方向和$\overrightarrow{M_1M_2}$一致的单位向量为

$$(\overrightarrow{M_1M_2})^0 = \{-\frac{4}{\sqrt{34}}, \frac{3}{\sqrt{34}}, \frac{3}{\sqrt{34}}\}.$$

例 2.8. 若点M的向径与x轴成$\frac{\pi}{4}$角，与y轴成$\frac{\pi}{3}$角，模为6，在z 轴上的投影是负值，求点M的坐标.

解: 已知$\alpha = \frac{\pi}{4}$, $\beta = \frac{\pi}{3}$, 由关系式$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$得

$$\cos^2\gamma = 1 - \cos^2\frac{\pi}{4} - \cos^2\frac{\pi}{3} = \frac{1}{4}.$$

由于在z轴上的投影是负值，故$\cos\gamma = -\frac{1}{2}$. 因此

$$\begin{aligned}\overrightarrow{OM} =& |\overrightarrow{OM}|(\overrightarrow{OM})^0 = |\overrightarrow{OM}|\{\cos\alpha, \cos\beta, \cos\gamma\}\\ =& 6\left\{\frac{\sqrt{2}}{2}, \frac{1}{2}, -\frac{1}{2}\right\} = \{3\sqrt{2}, 3, -3\}.\end{aligned}$$

所以点M的坐标为$(3\sqrt{2}, 3, -3)$.

7.3 数量积、向量积、混合积

7.3.1 向量的数量积

设向量$\boldsymbol{a}$和$\boldsymbol{b}$, 称数$|\boldsymbol{a}||\boldsymbol{b}|\cos(\widehat{\boldsymbol{a},\boldsymbol{b}})$ 为向量$\boldsymbol{a}$和$\boldsymbol{b}$的数量积, 记作$\boldsymbol{a}\cdot\boldsymbol{b}$, 即

$$\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos(\widehat{\boldsymbol{a},\boldsymbol{b}}).$$

设一物体在常力$\boldsymbol{F}$的作用下沿直线从点M_0移动到点M, 若用$\boldsymbol{s}$表示位移$\overrightarrow{M_0M}$, 则力$\boldsymbol{F}$所做的功为

$$W = |\boldsymbol{F}||\boldsymbol{s}|\cos\theta,$$

其中θ为$\boldsymbol{F}$与$\boldsymbol{s}$的夹角. 用数量积表示即为

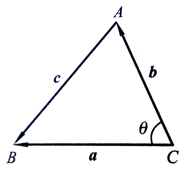
$$W = \boldsymbol{F}\cdot\boldsymbol{s}.$$

数量积的基本性质

(1) $\boldsymbol{a}\cdot\boldsymbol{0} = \boldsymbol{0}\cdot\boldsymbol{a} = 0$.

(2) $\boldsymbol{a}\cdot\boldsymbol{a} = |\boldsymbol{a}|^2$.

(3) $\boldsymbol{a}\perp\boldsymbol{b}$的充要条件为$\boldsymbol{a}\cdot\boldsymbol{b} = 0$.



例 3.1. 设向量$\boldsymbol{a}$与$\boldsymbol{b}$的夹角为$\dfrac{\pi}{3}$, $|\boldsymbol{a}| = 2$, $|\boldsymbol{b}| = 3$, 求$\boldsymbol{a}\cdot\boldsymbol{b}$.

解:

$$\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos(\widehat{\boldsymbol{a},\boldsymbol{b}}) = 2\times 3\times\cos\frac{\pi}{3} = 3.$$

例 3.2. 试用向量方法证明三角形的余弦定理.

证明: 在$\triangle ABC$中, 记$\theta = \angle BCA$, $a = |\boldsymbol{a}| = |\overrightarrow{CB}|$, $b = |\boldsymbol{b}| = |\overrightarrow{CA}|$, $c = |\boldsymbol{c}| = |\overrightarrow{AB}|$, 要证明

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

由于$\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b}$, 故

$$|\boldsymbol{c}|^2 = \boldsymbol{c}\cdot\boldsymbol{c} = (\boldsymbol{a}-\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b}) = \boldsymbol{a}\cdot\boldsymbol{a} + \boldsymbol{b}\cdot\boldsymbol{b} - 2\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\boldsymbol{a}||\boldsymbol{b}|\cos\theta,$$

即有

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

数量积的运算规律

(1) 交换律 $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a}$.

