10.4 对面积的曲面积分

10.4.1 对面积的曲面积分的定义与性质

曲面的质量

设Σ是一片光滑曲面, 面密度为$\rho(x,y,z)$, 则曲面的质量为

$$M=\lim_{\lambda\to 0}\sum_{i=1}^{n}\rho(\xi_i,\eta_i,\zeta_i)\Delta S_i.$$

定义 4.1. 设Σ是一片光滑曲面, 函数$f(x,y,z)$在Σ上有界. 将Σ任意分成n小块ΔS_1, ΔS_2, $\cdots$, ΔS_n, 仍用ΔS_i表示第i个小块的面积, 又在ΔS_i上任取一点$(\xi_i,\eta_i,\zeta_i)(i=1,2,\cdots,n)$, 作和

$$\sum_{i=1}^{n}f(\xi_i,\eta_i,\zeta_i)\Delta S_i,$$

如果当各小块曲面的直径的最大值$\lambda\to 0$时, 这和的极限存在, 则称此极限为函数$f(x,y,z)$在曲面Σ上对面积的曲面积分, 记作$\iint\limits_{\Sigma}f(x,y,z)\,\mathrm{d}S$, 即

$$\iint\limits_{\Sigma}f(x,y,z)\,\mathrm{d}S=\lim_{\lambda\to 0}\sum_{i=1}^{n}f(\xi_i,\eta_i,\zeta_i)\Delta S_i,$$

其中$f(x,y,z)$称为被积函数, $f(x,y,z)\,\mathrm{d}S$称为被积表达式, Σ称为积分曲面, $\mathrm{d}S$称为曲面面积微元.

对面积的曲面积分也常称为第一类曲面积分.

曲面的质量可表示为

$$M=\iint\limits_{\Sigma}\rho(x,y,z)\,\mathrm{d}S.$$

当被积函数为常数1时, $\iint\limits_{\Sigma}\mathrm{d}S$等于曲面$\Sigma$的面积.

如果Σ是分片光滑的(即Σ由有限片光滑曲面所组成), 则规定函数在Σ上的曲面积分等于函数在Σ的各光滑片上的曲面积分之和. 若Σ是闭曲面, 则曲面积分的符号常写作$\oiint\limits_{\Sigma}$.

当函数$f(x,y,z)$在积分曲面Σ上连续时, 对面积的曲面积分$\iint\limits_{\Sigma}f(x,y,z)\,\mathrm{d}S$存在.

对面积的曲面积分的性质

(1) 若$\iint\limits_{\Sigma}f(x,y,z)\,\mathrm{d}S$和$\iint\limits_{\Sigma}g(x,y,z)\,\mathrm{d}S$都存在, 则$\iint\limits_{\Sigma}[f(x,y,z)\pm g(x,y,z)]\,\mathrm{d}S$也存在, 且有

$$\iint\limits_{\Sigma}[f(x,y,z)\pm g(x,y,z)]\,\mathrm{d}S=\iint\limits_{\Sigma}f(x,y,z)\,\mathrm{d}S\pm\iint\limits_{\Sigma}g(x,y,z)\,\mathrm{d}S.$$

(2) 若$\iint\limits_{\Sigma}f(x,y,z)\,\mathrm{d}S$存在, k为常数时, 则$\iint\limits_{\Sigma}kf(x,y,z)\,\mathrm{d}S$也存在, 且有

$$\iint\limits_{\Sigma}kf(x,y,z)\,\mathrm{d}S=k\iint\limits_{\Sigma}f(x,y,z)\,\mathrm{d}S.$$

(3) 若$\Sigma = \Sigma_1 + \Sigma_2$, 且$\iint\limits_{\Sigma_1} f(x,y,z)\,\mathrm{d}S$和$\iint\limits_{\Sigma_2} f(x,y,z)\,\mathrm{d}S$都存在, 则$\iint\limits_{\Sigma} f(x,y,z)\,\mathrm{d}S$也存在, 且有

$$\iint\limits_{\Sigma} f(x,y,z)\,\mathrm{d}S = \iint\limits_{\Sigma_1} f(x,y,z)\,\mathrm{d}S + \iint\limits_{\Sigma_2} f(x,y,z)\,\mathrm{d}S.$$

10.4.2 对面积的曲面积分的计算

设光滑曲面Σ由方程$z = z(x,y)$给出, Σ在xOy面上的投影区域为D_{xy}, 函数$f(x,y,z)$在Σ上连续, 则

$$\iint\limits_{\Sigma} f(x,y,z)\,\mathrm{d}S = \iint\limits_{D_{xy}} f[x,y,z(x,y)]\sqrt{1 + z_x'^2(x,y) + z_y'^2(x,y)}\,\mathrm{d}x\,\mathrm{d}y.$$

如果积分曲面Σ由方程$x = x(y,z)$或$y = y(z,x)$给出, 也可类似地把对面积的曲面积分化为相应的二重积分.

