Solve::svrs: Equations may not give solutions for all "solve" variables. >>

$$\left\{ \left\{ x \rightarrow 0, y \rightarrow \frac{1}{2} + z \right\} \right\}$$

Thus we have

$$v^{(3)} = \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

We now have a set of generalized eigenvectors  $\mathcal{B} = \{v^{(1)}, v^{(2)}, v^{(3)}\}$ , where  $v^{(1)}$  is an actual eigenvector,  $v^{(2)}$  a generalized eigenvector of rank 2 and  $v^{(3)}$  a generalized eigenvector of rank 3.

## The Basis of Generalized Eigenvectors

Obviously,  $\mathcal{B}$  is a basis of  $\mathbb{R}^3$ . Let us find what form  $A$  takes in this basis.  
Setting

$$U = (v^{(1)}, v^{(2)}, v^{(3)}) = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 2 & 1 & 1 \\ 2 & -1 & 0 \end{pmatrix},$$

we calculate  $U^{-1}AU$ :

$$U = \begin{pmatrix} 0 & 1/2 & 0 \\ 1 & 1/2 & 1/2 \\ 1 & -1/2 & 0 \end{pmatrix};$$

**MatrixForm[Inverse[U].A.U]**

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

## A “Nearly-Diagonalization”

We see that

$$U^{-1}AU = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad (1.9.3)$$

a matrix that is “nearly diagonal”.

**Question.** Why is it obvious (without performing any matrix multiplication) that  $U^{-1}AU$  has the form (1.9.3)?

This shows why the generalized eigenvectors are useful for “nearly diagonalizing” a matrix. Before treating this near-diagonalization in more detail, we continue our discussion of how to obtain generalized eigenvectors. At this point we will call a matrix “nearly diagonal” if it has its eigenvalues on the diagonal, possibly some ones directly above the diagonal and zeroes everywhere else.

## Another Example for Generalized Eigenvectors

1.9.4. Example. Consider the matrix

$$A = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix}.$$

The eigenvalues of  $A$  are given through

$$\begin{aligned}\det(A - \lambda \mathbb{1}) &= \det \begin{pmatrix} 5 - \lambda & -3 & -2 \\ 8 & -5 - \lambda & -4 \\ -4 & 3 & 3 - \lambda \end{pmatrix} \\ &= 1 - 3\lambda + 3\lambda^2 - \lambda^3 = (\lambda - 1)^3 = 0.\end{aligned}$$

Hence  $A$  has the eigenvalue  $\lambda = 1$  (of algebraic multiplicity 3).

# The Eigenspace

To find eigenvectors, we solve

$$(A - \mathbb{I})x = \begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

giving the general solution

$$x_3 = 2x_1 - \frac{3}{2}x_2.$$

We can find two independent eigenvectors by choosing

$$v_1 = \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}. \quad (1.9.4)$$

Hence,  $\dim V_\lambda = 2$  and we need to find one generalized eigenvector.

## The Generalized Eigenvectors

However, our initial attempts to find a generalized eigenvector fail: neither

$$(A - \mathbb{1})v = v_1 \quad \text{nor} \quad (A - \mathbb{1})v = v_2$$

have a solution:

```
Solve[
 Thread[(A - IdentityMatrix[3]).{x, y, z} == {1, 0, 2}],
 {x, y, z}]
{}
```

```
Solve[
 Thread[(A - IdentityMatrix[3]).{x, y, z} == {0, 2, -3}],
 {x, y, z}]
{}
```

## The Generalized Eigenvectors

What went wrong? As mentioned earlier, we need to pick an eigenvector  $v^{(1)}$  that is actually in the range of  $A - \lambda \mathbb{1}$ , and this is not the case for either  $v_1$  or  $v_2$ . However, if we choose

$$v^{(1)} := v_2 + v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix},$$

then we obtain a generalized eigenvector:

```
Solve[
 Thread[(A - IdentityMatrix[3]).{x, y, z} == {1, 2, -1}],
 {x, y, z}]
```

Solve::svars : Equations may not give solutions for all "solve" variables. >>

$$\left\{ \left\{ z \rightarrow -\frac{1}{2} + 2x - \frac{3y}{2} \right\} \right\}$$

## The Generalized Eigenvectors

We choose a generalized eigenvector (of rank 2) to be

$$v^{(2)} := \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}.$$

Then the basis of generalized eigenvectors can be taken as

$$\mathcal{B} = \{v_1, v^{(1)}, v^{(2)}\}.$$

Question. If  $U = (v_1, v^{(1)}, v^{(2)})$ , what is  $U^{-1}AU$ ?

Question. It would also be permissible to choose  $\mathcal{B} = \{v_2, v^{(1)}, v^{(2)}\}$  but  $\mathcal{B} = \{v_1, v_2, v^{(2)}\}$  does **not** lead to a “nearly diagonal” matrix. Why?

## Problems with the “Bottom-up” Method

Example 1.9.4 illustrates a pitfall in the “bottom-up” method (see Slide 189) we have used so far: to find a generalized eigenvector  $w$  of rank  $r$ , we need to make sure that the generalized eigenvector  $v$  of rank  $r - 1$  is in the range of  $A - \lambda\mathbb{1}$ . The reason for this is that, by (1.9.1), for every  $w$  there exists a  $v$  such that

$$(A - \lambda\mathbb{1})w = v,$$

but it is not true that for every  $v$  there exists a  $w$  such that (1.9.1) holds.

As we have seen in Example 1.9.4, this means we may have to choose our eigenvectors in a certain, non-obvious way. Apart from either guessing that  $v^{(1)} = v_1 + v_2$  works or using a brute-force approach by setting  $v^{(1)} = \alpha v_1 + \beta v_2$  and finding  $\alpha, \beta \in \mathbb{R}$  such that  $v^{(1)} \in \text{ran}(A - \lambda\mathbb{1})$ , there is no obvious way to choose “correct” eigenvectors.

## Finding Generalized Eigenvectors “Top-down”

This problem is (partially) averted by constructing the generalized eigenvectors “top-down”.

**Top-down Method:** Instead of using the eigenvectors to find generalized eigenvectors of rank 2, then generalized eigenvectors of rank 3 and so on, we start by finding generalized eigenvectors of the highest rank first.

Given an eigenvalue  $\lambda$  of algebraic multiplicity  $a_\lambda$  and geometric multiplicity  $\dim V_\lambda$ , the highest-rank generalized eigenvector that may be needed is of rank  $m := a_\lambda - \dim V_\lambda + 1$ . Therefore, we attempt to find such a generalized eigenvalue first, by solving

$$(A - \lambda \mathbb{1})^m v = 0$$

under the condition  $(A - \lambda \mathbb{1})^{m-1} v \neq 0$ . From this solution, denoted  $v^{(m)}$  we find a generalized eigenvector of rank  $m - 1$  simply by setting

$$v^{(m-1)} := (A - \lambda \mathbb{1}) v^{(m)}.$$

## Finding Generalized Eigenvectors “Top-down”

Clearly,  $v^{(m-1)}$  satisfies

$$(A - \lambda \mathbb{1})^{m-1} v^{(m-1)} = 0 \quad \text{and} \quad (A - \lambda \mathbb{1})^{m-2} v^{(m-1)} \neq 0.$$

We iteratively set

$$\begin{aligned} v^{(m-1)} &:= (A - \lambda \mathbb{1})v^{(m)}, \\ v^{(m-2)} &:= (A - \lambda \mathbb{1})v^{(m-1)}, \\ &\vdots \\ v^{(1)} &:= (A - \lambda \mathbb{1})v^{(2)}. \end{aligned}$$

The set of generalized eigenvectors  $\{v^{(m)}, v^{(m-1)}, \dots, v^{(1)}\}$  is called a **chain of length  $m$**  of generalized eigenvectors. Note that  $v^{(1)}$  is always an eigenvector.

## Example for the “Top-down” Method

1.9.5. Example. Let us return to the matrix  $A$  of Example 1.9.3,

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 3 & -1 \\ 4 & -1 & 3 \end{pmatrix}.$$

We already know that  $A$  has a single eigenvalue  $\lambda = 2$  of algebraic multiplicity 3 and geometric multiplicity 1. Therefore, we need at most a generalized eigenvalue of rank 3 to extend  $V_\lambda$  to  $E_3$ . The top-down method requires us to first solve

$$(A - 2\mathbb{1})^3 v = 0 \quad \text{with} \quad (A - 2\mathbb{1})^2 v \neq 0.$$

We will again use Mathematica for all calculations.

## Example for the “Top-down” Method

```
MatrixForm[MatrixPower[A - 2 IdentityMatrix[3], 3]]
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
MatrixForm[MatrixPower[A - 2 IdentityMatrix[3], 2]]
```

$$\begin{pmatrix} 0 & 0 & 0 \\ -4 & 2 & -2 \\ -4 & 2 & -2 \end{pmatrix}$$

We see that any vector solves  $(A - 2\mathbb{1})^3 v = 0$  and we can easily choose a vector such that  $(A - 2\mathbb{1})^2 v \neq 0$ . We set

$$v^{(3)} := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

## Example for the “Top-down” Method

We obtain  $v^{(2)}$  by applying  $A - \lambda \mathbb{1}$  to  $v^{(3)}$ :

```
(A - 2 IdentityMatrix[3]).{0, 1, 0}  
{1, 1, -1}
```

Hence,

$$v^{(2)} := \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

## Example for the “Top-down” Method

Similarly,  $v^{(1)} = (A - \lambda \mathbb{1})v^{(2)}$ :

```
(A - 2 IdentityMatrix[3]).{1, 1, -1}  
{0, 2, 2}
```

Hence,

$$v^{(1)} := \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}.$$

We can easily verify that  $v^{(1)}$  is actually an eigenvector. We now have a cycle of generalized eigenvectors  $v^{(1)}, v^{(2)}, v^{(3)}$ . Setting  $U := (v^{(1)}, v^{(2)}, v^{(3)})$ , we obtain

$$U^{-1}AU = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

## Another Example for the “Top-down” Method

1.9.6. Example. The bottom-up method did not immediately work in Example 1.9.4. Let us apply the top-down method for

$$A = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix}.$$

We know that  $A$  has a single eigenvalue  $\lambda = 1$  of algebraic multiplicity 3 and geometric multiplicity 2. Hence, we need only a single generalized eigenvector, which will be of rank 2. We solve

$$(A - \mathbb{1})^2 v = 0 \quad \text{with} \quad (A - \mathbb{1})v \neq 0.$$

```
MatrixForm[MatrixPower[A - IdentityMatrix[3], 2]]
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

## Another Example for the “Top-down” Method

We see that we can take any vector that is not an eigenvector for our generalized eigenvector of rank 2. From (1.9.4) we find that

$$v^{(2)} := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

is not an eigenvector. Next, we have

$$v^{(1)} := (A - \mathbb{1})v^{(2)} = \begin{pmatrix} -3 \\ -6 \\ 3 \end{pmatrix}$$

which is an eigenvector of  $A$ . So now  $v^{(1)}, v^{(2)}$  is a cycle of generalized eigenvectors. Since the algebraic multiplicity is equal to 3, we need another independent eigenvector.

## Another Example for the “Top-down” Method

We may simply take  $v_1$  from (1.9.4) and set

$$U = (v_1, v^{(1)}, v^{(2)}) = \begin{pmatrix} 0 & -3 & 0 \\ 2 & -6 & 1 \\ -3 & 3 & 0 \end{pmatrix}.$$

Then  $U^{-1}AU$  has the desired form:

$$U = \begin{pmatrix} 0 & -3 & 0 \\ 2 & -6 & 1 \\ -3 & 3 & 0 \end{pmatrix};$$

**MatrixForm[Inverse[U].A.U]**

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

## Jordan Matrices

1.9.7. Definition. For  $\lambda \in \mathbb{C}$  we define the **Jordan block of size  $k \in \mathbb{N} \setminus \{0\}$**  by

$$J_k(\lambda) := \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & 1 \\ & & & \lambda \end{pmatrix} \in \text{Mat}(k \times k, \mathbb{C}).$$

A block matrix of the form

$$J = \begin{pmatrix} J_{k_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{k_m}(\lambda_m) \end{pmatrix}$$

with not necessarily distinct  $\lambda_1, \dots, \lambda_m \in \mathbb{C}$  and  $k_1, \dots, k_m \in \mathbb{N}$  is called a **Jordan matrix**.

# Jordan Matrices

## 1.9.8. Examples.

1. A diagonal matrix is a Jordan matrix where each block has size  $k = 1$ .
2. The matrix

$$J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

is a Jordan block of size  $k = 3$  and also a Jordan matrix consisting of a single Jordan block.

3. The matrix

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is a Jordan matrix consisting of two Jordan blocks:

$$J = \begin{pmatrix} J_1(1) & 0 \\ 0 & J_2(1) \end{pmatrix}.$$

## The Jordan Normal Form

It turns out that any matrix may be transformed into a Jordan matrix:

1.9.9. **Definition and Theorem.** For any matrix  $A \in \text{Mat}(\mathbb{C}, n)$  there exists a basis of  $\mathbb{C}^n$  consisting of generalized eigenvectors such that

$$J := U^{-1}AU$$

is a Jordan matrix, where  $U$  is the transformation to this basis. This Jordan matrix is unique and called the **Jordan normal form** of  $A$ .

Hence, the “nearly diagonal” form we have constructed previously is formally the Jordan normal form of a matrix  $A$ .

## The Jordan Normal Form

Mathematica has a built-in command to give both the Jordan normal form and a matrix  $U$  of generalized eigenvectors that yields this form:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 3 & -1 \\ 4 & -1 & 3 \end{pmatrix};$$

```
{U, J} = JordanDecomposition[A];  
{MatrixForm[U], MatrixForm[J]}
```

$$\left\{ \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{4} \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \right\}$$

## Nilpotent Matrices

1.9.10. **Definition.** We say that a matrix  $N$  is nilpotent if there exists some  $k \in \mathbb{N}$  such that  $N^k = 0$ .

It is easy to see that a matrix that is zero on and below the diagonal is nilpotent. Hence we can write a Jordan matrix as

$$J = D + N$$

where  $D$  is a diagonal and  $N$  is a nilpotent matrix.

## Functional Calculus Based on the Jordan Form

Ideally, we would like to write

$$e^J = e^{D+N} = e^D \cdot e^N,$$

because the exponential of a nilpotent matrix of degree can be calculated easily:

$$e^N = \sum_{i=0}^{\infty} \frac{1}{i!} N^i = \sum_{i=0}^{k-1} c_i N^i$$

if  $N^k = 0$ . However, there is a problem: for two matrices  $A, B \in \text{Mat}(n, \mathbb{R})$

$$e^{A+B} = e^A e^B \quad \Leftrightarrow \quad AB = BA.$$

and this is in general not the case for  $D$  and  $N$ .

## Functional Calculus Based on the Jordan Form

However, the Jordan matrix is a block matrix,

$$J = \begin{pmatrix} J_{k_1}(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J_{k_m}(\lambda_m) \end{pmatrix}$$

so that

$$e^J = \begin{pmatrix} e^{J_{k_1}(\lambda_1)} & & 0 \\ & \ddots & \\ 0 & & e^{J_{k_m}(\lambda_m)} \end{pmatrix}.$$

It is hence sufficient to calculate the exponential of a Jordan block. Since

$$J_k(\lambda) = \lambda \mathbb{1} + N$$

for some nilpotent matrix, this can be done easily.

# Functional Calculus Based on the Jordan Form

1.9.11. Example. Let

$$J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Then

$$e^J = \begin{pmatrix} e^{(2 \ 1)} & 0 \\ 0 & e^3 \end{pmatrix}.$$

Now

$$\begin{aligned} e^{(2 \ 1)} &= e^{(2 \ 0)} e^{(0 \ 1)} = \begin{pmatrix} e^2 & 0 \\ 0 & e^2 \end{pmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^k \\ &= \begin{pmatrix} e^2 & 0 \\ 0 & e^2 \end{pmatrix} \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{1!} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 + \dots \right) \\ &= \begin{pmatrix} e^2 & 0 \\ 0 & e^2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^2 & e^2 \\ 0 & e^2 \end{pmatrix} \end{aligned}$$

## Functional Calculus Based on the Jordan Form

Thus we have

$$e^J = \begin{pmatrix} e^2 & e^2 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^3 \end{pmatrix}$$

Once more, there is a dedicated command for this in Mathematica:

