8.3 全微分

8.3.1 多元函数全微分的定义及其计算

定义 3.1. 设D是$\mathbb{R}^2$中的一个区域，点$P_0(x_0,y_0)\in D$，函数$z=f(x,y)$在点P_0的某邻域内有定义，自变量x,y分别有增量Δx，Δy，若存在仅与点$P_0(x_0,y_0)$有关而与Δx，Δy无关的常数A,B，使得函数全增量$\Delta z=f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)$可表示为

$$\Delta z=A\Delta x+B\Delta y+o(\rho),$$

其中$\rho=\sqrt{\Delta x^2+\Delta y^2}$，则称函数$z=f(x,y)$在点$P_0(x_0,y_0)$可微，并称其线性主部$A\Delta x+B\Delta y$为函数$z=f(x,y)$在点$P_0(x_0,y_0)$的全微分，记作$\mathrm{d}z|_{(x_0,y_0)}$或$\mathrm{d}f|_{(x_0,y_0)}$，即

$$\mathrm{d}z|_{(x_0,y_0)}=A\Delta x+B\Delta y.$$

若函数在域D内各点都可微，则称此函数在D内可微，其全微分记为$\mathrm{d}z$.

注

- 由于$\Delta x = \mathrm{d}x$和$\Delta y = \mathrm{d}y$, 故函数$z = f(x,y)$在点(x_0,y_0)的全微分可表示为$\mathrm{d}z|_{(x_0,y_0)} = A\Delta x + B\Delta y = A\,\mathrm{d}x + B\,\mathrm{d}y$.

- 函数$z = f(x,y)$在点(x_0,y_0)的全增量Δz 与 $\mathrm{d}z|_{(x_0,y_0)} = A\Delta x + B\Delta y$ 之差为ρ的高阶无穷小量.

定理 3.1. 若区域D上的二元函数$z = f(x,y)$在点$P_0(x_0,y_0) \in D$可微, 其全微分为$\mathrm{d}z|_{(x_0,y_0)} = A\Delta x + B\Delta y$, 则函数$z = f(x,y)$在点$P_0(x_0,y_0)$的一阶偏导数都存在, 且

$$A = \frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)}, \quad B = \frac{\partial f}{\partial y}\bigg|_{(x_0,y_0)}.$$

此时

$$\mathrm{d}z|_{(x_0,y_0)} = \frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)} \mathrm{d}x + \frac{\partial f}{\partial y}\bigg|_{(x_0,y_0)} \mathrm{d}y.$$

定理 3.2 (**可微的必要条件**)**.** 若区域D上的二元函数$z = f(x,y)$在点(x_0,y_0)处可微, 则函数$z = f(x,y)$在点(x_0,y_0)处连续.

注 3.3. 函数连续与可偏导是可微的必要条件而非充分条件. 事实上, 函数可偏导并不能保证函数可微.

例 3.1. 求函数$z = x^2 + e^{xy}$在点$(2,1)$处的全微分.

解: 由于

$$z'_x = 2x + ye^{xy}, \quad z'_y = xe^{xy},$$

故全微分为

$$\mathrm{d}z = z'_x(2,1)\,\mathrm{d}x + z'_y(2,1)\,\mathrm{d}y = (4+e^2)\,\mathrm{d}x + 2e^2\,\mathrm{d}y.$$

例 3.2. 设$z = x^y$, 求全微分$\mathrm{d}z$.

解: 由于

$$z'_x = yx^{y-1}, \quad z'_y = x^y \ln x,$$

故全微分

$$\mathrm{d}z = yx^{y-1}\,\mathrm{d}x + x^y \ln x\,\mathrm{d}y.$$

例 3.3. 考察函数$f(x,y) = xy$在点(x_0,y_0)(除$(0,0)$外)处的可微性.

解: 由于

$$\Delta z = (x_0+\Delta x)(y_0+\Delta y) - x_0y_0 = y_0\Delta x + x_0\Delta y + \Delta x\Delta y,$$
$$|\Delta x\Delta y| \le \frac{1}{2}(\Delta x^2+\Delta y^2),$$

所以

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta z-(y_0\Delta x+x_0\Delta y)}{\sqrt{\Delta x^2+\Delta y^2}} = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta x\Delta y}{\sqrt{\Delta x^2+\Delta y^2}} = 0.$$

故$f(x,y)=xy$在点(x_0,y_0))处可微.