(3) 数乘结合律 $\lambda(\boldsymbol{a}\cdot\boldsymbol{b}) = (\lambda\boldsymbol{a})\cdot\boldsymbol{b} = \boldsymbol{a}\cdot(\lambda\boldsymbol{b})$.

(2) 分配律 $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})\cdot = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\boldsymbol{c}$.

例 3.3. 设$\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c} = \boldsymbol{0}$, $|\boldsymbol{a}| = 1$, $|\boldsymbol{b}| = 2$, $|\boldsymbol{c}| = 3$, 求$\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{b}\cdot\boldsymbol{c}+\boldsymbol{a}\cdot\boldsymbol{c}$.

解:

$$0 = |\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + |\boldsymbol{c}|^2 + 2(\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{b}\cdot\boldsymbol{c}+\boldsymbol{a}\cdot\boldsymbol{c}),$$

故

$$\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{b}\cdot\boldsymbol{c}+\boldsymbol{a}\cdot\boldsymbol{c} = -\frac{1}{2}(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + |\boldsymbol{c}|^2) = -7.$$

向量数量积的坐标表达式

设向量$\boldsymbol{a}=\{a_x,a_y,a_z\}$, $\boldsymbol{b}=\{b_x,b_y,b_z\}$, 即

$$\boldsymbol{a}=a_x\boldsymbol{i}+a_y\boldsymbol{j}+a_z\boldsymbol{k},\quad \boldsymbol{b}=b_x\boldsymbol{i}+b_y\boldsymbol{j}+b_z\boldsymbol{k},$$

则有向量数量积的坐标表示

$$\boldsymbol{a}\cdot\boldsymbol{b}=a_xb_x+a_yb_y+a_zb_z.$$

当$\boldsymbol{a},\boldsymbol{b}$为非零向量时, $\boldsymbol{a},\boldsymbol{b}$的夹角$\theta$满足公式

$$\cos\theta=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}=\frac{a_xb_x+a_yb_y+a_zb_z}{\sqrt{a_x^2+a_y^2+a_z^2}\sqrt{b_x^2+b_y^2+b_z^2}}.$$

两个向量$\boldsymbol{a},\boldsymbol{b}$垂直的充要条件是$\boldsymbol{a}\cdot\boldsymbol{b}=0$, 即

$$a_xb_x+a_yb_y+a_zb_z=0.$$

例 3.4. *已知点$M(1,1,1)$, $A(2,2,1)$, $B(2,1,2)$, 求$\angle AMB$.*

解: $\angle AMB$可以看作向量$\overrightarrow{MA}$与$\overrightarrow{MB}$的夹角, 而

$$\overrightarrow{MA}=\{1,1,0\},\quad \overrightarrow{MB}=\{1,0,1\},$$

故

$$\overrightarrow{MA}\cdot\overrightarrow{MB}=1\times1+1\times0+0\times1=1,|\overrightarrow{MA}|=\sqrt{2},|\overrightarrow{MB}|=\sqrt{2}.$$

从而

$$\cos\angle AMB=\frac{\overrightarrow{MA}\cdot\overrightarrow{MB}}{|\overrightarrow{MA}||\overrightarrow{MB}|}=\frac{1}{2},$$

所以

$$\angle AMB=\frac{\pi}{3}.$$

例 3.5. *已知向量$\boldsymbol{a}=\{1,1,1\}$, $\boldsymbol{b}=\{1,2,-2\}$, $\boldsymbol{c}=\{3,-5,4\}$, 求向量$\boldsymbol{d}=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}+(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$及$\boldsymbol{d}$在$\boldsymbol{a}$上的投影.*

解:

$$\boldsymbol{d}=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}+(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}=2\boldsymbol{b}+\boldsymbol{c}=(5,-1,0).$$

又

$$|\boldsymbol{a}|=\sqrt{3},\quad \boldsymbol{a}\cdot\boldsymbol{d}=4,$$

故

$$\mathrm{Prj}_a\boldsymbol{d}=\frac{\boldsymbol{a}\cdot\boldsymbol{d}}{|\boldsymbol{a}|}=\frac{4}{\sqrt{3}}.$$

7.3.2 向量的向量积

设向量$\boldsymbol{a}$, $\boldsymbol{b}$, 规定$\boldsymbol{a}$和$\boldsymbol{b}$的向量积是一个向量, 记作$\boldsymbol{a}\times\boldsymbol{b}$, 它的模$|\boldsymbol{a}\times\boldsymbol{b}|$满足

$$|\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|\sin(\widehat{\boldsymbol{a},\boldsymbol{b}}),$$

它的方向由以下方法确定：$\boldsymbol{a}\times\boldsymbol{b}$同时垂直于$\boldsymbol{a}$和$\boldsymbol{b}$, 并且$\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{a}\times\boldsymbol{b}$符合右手法则.

