Quantum Computing Exercise

Name:

1.

$$
\begin{aligned}
D_{\vec{n}}(\alpha) &= e^{-i\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma}} = Icos(\frac{\alpha}{2}) - isin(\frac{\alpha}{2})\vec{n}\cdot\vec{\sigma} \\
[D_{\vec{n}}(\alpha)]^{\dagger} &= D_{\vec{n}}(-\alpha) = e^{-i\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma}} = Icos(\frac{\alpha}{2}) + isin(\frac{\alpha}{2})\vec{n}\cdot\vec{\sigma} \\
\rho' &= [D_{\vec{n}}(\alpha)] \dagger \rho D_{\vec{n}}(\alpha) = D_{\vec{n}}(-\alpha)\rho D_{\vec{n}}(\alpha) = [D_{\vec{n}}(\alpha)](I + \vec{x}\cdot\vec{\sigma})D_{\vec{n}}(\alpha) \\
&= \frac{1}{2}I + \frac{1}{2}(Icos(\frac{\alpha}{2}) + isin(\frac{\alpha}{2})\vec{n}\cdot\vec{\sigma})(\vec{x}\cdot\vec{\sigma})(Icos(\frac{\alpha}{2}) - isin(\frac{\alpha}{2})\vec{n}\cdot\vec{\sigma}) \\
&= \frac{1}{2}I + \frac{1}{2}[(e^{-i\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma}} = Icos(\frac{\alpha}{2}))\vec{n}\cdot\vec{\sigma}(cos(\frac{\alpha}{2})\vec{x}\cdot\vec{\sigma}(\vec{n}\cdot\vec{\sigma}))] \\
&= \frac{1}{2}I + \frac{1}{2}[cos^2(\frac{\alpha}{2})\vec{x}\cdot\vec{\sigma} - isin(\frac{\alpha}{2})cos(\frac{\alpha}{2})(\vec{x}\cdot\vec{\sigma}\cdot\vec{n}\cdot\vec{\sigma}) \\
&+ isin(\frac{\alpha}{2})cos(\frac{\alpha}{2})\vec{n}\cdot\vec{\sigma}\cdot\vec{x}\cdot\vec{\sigma} + sin^2(\frac{\alpha}{2})(\vec{n}\cdot\vec{\sigma}\cdot\vec{x}\cdot\vec{\sigma}\vec{n}\cdot\vec{\sigma})] \\
&= \frac{1}{2}I + \frac{1}{2}[cos^2(\frac{\alpha}{2})\vec{x}\cdot\vec{\sigma} + isin(\frac{\alpha}{2})cos(\frac{\alpha}{2})(\vec{x}\cdot\vec{\sigma}\cdot\vec{n}\cdot\vec{\sigma}) \\
&+ isin(\frac{\alpha}{2})cos(\frac{\alpha}{2})(\vec{n}\cdot\vec{x} - \vec{x}\cdot\vec{n}) + sin^2(\frac{\alpha}{2})\vec{n}\cdot\vec{\sigma}\vec{x}\cdot\vec{\sigma}\vec{n}\cdot\vec{\sigma}] \\
Let\ Q &= \vec{n}\cdot\vec{\sigma}\cdot\vec{x}\cdot\vec{\sigma}\vec{n}\cdot\vec{\sigma} = [\vec{n}\cdot\vec{\sigma}, \vec{x}\cdot\vec{\sigma}] = 2i(\vec{n}\times\vec{x}), \\
&= \frac{1}{2}I + \frac{1}{2}[cos^2(\frac{\alpha}{2})\vec{x}\cdot\vec{\sigma} + isin(\frac{\alpha}{2})cos(\frac{\alpha}{2})(\vec{x}\cdot\vec{\sigma}\cdot\vec{n}\cdot\vec{\sigma}) \\
&+ isin(\frac{\alpha}{2})cos(\frac{\alpha}{2})Q + sin^2(\frac{\alpha}{2})\vec{n}\cdot\vec{\sigma}\vec{x}\cdot\vec{\sigma}\vec{n}\cdot\vec{\sigma}] \\
&= \frac{1}{2}I + \frac{1}{2}(cos^2(\frac{\alpha}{2})\vec{x}\cdot\vec{\sigma} + isin(\frac{\alpha}{2})cos(\frac{\alpha}{2})2i\vec{n}\times\vec{x}\cdot\vec{\sigma} + sin^2(\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma}\vec{x}\cdot\vec{\sigma}\vec{x}\cdot\vec{\sigma}) \\
&= \frac{1}{2}I + \frac{1}{2}[cos\alpha + (1 - cos\alpha)(\vec{n}\vec{x}\vec{n} - sin\alpha\vec{n}\times\vec{x}]\cdot\vec{\sigma} \\
&= \frac{1}{2} + \frac{1}{2}\vec{y}\cdot\vec{\sigma}
\end{aligned}
$$