例 4.1. 计算$\iint\limits_{S} \frac{\mathrm{d}S}{z}$, 其中$S$是球面$x^2 + y^2 + z^2 = a^2$被平面$z = h\ (0 < h < a)$所截的顶部.

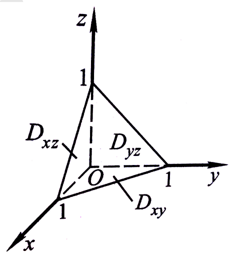
解: 曲面S的方程为$z = \sqrt{a^2 - x^2 - y^2}$, 定义域$D$为圆域$x^2 + y^2 \le a^2 - h^2$. 由于

$$\sqrt{1 + z_x^2 + z_y^2} = \frac{a}{\sqrt{a^2 - x^2 - y^2}},$$

所以

$$\begin{aligned}\iint_S \frac{\mathrm{d}S}{z} = \iint_D \frac{a}{a^2 - x^2 - y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi} d\theta \int_0^{\sqrt{a^2-h^2}} \frac{a}{a^2 - r^2} r\,\mathrm{d}r \\ = 2\pi a \int_0^{\sqrt{a^2-h^2}} \frac{r}{a^2 - r^2}\,\mathrm{d}r = 2\pi a \ln\frac{a}{h}.\end{aligned}$$

例 4.2. 计算$\oiint\limits_{\Sigma} \frac{\mathrm{d}S}{(1+x+y)^2}$, 其中$\Sigma$是由平面$x = 0$, $y = 0$, $z = 0$及$x + y + z = 1$所围成的四面体的整个边界曲面.



解: 将Σ在平面$x + y + z = 1$, $x = 0$, $y = 0$及$z = 0$上的部分依次记为Σ_1, Σ_2, Σ_3及Σ_4, 则所求曲面积分等于这四个部分上的曲面积分之和.

$$\iint\limits_{\Sigma_1} \frac{\mathrm{d}S}{(1+x+y)^2} = \int_0^1 \mathrm{d}x \int_0^{1-x} \frac{\sqrt{3}\,\mathrm{d}y}{(1+x+y)^2} = \sqrt{3}\left(\ln 2 - \frac{1}{2}\right),$$

$$\iint\limits_{\Sigma_2} \frac{\mathrm{d}S}{(1+x+y)^2} = \int_0^1 \mathrm{d}z \int_0^{1-z} \frac{\mathrm{d}y}{(1+y)^2} = 1 - \ln 2,$$

$$\iint\limits_{\Sigma_3} \frac{\mathrm{d}S}{(1+x+y)^2} = \int_0^1 \mathrm{d}z \int_0^{1-z} \frac{\mathrm{d}x}{(1+x)^2} = 1 - \ln 2,$$

$$\iint\limits_{\Sigma_4} \frac{\mathrm{d}S}{(1+x+y)^2} = \int_0^1 \mathrm{d}x \int_0^{1-x} \frac{\mathrm{d}y}{(1+x+y)^2} = \ln 2 - \frac{1}{2}.$$

所以

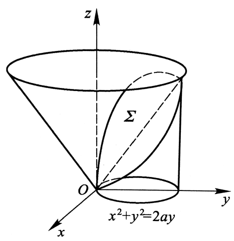
$$\iint\limits_{\Sigma} \frac{\mathrm{d}S}{(1+x+y)^2} = \frac{3-\sqrt{3}}{2} + (\sqrt{3}-1)\ln 2.$$

例 4.3. 计算$\iint\limits_{\Sigma}(x+y+z)\,\mathrm{d}S$, 其中$\Sigma$是上半球面$x^2+y^2+z^2=2$被旋转抛物面$z=x^2+y^2$截出的顶部.

解: Σ关于zOx, yOz坐标面对称, 故$\iint\limits_{\Sigma} y\,\mathrm{d}S = 0$, $\iint\limits_{\Sigma} x\,\mathrm{d}S = 0$.

$$\begin{aligned}\iint\limits_{\Sigma}(x+y+z)\,\mathrm{d}S = \iint\limits_{\Sigma} z\,\mathrm{d}S &= \iint\limits_{x^2+y^2\le 1} \sqrt{2-x^2-y^2}\cdot\frac{\sqrt{2}}{\sqrt{2-x^2-y^2}}\,\mathrm{d}\sigma \\ &= \sqrt{2}\iint\limits_{x^2+y^2\le 1} \mathrm{d}\sigma = \sqrt{2}\pi.\end{aligned}$$

例 4.4. 计算$\iint\limits_{\Sigma}(xy+yz+zx)\,\mathrm{d}S$, 其中$\Sigma$是锥面$z=\sqrt{x^2+y^2}$被圆柱面$x^2+y^2=2ay$ $(a>0)$所截下的部分.



