Quantum Computing Exercise

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1.

$$
\begin{aligned}
D_{\vec{n}}(\alpha) &= e^{-i\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma}} = Icos(\frac{\alpha}{2}) - isin(\frac{\alpha}{2})\vec{n}\cdot\vec{\sigma} \\
[D_{\vec{n}}(\alpha)]^{\dagger} &= D_{\vec{n}}(-\alpha) = e^{i\frac{\alpha}{2}\vec{n}\cdot\vec{\sigma}} = Icos(\frac{\alpha}{2}) + isin(\frac{\alpha}{2})\vec{n}\cdot\vec{\sigma} \\
\rho' &= [D_{\vec{n}}(\alpha)] \dagger \rho D_{\vec{n}}(\alpha) = D_{\vec{n}}(-\alpha)\rho D_{\vec{n}}(\alpha) = \frac{1}{2}[D_{\vec{n}}(-\alpha)](I + \vec{x}\cdot\vec{\sigma})D_{\vec{n}}(\alpha) \\
&= \frac{1}{2}I + \frac{1}{2}(Icos(\frac{\alpha}{2}) + isin(\frac{\alpha}{2})\vec{n}\cdot\vec{\sigma})(\vec{x}\cdot\vec{\sigma})(Icos(\frac{\alpha}{2}) - isin(\frac{\alpha}{2})\vec{n}\cdot\vec{\sigma}) \\
&= \frac{1}{2}I + \frac{1}{2}[cos^2(\frac{\alpha}{2})\vec{x}\cdot\vec{\sigma} - isin(\frac{\alpha}{2})cos(\frac{\alpha}{2})(\vec{x}\vec{\sigma}\cdot\vec{n}\vec{\sigma}) \\
&+ isin(\frac{\alpha}{2})cos(\frac{\alpha}{2})\vec{n}\cdot\vec{\sigma}\cdot\vec{x}\cdot\vec{\sigma} + sin^2(\frac{\alpha}{2})\vec{n}\vec{\sigma}\cdot\vec{x}\vec{\sigma}\cdot\vec{n}\vec{\sigma}] \\
&= \frac{1}{2}I + \frac{1}{2}[cos^2(\frac{\alpha}{2})\vec{x}\cdot\vec{\sigma} + isin(\frac{\alpha}{2})cos(\frac{\alpha}{2})(\vec{x}\vec{\sigma}\cdot\vec{n}\vec{\sigma}) \\
&+ isin(\frac{\alpha}{2})cos(\frac{\alpha}{2})(\vec{n}\cdot\vec{x} - \vec{x}\cdot\vec{n}) + sin^2(\frac{\alpha}{2})\vec{n}\vec{\sigma}\cdot\vec{x}\vec{\sigma}\cdot\vec{n}\vec{\sigma}] \\
Let\ Q &= \vec{n}\vec{\sigma}\cdot\vec{x}\vec{\sigma} - \vec{x}\vec{\sigma}\cdot\vec{n}\vec{\sigma} = [\vec{n}\cdot\vec{\sigma}, \vec{x}\cdot\vec{\sigma}] = 2i(\vec{n}\times\vec{x}), \\
&= \frac{1}{2}I + \frac{1}{2}[cos^2(\frac{\alpha}{2})\vec{x}\cdot\vec{\sigma} + isin(\frac{\alpha}{2})cos(\frac{\alpha}{2})Q + sin^2(\frac{\alpha}{2})\vec{n}\vec{\sigma}\cdot\vec{x}\vec{\sigma}\cdot\vec{n}\vec{\sigma}] \\
&= \frac{1}{2}I + \frac{1}{2}(cos^2(\frac{\alpha}{2})\vec{x}\cdot\vec{\sigma} + isin(\frac{\alpha}{2})cos(\frac{\alpha}{2})(2i)\vec{n}\times\vec{x}\cdot\vec{\sigma} + sin^2(\frac{\alpha}{2})\vec{n}\vec{\sigma}\cdot\vec{x}\vec{\sigma}\cdot\vec{x}\vec{\sigma}) \\
&= \frac{1}{2}I + \frac{1}{2}[cos\alpha + (1 - cos\alpha)(\vec{n}\vec{x}\cdot\vec{n} - sin\alpha(\vec{n}\times\vec{x})]\cdot\vec{\sigma} \\
&= \frac{1}{2} + \frac{1}{2}\vec{y}\cdot\vec{\sigma}
\end{aligned}
$$

where $y = cos\alpha\vec{x} + (1 - cos\alpha)(\vec{n}\vec{x}\cdot\vec{n}) - sin\alpha(\vec{n}\times\vec{x}) = O_{\vec{n}}\cdot\vec{x}$

and $O_{\vec{n}(\alpha)} = \begin{bmatrix} cos\alpha + (1-cos\alpha){n_1}^2 & (1-cos\alpha)n_1n_2 + sin\alpha n_3 & (1-cos\alpha)\ n_1n_3 - sin\alpha n_2 \\ (1-cos\alpha)n_1n_2 - sin\alpha n_3 & cos\alpha + (1-cos\alpha){n_2}^2 & (1-cos\alpha)n_2n_3 + sin\alpha n_1 \\ (1-cos\alpha)n_1n_3 + sin\alpha n_2 & (1-cos\alpha)n_2n_3 - sin\alpha n_1 & cos\alpha + (1-cos\alpha){n_3}^2 \end{bmatrix}$

2.

