8.6 隐函数的求导法则

8.6.1 单个方程的情形

引例

设方程$x^2+y^2=1$, $\forall y_0\neq 0$, 则一定存在相应的邻域, 在该邻域中, 可以将y写成x的函数. 当$y_0>0$时,

$$y=\sqrt{1-x^2};$$

当$y_0<0$时,

$$y=-\sqrt{1-x^2}.$$

定理 6.1 (**二元方程确定的一元隐函数存在定理**). *设二元函数$F(x,y)$满足下列条件:*

(1) $F(x_0,y_0)=0$;

(2) 在点$P_0(x_0,y_0)$的某个邻域$U(P_0)\subset\mathbb{R}^2$内, 函数$F(x,y)$连续且具有连续的偏导数;

(3) $F'_y(x_0,y_0)\neq 0$,

则

(i) 在点$P_0(x_0,y_0)$的某个邻域$V(P_0)\subset U(P_0)\subset\mathbb{R}^2$内, 方程$F(x,y)=0$唯一确定了一个定义在某区间$(x_0-\delta,x_0+\delta)$内的隐函数$y=f(x)$, 满足$y_0=f(x_0)$且$F(x,f(x))\equiv 0$;

(ii) $y=f(x)$在区间$(x_0-\delta,x_0+\delta)$内单值连续;

(iii) $y=f(x)$在区间$(x_0-\delta,x_0+\delta)$内具有连续的导数, 满足

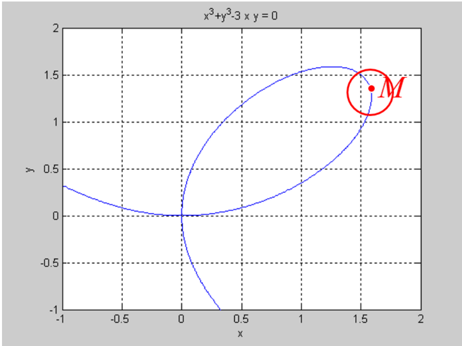
$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F'_x(x,y)}{F'_y(x,y)}.$$

注 6.2. *(1)* 定理中的条件"$F_y(x_0,y_0)\neq 0$"对定理结论的成立时很重要的. 在这一条件下, 由于F_y的连续性, 使得在点(x_0,y_0)的某个邻域内的每一点(x,y)处都有$F_y(x,y)\neq 0$. 于是对x_0近旁的每一固定的x值, 以"适合方程$F(x,y)=0$"为对应法则, 必定对应唯一的y 值. 相反, 如果$F_y(x_0,y_0)=0$, 则可能使方程在点(x_0,y_0)的任何邻域内都不能唯一地确定隐函数.

(2) 若把条件"$F_y(x_0,y_0)\neq 0$"改为"$F_x(x_0,y_0)\neq 0$", 则方程$F(x,y)=0$在点(x_0,y_0)的某邻域内确定唯一的有连续导数的一元函数$x=x(y)$, 它满足条件$x_0=x(y_0)$, 且有$\frac{dx}{dy}=-\frac{F_y}{F_x}$.

定理的条件充分不必要. 如$y^3+x^3=0$在点$(0,0)$处有$F'_y=0$,不满足条件(3), 但仍有$y=-x$.

定理 6.3 ($n+1$**元方程确定的**n**元隐函数存在定理**). *设$n+1$元函数$F(x_1^0,x_2^0,\cdots,x_n^0,y^0)$满足下列条件:*



(1) $F(x_1^0,x_2^0,\cdots,x_n^0,y^0)=0$;

(2) 在点$P_0(x_1^0,x_2^0,\cdots,x_n^0,y^0)$的某个邻域$U(P_0)\subset\mathbb{R}^{n+1}$内，函数$F(x_1,x_2,\cdots,x_n,y)$连续且具有连续的偏导数$F'_y$, F'_{x_i}, $i=1,2,\cdots,n$;

