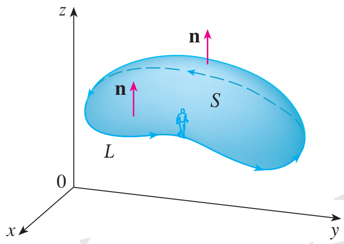
10.8 斯托克斯公式、环流量与旋度

10.8.1 斯托克斯公式

右手规则

设L是分段光滑的空间有向闭曲线, S是以L为边界的分片光滑的有向曲面, 当右手除拇指外的四指依L的绕行方向时, 拇指所指的方向与S上法向量的指向相同. 这时称L是有向曲面S 的正向边界曲线.



定理 8.1 (**斯托克斯**(Stokes)**公式**)**.** 设L是分段光滑的空间有向闭曲线, Σ是以L为边界的分片光滑的有向曲面, L的正向与Σ的侧符合右手规则. 若函数$P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$在包含曲面Σ在内的一个空间区域内具有一阶连续偏导数, 则有

$$\iint_{\Sigma}\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathrm{d}y\,\mathrm{d}z+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathrm{d}z\,\mathrm{d}x+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$
$$=\oint_{L}P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z.$$

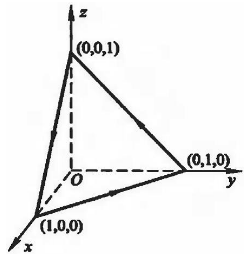
斯托克斯公式的另一种形式

$$\iint\limits_{\Sigma}\begin{vmatrix}\mathrm{d}y\,\mathrm{d}z & \mathrm{d}z\,\mathrm{d}x & \mathrm{d}x\,\mathrm{d}y\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}=\oint\limits_{L}P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z.$$

$$\iint\limits_{\Sigma}\begin{vmatrix}\cos\alpha & \cos\beta & \cos\gamma\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}\mathrm{d}S=\oint\limits_{L}P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z.$$

- 斯托克斯公式表达了有向曲面上的曲面积分与其边界曲线上的曲线积分之间的关系.
- 当Σ是xOy面的平面闭区域时, 斯托克斯公式就变成格林公式. 因此, 格林公式是斯托克斯公式的一个特殊情形.

例 8.1. 计算曲线积分$\oint_L(2y+z)\,\mathrm{d}x+(x-z)\,\mathrm{d}y+(y-x)\,\mathrm{d}z$, 其中$L$为平面$x+y+z=1$与各坐标面的交线, 它的正向与这个三角形上侧的法向量之间符合右手规则.



解: 由斯托克斯公式可得

$$\begin{aligned}&\oint_L (2y+z)\,\mathrm{d}x + (x-z)\,\mathrm{d}y + (y-x)\,\mathrm{d}z \\ =& \iint_\Sigma 2\,\mathrm{d}y\,\mathrm{d}z + 2\,\mathrm{d}z\,\mathrm{d}x - \,\mathrm{d}x\,\mathrm{d}y = 1 + 1 - \frac{1}{2} = \frac{3}{2}.\end{aligned}$$

例 8.2. 计算

$$I = \oint_L (y^2 - z^2)\,\mathrm{d}x + (z^2 - x^2)\,\mathrm{d}y + (x^2 - y^2)\,\mathrm{d}z,$$

其中L是用平面$x+y+z=\frac{3}{2}$截立方体: $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$的表面所得的截痕, 若从x轴的正向看去取逆时针方向.

解: 取Σ为平面$x+y+z=\frac{3}{2}$的上侧被L所围成的部分, Σ的单位法向量为$\frac{1}{\sqrt{3}}(1,1,1)$. 令$D_{xy}$为$\Sigma$在$xOy$平面上的投影区域. 由斯托克斯公式可得

$$\begin{aligned} I &= -2 \iint\limits_\Sigma (y+z)\,\mathrm{d}y\,\mathrm{d}z + (z+x)\,\mathrm{d}z\,\mathrm{d}x + (x+y)\,\mathrm{d}x\,\mathrm{d}y \\ &= -\frac{4}{\sqrt{3}} \iint\limits_\Sigma (x+y+z)\,\mathrm{d}S = -\frac{4}{\sqrt{3}} \cdot \frac{3}{2} \iint\limits_\Sigma \,\mathrm{d}S \\ &= -\frac{4}{\sqrt{3}} \cdot \frac{3}{2} \iint\limits_{D_{xy}} \sqrt{3}\,\mathrm{d}x\,\mathrm{d}y = -6 \iint\limits_{D_{xy}} \,\mathrm{d}x\,\mathrm{d}y = -6 \cdot \frac{3}{4} = -\frac{9}{2}. \end{aligned}$$

例 8.3. 计算曲线积分$\oint\limits_L y\,\mathrm{d}x + z\,\mathrm{d}y + x\,\mathrm{d}z$, 其中$L$为圆周$x^2+y^2+z^2=a^2$, $x+y+z=0$, 若从x轴的正向看去, 这圆周取逆时针方向.