- (1) & (2) Reduced states and Schmidt form of ρ_A, ρ_B
  In order to construct the basis of $|\psi\rangle \in \mathbb{H}^2 \times \mathbb{H}^2$ under the Schmidt decompostion, we can compute the reduced density operator of $|\psi\rangle$firstly:

$$
\begin{gathered}
|\psi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle) \\
|\psi\rangle\langle\psi| = \frac{1}{3}(|00\rangle + |01\rangle + |10\rangle)(\langle 00| + \langle 01| + \langle 10|) \\
\rho_A = tr_B(|\psi\rangle\langle\psi|) = \frac{1}{3}(|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + |0\rangle\langle 0|_A \otimes |0\rangle\langle 1|_B + |0\rangle\langle 1|_A \otimes |0\rangle\langle 0|_B \\
+|0\rangle\langle 0|_A \otimes |1\rangle\langle 0|_B + |0\rangle\langle 0|_A \otimes |1\rangle\langle 1|_B + |0\rangle\langle 1|_A \otimes |1\rangle\langle 0|_B \\
|1\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + |1\rangle\langle 0|_A \otimes |0\rangle\langle 1|_B + |1\rangle\langle 1|_A \otimes |0\rangle\langle 0|_B
\end{gathered}
$$

Since $tr_B(|i\rangle\langle j|) = \langle i|j\rangle = \delta_{ij}$,

$$\rho_A = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Apply the same computatation to ρ_B, $\rho_B = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.
Let ρ_A, ρ_B be in the eigen-basis: $\rho_A = \sum_i p_i|i\rangle\langle i|,\ \rho_B = \sum_j p_j|j\rangle\langle j|$,
and considering = the fact that ρ_A, ρ_B share common eigenvalues(that's: $p_i = p_j$), it helps with the Schmidt form:

$$|\psi\rangle = \sum_i \sqrt{p_i}|i_A\rangle|i_B\rangle$$

In this case, we get:

$$\rho_A = \rho_B = \begin{cases} p_1 = \frac{3+\sqrt{5}}{2}, & |i_{p_1}\rangle = \begin{bmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{bmatrix} \\ p_2 = \frac{3-\sqrt{5}}{2}, & |i_{p_2}\rangle = \begin{bmatrix} 1 \\ -\frac{\sqrt{5}+1}{2} \end{bmatrix} \end{cases}$$

hence,$|\psi\rangle = \frac{3+\sqrt{5}}{2} \cdot |i_{p_1}\rangle \otimes |i_{p_1}\rangle + \frac{3-\sqrt{5}}{2} \cdot |i_{p_2}\rangle \otimes |i_{p_2}\rangle$

$$= \frac{3+\sqrt{5}}{2} \cdot \begin{bmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{bmatrix}_A \otimes \begin{bmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{bmatrix}_B + \frac{3-\sqrt{5}}{2} \cdot \begin{bmatrix} 1 \\ -\frac{\sqrt{5}+1}{2} \end{bmatrix}_A \otimes \begin{bmatrix} 1 \\ -\frac{\sqrt{5}+1}{2} \end{bmatrix}_B$$

This is the Schmidt form of ρ_A, ρ_B .

- (3)

$$\rho = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \lambda_1 = \frac{3}{2}, \lambda_{2,3,4} = 0$$

$$Etr(\rho) = -tr(\rho log(\rho)) = -tr(\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} log\frac{3}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix})$$
$$= -\frac{3}{2}log\frac{3}{2}$$

3.