注 3.1. *(1) 向量积是一个向量而不是数.*

(2) 向量积不满足交换律.

(3) $\boldsymbol{a}\times\boldsymbol{b}\perp\boldsymbol{a}$, $\boldsymbol{a}\times\boldsymbol{b}\perp\boldsymbol{b}$.

(4) 向量积的几何意义: 向量$\boldsymbol{a}\times\boldsymbol{b}$*的模是以*$\boldsymbol{a},\boldsymbol{b}$*为邻边的平行四边形的面积.*

向量积的基本性质

(1) $\boldsymbol{a}\,/\!/\,\boldsymbol{b}$等价于$\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{0}$.

(2) $\boldsymbol{i}\times\boldsymbol{i}=\boldsymbol{j}\times\boldsymbol{j}=\boldsymbol{k}\times\boldsymbol{k}=\boldsymbol{0}$.

(3) $\boldsymbol{i}\times\boldsymbol{j}=\boldsymbol{k}$, $\boldsymbol{j}\times\boldsymbol{k}=\boldsymbol{i}$, $\boldsymbol{k}\times\boldsymbol{i}=\boldsymbol{j}$.

向量积的运算规律

(1) 反交换律 $\boldsymbol{b}\times\boldsymbol{a}=(-\boldsymbol{a})\times\boldsymbol{b}=-(\boldsymbol{a}\times\boldsymbol{b})$.

(3) 数乘结合律 $\lambda(\boldsymbol{a}\times\boldsymbol{b})=(\lambda\boldsymbol{a})\times\boldsymbol{b}=\boldsymbol{a}\times(\lambda\boldsymbol{b})$.

(2) 分配律 $(\boldsymbol{a}+\boldsymbol{b})\times\boldsymbol{c}=\boldsymbol{a}\times\boldsymbol{c}+\boldsymbol{b}\times\boldsymbol{c}$, $\boldsymbol{c}\times(\boldsymbol{a}+\boldsymbol{b})=\boldsymbol{c}\times\boldsymbol{a}+\boldsymbol{c}\times\boldsymbol{b}$.

注 3.2. *向量积不满足结合律, 即*

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$$

一般不成立.

例如

$$(\boldsymbol{i}\times\boldsymbol{i})\times\boldsymbol{j}=\boldsymbol{0}\times\boldsymbol{j}=\boldsymbol{0},$$

$$\boldsymbol{i}\times(\boldsymbol{i}\times\boldsymbol{j})=\boldsymbol{i}\times\boldsymbol{k}=-\boldsymbol{j}.$$

向量积的坐标表达式

设向量

$$\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}, \quad \boldsymbol{b} = b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k},$$

则有向量积的坐标表示

$$\boldsymbol{a} \times \boldsymbol{b} = (a_yb_z - a_zb_y)\boldsymbol{i} + (a_zb_x - a_xb_z)\boldsymbol{j} + (a_xb_y - a_yb_x)\boldsymbol{k}.$$

为方便记忆, 引入行列式记号, 则有

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix}\boldsymbol{i} + \begin{vmatrix} a_z & a_x \\ b_z & b_x \end{vmatrix}\boldsymbol{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}\boldsymbol{k}$$

或

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

例 3.6. 设向量$\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$, 求$\boldsymbol{a} \times \boldsymbol{b}$.

解:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 3\boldsymbol{i} - 7\boldsymbol{j} - 5\boldsymbol{k}.$$

例 3.7. 利用向量积证明正弦定理.