解: Σ在xOy面上的投影为$D_{xy}: x^2+y^2\le 2ay$. 又$\sqrt{1+z_x'^2+z_y'^2}=\sqrt{2}$, 所以

$$\iint\limits_{\Sigma}(xy+yz+zx)\,\mathrm{d}S = \iint\limits_{D_{xy}} \sqrt{2}(xy+(x+y)\sqrt{x^2+y^2})\,\mathrm{d}\sigma.$$

令$x=r\cos\theta$, $y=r\sin\theta$, 则$r\le 2a\sin\theta$, $0\le\theta\le\pi$. 于是

$$\begin{aligned}
\text{原式} &= \int_0^\pi \mathrm{d}\theta \int_0^{2a\sin\theta} \sqrt{2}(r^2\cos\theta\sin\theta + (r\cos\theta + r\sin\theta)r)r\,\mathrm{d}r \\
&= \frac{\sqrt{2}(2a)^4}{4}\int_0^\pi (\cos\theta\sin\theta + (\cos\theta+\sin\theta))\sin^4\theta\,\mathrm{d}\theta \\
&= 4a^4\sqrt{2}\int_0^\pi \sin^5\theta\,\mathrm{d}\theta = \frac{64\sqrt{2}}{15}a^4.
\end{aligned}$$

例 4.5. 计算$\iint\limits_S (x^2+y^2+z^2)\,\mathrm{d}S$, 其中

(1) $S: x^2+y^2+z^2=a^2$;

(2) $S: x^2+y^2+z^2=2az$.

解:

(1) $\iint\limits_S (x^2+y^2+z^2)\,\mathrm{d}S = \iint\limits_S a^2\,\mathrm{d}S = 4\pi a^2\cdot a^2 = 4\pi a^4$.

(2) $\iint\limits_S (x^2+y^2+z^2)\,\mathrm{d}S = \iint\limits_S 2az\,\mathrm{d}S = \iint\limits_{S_1} 2az\,\mathrm{d}S + \iint\limits_{S_2} 2az\,\mathrm{d}S$, 其中

$$S_1: z_1 = a + \sqrt{a^2-(x^2+y^2)}, (x,y)\in D,$$

$$S_2: z_2 = a - \sqrt{a^2-(x^2+y^2)}, (x,y)\in D,$$

$$D: x^2+y^2\le a^2.$$

由于$\sqrt{1+\left(\frac{\partial z_1}{\partial x}\right)^2+\left(\frac{\partial z_1}{\partial y}\right)^2} = \sqrt{1+\left(\frac{\partial z_2}{\partial x}\right)^2+\left(\frac{\partial z_2}{\partial y}\right)^2} = \frac{a}{\sqrt{a^2-(x^2+y^2)}}$, 故

$$\iint_{S_1} 2az\,\mathrm{d}S = \iint_D 2a(a+\sqrt{a^2-(x^2+y^2)})\frac{a}{\sqrt{a^2-(x^2+y^2)}}\,\mathrm{d}x\,\mathrm{d}y,$$

$$\iint_{S_2} 2az\,\mathrm{d}S = \iint_D 2a(a-\sqrt{a^2-(x^2+y^2)})\frac{a}{\sqrt{a^2-(x^2+y^2)}}\,\mathrm{d}x\,\mathrm{d}y,$$

所以

$$\begin{aligned}
\iint_S (x^2+y^2+z^2)\,\mathrm{d}S =& 4\iint_D \frac{a^3}{\sqrt{a^2-(x^2+y^2)}}\,\mathrm{d}x\,\mathrm{d}y \\
=& 4a^3\int_0^{2\pi}\mathrm{d}\theta\int_0^a \frac{r\,\mathrm{d}r}{\sqrt{a^2-r^2}} = 8a^4\pi.
\end{aligned}$$

例 4.6. 计算$\iint\limits_\Sigma z^2\,\mathrm{d}S$, 其中$\Sigma: x^2+y^2+z^2=R^2$.

