1) Let $\varrho : \mathbb{C}^2 \to \mathbb{C}^2$ be a mixed state of a qubit in the Bloch representation, that is,

$$\varrho = \frac{1}{2}\left(I + \bar{x}\cdot\bar{\sigma}\right)$$

where $\bar{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, $|\bar{x}| \le 1$ and $\bar{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices, $\bar{x}\cdot\bar{\sigma} = x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3$

Let $D_{\bar{n}}(\alpha) = e^{-i\frac{\alpha}{2}\bar{n}\cdot\bar{\sigma}}$ $\bar{n} \in \mathbb{R}^3$, $|\bar{n}| = 1$, Show that

$$\left[D_{\bar{n}}(\alpha)\right]^{+} \varrho\, D_{\bar{n}}(\alpha) = \frac{1}{2}\left(I + \bar{y}\,\bar{\sigma}\right) \quad \text{where} \quad \bar{y} = O_{\bar{n}}(\alpha)\,\bar{x}$$

and $O_{\bar{n}}(\alpha)$ is a rotation matrix by angle α along the axis $\bar{n}$

2) Let $|\psi\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + |01\rangle + |10\rangle\right)$ be a state of two qubits. Calculate:

a) Shmidt normal form of $|\psi\rangle$
b) Reduced states ϱ_1 and ϱ_2 of $|\psi\rangle$
c) Entanglement entropy of $|\psi\rangle$

3) Let $|\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$ For any $\bar{n}_1, \bar{n}_2 \in \mathbb{R}^3$ s.t. $|\bar{n}_1| = |\bar{n}_2| = 1$ show that

$$\langle \bar{n}_1\cdot\bar{\sigma} \otimes \bar{n}_2\cdot\bar{\sigma} \rangle_{|\psi^-\rangle} = -\,\bar{n}_1\cdot\bar{n}_2$$

4) Let $S_k = \bar{n}_k \cdot \bar{\sigma} : \mathcal{H}_A \to \mathcal{H}_A$ $k \in \{1,4\}$ and $S_j = \bar{n}_j \cdot \bar{\sigma} : \mathcal{H}_B \to \mathcal{H}_B$ $j \in \{2,3\}$, where $\mathcal{H}_A = \mathbb{C}^2 = \mathcal{H}_B$ and $\bar{n}_k, \bar{n}_j \in \mathbb{R}^3$, $|\bar{n}_k| = 1 = |\bar{n}_j|$ and $\bar{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices.
Let $|\psi\rangle$ be any separable state on $\mathcal{H}_A \otimes \mathcal{H}_B$, that is $|\psi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle$ where $|\varphi_i\rangle \in \mathbb{C}^2$. Show that for any choice of $\bar{n}_k, \bar{n}_j$ we have

$$\left| \langle S_1 \otimes S_2 \rangle_{|\psi\rangle} - \langle S_1 \otimes S_3 \rangle_{|\psi\rangle} + \langle S_4 \otimes S_2 \rangle_{|\psi\rangle} + \langle S_4 \otimes S_3 \rangle_{|\psi\rangle} \right| \leq 2$$