解: 设三角形ABC的三个内角为α, β, γ, 三边长为a, b, c. 因为

$$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB},$$

所以

$$\overrightarrow{AB} \times \overrightarrow{AB} = (\overrightarrow{AC} + \overrightarrow{CB}) \times \overrightarrow{AB},$$

即$\mathbf{0} = \overrightarrow{AC} \times \overrightarrow{AB} + \overrightarrow{CB} \times \overrightarrow{AB}$, 故

$$\overrightarrow{AC} \times \overrightarrow{AB} = -\overrightarrow{CB} \times \overrightarrow{AB},$$

两边取模得$|\overrightarrow{AC} \times \overrightarrow{AB}| = |\overrightarrow{CB} \times \overrightarrow{AB}|$, 于是$bc\sin\alpha = ac\sin\beta$, 故得

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}.$$

同理可证$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$. 证毕.

例 3.8. 设$|\boldsymbol{a}| = 2$, $|\boldsymbol{b}| = 3$, 且$\boldsymbol{a}$与$\boldsymbol{b}$垂直, 求$|\boldsymbol{a} \times \boldsymbol{b}|$及$|(\boldsymbol{a}+\boldsymbol{b}) \times (2\boldsymbol{a}-\boldsymbol{b})|$.

解:

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|\sin(\widehat{\boldsymbol{a},\boldsymbol{b}}) = 2 \times 3 \times \sin\frac{\pi}{2} = 6,$$

$$|(\boldsymbol{a}+\boldsymbol{b}) \times (2\boldsymbol{a}-\boldsymbol{b})| = |-\boldsymbol{a} \times \boldsymbol{b} + 2\boldsymbol{b} \times \boldsymbol{a}| = 3|\boldsymbol{a} \times \boldsymbol{b}| = 18.$$

例 3.9. 设$A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$为两已知点, 在直线AB上的点M分有向线段$\overrightarrow{AB}$为两个有向线段$\overrightarrow{AM}$与$\overrightarrow{MB}$, 且$\overrightarrow{AM} = \lambda\overrightarrow{MB}$ $(\lambda \neq -1)$, 求分点M的坐标x, y, z.

解: 易知$\overrightarrow{AM} = \overrightarrow{OM} - \overrightarrow{OA}$, $\overrightarrow{MB} = \overrightarrow{OB} - \overrightarrow{OM}$, 由$\overrightarrow{AM} = \lambda\overrightarrow{MB}$ 得

$$\overrightarrow{OM} - \overrightarrow{OA} = \lambda(\overrightarrow{OB} - \overrightarrow{OM}) \quad 或 \quad \overrightarrow{OM} = \frac{1}{1+\lambda}(\overrightarrow{OA} + \lambda\overrightarrow{OB}),$$

即

$$\begin{aligned}\{x, y, z\} &= \frac{1}{1+\lambda}(\{x_1, y_1, z_1\} + \lambda\{x_2, y_2, z_2\}) \\ &= \frac{1}{1+\lambda}\{x_1 + \lambda x_2, y_1 + \lambda y_2, z_1 + \lambda z_2\}.\end{aligned}$$

所以点M的坐标为

$$x = \frac{x_1 + \lambda x_2}{1+\lambda}, \quad y = \frac{y_1 + \lambda y_2}{1+\lambda}, \quad z = \frac{z_1 + \lambda z_2}{1+\lambda}.$$

点M叫做有向线段$\overrightarrow{AB}$的定比分点.

- 当$\lambda > 0$时, 点M在两点A与B之间.
- 当$\lambda < 0$ $(\lambda \neq -1)$时, 点M在两点A与B之外.
- 当$\lambda = 1$时, 点M是两点A与B的中间点, 得有向线段$\overrightarrow{AB}$的中点坐标为

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}, \quad z = \frac{z_1 + z_2}{2}.$$

例 3.10. 已知三角形ABC的顶点$A(1, -1, 2)$, $B(5, -6, 2)$和$C(1, 3-1)$. 求由顶点B到AC边高的长h.

解:

$$h = |\overrightarrow{AB}|\sin(\widehat{\overrightarrow{AB}, \overrightarrow{AC}}) = |\overrightarrow{AB}|\frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{|\overrightarrow{AC}|},$$

而$\overrightarrow{AB} = \{4, -5, 0\}$, $\overrightarrow{AC} = \{0, 4, -3\}$,

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 4 & -5 & 0 \\ 0 & 4 & -3 \end{vmatrix} = \{15, 12, 16\},$$

故

$$h = \frac{\sqrt{15^2 + 12^2 + 16^2}}{\sqrt{0^2 + 4^2 + (-3)^2}} = 5.$$

例 3.11. 计算$(\boldsymbol{a}+\boldsymbol{b}) \times (\boldsymbol{a}-\boldsymbol{b})$, 并解释它的几何意义.

