8.2 偏导数

8.2.1 偏导数的定义及其计算

定义 2.1. 设函数$z=f(x,y)$在点(x_0,y_0)的某邻域内有定义，当y固定在y_0，而x在x_0处取得增量Δx时，函数相应地取得增量$f(x_0+\Delta x,y_0)-f(x_0,y_0)$(称作函数对$x$ 的偏增量)．如果极限

$$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$

存在，则称此极限为函数$z=f(x,y)$在点(x_0,y_0)对x的偏导数，记作$\frac{\partial z}{\partial x}\Big|_{(x_0,y_0)}$，$z_x(x_0,y_0)$，$z'_x(x_0,y_0)$，$\frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}$，$f_x(x_0,y_0)$ 或$f'_x(x_0,y_0)$．即

$$\begin{aligned}f'_x(x_0,y_0)&=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}\\&=\lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}.\end{aligned}$$

类似地, 若极限

$$\lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$$

存在, 则称此极限为函数$z=f(x,y)$在点(x_0,y_0)对y的偏导数, 记作$\frac{\partial z}{\partial y}\Big|_{(x_0,y_0)}$, $z_y(x_0,y_0)$, $z'_y(x_0,y_0)$, $\frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}$, $f_y(x_0,y_0)$ 或$f'_y(x_0,y_0)$.

当函数$z=f(x,y)$在点(x_0,y_0)同时存在对x与对y的偏导数时, 简称$f(x,y)$在点(x_0,y_0)可偏导.

偏导数的几何意义

$\frac{\partial f}{\partial x}\Big|_{(a,b)}=\frac{\mathrm{d}}{\mathrm{d}x}f(x,b)\Big|_{x=a}$ 是曲线$C_1:\begin{cases}z=f(x,y),\\y=b\end{cases}$ 在点$P(a,b,f(a,b))$处的切线T_1对x轴的斜率. $\frac{\partial f}{\partial y}\Big|_{(a,b)}=\frac{\mathrm{d}}{\mathrm{d}x}f(a,y)\Big|_{y=b}$ 是曲线$C_2:\begin{cases}z=f(x,y),\\x=a\end{cases}$ 在点$P(a,b,f(a,b))$处的切线T_2对y轴的斜率.

例 2.1. 求曲线$\begin{cases}z=\frac{x^2+y^2}{4},\\y=4\end{cases}$ 在点$(2,4,5)$处的切线对于x轴的倾角θ.

解: 由于$z'_x(x,y)=\frac{x}{2}$, 而$z'_x(2,4)=1$, 故$\tan\theta=1$, 于是$\theta=\frac{\pi}{4}$.

如果函数$z=f(x,y)$在某平面区域D内的每一点(x,y)处都存在对x的偏导数, 那么这个偏导数仍然是x,y的函数, 称它为$f(x,y)$对x的偏导函数. 记作$\frac{\partial z}{\partial x}$, $\frac{\partial f}{\partial x}$, z_x, z'_x, $f_x(x,y)$或$f'_x(x,y)$. 类似地有$f(x,y)$对y的偏导函数. 在不致产生误解时, 偏导函数也简称为偏导数.

$f(x,y)$在点(x_0,y_0)处的偏导数$f_x(x_0,y_0)$和$f_y(x_0,y_0)$，就是偏导函数$f_x(x,y)$和$f_y(x,y)$ 在点(x_0,y_0)处的值.

n元函数偏导数

n元函数$u=f(x_1,x_2,\cdots,x_n)$在点$Q(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n$处关于$x_k(k=1,2,\cdots,n)$的偏导数$\left.\frac{\partial u}{\partial x_k}\right|_Q$定义为

$$\left.\frac{\partial u}{\partial x_k}\right|_Q=\lim_{\Delta x_k\to 0}\frac{f(x_1,\cdots,x_k+\Delta x_k,\cdots,x_n)-f(x_1,\cdots,x_k,\cdots,x_n)}{\Delta x_k}.$$

从偏导数的定义中可以看出，多元函数对某一变量的偏导数，实际上是固定其它变量后仅对该变量的导数. 因而在对某变量求导的过程中，只需把其它变量视为常数，对该变量用一元函数求导的方法，即可求出相应的导数.

例 2.2. 求函数$z=x^2-y^3+2xy$在点$(1,2)$处的偏导数.

解:

$$z_x=2x+2y,\quad z_y=-3y^2+2x,$$

所以

$$z_x(1,2)=6,\quad z_y(1,2)=-10.$$

例 2.3. 设$z=\arctan\frac{x}{y}$，求z_x,z_y.

