8.7 空间曲线的切线与空间曲面的切平面方程

8.7.1 空间曲线的切线与法平面

空间曲线的切线

设空间曲线C的参数方程为

$$x = x(t), \quad y = y(t), \quad z = z(t) \quad (\alpha \le t \le \beta).$$

若$t_0, t_1 \in (\alpha, \beta)$, $A(x(t_0), y(t_0), z(t_0))$与$B(x(t_1), y(t_1), z(t_1))$为曲线$C$上两点, A, B的连线AB称为曲线C的割线. 当$B \to A$时, 若割线AB趋于一条直线, 则此直线称为曲线C在点A的切线.

若$x = x(t)$, $y = y(t)$, $z = z(t)$对于t的导数都连续且不全为零, 称空间曲线C为光滑曲线. 若空间曲线C为光滑曲线, 则曲线C在点A处必存在切线.

当光滑曲线C由参数方程$x = x(t)$, $y = y(t)$, $z = z(t)$给出时, 由于$x'(t)$, $y'(t)$, $z'(t)$连续且不同时为零, 因此从几何上看曲线上的每点处都有切线, 且切线随着切点的移动而连续转动, 此即为光滑曲线的几何特征.

设空间曲线C为光滑曲线, 割线AB的方程为

$$\frac{x - x(t_0)}{x(t_1) - x(t_0)} = \frac{y - y(t_0)}{y(t_1) - y(t_0)} = \frac{z - z(t_0)}{z(t_1) - z(t_0)},$$

可改写为

$$\frac{x - x(t_0)}{\frac{x(t_1) - x(t_0)}{t_1 - t_0}} = \frac{y - y(t_0)}{\frac{y(t_1) - y(t_0)}{t_1 - t_0}} = \frac{z - z(t_0)}{\frac{z(t_1) - z(t_0)}{t_1 - t_0}}.$$

当$B \to A$时, 有$t \to t_0$, 可得切线的方向向量为$(x'(t_0), y'(t_0), z'(t_0))$, 切线方程为

$$\frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)} = \frac{z - z(t_0)}{z'(t_0)}.$$

过点$A(x(t_0),y(t_0),z(t_0))$且与切线垂直的平面称为空间曲线C在点$A(x(t_0),y(t_0),z(t_0))$的法平面，其法平面方程为

$$x'(t_0)(x-x_0)+y'(t_0)(y-y_0)+z'(t_0)(z-z_0)=0.$$

设空间曲线C的方程为

$$y=y(x),\quad z=z(x),$$

且$y'(x_0)$, $z'(x_0)$存在，则曲线C在点$A(x_0,y(x_0),z(x_0))$处的切线方程为

$$\frac{x-x_0}{1}=\frac{y-y(x_0)}{y'(x_0)}=\frac{z-z(x_0)}{z'(x_0)}.$$

法平面方程为

$$x-x_0+y'(x_0)(y-y(x_0))+z'(x_0)(z-z(x_0))=0.$$

设空间曲线C的方程为

$$\begin{cases}F(x,y,z)=0,\\G(x,y,z)=0.\end{cases}$$

假设$J=\left.\frac{\partial(F,G)}{\partial(y,z)}\right|_{(x_0,y_0,z_0)}\neq 0$，则上述方程组在点$A(x_0,y_0,z_0)$的某个邻域内能确定隐函数

$$y=y(x),\quad z=z(x)$$

满足$y_0=y(x_0)$, $z_0=z(x_0)$, $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,z)}$, $\frac{\mathrm{d}z}{\mathrm{d}x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,x)}$. 于是空间曲线$C$在点$A(x_0,y(x_0),z(x_0))$处的切线方程为

$$\frac{x-x_0}{1}=\frac{y-y_0}{\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_A}=\frac{z-z_0}{\left.\frac{\mathrm{d}z}{\mathrm{d}x}\right|_A},\quad 即\quad \frac{x-x_0}{\left.\frac{\partial(F,G)}{\partial(y,z)}\right|_A}=\frac{y-y_0}{\left.\frac{\partial(F,G)}{\partial(z,x)}\right|_A}=\frac{z-z_0}{\left.\frac{\partial(F,G)}{\partial(x,y)}\right|_A}.$$