(3) $F'_y(x_1^0,x_2^0,\cdots,x_n^0,y^0)\neq 0$,

则

(i) 在点$P_0(x_1^0,x_2^0,\cdots,x_n^0,y^0)$的某个邻域$V(P_0)\subset U(P_0)\subset\mathbb{R}^{n+1}$内，方程$F(x_1,x_2,\cdots,x_n,y)=0$唯一确定了一个定义在点$R_0(x_1^0,x_2^0,\cdots,x_n^0)$某邻域$U(R_0)\subset\mathbb{R}^n$内的隐函数$y=f(x_1,x_2,\cdots,x_n)$，满足$y_0=f(x_1^0,x_2^0,\cdots,x_n^0)$且$F(x_1,x_2,\cdots,x_n,f(x_1,x_2,\cdots,x_n))\equiv 0$;

(ii) $y=f(x_1,x_2,\cdots,x_n)$在邻域$U(R_0)\subset\mathbb{R}^n$内单值连续;

(iii) $y=f(x_1,x_2,\cdots,x_n)$在邻域$U(R_0)\subset\mathbb{R}^n$内具有连续的导数，满足

$$\frac{\partial y}{\partial x_i}=-\frac{F'_{x_i}(x_1,x_2,\cdots,x_n,y)}{F'_y(x_1,x_2,\cdots,x_n,y)},\quad i=1,2,\cdots,n.$$

例 6.1. 验证方程$x^3+y^3-3xy=0$在点$M(4^{\frac13},2^{\frac13})$的某个邻域内能唯一地确定有连续导数的函数$x=\varphi(y)$，并求$\frac{dx}{dy}$.

解: 令$F(x,y)=x^3+y^3-3xy$, 则函数$F(x,y)$具有连续偏导数, 且$F(4^{\frac13},2^{\frac13})=0$.

$$F'_x(x,y)=3x^2-3y,\quad F'_y(x,y)=3y^2-3x,$$

由于$F'_x(4^{\frac13},2^{\frac13})=3(4^{\frac23}-2^{\frac13})\neq 0$. 因此由隐函数存在定理知, 方程$x^3+y^3-3xy=0$在点$M$的某个邻域内能唯一地确定有连续导数的函数$x=\varphi(y)$, 满足$4^{\frac13}=\varphi(2^{\frac13})$, 且

$$\frac{dx}{dy}=-\frac{F'_y}{F'_x}=-\frac{y^2-x}{x^2-y}.$$

方程$x^3+y^3-3xy=0$所表示的平面曲线称为叶形线.

例 6.2. 设$x^2+y^2+z^2-4z=0$，求$\frac{\partial z}{\partial x}$与$\frac{\partial^2 z}{\partial x^2}$.

解: 令$F(x,y,z)=x^2+y^2+z^2-4z$, 则当$F'_z=2z-4\neq 0$时, 方程$F(x,y,z)=0$ 确定了隐函数$z=z(x,y)$. 由于

$$F'_x=2x,\quad F'_y=2y,$$

所以

$$\frac{\partial z}{\partial x}=-\frac{F_x'}{F_z'}=\frac{x}{2-z},$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{x}{2-z}\right)=\frac{2-z+xz_x}{(2-z)^2}=\frac{(2-z)^2+x^2}{(2-z)^3}.$$

例 6.3. 设$xy+yz+zx=1$，求$\frac{\partial^2 z}{\partial x^2}$与$\frac{\partial^2 z}{\partial x\partial y}$.

解: 令$F(x,y,z)=xy+yz+zx-1$, 则

$$F_x'=y+z,\quad F_y'=z+x,\quad F_z'=x+y,$$

$$\frac{\partial z}{\partial x}=-\frac{F_x'}{F_z'}=-\frac{y+z}{x+y},\quad \frac{\partial z}{\partial y}=-\frac{F_y'}{F_z'}=-\frac{z+x}{x+y},$$

所以

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(-\frac{y+z}{x+y}\right)=-\frac{(x+y)z_x-(y+z)}{(x+y)^2}=\frac{2(y+z)}{(x+y)^2},$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(-\frac{y+z}{x+y}\right)=-\frac{(x+y)(1+z_y)-(y+z)}{(x+y)^2}=\frac{2z}{(x+y)^2}.$$

例 6.4. 设$x+y+z=e^{xyz}$，求$\mathrm{d}z$.