解: 由一元复合函数的求导法则得

$$z_x=\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\frac{1}{y}=\frac{y}{x^2+y^2},$$

$$z_y=\frac{1}{1+\left(\frac{x}{y}\right)^2}\cdot\left(-\frac{x}{y^2}\right)=\frac{-x}{x^2+y^2}.$$

例 2.4. 求$r=\sqrt{x^2+y^2+z^2}$的偏导数.

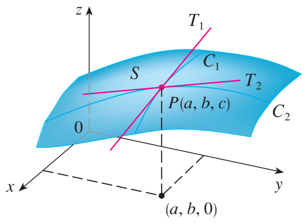

FIGURE 1
The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

Interpretations of Partial Derivatives

To give a geometric interpretation of partial derivatives, we recall that $z=f(x,y)$ represents a surface S (the graph of f). If $f(a,b)=c$, then the p lies on S. By fixing $y=b$, we are restricting our attention to the curve C vertical plane $y=b$ intersects S. (In other words, C_1 is the trace of S in the Likewise, the vertical plane $x=a$ intersects S in a curve C_2. Both of the C_2 pass through the point P. (See Figure 1.)

Note that the curve C_1 is the graph of the function $g(x)=f(x,b)$, so the s gent T_1 at P is $g'(a)=f_x(a,b)$. The curve C_2 is the graph of the function G so the slope of its tangent T_2 at P is $G'(b)=f_y(a,b)$.

Thus the partial derivatives $f_x(a,b)$ and $f_y(a,b)$ can be interpreted ge the slopes of the tangent lines at $P(a,b,c)$ to the traces C_1 and C_2 of S in the

As we have seen in the case of the heat index function, partial derivativ interpreted as *rates of change*. If $z=f(x,y)$, then $\partial z/\partial x$ represents the rat z with respect to x when y is fixed. Similarly, $\partial z/\partial y$ represents the rate of ch respect to y when x is fixed.

EXAMPLE 2 If $f(x,y)=4-x^2-2y^2$, find $f_x(1,1)$ and $f_y(1,1)$ and inte numbers as slopes.

SOLUTION We have

$$f_x(x,y)=-2x \qquad f_y(x,y)=-4y$$

$$f_x(1,1)=-2 \qquad f_y(1,1)=-4$$

The graph of f is the paraboloid $z=4-x^2-2y^2$ and the vertical plane sects it in the parabola $z=2-x^2$, $y=1$. (As in the preceding discussion C_1 in Figure 2.) The slope of the tangent line to this parabola at the point ($f_x(1,1)=-2$. Similarly, the curve C_2 in which the plane $x=1$ intersects loid is the parabola $z=3-2y^2$, $x=1$, and the slope of the tangent line a $f_y(1,1)=-4$. (See Figure 3.)

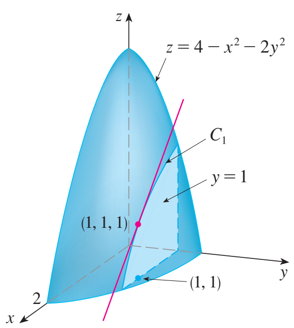

FIGURE 2

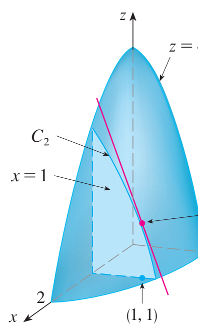

FIGURE 3

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解: 由一元复合函数的求导法则得

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}, \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}, \quad \frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}.$$

例 2.5. *设理想气体的物态方程$pV = RT$(R为常量), 求证*

$$\frac{\partial p}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial p} = -1.$$

证明:

$$p = \frac{RT}{V} \quad \Rightarrow \quad \frac{\partial p}{\partial V} = -\frac{RT}{V^2}, \qquad V = \frac{RT}{p} \quad \Rightarrow \quad \frac{\partial V}{\partial T} = \frac{R}{p},$$

$$T = \frac{pV}{R} \quad \Rightarrow \quad \frac{\partial T}{\partial p} = \frac{V}{R}.$$

故

$$\frac{\partial p}{\partial V} \cdot \frac{\partial V}{\partial T} \cdot \frac{\partial T}{\partial p} = -\frac{RT}{V^2} \cdot \frac{R}{p} \cdot \frac{V}{R} = -\frac{RT}{pV} = -1.$$

偏导数记号是一个整体记号, 不能看作分子与分母的商!

例 2.6. 设$f(x,y) = \begin{cases} \frac{x^3+y^2}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0, \\ 0, & x^2+y^2 = 0, \end{cases}$ 证明$f(x,y)$在点$(0,0)$处连续并求$f'_x(0,0)$和$f'_y(0,0)$.