法平面方程为

$$\left.\frac{\partial(F,G)}{\partial(y,z)}\right|_A(x-x_0)+\left.\frac{\partial(F,G)}{\partial(z,x)}\right|_A(y-y_0)+\left.\frac{\partial(F,G)}{\partial(x,y)}\right|_A(z-z_0)=0.$$

例 7.1. *求曲线$x=a\cos\theta$, $y=a\sin\theta$, $z=b\theta$在点$(-a,0,b\pi)$处的切线.*

解: 点$(-a,0,b\pi)$对应$\theta=\pi$. 切线的方向向量为

$$(-a\sin\theta,a\cos\theta,b)|_{\theta=\pi}=(0,-a,b).$$

于是切线方程为

$$\frac{x-(-a)}{0}=\frac{y-0}{-a}=\frac{z-b\pi}{b},$$

即

$$\begin{cases}x=-a,\\-\frac{y}{a}=\frac{z}{b}-\pi.\end{cases}$$

例 7.2. 求曲线$x = \frac{t^4}{4}$, $y = \frac{t^3}{3}$, $z = \frac{t^2}{2}$的切线方程, 使之平行于平面$x + 3y + 2z = 0$.

解: 设曲线上的点对应的参数为t_0, 则切向量为

$$\boldsymbol{\tau} = (t_0^3, t_0^2, t_0).$$

因切线和平面平行, 故切向量与平面的法向垂直, 所以

$$t_0^3 + 3t_0^2 + 2t_0 = 0,$$

解得$t_0 = -1$或$t_0 = -2$, 因而切点为$\left(\frac{1}{4}, -\frac{1}{3}, \frac{1}{2}\right)$, $\left(4, -\frac{8}{3}, 2\right)$, 切线方程为

$$\frac{4x-1}{-4} = \frac{3y+1}{3} = \frac{2z-1}{-2},$$
$$\frac{x-4}{-8} = \frac{3y+8}{12} = \frac{z-2}{-2}.$$

8.7.2 空间曲面的切平面与法线

光滑曲面

设曲面S的一般方程为

$$F(x, y, z) = 0,$$

$F(x, y, z)$的偏导数F'_x, F'_y, F'_z连续且不同时为零, 这时称S是光滑曲面.

切平面

设$P_0(x_0, y_0, z_0)$是曲面S上一点, 平面π称为曲面S在点P_0处的切平面, 如果它满足: 曲面上任一条过该点的光滑曲线在该点的切线总在该平面上.

切平面方程的建立

设C是曲面S上过点P_0的任一条光滑曲线, 参数方程为

$$x = x(t), \quad y = y(t), \quad z = z(t),$$

点P_0对应于$t = t_0$, C在点P_0的切向量为

$$\boldsymbol{\tau} = (x'(t_0), y'(t_0), z'(t_0)).$$

因曲线C在曲面S上, 故总有

$$F[x(t), y(t), z(t)] \equiv 0.$$

两边求导, 并取$t = t_0$处的导数, 有

$$F'_x(x_0, y_0, z_0)x'(t_0) + F'_y(x_0, y_0, z_0)y'(t_0) + F'_z(x_0, y_0, z_0)z'(t_0) = 0.$$

记

$$\boldsymbol{n}=(F'_x(x_0,y_0,z_0),F'_y(x_0,y_0,z_0),F'_z(x_0,y_0,z_0)),$$

则$\boldsymbol{\tau}\perp\boldsymbol{n}$.