解:

$$(\boldsymbol{a}+\boldsymbol{b}) \times (\boldsymbol{a}-\boldsymbol{b}) = \boldsymbol{a} \times \boldsymbol{a} - \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{b} \times \boldsymbol{a} - \boldsymbol{b} \times \boldsymbol{b} = -2(\boldsymbol{a} \times \boldsymbol{b}),$$

从而$|(\boldsymbol{a}+\boldsymbol{b}) \times (\boldsymbol{a}-\boldsymbol{b})| = 2|\boldsymbol{a} \times \boldsymbol{b}|$.

上式表明, 已知一平行四边形两邻边为$\boldsymbol{a}$与$\boldsymbol{b}$, 则以它的两条对角线为邻边的平行四边形的面积等于原平行四边形面积的两倍.

7.3.3 向量的混合积

设三个向量$\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, 称$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$为向量$\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$的混合积, 记为$[\boldsymbol{abc}]$.

混合积的坐标表达式

设向量$\boldsymbol{a} = \{a_x, a_y, a_z\}$, $\boldsymbol{b} = \{b_x, b_y, b_z\}$, $\boldsymbol{c} = \{c_x, c_y, c_z\}$, 则

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \boldsymbol{i} + \begin{vmatrix} a_z & a_x \\ b_z & b_x \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \boldsymbol{k},$$

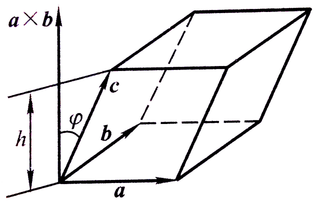
所以

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} c_x + \begin{vmatrix} a_z & a_x \\ b_z & b_x \end{vmatrix} c_y + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} c_z = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

由行列式的性质易知混合积的轮换对称性:

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = (\boldsymbol{b} \times \boldsymbol{c}) \cdot \boldsymbol{a} = (\boldsymbol{c} \times \boldsymbol{a}) \cdot \boldsymbol{b},$$

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = -(\boldsymbol{b} \times \boldsymbol{a}) \cdot \boldsymbol{c} = -(\boldsymbol{a} \times \boldsymbol{c}) \cdot \boldsymbol{b} = -(\boldsymbol{c} \times \boldsymbol{b}) \cdot \boldsymbol{a}.$$



混合积的几何意义

将向量$\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$看作一个平行六面体的相邻三棱, 则$|\boldsymbol{a}\times\boldsymbol{b}|$是该平行六面体的底面积. 又$\boldsymbol{a}\times\boldsymbol{b}$垂直于$\boldsymbol{a},\boldsymbol{b}$所在的底面, 若以$\varphi$表示向量$\boldsymbol{a}\times\boldsymbol{b}$与$\boldsymbol{c}$的夹角, 则当$0\le\varphi\le\frac{\pi}{2}$时, $|\boldsymbol{c}|\cos\varphi$就是该平行六面体的高$h$, 于是

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}=|\boldsymbol{a}\times\boldsymbol{b}||\boldsymbol{c}|\cos\varphi=|\boldsymbol{a}\times\boldsymbol{b}|h=V,$$

其中V表示平行六面体的体积. 当$\frac{\pi}{2}<\varphi\le\pi$时, $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}=-V$.

混合积$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$的绝对值是以$\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$为相邻三棱的平行六面体的体积.

三向量$\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$共面的充要条件是$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}=0$, 也即

$$\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}=0.$$

例 3.12. 求以点$A(1,1,1)$, $B(3,4,4)$, $C(3,5,5)$ 和$D(2,4,7)$为顶点的四面体$ABCD$的体积.

解: 由混合积的几何意义知

$$V=\frac{1}{6}\left|(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AD}\right|.$$

又$\overrightarrow{AB}=\{2,3,3\}$, $\overrightarrow{AC}=\{2,4,4\}$, $\overrightarrow{AD}=\{1,3,6\}$, 故

$$(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{AD}=\begin{vmatrix} 2 & 3 & 3 \\ 2 & 4 & 4 \\ 1 & 3 & 6 \end{vmatrix}=6.$$

于是

$$V_{ABCD}=\frac{1}{6}\times 6=1.$$

例 3.13. 问四个点$A(1,1,1)$, $B(4,5,6)$, $C(2,3,3)$ 和$D(10,15,17)$是否在同一平面上?