解: 令$F(x,y,z)=x+y+z-e^{xyz}$, 则

$$F_x'=1-yze^{xyz},\quad F_y'=1-zxe^{xyz},\quad F_z'=1-xye^{xyz},$$

$$\frac{\partial z}{\partial x}=-\frac{F_x'}{F_z'}=\frac{yze^{xyz}-1}{1-xye^{xyz}},\quad \frac{\partial z}{\partial y}=-\frac{F_y'}{F_z'}=\frac{zxe^{xyz}-1}{1-xye^{xyz}}.$$

所以

$$\mathrm{d}z=\frac{yze^{xyz}-1}{1-xye^{xyz}}\,\mathrm{d}x+\frac{zxe^{xyz}-1}{1-xye^{xyz}}\,\mathrm{d}y.$$

例 6.5. 设函数$z=z(x,y)$由方程$F\left(x+\frac{z}{y},y+\frac{z}{x}\right)=0$确定，证明函数$z=z(x,y)$满足方程

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy.$$

解: 令$G(x,y,z)=F\left(x+\frac{z}{y},y+\frac{z}{x}\right)$, 则

$$G_x'=F_1'-\frac{z}{x^2}F_2',\quad G_y'=-\frac{z}{y^2}F_1'+F_2',\quad G_z'=\frac{1}{y}F_1'+\frac{1}{x}F_2',$$

故

$$z_x'=-\frac{G_x'}{G_z'}=-\frac{y(x^2F_1'-zF_2')}{x(xF_1'+yF_2')},\quad z_y'=-\frac{G_y'}{G_z'}=-\frac{x(-zF_1'+y^2F_2')}{y(xF_1'+yF_2')}.$$

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例 6.6. 设$\Phi(u,v)$具有连续的偏导数，证明由方程$\Phi(cx-az,cy-bz)=0$所确定的函数$z=f(x,y)$满足$a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=c$.

解: 令$F(x,y,z)=\Phi(cx-az,cy-bz)$, 则

$$F_x'=c\Phi_1',\quad F_y'=c\Phi_2',\quad F_z'=-a\Phi_1'-b\Phi_2',$$

故

$$z_x'=-\frac{F_x'}{F_z'}=\frac{c\Phi_1'}{a\Phi_1'+b\Phi_2'},\quad z_y'=-\frac{F_y'}{F_z'}=\frac{c\Phi_2'}{a\Phi_1'+b\Phi_2'}.$$

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8.6.2 方程组情形

定理 6.4 (方程组情形的隐函数存在定理). *设$\begin{cases}F(x,y,u,v)=0,\\G(x,y,u,v)=0\end{cases}$ 是点$P_0(x_0,y_0,u_0,v_0)$的某个邻域$U(P_0)\subset\mathbb{R}^4$上函数组方程，若下列条件成立:*

(1) $F(x_0,y_0,u_0,v_0)=0$, $G(x_0,y_0,u_0,v_0)=0$;

(2) 函数$F(x,y,u,v)$, $G(x,y,u,v)$在$U(P_0)$上连续且具有连续的一阶偏导数;

(3) 函数$F(x,y,u,v)$, $G(x,y,u,v)$关于变量u,v的雅可比(*Jacobi*)行列式$J=\frac{\partial(F,G)}{\partial(u,v)}=\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}$在点$P_0(x_0,y_0,u_0,v_0)$不等于零,