证明: 当$0 < x^2+y^2 < 1$时,

$$\left|\frac{x^3+y^2}{\sqrt{x^2+y^2}}\right| \leq \frac{|x|^3+y^2}{\sqrt{x^2+y^2}} \leq \sqrt{x^2+y^2}.$$

又$\lim\limits_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0$, 故$\lim\limits_{(x,y)\to(0,0)} \frac{x^3+y^2}{\sqrt{x^2+y^2}} = 0 = f(0,0)$. 所以$f(x,y)$在点$(0,0)$处连续.

$$f'_x(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{x^3}{x|x|} = 0,$$

$$f'_y(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0} \frac{y^2}{y|y|} \quad (\text{不存在}).$$

例 2.7. 设$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & x^2+y^2 \neq 0, \\ 0, & x^2+y^2 = 0, \end{cases}$ 证明$f(x,y)$在点$(0,0)$处可偏导但不连续.

证明: 由偏导定义得

$$f_x(0,0) = \lim_{\Delta x\to 0} \frac{f(\Delta x,0)-f(0,0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{0}{\Delta x} = 0,$$

$$f_y(0,0) = \lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y} = \lim_{\Delta y\to 0}\frac{0}{\Delta y} = 0.$$

所以函数$f(x,y)$在点$(0,0)$处的两个偏导都存在, 但在上节中我们知道, 该函数在点$(0,0)$处不连续.

多元函数在一点处的可偏导性并不能保证函数在该点处连续, 这是多元函数与一元函数在性质上的不同之处.

8.2.2 高阶偏导数

高阶偏导数

定义 2.2. 设函数$z = f(x,y)$在区域D内处处存在偏导数$f_x(x,y)$与$f_y(x,y)$, 如果这两个偏导数仍可偏导, 则称它们的偏导数为函数$z = f(x,y)$的二阶偏导数, 按照求导次序的不同, 有下列四种不同的二阶偏导数.

函数$f(x,y)$关于x的二阶偏导数, 记作$\frac{\partial^2 z}{\partial x^2}$或$\frac{\partial^2 f}{\partial x^2}$ 或$f''_{xx}(x,y)$ 或z''_{xx}, 由下式定义

$$\frac{\partial^2 z}{\partial x^2} = \left(\frac{\partial^2 f}{\partial x^2} = f_{xx}(x,y) = z_{xx}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right),$$

类似地可定义其他三种二阶偏导数, 其记号和定义分别为

$$\frac{\partial^2 z}{\partial x\partial y} = \left(\frac{\partial^2 f}{\partial x\partial y} = f''_{xy}(x,y) = z''_{xy}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right),$$

$$\frac{\partial^2 z}{\partial y\partial x} = \left(\frac{\partial^2 f}{\partial y\partial x} = f''_{yx}(x,y) = z''_{yx}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right),$$

$$\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 f}{\partial y^2} = f''_{yy}(x,y) = z''_{yy}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right),$$

其中偏导数$\frac{\partial^2 z}{\partial x\partial y}$和$\frac{\partial^2 z}{\partial y\partial x}$称为函数$z = f(x,y)$的二阶混合偏导数.

如下递归定义二元函数的n阶导数

$$\frac{\partial^n f(x,y)}{\partial x^n} = \frac{\partial}{\partial x}\left(\frac{\partial^{n-1} f(x,y)}{\partial x^{n-1}}\right),\quad \frac{\partial^n f(x,y)}{\partial y^n} = \frac{\partial}{\partial y}\left(\frac{\partial^{n-1} f(x,y)}{\partial y^{n-1}}\right),$$

$$\frac{\partial^n f(x,y)}{\partial x^k\partial y^{n-k}} = \frac{\partial}{\partial y}\left(\frac{\partial^{n-1} f(x,y)}{\partial x^k\partial y^{n-k-1}}\right).$$

可以类似地对n元函数$f(x_1,x_2,\cdots,x_n)$定义各阶偏导数.

例 2.8. 求$z = x^3y^2 + xy$的四个二阶偏导数.

解:

$$\frac{\partial z}{\partial x} = 3x^2y^2 + y, \quad \frac{\partial z}{\partial y} = 2x^3y + x,$$

$$\frac{\partial^2 z}{\partial x^2} = 6xy^2, \quad \frac{\partial^2 z}{\partial x \partial y} = 6x^2y + 1, \quad \frac{\partial^2 z}{\partial y \partial x} = 6x^2y + 1, \quad \frac{\partial^2 z}{\partial y^2} = 2x^3.$$

例 2.9. 设$z = x^y$, 求所有二阶偏导数.