解: 四点共面等价于三向量共面. 现$\overrightarrow{AB} = \{3,4,5\}$, $\overrightarrow{AC} = \{1,2,2\}$, $\overrightarrow{AD} = \{9,14,16\}$, 故

$$(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} = \begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 2 \\ 9 & 14 & 16 \end{vmatrix} = 0.$$

因此$\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$共面, 即A, B, C, D四点在同一平面上.

例 3.14. *证明向量$\boldsymbol{m} = \boldsymbol{a} - \boldsymbol{b}$, $\boldsymbol{n} = \boldsymbol{b} - \boldsymbol{c}$, $\boldsymbol{p} = \boldsymbol{c} - \boldsymbol{a}$共面.*

证明: 易知$\boldsymbol{p} = -\boldsymbol{m} - \boldsymbol{n}$, 从而

$$(\boldsymbol{m} \times \boldsymbol{n}) \cdot \boldsymbol{p} = -(\boldsymbol{m} \times \boldsymbol{n}) \cdot \boldsymbol{m} - (\boldsymbol{m} \times \boldsymbol{n}) \cdot \boldsymbol{n} = 0.$$

因此三向量共面.

例 3.15. *设向量$\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{p}$两两垂直, 符合右手法则, 且$|\boldsymbol{m}| = 4$, $|\boldsymbol{n}| = 2$, $|\boldsymbol{p}| = 3$, 计算$(\boldsymbol{m} \times \boldsymbol{n}) \cdot \boldsymbol{p}$.*

证明:

$$\boldsymbol{m} \times \boldsymbol{n} = |\boldsymbol{m}||\boldsymbol{n}| \sin(\widehat{\boldsymbol{m}, \boldsymbol{n}}) = 4 \times 2 \times 1 = 8,$$

所以

$$(\boldsymbol{m} \times \boldsymbol{n}) \cdot \boldsymbol{p} = |\boldsymbol{m} \times \boldsymbol{n}||\boldsymbol{p}| \cos 0 = 8 \times 3 = 24.$$

7.3.4 思考与练习

练习 7.1. *设$\boldsymbol{m}$, $\boldsymbol{n}$为相互垂直的单位向量, 求$\boldsymbol{a} = 10\boldsymbol{m} + 2\boldsymbol{n}$在$\boldsymbol{b} = 5\boldsymbol{m} - 12\boldsymbol{n}$ 上的投影.*

解:

$$\boldsymbol{a} \cdot \boldsymbol{b} = (10\boldsymbol{m} + 2\boldsymbol{n}) \cdot (5\boldsymbol{m} - 12\boldsymbol{n}) = 50 - 24 = 26,$$

$$|\boldsymbol{b}| = \sqrt{169} = 13,$$

所以

$$\mathrm{Prj}_b \boldsymbol{a} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|} = \frac{26}{13} = 2.$$

练习 7.2. *设$a_i, b_i \in \boldsymbol{R} (i = 1,2,3)$, 证明不等式*

$$|a_1 b_1 + a_2 b_2 + a_3 b_3| \le (a_1^2 + a_2^2 + a_3^2)^{\frac{1}{2}} (b_1^2 + b_2^2 + b_3^2)^{\frac{1}{2}}.$$

证明: 设向量$\boldsymbol{a} = \{a_1, a_2, a_3\}$, $\boldsymbol{b} = \{b_1, b_2, b_3\}$. 由于

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta,$$

故

$$|\boldsymbol{a} \cdot \boldsymbol{b}| \leq |\boldsymbol{a}||\boldsymbol{b}|.$$

将$\boldsymbol{a}, \boldsymbol{b}$的坐标代入上式即得所要求证的不等式

$$|a_1b_1 + a_2b_2 + a_3b_3| \leq (a_1^2 + a_2^2 + a_3^2)^{\frac{1}{2}}(b_1^2 + b_2^2 + b_3^2)^{\frac{1}{2}}.$$

练习 7.3. 设向量$\boldsymbol{a}$, $\boldsymbol{b}$不共线, 向量$\boldsymbol{p} = \lambda\boldsymbol{a} + 5\boldsymbol{b}$, $\boldsymbol{q} = 3\boldsymbol{a} - \boldsymbol{b}$, 问$\lambda$为何值时, $\boldsymbol{p}$, $\boldsymbol{q}$共线.