则

(i) 在点$P_0(x_0,y_0,u_0,v_0)$的某个邻域$V(P_0)\subset U(P_0)\subset\mathbb{R}^4$内，方程组$\begin{cases}F(x,y,u,v)=0,\\G(x,y,u,v)=0\end{cases}$ 唯一确定了一个定义在点$R_0(x_0,y_0)$某邻域$U(R_0)\subset\mathbb{R}^2$内的两个二元函数

$$u=f(x,y),\quad v=g(x,y)$$

满足$u_0=f(x_0,y_0)$, $v_0=g(x_0,y_0)$且$\begin{cases}F(x,y,f(x,y),g(x,y))=0,\\G(x,y,f(x,y),g(x,y))=0;\end{cases}$

(ii) $u=f(x,y)$, $v=g(x,y)$在邻域$U(R_0)\subset\mathbb{R}^2$内单值连续;

(iii) $u=f(x,y)$, $v=g(x,y)$在邻域$U(R_0)\subset\mathbb{R}^2$内具有连续的一阶偏导数，满足

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)},\quad \frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)},$$

$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)},\quad \frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)}.$$

例 6.7. 设方程组$\begin{cases}x+y+z=0,\\x^2+y^2+z^2=1,\end{cases}$

(1) 求$\dfrac{\mathrm{d}x}{\mathrm{d}z}$, $\dfrac{\mathrm{d}y}{\mathrm{d}z}$;　　*(2)* 求$\dfrac{\mathrm{d}y}{\mathrm{d}x}$, $\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

解:

(1) 以z为自变量, 方程组关于z求导得

$$\begin{cases}\frac{\mathrm{d}x}{\mathrm{d}z}+\frac{\mathrm{d}y}{\mathrm{d}z}=-1,\\ x\frac{\mathrm{d}x}{\mathrm{d}z}+y\frac{\mathrm{d}y}{\mathrm{d}z}=-z.\end{cases}$$

当$\begin{vmatrix}1 & 1\\ x & y\end{vmatrix} = y - x \neq 0$时, 解得

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{y-z}{x-y}, \quad \frac{\mathrm{d}y}{\mathrm{d}z} = \frac{z-x}{x-y}.$$

(2) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x-z}{z-y}$, $\dfrac{\mathrm{d}z}{\mathrm{d}x} = \dfrac{y-x}{z-y}$.

例 6.8. *设函数$x = x(u,v)$, $y = y(u,v)$在点(u,v)的某邻域内连续且有连续偏导数, 又$\frac{\partial(x,y)}{\partial(u,v)} \neq 0$.*

(1) 证明方程组$\begin{cases} x = x(u,v), \\ y = y(u,v)\end{cases}$在点$(x,y,u,v)$的某邻域内唯一确定一组单值连续且有连续偏导数的反函数$u = u(x,y)$, $v = v(x,y)$;

(2) 求反函数$u = u(x,y)$, $v = v(x,y)$关于x, y的偏导数.

解: 将方程组改写成

$$\begin{cases} F(x,y,u,v) = x - x(u,v), \\ G(x,y,u,v) = y - y(u,v). \end{cases}$$

由假设

$$J = \frac{\partial(F,G)}{\partial(u,v)} = \frac{\partial(x,y)}{\partial(u,v)} \neq 0,$$

由隐函数组存在定理知结论成立.

将方程组$\begin{cases} x = x(u,v), \\ y = y(u,v)\end{cases}$确定的反函数$u = u(x,y)$, $v = v(x,y)$代入此方程组得

$$\begin{cases} x \equiv x[u(x,y), v(x,y)], \\ y \equiv y[u(x,y), v(x,y)]. \end{cases}$$

对x求偏导得

$$\begin{cases} 1 \equiv \frac{\partial x}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial x}{\partial v} \cdot \frac{\partial v}{\partial x}, \\ 0 \equiv \frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial x}. \end{cases}$$

解得

$$\frac{\partial u}{\partial x} = \frac{1}{J}\frac{\partial y}{\partial v}, \quad \frac{\partial v}{\partial x} = -\frac{1}{J}\frac{\partial y}{\partial u}.$$

同理可得

$$\frac{\partial u}{\partial y} = -\frac{1}{J}\frac{\partial x}{\partial v}, \quad \frac{\partial v}{\partial y} = \frac{1}{J}\frac{\partial x}{\partial u}.$$

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例 6.9. 设方程组$\begin{cases} x = u\cos v, \\ y = u\sin v \end{cases}$ 确定了函数$u = u(x,y)$与$v = v(x,y)$, 求$\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$与$\frac{\partial v}{\partial y}$.

