VV256 Recitation Class - Week 4

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Overview

1. Higher-order ODE
   - Introduction from linear second order
   - Variation of constants
   - Wronskian
   - Undetermined coefficients
   - Eulers equation
   - Homogeneity
   - Order reduction

Introduction from linear second order

Many second-order ordinary differential equations have the form

$$\frac{d^2y}{dt^2} = f(t, y, \frac{dy}{dt}) \tag{1}$$

We start with the simple case: linear

$$y'' + p(t)y' + q(t)y = g(t) \tag{2}$$

As said before, one important method to solve ODE is to reduce order. We set $y_1(t) = y(t), y_2(t) = y'(t)$. Then we have:

$$y_1'(t) = y_2(t)$$

$$y_2'(t) = -p(t)y_2(t) - q(t)y_1(t) + g(t)$$

Introduction from linear second order

Then we rewrite it in matrix form:

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

which is a linear normal system $\bar{y}' = A(t)\bar{y} + b(t)$,and it will own its solution. To solve it, we first solve its corresponding homogeneous linear system.

Remember that, for non-homogeneous ODE, its homogeneous part will lead to a general solution(usually with constant C) and its non-homogeneous part will lead to a "special" slution. The final answer is the sum of them.

FSS

Consider its homogeneous linear system $\bar{y}' = A(t)\bar{y}$.

There exists a fundamental system of solutions (FSS) if a system $\bar{\varphi}(t)$ of solutions of the homogeneous system is linearly independent.

Thus, any solution $y = \bar{\varphi}(t)$ of the system can be written in the form $y = C_1\varphi_1(t) + ... + C_n\varphi_n(t)$

Method of variation of parameters

Then we shall look for the solutions of the non-homogeneous system.

Based on the FSS of homogeneous system $\varphi_1(t), ..., \varphi_n(t)$, list the equation $C_1'(t)\varphi_1(t) + ... + C_n'(t)\varphi_n(t) = b(t)$ to solve each $C_i(t), i = 1, .., n$. This method is called variation of parameters.

Summary

So far we should:

1. Understand the method to manage higher-order ODE: convert it into a lower-order ODE. Such as convert a higher-order linear ODE into a first-order ODE linear system. (The idea is important insdead of solution of it.)

2. Know how to deal with non-homogeneous ODE: solve it homogeneous part first. Then use method of variation of parameters.

As said before, variation of constants method is used in non-homogeneous ODE to find a particular solution due to the non-homogeneous solution.

Practice: $y' + y = t + 1$

$y' + y = 0$

$\frac{dy}{y} = -dx$

$\ln y = -x + C$

$y = Ce^{-x}$

$y' = c'(t)e^{-x} - c(t)e^{-x}$

$c'(t)e^{-t} = t+1$

$c'(t) = (t+1)e^{t}$

$c(t) = \int (t+1)e^{t}\,dt$

$= \cdots e^{t} + C$

Wronskian

Consider two solutions $y_1(t)$ and $y_2(t)$ ($y = C_1y_1(t) + C_2y_2(t)$) of the second-order linear homogeneous ODE. If given one of them, we may want to find their relationship in order to find the other.

$$(\frac{y_2}{y_1})' = \frac{y_1y_2' - y_1'y_2}{y_1^2} = \frac{W[y_1, y_2]}{y_1^2}$$

$W[y_1, y_2]$ is the wronskian mentioned before. Here we can still use it to test the linear independence of solutions. More importantly, it can help us to deal with linear homogeneous second order ODE ($y" + a_1(t)y' + a_2(t)y = 0$) with folowing steps.

* Guess a solution of second ODE y_1(or given)
* Calculate the Wronskian of ODE $W = e^{-\int a_1(t)dt}$
* Represent the other solution $y_2 = y_1 \int \frac{e^{-\int a_1(t)dt}}{y_1^2} dt$

Practice: $(2t+1)y" + 4ty' - 4y = 0$

$(2t+1)y'' + 4ty' - 4y = 0 \quad y_1 = t$

$$y'' + \frac{4t}{2t+1}y' - \frac{4}{2t+1}y = 0$$

$$\frac{y_2}{y_1} = \int \frac{e^{\int -\frac{4t}{2t+1}dt}}{t^2} dt$$

$y'' + 2y' + 3y = 0$

$\underline{y = e^{\lambda t}} \quad \lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} - 3e^{\lambda t} = 0$

$\lambda^2 + 2\lambda - 3 = 0$

$\lambda = -3 \quad \lambda = 1$

$y = e^{-3t} \quad y = e^{t}$

$y = C_1 e^{-3t} + C_2 e^{t}$

Undetermined coefficients

When dealing with the linear higher order ODE with constant coefficient (e.g. $3y^{(3)} + y" - 2y' + y = 0$).

It has a very good form that we can guess its form of solution is $e^{\lambda t}$.

Think about it. $3\lambda^3 * e^{\lambda t} + \lambda^2 * e^{\lambda t} - 2\lambda * e^{\lambda t} + e^{\lambda t} = 0$

$3\lambda^3 + \lambda^2 - 2\lambda + 1 = 0$, solve it to obtain λ then get solutions. Notice that λ could be a complex number and there maybe multiple roots appear. How to deal with them will be introduced next week.
Practice: $y" + 2y' - 3y = 0$

Eulers equation

An Euler differential equation is a linear equation with variable coefficients of the form:

$$a_n t^n \frac{d^n y}{dt^n} + ... + a_1 t \frac{dy}{dt} + a_0 y = f(t)$$

For example, $t^2 y" + ty' + y = 0$.By observing its form, we can guess out the form of its solution is t^λ.
Think about it. $t^2 * \lambda * (\lambda - 1)t^{\lambda-2} + t * \lambda t^{\lambda-1} + t^\lambda = 0$
$\lambda^2 + \lambda + 1 = 0$, solve it to obtain λ then get solutions. Similar as previous slide.

①

$y''' = \cos t + \sin t$

$y'' = \sin t - \cos t + C_1$

$y' = -\cos t - \sin t + C_1 t + C_2$

$y = -\sin t + \cos t + \frac{1}{2} C_1 t^2 + C_2 t + C_3$

② $t(y''+1) + y' = 0$

let $y' = p(t)$ $y'' = p'(t)$

$t(p'(t)+1) + p = 0$

③ $y'' = \sqrt{1-(y')^2}$

☆ $y' = p(y)$ $y'' = \frac{dp}{dt} = \frac{dp}{dy} \cdot \frac{dy}{dt}$

$= \frac{dp}{dy} \cdot p = \sqrt{1-p^2}$

$p = \frac{dy}{dt}$

$y + C = -\sqrt{1-p^2}$

$(y+C)^2 = 1 - p^2$

$p^2 = \sqrt{1-(y+C)^2}$

④ $tyy' - t(y')^2 - yy' = 0$

☆ homo ⇒ $y' = y \cdot z(t)$

$$y'' = y' \cdot z(t) + y \cdot z'(t)$$

$$= y\left(z^2(t) + z'(t)\right)$$