解:

$$z_x = yx^{y-1}, \quad z_y = x^y \ln x,$$

$$z_{xx} = y(y-1)x^{y-2}, \quad z_{xy} = x^{y-1} + yx^{y-1}\ln x,$$

$$z_{yx} = yx^{y-1}\ln x + x^{y-1}, \quad z_{yy} = x^y \ln^2 x.$$

例 2.10. 求函数$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & x^2+y^2 \neq 0, \\ 0, & x^2+y^2 = 0 \end{cases}$在$(0,0)$处的二阶混合偏导数.

$$f_x'(0,0) = \lim_{x\to 0}\frac{f(x,0) - f(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0,$$

$$f_y'(0,0) = \lim_{y\to 0}\frac{f(0,y) - f(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0.$$

当$x^2 + y^2 \neq 0$时,

$$f_x'(x,y) = y\frac{x^2-y^2}{x^2+y^2} + y\frac{4x^2y^2}{(x^2+y^2)^2}, \quad f_y'(x,y) = x\frac{x^2-y^2}{x^2+y^2} - x\frac{4x^2y^2}{(x^2+y^2)^2}.$$

故

$$f_{xy}''(0,0) = \lim_{y\to 0}\frac{f_x'(0,y) - f_x'(0,0)}{y} = \lim_{y\to 0}\frac{-y}{y} = -1,$$

$$f_{yx}''(0,0) = \lim_{x\to 0}\frac{f_y'(x,0) - f_y'(0,0)}{x} = \lim_{x\to 0}\frac{x}{x} = 1,$$

可见$f_{xy}''(0,0) \neq f_{yx}''(0,0)$.

混合偏导有时是相同的. 但一般情况下, 混合偏导不总是相等的, 它与求导次序有关.

定理 2.1. 如果函数$z = f(x,y)$的两个二阶混合偏导数$f_{xy}(x,y)$ 与$f_{yx}(x,y)$ 在区域D内连续, 那么在该区域内$f_{xy}(x,y) = f_{yx}(x,y)$.

此定理对n元函数的高阶混合导数也成立.

例 2.11. 求$z = \arctan\frac{x}{y}$的所有二阶偏导数.

解:

$$z_x = \frac{y}{x^2+y^2}, \quad z_y = -\frac{x}{x^2+y^2},$$

$$z_{xx} = \frac{-2xy}{(x^2+y^2)^2},\ z_{yy} = \frac{2xy}{(x^2+y^2)^2},\ z_{xy} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

例 2.12. 验证函数$z = \ln\sqrt{x^2+y^2}$满足拉普拉斯方程

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

证明: 因

$$\frac{\partial z}{\partial x} = \frac{x}{x^2+y^2}, \quad \frac{\partial^2 z}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$

由对称性知

$$\frac{\partial^2 z}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2},$$

故有

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

例 2.13. 设函数$u = \frac{1}{r}$, 其中$r = \sqrt{x^2+y^2+z^2}$, 证明其满足拉普拉斯方程

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

证明: 因

$$\frac{\partial u}{\partial x} = -\frac{1}{r^2}\cdot\frac{\partial r}{\partial x} = -\frac{1}{r^2}\cdot\frac{x}{r} = -\frac{x}{r^3},$$

故

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{x}{r^3}\right) = -\frac{1}{r^3} + \frac{3x^2}{r^5},$$

由对称性知

$$\frac{\partial^2 u}{\partial y^2} = -\frac{1}{r^3} + \frac{3y^2}{r^5}, \quad \frac{\partial^2 u}{\partial z^2} = -\frac{1}{r^3} + \frac{3z^2}{r^5},$$

故有

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

8.2.3 思考与练习

练习 8.4. 设$z = x^y$ $(x > 0$, 且$x \neq 1)$, 求证

$$\frac{x}{y}\frac{\partial z}{\partial x} + \frac{1}{\ln x}\frac{\partial z}{\partial y} = 2z.$$

练习 8.5. 设$z = \frac{2}{3}\ln\frac{y}{x}$, 则$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} =$ ____.

0

练习 8.6. 设函数$F(x,y) = \int_0^{xy}\frac{\sin t}{1+t^2}\,\mathrm{d}t$, 则 $\left.\frac{\partial^2 F}{\partial x^2}\right|_{\substack{x=0\\y=2}} =$ ____.

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练习 8.7. 验证函数$z = \sin(x - ay)$满足波动方程

$$\frac{\partial^2 z}{\partial y^2} = a^2\frac{\partial^2 z}{\partial x^2}.$$

证明: 因

$$\frac{\partial z}{\partial x} = \cos(x - ay), \quad \frac{\partial^2 z}{\partial x^2} = -\sin(x - ay),$$
$$\frac{\partial z}{\partial y} = -a\cos(x - ay), \quad \frac{\partial^2 z}{\partial y^2} = -a^2\sin(x - ay),$$

故有

$$\frac{\partial^2 z}{\partial y^2} = a^2\frac{\partial^2 z}{\partial x^2}.$$