解: 在恒等式组$\begin{cases} x \equiv u\cos v, \\ y \equiv u\sin v \end{cases}$ 两端分别对x与y求偏导, 可得

$$\begin{cases} 1 = \cos v\frac{\partial u}{\partial x} - u\sin v\frac{\partial v}{\partial x}, \\ 0 = \sin v\frac{\partial u}{\partial x} + u\cos v\frac{\partial v}{\partial x} \end{cases} \quad 与 \quad \begin{cases} 0 = \cos v\frac{\partial u}{\partial y} - u\sin v\frac{\partial v}{\partial y}, \\ 1 = \sin v\frac{\partial u}{\partial y} + u\cos v\frac{\partial v}{\partial y}, \end{cases}$$

解得

$$\begin{cases} \frac{\partial u}{\partial x} = \cos v, \\ \frac{\partial v}{\partial x} = -\frac{\sin v}{u}, \end{cases} \qquad \begin{cases} \frac{\partial u}{\partial y} = \sin v, \\ \frac{\partial v}{\partial y} = \frac{\cos v}{u}. \end{cases}$$

8.6.3 思考与练习

练习 8.27. 验证方程$x^4 + y^4 = 1$在点$(0,1)$的某邻域内能唯一确定一个有连续导数的函数$y = y(x)$, 并求$y'(0)$与$y''(0)$的值.

解: 令$F(x,y) = x^4 + y^4 - 1$, 则$F(0,1) = 0$, $F_y(0,1) = 4 \neq 0$. 因此由隐函数存在定理知, 方程$x^4 + y^4 = 1$在点$(0,1)$的某邻域内能唯一确定一个有连续导数的函数$y = y(x)$, 它满足条件$y(0) = 1$, 且

$$y'(x) = -\frac{F'_x}{F'_y} = -\frac{x^3}{y^3},$$

$$y''(x) = \left(-\frac{x^3}{y^3}\right)' = -\frac{3x^2y^3 - 3y^2x^3y'}{y^6} = -\frac{3x^2y^4 + 3x^6}{y^7},$$

所以

$$y'(0) = y''(0) = 0.$$

练习 8.28. 设$y = y(x)$与$z = z(x)$是由方程组$\begin{cases} z = x^2 + 2y^2, \\ y = 2x^2 + z^2 \end{cases}$ 所确定的函数, 求$\frac{dy}{dx}$与$\frac{dz}{dx}$.

解: 由于方程组确定了函数$y = y(x)$与$z = z(x)$, 故有恒等式组

$$\begin{cases} z(x) = x^2 + 2y^2(x), \\ y(x) = 2x^2 + z^2(x). \end{cases}$$

在每个等式的两边对x求导, 可得

$$\begin{cases} \frac{dz}{dx} = 2x + 4y\frac{dy}{dx}, \\ \frac{dy}{dx} = 4x + 2z\frac{dz}{dx}, \end{cases} \quad 即 \quad \begin{cases} 4y\frac{dy}{dx} - \frac{dz}{dx} = -2x, \\ \frac{dy}{dx} - 2z\frac{dz}{dx} = 4x, \end{cases}$$

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解得

$$\frac{dy}{dx}=\frac{4x(z+1)}{1-8yz},\quad \frac{dz}{dx}=\frac{4x(8y+1)}{1-8yz}.$$

练习 8.29. 设$y=y(x)$与$z=z(x)$是由方程$z=xf(x+y)$和$F(x,y,z)=0$所确定的函数，求$\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

$$\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{(f+xf')F_y'-xf'F_x'}{F_y'+xf'F_z'}.$$

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