where $cos\alpha\vec{x} + (1 - cos\alpha)(\vec{n}\cdot\vec{x}\cdot\vec{n}) - sin\alpha(\vec{n}\times\vec{x}) = O_{\vec{n}}\cdot\vec{x}$
and $O_{\vec{n}(\alpha)} = \begin{bmatrix} cos\alpha + (1 - cos\alpha){n_1}^2 & (1 - cos\alpha)n_1n_2 + sin\alpha n_3 & (1 - cos\alpha)\ n_1 - n_3 - sin\alpha n_2 \\ (1 - cos\alpha)n_1n_2 - sin\alpha n_3 & cos\alpha + (1 - cos\alpha){n_2}^2 & (1 - cos\alpha)n_2n_3 + sin\alpha n_1 \\ (1 - cos\alpha)n_1n_3 + sin\alpha n_2 & (1 - cos\alpha)n_2n_3 - sin\alpha n_1 & cos\alpha + (1 - cos\alpha){n_3}^2 \end{bmatrix}$

2.

- (1) & (2) Reduced states and Schmidt form of ρ_A, ρ_B
  In order to construct the basis of $|\psi\rangle \in \mathbb{H}^2 \times \mathbb{H}^2$ under the Schmidt decompostion, we can compute the reduced density operator of $|\psi\rangle$firstly:

$$
\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle) \\
|\psi\rangle\langle\psi| &= \frac{1}{3}(|00\rangle + |01\rangle + |10\rangle)(\langle 00| + \langle 01| + \langle 10|)
\end{aligned}
$$

$$
\begin{gathered}
\rho_A = tr_B(|\psi\rangle\langle\psi|) = \frac{1}{3}(|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + |0\rangle\langle 0|_A \otimes |0\rangle\langle 1|_B + |0\rangle\langle 1|_A \otimes |0\rangle\langle 0|_B \\
+|0\rangle\langle 0|_A \otimes |1\rangle\langle 0|_B + |0\rangle\langle 0|_A \otimes |1\rangle\langle 1|_B + |0\rangle\langle 1|_A \otimes |1\rangle\langle 0|_B \\
|1\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + |1\rangle\langle 0|_A \otimes |0\rangle\langle 1|_B + |1\rangle\langle 1|_A \otimes |0\rangle\langle 0|_B
\end{gathered}
$$

Since $tr_B(|i\rangle\langle j|) = \langle i|j\rangle = \delta_{ij}$,

$$
\rho_A = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}
$$

Apply the same computatation to $\rho_B,\ \rho_B = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.
Let ρ_A, ρ_B be in the eigen-basis: $\rho_A = \sum_i p_i |i\rangle\langle i|,\ \rho_B = \sum_j p_j |j\rangle\langle j|$,
and consendering the fact that ρ_A, ρ_B share common eigenvalues(that's: $p_i = p_j$), it helps with the Schmidt form:

$$
|\psi\rangle = \sum_i \sqrt{p_i}|i_A\rangle|i_B\rangle
$$

In this case, we get:

$$
\rho_A = \rho_B = \begin{cases} p_1 = \frac{3+\sqrt{5}}{2}, & |i_{p_1}\rangle = \begin{bmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{bmatrix} \\ p_2 = \frac{3-\sqrt{5}}{2}, & |i_{p_2}\rangle = \begin{bmatrix} 1 \\ -\frac{\sqrt{5}+1}{2} \end{bmatrix} \end{cases}
$$

hence,$|\psi\rangle = \frac{3+\sqrt{5}}{2} \cdot |i_{p_1}\rangle \otimes |i_{p_1}\rangle + \frac{3-\sqrt{5}}{2} \cdot |i_{p_2}\rangle \otimes |i_{p_2}\rangle$

$$
= \frac{3+\sqrt{5}}{2} \cdot \begin{bmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{bmatrix}_A \otimes \begin{bmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{bmatrix}_B + \frac{3-\sqrt{5}}{2} \cdot \begin{bmatrix} 1 \\ -\frac{\sqrt{5}+1}{2} \end{bmatrix}_A \otimes \begin{bmatrix} 1 \\ -\frac{\sqrt{5}+1}{2} \end{bmatrix}_B
$$

This is the Schmidt form of ρ_A, ρ_B .

- (3)

3.

4.