解: 设Σ为平面$x+y+z=0$上L所围成的部分, Σ上侧的单位法向量为$\frac{1}{\sqrt{3}}(1,1,1)$. 由斯托克斯公式可得

$$\begin{aligned} \oint\limits_L y\,\mathrm{d}x + z\,\mathrm{d}y + x\,\mathrm{d}z &= -\iint\limits_\Sigma \,\mathrm{d}y\,\mathrm{d}z + \,\mathrm{d}z\,\mathrm{d}x + \,\mathrm{d}x\,\mathrm{d}y \\ &= -\sqrt{3} \iint\limits_\Sigma \,\mathrm{d}S = -\sqrt{3}\pi a^2. \end{aligned}$$

10.8.2 环流量与旋度

设向量场$\boldsymbol{A}=P(x,y,z)\boldsymbol{i}+Q(x,y,z)\boldsymbol{j}+R(x,y,z)\boldsymbol{k}$, 其中函数$P(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$在空间区域Ω内具有一阶连续偏导数, 称

$$\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\boldsymbol{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\boldsymbol{k}$$

为向量场$\boldsymbol{A}$的旋度, 记为$\operatorname{curl}\boldsymbol{A}$, 即

$$\begin{aligned}\operatorname{curl}\boldsymbol{A}&=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\boldsymbol{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\boldsymbol{k}\\&=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\P&Q&R\end{vmatrix}.\end{aligned}$$

由旋度可得到Stokes公式的向量型表达式

$$\iint\limits_{\Sigma}\operatorname{curl}\boldsymbol{A}\cdot\boldsymbol{n}\,\mathrm{d}S=\oint\limits_{L}\boldsymbol{A}\cdot\boldsymbol{\tau}\,\mathrm{d}s\quad\text{或}\quad\iint\limits_{\Sigma}(\operatorname{curl}\boldsymbol{A})_n\,\mathrm{d}S=\oint\limits_{L}A_\tau\,\mathrm{d}s,$$

其中$\boldsymbol{n}=(\cos\alpha,\cos\beta,\cos\gamma)$是曲面$\Sigma$上点$(x,y,z)$处的单位法向量, $\boldsymbol{\tau}=(\cos\lambda,\cos\mu,\cos\nu)$是$\Sigma$的正向边界曲线$L$上点$(x,y,z)$处的单位切向量,

$$(\operatorname{curl}\boldsymbol{A})_n=\operatorname{curl}\boldsymbol{A}\cdot\boldsymbol{n},\quad A_\tau=\boldsymbol{A}\cdot\boldsymbol{\tau}.$$

沿有向闭曲线L的曲线积分

$$\oint\limits_{L}P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z=\oint\limits_{L}A_\tau\,\mathrm{d}s$$

称为向量场$\boldsymbol{A}$沿有向闭曲线L的环流量.

Stokes公式可叙述为: 向量场$\boldsymbol{A}$沿有向闭曲线L的环流量等于向量场$\boldsymbol{A}$的旋度场通过L所张的曲面Σ的通量.

10.8.3 空间曲线积分与路线的无关性

定理 8.2. *设$\Omega\subset\mathbb{R}^3$为空间单连通区域. 若函数$P,Q,R$在$\Omega$上连续, 且有一阶连续偏导数, 则以下四个条件是等价的:*

(i) *对于Ω内任一逐段光滑的封闭曲线L有*

$$\oint_L P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z=0.$$

(ii) *对于Ω内任一逐段光滑的曲线L, 曲线积分*

$$\int_L P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z$$

与路径无关.

(iii) $P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$是$\Omega$内某一函数$u$的全微分, 即

$$\mathrm{d}u = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z.$$

(iv) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$, $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$在$\Omega$内处处成立.

例 8.4. 验证曲线积分

$$\int_L (y+z)\,\mathrm{d}x + (z+x)\,\mathrm{d}y + (x+y)\,\mathrm{d}z$$

与路径无关, 并求被积表达式的原函数$u(x,y,z)$.

解: 由于

$$P = y+z,\ \ Q = z+x,\ \ R = x+y,$$
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} = 1,$$

所以曲线积分与路径无关.

取$M_0(0,0,0)$, $M(x,y,z)$, 则原函数为

$$\begin{aligned} u(x,y,z) &= \int_{M_0M} (y+z)\,\mathrm{d}x + (z+x)\,\mathrm{d}y + (x+y)\,\mathrm{d}z + C \\ &= \int_0^x (0+0)\,\mathrm{d}s + \int_0^y (0+x)\,\mathrm{d}t + \int_0^z (x+y)\,\mathrm{d}r + C = xy + xz + yz + C. \end{aligned}$$

例 8.5. 验证曲线积分

$$\int_{(1,1,1)}^{(2,3,-4)} x\,\mathrm{d}x + y^2\,\mathrm{d}y - z^3\,\mathrm{d}z$$

与路径无关, 并计算其值.

解: 由于

$$P = x,\ \ Q = y^2,\ \ R = -z^3,$$
$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0 - 0 = 0,\quad \frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} = 0 - 0 = 0,\quad \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} = 0 - 0 = 0,$$

所以曲线积分与路径无关.

路径取为从$A(1,1,1)$到$B(2,3,-4)$的直线段

$$AB: x = 1+t,\ y = 1+2t,\ z = 1-5t,\quad 0 \le t \le 1,$$

则

$$\text{原式} = \int_0^1 [(1+t) + 2(1+2t)^2 + 5(1-5t)^3]\,\mathrm{d}t = -53\frac{7}{12}.$$