解: $\boldsymbol{p}$, $\boldsymbol{q}$共线等价于$\boldsymbol{p} \times \boldsymbol{q} = \boldsymbol{0}$. 由于

$$\begin{aligned}
\boldsymbol{p} \times \boldsymbol{q} =& (\lambda\boldsymbol{a} + 5\boldsymbol{b}) \times (3\boldsymbol{a} - \boldsymbol{b}) \\
=& 3\lambda\boldsymbol{a} \times \boldsymbol{a} - \lambda\boldsymbol{a} \times \boldsymbol{b} + 15\boldsymbol{b} \times \boldsymbol{a} - 5\boldsymbol{b} \times \boldsymbol{b} \\
=& -\lambda\boldsymbol{a} \times \boldsymbol{b} - 15\boldsymbol{a} \times \boldsymbol{b} = -(\lambda + 15)\boldsymbol{a} \times \boldsymbol{b},
\end{aligned}$$

所以$\lambda = -15$.

练习 7.4. 已知三角形$\triangle ABC$的顶点分别是$A(1,2,3)$, $B(3,4,5)$ 和$C(2,4,7)$, 求$\triangle ABC$的面积.

解: 所求$\triangle ABC$的面积

$$S_{\triangle ABC} = \frac{1}{2}|\overrightarrow{AB}||\overrightarrow{AC}| \sin\angle A = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|.$$

由于$\overrightarrow{AB} = (2,2,2)$, $\overrightarrow{AC} = (1,2,4)$, 因此

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 2 & 2 \\ 1 & 2 & 4 \end{vmatrix} = 4\boldsymbol{i} - 6\boldsymbol{j} + 2\boldsymbol{k}.$$

故

$$S_{\triangle ABC} = \frac{1}{2}|4\boldsymbol{i} - 6\boldsymbol{j} + 2\boldsymbol{k}| = \frac{1}{2}\sqrt{4^2 + (-6)^2 + 2^2} = \sqrt{14}.$$

练习 7.5. 设$\boldsymbol{l}$是过空间$A(1,1,1)$, $B(1,-1,2)$的直线, $C(2,1,1)$为直线外的一点, 求点C到直线的距离.

解: 由平行四边形的面积公式知所求距离

$$d = \frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{|\overrightarrow{AB}|}.$$

由于$\overrightarrow{AB} = (0, -2, 1)$, $\overrightarrow{AC} = (1, 0, 0)$, 因此

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & -2 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \boldsymbol{j} + 2\boldsymbol{k}.$$

故$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{5}$. 又$|\overrightarrow{AB}| = \sqrt{5}$, 所以$d = 1$.

练习 7.6. 设向量$\boldsymbol{a} = (3, 2, 2)$, $\boldsymbol{b} = (18, -22, -5)$. 已知$\boldsymbol{c} \perp \boldsymbol{a}$, $\boldsymbol{c} \perp \boldsymbol{b}$, $|\boldsymbol{c}| = 14$, 且$\boldsymbol{c}$与y轴正向的夹角为钝角, 求$\boldsymbol{c}$.

解: 由题意知$\boldsymbol{c} \parallel \boldsymbol{a} \times \boldsymbol{b}$, 又

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 2 & 2 \\ 18 & -22 & -5 \end{vmatrix} = 34\boldsymbol{i} + 51\boldsymbol{j} - 102\boldsymbol{k},$$

故可取$\boldsymbol{c} = \lambda(2, 3, -6)$. 因为$|\boldsymbol{c}| = 7|\lambda| = 14$, 所以$\lambda = \pm 2$. 又由$\cos\beta < 0$知$\lambda = -2$, 于是

$$\boldsymbol{c} = (-4, -6, 12).$$

7.4 空间平面及其方程

7.4.1 平面的点法式和一般方程

法向量

与平面垂直的直线称为该平面的法线. 垂直于平面的非零向量叫作该平面的法向量, 记作$\boldsymbol{n}$.

由于过空间一点可以作且只能作一个平面垂直于已知直线, 因此当给定平面上的一点及其法向量时, 该平面的位置就完全确定了.

平面的点法式和一般方程

设$M_0(x_0, y_0, z_0)$是平面Π上的一点, $\boldsymbol{n} = (A, B, C)$是$\Pi$的法向量, 则该平面的方程为

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \tag{7.4.1}$$