$\sigma_1|0\rangle = |1\rangle, \sigma_1|1\rangle = |0\rangle, \sigma_3|0\rangle = |0\rangle$
$\sigma_2|0\rangle = i|1\rangle, \sigma_2|1\rangle = -i|0\rangle, \sigma_3|1\rangle = -|1\rangle$
$\vec{n_1} = (n_{11}, n_{12}, n_{13}), \vec{n_2} = (n_{21}, n_{22}, n_{23})$
$\langle 0|F_1|0\rangle = n_{13}, \langle 0|F_2|0\rangle = n_{23}, \langle 1|F_2|1\rangle = -n_{23}$
$\langle 1|F_1|1\rangle = -n_{13}, \langle 1|F_1|0\rangle = n_{11} + in_{12}, \langle 1|F_2|0\rangle = n_{21} + in_{22}$
$\langle 0|F_2|1\rangle = n_{21} - in_{22}, \langle 0|F_1|1\rangle = n_{11} - in_{12}$
$F_1 = \vec{n}_1 \cdot \vec{\sigma}, F_2 = \vec{n}_2 \cdot \vec{\sigma}$

$$
\begin{aligned}
\langle F_1 \otimes F_2 \rangle_{|\psi\rangle} &= \langle \psi | F_1 \otimes F_2 | \psi \rangle = \frac{1}{\sqrt{2}} \langle \psi | F_1 0 \rangle \otimes F_2 1 \rangle - \frac{1}{\sqrt{2}} \langle \psi | F_1 1 \rangle \otimes F_2 0 \rangle \\
&= \frac{1}{2} \langle 01 | F_1 0 \otimes F_2 1 \rangle - \frac{1}{2} \langle 10 | F_1 0 \otimes F_2 1 \rangle - \frac{1}{2} \langle 01 | F_1 1 \otimes F_2 0 \rangle + \frac{1}{2} \langle 10 | F_1 1 \otimes F_2 0 \rangle \\
&= \frac{1}{2} \langle 0 | F_1 0 \rangle \langle 1 | F_2 1 \rangle - \frac{1}{2} \langle 1 | F_1 0 \rangle \langle 0 | F_2 1 \rangle - \frac{1}{2} \langle 0 | F_1 1 \rangle \langle 1 | F_2 0 \rangle + \frac{1}{2} \langle 1 | F_1 1 \rangle \langle 0 | F_2 0 \rangle \\
&= -\frac{n_{13} n_{23}}{2} - \frac{(n_{11} + i n_{12})(n_{21} - i n_{22})}{2} - \frac{(n_{11} - i n_{12})(n_{21} + i n_{22})}{2} + \frac{-n_{13} n_{23}}{2} \\
&= -n_{13} n_{23} - n_{11} n_{21} - n_{12} n_{22} = -\vec{n_1} \cdot \vec{n_2}
\end{aligned}
$$

4.

$$
\langle S_k \otimes S_j \rangle_{|\psi\rangle} = \langle \psi | S_k \otimes S_j | \psi \rangle = \langle \psi | S_k \otimes S_j | \rho_1 \otimes \rho_2 \rangle = \langle \psi | S_k \rho_1 \otimes S_j \rho_2 \rangle
$$

$$
= \langle \rho_1 \otimes \rho_2 | S_k \rho_1 \otimes S_j \rho_2 \rangle = \langle S_k \rangle_{|\rho_1\rangle} \cdot \langle S_j \rangle_{|\rho_2\rangle}
$$

$$
\forall \rho_1 \in \mathbb{C}^2, |\rho_1\rangle = e^{i\alpha} cos(\frac{\theta}{2})|0\rangle + e^{i\beta} sin(\frac{\theta}{2})|1\rangle
$$

$$
Suppose\ \exists\ |\uparrow_{\vec{n}_{(\phi,\psi)}}\rangle = e^{-i\frac{\phi}{2}} cos(\frac{\psi}{2})|0\rangle + e^{i\frac{\phi}{2}} sin(\frac{\psi}{2})|1\rangle,\ where\ \vec{n}_{(\phi,\psi)} = (sin\phi cos\psi, sin\phi sin\psi, cos\phi)
$$

$$
Finding\ \ m, \phi, \psi\ s.t.\ |\rho_1\rangle = e^{im} |\uparrow_{\vec{n}_{\rho_1}}\rangle = e^{im}(e^{-i\frac{\phi}{2}} cos(\frac{\psi}{2})|0\rangle + e^{i\frac{\phi}{2}} sin\frac{\psi}{2}|1\rangle)
$$

$$
= e^{i(m-\frac{\phi}{2})} cos(\frac{\psi}{2})|0\rangle + e^{i(m+\frac{\phi}{2})}|1\rangle
$$

$$
Since\ \ |\rho_1\rangle = e^{i\alpha} cos(\frac{\theta}{2})|0\rangle + e^{i\beta} sin(\frac{\theta}{2})|1\rangle
$$

$$
\begin{cases} e^{i(m-\frac{\phi}{2})} cos(\frac{\psi}{2}) = e^{i\alpha} cos(\frac{\theta}{2} \\ e^{i(m+\frac{\phi}{2})} sin(\frac{\psi}{2}) = e^{i\beta} sin(\frac{\theta}{2} \end{cases} \Rightarrow \begin{cases} \psi = \theta \\ m - \frac{\phi}{2} = \alpha \\ m + \frac{\phi}{2} = \beta \end{cases} \Rightarrow \begin{cases} \psi = \theta \\ m = \frac{\alpha+\beta}{2} \\ \phi = -\alpha + \beta \end{cases}
$$