```
MatrixForm[MatrixExp[{{2, 1, 0}, {0, 2, 0}, {0, 0, 3}}]]
```

$$\begin{pmatrix} e^2 & e^2 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^3 \end{pmatrix}$$

## Application to Systems of Equations

1.9.12. Example. We wish to solve the system of equations

$$\dot{x}_1 = x_1 - x_2, \quad \dot{x}_2 = x_1 + 3x_2.$$

In vector notation,  $\dot{x} = Ax$ , where  $x = (x_1, x_2)$  and

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}.$$

The general solution is given by  $x(t) = e^{At}x_0$  where  $x_0 = x(0)$  is an arbitrary initial condition. An easy calculation shows that the matrix  $A$  has eigenvalue  $\lambda = 2$  with algebraic multiplicity 2. However, the eigenvalue only has geometric multiplicity 1 and we choose the eigenvector

$$v^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

## Application to Systems of Equations

A generalized eigenvector is found by solving  $(A - \lambda \mathbb{1})v^{(2)} = v^{(1)}$ , i.e.,

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We choose the solution

$$v^{(2)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Setting

$$U = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \quad \text{and noting} \quad U^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix},$$

we have

$$J = U^{-1}AU = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

## Application to Systems of Equations

Then

$$e^{Jt} = e^{\begin{pmatrix} 2t & 0 \\ 0 & 2t \end{pmatrix}} e^{\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}.$$

and hence

$$\begin{aligned} e^{At} &= U e^{Jt} U^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} (1-t)e^{2t} & -te^{2t} \\ te^{2t} & (t+1)e^{2t} \end{pmatrix}. \end{aligned}$$

The general solution is therefore given by

$$x(t) = \begin{pmatrix} (1-t)e^{2t} & -te^{2t} \\ te^{2t} & (t+1)e^{2t} \end{pmatrix} x(0).$$

1. Separable Equations
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## Fundamental Systems of Solutions

Let us now discuss how to explicitly and efficiently construct fundamental systems of solutions (cf. Remark 1.6.11) for ***linear systems with constant coefficients***. Recall that a solution to the initial value problem

$$\dot{x} = Ax, \quad x(0) = x_0 \quad (1.10.1)$$

for constant  $A \in \text{Mat}(n \times n, \mathbb{R})$  is given by  $x(t) = e^{At}x_0$ . Then the following result follows immediately from Proposition 1.6.13.

**1.10.1. Lemma.** For any basis  $\mathcal{B} = \{v_1, \dots, v_n\}$  of  $\mathbb{R}^n$ , the system of functions

$$\mathcal{F} = \{e^{At}v_1, \dots, e^{At}v_n\}$$

is a fundamental system for  $\dot{x} = Ax$ .

## Constructing Fundamental Systems

In practice, we can obtain a fundamental system in several ways:

- (i) Take  $v_i = e_i$ ,  $i = 1, \dots, n$ , where  $e_i$  is the  $i$ th standard basis vector of  $\mathbb{R}^n$ . Then the columns of  $e^{At}$  are a fundamental system of solutions of  $\dot{x} = Ax$  and the fundamental matrix is given by  $X(t) = e^{At}$ .
- (ii) There exists a basis  $\{u_1, \dots, u_n\}$  of  $\mathbb{R}^n$  consisting of (generalized) eigenvectors  $u_i$ ,  $i = 1, \dots, n$ , such that  $J = U^{-1}AU$  is a Jordan matrix, where  $U = (u_1, \dots, u_n)$ . Thus,  $e^{At} = Ue^{Jt}U^{-1}$ .

If we set  $v_i = u_i$ , we obtain the fundamental matrix

$$X(t) = Ue^{Jt}.$$

(The columns of  $X(t)$  give the fundamental system.) Therefore, if we simply wish to obtain a fundamental solution, we can avoid calculating the inverse of  $U$ .

## Constructing Fundamental Systems

1.10.2. Example. In Example 1.9.12, we considered the equation  $\dot{x} = Ax$  with

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}.$$

We found

$$e^{Jt} = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

Then

$$X(t) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} = \begin{pmatrix} e^{2t} & te^{2t} \\ -e^{2t} & -(t+1)e^{2t} \end{pmatrix}$$

gives a fundamental matrix and

$$x^{(1)}(t) = \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}, \quad x^{(2)}(t) = \begin{pmatrix} te^{2t} \\ -(t+1)e^{2t} \end{pmatrix}$$

a fundamental system of solutions.

## Constructing Fundamental Systems

- (iii) If  $A$  is diagonalizable,  $J = \text{diag}(\lambda_1, \dots, \lambda_n)$ , where  $\lambda_i$  are the eigenvalues of  $A$ , counted with multiplicities. In that case,

$$X(t) = U e^{\text{diag}(\lambda_1, \dots, \lambda_n)t} = (e^{\lambda_1 t} u_1, \dots, e^{\lambda_n t} u_n).$$

Thus, we obtain a fundamental system of the simple form

$$\mathcal{F} = \{e^{\lambda_1 t} u_1, \dots, e^{\lambda_n t} u_n\},$$

where  $u_i$  is the eigenvector corresponding to the eigenvalue  $\lambda_i$ ,  
 $i = 1, \dots, n$ .

# Constructing Fundamental Systems

1.10.3. Example. In Example 1.7.14, the matrix

$$A = \begin{pmatrix} -1/2 & -1/2 \\ 3/2 & -5/2 \end{pmatrix}$$

is diagonalizable and has eigenvalues

- ▶  $\lambda_1 = -2$  with eigenvector  $v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ ,
- ▶  $\lambda_2 = -1$  with eigenvector  $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

A fundamental system is hence given by

$$x^{(1)}(t) = e^{-2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad x^{(2)}(t) = e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (1.10.2)$$

Note that the columns of

$$e^{At} = \frac{1}{2} \begin{pmatrix} 3e^{-t} - e^{-2t} & e^{-2t} - e^{-t} \\ 3e^{-t} - 3e^{-2t} & 3e^{-2t} - e^{-t} \end{pmatrix}.$$

also give a fundamental system, with each column being a linear combination of  $x^{(1)}$  and  $x^{(2)}$ .

## Solving the Inhomogeneous Equation

The key to finding a particular solution to the equation

$$\frac{dx}{dt} = Ax + b(t), \quad x(t_0) = 0 \quad (1.10.3)$$

for constant  $A \in \text{Mat}(n \times n, \mathbb{R})$  is to multiply the equation with  $e^{-At}$ , so (1.10.3) becomes

$$e^{-At} \frac{dx}{dt} = e^{-At} Ax + e^{-At} b(t)$$

and then re-write the equation as

$$\frac{d}{dt}(e^{-At}x) = e^{-At}b(t).$$

This procedure may be regarded as finding the integrating factor  $e^{-At}$  for the system (1.10.3).

## Solution of Initial Value Problems

Integrating, we obtain the inhomogeneous solution

$$x_{\text{inhom}}(t) = e^{At} \int_{t_0}^t e^{-As} b(s) ds$$

This solution generalizes (1.2.7) to explicit systems with constant coefficients. We summarize:

**1.10.4. Theorem.** Let  $A \in \text{Mat}(n \times n, \mathbb{R})$ ,  $b: \mathbb{R} \rightarrow \mathbb{R}^n$  be a continuous function,  $t_0 \in \mathbb{R}$  and  $x_0 \in \mathbb{R}^n$ . Then the **solution to the initial value problem**

$$\frac{dx}{dt} = Ax + b(t), \quad x(t_0) = x_0$$

is given by

$$x(t) = e^{A(t-t_0)} x_0 + e^{At} \int_{t_0}^t e^{-As} b(s) ds.$$

## General Solution of Linear, Constant-Coefficient Systems

1.10.5. Theorem. Let  $A \in \text{Mat}(n \times n, \mathbb{R})$  and  $b: \mathbb{R} \rightarrow \mathbb{R}^n$  be a continuous function. Then the **general solution of the system**

$$\frac{dx}{dt} = Ax + b(t)$$

is given by

$$x(t; c_1, \dots, c_n) = \sum_{k=1}^n c_k x^{(k)}(t) + e^{At} \int e^{-At} b(t) dt.$$

where  $\mathcal{F} = \{x^{(1)}, \dots, x^{(n)}\}$  a fundamental system for the associated homogeneous system  $\frac{dx}{dt} = Ax$  and  $c_1, \dots, c_n \in \mathbb{R}$  are arbitrary.

## Linear Systems with Variable Coefficients

In our extended discussion of systems with constant coefficients (from Slide 140 onwards), we have treated only systems of the form  $\frac{dx}{dt} = Ax$  where  $A$  is a constant matrix. But what of the case of a  $t$ -dependent matrix, i.e.,

$$\frac{dx}{dt} = A(t)x ?$$

It turns out that we can not generalize the solution for single equations to systems in the case of variable coefficients, because the cornerstone of our calculations is no longer true: in general,

$$\frac{d}{dt}e^{A(t)} \neq A'(t)e^{A(t)},$$

as can be seen when writing out the series expansion of the exponential. The problem is that  $A'(t)A(t) \neq A(t)A'(t)$ , in general.

## Linear Systems with Variable Coefficients

In fact, ***there is no general method to solve variable-coefficient homogeneous, linear systems in terms of elementary functions*** and some systems may not have elementary functions as solutions at all. This means that there is now algorithmic way to obtain a fundamental system

Nevertheless, given a fundamental systems of solutions to an associated homogeneous equation, a solution to an inhomogeneous equation can be found. In this and many other constructions of solutions for constant- and variable-coefficient systems, a central role is played by the so-called ***Wronskian***, which we now introduce.

## The Wronskian of $n$ Solutions of a System

1.10.6. Definition. Let  $I \subset \mathbb{R}$  be an interval and  $A(t)$  a matrix-valued function on  $I$ . Let  $x^{(1)}, \dots, x^{(n)}: I \rightarrow \mathbb{R}^n$  be  $n$  arbitrary solutions of the homogeneous system

$$\frac{dx}{dt} = A(t)x. \quad (1.10.4)$$

Then the determinant function  $W: \mathbb{R} \rightarrow \mathbb{R}$  given by

$$W_{x_1, \dots, x_n}(t) := \det(x^{(1)}(t), \dots, x^{(n)}(t))$$

is called the **Wronskian** of these solutions. For short, we often simply write  $W(t)$  when it is clear which solutions  $x^{(1)}, \dots, x^{(n)}$  are referred to.

# The Wronskian of $n$ Solutions of a System

We will now establish that the Wronskian vanishes either for all  $t \in I$  or never vanishes at all. Hence, it is always non-zero for a fundamental system of solutions.

1.10.7. Remark. Let us write

$$X(t) := (x^{(1)}(t), \dots, x^{(n)}(t))$$

for any  $n$  solutions. Then

$$\begin{aligned}\frac{dX}{dt} &= (\dot{x}^{(1)}(t), \dots, \dot{x}^{(n)}(t)) \\ &= (A(t)x^{(1)}, \dots, A(t)x^{(n)}) \\ &= A(t)X.\end{aligned}$$

because each column  $x^{(k)}$  satisfies  $\dot{x}^{(k)} = A(t)x^{(k)}$  and the matrix  $A(t)$  acts on  $X(t)$  “column by column”.

# The Wronskian of $n$ Solutions of a System

1.10.8. Lemma. The Wronskian of  $n$  solutions of (1.10.4) satisfies

$$\frac{dW}{dt} = a(t)W, \quad \text{where} \quad a(t) = \operatorname{tr} A(t).$$

Proof.

We calculate the derivative of  $W$  directly, using a result from the assignments in Vv285:

$$\begin{aligned} W(t+h) &= \det X(t+h) = \det(X(t) + h \cdot \dot{X}(t) + o(h)) \\ &= \det(X(t) + h \cdot A(t)X(t)) + o(h) \\ &= \det((\mathbb{1} + h \cdot A(t))X(t)) + o(h) \\ &= \det X(t) \det(\mathbb{1} + h \cdot A(t)) + o(h) \\ &= W(t)(1 + h \operatorname{tr} A(t) + o(h)) + o(h) \\ &= W(t) + h \cdot W(t) \operatorname{tr} A(t) + o(h) \quad \text{as } h \rightarrow 0. \end{aligned}$$



## Abel's Equation for the Wronskian

From Lemma 1.10.8 we immediately obtain

$$W(t) = W(t_0) e^{-\int_{t_0}^t a(s) ds}.$$

This is known as ***Abel's equation for the Wronskian***.

**1.10.9. Corollary.** Either  $W(t) = 0$  for all  $t$  or  $W(t) \neq 0$  for all  $t$ . In particular, for solutions constructed as in Slide 135, the Wronskian never vanishes.

The Wronskian plays an important role in finding particular solutions of inhomogeneous, linear systems.

## Variation of Parameters for Linear Systems

We wish to apply the method of variation of parameters to the equation

$$\frac{dx}{dt} = A(t)x + b(t), \quad A: \mathbb{R} \rightarrow \text{Mat}(n, \mathbb{R}), \quad b: \mathbb{R} \rightarrow \mathbb{R}^n. \quad (1.10.5)$$

Suppose that we have a fundamental system  $x^{(1)}, \dots, x^{(n)}$  of the associated homogeneous equation

$$\frac{dx}{dt} = A(t)x. \quad (1.10.6)$$

The general solution of (1.10.6) then has the form

$$x_{\text{hom}}(t) = c_1 x^{(1)}(t) + \cdots + c_n x^{(n)}(t), \quad c_1, \dots, c_n \in \mathbb{R}.$$

We now make the ansatz

$$x_{\text{part}}(t) = c_1(t)x^{(1)}(t) + \cdots + c_n(t)x^{(n)}(t)$$

which we insert into (1.10.5).

## Variation of Parameters for Linear Systems

Applying the product rule, we obtain

$$\frac{dx_{\text{part}}}{dt} = \sum_{k=1}^n (c'_k(t)x^{(k)}(t) + c_k(t)(x^{(k)})'(t)) = A(t)x_{\text{part}} + b(t)$$

leading to

$$\sum_{k=1}^n c'_k(t)x^{(k)}(t) = b(t).$$

We write

$$x^{(k)}(t) = \begin{pmatrix} x_1^{(k)} \\ \vdots \\ x_n^{(k)} \end{pmatrix}, \quad k = 1, \dots, n, \quad c(t) = \begin{pmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{pmatrix}$$

and denote by  $X = (x^{(1)}, \dots, x^{(n)})$  the associated fundamental matrix.

# Variation of Parameters for Linear Systems

Using

$$\begin{aligned}
 \sum_{k=1}^n c'_k(t)x^{(k)}(t) &= \begin{pmatrix} c'_1(t)x_1^{(1)}(t) + c'_2(t)x_1^{(2)}(t) + \cdots + c'_n(t)x_1^{(n)}(t) \\ \vdots \\ c'_1(t)x_n^{(1)}(t) + c'_2(t)x_n^{(2)}(t) + \cdots + c'_n(t)x_n^{(n)}(t) \end{pmatrix} \\
 &= \begin{pmatrix} x_1^{(1)}(t) & x_1^{(2)}(t) & \dots & x_1^{(n)}(t) \\ \vdots & & & \vdots \\ x_n^{(1)}(t) & x_n^{(2)}(t) & \dots & x_n^{(n)}(t) \end{pmatrix} \begin{pmatrix} c'_1(t) \\ \vdots \\ c'_n(t) \end{pmatrix} \\
 &= X \cdot c'(t)
 \end{aligned}$$

we obtain the following inhomogeneous linear system of equations for  $c'(t)$ ,

$$Xc'(t) = b(t).$$

## Variation of Parameters for Linear Systems

We solve  $Xc'(t) = b(t)$  using Cramer's rule. Then

$$c'_k(t) = \frac{\det X^{(k)}(t)}{\det X(t)} = \frac{W^{(k)}(t)}{W(t)}, \quad k = 1, \dots, n,$$

where

- ▶  $X^{(k)}$  is the fundamental matrix where the  $k$ th column has been replaced with  $b$ ,
- ▶  $W(t) = \det X(t)$  is the Wronskian,
- ▶  $W^{(k)}(t) = \det X^{(k)}(t)$ .

Therefore,

$$c_k(t) = \int \frac{W^{(k)}(t)}{W(t)} dt.$$

# Inhomogeneous Linear Systems

1.10.10. Theorem. The solution of the initial value problem

$$\frac{dx}{dt} = A(t)x + b(t), \quad x(t_0) = x_0, \quad (1.10.7)$$

is given by

$$x_{\text{inhom}}(t) = x_{\text{hom}}(t) + x_{\text{part}}(t)$$

where  $x_{\text{hom}}(t)$  is the solution of the associated homogeneous initial value problem

$$\frac{dx_{\text{hom}}}{dt} = A(t)x_{\text{hom}}, \quad x_{\text{hom}}(t_0) = x_0,$$

and

$$x_{\text{part}}(t) = \sum_{k=1}^n x^{(k)}(t) \int_{t_0}^t \frac{W^{(k)}(s)}{W(s)} ds, \quad (1.10.8)$$

for some fundamental system  $(x^{(1)}, \dots, x^{(n)})$ .

## Inhomogeneous Linear Systems

Recall that we showed in (1.6.3) how an explicit ODE of order  $n$  can be transformed into a system of first-order ODEs, so that our previously developed solution methods can be applied. This works particularly well if we have an ODE with constant coefficients, i.e.,

$$c_n y^{(n)} + \cdots + c_1 y' + c_0 y = b(t)$$

where  $c_0, \dots, c_n \in \mathbb{R}$  are constants. We will illustrate this in the following example, where we also show how the method of variation of parameters can be used to find a particular solution to an inhomogeneous system.

1.10.11. Example. Consider the second-order ODE

$$y'' + y = \cos t. \tag{1.10.9}$$

We are looking for the general solution to this equation.

## Inhomogeneous Linear Systems

Transforming (1.10.9) into a system of equations, we set  $x_1(t) := y(t)$ ,  $x_2(t) := y'(t)$ . Then

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{=:A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{=:x(t)} + \begin{pmatrix} 0 \\ \cos t \end{pmatrix} \quad (1.10.10)$$

The associated homogeneous equation is

$$\dot{x} = Ax$$

and has general solution

$$x_{\text{hom}}(t, c_1, c_2) = c_1 e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 e^{-it} \begin{pmatrix} i \\ 1 \end{pmatrix}. \quad (1.10.11)$$

## Inhomogeneous Linear Systems

Note that we originally defined  $x_1(t) = y(t)$ , where  $y$  solved (1.10.9).

From

$$xa_{\text{hom}}(t, c_1, c_2) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

we see that

$$y_{\text{hom}}(t) = c_1 e^{it} + i c_2 e^{-it}, \quad c_1, c_2 \in \mathbb{C},$$

is the general solution of the homogeneous equation  $y'' + y = 0$ .

## Inhomogeneous Linear Systems

We calculate the Wronskian of the homogeneous solution (1.10.11), setting  $c_1 = c_2 = 1$ :

$$W(t) = \det X(t) = \det \begin{pmatrix} e^{it} & ie^{-it} \\ ie^{it} & e^{-it} \end{pmatrix} = \det \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = 2.$$

Using the method of variation of parameters, a particular solution of the inhomogeneous equation (1.10.10) is given by

$$x_{\text{part}}(t) = c_1(t)e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} + c_2(t)e^{-it} \begin{pmatrix} i \\ 1 \end{pmatrix}$$

with

$$c'_1(t) = \frac{1}{W(t)} \det \begin{pmatrix} 0 & ie^{-it} \\ \cos t & e^{-it} \end{pmatrix} = -\frac{i}{2}e^{-it} \cos t = -\frac{i}{4}(1 + e^{-2it}).$$

# Inhomogeneous Linear Systems

It follows that

$$c_1(t) = -\frac{i}{4} \int (1 + e^{-2it}) dt = -\frac{i}{4}t + \frac{1}{8}e^{-2it}$$

Furthermore,

$$c'_2(t) = \frac{1}{W(t)} \det \begin{pmatrix} e^{it} & 0 \\ ie^{it} & \cos t \end{pmatrix} = \frac{1}{2} e^{it} \cos t = \frac{1}{4} (e^{2it} + 1),$$

so

$$c_2(t) = \frac{1}{4} \int (e^{2it} + 1) dt = \frac{1}{4}t - \frac{i}{8}e^{2it}$$

# Inhomogeneous Linear Systems

Inserting into our ansatz, we obtain

$$\begin{aligned}
 x_{\text{part}}(t) &= \frac{1}{8}(e^{-2it} - 2it)e^{it} \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{1}{8}(2t - ie^{2it})e^{-it} \begin{pmatrix} i \\ 1 \end{pmatrix} \\
 &= \frac{e^{-it} - 2ite^{it}}{8} \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{2te^{-it} - ie^{it}}{8} \begin{pmatrix} i \\ 1 \end{pmatrix} \\
 &= \frac{1}{8} \left( e^{-it} - 2ite^{it} + 2ite^{-it} + e^{it} \right) \\
 &\quad \left( ie^{-it} + 2te^{it} + 2te^{-it} - ie^{it} \right) \\
 &= \frac{1}{4} \begin{pmatrix} \cos t + 2t \sin t \\ \sin t + 2t \cos t \end{pmatrix} = \begin{pmatrix} y_{\text{part}}(t) \\ y'_{\text{part}}(t) \end{pmatrix}.
 \end{aligned}$$

Hence

$$y_{\text{part}}(t) = \frac{\cos t}{4} + \frac{t \sin t}{2}$$

is a particular solution of (1.10.9).

# Inhomogeneous Linear Systems

The general solution of (1.10.9) is given by

$$\begin{aligned}y_{\text{inhom}}(t) &= y_{\text{hom}}(t) + y_{\text{part}}(t) \\&= c_1 \cos t + c_2 \sin t + \frac{t \sin t}{2}.\end{aligned}$$

This long and painful method may be compared with Duhamel's method.

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# Linear Second-Order ODEs

We can apply our solution techniques for systems of linear first order ODEs to the solution of single linear ODEs of degree  $n \geq 2$ . We will focus here on the case  $n = 2$ .

1.11.1. Definition. A **linear differential equation of order 2** is an equation of the form

$$r(t)y'' + p(t)y' + q(t)y = g(t), \quad t \in I, \quad (1.11.1)$$

where  $I \subset \mathbb{R}$  is an interval and  $p, q, r, g \in C(I)$ . The equation

- ▶ (1.11.1) is **homogeneous** if  $g(t) = 0$  for all  $t \in I$ ,  
**inhomogeneous** otherwise.
- ▶ (1.11.1) has **constant coefficients** if  $p, q, r$  are constant functions.

If  $r(t) \neq 0$  for all  $t \in I$ , we can divide (1.11.1) by  $r(t)$  and thereafter assume that  $r \equiv 1$ . We will suppose this is the case for the moment.

Equations where  $r(t_0) = 0$  for some  $t_0 \in I$  are technically more difficult to treat and we postpone their discussion.

# Linear Second-Order ODEs

The equation

$$y'' + p(t)y' + q(t)y = g(t) \quad (1.11.2)$$

is equivalent to the system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix}x + \begin{pmatrix} 0 \\ g(t) \end{pmatrix} \quad (1.11.3)$$

with  $x_1 = y$  and  $x_2 = y'$ . From the Theorem of Picard-Lindelöf 1.6.5 we then immediately obtain the following result:

**1.11.2. Theorem.** Let  $I \subset \mathbb{R}$  be an interval,  $p, q, g \in C(I)$  continuous and  $t_0 \in I$ . Then the initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0 \quad (1.11.4)$$

has a unique solution  $y$  that exists throughout  $I$ .

## Homogeneous Linear Second Order Equations

Let  $y^{(1)}$  and  $y^{(2)}$  be any two solutions of the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \quad (1.11.5)$$

Then we define the **Wronskian of  $y^{(1)}$  and  $y^{(2)}$**  to be the Wronskian of the associated solutions of (1.11.3),

$$W(y^{(1)}, y^{(2)}) := \det \begin{pmatrix} y^{(1)} & y^{(2)} \\ (y^{(1)})' & (y^{(2)})' \end{pmatrix}.$$

The two solutions  $y^{(1)}$  and  $y^{(2)}$  are independent if  $W(y^{(1)}(t), y^{(2)}(t)) \neq 0$  for one  $t$  (then also for any  $t$ ). In that case, the general solution to the (1.11.5) is given by

$$y_{\text{hom}}(t; c_1, c_2) = c_1 y^{(1)}(t) + c_2 y^{(2)}(t).$$

## Particular Solutions of Inhomogeneous Equations

Supposing that two independent solutions  $y^{(1)}$  and  $y^{(2)}$  of (1.11.5) are known, we can find a particular solution of the inhomogeneous equation (1.11.2) by using our techniques from the theory of systems:

A particular solution of the inhomogeneous system (1.11.3) is given by

$$x_{\text{part}}(t) = c_1(t)x^{(1)}(t) + c_2(t)x^{(2)}(t), \quad c_k(t) = \int_{t_0}^t \frac{W^{(k)}(s)}{W(s)} ds,$$

where  $x^{(k)} = \begin{pmatrix} y^{(k)} \\ (y^{(k)})' \end{pmatrix}$  and

$$W^{(1)}(s) = \det \begin{pmatrix} 0 & y^{(2)}(s) \\ g(s) & (y^{(2)})'(s) \end{pmatrix} = -g(s)y^{(2)}(s),$$

$$W^{(2)}(s) = \det \begin{pmatrix} y^{(1)}(s) & 0 \\ (y^{(1)})'(s) & g(s) \end{pmatrix} = g(s)y^{(1)}(s).$$

## Particular Solutions of Inhomogeneous Equations

It follows that a particular solution to the inhomogeneous equation (1.11.2) is given by

$$\begin{aligned}y_{\text{part}}(t) &= -y^{(1)}(t) \int \frac{g(t)y^{(2)}(t)}{W(y^{(1)}(t), y^{(2)}(t))} dt \\&\quad + y^{(2)}(t) \int \frac{g(t)y^{(1)}(t)}{W(y^{(1)}(t), y^{(2)}(t))} dt.\end{aligned}$$

The general solution to (1.11.2) then has the form

$$y_{\text{inhom}}(t; c_1, c_2) = y_{\text{hom}}(t; c_1, c_2) + y_{\text{part}}(t)$$

where in the case of an initial value problem (1.11.4) with initial data at  $t_0$  the homogeneous solution is used to fit the initial conditions and the integrals for  $y_{\text{part}}(t)$  above are taken as  $\int_{t_0}^t$ .

## Reduction of Order

As with systems of ODEs with non-constant coefficients, there is no general solution formula for (1.11.5). However, if one solution is known, we can easily find a second independent solution. (This is a similar situation to that of the Riccati equation.) The corresponding procedure is called **reduction of order**. Suppose we have a non-trivial solution  $y_1$  of the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0.$$

Set

$$y_2(t) = v(t)y_1(t)$$

with some unknown function  $v$ . Inserting into the ODE, we obtain

$$0 = y_1v'' + (2y_1' + py_1)v' + \underbrace{(y_1'' + py_1' + qy_1)}_{=0}v = y_1v'' + (2y_1' + py_1)v'.$$

This is a linear first-order equation for  $v'$ , which we can solve using our known methods. Once  $v$  is determined,  $y_2 = vy_1$  is the second solution.

## Reduction of Order

1.11.3. Example. Given the ODE

$$ty'' - 2y' + (2 - t)y = 0, \quad t > 0, \quad (1.11.6)$$

we guess a solution  $y_1(t) = e^t$ . We now want to construct a second, independent solution. Set  $y(t) = v(t)e^t$ . Inserting into (1.11.6) we obtain

$$v'' + \frac{2(t-1)}{t}v' = 0.$$

This is a separable first-order ODE for  $v'$ . Solving, we find

$$v'(t) = c_1 t^2 e^{-2t},$$

so  $v(t) = -\frac{1}{4}c_1 e^{-2t}(1 + 2t + 2t^2) + c_2$  and hence

$$y_2(t) = -\frac{1}{4}c_1 e^{-t}(1 + 2t + 2t^2) + c_2 e^t, \quad c_1, c_2 \in \mathbb{R}. \quad (1.11.7)$$

Thus, we have the two independent solutions  $y_1(t) = e^t$  and  $y_2(t) = e^{-t}(1 + 2t + 2t^2)$ .

# Linear Second-Order ODEs with Constant Coefficients

We will now consider the constant-coefficient equation

$$ay'' + by' + cy = 0, \quad a, b, c \in \mathbb{R}, \quad a \neq 0, \quad (1.11.8)$$

As a system, (1.11.8) has the form

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Ax. \quad (1.11.9)$$

with  $x_1 = y$  and  $x_2 = y'$ . The eigenvalues of  $A$  are determined by

$$\det A = -\lambda(-b/a - \lambda) + c/a = 0$$

or

$$a\lambda^2 + b\lambda + c = 0. \quad (1.11.10)$$

# Linear Second-Order ODEs with Constant Coefficients

The roots of (1.11.10) are given by

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (1.11.11)$$

We will consider two cases:

1.  $b^2 \neq 4ac$ . Then there are two distinct eigenvalues  $\lambda_1 \neq \lambda_2 \in \mathbb{C}$  of  $A$  and two corresponding eigenvectors  $v_1, v_2 \in \mathbb{R}^2$ . We know that the general solution of (1.11.9) will have the form

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t},$$

where  $c_1, c_2 \in \mathbb{R}$  are arbitrary constants. Since  $x_1 = y$ ,  $x_2 = y'$ , we are actually only interested in  $x_1$  and can write

$$y(t; c_1, c_2) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad c_1, c_2 \in \mathbb{R},$$

for the general solution of (1.11.8).

# Linear Second-Order ODEs with Constant Coefficients

2.  $b^2 = 4ac$ . Then there is only one eigenvalue  $\lambda \in \mathbb{C}$  of

$$A = \begin{pmatrix} 0 & 1 \\ -b^2/(4a^2) & -b/a \end{pmatrix}.$$

It can be shown explicitly that in this case  $A$  is not diagonalizable (see exercises). The Jordan form of  $A$  has the form

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

so

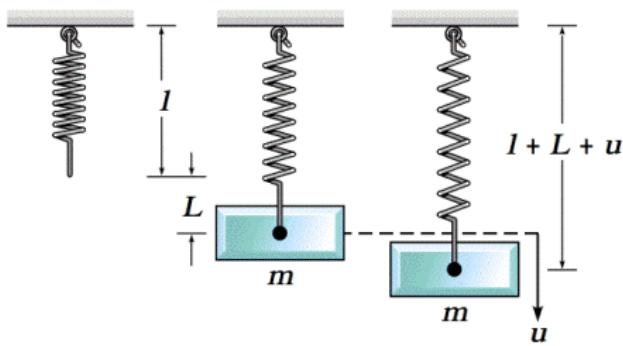
$$e^{Jt} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}.$$

A general solution of (1.11.8) is then given by

$$y(t; c_1, c_2) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}, \quad c_1, c_2 \in \mathbb{R}.$$

# Mechanical and Electrical Vibrations

Consider a spring-mass system as shown below:



Boyce, W.E. and DiPrima, R.C. Elementary Differential Equations and Boundary Value Problems, 7th Ed., Wiley 2001,

The spring (assumed weightless) has equilibrium length  $I$ , the mass  $m$  causes an extension by  $L$ . We will measure  $u = u(t)$ , the displacement of the mass from its equilibrium position  $I + L$ . We will assume that the spring satisfies Hooke's law with a **spring constant  $k$**  and that any oscillations of the mass are **linearly damped** (by air resistance or the mechanical properties of the spring) with **damping constant  $\gamma$** . We will also consider the case of an external, time-dependent force  $F = F(t)$  acting on the mass.

## Mechanical and Electrical Vibrations

By applying Newton's laws we then obtain the differential equation

$$mu'' + \gamma u' + ku = F(t), \quad m, \gamma, k \geq 0, \quad F(t) \in \mathbb{R}. \quad (1.11.12)$$

This type of equation can be used to model vibrations in general, not just the specific spring oscillations from which it was derived. If  $F(t) = 0$  for all  $t$ , we say the vibrations are **free**, otherwise they are **forced**. If  $\gamma \neq 0$  the vibrations are **damped**, otherwise **undamped**.

Clearly, (1.11.12) is an inhomogeneous second-order ODE with constant coefficients, so we can find specific solutions. Mathematically, there is nothing further to say. However, physically, there remains a great deal to analyze, and we will now proceed to look at (1.11.12) from a physical point of view.

## Undamped Free Vibrations

The simplest case is  $F \equiv 0$  and  $\gamma = 0$ , free and undamped vibrations.  
Then (1.11.12) becomes

$$u'' + \frac{k}{m}u = 0.$$

It is easily verified (as in Example 1.10.11 or by direct insertion) that the general solution can be written as

$$\begin{aligned} u(t) &= A \cos(\omega_0 t) + B \sin(\omega_0 t), & A, B \in \mathbb{R}, \\ &= R \cos(\omega_0 t - \delta), & R = \sqrt{A^2 + B^2}, \quad \delta = \arctan(B/A), \end{aligned}$$

where the **circular frequency** ("circular" means units of rad / s)

$$\omega_0 := \sqrt{\frac{k}{m}} \tag{1.11.13}$$

is the **natural frequency** of the undamped free vibrations. The **period** of the motion is  $T = 2\pi/\omega_0$ .

## Damped Free Vibrations

If  $\gamma \neq 0$  the solutions of

$$mu'' + \gamma u' + ku = 0$$

are determined by the eigenvalues,

$$\lambda = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = \frac{\gamma}{2m} \left( -1 \pm \sqrt{1 - \frac{4km}{\gamma^2}} \right)$$

cf. (1.11.11).

## Free Vibrations (Overdamping and Critical damping)

- (i) If  $\gamma^2 - 4km > 0$  there are two distinct real eigenvalues and the solution is given by

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}, \quad c_1, c_2 \in \mathbb{R}.$$

Note that in this case  $1 - 4km/\gamma^2 < 1$  so both eigenvalues are negative. Thus  $u(t) \searrow 0$  as  $t \rightarrow \infty$ . This is known as the **overdamped case**.

- (ii) If  $\gamma^2 - 4km = 0$  there is only the single eigenvalue  $\lambda = -\gamma/(2m)$  and the solution is given by

$$u(t) = (c_1 + c_2 t) e^{-\gamma t/(2m)}, \quad c_1, c_2 \in \mathbb{R}.$$

Here, too,  $u(t) \rightarrow 0$  as  $t \rightarrow \infty$  without any oscillations. This solution corresponds to the fastest non-oscillating decay to zero. This case is referred to as **critical damping**.

## Free Vibrations (Underdamping)

(iii) If  $\gamma^2 - 4km < 0$  both eigenvalues are complex. However, the real part will be negative:

$$\operatorname{Re} \lambda_1 = \operatorname{Re} \lambda_2 = -\frac{\gamma}{2m}.$$

The general solution then has the form

$$\begin{aligned} u(t) &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = e^{-\gamma/(2m)t} (c_1 e^{i\mu t} + c_2 e^{-i\mu t}) \\ &= e^{-\gamma/(2m)t} (A \cos(\mu t) + B \sin(\mu t)) \\ &= R e^{-\gamma/(2m)t} \cos(\mu t - \delta) \end{aligned}$$

for constants  $c_1, c_2, A, B \in \mathbb{R}$ ,  $R, \delta$  as before and

$$\mu = |\operatorname{Im} \lambda| = \frac{\sqrt{4km - \gamma^2}}{2m}.$$

The motion corresponds to oscillations with decreasing amplitude. Since it is not periodic,  $\mu$  is not a true frequency but rather called a **quasifrequency**. Similarly,  $T_d = 2\pi/\mu$  is called the **quasiperiod**.

## Free Vibrations (Small Damping)

The relationship between  $\mu$  and  $\omega_0$  and  $T_d$  and  $T$  is given by

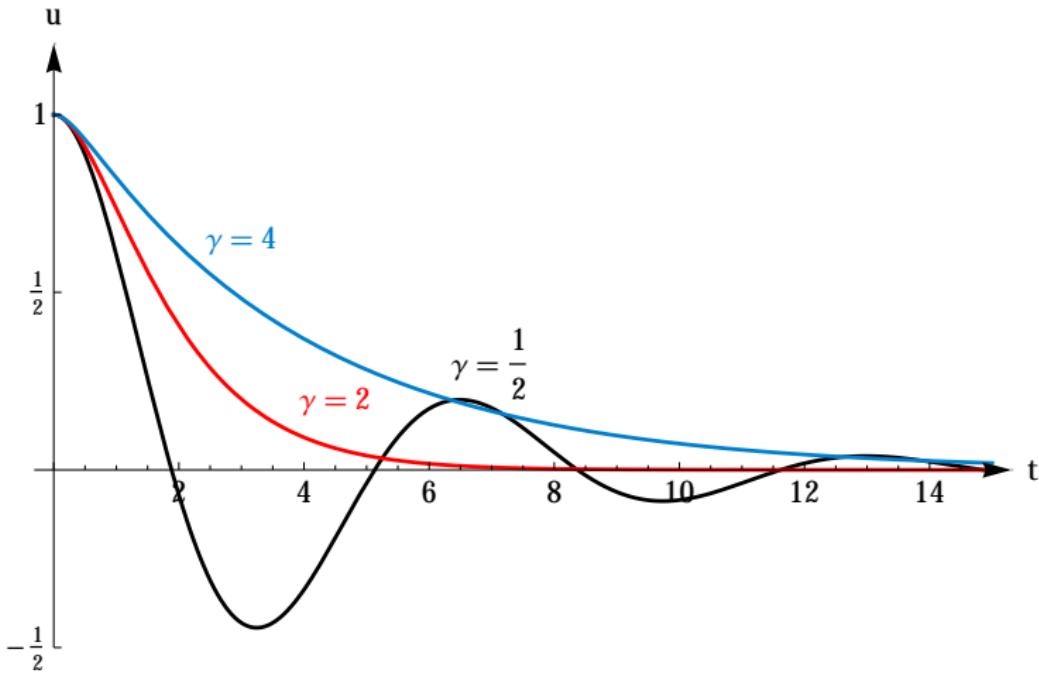
$$\frac{\mu}{\omega_0} = \frac{\sqrt{4km - \gamma^2}/(2m)}{\sqrt{k/m}} = \sqrt{1 - \frac{\gamma^2}{4km}} = 1 - \frac{\gamma^2}{8km} + O(\gamma^4/(km)^2),$$

$$\frac{T_d}{T} = \frac{\omega_0}{\mu} = \frac{1}{\sqrt{1 - \frac{\gamma^2}{4km}}} = 1 + \frac{\gamma^2}{4km} + O(\gamma^4/(km)^2)$$

as  $\gamma/(km) \rightarrow 0$ . For small damping  $\gamma \ll 4km$  the first two terms give a good approximation.

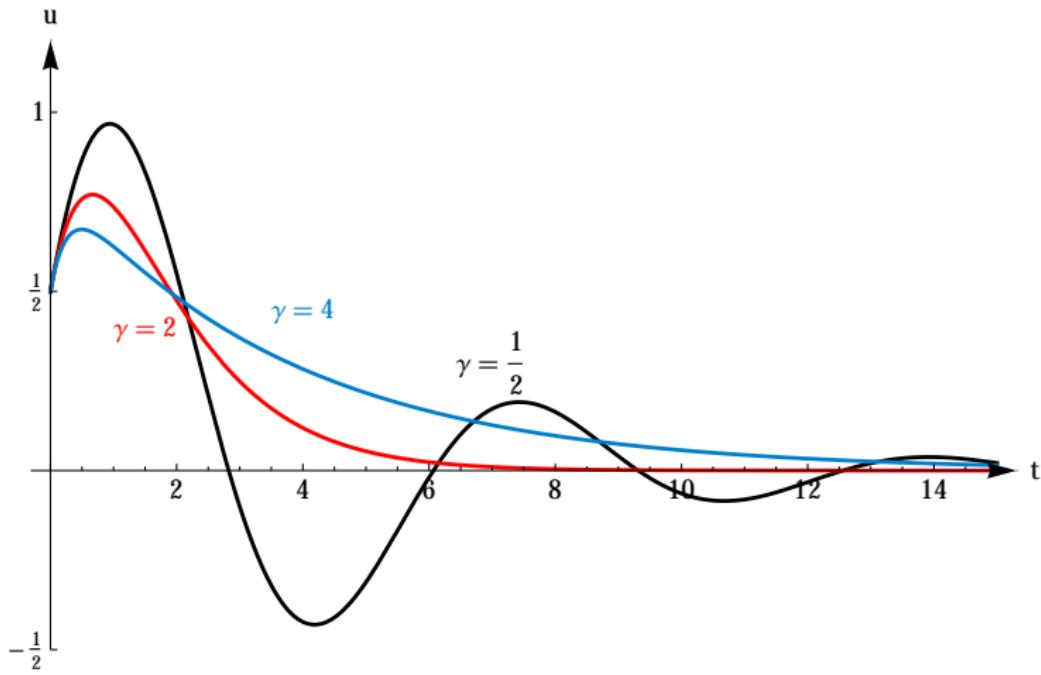
## Damped Free Vibrations

For comparison, we plot solutions of the equation  $u'' + \gamma u + u = 0$ ,  $u(0) = 1$ ,  $u'(0) = 0$ , for different values of  $\gamma$ . Critical damping is achieved for  $\gamma = 2$ .



## Damped Free Vibrations

Considering the problem  $u'' + \gamma u + u = 0$ ,  $u(0) = 1/2$ ,  $u'(0) = 1$ , it becomes even more obvious why critical damping is “ideal” for many applications.



# Undamped Forced Vibrations

In this situation we need to solve

$$mu'' + ku = F(t)$$

with some function  $F$ . The case we are most interested in concerns **periodic forcing**, i.e.,

$$F(t) = F_0 \cos(\omega t), \quad F_0, \omega \in \mathbb{R}.$$

Assuming that  $\omega_0 \neq \omega$  (recall  $\omega_0 = \sqrt{k/m}$ ) we have the general solution

$$u(t; c_1, c_2) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

Now assume we are given initial conditions  $u(0) = u'(0) = 0$ . Then the solution is given by

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos(\omega t) - \cos(\omega_0 t)) \quad (1.11.14)$$

## Beats

Now (1.11.14) can be rewritten as

$$u(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2} \sin \frac{(\omega_0 + \omega)t}{2}. \quad (1.11.15)$$

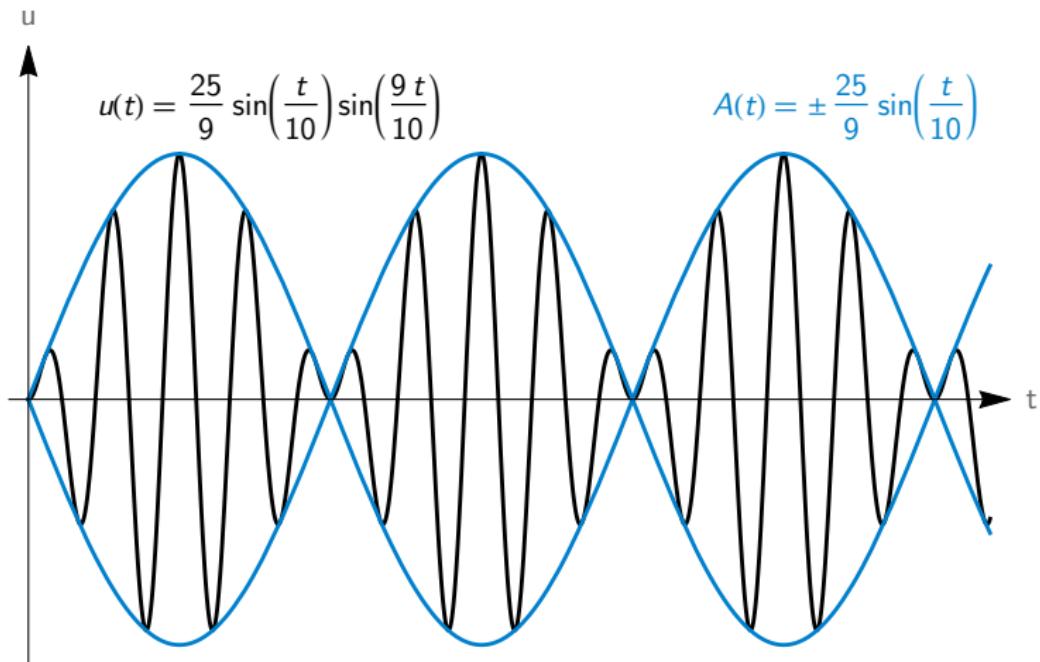
Consider the case where  $|\omega_0 - \omega|$  is small, i.e.,  $\omega_0 \approx \omega$ . Then (1.11.15) is essentially a cosine oscillation with circular frequency  $(\omega_0 + \omega)/2$  and **slowly varying amplitude**

$$A(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2}.$$

This type of oscillation  $u(t) = A(t) \sin \frac{(\omega_0+\omega)t}{2}$  is called a **beat** in mechanics or acoustics. In electronics the imposition of such a slowly varying amplitude is called **amplitude modulation**.

## Beats

A beat: solution of  $u'' + u = \frac{1}{2} \cos(4t/5)$ ,  $u(0) = u'(0) = 0$ .



## Undamped Resonance

We next consider the case  $\omega = \omega_0$  where the frequency of the periodic forcing is equal to the natural frequency of the system. Then the general solution of

$$mu'' + ku = F_0 \cos(\omega_0 t), \quad \omega_0 = \sqrt{\frac{k}{m}}$$

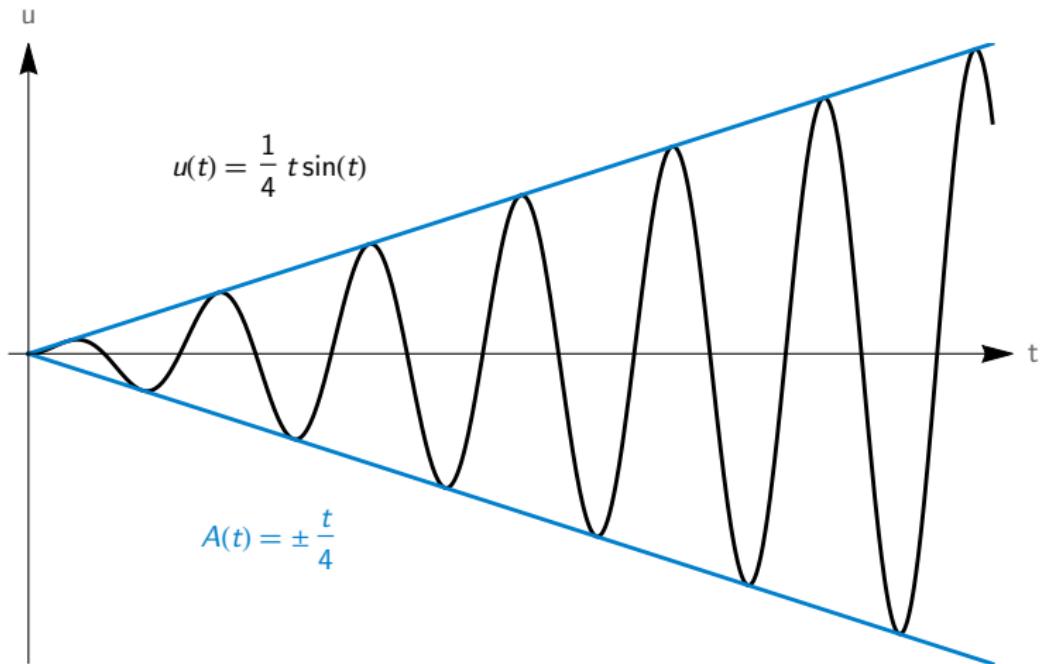
is given by

$$u(t; c_1, c_2) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t).$$

Note that due to the last term, the solution is unbounded. The amplitude of the oscillations increases without bound. (In practice, once the amplitude is large enough, the original equation will no longer be physically valid, and other effects start to dominate.) This phenomenon is known as **resonance**.

## Undamped Resonance

Resonance: solution of  $u'' + u = \frac{1}{2} \cos t$ ,  $u(0) = u'(0) = 0$ .



## Damped Forced Vibrations

We finally consider the full equation

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t).$$

Using our known methods, we can find the general solution to be

$$u(t; c_1, c_2) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + R \cos(\omega t - \delta), \quad \lambda_1 \neq \lambda_2. \quad (1.11.16)$$

if the eigenvalues associated with the homogeneous equation are distinct, and an analogous expression if there is only one eigenvalue. In any case, the real parts of the eigenvalue(s) will be negative, so the exponential functions will approach zero as  $t \rightarrow \infty$ :

$$\lim_{t \rightarrow \infty} (u(t; c_1, c_2) - R \cos(\omega t - \delta)) = 0$$

Thus the exponential functions are called the **transient solution** while  $R \cos(\omega t - \delta)$  is the **steady-state solution** or **forced response**.

## Damped Forced Vibrations

In (1.11.16) the constants  $R$  and  $\delta$  are given by

$$R = \frac{F_0}{\Delta}, \quad \cos \delta = \frac{m(\omega_0^2 - \omega^2)}{\Delta}, \quad \sin \delta = \frac{\gamma \omega}{\Delta}$$

where

$$\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}.$$

It is now interesting to consider the amplitude  $R$  of the forced response as a function of the forcing frequency  $\omega$ ,

$$R(\omega) = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}}.$$

As  $\omega \rightarrow \infty$ ,  $R(\omega) \rightarrow 0$ , so high-frequency forcing gives a small amplitude forced response. If  $\omega$  is very small,  $R(\omega) \approx F_0/k$ . The maximum of  $R$  occurs when  $R'(\omega) = 0$ , which gives

$$\omega_{\max}^2 = \omega_0^2 - \frac{\gamma^2}{2m^2} = \omega_0^2 \left(1 - \frac{\gamma^2}{2km}\right).$$

## Damped Forced Vibrations

Observe that  $\omega_{\max} < \omega_0$  and  $\omega_{\max} \approx \omega_0$  if  $\gamma$  is small. The value of the maximum amplitude is

$$R(\omega_{\max}) = \frac{F_0}{\gamma\omega_0\sqrt{1 - \gamma^2/(4mk)}}.$$

If  $\gamma^2 > 4mk$  this value is imaginary and there is no maximum at  $\omega_{\max}$ . Instead, the maximum occurs at  $\omega = 0$  and  $R$  is a decreasing function of  $\omega$ .

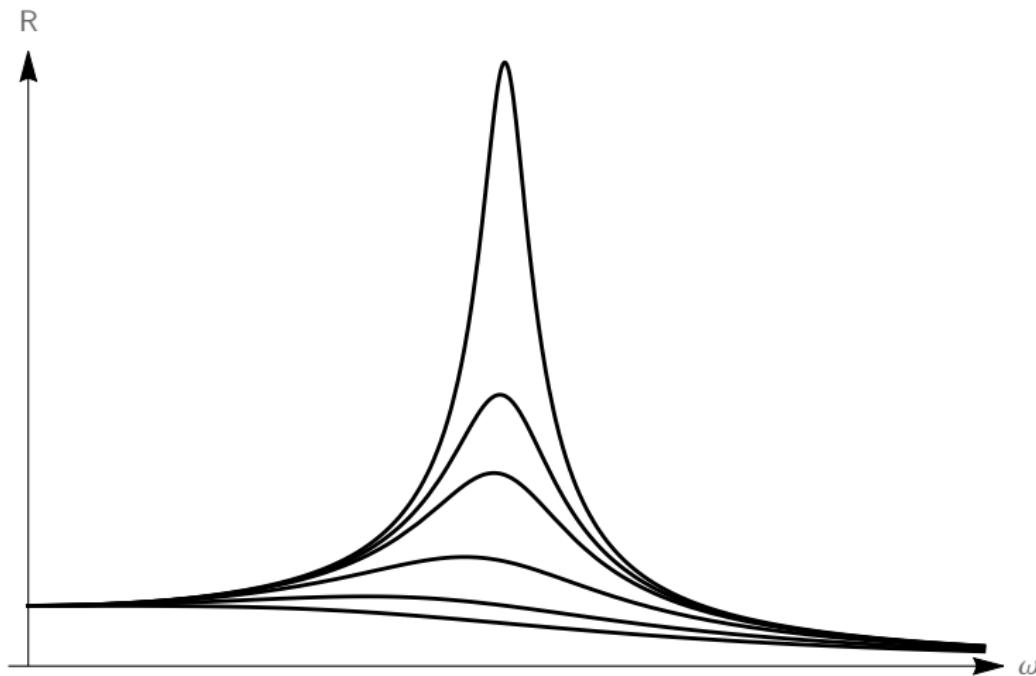
Otherwise, we have