因此$\boldsymbol{n}$为所求切平面π的一个法向量. 由此切平面方程为

$$F'_x(x_0,y_0,z_0)(x-x_0)+F'_y(x_0,y_0,z_0)(y-y_0)$$
$$+F'_z(x_0,y_0,z_0)(z-z_0)=0.$$

过点P_0且与切平面垂直的直线称为曲面S在点P_0处的法线, 法线方程为

$$\frac{x-x_0}{F'_x(x_0,y_0,z_0)}=\frac{y-y_0}{F'_y(x_0,y_0,z_0)}=\frac{z-z_0}{F'_z(x_0,y_0,z_0)}.$$

设曲面S的方程为$z=f(x,y)$, 则曲面S在点$P_0(x_0,y_0,z_0)$处的切平面方程为

$$f'_x(x_0,y_0)(x-x_0)+f'_y(x_0,y_0)(y-y_0)-(z-z_0)=0,$$

法线方程为

$$\frac{x-x_0}{f'_x(x_0,y_0)}=\frac{y-y_0}{f'_y(x_0,y_0)}=\frac{z-z_0}{-1}.$$

设曲面S的方程为

$$x=x(u,v),\quad y=y(u,v),\quad z=z(u,v).$$

设$x_0=x(u_0,v_0)$, $y_0=y(u_0,v_0)$, $z_0=z(u_0,v_0)$, $P_0(x_0,y_0,z_0)$为曲面S上一点, 且$J=\frac{\partial(x,y)}{\partial(u,v)}\Big|_{P_0}\neq 0$, 则上述方程在点$P_0(x_0,y_0,z_0)$处的某个邻域内能确定隐函数

$$u=u(x,y),\quad v=v(x,y)$$

满足$u_0=u(x_0,y_0)$, $v_0=v(x_0,y_0)$. 于是曲面S可表示为

$$z=f(x,y)=z(u(x,y),v(x,y)).$$

通过计算求得

$$\frac{\partial u}{\partial x}=\frac{\frac{\partial y}{\partial v}}{\frac{\partial(x,y)}{\partial(u,v)}},\quad \frac{\partial v}{\partial x}=-\frac{\frac{\partial y}{\partial u}}{\frac{\partial(x,y)}{\partial(u,v)}},\quad \frac{\partial u}{\partial y}=-\frac{\frac{\partial x}{\partial v}}{\frac{\partial(x,y)}{\partial(u,v)}},\quad \frac{\partial v}{\partial y}=\frac{\frac{\partial x}{\partial u}}{\frac{\partial(x,y)}{\partial(u,v)}},$$

故

$$f'_x=z'_uu'_x+z'_vv'_x=-\frac{\frac{\partial(y,z)}{\partial(u,v)}}{\frac{\partial(x,y)}{\partial(u,v)}},\quad f'_y=z'_uu'_y+z'_vv'_y=-\frac{\frac{\partial(z,x)}{\partial(u,v)}}{\frac{\partial(x,y)}{\partial(u,v)}}.$$

所以曲面S在点$P_0(x_0,y_0,z_0)$处的切平面方程为

$$\left.\frac{\partial(y,z)}{\partial(u,v)}\right|_{(u_0,v_0)}(x-x_0)+\left.\frac{\partial(z,x)}{\partial(u,v)}\right|_{(u_0,v_0)}(y-y_0)+\left.\frac{\partial(x,y)}{\partial(u,v)}\right|_{(u_0,v_0)}(z-z_0)=0,$$

法线方程为

$$\frac{x-x_0}{\left.\frac{\partial(y,z)}{\partial(u,v)}\right|_{(u_0,v_0)}}=\frac{y-y_0}{\left.\frac{\partial(z,x)}{\partial(u,v)}\right|_{(u_0,v_0)}}=\frac{z-z_0}{\left.\frac{\partial(x,y)}{\partial(u,v)}\right|_{(u_0,v_0)}}.$$

例 7.3. 求曲面$z=y+\ln\frac{x}{z}$在点$(1,1,1)$处的切平面与法线方程.