解: 因为积分曲面具有轮换对称性, 即

$$\iint_{\Sigma} z^2 \, \mathrm{d}S = \iint_{\Sigma} x^2 \, \mathrm{d}S = \iint_{\Sigma} y^2 \, \mathrm{d}S,$$

所以

$$\iint_{\Sigma} z^2 \, \mathrm{d}S = \frac{1}{3} \iint_{\Sigma} (x^2 + y^2 + z^2) \, \mathrm{d}S = \frac{R^2}{3} \iint_{\Sigma} \mathrm{d}S = \frac{4}{3}\pi R^4.$$

10.4.3 思考与练习

练习 10.15. *计算* $\iint\limits_{\Sigma} x^2y^2 \, \mathrm{d}S$*, 其中* $\Sigma : z = \sqrt{R^2 - x^2 - y^2}$.

解: 因为

$$z_x = \frac{-x}{\sqrt{R^2 - x^2 - y^2}}, \quad z_y = \frac{-y}{\sqrt{R^2 - x^2 - y^2}},$$

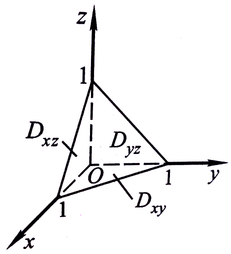
所以

$$\sqrt{1 + z_x^2 + z_y^2} = \frac{R}{\sqrt{R^2 - x^2 - y^2}}.$$

又D_{xy}为$\{(x,y) | x^2 + y^2 \le R^2\}$, 于是

$$\begin{aligned}\iint_{\Sigma} x^2y^2 \, \mathrm{d}S &= \iint_{D_{xy}} x^2y^2 \frac{R}{\sqrt{R^2 - x^2 - y^2}} \, \mathrm{d}\sigma \\ &= 4\int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^R r^5 \cos^2\theta \sin^2\theta \frac{R}{\sqrt{R^2 - r^2}} \, \mathrm{d}r = \frac{2}{15}\pi R^6.\end{aligned}$$

练习 10.16. *计算* $\oiint\limits_{\Sigma} z \, \mathrm{d}S$*, 其中*$\Sigma$*是由平面*$x = 0$, $y = 0$, $z = 0$*及*$x + y + z = 1$*所围成的四面体的整个边界曲面.*



解: 将Σ在平面$x = 0$, $y = 0$, $z = 0$及$x + y + z = 1$上的部分依次记为Σ_1, Σ_2, Σ_3及Σ_4, 则所求曲面积分等于这四个部分上的曲面积分之和.

Σ_1的方程是$x = 0$, 它在yOz面上的投影区域就是Σ_1自身, 即由直线$y = 0$, $z = 0$ 及$y + z = 1$所围成的三角形区域. 又在Σ_1上

$$\sqrt{1 + x_y^2 + x_z^2} = \sqrt{1 + 0^2 + 0^2} = 1,$$

因此

$$\begin{aligned}\iint_{\Sigma_1} z\,\mathrm{d}S &= \iint_{D_{yz}} z\sqrt{1+x_y^2+x_z^2}\,\mathrm{d}\sigma = \iint_{D_{yz}} z\,\mathrm{d}\sigma \\ &= \int_0^1 z\,\mathrm{d}z \int_0^{1-z} \mathrm{d}y = \int_0^1 z(1-z)\,\mathrm{d}z = \frac{1}{6}.\end{aligned}$$

同理可得$\iint\limits_{\Sigma_2} z\,\mathrm{d}S = \frac{1}{6}$.

因在Σ_3上, $z=0$，故$\iint\limits_{\Sigma_3} z\,\mathrm{d}S = 0$.

Σ_4的方程可写成$z=1-x-y$，它在xOy面上的投影区域D_{xy}是由直线$x=0$, $y=0$及$x+y=1$所围成的三角形区域. 又在Σ_4上

$$\sqrt{1+z_x^2+z_y^2} = \sqrt{1+(-1)^2+(-1)^2} = \sqrt{3},$$

因此

$$\begin{aligned}\iint_{\Sigma_4} z\,\mathrm{d}S &= \iint_{D_{xy}} (1-x-y)\sqrt{1+x_y^2+x_z^2}\,\mathrm{d}\sigma = \sqrt{3}\iint_{D_{xy}} (1-x-y)\,\mathrm{d}\sigma \\ &= \sqrt{3}\int_0^1 \mathrm{d}x \int_0^{1-x} (1-x-y)\,\mathrm{d}y = \sqrt{3}\int_0^1 \left(\frac{1}{2}-x+\frac{x^2}{2}\right)\mathrm{d}x = \frac{\sqrt{3}}{6}.\end{aligned}$$

于是

$$\oint\!\!\!\oint_{\Sigma} z\,\mathrm{d}S = \iint_{\Sigma_1} z\,\mathrm{d}S + \iint_{\Sigma_2} z\,\mathrm{d}S + \iint_{\Sigma_3} z\,\mathrm{d}S + \iint_{\Sigma_4} z\,\mathrm{d}S = \frac{2+\sqrt{3}}{6}.$$