$$R(\omega) = \frac{F_0}{\gamma\omega_0} \left( 1 + \frac{\gamma^2}{8mk} + O(\gamma^4/(mk)^2) \right) = \frac{F_0}{\gamma\omega_0} + O(\gamma), \quad \gamma \rightarrow 0,$$

and we see that small damping leads to potentially very large amplitudes  $R \gg F_0$ .

## Damped Forced Vibrations

Dependence of the response amplitude on the excitation frequency  $\omega$ . Here  $F_0$ ,  $m$  and  $\omega_0$  are fixed, while each curve is associated to a different value of  $\gamma$ .



## Damped Forced Vibrations

The **phase angle**  $\delta$  also has an interesting dependence on  $\omega$ . From

$$\delta(\omega) = \arctan \frac{\gamma\omega}{m(\omega_0^2 - \omega^2)}$$

we can draw the following conclusions:

1. If  $\omega$  is close to zero,  $\delta(\omega) \approx 0$ , so the response  $R \cos(\omega t - \delta)$  is nearly in phase with the excitation  $F_0 \cos(\omega t)$ .
2. If  $\omega \approx \omega_0$ ,  $\delta(\omega)$  approaches  $\pi/2$ , thus there is a phase difference of  $1/4$  period between the excitation and the forced response.
3. As  $\omega$  increases further, the phase difference continues to increase, until it reaches  $\pi$  (we are on the next branch of the arctan function, which has been chosen so that it is continuous at infinity). Thus the response has a half-period phase difference to the excitation.

# The Tacoma Narrows Bridge Collapse

Reference **Braun**, Section 2.6.1.

One of the most famous examples of **forced oscillations** is the collapse of the bridge of the Tacoma river in 1940. While this is often cited as an example of resonance, the true physical situation is more complicated and related to aerodynamical effects (see **Braun**). It is **not** an example of resonance in the form we have just discussed. However, it is an impressive example of the destructive power of forced oscillations in general.

There is a public-domain newsreel of this event, which can be downloaded from the Internet Archive at <http://www.archive.org/details/SF121>

(The film clip below can be viewed with Adobe reader 10 or newer.)

## First Midterm Exam

The preceding material completes the first part of the course material. It encompasses everything that will be the subject of the **First Midterm Exam**.

The exact exam date will be announced on Canvas.

No calculators or other aids will be permitted during the exam. A sample exam with solutions has been uploaded to Canvas. Please study it carefully, including the instructions on the cover page.

## Part II

# Integration Methods for Second-Order Equations

12. Complex Differentiability
13. Properties of Holomorphic Functions
14. Singularities and Poles
15. Residue Calculus
16. The Heaviside Operator Method
17. The Laplace Transform
18. The Fourier Transform

12. Complex Differentiability

13. Properties of Holomorphic Functions

14. Singularities and Poles

15. Residue Calculus

16. The Heaviside Operator Method

17. The Laplace Transform

18. The Fourier Transform

# Why Complex Analysis?

**Reference** We will use the first few pages of Stein and Shakarchi's *Complex Analysis* for this section. The subject of complex analysis is, however, large, fascinating and useful, so you are encouraged to use the book for further investigations.

We know that  $\mathbb{C} \cong \mathbb{R}^2$  and functions  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  for  $n, m \in \mathbb{N} \setminus \{0\}$  extensively studied in multivariable calculus. So why is there an entirely separate branch of analysis focusing on functions  $f: \mathbb{C} \rightarrow \mathbb{C}$ ? In fact, it turns out that the **complex derivative**, which is different from the derivative in  $\mathbb{R}^2$ , has properties that lead to astounding results.

**2.1.1. Definition.** We say that a function  $f: \mathbb{C} \rightarrow \mathbb{C}$  is **complex differentiable**, or **holomorphic**, at  $z \in \mathbb{C}$  if

$$f'(z) := \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{C}}} \frac{f(z + h) - f(z)}{h}, \quad (2.1.1)$$

exists.

# Why Complex Analysis?

Definition 2.1.1 will yield (among other things) that

- ▶ A holomorphic function is **automatically infinitely often differentiable**;
- ▶ A holomorphic function is **automatically analytic** (has a power series expansion);
- ▶ Any closed curve integral of a holomorphic function is vanishes.

Of course, we say that a function is **holomorphic on an open set**  $\Omega \subset \mathbb{C}$  if it is holomorphic at every  $z \in \Omega$ . A function that is holomorphic on  $\mathbb{C}$  is called **entire**.

**2.1.2. Remark.** Note that  $f'(z)$  defined in (2.1.1) is a **number**. If we considered a corresponding map  $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , the derivative (e.g., the Jacobian) would be a **matrix**. This is our first clue that complex differentiability encompasses some entirely new mathematics. We will proceed to investigate this further.

## Basic Properties of Holomorphic Functions

Of course, we can define complex differentiability through the existence of a number  $f'(z) \in \mathbb{C}$  such that

$$f(z + h) = f(z) + h \cdot f'(z) + o(h) \quad \text{as } h \rightarrow 0. \quad (2.1.2)$$

Of course, in the use of the Landau symbols in  $\mathbb{C}$  the complex modulus is used. As in real variable calculus, equations (2.1.1) and (2.1.2) are equivalent.

We have the usual formulas from real analysis: if  $f, g$  are holomorphic, then so are the pointwise sums and products  $f + g$  and  $f \cdot g$ , the multiple  $\lambda f$  for  $\lambda \in \mathbb{C}$ , and we also have the chain rule

$$(f \circ g)'(z) = f'(g(z)) \cdot g'(z).$$

The proof of these statements is also identical to the real case. We can also derive the usual quotient rule from the above.

# Basic Examples of Holomorphic Functions

## 2.1.3. Examples.

1. Any polynomial  $p(z) = a_0 + a_1z + \cdots + a_nz^n$  is holomorphic on  $\mathbb{C}$  (i.e., entire) and has derivative

$$p'(z) = a_1 + 2a_2z + \cdots + na_nz^{n-1}.$$

2. The function  $f(z) = 1/z$  is holomorphic on any set not containing the origin and

$$f'(z) = -\frac{1}{z^2}$$

3. The function  $f(z) = \bar{z}$  is not holomorphic at any  $z$ , since

$$\lim_{h \rightarrow 0} \frac{\overline{z+h} - \bar{z}}{h} = \lim_{h \rightarrow 0} \frac{\bar{h}}{h}$$

does not exist. (Take sequences  $h_n = i/n$  and  $\tilde{h}_n = 1/n$ )

## $\mathbb{C}$ as a Complex vs. a Real Vector Space

Definition 2.1.1 implies that we are treating  $\mathbb{C}$  as a one-dimensional complex vector space, and not as a two-dimensional real space isomorphic to  $\mathbb{R}^2$ . This simple fact clarifies that all our discussions of functions  $\mathbb{R}^n \rightarrow \mathbb{R}^m$  which are based on real vector spaces simply do not apply. A complex vector space of dimension  $n$  is actually different from a real vector space of dimension  $2n$ !

We will illustrate this by comparing the properties of the functions

$$f: \mathbb{C} \rightarrow \mathbb{C}, \quad f(x + iy) = u(x, y) + iv(x, y)$$

and

$$F: \mathbb{R}^2 \rightarrow \mathbb{C}, \quad F(x, y) = u(x, y) + iv(x, y)$$

which can be related to each other through  $\mathbb{C} \ni x + iy \leftrightarrow (x, y) \in \mathbb{R}^2$ . We assume here and in the future that  $u, v: \mathbb{R}^2 \rightarrow \mathbb{R}$ .

## $\mathbb{C}$ as a Complex vs. a Real Vector Space

Let us suppose that the partial derivatives of  $u$  and  $v$  exist (and are continuous). Then the derivative of  $F$  is given by its Jacobian

$$J_F(x, y) = \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right).$$

Assuming that  $f$  is complex differentiable, the derivative of  $f$  at  $z_0 = x_0 + iy_0 \in \mathbb{C}$  can be calculated as follows:

$$\begin{aligned} f'(z_0) &= \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{C}}} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{\substack{h_1 \rightarrow 0 \\ h_1 \in \mathbb{R}}} \frac{f(x_0 + h_1 + iy_0) - f(x_0 + iy_0)}{h_1} \\ &= \lim_{\substack{h_1 \rightarrow 0 \\ h_1 \in \mathbb{R}}} \frac{u(x_0 + h_1, y_0) - u(x_0, y_0)}{h_1} + i \lim_{\substack{h_1 \rightarrow 0 \\ h_1 \in \mathbb{R}}} \frac{v(x_0 + h_1, y_0) - v(x_0, y_0)}{h_1} \\ &= \left. \frac{\partial u}{\partial x} \right|_{(x_0, y_0)} + i \left. \frac{\partial v}{\partial x} \right|_{(x_0, y_0)} \end{aligned} \tag{2.1.3}$$

## $\mathbb{C}$ as a Complex vs. a Real Vector Space

But if  $f$  is complex differentiable, the limit can also be calculated in another way:

$$\begin{aligned} f'(z_0) &= \lim_{\substack{h \rightarrow 0 \\ h \in \mathbb{C}}} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{\substack{h_2 \rightarrow 0 \\ h_2 \in \mathbb{R}}} \frac{f(x_0 + i(y_0 + h_2)) - f(x_0 + iy_0)}{ih_2} \\ &= -i \left. \frac{\partial u}{\partial y} \right|_{(x_0, y_0)} + \left. \frac{\partial v}{\partial y} \right|_{(x_0, y_0)} \end{aligned} \quad (2.1.4)$$

If  $f$  is holomorphic, the limits (2.1.3) and (2.1.4) must coincide. This is only possible if the component functions  $u, v: \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfy the

**Cauchy-Riemann differential equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

## The Cauchy-Riemann Equations

We have seen that the assumption of complex differentiability of  $f: \mathbb{C} \rightarrow \mathbb{C}$  is much stronger than the assumption of simple differentiability of  $F: \mathbb{R}^2 \rightarrow \mathbb{C}$ .

For simplicity, we will often write  $u_x$  instead of  $\frac{\partial u}{\partial x}$ . The Cauchy-Riemann equations then become

$$u_x = v_y, \quad u_y = -v_x.$$

We define

$$\frac{\partial}{\partial z} := \frac{1}{2} \left( \frac{\partial}{\partial x} + \frac{1}{i} \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial x} - \frac{1}{i} \frac{\partial}{\partial y} \right)$$

where the differential operators are understood to act on  $f(x + iy) = u(x, y) + iv(x, y)$ .

## Real and Complex Differentiability

2.1.4. Proposition. If  $f = u + iv: \mathbb{C} \rightarrow \mathbb{C}$  is holomorphic at  $z \in \mathbb{C}$ , then

$$f'(z) = \frac{\partial f}{\partial z} = 2 \frac{\partial u}{\partial z} \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}} = 0.$$

Furthermore, suppose that the partial derivatives of  $u$  and  $v$  exist, are continuous and satisfy the Cauchy-Riemann equations. Then  $f$  is holomorphic.

### Proof.

If  $f$  is holomorphic, the partial derivatives of  $u$  and  $v$  exist and satisfy the Cauchy-Riemann equations. We then have

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{\partial u}{\partial \bar{z}} + i \frac{\partial v}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial u}{\partial x} - \frac{1}{i} \frac{\partial u}{\partial y} \right) + i \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{1}{i} \frac{\partial v}{\partial y} \right) \\ &= \frac{1}{2} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) + i \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) = 0\end{aligned}$$

# Real and Complex Differentiability

Proof (continued).

We have seen in (2.1.3) and (2.1.4) that for holomorphic  $f$ ,

$$f'(z) = \frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}$$

which yields

$$f'(z) = \frac{1}{2} \left( \frac{\partial f}{\partial x} + \frac{1}{i} \frac{\partial f}{\partial y} \right) = \frac{\partial f}{\partial z}.$$

We similarly find  $f' = 2 \frac{\partial u}{\partial z}$ .

# Real and Complex Differentiability

Proof (continued).

Now suppose that the partial derivatives of  $u$  and  $v$  exist and are continuous. The  $u$  and  $v$  are differentiable and we have, with  $z = x + iy$  and  $h = h_1 + ih_2$ ,

$$\begin{aligned}f(z+h) &= u(x+h_1, y+h_2) + iv(x+h_1, y+h_2) \\&= u(x, y) + iv(x, y) + u_x h_1 + u_y h_2 + i(v_x h_1 + v_y h_2) + o(h) \\&= u(x, y) + iv(x, y) + u_x h_1 - v_x h_2 + i(v_x h_1 + u_x h_2) + o(h) \\&= f(z) + (u_x + iv_x)(h_1 + ih_2) + o(h) \\&= f(z) + 2 \frac{\partial u}{\partial z} h + o(h).\end{aligned}$$

This establishes that  $f$  is holomorphic. The first part of the theorem shows that the derivative is actually  $f'(z)$ . □

## Sets in the Complex Plane

All our previous characterizations of open, closed and compact sets remain valid. We make/recall the following definitions:

1. A set  $\Omega \subset \mathbb{C}$  is called **open** if for every  $z \in \Omega$  there exists an  $\varepsilon > 0$  such that  $B_\varepsilon(z) = \{w \in \mathbb{C} : |w - z| < \varepsilon\} \subset \Omega$ . A set is called **closed** if its complement is open.
2. A set  $\Omega \subset \mathbb{C}$  is called **bounded** if  $\Omega \subset B_R(0)$  for some  $R > 0$ .
3. A set  $K \subset \mathbb{C}$  is called **compact** if every sequence in  $K$  has a subsequence that converges in  $K$ . A set  $K \subset \mathbb{C}$  is compact if and only if it is closed and bounded.
4. An open (closed) set  $\Omega \subset \mathbb{C}$  is called **disconnected** if there exist two open (closed) sets  $\Omega_1, \Omega_2 \subset \mathbb{C}$  such that  $\Omega_1 \cap \Omega_2 = \emptyset$  and  $\Omega = \Omega_1 \cup \Omega_2$ . If  $\Omega$  is not disconnected,  $\Omega$  is called **connected**.

A set  $\Omega \subset \mathbb{C}$  is connected if and only if for any two points in  $\Omega$  there exists a curve joining them.

5. A **region** or **domain** in  $\mathbb{C}$  is an open and connected set.

## Sets in the Complex Plane

We define the **diameter** of a set  $\Omega \subset \mathbb{C}$  by

$$\text{diam}(\Omega) := \sup_{z,w \in \Omega} |z - w|.$$

**2.1.5. Proposition.** If  $(\Omega_n)$  is a sequence of non-empty compact sets such that  $\Omega_{n+1} \subset \Omega_n$  for  $n \in \mathbb{N}$  and  $\text{diam } \Omega_n \rightarrow 0$  as  $n \rightarrow \infty$ , then there exists a unique point  $w \in \mathbb{C}$  such that  $w \in \Omega_n$  for all  $n$ .

**Proof.**

Choose a point  $z_n$  in each  $\Omega_n$ . The condition  $\text{diam } \Omega_n \rightarrow 0$  says precisely that  $(z_n)$  is a Cauchy sequence, therefore this sequence converges to a limit that we call  $w$ . Since each set  $\Omega_n$  is compact we must have  $w \in \Omega_n$  for all  $n$ . Finally,  $w$  is the unique point satisfying this property, for otherwise, if  $w'$  satisfied the same property with  $w' \neq w$  we would have  $|w - w'| > 0$  and the condition  $\text{diam } \Omega_n \rightarrow 0$  would be violated. □

## Power series

The main results on real power series remain valid for series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n. \quad (2.1.5)$$

In particular, the radius of convergence  $\varrho$  may be calculated as usual, and the series will converge for all  $z \in \mathbb{C}$  with  $|z| < \varrho$  and diverge if  $|z| > \varrho$ . Points with  $|z| = \varrho$  must be analyzed individually.

## Power series

2.1.6. **Theorem.** The power series (2.1.5) defines a holomorphic function in its disc of convergence. The (complex) derivative of  $f$  is also a power series obtained by differentiating term by term the series for  $f$ , that is,

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

Moreover,  $f'$  has the same radius of convergence as  $f$ .

For the proof, we refer to the textbook.

2.1.7. **Corollary.** A power series is infinitely complex differentiable in its disc of convergence, and the higher derivatives are also power series obtained by termwise differentiation.

## Analytic Functions

2.1.8. Definition. A function  $f$  defined on an open set  $\Omega \subset \mathbb{C}$  is said to be **analytic** (or have a power series expansion) at a point  $z_0 \in \Omega$  if there exists a power series  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  centered at  $z_0$ , with positive radius of convergence, such that

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

for all  $z$  in a neighborhood of  $z_0$ . If  $f$  has a power series expansion at every point in  $\Omega$ , we say that  $f$  is **analytic on  $\Omega$** .

2.1.9. Remark. The exponential, sine and cosine functions are (by our definition) analytic at 0 and have an infinite radius of convergence. They are automatically defined for all complex numbers.

12. Complex Differentiability

13. Properties of Holomorphic Functions

14. Singularities and Poles

15. Residue Calculus

16. The Heaviside Operator Method

17. The Laplace Transform

18. The Fourier Transform

## Curves in the Complex Plane

A **parametrized curve** is a set  $\mathcal{C} \subset \mathbb{C}$  such that there exists a **parametrization**

$$\gamma: I \rightarrow \mathcal{C}$$

for some interval  $I \rightarrow \mathbb{C}$ , where  $\gamma$  is locally injective. We will say that  $\mathcal{C}$  is **smooth** if there exists a parametrization  $\gamma$  that is differentiable with  $\gamma'(t) \neq 0$  for all  $t \in I$ .

We borrow from  $\mathbb{R}^2$  the concept of **positively** and **negatively oriented** curves, i.e., those parametrized in a counter-clockwise and clockwise fashion, respectively.

## Integrals along Complex Curves

We will now define the integral of a holomorphic function along a smooth curve.

**2.2.1. Definition.** Let  $\Omega \subset \mathbb{C}$  be an open set,  $f$  holomorphic on  $\Omega$  and  $\mathcal{C}^* \subset \Omega$  an oriented smooth curve. We then define the integral of  $f$  along  $\mathcal{C}^*$  by

$$\int_{\mathcal{C}^*} f(z) dz := \int_I f(\gamma(t)) \cdot \gamma'(t) dt. \quad (2.2.1)$$

**2.2.2. Remark.** Compare (2.2.1) with the definition of integrals of scalar functions and vector fields in, e.g.,  $\mathbb{R}^2$ :

$$\int_{\mathcal{C}^*} f dr = \int_I f(\gamma(t)) \cdot |\gamma'(t)| dt, \quad \int_{\mathcal{C}^*} \langle F, d\vec{r} \rangle = \int_I \langle F(\gamma(t)), \gamma'(t) \rangle dt.$$

Observe that (2.2.1) shares features with both of these formulas. We will see that this leads to some interesting combinations of properties of (2.2.1).

# Integrals along Complex Curves

In particular, we define the **curve length**

$$\ell(\mathcal{C}) := \left| \int_{\mathcal{C}} dz \right|,$$

we note that

$$\int_{-\mathcal{C}^*} f(z) dz = - \int_{\mathcal{C}^*} f(z) dz$$

where  $-\mathcal{C}^*$  has the opposite orientation of  $\mathcal{C}^*$  and finally

$$\left| \int_{\mathcal{C}^*} f(z) dz \right| \leq \ell(\mathcal{C}) \cdot \sup_{z \in \mathcal{C}} |f(z)|.$$

## Integrals along Complex Curves

2.2.3. Example. Denote the positively oriented unit circle in  $\mathbb{C}$  by  $S^1 = \{z \in \mathbb{C}: |z| = 1\}$ . A parametrization is given, for example, by

$$\gamma: [0, 2\pi) \rightarrow S^1, \quad \gamma(t) = \cos t + i \sin t = e^{it}.$$

We calculate the integral

$$\oint_{S^1} z^2 dz = \int_0^{2\pi} (e^{it})^2 \cdot ie^{it} dt = i \int_0^{2\pi} e^{3it} dt = 0.$$

2.2.4. Definition. Let  $\Omega \subset \mathbb{C}$  be an open set,  $f: \Omega \rightarrow \mathbb{C}$ . A **primitive** for  $f$  is a holomorphic function  $F: \Omega \rightarrow \mathbb{C}$  such that  $f(z) = F'(z)$  for all  $z \in \Omega$ .

## Curve Integrals for Functions with Primitives

2.2.5. **Theorem.** If a continuous function  $f$  has a primitive  $F$  in  $\Omega$ , and  $\mathcal{C}^*$  is a curve in  $\Omega$  that begins at  $w_1$  and ends at  $w_2$ , then

$$\int_{\mathcal{C}^*} f(z) dz = F(w_2) - F(w_1).$$

**Proof.**

Let  $\gamma: [a, b] \rightarrow \mathbb{C}$  be a parametrization of  $\mathcal{C}^*$ . Then

$$\begin{aligned}\int_{\mathcal{C}^*} f(z) dz &= \int_a^b f(\gamma(t)) \gamma'(t) dt = \int_a^b F'(\gamma(t)) \gamma'(t) dt \\ &= \int_a^b \frac{d}{dt}(F(\gamma(t))) dt = F(\gamma(b)) - F(\gamma(a)) \\ &= F(w_2) - F(w_1).\end{aligned}$$



## Curve Integrals for Functions with Primitives

2.2.6. Corollary. If  $\mathcal{C}$  is a closed curve in an open set  $\Omega$ , and  $f$  is continuous and has a primitive in  $\Omega$ , then

$$\oint_{\mathcal{C}} f(z) dz = 0.$$

2.2.7. Corollary. If  $f$  is holomorphic in a region  $\Omega$  and  $f' = 0$ , then  $f$  is constant.

Proof.

Choose some  $w_0 \in \Omega$ . We will show that  $f(w) = f(w_0)$  for all  $w \in \Omega$ . Let  $w \in \Omega$ . Since  $\Omega$  is connected, we can find some curve  $\mathcal{C}$  joining  $w$  to  $w_0$ . Furthermore,  $f$  is a primitive for  $f'$ , so

$$f(w) - f(w_0) = \int_{\mathcal{C}} f'(z) dz = 0.$$

□

## Curve Integrals for Functions with Primitives

2.2.8. Example. The function  $f(z) = 1/z$  does not have a primitive in the open set  $\mathbb{C} \setminus \{0\}$ , since if  $S^1$  is the unit circle parametrized by  $\gamma(t) = e^{it}$ ,  $0 \leq t \leq 2\pi$ , we have

$$\oint_{S^1} \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt = 2\pi i \neq 0.$$

Note that the functions  $z^{-n}$ ,  $n = 2, 3, \dots$  do have primitives,  
 $(-\frac{1}{n-1}z^{-n+1})' = z^{-n}$ , and in fact

$$\begin{aligned}\oint_{S^1} \frac{dz}{z^n} &= \int_0^{2\pi} \frac{ie^{it}}{e^{nit}} dt = i \int_0^{2\pi} e^{(1-n)it} dt \\ &= i \int_0^{2\pi} (\cos((n-1)t) - i \sin((n-1)t)) dt = 0.\end{aligned}$$

Thus among all functions  $f(z) = z^n$ ,  $n \in \mathbb{Z}$ , the function  $f(z) = 1/z$  has a special role. We will see evidence of this later in the **residue calculus**.

## Goursat's Theorem

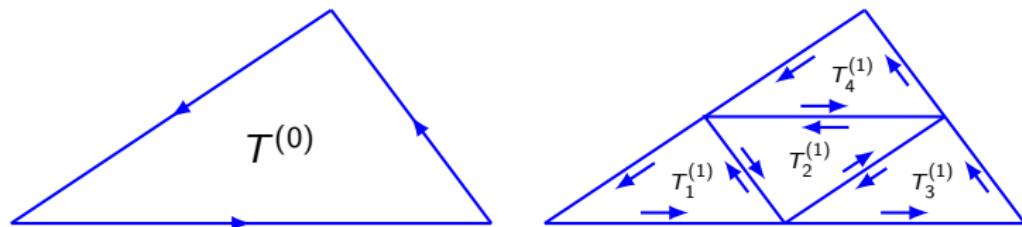
Corollary 2.2.6 says that the integral over a closed curve of  $f$  vanishes if  $f$  is continuous and has a primitive in an open set containing the curve. This is not too surprising and reminiscent of the corresponding property for potential fields.

One of the interesting features of complex function theory is that “having a primitive” is very closely related to “being differentiable.” Thus a similar statement to that of Corollary 2.2.6 is true even if  $f$  is just holomorphic. We will start with Goursat’s theorem, which considers triangles.

**2.2.9. Goursat’s Theorem.** Let  $\Omega \subset \mathbb{C}$  be open and  $f$  holomorphic on  $\Omega$ . Let  $T \subset \Omega$  be a triangle whose interior is also contained in  $\Omega$ . Then

$$\oint_T f(z) dz = 0.$$

# Goursat's Theorem



## Proof.

Denote the initial triangle by  $T = T^{(0)}$  and the region bounded by  $T$  by  $\mathcal{T}^{(0)}$ . After bisecting the sides and joining their midpoints as shown above, we obtain four smaller triangles  $T_i^{(1)}$ ,  $i = 1, 2, 3, 4$ . After canceling anti-parallel integrations along the same lines, we see that

$$\oint_{T^{(0)}} f(z) dz = \sum_{i=1}^4 \oint_{T_i^{(1)}} f(z) dz.$$

# Goursat's Theorem

Proof (continued).

For one of the smaller triangles,

$$\left| \oint_{T^{(0)}} f(z) dz \right| \leq 4 \left| \oint_{T_i^{(1)}} f(z) dz \right|.$$

We call this triangle  $T^{(1)}$  and denote by  $\mathcal{T}^{(1)}$  the region bounded by  $T^{(1)}$ .

Note that

$$\text{diam } \mathcal{T}^{(1)} = \frac{1}{2} \text{diam } \mathcal{T}^{(0)}, \quad \ell(T^{(1)}) = \frac{1}{2} \ell(T^{(0)}).$$

Continuing this process, we obtain a sequence of nested triangles  $T^{(n)}$  with

$$\text{diam } \mathcal{T}^{(n)} = \frac{1}{2^n} \text{diam } \mathcal{T}^{(0)}, \quad \ell(T^{(n)}) = \frac{1}{2^n} \ell(T^{(0)}).$$

# Goursat's Theorem

Proof (continued).

Here  $\mathcal{T}^{(n)}$ ,  $n \in \mathbb{N}$ , is the closed solid triangle with boundary  $\partial\mathcal{T}^{(n)} = T^{(n)}$ . Then for all  $n$ ,  $\mathcal{T}^{(n)}$  is compact and  $\mathcal{T}^{(n+1)} \subset \mathcal{T}^{(n)}$ . Furthermore,

$$\operatorname{diam} \mathcal{T}^{(n)} = \frac{1}{2^n} \operatorname{diam} \mathcal{T}^{(0)} \xrightarrow{n \rightarrow \infty} 0.$$

Therefore, by Proposition 2.1.5, there exists a unique point  $z_0$  that is contained in all  $\mathcal{T}^{(n)}$ .

Since  $f$  is holomorphic, we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0),$$

where  $\psi(z) \rightarrow 0$  as  $z \rightarrow z_0$ .

# Goursat's Theorem

Proof (continued).

Note that there exists a primitive for  $f(z_0) + f'(z_0)(z - z_0)$ , so by Corollary 2.2.6 we have

$$\begin{aligned} \left| \oint_{T^{(0)}} f(z) dz \right| &= \left| 0 + \oint_{T^{(0)}} \psi(z)(z - z_0) dz \right| \\ &\leq 4^n \left| \oint_{T^{(n)}} \psi(z)(z - z_0) dz \right| \\ &\leq 4^n \ell(T^{(n)}) \sup_{z \in T^{(n)}} |\psi(z)| \sup_{z \in T^{(n)}} |z - z_0| \\ &\leq 4^n \ell(T^{(n)}) \operatorname{diam} \mathcal{T}^{(n)} \sup_{z \in T^{(n)}} |\psi(z)| \\ &\leq \ell(T^{(0)}) \operatorname{diam} \mathcal{T}^{(0)} \sup_{z \in T^{(n)}} |\psi(z)| \end{aligned}$$

# Goursat's Theorem

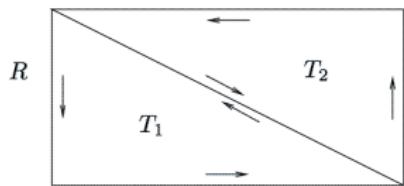
Proof (continued).

Noting that  $\sup_{z \in T^{(n)}} |\psi(z)| \rightarrow 0$  as  $n \rightarrow \infty$ , and that the left-hand side does

not depend on  $n$ , we can take the limit and obtain  $\oint_{T^{(0)}} f(z) dz = 0$ . □

**2.2.10. Corollary.** If  $f$  is holomorphic in an open set  $\Omega$  that contains a rectangle  $R$  and its interior, then

$$\oint_R f(z) dz = 0$$



## Local Existence of Primitives

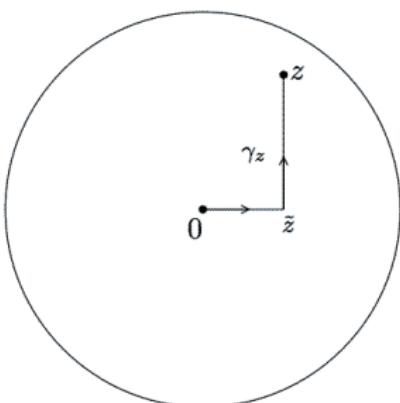
2.2.11. Theorem. A holomorphic function in an open disc has a primitive in that disc.

### Proof.

Given a function  $f$  defined on a disk  $B_R(z_0)$ , we are going to construct a primitive explicitly. The main idea will be the same as for the proof that a conservative vector field on a connected open set in  $\mathbb{R}^n$  has a potential function.

First, we can assume that  $z_0 = 0$ , i.e., the disk is centered at the origin. (Otherwise we first translate the disk there, find the primitive, then translate back.) For any  $z \in B_R(0)$  we define a curve by

$$\gamma_z(t) = \begin{cases} t \cdot \operatorname{Re} z & 0 \leq t \leq 1 \\ \operatorname{Re} z + (t - 1) \operatorname{Im} z & 1 < t \leq 2 \end{cases}$$



# Local Existence of Primitives

Proof (continued).

We then define

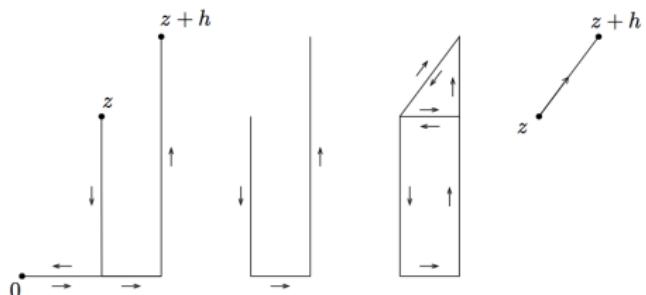
$$F(z) := \int_{\gamma_z} f(w) dw.$$

Our goal is now to show that  $F'(z) = f(z)$ .

We can show that

$$F(z+h) - F(z) = \int_{\eta} f(w) dw,$$

where  $\eta(t) = z + th$ ,  $0 \leq t \leq 1$ :



Now by the continuity of  $f$ ,  $f(w) = f(z) + \psi(w)$  where  $\psi(w) \rightarrow 0$  as  $w \rightarrow z$ . Thus

$$F(z+h) - F(z) = \int_{\eta} (f(z) + \psi(w)) dw = f(z)h + \int_{\eta} \psi(w) dw.$$

## Cauchy's Theorem for a Disc

Proof (continued).

It just remains to verify  $\int_{\eta} \psi(w) dw = o(h)$  as  $h \rightarrow 0$ . Now

$$\left| \int_{\eta} \psi(w) dw \right| \leq |z + h - z| \cdot \sup_{t \in [0,1]} |\psi(z + th)|.$$

Since  $\sup_{t \in [0,1]} |\psi(z + th)| \rightarrow 0$  as  $h \rightarrow 0$ , we are finished. □

Combining Theorem 2.2.11 and Corollary 2.2.6, we obtain Cauchy's Theorem for a disc:

**2.2.12. Cauchy's Theorem.** If  $f$  is holomorphic in a disc, then

$$\oint_{\mathcal{C}} f(z) dz = 0$$

for any closed curve  $\mathcal{C}$  in that disc.

## Cauchy's Theorem

2.2.13. Corollary. Suppose  $f$  is holomorphic in an open set  $\Omega \subset \mathbb{C}$  containing a circle  $C$  and its interior. Then

$$\oint_C f(z) dz = 0$$

Proof.

Since the boundary  $\partial\Omega$  of  $\Omega$  is a closed set and  $C$  is compact, there exists a minimum distance

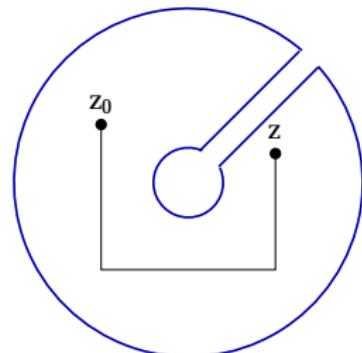
$$\varepsilon := \min_{\substack{z \in \partial\Omega \\ w \in C}} |z - w|$$

between  $C$  and  $\partial\Omega$ . (This was proven in the assignments of Vv285 last term.) If the circle  $C$  has radius  $r > 0$ , we see that the disk  $D$  of radius  $r + \varepsilon/2$  is contained in  $\Omega$  and contains  $C$  and its interior. The result then holds by Cauchy's Theorem for the disc  $D$ . □

## Cauchy's Theorem for General Contours

The essence of the proof of Cauchy's theorem for a disc was the construction of a primitive by joining every point in the disc to the center  $z_0$  of the disc. It is clear from the proof that instead of the center,  $z_0$  could have been any arbitrary, but fixed point.

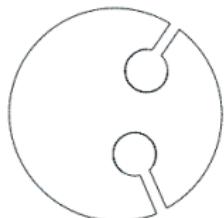
Furthermore, the same procedure can be used to prove Cauchy's theorem for other contours, as long as any point  $z$  in their interior can be joined to some selected  $z_0$  through straight lines (using some previously defined procedure). As an example, consider the so-called **disc with a keyhole**, shown at right.



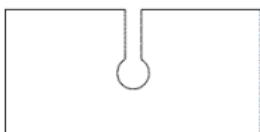
Clearly, Corollary 2.2.13 can be extended to such a contour. The main difficulty in formulating a general theorem lies in the existence of "pathological" contours which may not have a well-defined interior.

# Toy Contours

We will apply Cauchy's theorem to various contours whose geometry is uncomplicated but that are well-suited to the study of particular problems. Such curves are called ***toy contours***. Some examples are pictured below:



The multiple keyhole



Rectangular keyhole



Semicircle



Indented semicircle



Sector



Parallelogram

# Evaluation of Real Integrals using Cauchy's Theorem

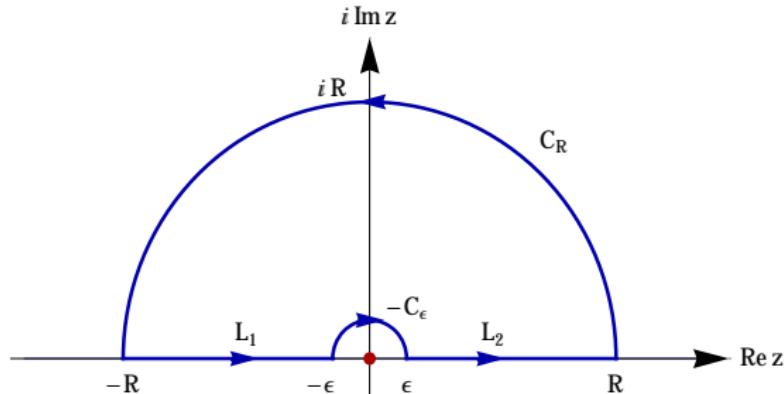
2.2.14. Example. We will show that

$$\int_0^\infty \frac{1 - \cos x}{x^2} dx = \frac{\pi}{2}.$$

We integrate the function

$$f: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}, \quad f(z) = \frac{1 - e^{iz}}{z^2}$$

along the indented semi-circle:



## Evaluation of Real Integrals using Cauchy's Theorem

Here we take  $\varepsilon > 0$  and  $R > \varepsilon$ . Since  $f$  is holomorphic in an open set containing the contour, Cauchy's theorem gives

$$\int_{L_1} f(z) dz + \int_{-C_\varepsilon} f(z) dz + \int_{L_2} f(z) dz + \int_{C_R} f(z) dz = 0.$$

for any such  $\varepsilon, R$ . We parametrize these curves as follows:

$$L_1: \gamma_1(t) = t, \quad -R \leq t \leq -\varepsilon,$$

$$L_2: \gamma_2(t) = t, \quad \varepsilon \leq t \leq R,$$

$$-C_\varepsilon: \gamma_3(t) = \varepsilon e^{-it}, \quad 0 \leq t \leq \pi,$$

$$C_R: \gamma_4(t) = Re^{it}, \quad 0 \leq t \leq \pi.$$

We then have

$$\int_{-R}^{-\varepsilon} f(x) dx + \int_{-C_\varepsilon} f(z) dz + \int_{\varepsilon}^R f(x) dx + \int_{C_R} f(z) dz = 0. \quad (2.2.2)$$

# Evaluation of Real Integrals using Cauchy's Theorem

The equation (2.2.2) holds for any  $R > \varepsilon > 0$ . Our aim is to let  $R \rightarrow \infty$  and  $\varepsilon \rightarrow 0$  in order to obtain an expression for  $\int_{\mathbb{R}} f(x) dx$ .

Consider first the integral

$$\int_{C_R} f(z) dz = \int_0^\pi f(Re^{it}) iRe^{it} dt.$$

We see that

$$\begin{aligned} \left| \int_{C_R} f(z) dz \right| &\leq R \int_0^\pi |f(Re^{it})| dt = R \int_0^\pi \frac{|1 - e^{iR(\cos t + i \sin t)}|}{|Re^{it}|^2} dt \\ &\leq \frac{1}{R} \int_0^\pi (1 + e^{-R \sin t}) dt \\ &\leq \frac{2\pi}{R} \end{aligned}$$

so that

$$\left| \int_{C_R} f(z) dz \right| \xrightarrow{R \rightarrow \infty} 0.$$

# Evaluation of Real Integrals using Cauchy's Theorem

Letting  $R \rightarrow \infty$ , we hence obtain

$$\int_{-\infty}^{-\varepsilon} f(x) dx + \int_{C_\varepsilon} f(z) dz + \int_{\varepsilon}^{\infty} f(x) dx + 0 = 0.$$

Now

$$f(z) = \frac{1 - e^{iz}}{z^2} = \frac{-iz - \sum_{j=2}^{\infty} (iz)^j / j!}{z^2} = \frac{-i}{z} + \underbrace{\sum_{j=0}^{\infty} \frac{(iz)^j}{(j+2)!}}_{=: E(z)}.$$

Then

$$\int_{C_\varepsilon} f(z) dz = \int_{\pi}^0 f(\varepsilon e^{it}) i\varepsilon e^{it} dt = \underbrace{\int_{\pi}^0 \frac{-i}{\varepsilon e^{it}} i\varepsilon e^{it} dt}_{=-\pi} + i\varepsilon \int_{-\pi}^0 E(\varepsilon e^{it}) e^{it} dt.$$

# Evaluation of Real Integrals using Cauchy's Theorem

Note that the series  $E(z)$  converges absolutely for all  $z \in \mathbb{C}$  and

$$\begin{aligned}|E(\varepsilon e^{it})| &= \left| \sum_{j=0}^{\infty} \frac{(i\varepsilon e^{it})^j}{(j+2)!} \right| \leq \sum_{j=0}^{\infty} \frac{|(i\varepsilon e^{it})^j|}{(j+2)!} \\&= \frac{1}{2} + \sum_{j=1}^{\infty} \frac{|\varepsilon|^j}{(j+2)!} \leq 1\end{aligned}$$

for  $\varepsilon$  small enough. It follows that

$$\varepsilon \left| \int_{-\pi}^0 E(\varepsilon e^{it}) e^{it} dt \right| \leq \varepsilon \pi \sup_{t \in [-\pi, 0]} |E(\varepsilon e^{it}) e^{it}| \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Hence,

$$\int_{C_\varepsilon} f(z) dz \xrightarrow{\varepsilon \rightarrow 0} -\pi$$

# Evaluation of Real Integrals using Cauchy's Theorem

Letting  $\varepsilon \rightarrow 0$ , we obtain

$$\int_{-\infty}^0 \frac{1 - e^{ix}}{x^2} dx + 0 + \int_0^\infty \frac{1 - e^{ix}}{x^2} dx = \pi.$$

Taking the real part on both sides, we see

$$\operatorname{Re} \left( \int_{-\infty}^0 \frac{1 - e^{ix}}{x^2} dx \right) = \int_{-\infty}^0 \frac{1 - \cos x}{x^2} dx$$

and thus

$$\int_{-\infty}^\infty \frac{1 - \cos x}{x^2} dx = \pi.$$

Since the integrand is even, we obtain the result.

More examples will be discussed in the assignments.

## Jordan's Lemma

Reference <http://mathworld.wolfram.com/JordansLemma.html>

The situation that we integrate over a semi-circle of radius  $R$  is quite common, so we present a useful general result here.

**2.2.15. Jordan's Lemma.** Assume that for some  $R_0 > 0$  the function  $g: \mathbb{C} \setminus \overline{B_{R_0}(0)} \rightarrow \mathbb{C}$  is holomorphic. Let

$$f(z) = e^{iaz} g(z), \quad \text{for some } a > 0.$$

Let

$$C_R = \{z \in \mathbb{C}: z = R \cdot e^{i\theta}, 0 \leq \theta \leq \pi\}$$

be a semi-circle segment in the upper half-plane and assume that

$$\sup_{0 \leq \theta \leq \pi} |g(Re^{i\theta})| \xrightarrow{R \rightarrow \infty} 0.$$

Then

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0. \quad (2.2.3)$$

# Jordan's Lemma

## Proof.

In order to show (2.2.3) it suffices to assume  $R > R_0$ . The integral is given by

$$\begin{aligned} I_R &= \int_{C_R} f(z) dz = \int_0^\pi f(Re^{i\theta}) iRe^{i\theta} d\theta \\ &= iR \int_0^\pi g(Re^{i\theta}) e^{iaR(\cos \theta + i \sin \theta)} e^{i\theta} d\theta. \end{aligned}$$

We have

$$\begin{aligned} |I_R| &\leq R \int_0^\pi |g(Re^{i\theta})| \underbrace{|e^{i(\theta+aR \cos \theta)}|}_{=1} e^{-aR \sin \theta} d\theta \\ &\leq 2R \sup_{0 \leq \theta \leq \pi} |g(Re^{i\theta})| \int_0^{\pi/2} e^{-aR \sin \theta} d\theta. \end{aligned}$$

# Jordan's Lemma

Proof (continued).

In order to estimate the remaining integral, we use that

$$x \leq \frac{\pi}{2} \sin x \quad \text{for } 0 \leq x \leq \pi/2.$$

Thus  $e^{-aR \sin \theta} \leq e^{-2aR\theta/\pi}$  for  $0 \leq \theta \leq \pi/2$  and hence

$$\begin{aligned} |I_R| &\leq 2R \sup_{0 \leq \theta \leq \pi} |g(Re^{i\theta})| \int_0^{\pi/2} e^{-aR \sin \theta} d\theta \\ &\leq 2R \sup_{0 \leq \theta \leq \pi} |g(Re^{i\theta})| \int_0^{\pi/2} e^{-2aR\theta/\pi} d\theta \\ &= 2R \sup_{0 \leq \theta \leq \pi} |g(Re^{i\theta})| \cdot \frac{\pi}{2aR} (1 - e^{-aR}) \\ &\leq \frac{\pi}{a} \sup_{0 \leq \theta \leq \pi} |g(Re^{i\theta})| \xrightarrow{R \rightarrow \infty} 0. \end{aligned}$$



# Cauchy Integral Formulas

2.2.16. **Cauchy's Integral Formula.** Suppose  $f$  is a holomorphic function in an open set  $\Omega \subset \mathbb{C}$ . If  $D$  is an open disc whose closure is contained in  $\Omega$ , then