解: 令$F(x,y,z)=y+\ln\frac{x}{z}-z$, 则

$$\boldsymbol{n}=\nabla F=\left(\frac{1}{x},1,-\frac{1}{z}-1\right),$$

故$\boldsymbol{n}|_{(1,1,1)}=(1,1,-2)$. 由此得切平面方程为

$$(x-1)+(y-1)-2(z-1)=0,$$

即

$$x+y-2z=0,$$

法线方程为

$$\frac{x-1}{1}=\frac{y-1}{1}=\frac{z-1}{-2}.$$

例 7.4. 求旋转抛物面$z=x^2+y^2-1$在点$(2,1,4)$处的切平面与法线方程.

解: 因

$$\boldsymbol{n}=(z_x,z_y,-1)|_{(2,1,4)}=(2x,2y,-1)|_{(2,1,4)}=(4,2,-1),$$

故所求切平面方程为

$$4(x-2)+2(y-1)-(z-4)=0,$$

即

$$4x+2y-z-6=0,$$

法线方程为

$$\frac{x-2}{4}=\frac{y-1}{2}=\frac{z-4}{-1}.$$

例 7.5. 求曲面$z=x^2+y^2$的一个切平面, 使该切平面与直线$l:\begin{cases}x+2z=1,\\ y+2z=2\end{cases}$垂直.

解: 直线l的方向向量为

$$\boldsymbol{s} = (1,0,2)\times(0,1,2) = (-2,-2,1).$$

曲面在点(x,y,z)处的法向量为

$$\boldsymbol{n} = (2x,2y,-1).$$

由题意知$\boldsymbol{n}\parallel\boldsymbol{s}$, 即

$$\frac{2x}{-2} = \frac{2y}{-2} = \frac{-1}{1},$$

解得$x = 1$, $y = 1$, 代入曲面方程得$z = 2$. 故切平面方程为

$$2x + 2y - z - 2 = 0.$$

例 7.6. *求曲面$z = xy$的一个切平面, 使之平行于平面$3x + 2y - z = 2$.*

解: 设切点为(x_0,y_0,z_0), 则切平面的法向

$$\boldsymbol{n} = \nabla f(x_0,y_0,z_0) = (y_0,x_0,-1).$$

已知平面的法向量为

$$\boldsymbol{n}_1 = \nabla f_1(x_0,y_0,z_0) = (3,2,-1).$$

又两平面平行, 得$\boldsymbol{n}\parallel\boldsymbol{n}_1$, 故有$x_0 = 2$, $y_0 = 3$, 代入曲面方程得$z_0 = 6$. 从而相应的切平面方程为

$$3x + 2y - z - 12 = 0.$$

8.7.3 思考与练习

练习 8.30. *求曲线$x = t$, $y = t^2$, $z = t^3$在点$(1,1,1)$处的切线与法平面方程.*

解: 曲线上的点对应$t = 1$. 由于

$$(x'(t),y'(t),z'(t)) = (1,2t,3t^2),$$

故曲线在点$(1,1,1)$处的切向量

$$\boldsymbol{\tau} = (1,2,3).$$

于是切线方程为

$$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z-1}{3},$$

法平面方程为

$$(x-1) + 2(y-1) + 3(z-1) = 0,$$

即

$$x + 2y + 3z = 6.$$

练习 8.31. 求曲面$e^{xy} + x^2 - z = e^2$在点$(1, 2, 1)$处的切平面与法线方程.

解: 令$F(x, y, z) = e^{xy} + x^2 - z - e^2$, 则

$$F_x = ye^{xy} + 2x, \quad F_y = xe^{xy}, \quad F_z = -1,$$

因而$\nabla F(1, 2, 1) = (2e^2 + 2, e^2, -1)$. 由此得切平面方程为

$$(2e^2 + 2)(x - 1) + e^2(y - 2) - (z - 1) = 0,$$

即

$$2e^2x + e^2y - z = 4e^2 + 1,$$

法线方程为

$$\frac{x - 1}{2e^2 + 2} = \frac{y - 2}{e^2} = \frac{z - 1}{-1}.$$