例 3.4. 证明二元函数$f(x,y)=\begin{cases}\frac{xy}{x^2+y^2}, & (x,y)\ne(0,0),\\ 0, & (x,y)=(0,0)\end{cases}$在点$(0,0)$处可偏导, 但不可微.

解: $f(x,y)$在点$(0,0)$处可偏导的证明见上节. 对任意的实数k, 有

$$\lim_{\substack{y=kx\\x\to0}} f(x,y) = \lim_{\substack{y=kx\\x\to0}} \frac{kx^2}{x^2+(kx)^2} = \frac{k}{1+k^2}.$$

所以函数$f(x,y)$在原点处的极限不存在, 故$f(x,y)$在点$(0,0)$处不可微.

定理 3.4 (**可微的充分条件**). *若函数$z=f(x,y)$在点(x_0,y_0)具有连续偏导数, 则$f(x,y)$在点(x_0,y_0)处可微分.*

例 3.5. 证明函数$f(x,y)=\begin{cases}xy\sin\frac{1}{\sqrt{x^2+y^2}}, & (x,y)\ne(0,0),\\ 0, & (x,y)=(0,0)\end{cases}$在点$(0,0)$处连续且偏导数存在, 但偏导数在点$(0,0)$处不连续, 而$f(x,y)$在点$(0,0)$处可微.

解: 因为$f(x,0)=0$, $f(0,y)=0$, 故$f'_x(0,0)=0$, $f'_y(0,0)=0$. 当$(x,y)\ne(0,0)$时,

$$f'_x(x,y) = y\sin\frac{1}{\sqrt{x^2+y^2}} - \frac{x^2y}{\sqrt{(x^2+y^2)^3}}\cos\frac{1}{\sqrt{x^2+y^2}}.$$

由于

$$\lim_{\substack{y=|x|\\x\to0}} f'_x(x,y) = \lim_{\substack{y=|x|\\x\to0}} \left(y\sin\frac{1}{\sqrt{x^2+y^2}} - \frac{x^2y}{\sqrt{(x^2+y^2)^3}}\cos\frac{1}{\sqrt{x^2+y^2}}\right)$$

不存在, 所以$f'_x(x,y)$在点$(0,0)$处不连续. 由函数的对称性知, $f'_y(x,y)$在点$(0,0)$处不连续.

$$\lim_{\substack{\Delta x\to0\\\Delta y\to0}} \frac{\Delta f - f_x(0,0)\Delta x - f_y(0,0)\Delta y}{\sqrt{\Delta x^2+\Delta y^2}} = \lim_{\substack{\Delta x\to0\\\Delta y\to0}} \frac{\Delta x\Delta y\sin\frac{1}{\sqrt{\Delta x^2+\Delta y^2}}}{\sqrt{\Delta x^2+\Delta y^2}} = 0.$$

所以$f(x,y)$在点$(0,0)$处可微, 进而连续且可偏导.

以上所作的讨论可以完全类似地推广到二元以上的多元函数.

对于多元函数$u=f(x_1,x_2,\cdots,x_n)$, 当$u=f(x_1,x_2,\cdots,x_n)$可微时, 其全微分

$$\mathrm{d}u = f'_{x_1}\,\mathrm{d}x_1 + f'_{x_2}\,\mathrm{d}x_2 + \cdots + f'_{x_n}\,\mathrm{d}x_n.$$

例 3.6. 设$u = \ln(x - y + z^2)$, 求全微分du.

解: 由于

$$u_x = \frac{1}{x - y + z^2}, \quad u_y = \frac{-1}{x - y + z^2}, \quad u_z = \frac{2z}{x - y + z^2},$$

故全微分

$$du = \frac{1}{x - y + z^2}(dx - dy + 2zdz).$$

二元函数可微分的几何意义

设二元函数$z = f(x, y)$在点(x_0, y_0)可微分, 则在点(x_0, y_0)的某邻域内有

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

记上式的右端为

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

它表示通过点$(x_0, y_0, f(x_0, y_0))$并以$(f_x(x_0, y_0), f_y(x_0, y_0), -1)$为法向量的一张平面. 这张平面就是曲面在该点的切平面.

若函数在点(x_0, y_0)可微分, 则曲面在点$(x_0, y_0, f(x_0, y_0))$近旁的一小部分, 可用曲面在该点的切平面来近似代替.