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta \quad \text{for all } z \in D, \quad (2.2.4)$$

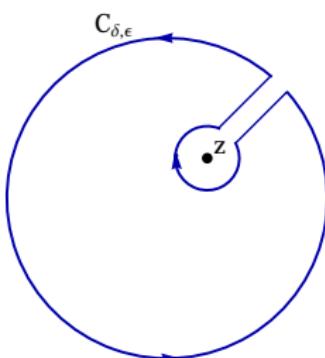
where  $C = \partial D$  is the (positively oriented) boundary circle of  $D$ .

## Proof.

Fix  $z \in D$  and consider the keyhole contour pictured at right. Here  $\delta > 0$  is the width of the “corridor” and  $\varepsilon > 0$  is the radius of the small circle segment centered at  $z$ . The large circle segment lies on  $C$ .

Since  $F(\zeta) = \frac{f(\zeta)}{\zeta - z}$  is holomorphic away from  $z$ ,

$$\oint_{C_{\delta,\varepsilon}} F(\zeta) d\zeta = 0.$$



# Cauchy Integral Formulas

Proof (continued).

When  $\delta \rightarrow 0$  the integrals along the corridor will cancel each other, so that

$$0 = \oint_{C_{\delta,\varepsilon}} F(\zeta) d\zeta \xrightarrow{\delta \rightarrow 0} \oint_C F(\zeta) d\zeta - \oint_{C_\varepsilon(z)} F(\zeta) d\zeta$$

where  $C_\varepsilon(z) = \{w \in \mathbb{C} : |w - z| = \varepsilon\}$ . Observe the orientation of the integrations!

We will analyze the integral over  $C_\varepsilon(z)$ . Note that

$$F(\zeta) = \frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta) - f(z)}{\zeta - z} + \frac{f(z)}{\zeta - z}.$$

Since  $f$  is holomorphic,  $\lim_{\zeta \rightarrow z} \frac{f(\zeta) - f(z)}{\zeta - z} = f'(z)$  exists, so the quotient must be bounded in some neighborhood of  $z$ .

# Cauchy Integral Formulas

Proof (continued).

Thus, if we let  $\varepsilon \rightarrow 0$ , we have

$$\oint_{C_\varepsilon(z)} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta \leq 2\pi\varepsilon \sup_{\zeta \in C_\varepsilon(z)} \underbrace{\left| \frac{f(\zeta) - f(z)}{\zeta - z} \right|}_{\text{bounded indep. of } \varepsilon} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Given that  $C_\varepsilon(z)$  can be parametrized by  $\gamma(t) = z + \varepsilon e^{it}$ ,  $0 \leq t \leq 2\pi$ , we can also calculate

$$\oint_{C_\varepsilon(z)} \frac{f(z)}{\zeta - z} d\zeta = f(z) \int_0^{2\pi} \frac{i\varepsilon e^{it}}{\varepsilon e^{it}} dt = 2\pi i f(z).$$

Thus,

$$\oint_C F(\zeta) d\zeta = \oint_{C_\varepsilon(z)} F(\zeta) d\zeta \xrightarrow{\varepsilon \rightarrow 0} 2\pi i f(z),$$

yielding the result. □

## Cauchy Integral Formulas

Cauchy's integral formula tells us that the values of a holomorphic function within a disc are fixed by the values of the function on the boundary; if we know  $f$  on  $C$ , then we know everything there is to know about  $f$  within  $D$ . This fact has particular applications in the theory of partial differential equations, where instead of initial values one often prescribes boundary values, e.g., on the surface of a sphere or a circle.

**2.2.17. Remark.** Cauchy's integral formula is of course also valid for all of our toy contours; we may repeat the above proof using a suitable "keyholed" version of each toy contour.

**2.2.18. Corollary.** If  $f$  is a holomorphic function in an open set  $\Omega \subset \mathbb{C}$ , then  $f$  has infinitely many complex derivatives in  $\Omega$ . Moreover, if  $D$  is an open disc whose closure is contained in  $\Omega$ ,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \quad \text{for all } z \in D \quad (2.2.5)$$

where  $C = \partial D$  is the (positively oriented) boundary circle of  $D$ .

# Cauchy Integral Formulas

## Proof.

We proceed by induction in  $n \in \mathbb{N}$ . The case  $n = 0$  is simply the Cauchy integral formula (2.2.4) which we have established. Now let  $n \geq 1$  and assume that  $f$  is  $n - 1$  times differentiable on  $\Omega$  and

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^n} d\zeta.$$

Note that for any  $x \in \mathbb{C} \setminus \{0\}$  we have ( $n \geq 1$ )

$$\frac{1}{(x+h)^n} = \frac{1}{x^n} - \frac{n}{x^{n+1}} h + o(h), \quad h \rightarrow 0.$$

This follows from the chain rule and Example 2.1.3 2.

# Cauchy Integral Formulas

Proof (continued).

This gives

$$\begin{aligned} f^{(n-1)}(z+h) &= \frac{(n-1)!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z - h)^n} d\zeta \\ &= \frac{(n-1)!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^n} d\zeta + h \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta + o(h) \end{aligned}$$

as  $h \rightarrow 0$ . (Since  $z$  is in the interior of  $C$ ,  $\zeta - z \neq 0$ ; since  $f$  is holomorphic the integral over the circle is finite and the integral over  $o(h)$  is still  $o(h)$ .) □

**2.2.19. Remark.** We have now proven one of the remarkable facts that we announced at the beginning of this section: a complex differentiable function is actually differentiable any number of times! We will next establish another “miracle:” a holomorphic function is, moreover, analytic.

## Holomorphic Functions are Analytic

2.2.20. Theorem. Suppose  $f$  is a holomorphic function in an open set  $\Omega$ . If  $D$  is an open disc centered at  $z_0$  and whose closure is contained in  $\Omega$ , then  $f$  has a power series expansion at  $z_0$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all  $z \in D$  and the coefficients are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!}, \quad n \in \mathbb{N}. \quad (2.2.6)$$

# Holomorphic Functions are Analytic

Proof.

Let  $z_0$  be the center of the open disc  $D$  and fix  $z \in D$ . Then

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

where  $C = \partial D$  denotes the boundary of the disc  $D$ . For any  $\zeta \in C$  write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z-z_0}{\zeta-z_0}}.$$

Since  $z$  is fixed and  $\zeta \in C$ , we have

$$\sup_{\zeta \in C} \left| \frac{z - z_0}{\zeta - z_0} \right| =: r < 1.$$

# Holomorphic Functions are Analytic

Proof (continued).

Thus, the following geometric series expansion converges uniformly for  $\zeta \in C$ :

$$\frac{1}{1 - \frac{z-z_0}{\zeta-z_0}} = \sum_{n=0}^{\infty} \left( \frac{z-z_0}{\zeta-z_0} \right)^n.$$

We may therefore exchange integration and summation to obtain

$$f(z) = \sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta-z_0)^{n+1}} d\zeta \right) (z-z_0)^n.$$

This proves the power series expansion; differentiating this series gives the formulas for the derivatives. □

## Uniqueness of Analytic Functions

The Cauchy Integral Formula 2.2.16 implies that an analytic function, defined on a region  $\Omega$ , is determined uniquely in the interior of a toy contour merely by its values on the contour itself. This is in itself remarkable and the integral formula has given rise to the further, remarkable theorems we just proved.

However, it also poses a question:

*How large must a set  $S \subset \Omega$  be so that a holomorphic function defined on the region  $\Omega$  is completely determined by its values on  $S$ ?*

The answer is surprisingly simple; let us first recall a basic concept.

## Accumulation Points of Complex Sets

2.2.21. **Definition.** A point  $a \in \mathbb{C}$  is said to be an *accumulation point* of a set  $S \subset \mathbb{C}$  if for every  $\varepsilon > 0$

$$(B_\varepsilon(a) \setminus \{a\}) \cap S \neq \emptyset$$

where  $B_\varepsilon(a) = \{z \in \mathbb{C} : |z - a| < \varepsilon\}$ .

2.2.22. **Lemma.** A point  $a \in \mathbb{C}$  is an accumulation point of  $S \subset \mathbb{C}$  if and only if there exists a complex sequence  $(z_n)$  in  $S$  with pairwise distinct elements (i.e.,  $z_n \neq z_m$  for  $n \neq m$ ) such that  $z_n \rightarrow a$  as  $n \rightarrow \infty$ .

The proof is left to you!

## Uniqueness of Holomorphic Functions

2.2.23. Theorem. Let  $\Omega \subset \mathbb{C}$  be a region and  $f, g: \Omega \rightarrow \mathbb{C}$  two holomorphic functions. Suppose that  $S \subset \Omega$  has an accumulation point that is contained in  $\Omega$  and that

$$f(z) = g(z) \quad \text{for all } z \in S.$$

Then  $f(z) = g(z)$  for all  $z \in \Omega$ .

By Lemma 2.2.22, we could also formulate the result as follows:

Let  $(z_n)$  be a sequence of points in  $\Omega$  such that  $z_n \neq z_m$  for  $n \neq m$  and such that the limit of  $(z_n)$  is contained in  $\Omega$ . Suppose that  $f(z_n) = g(z_n)$  for all  $n \in \mathbb{N}$ . Then  $f(z) = g(z)$  for all  $z \in \Omega$ .

# Uniqueness of the Analytic Continuation

Proof.

It is sufficient for us to prove the theorem for the case  $g = 0$ , i.e., to show that  $f = 0$  on  $\Omega$  if  $f(z_n) = 0$  for a sequence of points  $(z_n)$  with limit  $w \in \Omega$ .

Choose  $\varepsilon > 0$  small enough so that  $B_\varepsilon(w) \subset \Omega$ . We first show that  $f = 0$  on  $B_\varepsilon(w)$ .

Since  $f$  is analytic on  $\Omega$ , we can write

$$f(z) = \sum_{n=0}^{\infty} a_n(z - w)^n \quad \text{for } z \in B_\varepsilon(w). \quad (2.2.7)$$

If there exists a  $z \in B_\varepsilon(w)$  such that  $f(z) \neq 0$ , then there is smallest integer  $m \in \mathbb{N}$  with  $a_m \neq 0$ . This implies

$$f(z) = a_m(z - w)^m(1 + g(z - w)) \quad \text{for } z \in B_\varepsilon(w), \quad (2.2.8)$$

where  $g(z - w) \rightarrow 0$  as  $z \rightarrow w$ .

# Uniqueness of the Analytic Continuation

Proof (continued).

Choosing  $z = z_n$  for some  $n \in \mathbb{N}$  with  $z_n \neq w$ , we see that both  $z_n - w \neq 0$  and

$$|1 + g(z_n - w)| \geq |1 - |g(z_n - w)|| > 0$$

for  $n$  large enough. However, this contradicts  $f(z_n) = 0$ . We conclude that  $f = 0$  on  $B_\varepsilon(w)$ .

Let  $U = \text{int}\{z \in \Omega : f(z) = 0\}$  be the interior of the set of points where  $f$  vanishes. Then  $U$  is open by definition and non-empty, since  $B_\varepsilon(w) \subset U$ .

The set  $U$  is also closed in  $\Omega$ , which we establish as follows: if  $(z_n)$  is a sequence in  $U$  converging to  $z \in \Omega$ , we show that  $z \in U$ . To see this, note that  $f(z_n) = 0$  implies  $f(z) = 0$  by the continuity of  $f$ . Just as in the previous argument, we see that  $f$  must vanish on  $B_\varepsilon(z) \subset \Omega$  for some  $\varepsilon > 0$ . But that implies  $z \in U$  and  $U$  is closed.

# Uniqueness of the Analytic Continuation

Proof of Theorem 2.2.23 (continued).

Let  $V = \Omega \setminus U$ . Then  $V$  is open and  $\Omega = V \cup U$ ,  $V \cap U = \emptyset$ . Since both  $U$  and  $V$  are open in  $\Omega$  and  $\Omega$  is connected, either  $V$  or  $U$  is empty. Since  $U$  is non-empty, we must have  $V = \emptyset$  and hence

$$\Omega = U = \text{int}\{z \in \Omega : f(z) = 0\}.$$

□

## Analytic Continuation

2.2.24. Remark. Since all holomorphic functions are analytic (and all analytic functions are holomorphic by definition), we will from now on use the terms “holomorphic” and “analytic” interchangeably when discussing complex functions.

Consider the following problem: we wish to extend the map

$$f: \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = x^2$$

into the complex plane, i.e., to a function  $\tilde{f}: \mathbb{C} \rightarrow \mathbb{C}$  such that

- (i)  $\tilde{f}(x) = f(x)$  for  $x \in \mathbb{R}$  and
- (ii)  $\tilde{f}$  is holomorphic on some region  $\Omega \subset \mathbb{C}$  that includes  $\mathbb{R}$ .

Such a function  $\tilde{f}$  is called an ***analytic continuation*** of  $f$  into the region  $\Omega$ .

## Analytic Continuation

Conceivably, we have at least two candidates for  $\tilde{f}$ :

$$g_1(z) = z^2 \quad \text{and} \quad g_2(z) = |z|^2.$$

Both functions are defined on  $\Omega = \mathbb{C}$  and satisfy condition (i):

$$g_1(x) = g_2(x) = x^2 \quad \text{for } x \in \mathbb{R}.$$

However, it is easy to check that only  $g_1$  is holomorphic on  $\mathbb{C}$ , since, for any  $x \in \mathbb{R}$ ,

$$\lim_{h \rightarrow 0} \frac{g_2(x + ih) - g_2(x)}{ih} = \lim_{h \rightarrow 0} \frac{x^2 + h^2 - x^2}{ih} = 0$$

while

$$\lim_{h \rightarrow 0} \frac{g_2(x + h) - g_2(x)}{h} = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = 2x.$$

## Uniqueness of the Analytic Continuation

It turns out that this example is typical: under some quite general conditions, the analytic continuation of a function will be unique.

**2.2.25. Definition.** Let  $M \subset \mathbb{C}$  be a any set and  $f: M \rightarrow \mathbb{C}$  any function. Let  $\Omega$  be a region with  $M \subset \Omega$  and  $g: \Omega \rightarrow \mathbb{C}$  a holomorphic function such that  $g(z) = f(z)$  for  $z \in M$ . Then  $g$  is called an analytic continuation of  $f$  to  $\Omega$ .

Theorem 2.2.23 guarantees the uniqueness of the analytic continuation in practical applications:

## Uniqueness of the Analytic Continuation

In the context of our initial example,  $f(x) = x^2$  for  $x \in \mathbb{R}$  and  $g_1(z) = z^2$  is an analytic continuation of  $f$ . Any other analytic continuation of  $f$  must equal  $f$  on  $\mathbb{R}$ . In particular, if  $g$  is any analytic continuation,  $g(1/n) = f(1/n) = g_1(1/n)$  for all  $n \in \mathbb{N} \setminus \{0\}$ .

Since the sequence  $z_n = 1/n$  has the limit  $w = 0$ , which is contained in any domain including  $\mathbb{R}$ , Theorem 2.2.23 allows us to conclude that  $g(z) = g_1(z)$  on its domain. Therefore, the function  $g_1$  is the **only** analytic continuation of  $f$ .

12. Complex Differentiability

13. Properties of Holomorphic Functions

14. Singularities and Poles

15. Residue Calculus

16. The Heaviside Operator Method

17. The Laplace Transform

18. The Fourier Transform

# Singularities of Non-holomorphic Functions

In many of our applications we will encounter functions that are not holomorphic, but rather have one or more ***point singularities***:

**2.3.1. Definition.** Let  $\Omega \subset \mathbb{C}$  be open,  $z_0 \in \Omega$  and  $f: \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$  holomorphic. Then  $f$  is said to have a ***point singularity*** or ***isolated singularity*** at  $z_0$ .

Isolated singularities are hence single points that prevent  $f$  from being holomorphic. These come in three general types:

- ▶ removable singularities,
- ▶ poles,
- ▶ essential singularities.

## Classification of Singularities

2.3.2. Definition. Let  $\Omega \subset \mathbb{C}$  be open,  $z_0 \in \Omega$  and  $f: \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$  be holomorphic (i.e.,  $f$  has an isolated singularity at  $z_0$ ).

- (i) The singularity is said to be **removable** if there exists an analytic continuation  $\tilde{f}: \Omega \rightarrow \mathbb{C}$ . (The function  $\tilde{f}$  will be unique by Theorem 2.2.23.)
- (ii) The singularity is said to be a **pole** if  $g = 1/f$  is holomorphic on  $\Omega \setminus \{z_0\}$  and has a removable singularity at  $z_0$  such that the analytic continuation  $\tilde{g}$  of  $g$  satisfies  $\tilde{g}(z_0) = 0$ .
- (iii) The singularity is said to be **essential** if it is neither removable nor a pole.

# Classification of Singularities

## 2.3.3. Examples.

- (i) The function  $f(z) = \sin(z)/z$  defined on the punctured plane  $\mathbb{C} \setminus \{0\}$  has a removable singularity at  $z = 0$ . The analytic continuation  $\tilde{f}: \mathbb{C} \rightarrow \mathbb{C}$  takes the form of the power series

$$\tilde{f}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}$$

which is holomorphic for  $z \in \mathbb{C}$  by Theorem 2.1.6.

- (ii) The function  $f(z) = 1/z$  defined on  $\mathbb{C} \setminus \{0\}$  has a pole at  $z = 0$ , since  $g(z) = (1/f)(z) = z$  can be extended to a holomorphic function  $\tilde{g}: \mathbb{C} \rightarrow \mathbb{C}$  with  $\tilde{g}(0) = 0$ .
- (iii) The function  $f(z) = e^{1/z}$  defined on  $\mathbb{C} \setminus \{0\}$  has an essential singularity at  $z = 0$ . This can be seen by considering, e.g., the limit  $\lim_{x \rightarrow 0} (1/f)(ix)$ .

# Poles

## 2.3.4. Example. The function

$$f: \mathbb{C} \setminus \{i(1+2n)\pi : n \in \mathbb{Z}\} \rightarrow \mathbb{C}, \quad f(z) = \frac{e^{az}}{1+e^z}$$

has a pole at  $z = i\pi$ , because

$$g: \mathbb{C} \setminus \{i(1+2n)\pi : n \in \mathbb{Z}\} \rightarrow \mathbb{C}, \quad g(z) = (1/f)(z) = e^{-az}(1 + e^z),$$

can be extended to a holomorphic function

$$\tilde{g}: \mathbb{C} \rightarrow \mathbb{C}, \quad \tilde{g}(z) = e^{-az}(1 + e^z)$$

with  $\tilde{g}(i\pi) = 0$ .

Removable singularities are uninteresting; if required, we just remove them. Essential singularities are too difficult to deal with; we will not discuss them. However, poles occur frequently and can be treated analytically; we will focus on them.

## Representation Near Zeroes

**2.3.5. Theorem.** Suppose that  $f$  is holomorphic in a connected open set  $\Omega$ , has a zero at a point  $z_0 \in \Omega$ , and does not vanish identically in  $\Omega$ . Then there exists a neighborhood  $U \subset \Omega$  of  $z_0$ , a non-vanishing holomorphic function  $g$  on  $U$ , and a unique positive integer  $n$  such that

$$f(z) = (z - z_0)^n g(z) \quad \text{for all } z \in U. \quad (2.3.1)$$

### Proof.

We can assume that there exists a neighborhood of  $z_0$  where  $f$  does not vanish identically (otherwise, by Theorem 2.2.23  $f$  would have to vanish on all of  $\Omega$ ). In a small disc centered at  $z_0$ ,  $f$  has the expansion

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

Since  $f$  is not identically zero, there exists a smallest  $n$  such that  $a_n \neq 0$ .

## Representation Near Zeroes

Proof (continued).

Then

$$f(z) = (z - z_0)^n \underbrace{(a_n + a_{n+1}(z - z_0) + \dots)}_{=:g(z)}$$

where  $g$  is holomorphic and  $g(z) \neq 0$  near  $z_0$  (since  $a_n \neq 0$ ).

To prove that the integer  $n$  is unique, suppose that

$$f(z) = (z - z_0)^n g(z) = (z - z_0)^m h(z).$$

where  $h$  also does not vanish in a neighborhood of  $z_0$ . If  $m > n$ , we may divide by  $(z - z_0)^n$  to obtain

$$g(z) = (z - z_0)^{m-n} h(z) \xrightarrow{z \rightarrow z_0} 0,$$

which is a contradiction. The same argument shows that  $m \not< n$ . □

## Multiplicity of Zeroes

2.3.6. Definition. The integer  $n$  in Theorem 2.3.5 is called the **multiplicity** or **order** of the zero of  $f$ . If  $n = 1$ , we say that the zero is **simple**.

2.3.7. Example. Consider the function  $f: \mathbb{C} \rightarrow \mathbb{C}$ ,  $f(z) = e^{-az}(1 + e^z)$ , which has a zero at  $z = i\pi$ . We can represent  $f$  as

$$\begin{aligned} f(z) &= e^{-az}(1 + e^z) = e^{-az}e^{i\pi}(e^{-i\pi} + e^{z-i\pi}) \\ &= e^{-az}e^{i\pi}\left(-1 + 1 + \sum_{n=1}^{\infty} \frac{(z - i\pi)^n}{n!}\right) \\ &= (z - i\pi) \underbrace{e^{-az}e^{i\pi} \sum_{n=0}^{\infty} \frac{(z - i\pi)^n}{(n+1)!}}_{=:g(z)} \end{aligned}$$

where  $g(i\pi) \neq 0$ . Thus,  $f$  has a simple zero at  $z = i\pi$ .

## Multiplicity of Poles

We can use this description of zeroes of  $f$  to describe the poles of  $1/f$ :

**2.3.8. Theorem.** If  $f: \Omega \rightarrow \mathbb{C}$  has a pole at  $z_0 \in \Omega$ , then in a neighborhood  $U$  of that point there exist a non-vanishing holomorphic function  $h$  and a unique positive integer  $n$  such that

$$f(z) = (z - z_0)^{-n} h(z) \quad \text{for all } z \in U. \quad (2.3.2)$$

### Proof.

Since  $f$  has a pole at  $z_0$ , the holomorphic extension of  $1/f$  has a zero at  $z_0$ . By Theorem 2.3.5 we have  $1/f(z) = (z - z_0)^n g(z)$ , where  $g$  is holomorphic and non-vanishing in some neighborhood  $U$  of  $z_0$ . Then (2.3.2) follows with  $h(z) = 1/g(z)$ . □

**2.3.9. Definition.** The integer  $n$  in Theorem 2.3.8 is called the **multiplicity** or **order** of the pole of  $f$ . If  $n = 1$ , we say that the pole is **simple**.

# Representation Near Poles

2.3.10. Example. The function

$$f: \mathbb{C} \setminus \{i(1+2n)\pi : n \in \mathbb{Z}\} \rightarrow \mathbb{C}, \quad f(z) = \frac{e^{az}}{1 + e^z}$$

has a simple pole at  $z = i\pi$ .

An immediate consequence of Theorem 2.3.8 is the following result:

2.3.11. Theorem. If  $f: \Omega \rightarrow \mathbb{C}$  has a pole of order  $n$  at  $z_0 \in \Omega$ , then there exists a neighborhood  $U \subset \Omega$  of  $z_0$ , numbers  $a_{-n}, \dots, a_{-1} \in \mathbb{C}$  and a holomorphic function  $G: U \rightarrow \mathbb{C}$  such that

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0} + G(z) \quad (2.3.3)$$

for all  $z \in U$ .

## Local Description of Functions with Poles

Proof.

We use the representation (2.3.2) of  $f$  in a neighborhood  $U$  of  $z_0$  where the function  $h$  has a series representation

$$h(z) = \sum_{m=0}^{\infty} b_m(z - z_0)^m$$

for  $z \in U$ . Setting  $a_{-k} := b_{n-k}$ ,  $k = 1, \dots, n$  and

$$G(z) := \sum_{m=0}^{\infty} b_{n+m}(z - z_0)^m$$

we obtain (2.3.3). □

# The Principal Part

2.3.12. Definition. The term

$$P(z) := \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0}$$

of (2.3.3) is called the **principal part** of  $f$  at the pole  $z_0$ .

2.3.13. Example. The function

$$f: \mathbb{C} \setminus \{0, 2\} \rightarrow \mathbb{C}, \quad f(z) = \frac{1}{z^2(z - 2)}$$

clearly has a simple pole at  $z = 2$  and a pole of order 2 at  $z = 0$ . To find a representation of the form (2.3.3) near  $z = 0$  we write  $f$  as

$$f(z) = \frac{1}{z^2(z - 2)} = \frac{-1}{2z^2(1 - (z/2))} = \frac{-1}{2z^2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n \quad (2.3.4)$$

$$= \underbrace{-\frac{1}{2z^2} - \frac{1}{4z}}_{\text{principal part}} - \sum_{n=0}^{\infty} \frac{1}{2^{n+3}} z^n \quad (2.3.5)$$

## The Principal Part

Note that the power series expansion (2.3.5) converges for  $|z| < 2$ , so it is valid in the neighborhood  $B_2(0) = \{z \in \mathbb{C} : |z| < 2\}$ .

Near  $z = 2$  we have the representation

$$f(z) = \frac{1}{z^2(z-2)} = \frac{1}{z-2} \sum_{n=0}^{\infty} a_n (z-2)^n,$$

where, by Theorem 2.2.20,

$$a_n = \frac{1}{n!} \left. \frac{d^n}{dz^n} z^{-2} \right|_{z=2} = (-1)^n \frac{(n+1)!}{n!} \frac{1}{2^{n+2}} = (-1)^n \frac{n+1}{4 \cdot 2^n}.$$

It follows that

$$f(z) = \underbrace{\frac{1}{4} \frac{1}{z-2}}_{\text{principal part}} + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{n+2}{8 \cdot 2^n} (z-2)^n. \quad (2.3.6)$$

The series converges for  $|z-2| < 2$ , so (2.3.6) is valid in the neighborhood  $B_2(2) = \{z \in \mathbb{C} : |z-2| < 2\}$ .

# The Principal Part

2.3.14. Example. The function

$$f: \mathbb{C} \setminus \{i(1+2n)\pi : n \in \mathbb{Z}\} \rightarrow \mathbb{C}, \quad f(z) = \frac{e^{az}}{1+e^z}$$

has a simple pole at  $z = i\pi$ . To obtain a representation of the form (2.3.3) we write

$$\begin{aligned} f(z) &= e^{az} \frac{1}{1+e^z} = e^{az} \frac{1}{1-e^{z-i\pi}} \\ &= e^{az} \frac{1}{-\sum_{n=1}^{\infty} (z-i\pi)^n/n!} = \frac{-e^{az}}{z-i\pi} \underbrace{\frac{1}{1+\sum_{n=1}^{\infty} (z-i\pi)^n/(n+1)!}}_{=:g(z)}. \end{aligned}$$

The function  $g$  is holomorphic near  $z = i\pi$ , so it has a series expansion according to Theorem 2.2.20 of the form

$$g(z) = \sum_{n=0}^{\infty} a_n (z - i\pi)^n = 1 + \sum_{n=1}^{\infty} a_n (z - i\pi)^n.$$

# The Principal Part

Furthermore, we can write

$$e^{az} = e^{a\pi i} e^{a(z-i\pi)} = e^{a\pi i} + e^{a\pi i} \sum_{n=1}^{\infty} \frac{a^n}{n!} (z - i\pi)^n.$$

Thus, near  $z = i\pi$ ,

$$f(z) = \underbrace{\frac{e^{a\pi i}}{z - i\pi}}_{\text{principal part}} + \sum_{n=0}^{\infty} c_n (z - i\pi)^n \quad (2.3.7)$$

for suitable coefficients  $c_n \in \mathbb{C}$ .

## The Residue

2.3.15. Definition. The coefficient  $a_{-1}$  of  $1/(z - z_0)$  is called the **residue of  $f$  at the pole  $z_0$** , written

$$\text{res}_{z_0} f = a_{-1}.$$

The residue is important, because all other terms of the principal part have primitives, so for any contour  $\mathcal{C}$  whose interior contains only the pole at  $z_0$

$$\frac{1}{2\pi i} \int_{\mathcal{C}} f(z) dz = 0 + \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{a_{-1}}{z - z_0} dz = a_{-1}, \quad (2.3.8)$$

see the calculations of Example 2.2.8.

# The Residue

## 2.3.16. Examples.

1. For the function

$$f: \mathbb{C} \setminus \{0, 2\} \rightarrow \mathbb{C}, \quad f(z) = \frac{1}{z^2(z-2)}$$

we can read off from (2.3.5) and (2.3.6) that

$$\text{res}_2 f = \frac{1}{4}, \quad \text{res}_0 f = -\frac{1}{4}.$$

2. For

$$f: \mathbb{C} \setminus \{i(1+2n)\pi: n \in \mathbb{Z}\} \rightarrow \mathbb{C}, \quad f(z) = \frac{e^{az}}{1 + e^z}$$

we find from (2.3.7) that

$$\text{res}_{i\pi} f = -e^{a\pi i}.$$

## The Residue

In the case where  $f$  has a simple pole at  $z_0$  it is clear that

$$\text{res}_{z_0} f = \lim_{z \rightarrow z_0} (z - z_0) f(z). \quad (2.3.9)$$

This can be generalized as follows:

**2.3.17. Theorem.** Let  $\Omega \subset \mathbb{C}$  be a domain and  $f: \Omega \setminus \{z_0\} \rightarrow \mathbb{C}$  have a pole of order  $n$  at  $z_0$ . Then

$$\text{res}_{z_0} f = \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} ((z - z_0)^n f(z)). \quad (2.3.10)$$

The proof involves a straightforward insertion of (2.3.3) into the right-hand side of (2.3.10). Multiplication with  $(z - z_0)^n$  and term-by-term differentiation then yields the coefficient  $a_{-1}$ .

# The Residue

2.3.18. Example. The function

$$f: \mathbb{C} \setminus \{0, 2\} \rightarrow \mathbb{C}, \quad f(z) = \frac{1}{z^2(z-2)}$$

has a pole of order 1 at  $z = 2$  and a pole of order 2 at  $z = 0$ . Applying (2.3.10) we find that

$$\text{res}_2 f = \lim_{z \rightarrow 2} (z-2)f(z) = \lim_{z \rightarrow 2} \frac{1}{z^2} = \frac{1}{4},$$

$$\text{res}_0 f = \frac{1}{1!} \lim_{z \rightarrow 0} \frac{d}{dz} (z^2 f(z)) = \lim_{z \rightarrow 0} \frac{-1}{(z-2)^2} = -\frac{1}{4}.$$

# The Residue

2.3.19. Example. For

$$f: \mathbb{C} \setminus \{i(1+2n)\pi : n \in \mathbb{Z}\} \rightarrow \mathbb{C}, \quad f(z) = \frac{e^{az}}{1+e^z}$$

we find from (2.3.10) that

$$\text{res}_{i\pi} f = \lim_{z \rightarrow i\pi} (z - i\pi) \frac{e^{az}}{1 + e^z} = e^{a\pi i} \lim_{z \rightarrow i\pi} \frac{z - i\pi}{e^z - e^{i\pi}}.$$

Now

$$\lim_{z \rightarrow \pi i} \frac{e^z - e^{\pi i}}{z - \pi i} = (e^z)'|_{z=\pi i} = e^{\pi i} = -1,$$

so  $\text{res}_{i\pi} f = -e^{a\pi i}$ .

Observe that (2.3.10) allows us to calculate the residue without having to determine the principal part or any other aspect of the series representation (2.3.3).

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# The Residue Theorem

**2.4.1. Residue Theorem.** Suppose that  $f$  is holomorphic in an open set containing a toy contour  $\mathcal{C}$  and its interior, except for a pole at  $z_0$  inside  $\mathcal{C}$ . Then

$$\int_{\mathcal{C}} f(z) dz = 2\pi i \operatorname{res}_{z_0} f. \quad (2.4.1)$$

## Proof.

We choose a keyhole contour that avoids  $z_0$  and let the width of the corridor go to zero. Then

$$\int_{\mathcal{C}} f(z) dz = \int_{C_\varepsilon} f(z) dz$$

where  $C_\varepsilon$  is a circle of radius  $\varepsilon$  centered at  $z_0$ . The result then follows from the representation (2.3.3) together with (2.3.8). □

# The Residue Theorem

Using appropriate multiple keyhole contours, we can easily generalize this to multiple poles:

**2.4.2. Residue Theorem.** Suppose that  $f$  is holomorphic in an open set containing a (positively oriented) toy contour  $\mathcal{C}$  and its interior, except for poles at the points  $z_1, \dots, z_N$  inside  $\mathcal{C}$ . Then

$$\int_{\mathcal{C}} f(z) dz = 2\pi i \sum_{k=1}^N \text{res}_{z_k} f. \quad (2.4.2)$$

**2.4.3. Example.** We calculate the following integral:

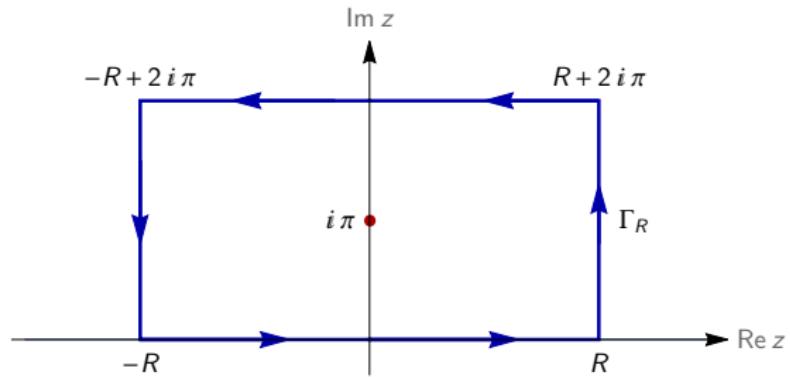
$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1 + e^x} dx = \frac{\pi}{\sin(\pi a)}, \quad 0 < a < 1.$$

# The Residue Theorem

We take

$$f(z) = \frac{e^{az}}{1 + e^z}$$

and integrate along the following contour:



We have seen that  $\text{res}_{\pi i} f = -e^{a\pi i}$  so by the residue formula (2.4.1),

$$\oint_{\Gamma_R} f(z) dz = -2\pi i e^{a\pi i}.$$

# The Residue Theorem

We now treat the individual integrals that make up the rectangle  $\gamma_R$ . For  $0 < R \leq \infty$  let

$$I_R = \int_{-R}^R f(x) dx.$$

be the integral along the side lying on the real axis. Then the integral along the top of the rectangle  $\Gamma_R$ ,  $R < \infty$ , from right to left, is given by

$$\int_R^{-R} f(x + 2\pi i) dx = \int_R^{-R} \frac{e^{a(x+2\pi i)}}{1 + e^{x+2\pi i}} dx = -e^{2\pi ai} I_R.$$

The integral along the right edge of the rectangle can be estimated by

$$\begin{aligned} \left| \int_0^{2\pi} \frac{e^{a(R+it)}}{1 + e^{R+it}} \cdot i dt \right| &\leq \int_0^{2\pi} \left| \frac{e^{a(R+it)}}{1 + e^{R+it}} \right| dt = e^{(a-1)R} \int_0^{2\pi} \frac{1}{|e^{-R} + e^{it}|} dt \\ &\leq e^{(a-1)R} \frac{2\pi}{1 - e^{-R}} \xrightarrow{(a<1)} 0. \end{aligned}$$

# The Residue Theorem

Similarly, the integral along the left edge of the rectangle can be estimated by

$$\begin{aligned} \left| \int_{2\pi}^0 \frac{e^{a(-R+it)}}{1+e^{-R+it}} i \, dt \right| &\leq \int_0^{2\pi} \left| \frac{e^{a(-R+it)}}{1+e^{-R+it}} \right| dt = e^{-aR} \int_0^{2\pi} \frac{1}{|1+e^{-R+it}|} dt \\ &\leq e^{-aR} \frac{2\pi}{1-e^{-R}} \xrightarrow[(a>0)]{R\rightarrow\infty} 0. \end{aligned}$$

It follows that

$$-2\pi ie^{a\pi i} = I_\infty + 0 - e^{2\pi ai} I_\infty + 0,$$

so we obtain

$$I_\infty = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = -2\pi i \frac{e^{a\pi i}}{1-e^{2a\pi i}} = \frac{\pi}{\sin(\pi a)}$$

# The Residue Theorem

2.4.4. Example. We want to calculate

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx.$$

We will regard it as the real part of the integral  $\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2} dx$  which we will calculate by integrating along a semi-circle toy contour. The function

$$f(z) = \frac{e^{iz}}{1+z^2}$$

clearly satisfies the conditions of Jordan's lemma since

$$\sup_{0 \leq \theta \leq \pi} \left| \frac{1}{1+R^2 e^{2i\theta}} \right| \leq \frac{1}{R^2 - 1} \xrightarrow{R \rightarrow \infty} 0.$$

Thus the integral over the upper semi-circle will vanish as  $R \rightarrow \infty$  and we obtain

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2} dx = 2\pi i \sum_{\operatorname{Im} z_0 \geq 0} \operatorname{res}_{z_0} f.$$

# The Residue Theorem

The only singular points of  $f$  are  $\pm i$ , of which only the residue at  $+i$  contributes to the integral. We have

$$f(z) = \frac{1}{(z-i)(z+i)} e^{iz} = \frac{e^{iz}}{2i} \left( \frac{1}{z-i} - \frac{1}{z+i} \right).$$

Now in a neighborhood of  $z = i$  the function  $\frac{e^{iz}}{2i} \frac{1}{z+i}$  is holomorphic and hence has a series expansion of the form  $\sum_{n=0}^{\infty} a_n(z-i)^n$ . Thus we have

$$\begin{aligned} f(z) &= \frac{1}{2i} \frac{1}{z-i} e^{iz} + \sum_{n=0}^{\infty} a_n(z-i)^n \\ &= \frac{1}{2ei} \frac{1}{z-i} e^{i(z-i)} + \sum_{n=0}^{\infty} a_n(z-i)^n \\ &= \frac{1}{2ei} \frac{1}{z-i} + \frac{1}{2ei} \sum_{n=1}^{\infty} \frac{i^n}{n!} (z-i)^{n-1} + \sum_{n=0}^{\infty} a_n(z-i)^n \end{aligned}$$

## The Residue Theorem

It follows that  $\text{res}_i f = 1/(2ie)$ . Note that we could also have argued differently: since  $1/f = (z - i)(z + i)e^{-iz}$  has a zero of order 1 at  $i$ ,  $f$  has a simple pole there. By (2.3.9) we then have

$$\text{res}_i f = \lim_{z \rightarrow i} (z - i)f(z) = \lim_{z \rightarrow i} \frac{1}{(z + i)} e^{iz} = \frac{1}{2ie}.$$

From this we obtain

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2} dx = 2\pi i \text{res}_i f = \frac{\pi}{e}.$$

Hence

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx = \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2} dx = \frac{\pi}{e}$$

## The Complex Logarithm

As we have previously mentioned, the concept of the logarithm is somewhat subtle in  $\mathbb{C}$ . For example, we know (see Example 2.2.8) that there can exist no holomorphic function  $f$  on  $\mathbb{C} \setminus \{0\}$  with  $f'(z) = 1/z$ , which is one of the properties of the real logarithm.

Let us try a naive approach to defining the logarithm. If  $z = re^{i\varphi}$  we would expect that

$$\ln z = \ln(re^{i\varphi}) = \ln r + i\varphi,$$

where  $\ln r$  is simply the usual real logarithm. The problem we encounter here is that  $\varphi$  is only determined up to  $2\pi$ , so

$$\ln(re^{i\varphi}) = \ln(re^{i(\varphi+2\pi)}) = \ln r + i\varphi + 2\pi i,$$

which means that the logarithm is not single-valued. A general treatment of the multi-valued logarithm is a springboard onto many deep subjects of complex analysis, such as Riemann surfaces, homotopy of curves, analytic continuation along curves etc. We will give a “quick and dirty” definition that suffices for all practical purposes.

## The Complex Logarithm

When we define the logarithm for complex numbers, we want it to agree with the logarithm that we know for the positive real numbers, so in particular we require that

$$\ln 1 = 0.$$

We can now define the logarithm on any open set  $\Omega$  that is simply connected, includes 1 and that does not include 0 by setting

$$\ln z := \int_{\mathcal{C}} \frac{dz}{z}, \quad (2.4.3)$$

where  $\mathcal{C} \subset \Omega$  is any simple curve joining  $1 \in \Omega$  to  $z \in \Omega$ .

Since  $1/z$  is holomorphic in  $\mathbb{C} \setminus \{0\}$  it is holomorphic in  $\Omega$  and because  $\Omega$  is simply connected we can apply Cauchy's integral theorem to see that the definition (2.4.3) is independent of the choice of  $\gamma$ . Furthermore, by Theorem 2.2.23, this extension is unique.

## Branches of the Complex Logarithm

We can hence define the complex logarithm for

$$\Omega = \mathbb{C} \setminus \{x \in \mathbb{R}: x \leq 0\},$$

in which case

$$\ln(re^{i\varphi}) = \ln r + \varphi i \quad (r > 0, -\pi < \varphi < \pi).$$

This is called the **principal branch** of the logarithm. Note that  $\Omega$  does not include any closed curve around the origin. For brevity, we set

$$\mathbb{R}_-^0 := \{x \in \mathbb{R}: x \leq 0\}, \quad \mathbb{R}_+^0 := \{x \in \mathbb{R}: x \geq 0\}.$$

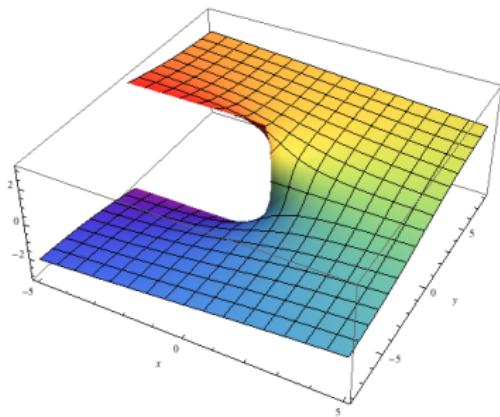
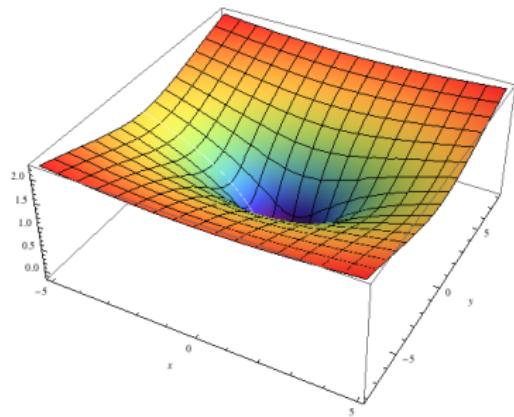
We can also define other branches of the logarithm. Often, a convenient choice (especially for residue calculus) is

$$\ln: \mathbb{C} \setminus \mathbb{R}_+^0 \rightarrow \mathbb{C}, \quad \ln z = \ln r + \varphi i$$

where  $r > 0$  and  $0 < \varphi < 2\pi$ .

# The Principal Branch of the Complex Logarithm

The real part (left) and imaginary part (right) of the principal branch of the complex logarithm.



## The Principal Branch of the Complex Logarithm

One of the important things to note for the principal branch of the logarithm is that, in general,

$$\ln(z_1 z_2) \neq \ln z_1 + \ln z_2.$$

For example, take  $z_1 = z_2 = e^{2\pi i/3}$ . Then

$$\ln z_1 = \ln z_2 = \frac{2\pi}{3}$$

but

$$\ln(z_1 z_2) = \ln e^{-2\pi i/3} = -\frac{2\pi}{3} \neq \ln z_1 + \ln z_2.$$

If we had taken the logarithm on  $\mathbb{C} \setminus \mathbb{R}_+^0$  this problem would have been avoided for the chosen  $z_1, z_2$ , because then

$$\ln(z_1 z_2) = \ln e^{4\pi i/3} = \frac{4\pi}{3} = \ln z_1 + \ln z_2$$

## Complex Powers and Roots

We can now define complex powers (this was not possible previously). We set