$$
Similar\ process\ for\ \rho_2 \Rightarrow \begin{cases} |\rho_1\rangle = e^{i(\frac{\alpha+\beta}{2})} |\uparrow_{\vec{n}_{\rho_1}}\rangle,\ where\ |\uparrow_{\vec{n}_{\rho_1}}\rangle = |\uparrow_{\vec{n}_{-\alpha+\beta,\theta}}\rangle \\ |\rho_2\rangle = e^{i(\frac{\alpha+\beta}{2})} |\uparrow_{\vec{n}_{\rho_2}}\rangle,\ where\ |\uparrow_{\vec{n}_{\rho_2}}\rangle = |\uparrow_{\vec{n}_{-\alpha+\beta,\theta}}\rangle \end{cases}
$$

$$
k \in \{1, 4\}, \langle S_k \rangle_{|\rho_1\rangle} = \langle \rho_1 | \vec{n}_k \cdot \vec{\sigma} | \rho_1 \rangle = \langle \uparrow \vec{n}_{\rho_1} | \vec{n}_k \cdot \vec{\sigma} | \uparrow_{\vec{n}_{\rho_1}} \rangle \quad (*1)
$$

$$
Since\ |\uparrow_{\vec{n}_{\rho_2}}\rangle\ is\ eigenvector\ with\ eigenvalue\ 1\ of\ Hermitian\ (\vec{n}_{\rho_1} \cdot \vec{\sigma})
$$

$$
(*1) = \langle (\vec{n_k}_{\rho_1} \vec{\sigma}) \uparrow_{\vec{n}_{\rho_1}} | \vec{n} \cdot \vec{\sigma} | \vec{n}_{\rho_1} \cdot \vec{\sigma} \uparrow_{\vec{n}_{\rho_1}} \rangle = \vec{n}_{\rho_1} \cdot \vec{n}_k \Rightarrow \begin{cases} \langle S_k \rangle_{|\rho_1\rangle} = \vec{n}_{\rho_1} \cdot \vec{n}_k \\ \langle S_j \rangle_{|\rho_2\rangle} = \vec{n}_{\rho_2} \cdot \vec{n}_j \end{cases}
$$

$$
|\langle S_1 \otimes S_2 \rangle - \langle S_1 \otimes S_3 \rangle + \langle S_4 \otimes S_2 \rangle + \langle S_4 \otimes S_3 \rangle|
$$

$$
= |(\vec{n}_{\rho_1} \cdot \vec{n}_1)(\vec{n}_{\rho_2} \cdot \vec{n}_2 - \vec{n}_{\rho_2} \cdot \vec{n}_3) + (\vec{n}_{\rho_1} \cdot \vec{n}_4)(\vec{n}_{\rho_2} \cdot \vec{n}_2 - \vec{n}_{\rho_2} \cdot \vec{n}_3)| = (*2)
$$

$$
\because |\vec{n}_{\rho_1}| \leq 1, |\vec{n}_1| \leq 1, |\vec{n}_{\rho_1} \cdot \vec{n}_1| = |\vec{n}_{\rho_1}||\vec{n}_1| \cdot cos\theta
$$

$$
\therefore (*2) \leq |\vec{n}_{\rho_2} \cdot \vec{n}_2 - \vec{n}_{\rho_2} \vec{n}_3| + |\vec{n}_{\rho_2} \cdot \vec{n}_{\rho_2} + \vec{n}_{\rho_2} \cdot \vec{n}_3| = (*3)
$$

$$
Let\ \ \begin{cases} A = \vec{n}_{\rho_2} \cdot \vec{n}_2 \\ B = \vec{n}_{\rho_2} \cdot \vec{n}_3 \end{cases}, (*3) = |A - B| + |A + B|,\ |A|, |B| \leq 1
$$

$$
\begin{cases} A > B, A + B > 0. (*3) = A - B + A + B = 2A \leq 2 \\ A < B, A + B > 0. (*3) = B - A + A + B = 2B \leq 2 \\ A > B, A + B < 0. (*3) = A - B - A - B = -2B \leq 2 \\ A < B, A + B < 0. (*3) = B - A - A - B = -2A \leq 2 \end{cases}
$$

$$
\Rightarrow |\langle S_1 \otimes S_2 \rangle - \langle S_1 \otimes S_3 \rangle + \langle S_4 \otimes S_2 \rangle + \langle S_4 \otimes S_4 \rangle| \leq 2
$$