8.3.2 近似计算

近似计算

由全微分定义

$$\Delta z|_{(x_0, y_0)} = f_x'(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + o(\rho).$$

可知当$|\Delta x|$和$|\Delta y|$都很小时, 有近似公式

$$\Delta z \approx \mathrm{d}z|_{(x_0, y_0)} = f_x'(x_0, y_0)\Delta x + f_y'(x_0, y_0)\Delta y,$$

$$f(x_0 + \Delta x, y_0 + \Delta y) \approx f(x_0, y_0) + f_x'(x_0, y_0)\Delta x + f_y'(x_0, y_0)\Delta y.$$

例 3.7. 计算$(1.04)^{2.02}$的近似值.

解: 设$z = f(x, y) = x^y$, 利用函数$f(x, y) = x^y$在点$(1, 2)$处的可微性, 可得

$$\Delta z \approx dz,$$

即

$$f(1.04,2.02)-f(1,2)\approx f'_x(1,2)\Delta x+f'_y(1,2)\Delta y,$$

其中$\Delta x=0.04$, $\Delta y=0.02$. 于是

$$\begin{aligned}(1.04)^{2.02}=&f(1.04,2.02)\approx f(1,2)+f'_x(1,2)\Delta x+f'_y(1,2)\Delta y\\=&1+2\times 0.04+0\times 0.02=1.08.\end{aligned}$$

例 3.8. *计算*$1.002\times 2.003^2\times 3.004^3$*的近似值.*

解: 设$f(x,y,z)=(1+x)^m(1+y)^n(1+z)^t$, 则当$|x|,|y|,|z|$很小时, 有近似公式

$$\begin{aligned}f(x,y,z)&\approx f(0,0,0)+f'_x(0,0,0)x+f'_y(0,0,0)y+f'_z(0,0,0)z\\&=1+mx+ny+tz.\end{aligned}$$

于是

$$\begin{aligned}&1.002\times 2.003^2\times 3.004^3\\=&(1+0.002)\times 2^2(1+0.003/2)^2\times 3^3(1+0.004/3)^3\\\approx&2^2\times 3^3(1+0.002+0.003+0.004)=108.972.\end{aligned}$$

8.3.3 思考与练习

练习 8.8. *函数*$f(x,y)$*在*(x_0,y_0)*处满足(　　), 则*$f(x,y)$*在*(x_0,y_0)*处可微.*

(A) $\lim\limits_{\rho\to 0}(\Delta z-f'_x(x_0,y_0)\Delta x-f'_y(x_0,y_0)\Delta y)=0$

(B) $\lim\limits_{\rho\to 0}\frac{\Delta z-f'_x(x_0,y_0)\Delta x-f'_y(x_0,y_0)\Delta y}{\rho}=0$

(C) 连续且偏导数存在

(D) 二阶偏导数存在

B

练习 8.9. *若*$f'_x(x_0,y_0)=0$, $f'_y(x_0,y_0)=0$*, 则函数*$f(x,y)$ *在点*(x_0,y_0)*处一定(　　)*

(A) 连续　　*(B) 不连续*

(C) 可微　　*(D) 以上均不正确*

D

练习 8.10. 设$f(x) = \dfrac{x\cos y + y\cos z + z\cos x}{1+\cos x+\cos y+\cos z}$, 求$\mathrm{d}f|_{(0,0,0)}$.

利用轮换对称性, $\frac{1}{4}(\mathrm{d}x + \mathrm{d}y + \mathrm{d}z)$

练习 8.11. 设$z = x^2 + 3xy - y^2$, x从2变到2.05, y从3变到2.96, 试求Δz与$\mathrm{d}z$的差.

解:

$$\Delta z = (2.05^2 + 3\times 2.05\times 2.96 - 2.96^2) - (2^2 + 3\times 2\times 3 - 3^2) = 0.6449,$$

而

$$z'_x(2,3) = (2x+3y)|_{(2,3)} = 13,\ z'_y(2,3) = (3x-2y)|_{(2,3)} = 0,$$

$$\Delta x = 0.05, \quad \Delta y = -0.04,$$

故

$$\mathrm{d}z = z'_x(2,3)\Delta x + z'_y(2,3)\Delta y = 0.65,$$

于是

$$\Delta z - \mathrm{d}z = -0.0051.$$