$$z^\alpha := e^{\alpha \ln z}, \quad \alpha \in \mathbb{C}, z \in \mathbb{C} \setminus \mathbb{R}_-$$

Setting

$$\sqrt[n]{z} := z^{1/n}$$

we then have

$$(z^{1/n})^n = \prod_{k=1}^n e^{\frac{1}{n} \ln z} = e^{\frac{n}{n} \ln z} = e^{\ln z} = z.$$

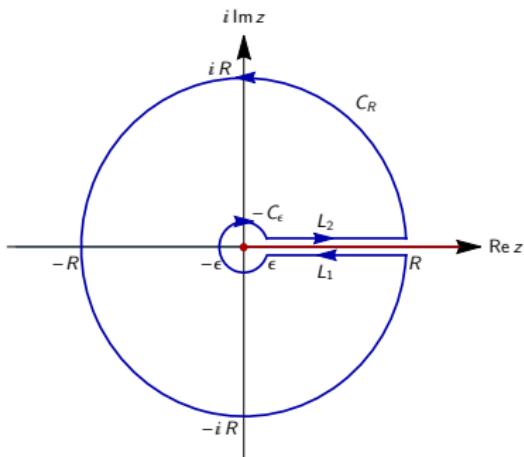
Note that the “other roots”  $\sqrt[n]{z}$  correspond to taking different branches of the logarithm. Just as the logarithm, the “root functions” have natural branch cuts at the negative real numbers but a different branch can be chosen as convenient.

# Residue Calculus for Functions with Branch points

We can apply these definitions to evaluate integrals such as

$$\int_0^\infty \frac{\sqrt{x}}{x(1+x)} dx.$$

In these cases, it is usually convenient to choose a contour of the shape shown below:



When integrating a function that includes branch points, it is thus best to choose a branch with branch cut at  $\mathbb{R}_+^0$ .

This toy contour is called the **slit circle**.

# Residue Calculus for Functions with Branch points

## Reference

<http://math.fullerton.edu/mathews/c2003/IntegralsBranchPointsMod.html>

**2.4.5. Theorem.** Let  $P$  and  $Q$  be polynomials of degree  $m$  and  $n$ , respectively, where  $n \geq m + 2$ . If  $Q(x) \neq 0$  for  $x > 0$ , if  $Q$  has a zero of order at most 1 at the origin and if

$$f(z) = \frac{z^\alpha P(z)}{Q(z)}, \quad 0 < \alpha < 1,$$

then

$$\text{p.v.} \int_0^\infty \frac{x^\alpha P(x)}{Q(x)} dx = \frac{2\pi i}{1 - e^{2\pi\alpha i}} \sum_{j=1}^k \text{res}_{z_j} f \quad (2.4.4)$$

where  $z_1, \dots, z_k$  are the nonzero poles of  $P/Q$ .

We omit the proof of this theorem.

# Residue Calculus for Functions with Branch points

2.4.6. Example. We calculate the integral

$$\int_0^\infty \frac{x^\alpha}{x(1+x)} dx, \quad 0 < \alpha < 1.$$

By Theorem 2.4.5 we need to calculate the residues of the integrand. There are evidently poles at  $z = 0$  and  $z = -1$  and

$$\text{res}_0 \frac{z^\alpha}{z(1+z)} = \lim_{z \rightarrow 0} \frac{z^\alpha}{1+z} = 0,$$

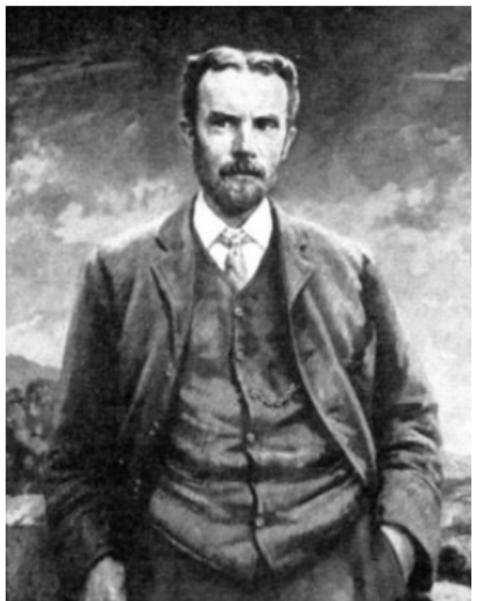
$$\text{res}_{-1} \frac{z^\alpha}{z(1+z)} = \lim_{z \rightarrow -1} \frac{z^\alpha}{z} = - \lim_{z \rightarrow -1} e^{\alpha \ln z} = -e^{\alpha \pi i}$$

By (2.4.4), we then have

$$\text{p.v.} \int_0^\infty \frac{x^\alpha}{x(1+x)} dx = \frac{2\pi i}{1 - e^{2\pi\alpha i}} (-e^{\alpha\pi i}) = \frac{\pi}{\sin(\alpha\pi)}.$$

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# Oliver Heaviside



Oliver Heaviside (1850-1925) was an English electrical engineer, physicist and mathematician who worked for the ***Great Northern Telegraph Company***. His work on electrical circuits was extensive, and he developed many innovative mathematical methods.

Among other things, he greatly simplified Maxwell's equations to their current form of four equations with four unknowns (from originally 20 equations with 20 unknowns written as ***quaternions***) and he developed his ***operator method*** for solving ODEs, which he applied to problems in circuits.

## References

- <http://myreckonings.com/wordpress/2007/12/07/heavisides-operator-calculus/>
- [http://en.wikipedia.org/wiki/Oliver\\_Heaviside](http://en.wikipedia.org/wiki/Oliver_Heaviside)

## The Heaviside Function

Heaviside considered initial value problems such as

$$y''(t) - y(t) = 1 \quad \text{for } t > 0, \quad y(0) = 0, \quad y'(0) = 0. \quad (2.5.1)$$

He was interested in situations where a voltage is “switched on” at time  $t = 0$ , so he defined the function

$$H: \mathbb{R} \rightarrow \mathbb{R}, \quad H(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}.$$

This function is still called the **Heaviside function** in his honor. If we write (2.5.1) as

$$y''(t) - y(t) = H(t) \quad \text{for } t \in \mathbb{R}, \quad y(0) = 0, \quad y'(0) = 0. \quad (2.5.2)$$

then the initial conditions guarantee that  $y = 0$  for  $t \leq 0$ , a physically realistic situation (no current before the voltage is “switched on” at  $t = 0$ ).

## The Heaviside Operator Method

In order to tackle the equation  $y''(t) - y(t) = H(t)$ , Heaviside wrote the differential operator  $d/dt$  as  $D$ , so that the equation became

$$D^2y - y = H(t). \quad (2.5.3)$$

The “operator method” is now based on working with  $D$  as with a number. In particular,

$$Df(t) = f'(t), \quad \frac{1}{D}f(t) = \int_0^t f(s) ds + f(0) \quad (2.5.4)$$

because then  $D(1/D)f = (1/D)Df = f$  for a function  $f$ . (Here,  $f(0)$  is actually  $\lim_{t \searrow 0} \int_0^t f(s) ds$ , but we will dispense with such subtleties for now.)

We are assuming that all functions we consider are zero for  $t < 0$ , otherwise we multiply with  $H(t)$ . In particular, we have

$$\frac{1}{D}H(t) = t.$$

# The Heaviside Operator Method

Heaviside solved (2.5.3) by writing

$$y(t) = \frac{1}{D^2 - 1} H(t) = \frac{1/D^2}{1 - 1/D^2} H(t) = \left( \frac{1}{1 - 1/D^2} - 1 \right) H(t)$$

Expanding  $1/(1 - (1/D)^2)$  using the geometric series, we obtain

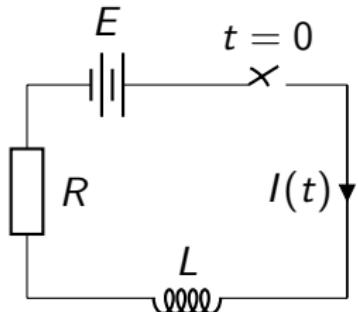
$$\begin{aligned} y(t) &= \left( \frac{1}{1 - 1/D^2} - 1 \right) H(t) = \sum_{n=1}^{\infty} (1/D^2)^n H(t) \\ &= \sum_{n=1}^{\infty} (1/D)^{2n} H(t) = \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!} \\ &= \frac{1}{2}(e^t + e^{-t}) - 1 = \cosh t - 1 \end{aligned}$$

It is easily checked that this is in fact the solution to the initial value problem (2.5.1).

# The Heaviside Operator Method

Let us look at another example.

## 2.5.1. Example.



A circuit consists of a resistor with resistance  $R$  and a spool with inductance  $L$ . At time  $t = 0$  the circuit is closed and a voltage  $E$  applied. We seek to determine the current  $I$  as a function of time.

By Kirchoff's laws, we have

$$L \frac{dI}{dt} + RI = E \cdot H(t), \quad I(0) = 0. \quad (2.5.5)$$

We abbreviate  $k := R/L$  and write  $D = d/dt$ . Then (2.5.5) becomes

$$I(t) = \frac{E}{L} \frac{1}{D + k} H(t) = \frac{E}{L} \frac{1}{D} \frac{1}{1 + (k/D)} H(t)$$

# The Heaviside Operator Method

Again using the geometric series, we obtain

$$\begin{aligned} I(t) &= \frac{E}{L} \frac{1}{D} \frac{1}{1 + (k/D)} H(t) = \frac{E}{L} \sum_{n=0}^{\infty} (-1)^n k^n \frac{1}{D^{n+1}} H(t) \\ &= \frac{E}{L} \sum_{n=0}^{\infty} (-1)^n k^n \frac{t^{n+1}}{(n+1)!} \\ &= \frac{E}{R} (1 - e^{-Rt/L}) \end{aligned}$$

Again, this is a correct solution to (2.5.5).

To a trained mathematician (and you, of course!) it is obvious that there are many difficulties with this procedure. For a start, operator like  $d/dt$  do not behave like numbers, since they are (e.g.) not commutative.

Furthermore, we are essentially playing around with series without regard to convergence. (For a matrix, we could define this type of series by using the operator norm to show absolute convergence; but for a differential operator, the operator norm does not exist - it is infinite!)

## The Heaviside Operator Method

Heaviside had only contempt for people who criticized his methods on such ground. He famously said,

*Why should I refuse a good dinner simply because I don't understand the digestive processes involved?*

Heaviside used various tricks to refine his methods, deriving fractional powers of  $D = d/dt$  to deal with circuits with continuous, distributed impedances such as in telegraph lines. He also solved problems using his "operator calculus" that could not be solved by any other means at the time.

However, his disregard for the niceties of mathematical proofs limited his theory, and of a three part series of publications in a journal meant to present his theories, the third part was rejected.

## The Heaviside Operator Method

As expected, the fact that the operator method is not rigorous means that it does not always work. Consider the equation

$$y' - y = -\sin t.$$

Writing  $(D - 1)y = \sin t$  gives

$$\begin{aligned} y(t) &= (1 - D)^{-1} \sin t = (1 + D + D^2 + \dots) \sin t \\ &= \sin t + \cos t - \sin t - \cos t + - \dots \end{aligned}$$

This series does not converge, and hence we do not find the solution. It becomes obvious that the method needs further refinement.

Another issue with the operator method is that it provides (if it works) only a particular solution to an ODE, not the general solution.

## A Transformation to Implement the Operator Method

The root problem with the operator method is simply that the operator  $D = d/dt$  can not be treated as a number. However, the underlying idea, that derivatives be solved algebraically, must have some potential, because the operator method does work in many cases.

Now  $d/dt$  does not act like a number on  $f$ . But perhaps we can transform  $f$  into another function (which we denote by  $\{f\}$ ) and find an operator  $D$  such that  $D\{f\} = \{f'\}$ , so the following diagram commutes:

$$\begin{array}{ccc} f & \xrightarrow{\hspace{2cm}} & \{f\} \\ \downarrow \frac{d}{dt} & & \downarrow D \\ f' & \xrightarrow{\hspace{2cm}} & \{f'\} = D\{f\} \end{array}$$

We hope that then  $D$  “can be treated like a number” and Heaviside’s calculus can be justified in this sense.

## A Transformation to Implement the Operator Method

We do not yet know what shape the transformation  $f \mapsto \{f\}$  will take, except that we want  $\{f\}$  to be a function of some kind. Let us, however, make a few assumptions:

- (i) We want the transformation to be linear, i.e.,

$$\{\alpha f + \beta g\} = \alpha\{f\} + \beta\{g\}.$$

- (ii) We want the operator  $D$  to have the property that

$$D\{f\}(p) = \{f'\}(p) + f(0).$$

The above properties imply that  $D$  is linear, i.e.,

$$D\{\alpha f + \beta g\} = \alpha D\{f\} + \beta D\{g\}. \quad (2.5.6)$$

and, in particular,

$$\{0\} = 0.$$

## Properties of the Transformation

We hence obtain

$$D\{1\} = \{0\} + 1 = 1. \quad (2.5.7)$$

and

$$D^n \left\{ \frac{t^n}{n!} \right\} = \{1\}. \quad (2.5.8)$$

The goal of this whole construction is to be able to treat  $D$  "like a number". Given that  $\{f\}$  is assumed to be a function, we can ask what that might mean. In view of the linearity (2.5.6) it is clear that it makes no sense to take this literally; defining  $D = 5$  (an actual number) so that

$$\{f'\}(p) = 5\{f\}(p) - f(0)$$

is just not going to work. The next simplest thing is to take

$$D\{f\}(p) = p \cdot \{f\}(p), \quad (2.5.9)$$

so that  $D$  is a multiplication operator.

## An Explicit Formula

Let us suppose that a transformation  $f \rightarrow \{f\}$  can be found such that (2.5.9) holds and the properties (i) and (ii) of Slide 401 are satisfied. We will try to find an explicit formula for this transformation.

Let  $f$  be an analytic function. Then

$$\{f\} = \left\{ \sum_{n=0}^{\infty} f^{(n)}(0) \frac{t^n}{n!} \right\} = \sum_{n=0}^{\infty} f^{(n)}(0) \left\{ \frac{t^n}{n!} \right\}$$

From (2.5.7), (2.5.8) and (2.5.9) we obtain

$$p^{n+1} \left\{ \frac{t^n}{n!} \right\} = 1$$

so

$$\{f\} = \frac{1}{p} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{p^n}.$$

# An Explicit Formula

Now recall the definition of the Euler gamma function,

$$\Gamma(t) = \int_0^\infty z^{t-1} e^{-z} dz$$

which has the property that  $\Gamma(n+1) = n!$ . Then

$$\begin{aligned}\{f\}(p) &= \frac{1}{p} \sum_{n=0}^{\infty} \int_0^\infty z^n e^{-z} dz \frac{f^{(n)}(0)}{n! p^n} \\ &= \frac{1}{p} \int_0^\infty \sum_{n=0}^{\infty} \left(\frac{z}{p}\right)^n \frac{f^{(n)}(0)}{n!} e^{-z} dz \\ &= \frac{1}{p} \int_0^\infty f(z/p) e^{-z} dz \\ &= \int_0^\infty f(z) e^{-pz} dz.\end{aligned}$$

## An Explicit Formula

We can easily check that by defining

$$\{f\}(p) := \int_0^\infty f(z)e^{-pz} dz \quad (2.5.10)$$

the properties (i) and (ii) of Slide 401 are satisfied:

- (i) It is clear that (2.5.10) is linear in  $f$ ,
- (ii) We have

$$\begin{aligned}\{f'\}(p) &= \int_0^\infty f'(z)e^{-pz} dz = f(z)e^{-pz} \Big|_0^\infty + p \int_0^\infty f(z)e^{-pz} dz \\ &= -f(0) + p \cdot \{f\}(p)\end{aligned}$$

so (ii) holds with  $D\{f\}(p) = p \cdot \{f\}(p)$ .

Since we have shown that any transformation satisfying (i) and (ii) must have the form (2.5.10), it follows that (2.5.10) is the **only** transformation with these properties.

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# The Laplace Transform

The transformation (2.5.10) was invented and studied nearly exactly 100 years before Heaviside's calculus (around 1815) by Laplace and Poisson and carries Laplace's name.

2.6.1. Definition. Let  $f: [0, \infty) \rightarrow \mathbb{R}$  be a continuous function such that

$$\sup_{t \in [0, \infty)} e^{-\beta t} |f(t)| < \infty \quad \text{for some } \beta \geq 0. \quad (2.6.1)$$

Then the function  $F: (\beta, \infty) \rightarrow \mathbb{R}$ ,

$$F(p) := (\mathcal{L}f)(p) := \int_0^\infty e^{-pt} f(t) dt$$

is called the **Laplace transform** of  $f$ .

# The Laplace Transform

We recall from the previous section that

$$\begin{aligned}(\mathcal{L}f')(p) &= \int_0^\infty e^{-pt} f'(t) dt = \int_0^\infty p e^{-pt} f(t) dt - f(0) \\&= p \cdot (\mathcal{L}f)(p) - f(0).\end{aligned}$$

In the same manner,

$$(\mathcal{L}f'')(p) = p^2(\mathcal{L}f)(p) - p \cdot f(0) - f'(0).$$

For the Heaviside function  $H$  (which on  $(0, \infty)$  is indistinguishable from the constant function  $f(x) = 1$ ) we have

$$(\mathcal{L}H)(p) = \int_0^\infty e^{-pt} H(t) dt = \int_0^\infty e^{-pt} dt = \frac{1}{p}. \quad (2.6.2)$$

We will now see how the Laplace transform can be used to solve differential equations.

# Solving an ODE with the Laplace Transform

2.6.2. Example. Let us revisit Example 2.5.1: we wanted to solve

$$L \frac{dI}{dt} + RI = E \cdot H(t), \quad I(0) = 0$$

For  $t \leq 0$  the equation is homogeneous and has vanishing initial condition, so the solution also vanishes. Now let us consider the solution for  $t > 0$ , i.e., let us consider  $I$  as a function  $(0, \infty) \rightarrow \mathbb{R}$ .

We apply the Laplace transform to this equation. Then

$$\begin{aligned} L \cdot \mathcal{L}(I')(p) + R \cdot \mathcal{L}(I)(p) &= E \cdot \mathcal{L}(H)(p) \\ \Leftrightarrow Lp(\mathcal{L}I)(p) - L \cdot I(0) + R(\mathcal{L}I)(p) &= \frac{E}{p} \\ \Leftrightarrow (\mathcal{L}I)(p) &= \frac{E}{L} \frac{1}{(p+k)p} \end{aligned} \tag{2.6.3}$$

where  $k = R/L$ .

## Inverting the Laplace Transform

The equation (2.6.3) gives the Laplace transform  $\mathcal{L}I$  of the current  $I$ . We now need to try to regain the original function  $I$ . We do not yet have a systematic procedure for this, so we simply seek a function  $I$  whose Laplace transform is given by (2.6.3).

We know from (2.6.2) that  $(\mathcal{L}H)(p) = 1/p$ , so we may ask: which function has the Laplace transform

$$(\mathcal{L}f)(p) = \frac{1}{p + \alpha}?$$

We find the answer by writing the right-hand side as an integral:

$$\frac{1}{p + \alpha} = \int_0^\infty e^{-t(p+\alpha)} dt = \int_0^\infty e^{-t\alpha} e^{-tp} dt = \mathcal{L}(e^{-t\alpha})$$

Hence,

$$\mathcal{L}(e^{-(\cdot)\alpha})(p) = \frac{1}{p + \alpha}. \quad (2.6.4)$$

## Solving an ODE with the Laplace Transform

In our example, we have

$$(\mathcal{L}I)(p) = \frac{E}{L} \frac{1}{(p+k)p} = \frac{E}{Lk} \left( \frac{1}{p} - \frac{1}{p+k} \right), \quad (2.6.5)$$

Using  $k = R/L$  and applying (2.6.4) we see that

$$I(t) = \frac{E}{R} (1 - e^{-Rt/L}),$$

satisfies (2.6.5). This is also precisely the solution obtained in Example 2.5.1 using the Heaviside operator method.

Observe that the Laplace transform in this example automatically incorporated the initial condition  $I(0) = 0$ . This is a general feature of solutions using the Laplace transform: we can obtain particular solutions to concrete initial value problems. Finding general solutions is not as easy using this method.

## Inverting the Laplace Transform

Finding the “inverse Laplace transform” is an essential part of the solution process. Before we introduce the formal process, we note that with a couple of “tricks” it is possible to deduce the inverse  $\mathcal{L}^{-1}$  from known transforms. One crucial property is that

$$g(t) = e^{at} f(t) \quad \text{implies} \quad \mathcal{L}g(p) = \mathcal{L}f(p-a) \quad \text{for } a \in \mathbb{R}.$$

This follows from

$$(\mathcal{L}g)(p) = \int_0^\infty e^{at} f(t) e^{-pt} dt = \int_0^\infty f(t) e^{-(p-a)t} dt = (\mathcal{L}f)(p-a).$$

# Inverting the Laplace Transform

Hence, if we know that

$$(\mathcal{L}^{-1}F)(t) = f(t)$$

then

$$(\mathcal{L}^{-1}F(\cdot - a))(t) = e^{at}f(t).$$

The previously calculated example that  $\mathcal{L}1(p) = 1/p$  implies  $\mathcal{L}(e^{at})(p) = 1/(p - a)$  is a special case of the use of this relation.

We can then establish a table of Laplace transforms and use it to find inverses. This is the way most engineers work in practice and also the method that Heaviside himself used.

## Table of Laplace Transforms

Through direct integration, it is not difficult to establish the following table of Laplace transforms:

$f(t)$	$(\mathcal{L}f)(p)$	Comment / Domain of $\mathcal{L}f$
1	$\frac{1}{p}$	$p > 0$
$t^n$	$\frac{n!}{p^{n+1}}$	$n \in \mathbb{N}, p > 0$
$e^{at}$	$\frac{1}{p - a}$	$p > a$
$\sin(bt)$	$\frac{b}{p^2 + b^2}$	$b \in \mathbb{R}, p > 0$
$\cos(bt)$	$\frac{p}{p^2 + b^2}$	$b \in \mathbb{R}, p > 0$

## Table of Laplace Transforms

The previous table can be used together with the following relations to find the Laplace transforms of elementary functions.

$f(t)$	$(\mathcal{L}f)(p)$	Comment
$H(t - a)$	$e^{-ap}/p$	$a, p > 0$
$g(t - a)H(t - a)$	$e^{-ap}(\mathcal{L}g)(p)$	$a > 0$
$e^{at}g(t)$	$(\mathcal{L}g)(p - a)$	$a \in \mathbb{R}$
$g(at)$	$\frac{1}{a}(\mathcal{L}g)\left(\frac{p}{a}\right)$	$a > 0$
$g^{(n)}(t)$	$p^n(\mathcal{L}g)(p) - p^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	$n \in \mathbb{N}$
$(-t)^n g(t)$	$(\mathcal{L}g)^{(n)}(p)$	$n \in \mathbb{N}$

# Applying the Laplace Transform

References **Braun**, Section 2.9-2.11.

We will mainly use the Laplace transform to solve **linear ODEs with constant coefficients**. The procedure is always as follows:

1. Apply the Laplace transform to both sides of the ODE/IVP;
2. The transformed equation is algebraic; solve for the Laplace transform of the unknown function;
3. Find the inverse Laplace transform of the unknown function. This is the solution to the original IVP.

The basic procedure was demonstrated in Example 2.6.2. You might very well ask what the advantages are, compared with alternative approaches, such as transforming the ODE to a system of first-order ODEs.

The main feature of the Laplace transform is that we can deal with discontinuous inhomogeneities and even inhomogeneities that are not functions at all!

## Applying the Laplace Transform

Due to the integral transform, discontinuities are “smoothed out” so that we can obtain continuous and even differentiable solutions even if the inhomogeneity has a jump discontinuity.

Such jump discontinuities (or, in general, piecewise defined functions) are modeled with the Heaviside function; for example, we can express

$$f(x) = \begin{cases} 2 & x < 1 \\ x & x > 1 \end{cases}$$

as

$$f(x) = 2H(1 - x) + xH(x - 1).$$

If we were to solve a constant-coefficient ODE with piecewise-continuous inhomogeneity we would actually have to solve several problems, each on a domain where the inhomogeneity is continuous and then join the solutions. The Laplace transform method allows us to calculate the entire solution directly.

# Piecewise-Continuous Inhomogeneities

2.6.3. Example. Consider the equation

$$q''(t) + \omega^2 q(t) = E \cdot (H(t - 2) - H(t - 5)), \quad q(0) = q_0, \quad q'(0) = 0.$$

where  $q_0, \omega, E \in \mathbb{R}$ . Here, the inhomogeneity is “switched on” at  $t = 2$  and “switched off” again at  $t = 5$ .

We write  $Q = \mathcal{L}q$ . Now

$$(\mathcal{L}q'')(s) = s^2 Q(s) - sq(0) - q'(0) = s^2 Q(s) - sq_0$$

and

$$(\mathcal{L}H(\cdot - c))(s) = e^{-cs} \frac{1}{s}$$

so the transformed equation is

$$s^2 Q(s) - sq_0 + \omega^2 Q(s) = E \cdot (e^{-2s} - e^{-5s}) \frac{1}{s}.$$

# Piecewise-Continuous Inhomogeneities

Hence

$$Q(s) = \frac{E}{s^2 + \omega^2} (e^{-2s} - e^{-5s}) \frac{1}{s} + \frac{sq_0}{s^2 + \omega^2}$$

Now

$$\frac{\omega^2}{s(s^2 + \omega^2)} = \frac{1}{s} - \frac{s}{s^2 + \omega^2}$$

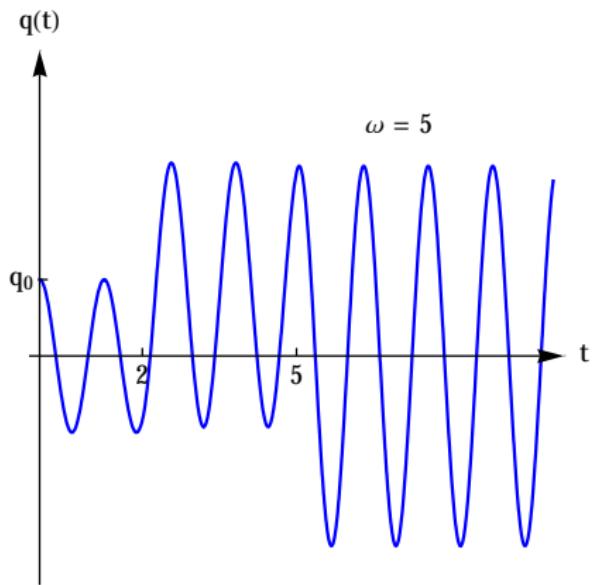
gives

$$\begin{aligned} Q(s) &= \frac{E}{\omega^2} \left( \frac{1}{s} - \frac{s}{s^2 + \omega^2} \right) (e^{-2s} - e^{-5s}) + \frac{sq_0}{s^2 + \omega^2} \\ &= \frac{E}{\omega^2} \left( \frac{e^{-2s}}{s} - \frac{e^{-5s}}{s} - e^{-2s} \frac{s}{s^2 + \omega^2} + e^{-5s} \frac{s}{s^2 + \omega^2} \right) + \frac{sq_0}{s^2 + \omega^2}. \end{aligned}$$

## Piecewise-Continuous Inhomogeneities

Applying the inverse transform,  $q(t) = (\mathcal{L}^{-1}Q)(t)$ , so

$$\begin{aligned} q(t) = & \frac{E}{\omega^2} (H(t-2)(1 - \cos(\omega(t-2))) + H(t-5)(\cos(\omega(t-5)) - 1)) \\ & + q_0 \cos(\omega t). \end{aligned}$$



## The Bilateral Laplace Transform

2.6.4. Remark. The Laplace transform as we have defined it is more properly called the ***unilateral Laplace transform***. More generally, the ***bilateral Laplace transform*** is defined as

$$(\tilde{\mathcal{L}}f)(p) := \int_{-\infty}^{\infty} f(t)e^{-pt} dt.$$

Of course, we have

$$\mathcal{L}f = \tilde{\mathcal{L}}(Hf)$$

where  $H$  is the Heaviside function. The bilateral Laplace transform is used less frequently than the (unilateral) Laplace transform, both play an important role in engineering applications, as you will see in next term's Signals & Systems course.

## Analyticity of the Laplace Transform of a Function

For a deeper understanding of the Laplace transform, it is useful to consider  $\mathcal{L}f$  as a function on the complex plane, i.e., to analyze the properties of

$$F(z) = (\mathcal{L}f)(z) \quad \text{for } z = p + iq \in \mathbb{C}.$$

A function satisfying (2.6.1) has the property that there exists some  $M > 0$ ,  $\beta \in \mathbb{R}$  such that

$$|f(t)| \leq Me^{\beta t} \quad \text{for all } t > 0. \quad (2.6.6)$$

Then, for  $z = p + iq$ , we have

$$\begin{aligned} |F(z)| &\leq \int_0^\infty |f(t)e^{-zt}| dt = \int_0^\infty |f(t)|e^{-pt} dt \\ &\leq M \int_0^\infty e^{-(p-\beta)t} dt = \frac{M}{p-\beta} \end{aligned} \quad (2.6.7)$$

for  $p > \beta$ .

## Analyticity of the Laplace Transform of a Function

The estimate (2.6.7) implies that

- (i)  $F(z) = \mathcal{L}f(z)$  exists at least for all  $z \in \mathbb{C}$  with  $\operatorname{Re} z > \beta$  and
- (ii)  $F(z)$  is holomorphic for all  $z \in \mathbb{C}$  with  $\operatorname{Re} z > \beta$ , since the integral converges absolutely and the function  $g(z) = e^{zt}$  is complex differentiable for any  $t \in [0, \infty)$ .

In particular, if  $F$  has poles, they must lie in the region  $\operatorname{Re} z \leq \beta$ .

Viewing  $\mathcal{L}f$  as a function on the complex plane is useful for extending the Laplace transform:

**2.6.5. Example.** The Laplace transform  $\mathcal{L}f$  of the function given by

$$f(t) = e^{at} \quad \text{for some } a > 0$$

is

$$(\mathcal{L}f)(p) = \int_0^\infty e^{(a-p)t} dt = \frac{1}{p-a} \quad \text{for } p > a. \quad (2.6.8)$$

## Analytic Continuation of the Laplace Transform

For  $p \leq a$  the integral (2.6.8) diverges, so the Laplace transform of  $f$  is only defined for  $p > a$ . (Compare with the table on Slide 414.)

However, the Laplace transform can be analytically continued into the complex plane by setting

$$F(z) := \frac{1}{z-a} \quad \text{for } z \neq a.$$

The function  $F$  is holomorphic on the region  $\mathbb{C} \setminus \{a\}$  and equals  $\mathcal{L}f$  on the real interval  $(a, \infty)$ . By Theorem 2.2.23,  $F$  is the only such function. We may therefore define

$$(\mathcal{L}f)(p) = \frac{1}{p-a} \quad \text{for } p \neq a.$$

where the Laplace transform exists by definition for  $p > a$  and is given by analytic continuation for  $p < a$ .

# The Bromwich Integral

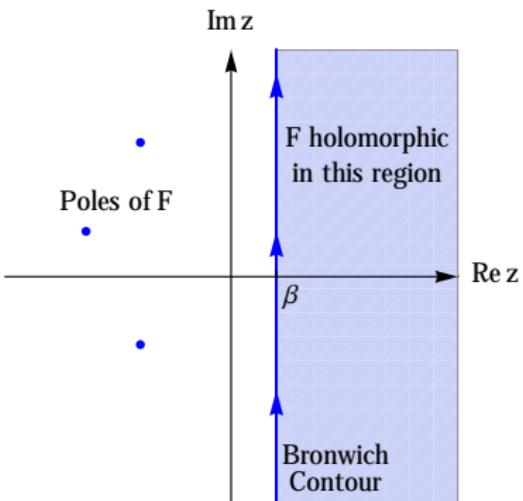
2.6.6. Definition. Let  $\Omega \subset \mathbb{C}$  be an open set,  $\beta \in \mathbb{R}$  and  $F: \Omega \rightarrow \mathbb{C}$  analytic for all  $z \in \mathbb{C}$  with  $\operatorname{Re} z \geq \beta$ . Then the **Bromwich integral of  $F$**  is defined as

$$(\mathcal{M}F)(t) = \frac{1}{2\pi i} \int_{\mathcal{C}^*} e^{pt} F(p) dp, \quad (2.6.9)$$

where  $\mathcal{C} = \{z \in \mathbb{C}: \operatorname{Re} z = \beta\}$  is the **Bromwich contour**, oriented positively if the contour is closed on the left (i.e., the line is traversed in the direction of positive imaginary part.)

Often, the integral (2.6.9) is written

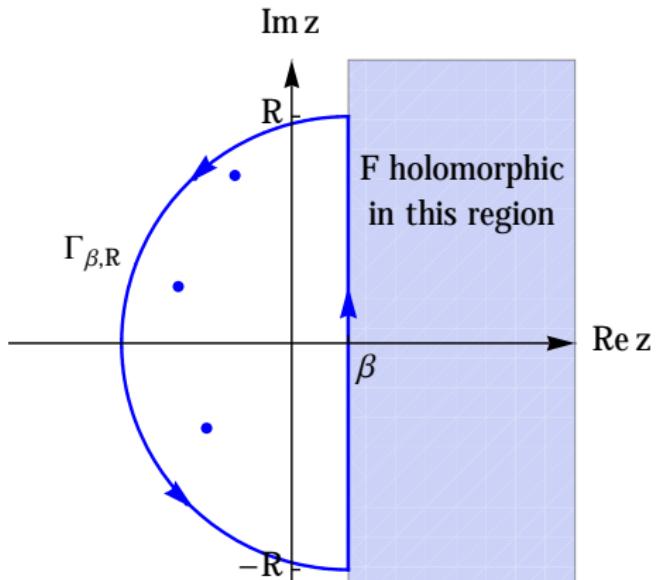
$$(\mathcal{M}F)(t) = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} e^{pt} F(p) dp.$$



# Evaluating the Bromwich Integral

Reference Section 4.6, G. Evans, J. Blackledge and P. Yardley, **Analytic Methods for Partial Differential Equations**, Springer 2001 (excerpt uploaded to SAKAI).

For  $t > 0$ , the Bromwich integral  $\mathcal{F}(t)$  is usually calculated by closing the contour on the left and applying the residue theorem.



## Evaluating the Bromwich Integral

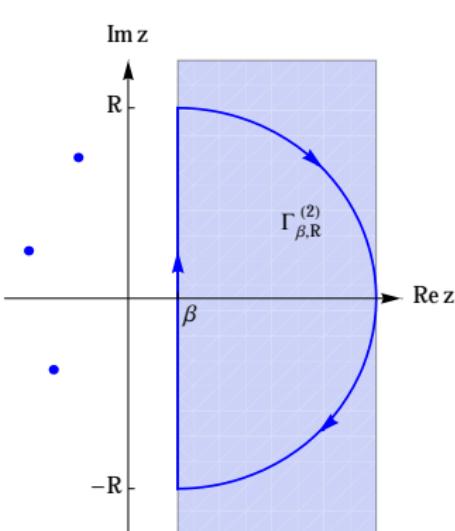
Often, but not in general, the integral over the curve  $\Gamma_{\beta,R}$  will vanish as  $R \rightarrow \infty$ . For the analysis it is useful to note that the curve  $\Gamma_{\beta,R}$  can be parametrized by  $\gamma_{\beta,R}(s) = \beta + Re^{is}$ ,  $\pi/2 \leq s \leq 3\pi/2$ . The integral then becomes

$$\begin{aligned}\int_{\Gamma_{\beta,R}} e^{pt} F(p) dp &= \int_{\pi/2}^{3\pi/2} e^{t(\beta + R \exp(is))} F(\beta + Re^{is}) iRe^{is} ds \\&= e^{\beta t} \int_0^\pi e^{tR \exp(is+i\pi/2)} F(\beta + Re^{i(s+\pi/2)}) iRe^{is+i\pi/2} ds \\&= ie^{\beta t} \int_0^\pi e^{itR \exp(is)} F(\beta + iRe^{is}) iRe^{is} ds \\&= ie^{\beta t} \int_{C_R} e^{itp} F(\beta + ip) dp\end{aligned}$$

where  $C_R$  is a semi-circle of radius  $R$  in the upper half-plane. In most cases, Jordan's Lemma 2.2.15 can then be used to show that the integral vanishes as  $R \rightarrow \infty$ .

# Evaluating the Bromwich Integral

For  $t < 0$  we close the contour on the right. We can use a similar argument as before to treat the auxiliary contour  $\Gamma_{\beta,R}^{(2)}$ .



$$\begin{aligned}
 & \int_{\Gamma_{\beta,R}^{(2)}} e^{pt} F(p) dp \\
 &= - \int_{-\pi/2}^{\pi/2} e^{t\gamma_{\beta,R}(s)} F(\beta + Re^{is}) iRe^{is} ds \\
 &= - \int_0^\pi e^{t\gamma_{\beta,R}(s-\pi/2)} F(\beta + Re^{i(s-\pi/2)}) \\
 &\quad \times iRe^{i(s-\pi/2)} ds \\
 &= -e^{\beta t} \int_0^\pi e^{-itR \exp(is)} F(\beta - iRe^{is}) Re^{is} ds \\
 &= ie^{\beta t} \int_{C_R} e^{i|t|p} F(\beta - ip) dp
 \end{aligned}$$

# Evaluating the Bromwich Integral

2.6.7. Example. We want to calculate  $\mathcal{M}F(t)$  for  $t \in \mathbb{R}$  and

$$F(s) = \frac{1}{s^4 - 1}.$$

We need to evaluate

$$(\mathcal{M}F)(t) = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} e^{pt} F(p) dp.$$

Note that  $F(z)$  has simple poles at  $z = 1, -1, i, -i$ . For our integral, we choose  $\beta = 2$ . The choice of  $\beta$  is arbitrary as long as  $\operatorname{Re} \beta > 1$ . It is not difficult to see that

$$\sup_{\theta \in [0, \pi]} |F(2 + iR\operatorname{e}^{i\theta})| = O(1/R^4) \quad \text{as } R \rightarrow \infty.$$

For  $t > 0$  we can therefore apply Jordan's Lemma and find that

$$\mathcal{M}F(t) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} e^{pt} F(p) dp = \sum_{z=1, -1, i, -i} \operatorname{res}_z e^{pt} F(p)$$

# Evaluating the Bromwich Integral

Evaluating the residues, we obtain

$$\mathcal{M}F(t) = \frac{e^t}{4} - \frac{e^{-t}}{4} - \frac{e^{it}}{4i} + \frac{e^{-it}}{4i} = \frac{\sinh t - \sin t}{2}$$

for  $t > 0$ . For  $t < 0$  we close the contour on the right. As before, we can see that the integral along the auxiliary contour  $\Gamma_{2,R}^{(2)}$  vanishes as  $R \rightarrow \infty$ . Since  $F$  is holomorphic to the right of  $\beta = 2$ , we find

$$\mathcal{M}F(t) = 0$$

for  $t < 0$  by Cauchy's theorem. We obtain the value of the integral at  $t = 0$  through continuity, so we have

$$\mathcal{M}F(t) = \begin{cases} \frac{1}{2}(\sinh t - \sin t), & t > 0, \\ 0, & t \leq 0. \end{cases}$$

## The Mellin Inversion Formula

It now turns out that the Bromwich integral is actually the inverse of the Laplace transform:

**2.6.8. Theorem.** The integral (2.6.9) is the inverse of the Laplace transform. In particular, if  $f$  is continuous on  $[0, \infty)$ , continuously differentiable on  $(0, \infty)$  and satisfies (2.6.1), then

$$f(s) = [\mathcal{M}(\mathcal{L}f)](s) \quad \text{for all } s \in [0, \infty).$$

For this reason, (2.6.9) is called the **Mellin inversion formula** for the Laplace transform.

We will defer the proof until the next section.

# ODEs with Variable Coefficients

The Laplace transform is of limited help when treating ODEs with variable coefficients. The reason is that even in simple cases, a product of the form  $t^n x(t)$  becomes a derivative of  $\mathcal{L}x$  (see the table of Laplace transforms).

2.6.9. Example. **Bessel's differential equation of order 0** is given by

$$tx'' + x' + tx = 0.$$

We will search for a solution with  $x(0) = 1$ . Taking the Laplace transform and writing  $X := \mathcal{L}x$ , we see that

$$\mathcal{L}(x')(p) = pX(p) - x(0),$$

$$\mathcal{L}((\cdot) \cdot x)(p) = -[\mathcal{L}(x)]'(p) = -X'(p),$$

$$\begin{aligned}\mathcal{L}((\cdot) \cdot x'')(p) &= -[\mathcal{L}(x'')]'(p) \\ &= -[p^2 X(p) - px(0) - x'(0)]' \\ &= -2pX(p) - p^2 X'(p) + x(0)\end{aligned}$$

# ODEs with Variable Coefficients

We hence obtain the equation

$$\begin{aligned} 0 &= -2pX(p) - p^2X'(p) + 1 - X'(p) + pX(p) - 1 \\ &= -(p^2 + 1)X' - pX \end{aligned}$$

or

$$X' + \frac{p}{p^2 + 1}X = 0.$$

This is a linear, homogeneous first-order differential equation. The general solution is

$$X(p; c) = \frac{c}{\sqrt{p^2 + 1}}, \quad c \in \mathbb{R}.$$

We now need to find the inverse Laplace transform of  $X(p; c)$ . It turns out (homework!) that it is given by

$$x(t; c) = \frac{c}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{e^{pt}}{\sqrt{p^2 + 1}} dp = \frac{c}{\pi} \int_0^\pi \cos(t \sin \theta) d\theta.$$

## ODEs with Variable Coefficients

The solution with  $c = 1$  satisfies  $x(0; 1) = 1$ . This solution is called the **Bessel function of the first kind of order 0** and is written

$$J_0(t) := \frac{1}{\pi} \int_0^\pi \cos(t \sin \theta) d\theta \quad (2.6.10)$$

This integral representation can not be simplified in any essential way; we will later derive a series representation of  $J_0$ , but it remains an inescapable fact that  $J_0$  can not be written as an elementary function.

In general, the Bessel function  $J_n$  of the first kind of order  $n > 0$  is a solution of

$$t^2 x'' + t x' + (t^2 - n^2)x = 0, \quad x(0) = 0.$$

(Note the difference in the initial value!)

## A Convolution Product for the Laplace Transform

In Example 2.6.9 we have dealt with the product of two functions that were transformed, i.e., we calculated  $\mathcal{L}(t^n y(t))$ . In practice, we are often faced with having to evaluate the inverse Laplace transform of a product of two functions. The problem we would like to discuss now is: does there exist a “product” of functions, denoted by “ $*$ ” such that

$$\mathcal{L}(f * g) = (\mathcal{L}f) \cdot (\mathcal{L}g)?$$

Interestingly, the answer is affirmative; the product is given by

$$(f * g)(t) := \int_0^t f(t-s)g(s) ds. \quad (2.6.11)$$

The product defined in (2.6.11) is called the **convolution** of  $f$  and  $g$ . (Compare it to the convolution of sequences we defined in Honors Math II in the first term.) It is commutative, associative and distributive with the pointwise addition of functions. We will discuss the existence of a unit element for the convolution later.

# A Convolution Product for the Laplace Transform

2.6.10. Theorem. The product defined in (2.6.11) satisfies

$$\mathcal{L}(f * g) = (\mathcal{L}f) \cdot (\mathcal{L}g).$$

Proof.

We simply write out the definition,

$$\begin{aligned}\mathcal{L}(f * g)(p) &= \int_0^\infty \int_0^t e^{-pt} f(t-s)g(s) ds dt \\ &= \iint_R e^{-pt} f(t-s)g(s) dA.\end{aligned}$$

Here  $R$  is the “triangular” region

$$\begin{aligned}R &= \{(s, t) \in \mathbb{R}^2 : t \in [0, \infty), 0 \leq s \leq t\} \\ &= \{(s, t) \in \mathbb{R}^2 : s \in [0, \infty), s \leq t < \infty\}.\end{aligned}$$

# A Convolution Product for the Laplace Transform

Proof (continued).

It follows that

$$\begin{aligned}\mathcal{L}(f * g)(p) &= \iint_R e^{-pt} f(t-s)g(s) dA \\ &= \int_0^\infty g(s) \int_s^\infty e^{-pt} f(t-s) dt ds\end{aligned}$$

We substitute  $\tau = t - s$  in the inner integral to obtain

$$\begin{aligned}\mathcal{L}(f * g)(p) &= \int_0^\infty g(s) \int_0^\infty e^{-p(\tau+s)} f(\tau) d\tau ds \\ &= \int_0^\infty g(s) e^{-ps} ds \cdot \int_0^\infty e^{-p\tau} f(\tau) d\tau \\ &= (\mathcal{L}f) \cdot (\mathcal{L}g).\end{aligned}$$



# A Convolution Product for the Laplace Transform

2.6.11. Example. We calculate the inverse Laplace transform of the function

$$F(p) = \frac{a}{p^2(p^2 + a^2)}.$$

Since

$$(\mathcal{L}t)(p) = \frac{1}{p^2}, \quad (\mathcal{L}\sin(at))(p) = \frac{a}{p^2 + a^2}$$

we have

$$\mathcal{L}(t * \sin(at)) = (\mathcal{L}t)(p) \cdot (\mathcal{L}\sin(at))(p) = \mathcal{L}F(p).$$

It follows that

$$f(t) = \mathcal{L}^{-1}F(t) = t * \sin(at) = \int_0^t (t-s) \sin(as) ds = \frac{at - \sin(at)}{a^2}.$$

## A Green's Function for a 2<sup>nd</sup> Order Linear ODE

We consider the linear, second order, inhomogeneous ODE with constant coefficients together with initial values at  $t = 0$ ,

$$ay'' + by' + cy = f(t), \quad y(0) = y_0, \quad y'(0) = y_1.$$

Taking the Laplace transform of this equation and writing  $Y = \mathcal{L}y$  and  $F = \mathcal{L}f$  we obtain

$$(ap^2 + bp + c)Y - (ap + b)y_0 - ay_1 = F(p).$$

We can write the solution of this algebraic equation as

$$Y(p) = \Phi(p) + \Psi(p)$$

with

$$\Phi(p) = \frac{(ap + b)y_0 + ay_1}{ap^2 + bp + c} \quad \text{and} \quad \Psi(p) = \frac{F(p)}{ap^2 + bp + c}.$$

# A Green's Function for a 2<sup>nd</sup> Order Linear ODE

Setting

$$\phi = \mathcal{L}^{-1}\Phi \quad \text{and} \quad \psi = \mathcal{L}^{-1}\Psi$$

we see that  $\phi$  solves the homogeneous problem

$$ay'' + by' + cy = 0, \quad y(0) = y_0, \quad y'(0) = y_1,$$

and  $\psi$  solves the inhomogeneous problem with vanishing initial conditions,

$$ay'' + by' + cy = f(t), \quad y(0) = 0, \quad y'(0) = 0.$$

The general solution of the inhomogeneous initial value problem is given by

$$y(t) = \phi(t) + \psi(t) = y_{\text{hom}}(t) + y_{\text{part}}(t).$$

We are particularly interested in  $y_{\text{part}}(t)$ , because this is usually the cumbersome part of the solution.

# A Green's Function for a 2<sup>nd</sup> Order Linear ODE

We have

$$\begin{aligned}y_{\text{part}}(t) &= \psi(t) = \mathcal{L}^{-1} \left( \frac{F(p)}{ap^2 + bp + c} \right) (t) \\&= f * g(t)\end{aligned}$$

where

$$\mathcal{L}g(p) = \frac{1}{ap^2 + bp + c}. \quad (2.6.12)$$

The function  $g$  is called a ***Green's function, fundamental solution, propagator*** or (sometimes) a ***transfer function*** for the differential equation. Here, we see that this function is very useful, because once we have determined it, we can immediately write down the solution for any inhomogeneous version of the differential equation as an integral. This has great analytic advantages in the analysis of the properties of solutions. The theory of Green's functions is extremely important for the solution of partial differential equations.

# A Green's Function for a 2<sup>nd</sup> Order Linear ODE

2.6.12. Example. The Green's function for the equation

$$y'' + 4y = f(t)$$

is given by

$$\mathcal{L}g(p) = \frac{1}{p^2 + 4} \quad \Rightarrow \quad g(t) = \frac{1}{2} \sin(2t).$$

A particular solution to the inhomogeneous equation is given by

$$y_{\text{part}}(t) = f * g(t) = \frac{1}{2} \int_0^t \sin(2s)f(t-s) ds.$$

Evaluating this integral is often more convenient than using the variation-of-parameters formula we have developed earlier.

## Impulses

We now consider the following problem: a force  $F(t)$  is to act on a system over a certain time interval, conferring an **impulse** (change of momentum)  $I = \int F(t) dt$  on the system. For example, this could take the form of a force acting on a damped harmonic oscillator,

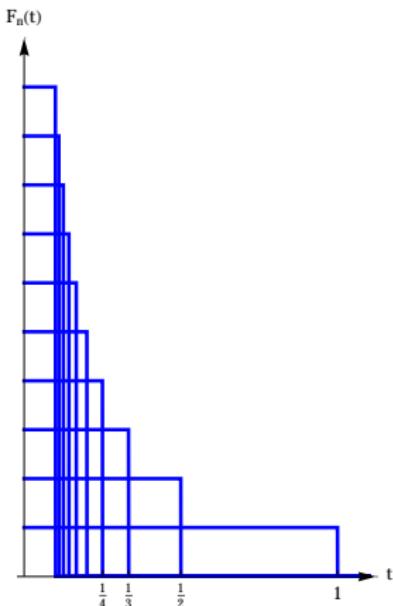
$$ax'' + bx' + cx = F(t). \quad (2.6.13)$$

If we keep the impulse  $I = 1$  constant and let the force remain constant during the time that it acts, we can take any function of the form

$$F_n(t) = \begin{cases} n & 0 \leq t \leq 1/n \\ 0 & \text{otherwise} \end{cases}$$

In this way, by choosing  $n$  large, the same impulse is transferred in an arbitrarily short time span. We would like to be able to analyze the behaviour of the system (2.6.13) with  $F = F_n$  when  $n \rightarrow \infty$ .

# Impulses



The plot shows the graph of  $F_n$  for  $n = 1, 2, \dots, 11$ . It is clear that

$$\begin{aligned} F_n(t) &\xrightarrow{n \rightarrow \infty} 0 & \text{for } t \neq 0, \\ F_n(0) &\xrightarrow{n \rightarrow \infty} \infty. \end{aligned}$$

The sequence of functions  $(F_n)$  does not converge to a function in a useful sense; in particular, it does not converge in the norms

$$\|f\|_\infty := \sup_{t \in \mathbb{R}} |f(t)|, \quad \|f\|_1 := \int_{-\infty}^{\infty} |f(t)| dt.$$

However, physically the limit of the functions  $F_n$  is understood to represent an **instantaneous impulse** that transfers the impulse  $I$  to the system. Such an instantaneous impulse occurs extremely often in physics (solid body mechanics is based on it, for example), and we would like to be able to incorporate it in our ODE theory.

## Impulses

From our previous discussion of Green's function, we know that a particular solution to the damped oscillator equation (2.6.13) with  $F = F_n$  is given by

$$x_{\text{part}}(t) = g * F_n(t)$$

where  $g$  is the Green's function and

$$g * F_n(t) = \int_0^t F_n(s)g(t-s) ds.$$

We fix  $t > 0$ . For sufficiently large  $n$ ,  $t > 1/n$  and

$$g * F_n(t) = n \int_0^{1/n} g(t-s) ds.$$

Assuming that  $g$  is continuous at  $t$ , we have  $g(t-s) = g(t) + r_t(s)$  with  $r_t(s) \rightarrow 0$  as  $s \rightarrow 0$ .

$$g * F_n(t) = g(t) + n \int_0^{1/n} r_t(s) ds$$

# Impulses

Now

$$\left| n \int_0^{1/n} r_t(s) ds \right| \leq \sup_{0 \leq s \leq 1/n} |r_t(s)| \xrightarrow{n \rightarrow \infty} 0$$

so for every fixed  $t > 0$  we have

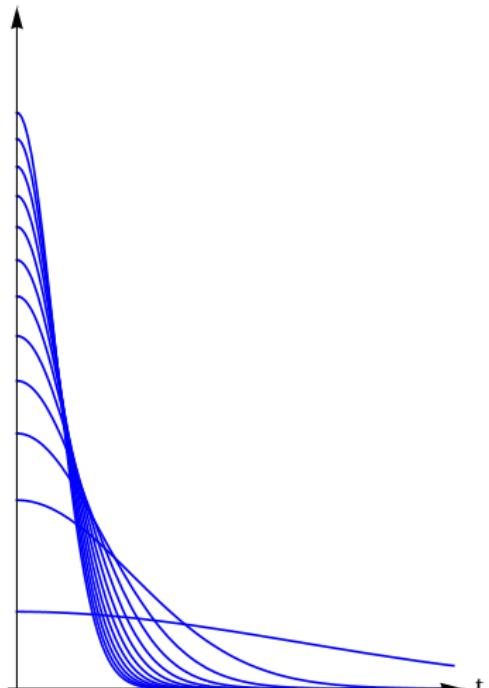
$$g * F_n(t) \xrightarrow{n \rightarrow \infty} g(t).$$

Thus, while the limit of the  $F_n$  does not exist, the limit of the corresponding solutions to the ODE does, and is actually equal to the Green's function. For this reason, the Green's function is also known as the **impulse response** of the physical system.

We may ask whether we would get the same result using a different set of impulses, e.g.,

$$F_n(t) = 2 \sqrt{\frac{n}{\pi}} e^{-nt^2}$$

# Impulses

 $F_n(t)$ 

In fact, for this sequence  $(F_n)$  we also obtain that

$$g * F_n(t) \rightarrow g(t)$$

for all  $t$ . One approach to analyzing an “instantaneous impulse” would be to regard it as the limit of any sequence of impulses such that

$$F_n(t) \xrightarrow{n \rightarrow \infty} 0 \quad \text{for } t \neq 0, n \rightarrow \infty,$$

$$\int_{\mathbb{R}} F_n(t) dt = 1 \quad \text{for all } n \in \mathbb{N}.$$

In this approach an “instantaneous impulse” would **be** such a sequence  $(F_n)$ , or rather the equivalence class of all such sequences.

## The “Delta Function”

Since physicists (and engineers) like names for such objects, one generally denotes this equivalence class of sequences by

$$[(F_n)] =: \delta,$$

so that one writes

$$g * \delta(t) = g(t) \quad \text{instead of} \quad \lim_{n \rightarrow \infty} g * F_n(t) = g(t).$$

This notation was first introduced by P.A.M. Dirac, a physicist who made great contributions to quantum theory. The “delta function”  $\delta(t)$  is of course not a function at all, and has no meaning except within an integral. Even so, we can symbolically write the Green’s function as the solution to

$$ax'' + bx' + cx = \delta(t)$$

and think of  $\delta(t)$  as denoting an instantaneous impulse at time  $t$ .

## Generalized Functions (Distributions)

All of this is fine as far as it goes, but of course physicists and engineers started using the delta function in ways which are hardly well-defined. For example, in quantum field theory, one encounters  $\delta(t)^2$  - this has no mathematical meaning!

It was not until the 1940's that a mathematical framework for the delta function and similar objects was found. This led to an extraordinarily fruitful area of mathematics and opened doors to many applications, such as "weak solutions" to differential equations and an extension of the concept of differentiability. This subject will be studied in greater depth in the graduate course "Vv557 Methods of Applied Mathematics".

# The Laplace Transform

For our purposes, we will pretend that  $\delta$  represents a function (or other mysterious object) such that

$$\delta(x) = 0 \quad \text{for } x \neq 0,$$

$$\int_{\Omega} \delta(x)f(x) dx = f(0) \quad \text{for any } \Omega \ni 0 \text{ and any function } f \text{ defined at } x = 0.$$

These properties can be embedded in a rigorous mathematical framework that extends the familiar calculus of functions to include (among other objects) the  $\delta$ -“function”. For now, we will content ourselves with solving ODEs where  $\delta$  appears using the Laplace transform:

$$(\mathcal{L}\delta)(p) = \int_0^{\infty} e^{-pt} \delta(t) dt = e^{-pt} \Big|_{t=0} = 1.$$

# Impulsive Differential Equations

2.6.13. Example. We want to solve the initial value problem

$$x'' + 2x' + x = \delta(t - 1) + 2\delta(t - 3), \quad x(0) = 0, \quad x'(0) = 0.$$

Applying the Laplace transform and writing  $X = \mathcal{L}x$ , we obtain

$$p^2X + 2pX + X = e^{-p} + 2e^{-3p}$$

yielding

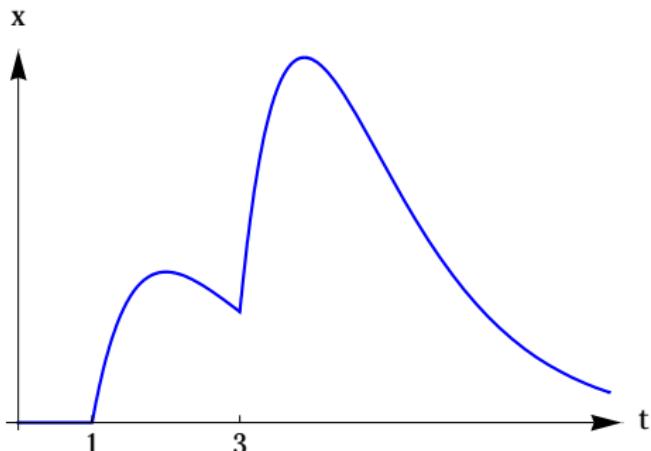
$$X(p) = \frac{e^{-p}}{(p+1)^2} + \frac{2e^{-3p}}{(p+1)^2}.$$

The inverse transform is then

$$x(t) = (t-1)e^{-(t-1)}H(t-1) + (t-3)e^{-(t-3)}H(t-3)$$

## Impulsive Differential Equations

The solution is plotted below. We can see clearly where the damped oscillator is given a “kick” at  $t = 1$  and  $t = 3$ .



Note that the solution is continuous, but not differentiable. In other words, the left-hand side of the ODE which contains derivatives of first and second order is not defined in a “classical” sense.

Such a solution is called a **weak solution** to a differential equation. In practical applications, we will often have linear inhomogeneous equations with discontinuous or impulsive inhomogeneities. In such cases, the solutions will in general only be weak solutions.

## Green's Functions

Finally, we note that the Green's function for a differential equation of the form

$$ax'' + bx' + cx = f(t)$$

is indeed found by solving

$$ax'' + bx' + cx = \delta(t), \quad x(0) = 0, \quad x'(0) = 0, \quad (2.6.14)$$

using the Laplace transform. In fact, the solution of (2.6.14) is by definition the Green's function for the differential operator

$$L = a \frac{d^2}{dx^2} + b \frac{d}{dx} + c.$$

12. Complex Differentiability

13. Properties of Holomorphic Functions

14. Singularities and Poles

15. Residue Calculus

16. The Heaviside Operator Method

17. The Laplace Transform

18. The Fourier Transform

# The Fourier Transform

The Laplace transform is not the only integral transformation of practical use. Another, even more widely applied example, is given by the **Fourier transform** of a function  $f: \mathbb{R} \rightarrow \mathbb{C}$ ,

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx. \quad (2.7.1)$$

We see immediately that  $\hat{f}$  exists for all  $\xi \in \mathbb{R}$  if

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty,$$

i.e., if  $f$  is absolutely integrable. While (2.7.1) appears similar to the laplace transform, there are two important differences:

- (i) The Fourier transform integrates  $f$  over all of  $\mathbb{R}$ , not just  $[0, \infty)$ .
- (ii) The exponential term contains the complex  $i$  and is therefore an oscillating, not a decaying function.

## Properties of the Fourier Transform

A physical motivation for the Fourier transform will be given in a later part of the course. For now, we will investigate the Fourier transform purely on its own merits. We first discuss some basic properties.

If  $f$  is absolutely integrable, then

$$\sup_{\xi \in \mathbb{R}} |\widehat{f}(\xi)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| \cdot |e^{-i\xi x}| dx =: C < \infty,$$

so the Fourier transform of an absolutely integrable function is bounded.

We will assume from here on that a function  $f$  of which we calculate the Fourier transform is absolutely integrable.

# Properties of the Fourier Transform

If  $f$  is differentiable, then

$$\begin{aligned}
 \widehat{(f')}(ξ) &= \frac{1}{\sqrt{2π}} \int_{-∞}^{∞} f'(x) e^{-iξx} dx \\
 &= 0 - \frac{1}{\sqrt{2π}} \int_{-∞}^{∞} f(x) (-iξ) e^{-iξx} dx \\
 &= iξ \cdot \widehat{f}(ξ),
 \end{aligned} \tag{2.7.2}$$

where we have used that  $f$  is absolutely integrable, so  $\lim_{x \rightarrow \pm\infty} f(x) = 0$ .

Similarly,

$$\begin{aligned}
 \frac{d}{dξ} \widehat{f}(ξ) &= \frac{1}{\sqrt{2π}} \int_{-∞}^{∞} f(x) \frac{d}{dξ} e^{-iξx} dx \\
 &= \frac{1}{\sqrt{2π}} \int_{-∞}^{∞} (-ix) f(x) e^{-iξx} dx \\
 &= \widehat{(-ix)f}(ξ),
 \end{aligned} \tag{2.7.3}$$

if the last integral exists, e.g., if  $x \cdot f(x)$  is absolutely integrable.

# The Fourier Transform

2.7.1. Example. Consider the function  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = e^{-x^2/2}$ . The Fourier transform of  $f$  is given by

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} e^{-x^2/2} dx.$$

This integral can be calculated directly, e.g., through contour integration in the complex plane. We can, however, also find  $\widehat{f}$  through the properties of the Fourier transform: consider

$$g(x) = -xe^{-x^2/2} = -i(-ix)f(x) = \frac{d}{dx}f(x).$$

By (2.7.2) and (2.7.3),

$$\widehat{(f')}(\xi) = i\xi\widehat{f}(\xi), \quad \widehat{(-ix)f}(\xi) = \widehat{f}'(\xi).$$

# The Fourier Transform

It follows that

$$\hat{f}'(\xi) = -\xi \hat{f}(\xi) \quad (2.7.4)$$

Furthermore,

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix \cdot 0} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-x^2/2} dx = 1$$

The ODE (2.7.4) with initial condition  $\hat{f}(0) = 1$  has the unique solution

$$\hat{f}(\xi) = e^{-\xi^2/2}.$$

Thus the function  $f$  satisfies  $\hat{f} = f$ ; it is a **fixed point** of the Fourier transform.

## Decay of the Fourier Transform

Suppose that a function  $f$  is absolutely integrable,  $n$  times differentiable and that  $f^{(n)}$  is absolutely integrable. Then we can apply (2.7.2) repeatedly to see that the Fourier transform of the  $n$ th derivative of  $f$  is given by

$$\widehat{f^{(n)}}(\xi) = (i\xi)^n \cdot \widehat{f}(\xi).$$

Since the Fourier transform of  $f^{(n)}$  is bounded, this implies that

$$|\widehat{f}(\xi)| = |\xi|^{-n} \underbrace{|\widehat{f^{(n)}}(\xi)|}_{\leq C < \infty} = O(|\xi|^{-n}) \quad \text{as } |\xi| \rightarrow \infty. \quad (2.7.5)$$

This illustrates a general principle:

“smoothness of  $f$  is equivalent to decay of  $\widehat{f}$ .”

## Decay Behavior at Infinity

2.7.2. Definition. Let  $\Omega \subset \mathbb{R}$  be bounded and  $f: \mathbb{R} \setminus \Omega \rightarrow \mathbb{C}$ .

- (i) If  $f(x) = O(x^{-n})$  as  $|x| \rightarrow \infty$  for some  $n > 0$ , then  $f$  is said to have **polynomial decay** at infinity.
- (ii) If  $f(x) = O(x^{-n})$  as  $|x| \rightarrow \infty$  for all  $n > 0$ , then  $f$  is said to have **faster-than-polynomial decay** at infinity.
- (iii) If  $f(x) = O(e^{-b|x|})$  as  $|x| \rightarrow \infty$  for some  $b > 0$ , then  $f$  is said to have **exponential decay** at infinity.

## 2.7.3. Examples.

- (i) The function  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = 1/(1 + x^2)$  has polynomial decay at infinity.
- (ii) The function  $f: \mathbb{R} \setminus \{0\}$ ,  $f(x) = |x|^{-\ln|x|} = e^{-(\ln|x|)^2}$  has faster-than-polynomial decay at infinity, but does not decay exponentially.
- (iii) The function  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = e^{-x^2}$  has exponential decay at infinity.

## Smoothness and Decay of the Fourier Transform

If  $f \in C^\infty(\mathbb{R})$  and all derivatives of  $f$  are absolutely integrable, then (2.7.5) implies that  $\hat{f}$  has faster-than-polynomial decay at infinity.

We now ask what properties  $f$  needs to have to ensure that  $\hat{f}$  has exponential decay at infinity. From our previous discussion, we would need to require  $f$  to be “more smooth” than  $C^\infty$ .

Recall that a function  $f$  of a real variable is called (real) analytic at  $x_0 \in \mathbb{R}$  if it has a power series representation near  $x_0$  with positive radius of convergence. An analytic function is  $C^\infty$ , but the converse is not necessarily true, so analytic functions might be considered “more smooth” than simple  $C^\infty$  functions.

It turns out that discussing analyticity is best done in the context of complex variables.

## Complex Analytic Functions

If a function  $f$  has a power series representation in the vicinity of some  $x_0 \in \mathbb{R}$ , then the power series has a radius of convergence  $\varrho$  that extends into the complex plane. The power series is analytic for all  $|z - x_0| < \varrho$  and equal to  $f$  for all real  $x$  with  $|x - x_0| < \varrho$ . By Theorem 2.2.23, this gives a unique analytic continuation of  $f$  into the complex plane.

If  $f$  is (real) analytic for all  $x_0 \in \mathbb{R}$ , we can find such a disk of analyticity in the complex plane centered at each  $x_0$ . However, it is possible that the radius of convergence  $\varrho$  (which of course depends on the point  $x_0$ ) becomes progressively smaller as  $|x_0|$  becomes larger.

We will discuss functions that are **complex analytic**, i.e., analytic within a “strip” of fixed width centered on the real axis. In addition, we will assume a moderate decay condition for these functions that ensures their integrability.

Reference E. M. Stein and R. Shakarchi, **Complex Analysis**, Section 4.1.

## Functions Analytic on a Strip

2.7.4. Definition. Let  $a > 0$  be some constant. We define the set  $\mathcal{F}_a$  of functions  $f: S_a \rightarrow \mathbb{C}$ ,

$$S_a := \{z \in \mathbb{C}: |\operatorname{Im} z| < a\},$$

such that

- (i)  $f$  is analytic on  $S_a$  and
- (ii) there exists a constant  $A > 0$  such that

$$|f(x + iy)| \leq \frac{A}{1 + x^2} \quad \text{for all } x \in \mathbb{R} \text{ and } |y| < a.$$

## 2.7.5. Examples.

- (i)  $e^{-z^2} \in \mathcal{F}_a$  for all  $a > 0$ .
- (ii)  $\frac{1}{z^2 + c^2} \in \mathcal{F}_a$  for all  $a < c$ .
- (iii)  $\frac{1}{\cosh(\pi z)} \in \mathcal{F}_a$  for all  $a < 1/2$ .

# Exponential Decay of the Fourier Transform

Our first main result is the following:

**2.7.6. Theorem.** Let  $f \in \mathcal{F}_a$  for some  $a > 0$ . Then for any  $0 \leq b < a$  there exists a constant  $B > 0$  such that

$$|\widehat{f}(\xi)| \leq Be^{-b|\xi|} \quad \text{for all } \xi \in \mathbb{R}. \quad (2.7.6)$$

Thus, complex analyticity of  $f$  implies exponential decay of  $\widehat{f}$ . The decay constant  $b$  is given by the width of the strip in which  $f$  is complex analytic.

**Proof.**

The statement is obvious for  $b = 0$ :

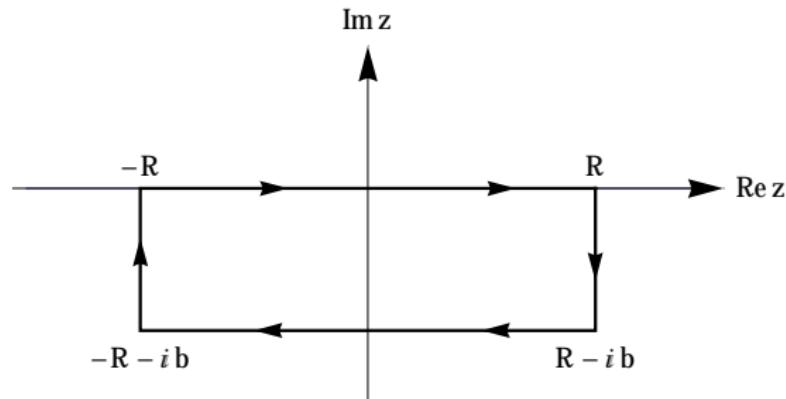
$$|\widehat{f}(\xi)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| \underbrace{|e^{-ix\xi}|}_{=1} dx \leq \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dx}{1+x^2} \leq B$$

where  $A, B > 0$  are suitable real constants.

# Exponential Decay of the Fourier Transform

Proof (continued).

We now prove (2.7.6) for  $0 < b < a$ . First, suppose that  $\xi > 0$ . We define the contour  $\mathcal{C}$  as shown below:



We calculate

$$\int_{\mathcal{C}} g(z) dz \quad \text{where} \quad g(z) = \frac{1}{\sqrt{2\pi}} f(z) e^{-i\xi z}.$$

# Exponential Decay of the Fourier Transform

Proof (continued).

The integrals over the sides of the contour go to zero as  $R \rightarrow \infty$ . For example, the left side gives

$$\begin{aligned} \left| \int_{-R-ib}^{-R} g(z) dz \right| &= \left| \int_b^0 g(-R-it) \cdot (-i) dt \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \int_0^b |f(-R-it)e^{-i\xi(-R-it)}| dt \\ &\leq \frac{1}{\sqrt{2\pi}} \int_0^b \frac{A}{1+R^2} e^{-\xi t} dt \xrightarrow{R \rightarrow \infty} 0. \end{aligned}$$

A similar estimate applies to the right side. By Cauchy's theorem we then have

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-ib) e^{-i\xi(x-ib)} dx.$$

# Exponential Decay of the Fourier Transform

Proof (continued).

This implies that

$$|\hat{f}(\xi)| \leq e^{-b\xi} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{A}{1+x^2} dx}_{=:B}$$

for  $\xi > 0$ . A similar method shows the corresponding estimate for  $\xi < 0$ .  
The only difference is that the line of integration must be shifted up by  $b$ .



## Analytic Theory of the Fourier Transform

2.7.7. Definition. We denote by  $\mathcal{F}$  the set of all complex functions  $f$  such that  $f \in \mathcal{F}_a$  for some  $a > 0$ .

We will now prove a result on the existence of an inverse of the Fourier transform when defined on  $\mathcal{F}$ . This is the basis of the “analytic theory” of the Fourier transform.

2.7.8. Fourier Inversion Theorem. If  $f \in \mathcal{F}$ , then  $\widehat{f}$  exists and

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{i\xi x} d\xi \quad \text{for all } x \in \mathbb{R}. \quad (2.7.7)$$

Before we begin the proof, we note that if  $A > 0$  and  $B \in \mathbb{R}$ , then

$$\int_0^{\infty} e^{-(A+iB)\xi} d\xi = \frac{1}{A+Bi}.$$

# Analytic Theory of the Fourier Transform

## Proof of Theorem 2.7.8.

We write  $d\xi$  for  $(2\pi)^{-1/2} d\xi$  and split the integral,

$$\int_{-\infty}^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi = \int_{-\infty}^0 \hat{f}(\xi) e^{i\xi x} d\xi + \int_0^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi.$$

We first treat the second integral, where we can assume  $\xi > 0$ . Suppose that  $f \in \mathcal{F}_a$  for some  $a > 0$ . Then we choose some  $0 < b < a$  and as in the proof of Theorem 2.7.6 deduce

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(y - ib) e^{-i\xi(y - ib)} dy \quad \text{for } \xi > 0.$$

We then have

$$\int_0^{\infty} \hat{f}(\xi) e^{i\xi x} d\xi = \int_0^{\infty} \int_{-\infty}^{\infty} f(y - ib) e^{-i\xi(y - ib)} e^{i\xi x} dy d\xi$$

# Analytic Theory of the Fourier Transform

Proof of Theorem 2.7.8 (continued).

Exchanging the order of integration, we obtain

$$\begin{aligned}\int_0^\infty \widehat{f}(\xi) e^{i\xi x} d\xi &= \int_{-\infty}^\infty f(y - ib) \int_0^\infty e^{-i\xi(y - ib - x)} d\xi dy \\&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(y - ib) \frac{1}{b + i(y - x)} dy \\&= \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{f(y - ib)}{y - ib - x} dy \\&= \frac{1}{2\pi i} \int_{L_1^*} \frac{f(\zeta)}{\zeta - x} d\zeta\end{aligned}$$

where

$$L_1 = \{z \in \mathbb{C}: z = y - ib, y \in \mathbb{R}\}$$

and the integration along the line is “from left to right.”

# Analytic Theory of the Fourier Transform

Proof of Theorem 2.7.8 (continued).

In the same way, we can prove that

$$\int_{-\infty}^0 \hat{f}(\xi) e^{i\xi x} d\xi = \frac{1}{2\pi i} \int_{L_2^*} \frac{f(\zeta)}{\zeta - x} d\zeta$$

where

$$L_2 = \{z \in \mathbb{C}: z = y + ib, y \in \mathbb{R}\}$$

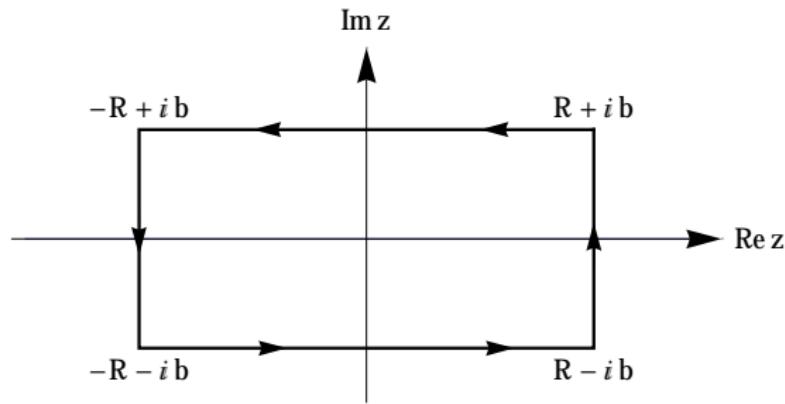
and the integration along the line is “from right to left.” Now consider

$$\frac{1}{2\pi i} \int_{\mathcal{C}_R^*} \frac{f(\zeta)}{\zeta - x} d\zeta$$

where  $\mathcal{C}_R^*$  is the oriented rectangle contour pictured below.

# Analytic Theory of the Fourier Transform

Proof of Theorem 2.7.8 (continued).



The function  $\frac{f(\zeta)}{\zeta - x}$  has a simple pole at  $\zeta = x$  with residue  $f(x)$ . Hence,

$$\frac{1}{2\pi i} \int_{\mathcal{C}_R^*} \frac{f(\zeta)}{\zeta - x} d\zeta = f(x).$$

# Analytic Theory of the Fourier Transform

Proof of Theorem 2.7.8 (continued).

When  $R \rightarrow \infty$ , the integrals along the top and bottom converge to the integrals over  $L_2^*$  and  $L_1^*$ , respectively, while the integrals along the sides vanish. It follows that

$$\begin{aligned} f(x) &= \frac{1}{2\pi i} \int_{L_2^*} \frac{f(\zeta)}{\zeta - x} d\zeta + \frac{1}{2\pi i} \int_{L_1^*} \frac{f(\zeta)}{\zeta - x} d\zeta \\ &= \int_{-\infty}^0 \hat{f}(\xi) e^{i\xi x} d\xi + \int_0^\infty \hat{f}(\xi) e^{i\xi x} d\xi \\ &= \int_{-\infty}^\infty \hat{f}(\xi) e^{i\xi x} d\xi, \end{aligned}$$

completing the proof. □

## The Complex Fourier Transform

We now want to show converses to Theorems 2.7.6 and 2.7.8: if  $f$  decays exponentially, we expect  $\hat{f}$  to be analytic and the inversion formula to hold. To establish this, we need to define the Fourier transform of a complex function:

For  $f: \mathbb{C} \rightarrow \mathbb{C}$  we define, formally, the Fourier transform of  $f$  at  $\xi + i\eta \in \mathbb{C}$  by

$$\hat{f}(\xi + i\eta) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i(\xi+i\eta)x} dx, \quad (2.7.8)$$

where in the integral  $f$  is evaluated on the real axis only. If  $f$  is analytic, this is sufficient to define  $\hat{f}$  uniquely as the values of  $f$  on  $\mathbb{R}$  determine  $f$  uniquely.

## Existence of the Complex Fourier Transform

2.7.9. Theorem. Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  satisfy  $f(x) = O(e^{-b|x|})$  as  $|x| \rightarrow \infty$  for some  $b > 0$ . Then  $\widehat{f}$  exists and is analytic in the strip  $S_b = \{z \in \mathbb{C}: |\operatorname{Im} z| < b\}$ .

Proof.

The integral (2.7.8) will exist if

$$|\widehat{f}(\xi + i\eta)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| e^{\eta x} dx < \infty, \quad (2.7.9)$$

which will be the case if  $|\eta| < b$ , i.e.,  $\xi + i\eta \in S_b$ . Since

$$\frac{d}{dz} \widehat{f}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \frac{d}{dz} e^{-izx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) (-ix) e^{-izx} dx$$

we see that  $\widehat{f}$  is analytic in  $S_b$ . □

## The Fourier Inversion Formula

If we assume only the exponential decay of  $f$  and make no assumption on smoothness whatsoever, then the Fourier inversion formula will not hold in general. There are two main reasons for this:

- (i) If we assume only (exponential) decay for  $f$ , then  $\hat{f}$  will be smooth, but it is quite possible that  $\hat{f}$  does not decay at all, i.e.,  $\hat{f}$  does not need to be integrable. Then the inverse transform does not exist.
- (ii) Two integrable functions that agree almost everywhere (except on a set of measure zero) will have the same integral, i.e., the same Fourier transform. Hence the Fourier transform is not injective if defined on a domain of non-smooth functions. An example of this effect will be explored in the assignments.

Nevertheless, the Fourier inversion formula holds (with suitable modifications) in a wide variety of cases. Generally speaking, if  $f$  is piece-wise continuously differentiable, then the Fourier inversion formula will hold wherever  $f$  is continuous.

# The Fourier Inversion Formula

We state, without proof, the following result:

**2.7.10. Fourier Inversion Theorem.** Suppose that  $f: \mathbb{R} \rightarrow \mathbb{R}$  is absolutely integrable and satisfies the following condition: there exists a set of numbers  $\{a_1, \dots, a_n\}$ ,  $n \in \mathbb{N}$ , such that  $f$  is continuously differentiable on the intervals  $(a_k, a_{k+1})$ , where we set  $a_0 = -\infty$ ,  $a_{n+1} = \infty$ , and such that the one-sided limits of  $f$  and  $f'$  at  $a_1, \dots, a_n$  exist.

Then  $\widehat{f}$  exists and for all  $x \in \mathbb{R}$

$$\frac{f(x^+) + f(x^-)}{2} = \frac{1}{\sqrt{2\pi}} \lim_{R \rightarrow \infty} \int_{-R}^R \widehat{f}(\xi) e^{ix\xi} d\xi,$$

where the limit on the right exists for all  $x \in \mathbb{R}$  and

$$f(x^+) := \lim_{y \searrow x} f(y), \quad f(x^-) := \lim_{y \nearrow x} f(y).$$

## Fourier Transform of Functions with Insufficient Decay

Now suppose that  $f$  is not integrable but satisfies the smoothness conditions of Theorem 2.7.10, i.e.,  $f$  does not decay fast enough at infinity. We then treat the behavior of  $f$  separately as  $x \rightarrow \pm\infty$ , so we set

$$f_+(x) := \begin{cases} f(x) & x \geq 0, \\ 0 & x < 0, \end{cases} \quad f_-(x) := \begin{cases} 0 & x \geq 0, \\ f(x) & x < 0. \end{cases}$$

Suppose that for some  $b > 0$  the function  $g$  defined by

$$g_+(x) := f_+(x)e^{-bx}$$

is integrable. Then, for any  $\xi \in \mathbb{R}$ ,

$$\begin{aligned}\widehat{g}_+(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_+(x)e^{-bx} e^{-ix\xi} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_+(x)e^{-ix(\xi - ib)} dx \\ &= \widehat{f}_+(\xi - ib).\end{aligned}$$

## Fourier Transform of Functions with Insufficient Decay

In fact,  $\widehat{f}_+(z)$  is defined for any  $z \in \mathbb{C}$  with  $\operatorname{Im} z < -b$ : let  $z = \xi + \eta i$ ,  $\eta < -b$ . Then

$$\begin{aligned} |\widehat{f}_+(z)| &\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f_+(x)e^{-ix(\xi+\eta i)}| dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f_+(x)e^{\eta x}| dx \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |g_+(x)| dx < \infty \end{aligned}$$

Similarly, suppose that for  $b > 0$

$$g_-(x) := f_-(x)e^{bx}$$

is integrable. Then

$$\widehat{g}_-(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_-(x)e^{bx}e^{-ix\xi} dx = \widehat{f}_-(\xi + ib).$$

and  $\widehat{f}_-(z)$  is defined for any  $z \in \mathbb{C}$  with  $\operatorname{Im} z > b$ .

## Fourier Transform of Functions with Insufficient Decay

2.7.11. Example. Let  $f(x) = e^{|x|}$ . Then for any  $\varepsilon > 0$ , setting  $b = 1 + \varepsilon$ , we have

$$e^{-b|x|} f(x) = e^{-\varepsilon|x|}$$

which is integrable. Then

$$\widehat{f}_+(\xi + i\eta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^x e^{-i(\xi+i\eta)x} dx = -\frac{1}{\sqrt{2\pi}} \frac{1}{1 - i\xi + \eta}$$

exists when  $\eta < -1 - \varepsilon$  and

$$\widehat{f}_-(\xi + i\eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{-x} e^{-i(\xi+i\eta)x} dx = \frac{1}{\sqrt{2\pi}} \frac{1}{-1 - i\xi + \eta}$$

when  $\eta > 1 + \varepsilon$ .

# Fourier Inversion for Functions with Insufficient Decay

Our assumptions for  $f$  guarantee that Theorem 2.7.10 holds for  $g_+$ , i.e.,

$$\frac{g_+(x^+) + g_-(x^-)}{2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}_+(\xi) e^{ix\xi} d\xi$$

for all  $x \in \mathbb{R}$ . Then

$$\begin{aligned} \frac{f_+(x^+) + f_-(x^-)}{2} &= e^{bx} \frac{g_+(x^+) + g_-(x^-)}{2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}_+(\xi) e^{ix(\xi - ib)} d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}_+(\xi - ib) e^{ix(\xi - ib)} d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty - ib}^{\infty - ib} \hat{f}_+(z) e^{ixz} dz \end{aligned} \tag{2.7.10}$$

where the integration in the last integral is along the line

$$L = \{z \in \mathbb{C} : z = \xi - ib, \xi \in \mathbb{R}\}$$

oriented from “left to right”.

# Fourier Inversion for Functions with Insufficient Decay

Similarly, we obtain

$$\frac{f_-(x^+) + f_-(x^-)}{2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty+ib}^{\infty+ib} \hat{f}_-(z) e^{ixz} dz.$$

We hence have the inversion formula

$$\frac{f(x^+) + f(x^-)}{2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty-ib}^{\infty-ib} \hat{f}_+(z) e^{ixz} dz + \frac{1}{\sqrt{2\pi}} \int_{-\infty+ib}^{\infty+ib} \hat{f}_-(z) e^{ixz} dz$$

for all  $x \in \mathbb{R}$ , where  $b$  is chosen such that  $e^{-b|x|} f(x)$  is absolutely integrable.

## Relation to Laplace Transform

Suppose we have a function  $f: [0, \infty) \rightarrow \mathbb{C}$  satisfying

$$|f(t)| \leq M e^{bt}$$

for some  $b, M > 0$ . Let us extend  $f$  to  $\mathbb{R}$  by setting  $f(t) = 0$  for  $t < 0$  and suppose that  $f$  satisfies the smoothness conditions of Theorem 2.7.10.

Then we have seen that we can define the complex Fourier transform  $\hat{f}(z)$  for all  $z$  with  $\operatorname{Im} z < -b$ . In particular, for  $z = -ip$ ,  $p > b$ , we have

$$\hat{f}(-ip) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(s) e^{-ps} ds = \frac{1}{\sqrt{2\pi}} (\mathcal{L}f)(p). \quad (2.7.11)$$

Hence the Laplace transform is just a special case of the complex Fourier transform! We can now prove the Mellin inversion formula for the Laplace transform, Theorem 2.6.8.

# The Laplace Transform

The inversion formula (2.7.10) immediately yields

$$f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty - ib}^{\infty - ib} \hat{f}(z) e^{izs} dz.$$

As in Theorem 2.7.9, we can easily see that  $\hat{f}(z)$  is holomorphic at  $z = x + iy$  when  $y < b$ , i.e., all singularities of  $\hat{f}$  lie above the line of integration. Furthermore,

$$\begin{aligned} f(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty - ib}^{\infty - ib} \hat{f}(z) e^{izs} dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(t - ib) e^{i(t - ib)s} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(-i(it + b)) e^{(it+b)s} dt \\ &= \frac{1}{\sqrt{2\pi}i} \int_{b-i\infty}^{b+i\infty} \hat{f}(-ip) e^{ps} dp = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} (\mathcal{L}f)(p) e^{ps} dp \end{aligned}$$

which is just the Bromwich integral of  $\mathcal{L}f$ . (Note that the singularities of  $(\mathcal{L}f)(p)$  lie to the left of the line  $\operatorname{Re} z = b$ .)

## Second Midterm Exam

The preceding material completes the second part of the course material. It encompasses everything that will be the subject of the **Second Midterm Exam**.

The exact exam date will be announced on Canvas.

No calculators or other aids will be permitted during the exam. A sample exam with solutions has been uploaded to Canvas. Please study it carefully, including the instructions on the cover page.

## Part III

# Series Methods for Second-Order Equations

19. Power Series Solutions

20. Fourier Series

21. Separation of Variables for PDEs

19. Power Series Solutions

20. Fourier Series

21. Separation of Variables for PDEs

## The Power Series Ansatz

We have seen that higher order ODEs with variable coefficients can be surprisingly difficult to solve. Example 2.6.9 demonstrates that even the harmless-looking equation

$$tx'' + x' + tx = 0$$

has solutions that can not be expressed in terms of elementary functions.

In this section, we will discuss finding a solution to such an equation through a **power series ansatz**. In particular, we try to find a solution of the form

$$x(t) = \sum_{k=0}^{\infty} a_k t^k.$$

by plugging this expression into the equation and determining the coefficient sequence  $(a_k)$ .

# The Power Series Ansatz

3.1.1. Example. We make a power series ansatz to solve the equation

$$x'' - 2tx' - 2x = 0, \quad t \in \mathbb{R}. \quad (3.1.1)$$

We have

$$x(t) = \sum_{k=0}^{\infty} a_k t^k = a_0 + a_1 t + a_2 t^2 + \dots, \quad (3.1.2)$$

$$x''(t) = \sum_{k=0}^{\infty} (k+1)(k+2)a_{k+2}t^k = 2a_2 + 2 \cdot 3a_3 t + \dots,$$

$$tx'(t) = \sum_{k=1}^{\infty} k a_k t^k = a_1 t + 2a_2 t^2 + 3a_3 t^3 + \dots$$

Thus a power series of the form (3.1.2) solves (3.1.1) if and only if

$$2a_2 - 2a_0 + \sum_{k=1}^{\infty} [(k+1)(k+2)a_{k+2} - 2ka_k - 2a_k] t^k = 0$$

## Recurrence Relation and Initial Conditions

Since the power series must vanish for all  $x$ , each coefficient must be zero.  
We hence obtain the sequence of recurrence equations

$$a_2 = a_0, \quad (k+1)(k+2)a_{k+2} - 2(k+1)a_k = 0, \quad k \geq 1.$$

We can rewrite this as

$$a_{k+2} = \frac{2}{k+2}a_k, \quad k \in \mathbb{N}. \quad (3.1.3)$$

The first two coefficients  $a_0$  and  $a_1$  are arbitrary, but their choice determines all other terms. This is not surprising, because their choice will furnish us with two independent solutions, as we would expect from a second order linear ODE. We set

$$a_0 = c_1, \quad a_1 = 0 \quad \text{for a solution } x_1(t),$$

$$a_0 = 0, \quad a_1 = c_2 \quad \text{for a solution } x_2(t),$$

where  $c_1, c_2 \in \mathbb{R}$ .

## Solution of the Recurrence Relations

Setting  $a_1 = 0$  immediately yields  $a_{2k+1} = 0$  for  $k \in \mathbb{N}$  from (3.1.3). Furthermore, we have

$$a_0 = c_1, \quad a_2 = a_0 = c_1, \quad a_4 = \frac{1}{2}a_2 = \frac{c_1}{2}, \dots$$

We now transform the recursive formula (3.1.3) for the coefficients into an explicit formula. We are only interested in the case where  $k = 2j$  is even, so we have

$$a_{2j+2} = \frac{2}{2j+2} a_{2j}$$

or

$$a_{2(j+1)} = \frac{1}{j+1} a_{2j} = \frac{1}{j+1} \frac{1}{j} a_{2(j-1)} = \dots = \frac{1}{(j+1)!} a_0.$$

Thus,

$$a_{2j} = \frac{1}{j!} a_0,$$

as we can verify by induction.

## Solution of the Recurrence Relations

It follows that one solution is given by

$$x_1(t) = c_1 \sum_{j=0}^{\infty} \frac{1}{j!} t^{2j} = c_1 \cdot e^{t^2}.$$

In this instance we were lucky, because we could identify the series solution as being a simple elementary function.

We obtain an independent second solution  $x_2$  from  $a_0 = 0$  and  $a_1 = c_2$ . Here, we see that  $a_{2k} = 0$  for  $k \in \mathbb{N}$  and

$$a_{2j+1} = \frac{2}{2j+1} a_{2j-1} = \frac{2}{2j+1} \frac{2}{2j-1} a_{2j-3} = \cdots = \frac{2^j}{(2j+1)!!} a_1,$$

where  $(2j+1)!! = \prod_{i=1}^j (2i+1)$ .

## General Solution

Our second solution is hence given by

$$x_2(t) = c_2 \cdot \sum_{j=0}^{\infty} \frac{2^j}{(2j+1)!!} t^{2j+1}$$

and the general solution is

$$x(t; c_1, c_2) = c_1 e^{t^2} + c_2 \cdot \sum_{j=1}^{\infty} \frac{2^j}{(2j+1)!!} t^{2j+1}.$$

Note that it is not immediately apparent if the second series corresponds to some elementary function. This is usually the case when we solve an equation with such an ansatz. It is, however, easy to check that in this case the power series has an infinite radius of convergence, so at least we have two independent solutions that exist for all  $t$ .

## Remarks on the Power Series Approach

Of course, given the first solution  $x_1$ , we could use reduction of order to come up with the second solution. It turns out that

$$x_2(t) = c_2 \cdot e^{t^2} \int_{-\infty}^t e^{-s^2} ds,$$

which is not expressible as an elementary function.

This example has shown the typical and ideal case of this approach. In practice, we may often get solutions that can only be represented as power series, and we may have trouble finding an explicit representation of the coefficients from a recursive one.

**3.1.2. Remark.** In this section we will only treat **homogeneous** linear ODEs with variable coefficients. Once the general solution to a homogeneous ODE is found, a particular solution to an associated inhomogeneous equation may be constructed using variation of parameters; cf. Slide 257.

# ODEs with Variable Coefficients

In principle, we would like to treat homogeneous equations of the form

$$P(t)x'' + Q(t)x' + R(t)x = 0, \quad t \in \mathbb{R}. \quad (3.1.4)$$

The main problem occurs if  $P(t_0) = 0$  for some  $t_0$ . Then the equation may be solved for  $t > t_0$  and for  $t < t_0$ . In that way one obtains two solutions, but there is no immediate way to connect them to a single solution on  $\mathbb{R} \setminus \{t_0\}$ .

## 3.1.3. Example. The equation

$$t^2x'' + 4tx' + 2x = 0$$

has solutions

$$x_1(t) = \frac{c_1}{t}, \quad x_2(t) = \frac{c_2}{t^2}, \quad c_1, c_2 \in \mathbb{R}.$$

These solutions are not defined at  $t = 0$  (where the coefficient of  $x''$  vanishes) and while an initial value problem can be solved

where  $P, Q, R$  are suitable functions. Even if  $P, Q, R$  are polynomials, we will experience problems whenever  $P(t) = 0$ . At these points (3.1.4) is no

## ODEs with Analytic Coefficients

Therefore, instead of treating (3.1.4) with functions (polynomials)  $P, Q, R$  we instead consider a “standard form”, which we get after dividing (3.1.4) by  $P(t)$ :

$$x'' + p(t)x' + q(t)x = 0 \quad (3.1.5)$$

The properties of  $p$  and  $q$  (in particular, whether they have singularities) will determine the properties of the solutions.

For a full analysis of the properties of solutions to such equations we would have to use some powerful methods from complex analysis and consider the ODEs as equations for functions of complex variables (i.e.,  $t \in \mathbb{C}$ ). For reasons of time we can not go into the theory here and merely quote the relevant results.

**3.1.4. Theorem.** If the function  $p$  and  $q$  in (3.1.5) have power series expansions with radius of convergence  $\varrho_1$  and  $\varrho_2$ , respectively, then the solution  $x$  of (3.1.5) also has a power series expansion with radius of convergence greater or equal to  $\min(\varrho_1, \varrho_2)$ .

# ODEs with Analytic Coefficients

3.1.5. Example. We consider the initial value problem

$$x'' + \frac{3t}{1+t^2}x' + \frac{1}{1+t^2}x = 0, \quad x(0) = 2, \quad x'(0) = 3.$$

Here the coefficients  $3t/(1+t^2)$  and  $1/(1+t^2)$  have power series expansions with radius of convergence  $\varrho = 1$ , so we expect the solution to have a series expansion at least for  $t \in (-1, 1)$  and possibly within a larger region. Of course, we could make our power series ansatz for  $x$  and also expand the coefficients in power series. But this is unnecessarily complicated. Instead, we multiply the equation with  $1+t^2$  to obtain

$$(1+t^2)x'' + 3tx' + x = 0.$$

# ODEs with Analytic Coefficients

We have

$$x(t) = \sum_{k=0}^{\infty} a_k t^k,$$

$$\begin{aligned}(1+t^2)x''(t) &= \sum_{k=0}^{\infty} (k+1)(k+2)a_{k+2}t^k + \sum_{k=2}^{\infty} k(k-1)a_k t^k \\&= 2a_2 + 6a_3 t + \sum_{k=2}^{\infty} [k(k-1)a_k + (k+1)(k+2)a_{k+2}] t^k\end{aligned}$$

$$tx'(t) = \sum_{k=1}^{\infty} k a_k t^k$$

# ODEs with Analytic Coefficients

Inserting this into the ODE, we have

$$\begin{aligned} 0 &= 2a_2 + 6a_3t + \sum_{k=2}^{\infty} [k(k-1)a_k + (k+1)(k+2)a_{k+2} + (3k+1)a_k] t^k \\ &\quad + 3a_1t + a_0 + a_1t \\ &= (2a_2 + a_0) + 2(2a_1 + 3a_3)t + \sum_{k=2}^{\infty} [(k+1)^2 a_k + (k+1)(k+2)a_{k+2}] t^k \\ &= \sum_{k=0}^{\infty} [(k+1)^2 a_k + (k+1)(k+2)a_{k+2}] t^k \end{aligned}$$

Thus we obtain the recurrence formula

$$a_{k+2} = -\frac{k+1}{k+2} a_k. \tag{3.1.6}$$

## ODEs with Analytic Coefficients

We want to solve the initial value problem  $x(0) = 2$ ,  $x'(0) = 3$ . Note that if  $x(t) = \sum a_k t^k$ , then

$$x(0) = a_0 = 2, \quad x'(0) = a_1 = 3.$$

We proceed to find two independent solutions, setting

$$\begin{array}{lll} a_0 = 2, & a_1 = 0 & \text{for a solution } x_1(t), \\ a_0 = 0, & a_1 = 3 & \text{for a solution } x_2(t). \end{array}$$

If  $a_1 = 0$  the recurrence formula (3.1.6) gives  $a_{2k+1} = 0$  for  $k \in \mathbb{N}$ . If  $k = 2j$  is even, we have

$$\begin{aligned} a_{2(j+1)} &= -\frac{2j+1}{2(j+1)} a_{2j} = \frac{2j+1}{2(j+1)} \frac{2j-1}{2j} a_{2(j-1)} \\ &= (-1)^{j+1} \frac{(2j+1)!!}{2^{j+1}(j+1)!} a_0 \end{aligned}$$

## ODEs with Analytic Coefficients

We thus obtain

$$x_1(t) = 2 + 2 \sum_{j=0}^{\infty} (-1)^{j+1} \frac{(2j+1)!!}{2^{j+1}(j+1)!} t^{2j+2}.$$

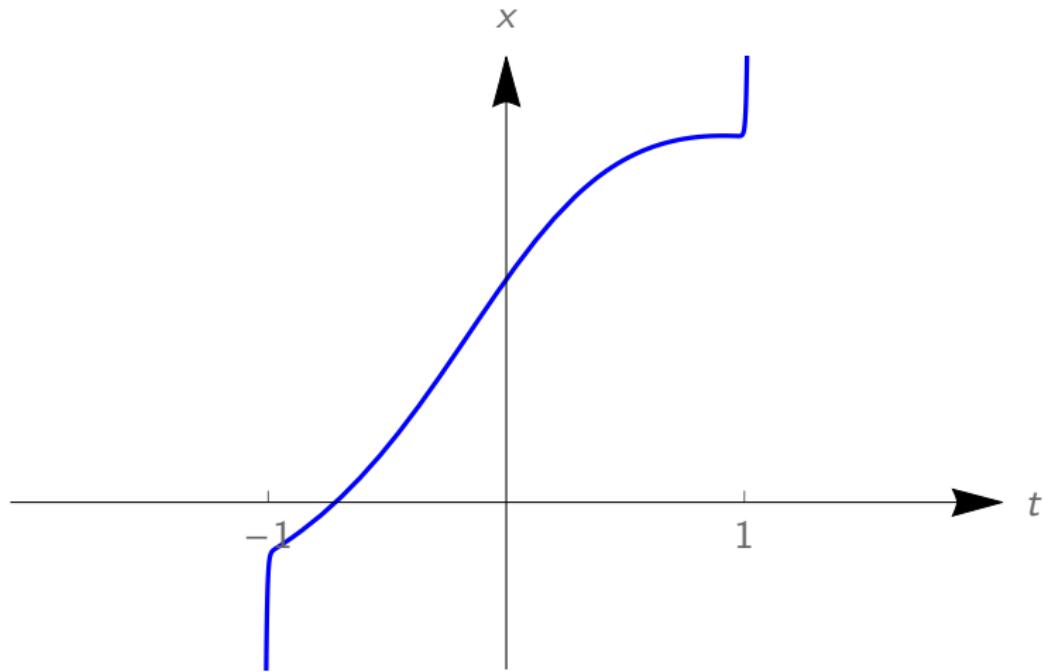
Note that this series has radius of convergence  $\varrho = 1$ . In a similar manner, we find the second solution to be

$$x_2(t) = 3 \sum_{j=0}^{\infty} (-1)^j \frac{2^j j!}{(2j+1)!!} t^{2j+1}.$$

The initial value problem is then solved by  $x(t) = x_1(t) + x_2(t)$  for  $|t| < 1$  (both solutions have radius of convergence  $\varrho = 1$ ).

# ODEs with Analytic Coefficients

An approximation using 200 terms of the series solution is plotted below.



## ODEs with Analytic Coefficients

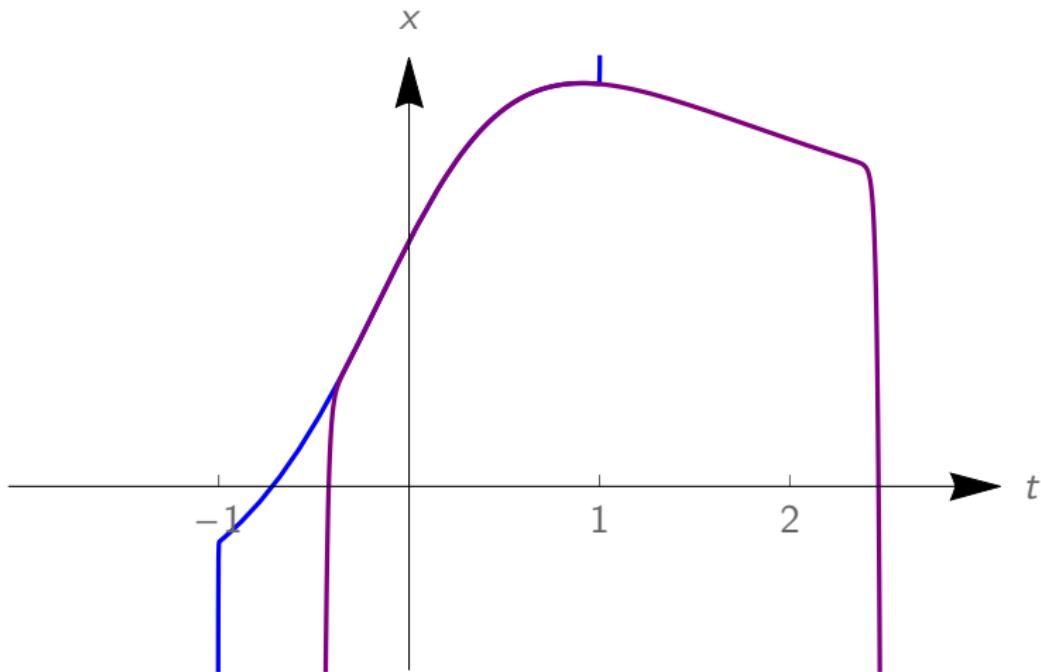
The fact that the series converges for  $-1 < x < 1$  is obvious, but at the same time, the solution of the differential equation does not appear to behave in a “bad” manner close to the boundary of that interval. In order to obtain a series solution that works in a neighborhood of  $x = 1$ , we could try an ansatz of the form

$$x(t) = \sum_{k=0}^{\infty} a_k (t - 1)^k$$

as before. This time the resulting recurrence relation is very messy, but an approximation can be obtained and plotted as follows.

## ODEs with Analytic Coefficients

Two approximations using 200 terms of the series solutions at  $x = 0$  (blue) and  $x = 1$  (purple):



## ODEs with Analytic Coefficients

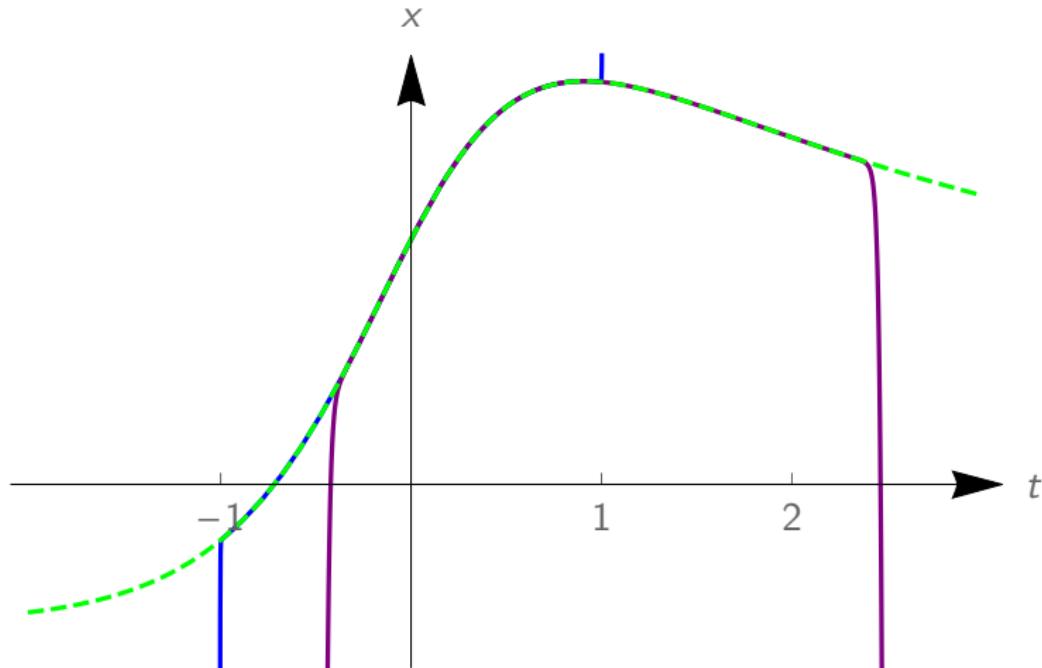
The radius of convergence of the series centered at  $x = 1$  is clearly larger. In fact, the limited radius of convergence appears to be solely an artefact of the fact that the function  $1/(1+t^2)$  has a pole at  $\pm i$  in the complex plane.

This can be confirmed by finding the exact solution to the initial value problem, which happens to be

$$x(t) = \frac{2 + 3 \operatorname{Arsinh}(t)}{\sqrt{1+t^2}}.$$

# ODEs with Analytic Coefficients

Exact solution (green, dashed) and series solutions (blue, purple) to the ODE.



# Singular Points

## 3.1.6. Definition. The equation

$$P(t)x'' + Q(t)x' + R(t)x = 0 \quad (3.1.7)$$

is said to have a **singular point at  $t_0$**  if  $P(t_0) = 0$ .

The method outlined above will in general fail if the power series ansatz is centered at a singular point, i.e., if we attempt to set

$$x(t) = \sum_{k=0}^{\infty} a_k(t - t_0)^n$$

and  $t_0$  is a singular point. Our goal is to find a class of singular points for which we may still get workable results using this approach. One particularly benign differential equations that is nevertheless treatable is **Euler's equation**, which we will now analyze.

# Euler's Equation

## 3.1.7. Definition. The equation

$$t^2x'' + \alpha tx' + \beta x = 0, \quad \alpha, \beta \in \mathbb{R}, \quad (3.1.8)$$

is called **Euler's equation**.

Euler's equation has a singular point at  $t = 0$ . We will initially look at solutions for  $t > 0$ . The form of the equation (terms with  $t^2x''$ ,  $tx'$ ,  $t^0x$ ) suggests that we try an ansatz

$$x(t) = t^r$$

for some  $r \in \mathbb{R}$ . Inserting into (3.1.8), we find

$$(r^2 + (\alpha - 1)r + \beta)t^r = 0.$$

which gives

$$r = -\frac{\alpha - 1}{2} \pm \frac{1}{2}\sqrt{(\alpha - 1)^2 - 4\beta}.$$

## Euler's Equation

We consider the usual array of cases:

1.  $(\alpha - 1)^2 - 4\beta > 0$ . There are two distinct roots  $r_1, r_2 \in \mathbb{R}$  and the general solution of Euler's equation is

$$x(t; c_1, c_2) = c_1 t^{r_1} + c_2 t^{r_2}, \quad c_1, c_2 \in \mathbb{R}.$$

2.  $(\alpha - 1)^2 - 4\beta = 0$ . In that case

$$r_1 = r_2 = \frac{1 - \alpha}{2}$$

and we obtain a single solution  $x_1(t) = t^{r_1}$ . We can then use the method of reduction of order to find a second solution,  $x_2(t) = t^{r_1} \ln t$  (homework!). Thus, the general solution is

$$x(t; c_1, c_2) = c_1 t^{r_1} + c_2 t^{r_1} \ln t, \quad c_1, c_2 \in \mathbb{R}.$$

## Euler's Equation

3.  $(\alpha - 1)^2 - 4\beta < 0$ . There are two distinct complex roots  $r_1, r_2 \in \mathbb{C}$  given by

$$r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu,$$

with

$$\lambda = \frac{1 - \alpha}{2}, \quad \mu = \frac{1}{2} \sqrt{4\beta - (\alpha - 1)^2}.$$

We obtain two solutions,

$$x_1(t) = t^{r_1} = t^\lambda (\cos(\mu \ln t) + i \sin(\mu \ln t)),$$
$$x_2(t) = t^{r_2} = t^\lambda (\cos(\mu \ln t) - i \sin(\mu \ln t)).$$

The general real-valued solution is then

$$x(t; c_1, c_2) = c_1 t^\lambda \cos(\mu \ln t) + c_2 t^\lambda \sin(\mu \ln t), \quad c_1, c_2 \in \mathbb{R}.$$

## Euler's Equation

We now turn to the solutions of the Euler equation for  $t < 0$ . This is a good opportunity for us to review how to change variables in differential equations. We define

$$\tau(t) = -t, \quad \xi = x \circ \tau^{-1}.$$

Thus  $\xi(\tau(t)) = x(t)$  and

$$\frac{dx}{dt} = \frac{d}{dt}\xi(\tau(t)) = \xi'(\tau(t)) \cdot \tau'(t).$$

For short, we write this as

$$\xi(\tau) = x(t) \quad \text{and} \quad \frac{d}{dt} = \frac{d\tau}{dt} \frac{d}{d\tau} = -\frac{d}{d\tau}.$$

In this vein,

$$\frac{d^2}{dt^2} = \frac{d}{dt} \frac{d}{dt} = \frac{d\tau}{dt} \frac{d}{d\tau} \left( \frac{d\tau}{dt} \frac{d}{d\tau} \right) = -\frac{d}{d\tau} \left( -\frac{d}{d\tau} \right) = \frac{d^2}{d\tau^2}$$

# Euler's Equation

Hence, we have

$$\frac{dx}{dt} = -\frac{d\xi}{d\tau}, \quad \frac{d^2x}{dt^2} = (-1)^2 \frac{d^2\xi}{d\tau^2}$$

and Euler's equation

$$t^2 x'' + \alpha t x' + \beta x = 0 \quad (3.1.9)$$

becomes

$$(-\tau)^2 \xi'' + \alpha(-\tau)(-\xi') + \beta \xi = 0. \quad (3.1.10)$$

Any solution  $\xi(\tau)$  of (3.1.10) induces a solution  $x(t) = \xi(\tau(t)) = \xi(-t)$  of (3.1.9). However, (3.1.9) and (3.1.10) are actually the same equation!

# Euler's Equation

We thus have general solutions

$$\xi(\tau; c_1, c_2) = \begin{cases} c_1\tau^{r_1} + c_2\tau^{r_2} & \text{if } (\alpha - 1)^2 - 4\beta > 0, \\ c_1\tau^{r_1} + c_2\tau^{r_1} \ln \tau & \text{if } (\alpha - 1)^2 - 4\beta = 0, \\ c_1\tau^\lambda \cos(\mu \ln \tau) + c_2\tau^\lambda \sin(\mu \ln \tau) & \text{if } (\alpha - 1)^2 - 4\beta < 0, \end{cases}$$

for  $\tau > 0$  and  $c_1, c_2 \in \mathbb{R}$ . Since  $t = -\tau$ , this means that we obtain solutions

$$x(t; c_1, c_2) = \xi(-t; c_1, c_2) = \begin{cases} c_1|t|^{r_1} + c_2|t|^{r_2}, \\ c_1|t|^{r_1} + c_2|t|^{r_1} \ln|t|, \\ c_1|t|^\lambda \cos(\mu \ln|t|) + c_2|t|^\lambda \sin(\mu \ln|t|), \end{cases}$$

for  $t < 0$  and  $c_1, c_2 \in \mathbb{R}$ . This solution coincides with the solution for  $t > 0$ , so it is valid for any  $t \in \mathbb{R} \setminus \{0\}$ . Note that the solutions may not be defined at  $t = 0$ , the singular point of Euler's equation.

## Regular Singular Points

We have seen that Euler's equation, despite its singular point at  $t = 0$  is treatable with an exponential ansatz. If we rewrite Euler's equation as

$$x'' + \frac{\alpha}{t}x' + \frac{\beta}{t^2}x = 0$$

we see that the coefficients of  $x'$  and  $x$  have poles of order 1 and 2, respectively, at  $x = 0$ . It turns out that this is precisely the situation we will be able to treat.

3.1.8. Definition. The equation

$$x'' + p(t)x' + q(t)x = 0 \tag{3.1.11}$$

is said to have a **regular singular point** at  $t_0$  if the functions  $(t - t_0)p(t)$  and  $(t - t_0)^2q(t)$  are analytic in a neighborhood of  $t_0$ . A singular point which is not regular is said to be **irregular**.

## Regular Singular Points

Definition 3.1.8 is equivalent to stating that (3.1.11) has a regular singular point at  $t_0$  if we have convergent power series such that

$$p(t) = \frac{p_{-1}}{t - t_0} + \sum_{j=0}^{\infty} p_j (t - t_0)^j,$$

$$q(t) = \frac{q_{-2}}{(t - t_0)^2} + \frac{q_{-1}}{t - t_0} + \sum_{j=0}^{\infty} q_j (t - t_0)^j$$

in a neighborhood of  $t_0$ .

### 3.1.9. Examples.

1. Bessel's differential equation of order  $\nu \in \mathbb{R}$ ,

$$t^2 x'' + tx' + (t^2 - \nu^2)x = 0$$

has a regular singular point at  $t = 0$ .

## Regular Singular Points

2. The Legendre differential equation of order  $\alpha \in \mathbb{R}$ ,

$$(1 - t^2)x'' - 2tx' + \alpha(\alpha + 1)x = 0$$

has regular singular points at  $x = \pm 1$ .

3. The differential equation

$$t^2x'' + 3x' + tx = 0$$

has an irregular singular point at  $t = 0$ .

# The Method of Frobenius

The equation

$$x'' + p(t)x' + q(t)x = 0$$

can be written as

$$t^2x'' + t(tp(t))x' + t^2q(t)x = 0.$$

This equation resembles Euler's equation, with  $\alpha$  and  $\beta$  replaced by the analytic functions  $tp(t)$  and  $t^2q(t)$ , respectively. This corresponds to adding higher order terms in  $t$  to  $\alpha$  and  $\beta$ , so it is natural to try a modification of the ansatz we used for Euler's equation. We set

$$x(t) = t^r \sum_{k=0}^{\infty} a_k t^k, \quad a_0 \neq 0. \quad (3.1.12)$$

This is known as the **Frobenius ansatz**.

## The Method of Frobenius

We will look at the equation

$$t^2x'' + t(tp(t))x' + t^2q(t)x = 0, \quad t > 0,$$

assuming that there is a singular point at  $t = 0$  so that

$$tp(t) = \sum_{j=0}^{\infty} p_j t^j, \quad t^2q(t) = \sum_{j=0}^{\infty} q_j t^j.$$

We insert the Frobenius ansatz (3.1.12) directly into our differential equation. We have

$$\begin{aligned} x(t) &= \sum_{k=0}^{\infty} a_k t^{r+k}, & x'(t) &= \sum_{k=0}^{\infty} (r+k)a_k t^{r+k-1}, \\ x''(t) &= \sum_{k=0}^{\infty} (r+k)(r+k-1)a_k t^{r+k-2}. \end{aligned}$$

# The Method of Frobenius

This gives

$$\begin{aligned} 0 &= t^2 x'' + t(tp(t))x' + t^2 q(t)x \\ &= t^2 x'' + \left( \sum_{j=0}^{\infty} p_j t^j \right) tx' + \left( \sum_{j=0}^{\infty} q_j t^j \right) x \\ &= \sum_{k=0}^{\infty} (r+k)(r+k-1)a_k t^{r+k} + \left( \sum_{j=0}^{\infty} p_j t^j \right) \sum_{k=0}^{\infty} (r+k)a_k t^{r+k} \\ &\quad + \left( \sum_{j=0}^{\infty} q_j t^j \right) \sum_{k=0}^{\infty} a_k t^{r+k} \end{aligned}$$

We divide the equation by  $t^r$  and apply the Cauchy product for series:

$$\left( \sum_{j=0}^{\infty} a_j \right) \left( \sum_{k=0}^{\infty} b_k \right) = \sum_{m=0}^{\infty} \sum_{j+k=m} a_j b_k$$

# The Method of Frobenius

Then

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} (r+k)(r+k-1)a_k t^k + \left( \sum_{j=0}^{\infty} p_j t^j \right) \sum_{k=0}^{\infty} (r+k)a_k t^k \\ &\quad + \left( \sum_{j=0}^{\infty} q_j t^j \right) \sum_{k=0}^{\infty} a_k t^k \\ &= \sum_{m=0}^{\infty} (r+m)(r+m-1)a_m t^m + \sum_{m=0}^{\infty} t^m \sum_{j+k=m} (r+k)p_j a_k \\ &\quad + \sum_{m=0}^{\infty} t^m \sum_{j+k=m} q_j a_k \\ &= \sum_{m=0}^{\infty} \left( (r+m)(r+m-1)a_m + \sum_{j+k=m} (q_j + (r+k)p_j)a_k \right) t^m \end{aligned}$$

## The Method of Frobenius

Setting each coefficient in the power series equal to zero yields

$$\begin{aligned} 0 &= (r+m)(r+m-1)a_m + \sum_{j+k=m} (q_j + (r+k)p_j)a_k \\ &= (r+m)(r+m-1)a_m + \sum_{k=0}^m (q_{m-k} + (r+k)p_{m-k})a_k \end{aligned}$$

for all  $m \in \mathbb{N}$ . Then

$$\begin{aligned} 0 &= ((r+m)(r+m-1) + q_0 + (r+m)p_0)a_m + \\ &\quad + \sum_{k=0}^{m-1} (q_{m-k} + (r+k)p_{m-k})a_k \end{aligned}$$

for  $m \geq 1$  and

$$(r(r-1) + q_0 + rp_0)a_0 = 0.$$

# The Method of Frobenius

Setting

$$F(x) := x(x - 1) + p_0x + q_0$$

for  $x \in \mathbb{R}$  and noting that  $a_0 \neq 0$  we can write this as

$$F(r) = 0, \quad (3.1.13)$$

$$a_m F(r + m) = - \sum_{k=0}^{m-1} (q_{m-k} + (r+k)p_{m-k}) a_k, \quad m \geq 1. \quad (3.1.14)$$

The equation (3.1.13) is known as the **indicial equation**. The equations (3.1.14) are recurrence equations that allow the determination of the coefficients  $a_k$ .

The indicial equation is a quadratic equation in  $r$ , that ideally yields two independent solutions  $r_1$  and  $r_2$ . Each of these values then induces different recurrence equations (3.1.14), yielding two independent solutions.

# The Method of Frobenius

What could possibly go wrong?

- ▶ If the indicial equation has two equal roots,  $r_1 = r_2$ , then we obtain only a single solution. A second solution requires further work, which we discuss later.
- ▶ If the roots of the indicial equation differ by a positive integer, i.e.,  $r_1 = r_2 + N$  for some  $N \in \mathbb{N}$ , then  $F(r_2 + N)$  will vanish at some point. This means that the root  $r_1$  generates a solution, but for the second solution,  $a_N$  can not in general be determined by (3.1.14). If by fortunate coincidence the right-hand side of (3.1.14) also vanishes for  $m = N$ , however, then  $a_N$  is arbitrary and we are able to find an independent second solution.

We will see later that each of these cases leads to a second solution which includes a logarithmic term.

# The Bessel Equation of Order 1/2

3.1.10. Example. Bessel's equation of order 1/2 is

$$t^2x'' + tx' + \left(t^2 - \frac{1}{4}\right)x = 0.$$

Here  $tp(t) = 1$ ,  $t^2q(t) = t^2 - 1/4$  are analytic at  $t = 0$  and

$$p_0 = 1, \quad p_k = 0 \quad (k \geq 1),$$

$$q_0 = -\frac{1}{4}, \quad q_1 = 0, \quad q_2 = 1, \quad q_k = 0 \quad (k \geq 3).$$

Thus the indicial equation is

$$\begin{aligned} F(r) &= r(r-1) + p_0r + q_0 = r(r-1) + r - \frac{1}{4} \\ &= (r+1/2)(r-1/2) = 0. \end{aligned}$$

The roots are  $r_1 = 1/2$  and  $r_2 = -1/2$ .

# The Bessel Equation of Order 1/2

The recurrence relation is

$$a_m F(r + m) = - \sum_{k=0}^{m-1} (q_{m-k} + (r+k)p_{m-k}) a_k, \quad m \geq 1.$$

We obtain

$$\begin{aligned} -a_1 F(r+1) &= (q_1 + rp_1) a_0 = 0, \\ -a_m F(r+m) &= (q_1 + (r+m-1)p_1) a_{m-1} \\ &\quad + (q_2 + (r+m-2)p_2) a_{m-2} + \cdots + (q_m + rp_m) a_0 \\ &= a_{m-2} \end{aligned}$$

for  $m \geq 2$ . The coefficient  $a_0$  is arbitrary; we set  $a_0 = 1$ . We first treat the solution associated with  $r_1 = 1/2$ . Since  $F(1/2 + 1) \neq 0$ , the coefficient  $a_1$  must vanish.

## The Bessel Equation of Order 1/2

For  $m \geq 2$  we have

$$a_m = -\frac{a_{m-2}}{F(1/2 + m)} = -\frac{a_{m-2}}{m(m+1)}.$$

For  $m = 2k + 1$  we see that  $a_{2k+1} = 0$  since  $a_1 = 0$ . If  $m = 2k$  we have

$$a_{2k} = -\frac{1}{2k(2k+1)} a_{2(k-1)} = \frac{1}{2k(2k+1)} \frac{1}{(2k-2)(2k-1)} a_{2(k-2)}.$$

We can then guess (and verify by induction) that

$$a_{2k} = \frac{(-1)^k}{(2k+1)!}.$$

It follows that

$$x_1(t) = t^{r_1} \sum_{m=0}^{\infty} a_m t^m = t^{1/2} \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k+1)!} = \frac{\sin t}{\sqrt{t}}.$$

## The Bessel Equation of Order 1/2

The second solution is associated to  $r_2 = -1/2$ . We expect a problem, because

$$F(r_2 + 1) = F(1/2) = F(r_1) = 0.$$

However, the right-hand side of the equation for  $a_1$ ,

$$-a_1 F(r_2 + 1) = (q_1 + r_2 p_1) a_0 = 0,$$

vanishes also, so  $a_1$  is arbitrary. We set  $a_1 = 0$  for simplicity. (If  $a_1 \neq 0$ , our solution will contain a multiple of the solution  $x_1$  constructed previously.) We again set  $a_0 = 1$  and obtain

$$a_m = -\frac{a_{m-2}}{F(-1/2 + m)} = -\frac{a_{m-2}}{m(m-1)}.$$

This recurrence relation leads to

$$a_{2k} = \frac{(-1)^k}{(2k)!}, \quad a_{2k+1} = 0, \quad k \in \mathbb{N}.$$

# The Bessel Equation of Order 1/2

We obtain a second independent solution as

$$x_2(t) = t^{r_2} \sum_{m=0}^{\infty} a_m t^m = t^{-1/2} \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} = \frac{\cos t}{\sqrt{t}}.$$

The two independent solutions of the Bessel equation of order 1/2 are

$$x_1 = \frac{\sin t}{\sqrt{t}}, \quad x_2 = \frac{\cos t}{\sqrt{t}}.$$

Note that  $x_1(0) = 0$  but  $x_2$  diverges at  $t = 0$ . The definition of the Bessel function  $J_\nu$  of order  $\nu \geq 0$  is (up to a constant) the solution which vanishes at  $t = 0$ . In particular

$$J_{1/2}(t) = \sqrt{\frac{2}{\pi t}} \sin t.$$

## The Bessel Equation of Order 1/2

The other independent solution is called a ***Neumann function*** or a ***Bessel function of the second kind*** and is denoted by  $Y_\nu$ . We set

$$Y_{1/2}(t) = \sqrt{\frac{2}{\pi t}} \cos t.$$

It turns out that all Bessel functions of the second kind are singular at the origin.

In the previous example we were lucky in obtaining two independent solutions despite the roots of the indicial equation differing by an integer. We now outline the general procedure, but omit the proof.

## Indicial Roots Differing by an Integer

Assume that the two roots  $r_1 \geq r_2$  of the indicial equation differ by an integer, i.e.,  $r_1 - r_2 \in \mathbb{Z}$ . The Frobenius ansatz supplies a solution

$$x_1(t) = t^{r_1} \sum_{k=0}^{\infty} a_k(r_1) t^k,$$

where  $a_k(r)$  is the solution to the recurrence equation

$$a_m(r)F(r+m) = - \sum_{k=0}^{m-1} (q_{m-k} + (r+k)p_{m-k})a_k(r).$$

Assume that further that the Frobenius ansatz does not supply a second independent solution  $x_2$ . Then

$$x_2(t) = \frac{\partial}{\partial r} \left( t^r \sum_{k=0}^{\infty} a_k(r) t^k \right) \Bigg|_{r=r_2} = c \cdot x_1(t) \ln t + t^{r_2} \sum_{k=0}^{\infty} a'_k(r_2) t^k$$

where the constant  $c \in \mathbb{R}$  may vanish. If  $r_1 = r_2$ , then  $c = 1$ .

# The Bessel Equation of Order 0

3.1.11. Example. Bessel's equation of order zero is

$$t^2x'' + tx' + t^2x = 0.$$

Here  $tp(t) = 1$ , and  $t^2q(t) = t^2$ , so

$$\begin{aligned} p_0 &= 1, & p_1 &= 0, & p_k &= 0 \quad (k \geq 2), \\ q_0 &= 0, & q_1 &= 0, & q_2 &= 1, & q_k &= 0 \quad (k \geq 3). \end{aligned}$$

Thus the indicial equation is

$$F(r) = r(r - 1) + p_0r + q_0 = r(r - 1) + r = r^2 = 0.$$

There is only a single root,  $r_1 = 0$ . The recurrence relation is

$$(r + m)^2 a_m = - \sum_{k=0}^{m-1} (q_{m-k} + (r + k)p_{m-k}) a_k, \quad m \geq 1.$$

# The Bessel Equation of Order 0

For  $m = 1$  the recurrence relation becomes

$$(r + 1)^2 a_1(r) = 0$$

and for  $m \geq 2$  we have

$$(r + m)^2 a_m(r) = -a_{m-2}(r).$$

The first equation implies  $a_1 = 0$  and the second shows that  $a_{2k+1} = 0$  for  $k \in \mathbb{N}$ . Setting  $a_0 = 1$ , we obtain

$$a_{2k}(r) = -\frac{1}{(r + 2k)^2} a_{2k-2}(r) = \frac{1}{(r + 2k)^2} \frac{1}{(r + 2k - 2)^2} a_{2k-4}(r)$$

leading to the general formula

$$a_{2k}(r) = \frac{(-1)^k}{(2+r)^2(4+r)^2 \dots (2k+r)^2}$$

# The Bessel Equation of Order 0

The indicial equation implies  $r_1 = 0$ , so we have

$$a_{2k}(r_1 = 0) = \frac{(-1)^k}{2^2 4^2 \dots (2k)^2} = \frac{(-1)^k}{2^{2k} (k!)^2}$$

and hence

$$x_1(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} (k!)^2} t^{2k}. \quad (3.1.15)$$

You can check (homework!) that this is actually  $J_0(x)$ , the Bessel function of the first kind of order zero (see (2.6.10)).

A second solution is given by

$$\begin{aligned} x_2(t) &= \left. \frac{d}{dr} \left( t^r \sum_{k=0}^{\infty} a_k(r) t^k \right) \right|_{r=r_2=0} \\ &= x_1(t) \ln t + \sum_{k=0}^{\infty} a'_k(0) t^k. \end{aligned}$$

# The Bessel Equation of Order 0

In order to calculate the derivative of the coefficients

$$a_{2k}(r) = \frac{(-1)^k}{(2+r)^2(4+r)^2 \dots (2k+r)^2}.$$

we use a trick to convert the product into a sum. We have

$$\begin{aligned}\frac{a'_{2k}(r)}{a_{2k}(r)} &= \frac{d}{dr} \ln|a_{2k}(r)| = -2 \frac{d}{dr} \sum_{j=1}^k \ln(2j+r) \\ &= -2 \sum_{j=1}^k \frac{1}{2j+r}.\end{aligned}$$

At  $r_2 = 0$  we then have

$$a'_{2k}(0) = - \left( 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} \right) a_{2n}(0).$$

# The Bessel Equation of Order 0

Setting

$$H_k := \sum_{j=1}^k \frac{1}{j},$$

we then have

$$a'_{2k}(0) = \frac{(-1)^{k+1} H_k}{2^{2k} (k!)^2}$$

and hence

$$x_2(t) = x_1(t) \ln t + \sum_{k=0}^{\infty} \frac{(-1)^{k+1} H_k}{2^{2k} (k!)^2} t^{2k}.$$

Up to a constant factor, this is the **Bessel function of the second kind of order 0**,  $Y_0$ . Note that it is singular at  $t = 0$  due to the logarithmic term.

The Bessel equation of order 1 will be discussed in the recitation class, while many more properties of the Bessel functions will be explored in the assignments.

19. Power Series Solutions

20. Fourier Series

21. Separation of Variables for PDEs

## Daniel Bernoulli



Daniel Bernoulli (1700-1782)

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File:Danielbernoulli.jpg](http://commons.wikimedia.org/wiki/File:Danielbernoulli.jpg).

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Daniel Bernoulli was the youngest and also the most productive of the famous Bernoulli family. He is best known for his numerous applications of mathematics to mechanics. The **Bernoulli principle** in fluid dynamics (the pressure of a fluid decreases as its velocity increases) was named after him, and he conducted intensive studies of vibrations, ocean tides and kinetic gas theory. He was also interested in statistics, where he formulated the **St. Petersburg paradox** and used it as a starting point for a discussion of risk.

Daniel Bernoulli was the first mathematician to encounter Bessel functions in his study of vibrating strings and chains.

## The Suspended Chain

Consider a flexible cable or chain suspended from a single point. Assume that the weight of the cable is not negligible. If the cable is excited to perform small oscillations, one notices that it quickly “locks” into one of several stable configurations:

First Mode

Second Mode

Third Mode

# The Suspended Chain

Reference The animations on the previous page were taken from

<http://www.kettering.edu/physics/drussell/Demos/HangChain/HangChain.html>. The

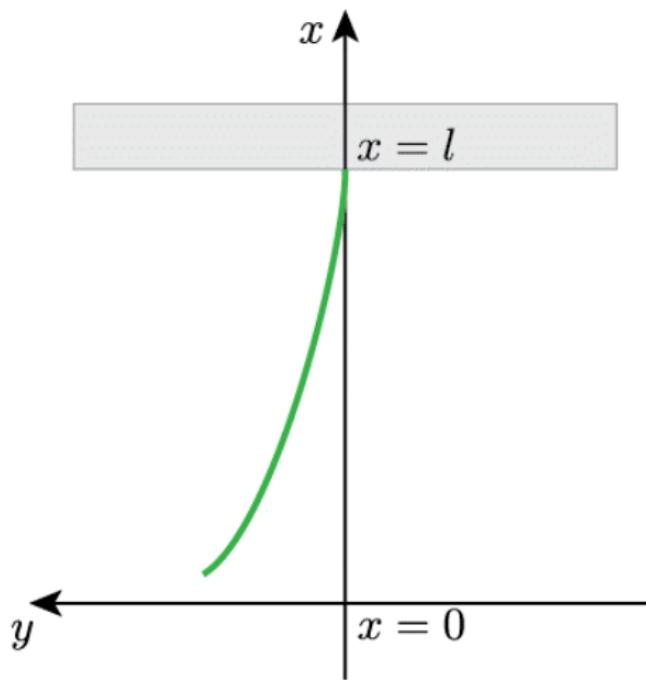
following material is based on various sources, such as the article at

<http://mysite.du.edu/~jcalvert/math/hchain.htm> and the project description here.

Let us try to model this problem. We introduce coordinates  $(x, y)$ , where the  $x$ -coordinate is the vertical axis (ordinate) and the  $y$ -coordinate axis is horizontal (the abscissa). The origin is taken to be at the lower end of the vertically hanging chain when it is in equilibrium. Suppose that the chain has a constant density  $\varrho$  and length  $l$ .

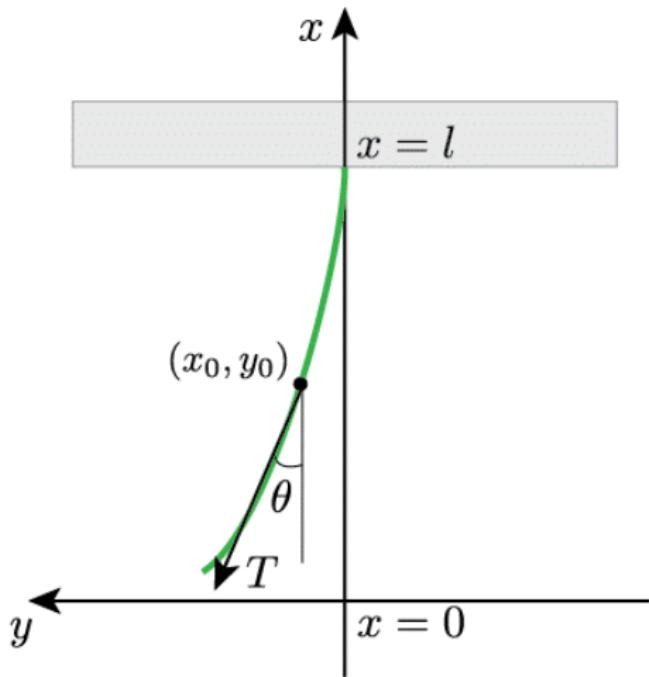
We assume that the ***oscillations of the chain are small***. This means that we can assume the  $x$ -coordinate to be approximately equal to the curve length parameter, i.e., a physical point  $(x, 0)$  at equilibrium is given by  $(x, y)$  on the displaced chain.

# The Suspended Chain



## Tension in the Suspended Chain

We now consider a point  $(x_0, y_0)$  on the displaced chain, where the tangent to the chain has angle  $\theta$  to the vertical axis.



The tension  $T$  at this point is

$$T(x_0, y_0) = T(x_0) = \rho \cdot g \cdot x_0,$$

where  $g = 9.80665 \text{ ms}^{-2}$ . The horizontal component  $T_h$  of the tension is responsible for the oscillations of the chain. It is given by

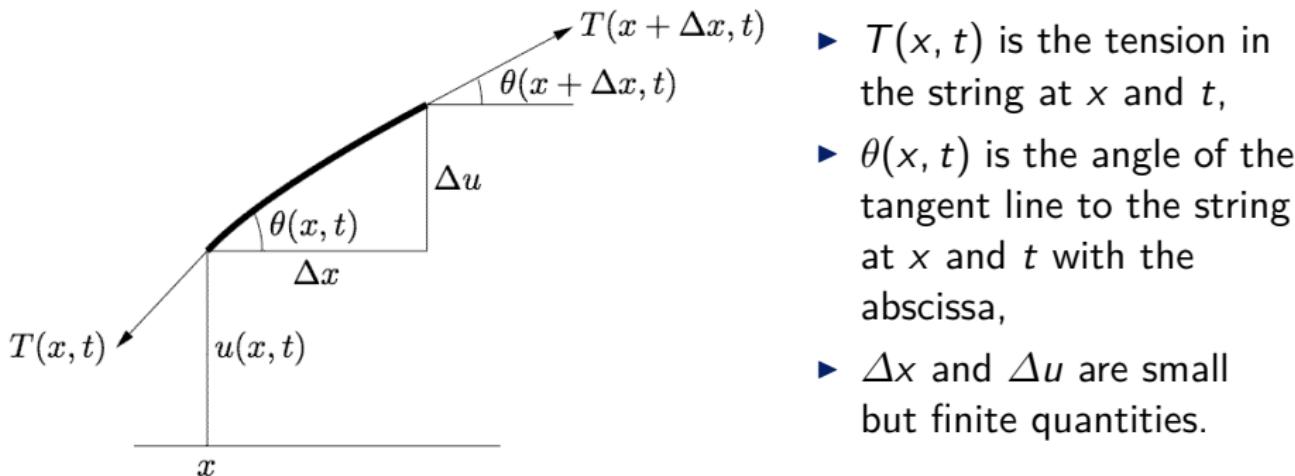
$$T_h = T \cdot \sin \theta \approx T \cdot \tan \theta$$

for small angles  $\theta$ .

## Derivation of the Model Equation

Assuming a flexible chain/cable/string and small oscillations, we derive the general wave equation.

- ▶  $x$  is the coordinate of the longitudinal axis,  $t$  is time and  $u(x, t)$  denotes the position on the transverse axis



- ▶ The mass density of the string at  $x$  is  $\varrho(x)$ .

## Derivation of the Model Equation

We assume that the string element is a curve that can be described through the parametrization  $y = u(x, t)$  at any time  $t$ . The mass-acceleration product of the string element is then

$$MA(x, \Delta x) = \int_x^{x+\Delta x} \varrho(\xi) u_{tt}(\xi, t) \sqrt{1 + u_\xi^2(\xi, t)} d\xi.$$

We will first consider Newton's law for the transversal forces. Then

$$\begin{aligned} MA(x, \Delta x) &= T(x + \Delta x, t) \sin \theta(x + \Delta x, t) - T(x, t) \sin \theta(x, t) \\ &\quad + \int_x^{x+\Delta x} F(\xi, t) \sqrt{1 + u_\xi^2(\xi, t)} d\xi \end{aligned} \tag{3.1.16}$$

where  $F(x, t)$  describes an external transversal force acting on the string at  $x$  and  $t$ . We will divide both sides of (3.1.16) by  $\Delta x$  and then let  $\Delta x \rightarrow 0$ . Although we are interested in the string element between  $x$  and  $x + \Delta x$ , we assume that the entire string is parametrized by a function  $u(x, t)$  for  $x \in [0, l]$  for some  $l > 0$ .

## Derivation of the Model Equation

We set

$$ma(x) = \int_0^x \varrho(\xi) u_{tt}(\xi, t) \sqrt{1 + u_\xi^2(\xi, t)} d\xi.$$

Thus for our string element,  $MA = ma(x + \Delta x) - ma(x)$  and

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{MA(x, \Delta x)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{ma(x + \Delta x) - ma(x)}{\Delta x} \\ &= ma'(x) = \varrho(x) u_{tt}(x, t) \sqrt{1 + u_x^2(x, t)} \end{aligned}$$

Using an analogous argument for the other terms, (3.1.16) becomes

$$\begin{aligned} \varrho(x) \sqrt{1 + u_x^2(x, t)} u_{tt}(x, t) &= \frac{\partial}{\partial x} (\tau(x, t) \sin \theta(x, t)) \\ &\quad + F(x, t) \sqrt{1 + u_x^2(x, t)} \end{aligned} \tag{3.1.17}$$

after dividing by  $\Delta x$  and letting  $\Delta x \rightarrow 0$ .

## Derivation of the Model Equation

Now, by definition,

$$\tan \theta(x, t) = \frac{\partial u(x, t)}{\partial x} \quad (3.1.18)$$

which gives

$$\sin \theta = \frac{u_x}{\sqrt{1 + u_x^2}}, \quad \cos \theta = \frac{1}{\sqrt{1 + u_x^2}}, \quad \theta_x = \frac{u_{xx}}{1 + u_x^2}.$$

Then (3.1.17) becomes

$$\begin{aligned} \varrho \sqrt{1 + u_x^2} \cdot u_{tt} &= \frac{\partial}{\partial x} (T \sin \theta) + F \sqrt{1 + u_x^2} \\ &= T_x \sin \theta + T \theta_x \cos \theta + F \sqrt{1 + u_x^2} \\ \Rightarrow \varrho(1 + u_x^2) u_{tt} &= T_x u_x + T \frac{u_{xx}}{1 + u_x^2} + (1 + u_x^2) F. \end{aligned} \quad (3.1.19)$$

Obviously, this is a quite complex differential equation. We can simplify it considerably by only considering "small oscillations," which we take to mean that  $\theta(x, t) \ll 1$  for all  $x, t$ . By (3.1.18) this is equivalent to  $u_x \ll 1$ .

## Small Oscillations and the Classical Wave Equation

Mathematically, we discard all terms in (3.1.19) that are of order  $o(u_x)$  as  $u_x \rightarrow 0$ . Since

$$\frac{1}{1+y} = 1 + O(y) \quad \text{and} \quad \sqrt{1+y} = 1 + O(y) \quad \text{as } y \rightarrow 0$$

we obtain

$$\varrho u_{tt} = T_x u_x + Tu_{xx} + F = \frac{\partial}{\partial x}(Tu_x) + F \quad (3.1.20)$$

to first order in  $u_x$ . If  $T$  is constant (no influence of gravity) and in the absence of external forces ( $F = 0$ ) we obtain the **classical wave equation**

$$c^2 u_{xx} = u_{tt}, \quad c^2 = \frac{T}{\varrho}. \quad (3.1.21)$$

We will discuss solutions of (3.1.21) in depth in a later section.

## The Suspended Chain

For a suspended chain of constant density, the tension is given by

$$T(x) = \varrho \cdot g \cdot x$$

from gravity and we have no additional external forces, so the model equation is

$$\varrho u_{tt}(x, t) = \frac{\partial}{\partial x}(T u_x) = \varrho \cdot g \cdot \frac{\partial}{\partial x}(x \cdot u_x(x, t))$$

or

$$\frac{1}{g} u_{tt}(x, t) = x \cdot u_{xx}(x, t) + u_x(x, t). \quad (3.1.22)$$

This is a partial differential equation of second order. We simplify it by supposing that the function  $u$  is periodic with respect to time, i.e.,

$$u(x, t) = y(x) \cdot e^{i\omega t} \quad (3.1.23)$$

for some frequency  $\omega$ .

## The Suspended Chain

Plugging (3.1.23) into (3.1.22), we obtain the ordinary differential equation

$$\frac{\partial}{\partial x}(x \cdot y') + \frac{\omega^2}{g}y = 0.$$

Substituting  $x = gz^2/(4\omega^2)$ ,  $z = 2\omega\sqrt{x/g}$  and taking  $w(z(x)) = y(x)$ , this equation becomes

$$zw'' + w' + zw = 0.$$

This is precisely the Bessel equation of order zero, cf. Examples 2.6.9 and 3.1.11. We are looking for solutions that are finite at  $x = 0$  (the end of the chain), so we obtain

$$w(z) = c \cdot J_0(z).$$

Substituting back,

$$y(x) = w(z(x)) = c \cdot J_0(2\omega\sqrt{x/g}).$$

## Fundamental Frequencies

The boundary condition at the upper end implies

$$y(l) = 0 = c \cdot J_0(2\omega\sqrt{l/g})$$

so we see that

$$\omega = \frac{1}{2} \sqrt{g/l} \cdot \alpha_{0,n},$$

where  $\alpha_{0,n} > 0$ ,  $n = 1, 2, \dots$ , is the  $n$ th zero of the Bessel function of the first kind of order zero, i.e.,  $J_0(\alpha_{0,n}) = 0$ . We hence see that only certain **fundamental frequencies**  $\omega$  lead to periodic motion of the chain and in those cases the shape of the chain follows that of a dilated Bessel function.

Such fundamental frequencies are fundamental to many physical systems. We will discuss their role more closely in a later section.

## Zeroes of the Bessel Functions

The obvious question that arises is that of how many zeroes a Bessel function has. The integral formula (2.6.10) and the series expression (3.1.15) are not really helpful in answering this question.

A heuristic approach is as follows: in the Bessel equation of order  $\nu \geq 0$ ,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

transform  $w(x) = x^{1/2}y(x)$ . The equation becomes

$$w'' + w + \underbrace{\frac{1}{x^2} \left( \frac{1}{4} - \nu^2 \right)}_{\text{small for large } x} w = 0.$$

This equation is close to the standard harmonic oscillator equation for large  $x$ , so that  $w(x) \approx c \cdot \cos(x + \delta)$  for some  $\delta \geq 0$  and  $c > 0$ .

## Zeroes of the Bessel Functions

This means that

$$y(x) \approx \frac{1}{\sqrt{x}} \cos(x + \delta)$$

when  $x$  is large. In fact, finer asymptotic analysis ("method of steepest descent") shows that, for  $n \in \mathbb{N}$ ,

$$J_n(x) \sim \frac{1}{\sqrt{x}} \cos\left(x - \frac{(2n+1)\pi}{4}\right) \quad \text{as } x \rightarrow \infty.$$

Hence, the Bessel functions decrease to zero as  $1/\sqrt{x}$  when  $x$  becomes large and, furthermore, have an infinite number of zeroes at a distance of approximately  $\pi$  from each other.

Mathematica gives the first 20 zeroes of  $J_0$  as

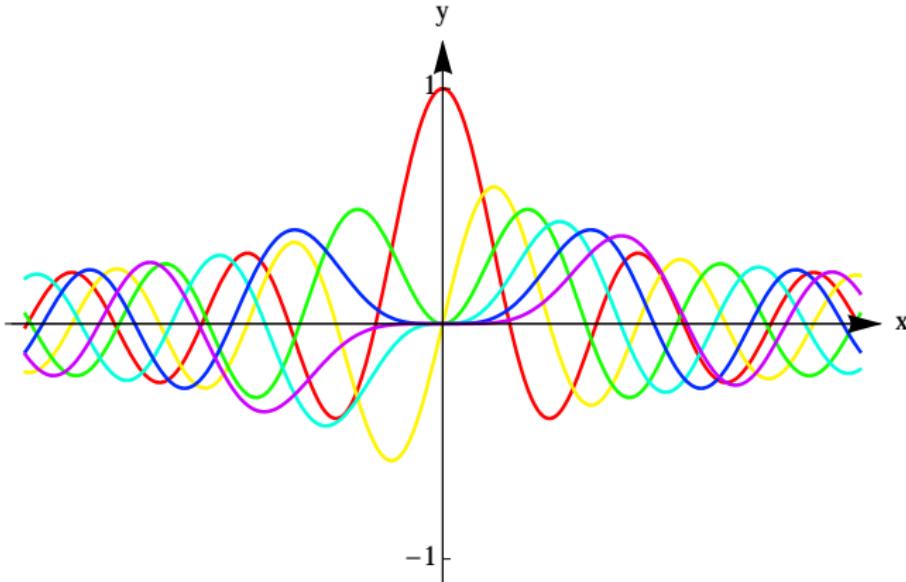
```
N[Table[BesselJZero[0, k], {k, 1, 20}]]  
  
{2.40483, 5.52008, 8.65373, 11.7915, 14.9309,  
18.0711, 21.2116, 24.3525, 27.4935, 30.6346,  
33.7758, 36.9171, 40.0584, 43.1998, 46.3412,  
49.4826, 52.6241, 55.7655, 58.907, 62.0485}
```

# Bessel Functions of the First Kind

The graph below shows some of the Bessel functions:

The graph shows

- ▶  $J_0$  (red),
- ▶  $J_1$  (yellow),
- ▶  $J_2$  (green),
- ▶  $J_3$  (light blue),
- ▶  $J_4$  (dark blue),
- ▶  $J_5$  (violet).



19. Power Series Solutions

20. Fourier Series

21. Separation of Variables for PDEs

## Approximation of Functions

The decomposition of an FM signal into harmonic waves can serve as a inspiration for a more general project: can any periodic function be written as an “infinite linear combination” of elementary (“basis”) functions?

Let us attempt a construction: Consider the (infinite-dimensional) space of continuous complex-valued functions  $C([a, b])$  on an interval  $[a, b] \subset \mathbb{R}$ . We aim to find an “infinite basis” for this space.

However, it soon turns out that the concept of linear independence is impractical to implement, as infinite linear combinations are difficult to handle unambiguously.

The way forward is to define ***orthonormal sets***, which are automatically linear independent. This requires the definition of an inner product.

## Approximation of Functions

We define an inner product on  $C([a, b])$  by

$$\langle f, g \rangle := \int_a^b \overline{f(x)}g(x) dx, \quad f, g \in C([a, b]). \quad (3.2.1)$$

It is clear that

- (i)  $\langle f, f \rangle = \int_a^b |f(x)|^2 dx \geq 0,$
- (ii)  $\langle f, g \rangle = \overline{\langle g, f \rangle},$
- (iii)  $\langle f, \lambda g + \mu h \rangle = \lambda \langle f, g \rangle + \mu \langle f, h \rangle.$

A crucial property that is perhaps not obvious is

$$\langle f, f \rangle = 0 \quad \stackrel{?}{\implies} \quad f = 0.$$

If  $f$  is continuous, it is easy to show that this property holds and  $\langle \cdot, \cdot \rangle$  defines an inner product on  $C([a, b])$ .

## Approximation of Functions

The norm induced by this inner product is

$$\|f\|_2 := \sqrt{\int_a^b |f(x)|^2 dx}, \quad f \in C([a, b]).$$

Since it is an induced norm, it satisfies all the required properties of a norm. In particular, the triangle inequality follows from the Cauchy-Schwarz inequality for  $\langle \cdot, \cdot \rangle$ . Two other important norms are

$$\|f\|_1 := \int_a^b |f(x)| dx, \quad \|f\|_\infty := \sup_{x \in [a, b]} |f(x)|.$$

These norms are not equivalent: we do have

$$\|f\|_1 \leq (b - a)\|f\|_\infty, \quad \|f\|_2 \leq \sqrt{b - a}\|f\|_\infty$$

but the reverse inequalities do not hold in general.

## Uniform, Mean and Mean Square Convergence

In fact, it is easy to find a sequence of functions  $(f_n)$  and a function  $f$  such that

$$\|f_n - f\|_2 \rightarrow 0 \quad \text{but} \quad \|f_n - f\|_\infty \not\rightarrow 0.$$

3.2.1. Example. Consider the sequence given by  $f_n: [0, 1] \rightarrow \mathbb{R}$ ,

$$f_n(x) = \begin{cases} 1 - nx & 0 \leq x \leq 1/n, \\ 0 & 1/n < x \leq 1, \end{cases} \quad (3.2.2)$$

which converges pointwise to

$$f(x) = \begin{cases} 1 & x = 0, \\ 0 & 0 < x \leq 1. \end{cases}$$

The sequence does not converge **uniformly** ( $\|f_n\|_\infty \not\rightarrow 0$ ) but it does converge in the **mean** ( $\|f_n\|_1 \rightarrow 0$ ) and in the **mean square** ( $\|f_n\|_2 \rightarrow 0$ ).

## Uniform, Mean and Mean Square Convergence

From Example 3.2.1 it follows that the sequence (3.2.2) is a Cauchy sequence with respect to  $\|\cdot\|_2$  and we immediately obtain:

**3.2.2. Lemma.** The space  $(C([a, b]), \|\cdot\|_2)$  is not complete.

For various reasons, the completeness of an infinite-dimensional vector space is important for constructing orthonormal bases. Therefore, we make the following definition:

**3.2.3. Definition.** The space  $L^2([a, b])$  is defined as the completion of  $C([a, b])$  with respect to the norm  $\|\cdot\|_2$ .

This is a simple and clear definition, but it is a bit abstract. For example,  $L^2([a, b])$  contains all piecewise continuous functions on  $[a, b]$ , but this is not obvious. We would like to give a more intuitive definition.

## Approximation of Functions

We would like to say something like

$$L^2([a, b]) := \left\{ f: [a, b] \rightarrow \mathbb{C}: \int_a^b |f(x)|^2 dx < \infty \right\} \quad (3.2.3)$$

and define the inner product (3.2.1) on  $L^2([a, b])$ . The Cauchy-Schwarz inequality then guarantees that this is well-defined:

$$\langle f, g \rangle = \int_a^b \overline{f(x)} g(x) dx \leq \sqrt{\int_a^b |f(x)|^2 dx} \sqrt{\int_a^b |g(x)|^2 dx}$$

However, a crucial property of the inner product is not given:

$$\langle f, f \rangle = 0 \quad \stackrel{?}{\implies} \quad f = 0.$$

As an example, consider

$$f(x) = \begin{cases} 1 & x = a, \\ 0 & a < x \leq b. \end{cases} \quad (3.2.4)$$

## Functions Vanishing Almost Everywhere

It is clear that although  $f$  in (3.2.4) vanishes “almost everywhere” but  $f \neq 0$  as a function. The crucial idea is now to identify functions that are the same “almost everywhere”. Let us try to define this formally:

We first introduce the concept of “negligible subsets” of the real numbers as follows:

**3.2.4. Definition.** Let  $\varepsilon > 0$ . A set  $\Omega \subset \mathbb{R}$  is said to have **measure less than  $\varepsilon$**  if there exists a (possibly countably infinite) family of intervals  $\{I_k\}$  such that  $\Omega \subset \bigcup I_k$  and the total length of the intervals is less than  $\varepsilon$ .

A set  $\Omega \subset \mathbb{R}$  is said to have **measure zero** if it has measure less than  $\varepsilon$  for any  $\varepsilon > 0$ . A property is said to hold **almost everywhere** (abbreviated by **a.e.**) on a subset  $D \subset \mathbb{R}$  if the set of all points of  $D$  where it does not hold has measure zero.

## Rational Numbers have Measure Zero

3.2.5. Example. The set  $\Omega = \mathbb{Q}$  of rational numbers has measure zero, seen as follows: The rational numbers are countable, so we write  $\mathbb{Q} = \{a_k\}_{k=1}^{\infty}$ . Then we define the intervals

$$I_k(\varepsilon) := \left( a_k - \frac{\varepsilon}{2^{k+1}}, a_k + \frac{\varepsilon}{2^{k+1}} \right) \quad \text{for } \varepsilon > 0.$$

Obviously,  $\Omega \subset I(\varepsilon) := \bigcup_{k=1}^{\infty} I_k(\varepsilon)$ . Furthermore, the total length of  $I(\varepsilon)$  is given by

$$|I(\varepsilon)| = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

Hence, for any  $\varepsilon > 0$ , the set  $\Omega$  has measure less than  $\varepsilon$ , so  $\Omega$  has measure zero.

Of course, any subset of a set of measure zero also has measure zero. In particular, any finite set of numbers has measure zero.

# Functions Vanishing A.E.

## 3.2.6. Example. The *Dirichlet function*

$$\chi: [0, 1] \rightarrow \mathbb{R}, \quad \chi(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

is equal to zero almost everywhere. We may hence write  $\chi \equiv 0$  a.e.

As in the above example, our goal is now to identify functions that have the same value almost everywhere, i.e., to consider  $f, g: I \rightarrow \mathbb{C}$ ,  $I \subset \mathbb{R}$  an interval, **equivalent** if  $f(x) = g(x)$  for all  $x \in I \setminus \Omega$ , where  $\Omega$  is a set of measure zero. In particular, we would like to have

$$\int_I |f(x) - g(x)| dx = 0.$$

if  $f = g$  a.e. This will be the case for the Riemann integral if  $\Omega$  contains only a finite number of points, but in general there will be problems. For instance, the Dirichlet function is not Riemann-integrable.

## Functions Agreeing A.E.

The technical solution is to introduce an extension of the Riemann integral, called the **Lebesgue integral**, for which, e.g.,

$$\int_0^1 \chi(x) dx = 0$$

and in general any integration over a set of measure zero vanishes. We will not construct the Lebesgue integral here, since for all “common” functions there is no difference in its application. We will just keep in mind that from now on, the integral symbol will denote the Lebesgue integral. This will not make any difference at all to our calculations; we only mention this for completeness.

We will from now on consider two functions to be ***the same function*** if they agree on a set of measure zero.

## Functions Agreeing A.E.

3.2.7. Example. The functions  $f_1, f_2: \mathbb{R} \rightarrow \mathbb{R}$

$$f_1(x) = \begin{cases} 0 & x \leq 0, \\ 1 & 0 < x < 1, \\ 0 & x \geq 1, \end{cases} \quad f_2(x) = \begin{cases} 0 & x < 0, \\ 1 & 0 < x < 1, \\ 0 & x > 1, \\ 1/2 & x = 0 \text{ or } x = 1, \end{cases}$$

differ on a set of measure zero. We have

$$\int_{\mathbb{R}} f_1 = \int_{\mathbb{R}} f_2 = 1$$

and we will not distinguish between  $f_1$  and  $f_2$ .

## Square-integrable Functions

With these two conventions,

- (i) We will not distinguish between functions that differ on a set of measure zero and
- (ii) The integral we use is an extension of the Riemann integral, the definition (3.2.3) then makes sense. We do not need to worry about the second statement in practice, but we should keep in mind that we now consider functions equivalent if they are the same almost everywhere.

Now that we have an inner product on a space of square-integrable functions, we can actually use it in some pretty interesting applications.

# Orthogonality and Orthonormal Systems

3.2.8. Definition. Let  $(V, \langle \cdot, \cdot \rangle)$  be an inner product vector space.

- (i) A vector  $v \in V$  is said to be **normed** (or **normalized**) if  $\langle v, v \rangle = 1$ .  
This is equivalent to  $\|v\| = 1$ .
- (ii) Two vectors  $u, v \in V$  are called **orthogonal** or **perpendicular** if  $\langle u, v \rangle = 0$ . We then write  $u \perp v$ .
- (iii) A family of vectors  $\{v_k\}_{k \in I} \subset V$ ,  $I \subset \mathbb{N}$ , is called an **orthonormal system** if

$$\langle v_j, v_k \rangle = \delta_{jk} = \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{for } j \neq k, \end{cases} \quad j, k \in I,$$

i.e., if  $\|v_k\| = 1$  and  $v_j \perp v_k$  for  $j \neq k$ .

# Orthogonal Polynomials

3.2.9. Example. Consider the space  $L^2([-1, 1])$ . Let us regard the polynomials

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = x^2.$$

Then

$$\langle p_0, p_1 \rangle = \int_{-1}^1 p_0(x)p_1(x) dx = \int_{-1}^1 1 \cdot x dx = 0,$$

$$\langle p_0, p_2 \rangle = \int_{-1}^1 1 \cdot x^2 dx = \frac{2}{3},$$

$$\langle p_1, p_2 \rangle = \int_{-1}^1 x \cdot x^2 dx = 0.$$

Thus,  $p_0 \perp p_1$  and  $p_1 \perp p_2$  but  $p_0 \not\perp p_2$ . Hence,  $(p_0, p_1, p_2)$  is not an orthogonal system.

## Gram-Schmidt Orthonormalization

The goal of Gram-Schmidt orthonormalization is to obtain an orthonormal system from any ordinary set of vectors.

Assume that we have a system of vectors  $(v_1, \dots, v_n)$  in an inner product space  $V$ . We wish to construct a new system  $(w_1, \dots, w_n)$  that is orthonormal. We start with  $v_1$  and norm it, defining

$$w_1 := \frac{v_1}{\|v_1\|}$$

Next, we want to obtain from  $v_2$  a vector  $w_2$  such that  $w_1 \perp w_2$ . By defining

$$w_2 := \frac{v_2 - \langle w_1, v_2 \rangle w_1}{\|v_2 - \langle w_1, v_2 \rangle w_1\|}.$$

we see that  $\|w_2\| = 1$  and  $\langle w_1, w_2 \rangle = 0$ , as desired.

## Gram-Schmidt Orthonormalization

Furthermore,

$$w_3 := \frac{v_3 - \langle w_2, v_3 \rangle w_2 - \langle w_1, v_3 \rangle w_1}{\|v_3 - \langle w_2, v_3 \rangle w_2 - \langle w_1, v_3 \rangle w_1\|}$$

will be normed and orthogonal to  $w_1$  and  $w_2$ . Proceeding in this way, we set

$$w_1 := \frac{v_1}{\|v_1\|}$$

$$w_k := \frac{v_k - \sum_{j=1}^{k-1} \langle w_j, v_k \rangle w_j}{\|v_k - \sum_{j=1}^{k-1} \langle w_j, v_k \rangle w_j\|}, \quad k = 2, \dots, n,$$

and hence obtain an orthonormal system as desired.

# Orthogonal Polynomials

3.2.10. Example. Following on from Example 3.2.9, we consider the polynomials

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = x^2$$

in  $L^2([-1, 1], dx)$ . We apply the orthonormalization procedure to obtain

$$q_0(x) = \frac{p_0(x)}{\|p_0\|_{L^2([-1,1], dx)}} = \frac{1}{\sqrt{\int_{-1}^1 |1|^2 dx}} = \frac{1}{\sqrt{2}},$$

$$q_1(x) = \frac{p_1(x) - \langle q_0, p_1 \rangle q_0(x)}{\|p_1 - \langle q_0, p_1 \rangle q_0\|_{L^2([-1,1], dx)}} = \frac{x}{\sqrt{\int_{-1}^1 |x|^2 dx}} = \sqrt{\frac{3}{2}}x,$$

$$\begin{aligned} q_2(x) &= \frac{p_2(x) - \langle q_1, p_2 \rangle q_1(x) - \langle q_0, p_2 \rangle q_0(x)}{\|p_2 - \langle q_1, p_2 \rangle q_1 - \langle q_0, p_2 \rangle q_0\|_{L^2([-1,1], dx)}} \\ &= \frac{x^2 - 1/3}{\sqrt{\int_{-1}^1 |x^2 - 1/3|^2 dx}} = \frac{3\sqrt{5}}{2\sqrt{2}}(x^2 - 1/3). \end{aligned}$$

## Theorem of Pythagoras

3.2.11. Pythagoras's Theorem. Let  $(V, \langle \cdot, \cdot \rangle)$  be an inner product space and  $z = x + y \in V$ , where  $\langle x, y \rangle = 0$ . Then

$$\|z\|^2 = \|x\|^2 + \|y\|^2.$$

### Proof.

We see directly that

$$\begin{aligned}\|z\|^2 &= \langle z, z \rangle = \langle x + y, x + y \rangle \\&= \langle x, x \rangle + \underbrace{\langle x, y \rangle}_{=0} + \underbrace{\langle y, x \rangle}_{=0} + \langle y, y \rangle \\&= \|x\|^2 + \|y\|^2.\end{aligned}$$

□

## Bessel's Inequality

As a consequence of Pythagoras's Theorem 3.2.11 we obtain the following important result:

**3.2.12. Bessel Inequality.** Let  $(V, \langle \cdot, \cdot \rangle)$  be an inner product space and  $\{e_k\}_{k \in I} \subset V$ ,  $I \subset \mathbb{N}$ , be an orthonormal system in  $V$ . Then, for any  $v \in V$ ,

$$\sum_{k \in I} |\langle e_k, v \rangle|^2 \leq \|v\|^2. \quad (3.2.5)$$

## Bessel's Inequality

Proof.

We prove only the case  $I = \mathbb{N}$ . Let  $\mathcal{B}_N = (e_1, \dots, e_N)$ . Then

$$w := \sum_{n=0}^N \langle e_n, v \rangle e_n \in \text{span } \mathcal{B}_N$$

and we now show that  $w \perp (v - w)$ . Note first that

$$\begin{aligned} \|w\|^2 &= \langle w, w \rangle = \left\langle \sum_{n=0}^N \langle e_n, v \rangle e_n, \sum_{m=0}^N \langle e_m, v \rangle e_m \right\rangle \\ &= \sum_{n=0}^N \sum_{m=0}^N \overline{\langle e_n, v \rangle} \langle e_m, v \rangle \underbrace{\langle e_n, e_m \rangle}_{=\delta_{nm}} = \sum_{n=0}^N |\langle e_n, v \rangle|^2. \end{aligned}$$

## Bessel's Inequality

Proof (continued).

It then follows that

$$\begin{aligned}\langle v - w, w \rangle &= \langle v, w \rangle - \|w\|^2 = \left\langle v, \sum_{n=0}^N \langle e_n, v \rangle e_n \right\rangle - \|w\|^2 \\ &= \sum_{n=0}^N \langle e_n, v \rangle \overline{\langle e_n, v \rangle} - \sum_{n=0}^N |\langle e_n, v \rangle|^2 = 0.\end{aligned}$$

By Pythagoras's Theorem 3.2.11 we then have  $\|v - w\|^2 + \|w\|^2 = \|v\|^2$  or

$$0 \leq \|v - w\|^2 = \|v\|^2 - \|w\|^2 = \|v\|^2 - \sum_{n=0}^N |\langle e_n, v \rangle|^2.$$

Since the inequality holds for all  $N \in \mathbb{N}$  we obtain (3.2.5). □

## Best Approximation

Let  $(V, \langle \cdot, \cdot \rangle)$  be an inner product space,  $v \in V$  and  $\mathcal{B} = \{e_k\}$  an orthonormal system in  $V$ . We seek to approximate  $v$  using a linear combination of the first  $N \in \mathbb{N}$  elements of the orthonormal system,

$$v \approx \sum_{i=1}^N \lambda_i e_i, \quad \lambda_1, \dots, \lambda_N \in \mathbb{F}. \quad (3.2.6)$$

The question is how to choose the coefficients  $\lambda_1, \dots, \lambda_N$  to make the approximation “as good as possible”. We note that

$$\begin{aligned} \left\| v - \sum_{i=1}^N \lambda_i e_i \right\|^2 &= \|v\|^2 + \sum_{i=1}^N |\lambda_i|^2 - \sum_{i=1}^N \lambda_i \langle v, e_i \rangle - \sum_{i=1}^N \bar{\lambda}_i \langle e_i, v \rangle \\ &= \|v\|^2 + \sum_{i=1}^N |\langle e_i, v \rangle - \lambda_i|^2 - \sum_{i=1}^N |\langle v, e_i \rangle|^2 \end{aligned} \quad (3.2.7)$$

It is clear that (3.2.7) is minimal if  $\lambda_i = \langle e_i, v \rangle$ , i.e., the coefficients in (3.2.6) are just the Fourier coefficients.

## Best Approximation

From (3.2.7) we also see that

$$\left\| v - \sum_{i=1}^n \langle e_i, v \rangle e_i \right\| \leq \left\| v - \sum_{i=1}^N \langle e_i, v \rangle e_i \right\| \quad \text{for } n > N, \quad (3.2.8)$$

so the approximation can only improve when we add further elements of the orthonormal system  $\mathcal{B}$  to the approximation.

# Orthogonal Trigonometric Polynomials

3.2.13. Example. In  $L^2([-\pi, \pi])$  we consider the family of functions

$$\mathcal{B}_{\mathcal{F}} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin(nx) \right\}_{n=1}^{\infty}. \quad (3.2.9)$$

These functions are sometimes called ***trigonometric polynomials***. You will check in the assignments that they are orthonormal, i.e., for  $m, n \in \mathbb{N}$ ,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin(mx) \sin(nx) dx = \begin{cases} 0 & n \neq m, \\ 1 & n = m, \end{cases}$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos(mx) \cos(nx) dx = \begin{cases} 0 & n \neq m, \\ 1 & n = m \neq 0, \\ 2 & n = m = 0, \end{cases}$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin(mx) \cos(nx) dx = 0$$

## Orthogonal Trigonometric Polynomials

The best approximation of a function  $f \in L^2([-\pi, \pi])$  using trigonometric polynomials of order  $\leq N$  for some  $N \setminus \{0\}$  is then given by

$$\begin{aligned} f(x) &\approx \left\langle \frac{1}{\sqrt{2\pi}}, f \right\rangle_{L^2} \frac{1}{\sqrt{2\pi}} + \sum_{n=1}^N \left\langle \frac{1}{\sqrt{\pi}} \cos(nx), f \right\rangle_{L^2} \frac{1}{\sqrt{\pi}} \cos(nx) \\ &\quad + \sum_{n=1}^N \left\langle \frac{1}{\sqrt{\pi}} \sin(nx), f \right\rangle_{L^2} \frac{1}{\sqrt{\pi}} \sin(nx) \\ &= \frac{\langle f, 1 \rangle_{L^2}}{2\pi} + \sum_{n=1}^N \frac{\langle \cos(nx), f \rangle_{L^2}}{\pi} \cos(nx) + \sum_{n=1}^N \frac{\langle \sin(nx), f \rangle_{L^2}}{\pi} \sin(nx) \end{aligned}$$

This approximation is called a **Fourier-Euler expansion** of  $f$  to  $N$  terms. Since the trigonometric polynomials are themselves periodic, such an expansion can be used to approximate periodic functions very well.

## Fourier-Euler Expansion

3.2.14. Example. We wish to approximate the  $2\pi$ -periodic function  $f: \mathbb{R} \rightarrow \mathbb{R}$  given by

$$f(x) = |x| \quad \text{for } -\pi < x \leq \pi.$$

We thus have to evaluate the scalar products:

$$\langle f, 1 \rangle_{L^2} = \int_{-\pi}^{\pi} f(x) \cdot 1 \, dx = \int_{-\pi}^{\pi} |x| \, dx = 2 \int_0^{\pi} x \, dx = \pi^2,$$

$$\begin{aligned} \langle f, \cos(nx) \rangle_{L^2} &= \int_{-\pi}^{\pi} |x| \cos(nx) \, dx = 2 \int_0^{\pi} x \cos(nx) \, dx \\ &= \underbrace{\frac{2}{n} x \sin(nx) \Big|_0^\pi}_{=0} - \frac{2}{n} \int_0^{\pi} \sin(nx) \, dx \end{aligned}$$

$$\begin{aligned} &= \frac{2}{n^2} (\cos(n\pi) - 1) = \frac{2}{n^2} ((-1)^n - 1) = \begin{cases} \frac{-4}{n^2} & n \text{ odd}, \\ 0 & n \text{ even}. \end{cases} \end{aligned}$$

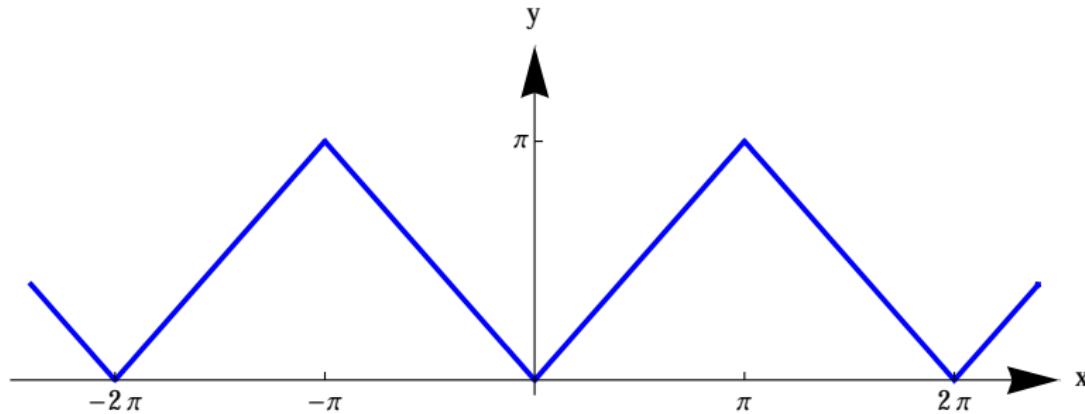
## Fourier-Euler Expansion

Note that

$$\langle f, \sin(nx) \rangle_{L^2} = \int_{-\pi}^{\pi} |x| \sin(nx) dx = 0$$

because  $f$  is an even function. The Fourier expansion is then

$$|x| \approx \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^N \frac{1}{(2k+1)^2} \cos((2k+1)x)$$

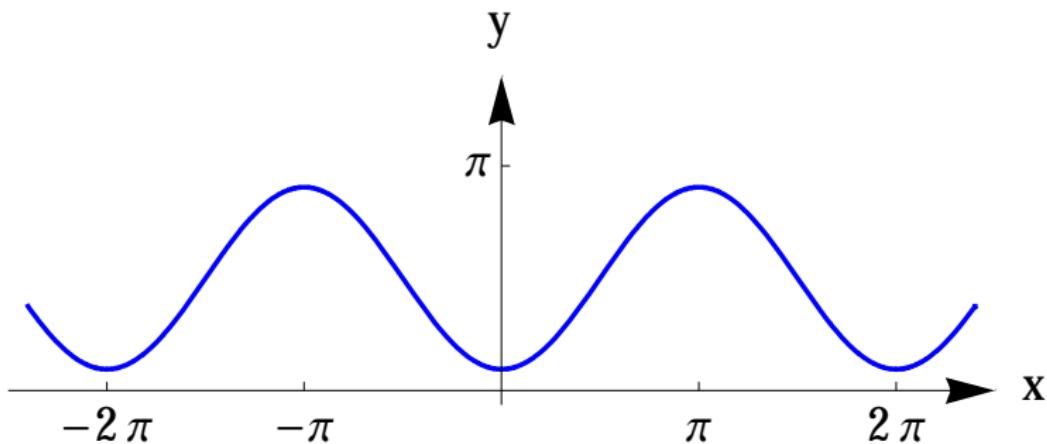


This is the original function  $f(x) = |x|$ ,  $x \in [-\pi, \pi]$ , continued periodically.

## Fourier-Euler Expansion

The Fourier expansion with one term is shown below:

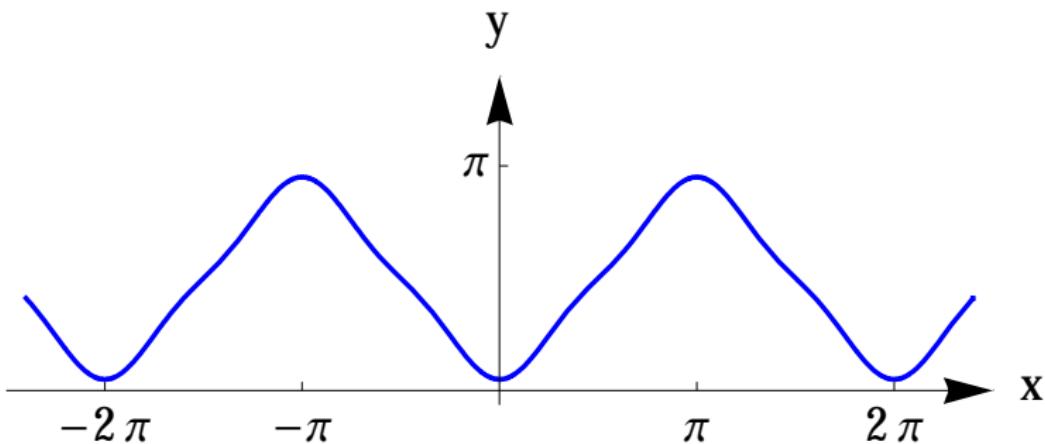
$$|x| \approx \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^1 \frac{1}{(2k+1)^2} \cos((2k+1)x)$$



## Fourier-Euler Expansion

The Fourier expansion with two terms is shown below:

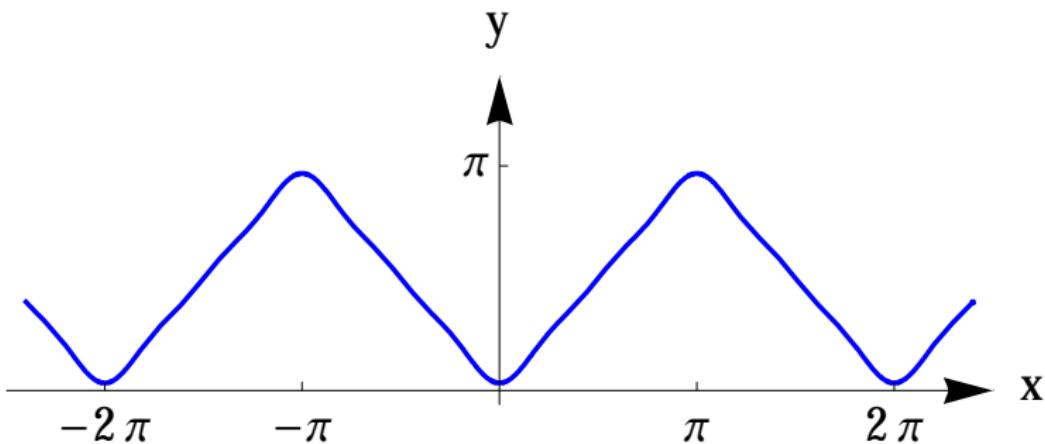
$$|x| \approx \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^2 \frac{1}{(2k+1)^2} \cos((2k+1)x)$$



# Fourier-Euler Expansion

The Fourier expansion with three terms is shown below:

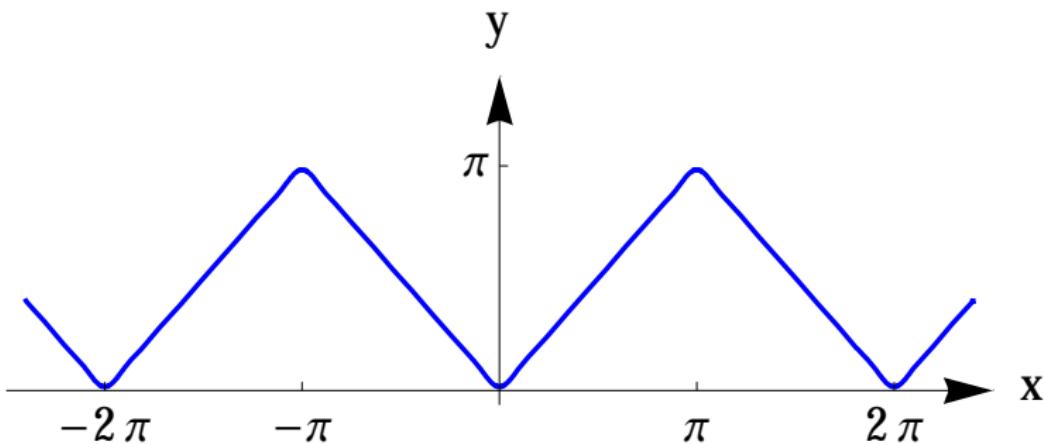
$$|x| \approx \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^3 \frac{1}{(2k+1)^2} \cos((2k+1)x)$$



# Fourier-Euler Expansion

The Fourier expansion with six terms is shown below:

$$|x| \approx \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^6 \frac{1}{(2k+1)^2} \cos((2k+1)x)$$



## Convergence of the Fourier Expansion

We see that the Fourier expansion provides a very good approximation to our function. Of course, the immediate question is whether we may actually write

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos((2k+1)x) \quad (3.2.10)$$

for all  $x \in \mathbb{R}$ . It turns out that the orthonormal system (3.2.9) actually is a basis, so in a certain (square-integrable) sense, this is the case.

# Fourier-Euler Series

3.2.15. Theorem. The orthonormal system

$$\mathcal{B}_{\mathcal{F}} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin(nx) \right\}_{n=1}^{\infty}.$$

is a basis of  $L^2([-\pi, \pi])$ . This means that every  $f \in L^2([-\pi, \pi])$  has a representation as

$$f = \lim_{n \rightarrow \infty} S_N \quad (3.2.11)$$

where  $S_N \in L^2([-\pi, \pi])$  is given by

$$S_N(x) = \frac{\langle f, 1 \rangle_{L^2}}{2\pi} + \sum_{n=1}^N \frac{\langle \cos(nx), f \rangle_{L^2}}{\pi} \cos(nx) + \sum_{n=1}^N \frac{\langle \sin(nx), f \rangle_{L^2}}{\pi} \sin(nx).$$

The limit (3.2.11) is understood in the mean-square sense, i.e.,

$$\|S_N - f\|_{L^2([-\pi, \pi])}^2 = \int_{-\pi}^{\pi} |S_N(x) - f(x)|^2 dx \xrightarrow{N \rightarrow \infty} 0. \quad (3.2.12)$$

## Fourier-Euler Series

3.2.16. Definition. Let  $f \in L^2([-\pi, \pi])$ . Then the series

$$S(x) := \frac{\langle f, 1 \rangle_{L^2}}{2\pi} + \sum_{n=1}^{\infty} \frac{\langle \cos(nx), f \rangle_{L^2}}{\pi} \cos(nx) + \sum_{n=1}^{\infty} \frac{\langle \sin(nx), f \rangle_{L^2}}{\pi} \sin(nx)$$

is called the **Fourier-Euler series** (or just **Fourier series**) of  $f$ . The basis  $\mathcal{B}_{\mathcal{F}}$  is called the **Fourier-Euler basis** of  $L^2([-\pi, \pi])$ ,

### 3.2.17. Remarks.

- (i) The Fourier series of  $f$  does not need to converge uniformly or even pointwise to  $f$ , i.e., it is not necessarily true that  $f(x) = S(x)$  for all  $x \in [-\pi, \pi]$ . We will give an example to illustrate this and then a theorem that gives a useful convergence criterion.

# Fourier-Euler Series

## 3.2.18. Remarks.

- (ii) The Fourier series of an even function on the interval  $[-\pi, \pi]$  will not contain sine terms, the Fourier series of an odd functions will not contain cosine terms. This is due to the scalar products (integrals) over the the symmetric interval and the fact that the sine/cosine functions are odd/even.
- (iii) We can represent functions on arbitrary closed intervals as Fourier series if we scale the Fourier-Euler basis so that the functions are orthonormal and pass through one period on that interval. For example, the Fourier-Euler basis of  $L^2([0, L])$  is

$$\mathcal{B}_{\mathcal{F}} = \left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}} \cos\left(\frac{2\pi nx}{L}\right), \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi nx}{L}\right) \right\}_{n=1}^{\infty} \quad (3.2.13)$$

## Fourier-Euler Series

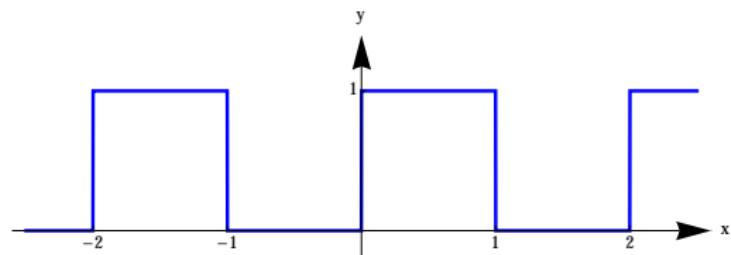
3.2.19. Example. We calculate the Fourier series for the function  $f$  given by

$$f(x) = \begin{cases} 1 & 0 \leq x < 1, \\ 0 & 1 \leq x \leq 2. \end{cases}$$

The representation of  $f$  as a Fourier series is

$$\begin{aligned} f(x) &= \left\langle \frac{1}{\sqrt{2}}, f \right\rangle_{L^2} \frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty} \langle \cos(n\pi x), f \rangle_{L^2} \cos(n\pi x) \\ &\quad + \sum_{n=1}^{\infty} \langle \sin(n\pi x), f \rangle_{L^2} \sin(n\pi x) \\ &= \frac{\langle f, 1 \rangle}{2} + \sum_{n=1}^{\infty} \langle \cos(n\pi x), f \rangle_{L^2} \cos(n\pi x) + \sum_{n=1}^{\infty} \langle \sin(n\pi x), f \rangle_{L^2} \sin(n\pi x) \end{aligned}$$

# Fourier-Euler Series



$$\langle f, 1 \rangle_{L^2} = 1,$$

$$\begin{aligned}\langle f, \cos(n\pi x) \rangle_{L^2} &= \int_0^1 1 \cdot \cos(n\pi x) dx = \frac{1}{n\pi} \sin(n\pi \cdot 1) - \frac{1}{n\pi} \sin(n\pi \cdot 0) \\ &= 0,\end{aligned}$$

$$\begin{aligned}\langle f, \sin(nx) \rangle_{L^2} &= \int_0^1 1 \cdot \sin(n\pi x) dx = \frac{1}{n\pi} \cos(n\pi \cdot 0) - \frac{1}{n\pi} \cos(n\pi \cdot 1) \\ &= \frac{1 - (-1)^n}{n\pi}.\end{aligned}$$

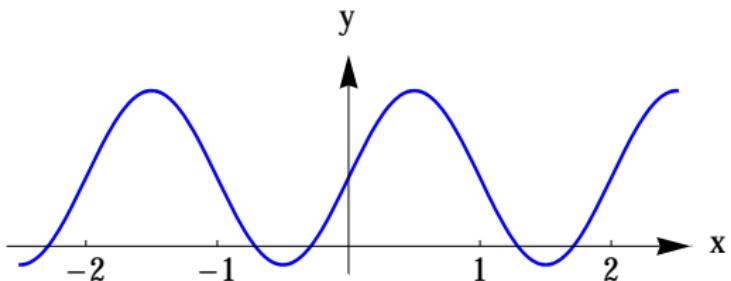
## Fourier-Euler Series

It follows that

$$f(x) = \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin(n\pi x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi x)}{2k+1}$$

For  $x = 1/2$  we obtain the well-known formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - + \dots$$

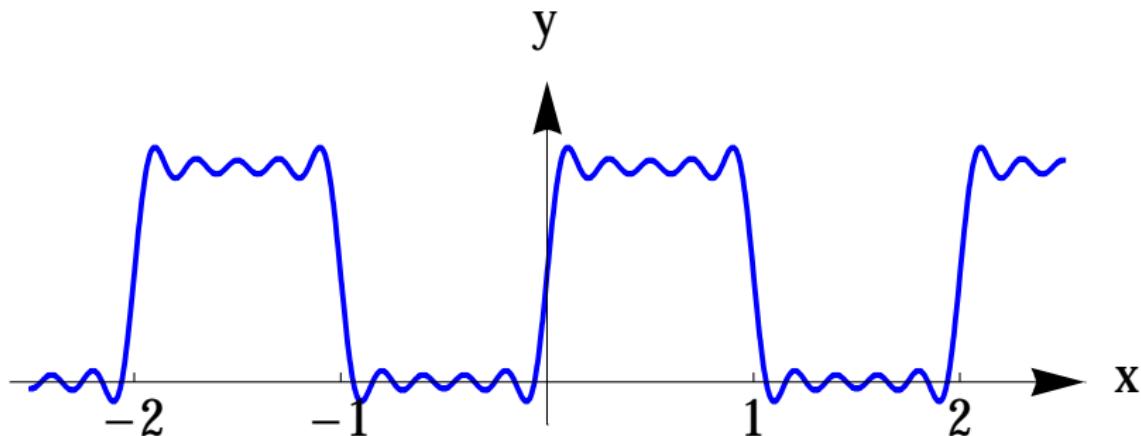


The Fourier expansion with just one term in the series.

## Fourier-Euler Series

It follows that

$$f(x) = \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin(n\pi x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi x)}{2k+1}$$

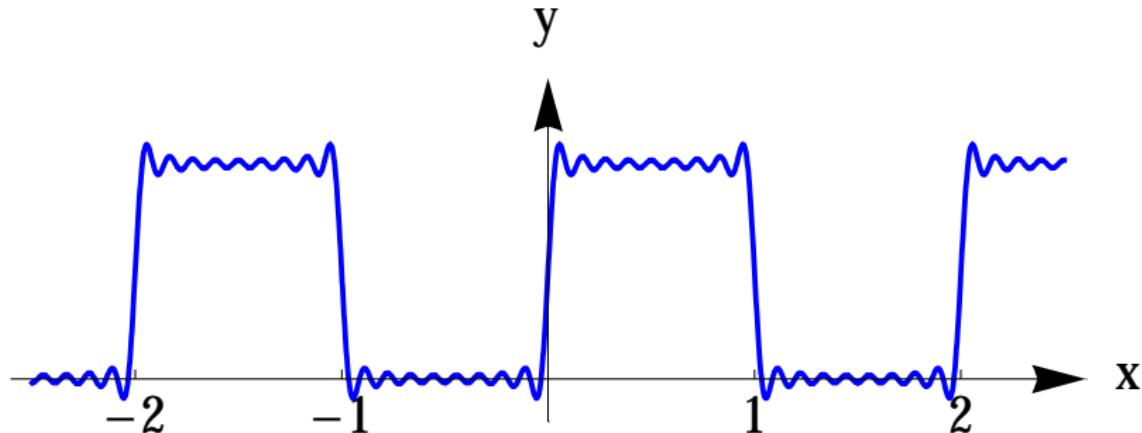


The Fourier expansion with five terms in the series.

## Fourier-Euler Series

It follows that

$$f(x) = \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin(n\pi x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi x)}{2k+1}$$

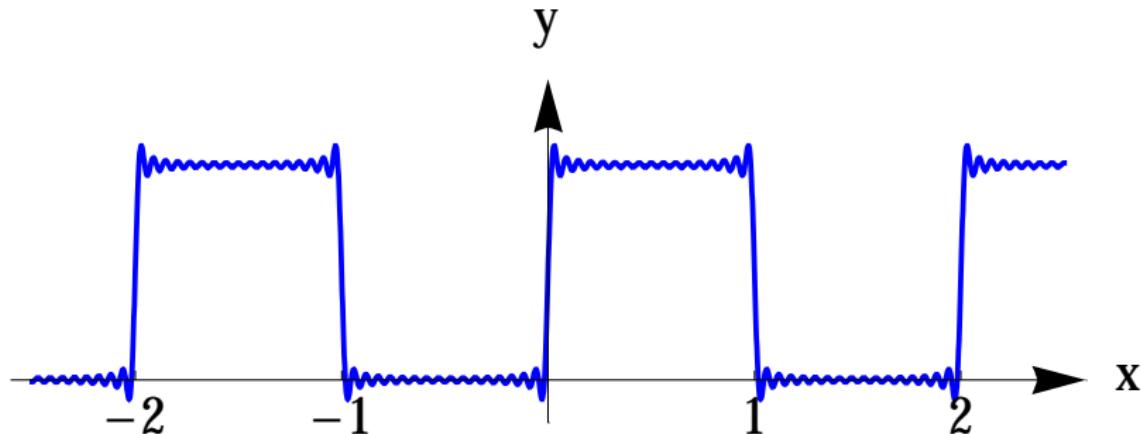


The Fourier expansion with nine terms in the series.

## Fourier-Euler Series

It follows that

$$f(x) = \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin(n\pi x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi x)}{2k+1}$$

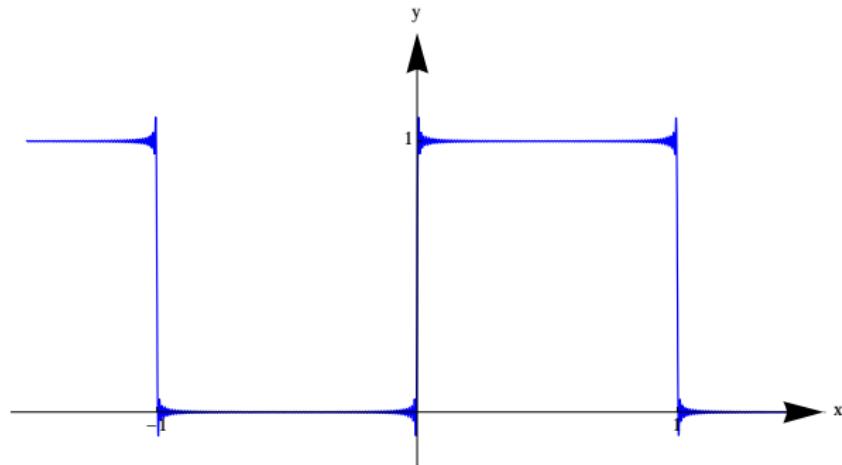


The Fourier expansion with nineteen terms in the series.

## Fourier-Euler Series

It follows that

$$f(x) = \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \sin(n\pi x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi x)}{2k+1}$$



The Fourier expansion with one hundred terms in the series.

## The Gibbs Phenomenon

It becomes obvious from this example that a Fourier series does not need to converge uniformly; the height of the “peaks” near the jump discontinuities does not decrease. (The occurrence of these peaks is known as the **Gibbs phenomenon**.)

## Convergence of Fourier Series

The following result (which we will not prove) clarifies the question of convergence for many applications:

**3.2.20. Theorem.** Let  $f \in L^2([a, b])$  be piecewise continuously differentiable. Then

1. On any subinterval  $[a', b'] \subset [a, b]$  with  $a' > a, b' < b$  on which  $f$  is continuous the Fourier series converges uniformly towards  $f$ .
2. At any point  $x \in [a, b]$ , we have the pointwise limit

$$S_N(x) \xrightarrow{N \rightarrow \infty} \frac{\lim_{y \nearrow x} f(y) + \lim_{y \searrow x} f(y)}{2}.$$

(This is known as **Dirichlet's rule**.)

Thus, at jump discontinuities of  $f$  the Fourier series converges pointwise towards the “mean value” of  $f$  near this point. This is precisely what we have observed in the previous example.

## Pure Sine and Cosine Fourier Series

Interestingly, it is also possible to obtain Fourier series that consist purely of sine or cosine terms (where the absolute term is counted as a cosine term). If a function  $f$  is defined on the interval  $[0, L]$ , then we can extend  $f$  to a function on  $[-L, L]$  and find the Fourier series for  $f$  in the larger interval. How we extend  $f$  depends on which type series we would like to get. If  $f$  is extended as an even function, we get a pure cosine series. If  $f$  is extended as an odd function, we get a pure sine series.

We will make the above arguments precise. Assume that  $f \in L^2([0, 1])$  (we choose this interval for simplicity). Then we can find a “standard” Fourier series of the form

$$\begin{aligned} f(x) &= \langle 1, f \rangle_{L^2} + 2 \sum_{n=1}^{\infty} \langle \cos(2n\pi x), f \rangle_{L^2} \cos(2n\pi x) \\ &\quad + 2 \sum_{n=1}^{\infty} \langle \sin(2n\pi x), f \rangle_{L^2} \sin(2n\pi x) \end{aligned}$$

## Pure Sine and Cosine Fourier Series

Let us now define

$$\tilde{f} = \begin{cases} f(x) & 0 \leq x \leq 1, \\ f(-x) & -1 \leq x \leq 0. \end{cases}$$

This is the **even extension** of  $f$ . Then  $\tilde{f} \in L^2([-1, 1])$  and we can find a Fourier series of the form

$$\begin{aligned}\tilde{f}(x) &= \left\langle \frac{1}{\sqrt{2}}, \tilde{f} \right\rangle_{L^2} \frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty} \left\langle \cos(n\pi x), \tilde{f} \right\rangle_{L^2} \cos(n\pi x) \\ &\quad + \sum_{n=1}^{\infty} \left\langle \sin(n\pi x), \tilde{f} \right\rangle_{L^2} \sin(n\pi x) \\ &= \frac{\langle \tilde{f}, 1 \rangle}{2} + \sum_{n=1}^{\infty} \langle \cos(n\pi x), \tilde{f} \rangle \cos(n\pi x)\end{aligned}$$

There are no sine terms in this series because  $\tilde{f}$  is an even function.

# Pure Sine and Cosine Fourier Series

Note that

$$\langle \tilde{f}, 1 \rangle = \int_{-1}^1 \tilde{f}(x) dx = 2 \int_0^1 f(x) dx,$$

$$\langle \cos(n\pi x), \tilde{f} \rangle = \int_{-1}^1 \cos(n\pi x) \tilde{f}(x) dx = 2 \int_0^1 \cos(n\pi x) f(x) dx.$$

Thus, we can express  $f$  on  $[0, 1]$  either as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2n\pi x) + \sum_{n=1}^{\infty} b_n \sin(2n\pi x)$$

or as

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x).$$

## Pure Sine and Cosine Fourier Series

In these series,

$$a_0 = A_0 = \int_0^1 f(x) dx$$

$$a_n = 2 \langle \cos(2n\pi x), f \rangle_{L^2} = 2 \int_0^1 f(x) \cos(2n\pi x) dx,$$

$$b_n = 2 \langle \sin(2n\pi x), f \rangle_{L^2} = 2 \int_0^1 f(x) \sin(2n\pi x) dx,$$

$$A_n = \left\langle \cos(n\pi x), \tilde{f} \right\rangle_{L^2} = 2 \int_0^1 f(x) \cos(n\pi x) dx.$$

Similarly, we can derive a pure sine series for  $f$  by finding the Fourier series for the **odd extension** of  $f$ ,

$$\tilde{\tilde{f}} = \begin{cases} f(x) & 0 \leq x \leq 1, \\ -f(-x) & -1 \leq x \leq 0. \end{cases}$$

## Pure Sine and Cosine Fourier Bases

For comparison, we list the following bases for  $L^2([0, L])$ :

(i) The Fourier-Euler Basis:

$$\mathcal{B}_1 := \left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}} \cos\left(\frac{2\pi nx}{L}\right), \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi nx}{L}\right) \right\}_{n=1}^{\infty}$$

(ii) The Fourier-Cosine Basis:

$$\mathcal{B}_2 := \left\{ \frac{1}{\sqrt{L}}, \sqrt{\frac{2}{L}} \cos\left(\frac{\pi nx}{L}\right) \right\}_{n=1}^{\infty}$$

(iii) The Fourier-Sine Basis:

$$\mathcal{B}_3 := \left\{ \sqrt{\frac{2}{L}} \sin\left(\frac{\pi nx}{L}\right) \right\}_{n=1}^{\infty}$$

## Pure Sine and Cosine Fourier Bases

3.2.21. Example. Let  $f \in L^2([0, 1])$  be given by

$$f(x) = \frac{1}{2} - |x - 1/2| = \begin{cases} x & 0 \leq x \leq 1/2, \\ 1 - x & 1/2 \leq x \leq 1. \end{cases}$$

Then we can express  $f$  in the Fourier-sine basis  $\mathcal{B} = \{\sqrt{2} \sin(n\pi x)\}_{n=1}^{\infty}$  as

$$f(x) = \sum_{n=1}^{\infty} 2 \int_0^1 f(x) \sin(n\pi x) dx \cdot \sin(n\pi x).$$

We calculate

$$\begin{aligned} \int_0^1 f(x) \sin(n\pi x) dx &= \int_0^{1/2} x \sin(n\pi x) dx + \int_{1/2}^1 (1-x) \sin(n\pi x) dx \\ &= \frac{2}{n^2 \pi^2} \sin(n\pi/2) = \begin{cases} 0 & n = 2k \\ \frac{2(-1)^k}{(2k+1)^2 \pi^2} & n = 2k+1 \end{cases} \end{aligned}$$

# Pure Sine and Cosine Fourier Bases

It follows that

$$f(x) = \sum_{k=0}^{\infty} \frac{4(-1)^k}{(2k+1)^2\pi^2} \sin((2k+1)\pi x)$$

## Exponential Fourier Series

Sometimes, we are interested in complex-valued functions of a real variable. The corresponding Fourier basis has a much more elegant form:

3.2.22. Definition. The sequence

$$\mathcal{B}_{\mathcal{F}} = \left\{ \frac{1}{\sqrt{2\pi}} e^{inx} \right\}_{n=-\infty}^{\infty} \quad (3.2.14)$$

is called a **complex Fourier-Euler basis** or just the **complex Fourier basis** of the space  $L^2([-\pi, \pi])$ .

Clearly, the elements of the real Fourier basis (3.2.9) are linear combinations of the complex basis elements (3.2.14) and vice-versa. It is easy to see that the elements of the complex Fourier basis are orthonormal:

$$\langle e_n, e_m \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} e^{imx} dx = \delta_{nm}.$$

Again, we do not prove that the orthonormal system (3.2.14) is actually a basis.

## Exponential Fourier Series

3.2.23. Example. We find the complex Fourier series for the function  $f(x) = x$ : The Fourier series is

$$\begin{aligned} f(x) &= \sum_{n \in \mathbb{Z}} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} x \, dx \cdot e^{inx} \\ &= \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{(-1)^{|n|} i}{n} e^{inx} \end{aligned} \tag{3.2.15}$$

$$\begin{aligned} &= \sum_{n=1}^{\infty} \frac{(-1)^n i}{n} e^{inx} - \sum_{n=1}^{\infty} \frac{(-1)^n i}{n} e^{-inx} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) \end{aligned} \tag{3.2.16}$$

The complex Fourier series is given by (3.2.15) and we see how it is related to the real Fourier series (3.2.16)

19. Power Series Solutions

20. Fourier Series

21. Separation of Variables for PDEs

# Boundary Value Problems

In contrast to a second-order ***initial value problem (IVP)***

$$y'' = f(y', y, x), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

a ***boundary value problem (BVP)*** for an ODE may have a form such as

$$y'' = f(y', y, x), \quad y(x_0) = y_0, \quad y(x_1) = y_1$$

or

$$y'' = f(y', y, x), \quad y'(x_0) = y_0, \quad y'(x_1) = y_1$$

or a mixture of these. BVPs occur often in physical applications, since one often fixes certain quantities at specific locations (e.g., a vibrating string with fixed ends or a cooling heat reservoir in a constant-temperature environment).

## The Sturm-Liouville Equation

We will make a small excursion into the field of so-called **Sturm-Liouville** problems. These are eigenvalue problems for the Sturm-Liouville operator

$$L := -\frac{1}{r(x)} \left( \frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + q(x) u \right) \quad (3.3.1)$$

defined on a suitable subset of a weighted space of square-integrable functions on an interval  $I \subset \mathbb{R}$ . The Sturm-Liouville equation is

$$Lu = \lambda u, \quad x \in I, \quad (3.3.2)$$

or

$$\frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + (q(x) + \lambda r(x)) u = 0, \quad x \in I, \quad (3.3.3)$$

with specified functions  $p, q, r$ .

## Generality of the Sturm-Liouville Equation

At first glance, it may appear that the form of the equation (3.3.3) is quite restrictive. However, any linear ODE of 2nd order can be written in the form (3.3.3): Suppose

$$L = a_2(x) \frac{d^2}{dx^2} + a_1(x) \frac{d}{dx} + a_0(x)$$

is given. Set

$$p(x) = e^{\int \frac{a_1}{a_2}}, \quad r(x) = -\frac{p(x)}{a_2(x)}, \quad q(x) = -a_0(x)r(x). \quad (3.3.4)$$

Then  $L$  is given by (3.3.1) with  $p, q, r$  as in (3.3.4).

## Regular Sturm-Liouville Equations

3.3.1. Example. Let  $L = x^2 \frac{d^2}{dx^2} + \frac{d}{dx} + x^3$ . Calculating

$$p(x) = e^{\int \frac{a_1}{a_2}} = e^{-\frac{1}{x}}, \quad r(x) = -\frac{1}{x^2} e^{-\frac{1}{x}}, \quad q(x) = x e^{-\frac{1}{x}},$$

we can write

$$L = x^2 e^{\frac{1}{x}} \left( \frac{d}{dx} \left( e^{-\frac{1}{x}} \frac{d}{dx} \right) + x e^{-\frac{1}{x}} \right)$$

and  $L$  now has Sturm-Liouville form.

3.3.2. Definition. The equation (3.3.3) and the operator (3.3.1) are said to be **regular** if

- ▶  $I = (a, b)$  is a finite interval,
- ▶  $p, p', q, r \in C([a, b])$ ,
- ▶  $p(x) > 0$  and  $r(x) > 0$  for all  $x \in [a, b]$ .

# Sturm-Liouville Problems (ODE Point of View)

A **regular Sturm-Liouville boundary value problem** on an interval  $[a, b]$  consists of the regular Sturm-Liouville equation

$$\frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + (q(x) + \lambda r(x)) u = 0, \quad x \in (a, b), \quad (3.3.5a)$$

together with boundary conditions

$$\begin{aligned} B_a u &:= \alpha_1 u(a) + \beta_1 u'(a) = 0, \\ B_b u &:= \alpha_2 u(b) + \beta_2 u'(b) = 0, \end{aligned} \quad (3.3.5b)$$

where  $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$  and  $|\alpha_1| + |\beta_1| \neq 0$ ,  $|\alpha_2| + |\beta_2| \neq 0$ . We sometimes refer to  $B_a$  and  $B_b$  as **boundary operators**.

## Solution of Sturm-Liouville Problems

A **solution** of the regular Sturm-Liouville problem (3.3.5) consists of

- ▶ numbers (eigenvalues)  $\lambda$  and
- ▶ (eigen-) functions  $u_\lambda$

such that (3.3.5) holds.

The eigenvalue problem for  $L$  is formulated as follows:  $L$  is regarded as an operator on the Hilbert space

$$L^2([a, b]; r(x) dx) = \left\{ u: [a, b] \rightarrow \mathbb{C}: \int_a^b |u(x)|^2 r(x) dx < \infty \right\}$$

defined on the domain  $U \subset L^2([a, b]; r(x) dx)$  given by

$$U := \{u \in C^2([a, b]): B_a u = B_b u = 0\}. \quad (3.3.6)$$

The eigenvalues  $\lambda$  and eigenfunctions  $u_\lambda$  of  $L$  then give the solution of the Sturm-Liouville problem.

# Spectral Theorem for Sturm-Liouville Operators

It is possible to prove the following fundamental theorem for the Sturm-Liouville operator  $L$ :

## 3.3.3. Theorem. The eigenvalue problem

$$Lu = \lambda u, \quad u \in U,$$

has at least one eigenvalue  $\lambda$ . Furthermore,

- ▶ All the eigenvalues of  $L$  are real numbers;
- ▶ There exists an infinite sequence of eigenvalues  $(\lambda_n)$  which can be arranged in an increasing sequence such that  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ ;
- ▶ The eigenfunctions of  $L$  are an orthonormal basis of the space  $L^2([a, b]; r(x) dx)$ .

# Solving a Sturm-Liouville Problem

3.3.4. Example. Consider the Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad 0 < x < l, \quad y(0) = 0, \quad y(l) = 0. \quad (3.3.7)$$

This corresponds to (3.3.5) with  $[a, b] = [0, l]$  and  $p(x) = r(x) = 1$ ,  $q(x) = 0$  for all  $x \in (0, l)$ .

We first try to find the possible eigenvalues. Note that, as for eigenvalue problems in general, we do not allow the function  $u(x) = 0$  for all  $x \in [a, b]$  to be an eigenfunction.

Since the eigenvalues of  $L$  are known to be real, we may discuss the cases of positive, negative and zero eigenvalues separately.

## Solving a Sturm-Liouville Problem

In another similarity with the treatment for matrices, we will often make an ***ansatz*** of the form

$$y_\lambda(x) = e^{\rho(\lambda)x} \quad (3.3.8)$$

in order to obtain a characteristic polynomial.

1.  $\lambda > 0$ : We insert (3.3.8) into (3.3.7). We obtain

$$(\rho(\lambda)^2 + \lambda)y_\lambda(x) = 0, \quad x \in [0, l]. \quad (3.3.9)$$

Since  $y_\lambda$  can not vanish everywhere, we obtain the characteristic polynomial  $\rho(\lambda)^2 + \lambda = 0$ , which for  $\lambda > 0$  has the solutions

$$\rho(\lambda) = \pm i\sqrt{\lambda}.$$

We hence have a general solution of the form

$$\begin{aligned} y_\lambda(x) &= c_1 e^{i\sqrt{\lambda}x} + c_2 e^{-i\sqrt{\lambda}x} \\ &= C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x) \end{aligned}$$

using Euler's relation  $e^{ix} = \cos x + i \sin x$ .

## Solving a Sturm-Liouville Problem

1.  $\lambda > 0$ : Using the boundary condition  $y(0) = 0$  we see that in fact  $C_1 = 0$ :

$$0 = y_\lambda(0) = C_1 \cos(\sqrt{\lambda}0) + C_2 \sin(\sqrt{\lambda}0) = C_1.$$

Thus, the general solution must have the form

$$y_\lambda(x) = C_2 \sin(\sqrt{\lambda}x), \quad C_2 \neq 0.$$

The other boundary condition states that  $y(l) = 0$ , leading to

$$0 = y_\lambda(\pi) = C_2 \sin(\sqrt{\lambda}\pi),$$

so we need  $\sqrt{\lambda}l$  to be an integer multiple of  $\pi$ . Thus we obtain the eigenvalues

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n = 1, 2, 3, \dots$$

with associated eigenfunctions  $y_n(x) = C \cdot \sin(n\pi x/l)$ .

## Solving a Sturm-Liouville Problem

2.  $\lambda < 0$ : We again obtain (3.3.9) and the characteristic equation  $\rho(\lambda)^2 + \lambda = 0$ . However, now  $\lambda < 0$ , so we have the solutions

$$\rho(\lambda) = \pm\sqrt{|\lambda|}.$$

The general solution can then be written as

$$\begin{aligned}y_\lambda(x) &= c_1 e^{\sqrt{|\lambda|}x} + c_2 e^{-\sqrt{|\lambda|}x} \\&= C_1 \cosh(\sqrt{|\lambda|}x) + C_2 \sinh(\sqrt{|\lambda|}x)\end{aligned}$$

since  $2 \cosh x = e^x + e^{-x}$  and  $2 \sinh x = e^x - e^{-x}$ . The condition  $y(0) = 0$  requires  $C_1 = 0$ , while the condition  $y(l) = 0$  can only be satisfied if  $C_2 = 0$ . Thus there are no strictly negative eigenvalues.

## Solving a Sturm-Liouville Problem

3.  $\lambda = 0$ : In this case, the equation (3.3.7) becomes  $y'' = 0$ , so we obtain

$$y_0(x) = c_1 x + c_2.$$

Now  $y_0(0) = 0$  implies  $c_2 = 0$  and  $y_0(l) = 0$  implies  $c_1 = 0$ . There is no eigenfunction for  $\lambda = 0$ .

Summarizing, the Sturm-Liouville problem (3.3.7) has only strictly positive eigenvalues

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad n \in \mathbb{N} \setminus \{0\},$$

with eigenfunctions

$$y_n(x) = C \cdot \sin(n\pi x/l).$$

Note how this is in accordance with the statements of Theorem 3.3.3.

## Separation of Variables

In many elementary situations, the following idea is promising: if a function  $u = u(x_1, \dots, x_n)$  is sought as a solution to a quasi-linear second-order PDE in  $n$  variables, we try to find solutions of the form

$$u(x_1, \dots, x_n) = u_1(x_1) \cdot u_2(x_2) \cdots u_n(x_n).$$

We thus restrict ourselves to solutions that are the product of functions of just one variable. If we find such a solution, it may be possible to choose it in such a way that prescribed boundary conditions are met and a BVP associated to the PDE is solved. This generally involves solving a Sturm-Liouville BVP for an ODE. However, there are of course many PDEs that do not have suitable or even any solutions of this form. In these cases, the separation of variables technique simply fails and other methods must be resorted to.

## The Wave Equation for a Finite String

We will illustrate the method of separation of variables through the wave equation for a finite string,

$$c^2 u_{xx} = u_{tt}, \quad 0 < x < l, \quad t > 0. \quad (3.3.10)$$

(This equation was derived in (3.1.21).) We typically impose **initial conditions**

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 \leq x \leq l, \quad (3.3.11)$$

where  $f, g \in L^2([0, l])$ . Since we have a finite string, we must also specify the behavior of the solution at the ends. We impose the **boundary conditions**

$$u(0, t) = u(l, t) = 0, \quad t > 0, \quad (3.3.12)$$

which corresponds to a string with fixed ends.

## The Wave Equation for a Finite String

We make the ansatz  $u(x, t) = X(x)T(t)$  to obtain

$$\frac{1}{c^2 T} T_{tt} = \frac{1}{X} X_{xx}.$$

Since the left hand side depends only on  $t$  and the right-hand side depends only on  $x$ , they must both be constant. We call this constant  $-\lambda \in \mathbb{C}$  and hence obtain the ODEs

$$-X'' = \lambda X, \quad T'' = -\lambda c^2 T.$$

The boundary conditions (3.3.12) translate directly into conditions for  $X$ : From  $u(0, t) = u(l, t) = 0$  we obtain

$$X(0)T(t) = X(l)T(t) = 0,$$

so either  $T(t) = 0$  for all  $t$  (this would imply the trivial solution  $u(x, t) = 0$ ) or  $X(0) = X(l) = 0$ . We thus have the Sturm-Liouville problem

$$X'' + \lambda X = 0, \quad X(0) = X(l) = 0. \quad (3.3.13)$$

## The Wave Equation for a Finite String

We have already solved this problem in Example 3.3.4. The eigenvalues  $\lambda$  are real, and we see that only the case  $\lambda > 0$  yields non-trivial solutions. In particular, we obtain eigenvalues

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \dots$$

and eigenfunctions

$$X_n(x) = C_n \sin\left(\sqrt{\lambda_n}x\right) = C_n \sin\left(\frac{n\pi x}{L}\right), \quad C_n \in \mathbb{R}. \quad (3.3.14)$$

We next need to solve

$$T'' = -\left(\frac{n\pi}{L}\right)^2 c^2 T$$

for  $n = 1, 2, 3, \dots$ . There are no boundary values prescribed for this ODE, so we find the general solution

$$T_n(t) = D_n \cos\left(\frac{cn\pi t}{L}\right) + E_n \sin\left(\frac{cn\pi t}{L}\right), \quad D_n, E_n \in \mathbb{R}.$$

## Normal Modes

Thus, we obtain a family of solutions given by

$$\begin{aligned} u_n(x, t) &= X_n(x) T_n(t) \\ &= \left( F_n \cos \left( \frac{cn\pi t}{l} \right) + G_n \sin \left( \frac{cn\pi t}{l} \right) \right) \sin \left( \frac{n\pi x}{l} \right) \end{aligned} \quad (3.3.15)$$

for  $F_n, G_n \in \mathbb{R}$  and  $n = 1, 2, 3, \dots$ . Each function  $u_n$  is called a **normal mode** and the corresponding frequency

$$\omega_n := \frac{cn\pi}{l}$$

a **normal frequency**. The **qualitative mode shape** is given by

$$u_n(x, t_0) = c_0 \sin \left( \frac{n\pi x}{l} \right)$$

where  $t_0$  is arbitrarily chosen so that  $c_0 \neq 0$ .

## The General Solution

The wave equation (3.3.10) together with the boundary conditions (3.3.12) is a linear problem, so that any linear combination of normal modes (3.3.15) solves the equation.

Without any regard to convergence, let us write the general solution as an infinite superposition,

$$u(x, t) = \sum_{n=1}^{\infty} \left( F_n \cos \left( \frac{cn\pi t}{L} \right) + G_n \sin \left( \frac{cn\pi t}{L} \right) \right) \sin \left( \frac{n\pi x}{L} \right) \quad (3.3.16)$$

with coefficients  $F_n, G_n \in \mathbb{R}$ ,  $n \in \mathbb{N} \setminus \{0\}$ .

We may now consider the initial conditions (3.3.11),

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad 0 \leq x \leq L,$$

## The Initial Conditions

Formally, we have from (3.3.16)

$$u(x, 0) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi x}{l}\right), \quad u_t(x, 0) = \sum_{n=1}^{\infty} G_n \frac{cn\pi}{l} \sin\left(\frac{n\pi x}{l}\right).$$

Expanding the function  $f$  in  $u(x, 0) = f(x)$  into a Fourier-sine series, we see that

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx \cdot \sin\left(\frac{n\pi x}{l}\right).$$

Therefore, to match the first initial condition, we need to choose

$$F_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

where our solution converges in the  $L^2$ -mean to  $f$  (and perhaps even uniformly; see Theorem 3.2.20).

## The Initial Conditions

In order to satisfy  $u_t(x, 0) = g(x)$  we must similarly choose

$$\frac{c\pi n}{l} G_n = \frac{2}{l} \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx.$$

The solution to the Cauchy problem is therefore given by

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx \cos\left(\frac{cn\pi t}{l}\right) \sin\left(\frac{n\pi x}{l}\right) + \\ &\quad + \sum_{n=1}^{\infty} \frac{2}{n\pi c} \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx \sin\left(\frac{cn\pi t}{l}\right) \sin\left(\frac{n\pi x}{l}\right) \end{aligned}$$

## Finite String with Fixed Ends

3.3.5. Example. Consider the wave equation  $u_{xx} = u_{tt}$  on  $[0, 1]$  with fixed ends  $u(0, t) = u(1, t) = 0$  and initial displacement

$$u(x, 0) = f(x) = \frac{1}{2} - |x - 1/2| = \begin{cases} x & 0 \leq x \leq 1/2 \\ 1 - x & 1/2 \leq x \leq 1 \end{cases}$$

and  $u_t(x, 0) = 0$  (no initial velocity).

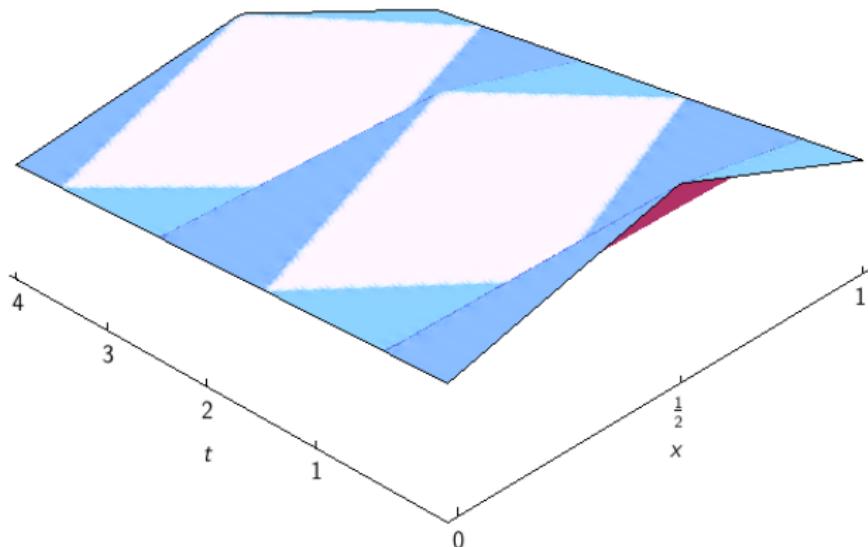
Here  $l = 1$  and  $g(x) = 0$  so the problem reduces to calculating

$$\begin{aligned} \int_0^1 f(x) \sin(n\pi x) dx &= \int_0^{1/2} x \sin(n\pi x) dx + \int_{1/2}^1 (1-x) \sin(n\pi x) dx \\ &= \frac{2}{n^2\pi^2} \sin(n\pi/2) \\ &= \begin{cases} 0 & n = 2k \\ \frac{2(-1)^k}{(2k+1)^2\pi^2} & n = 2k+1 \end{cases} \end{aligned}$$

# Finite String with Fixed Ends

The solution is therefore given by

$$u(x, t) = \sum_{k=0}^{\infty} \frac{2(-1)^k}{(2k+1)^2\pi^2} \cos((2k+1)\pi t) \sin((2k+1)\pi x) \quad (3.3.17)$$



The animation below can be viewed with Adobe reader 10 or newer:

## Final Exam

The preceding material completes the final third of the course material. It encompasses everything that will be the subject of the **Final Exam**.

The exact exam date will be announced on Canvas.

No calculators or other aids will be permitted during the exam. A sample exam with solutions has been uploaded to Canvas. Please study it carefully, including the instructions on the cover page.