111550113 謝詠晴

Numerical Methods_ Assignment 4

1. (a) Find f ′ (0.72) from a cubic polynomial.

$$f(x) \approx P_3'(x) = f[x_i, x_{i+1}] + f[x_i, x_{i+1}, x_{i+2}][(x - x_i) + (x - x_{i+1})]$$

$$+ f[x_i, x_{i+1}, x_{i+2}, x_{i+3}][(x - x_i)(x - x_{i+1})$$

$$+ (x - x_{i+1})(x - x_{i+2}) + (x - x_i)(x - x_{i+2})]$$

Choose starting value $i = 1$ and let $x_i = 0.5, x_{i+1} = 0.7, x_{i+2} = 0.9, x_{i+3} = 1.1$

$$f[x_i, x_{i+1}] = \frac{0.2549}{0.2}$$

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{-0.0086}{2(0.2)^2}$$

$$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}] = \frac{-0.0018}{6(0.2)^3}$$

q1.m

```
% problem 1-a
h = 0.2;
a1 = 0.2549;
a2 = -0.0086;
a3 = -0.0018;

diff_poly = @(x) (a1/h) + (a2/(2*h^2)) * ((x-0.5)+(x-0.7)) + ...
    (a3/(6*h^3)) * ((x-0.5)*(x-0.7)+(x-0.7)*(x-0.9)+(x-0.5)*(x-0.9));

x_val = 0.72;
result = diff_poly(x_val);
disp(result)
```

ANS: $\underline{f'(0.72) \approx P_3'(0.72) = 1.2502}$

(b) Find f ′ (1.33) from a quadratic.

$$f(x) \approx P_2'(x) = f[x_i, x_{i+1}] + f[x_i, x_{i+1}, x_{i+2}][(x - x_i) + (x - x_{i+1})]$$

Choose starting value $i = 4$ and let $x_i = 1.1, x_{i+1} = 1.3$

$$f[x_i, x_{i+1}] = \frac{0.2241}{0.2}$$

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{-0.0128}{2(0.2)^2}$$

q1.m

```
% problem 1-b
h = 0.2;
a1 =  0.2241;
a2 = -0.0128;

diff_poly = @(x) (a1/h) + (a2/(2*h^2)) * ((x-1.1)+(x-1.3));
x_val = 1.33;
result = diff_poly(x_val);
disp(result)
```

ANS: $\underline{f'(1.33) \approx P_2'(1.33) = 1.0789}$

(c) Find f ′ (0.50) from a fourth-degree polynomial.

$$f(x) \approx P_4'(x) = f[x_i, x_{i+1}] + f[x_i, x_{i+1}, x_{i+2}][(x - x_i) + (x - x_{i+1})]$$

$$+ f[x_i, x_{i+1}, x_{i+2}, x_{i+3}][(x - x_i)(x - x_{i+1})$$

$$+ (x - x_{i+1})(x - x_{i+2}) + (x - x_i)(x - x_{i+2})]$$

$$+ f[x_i, x_{i+1}, \dots, x_{i+4}] \sum_{j=0}^{j=3} \frac{(x - x_i) \dots (x - x_{i+3})}{x - x_{i+j}}$$

Choose starting value $i = 0$ and let $x_i = 0.3$

$$f[x_i, x_{i+1}] = \frac{0.2613}{0.2}, \qquad f[x_i, x_{i+1}, x_{i+2}] = \frac{-0.0064}{2(0.2)^2}$$

$$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}] = \frac{-0.0022}{6(0.2)^3}, \qquad f[x_i, x_{i+1} \dots x_{i+4}] = \frac{0.0003}{24(0.2)^4}$$

q1.m

```
% problem 1-c
h = 0.2;
a1 =  0.2613;
a2 = -0.0064;
a3 = -0.0022;
a4 = 0.0003;

diff_poly = @(x) (a1/h) + (a2/(2*h^2)) * ((x-0.3)+(x-0.5)) + ...
    (a3/(6*h^3)) * ((x-0.3)*(x-0.5)+(x-0.5)*(x-0.7)+(x-0.3)*(x-0.7)) + ...
    (a4/(24*h^4)) * ((x-0.3)*(x-0.5)*(x-0.7)+(x-0.5)*(x-0.7)*(x-0.9)+ ...
    (x-0.3)*(x-0.7)*(x-0.9)+(x-0.3)*(x-0.5)*(x-0.9));

x_val = 0.50;
result = diff_poly(x_val);
disp(result)
```

ANS: $\underline{f'(0.50) \approx P_4'(0.50) = 1.2925}$

2. Use the method of undetermined coefficients to obtain the formulas for $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$ at x_0 using five evenly spaced points from x_2 to x_{-2}, together with their error terms.

(a) $f''(x_0) = C_{-2}f_{-2} + C_{-1}f_{-1} + C_0 f_0 + C_1 f_1 + C_2 f_2$

- **$\underline{P(u) = 1}$:**

  $f_{-2} = f_{-1} = f_0 = f_1 = f_2 = P(u) = 1$

  $f''(x_0) = C_{-2} + C_{-1} + C_0 + C_1 + C_2 = P''(u) = 0$

- **$\underline{P(u) = u}$:**

  $f_{-2} = P(-2h) = -2h, f_{-1} = -h, f_0 = 0, f_1 = h, f_2 = 2h$

  $f''(x_0) = C_{-2}(-2h) + C_{-1}(-h) + C_0(0) + C_1(h) + C_2(2h) = P''(u) = 0$

- **$\underline{P(u) = u^2}$:**

  $f_{-2} = P(-2h) = 4h^2, f_{-1} = h^2, f_0 = 0, f_1 = h^2, f_2 = 4h^2$

  $f''(x_0) = C_{-2}(4h^2) + C_{-1}(h^2) + C_0(0) + C_1(h^2) + C_2(4h^2) = P''(u) = 2$

- **$\underline{P(u) = u^3}$:**

  $f_{-2} = P(-2h) = -8h^3, f_{-1} = -h^3, f_0 = 0, f_1 = h^3, f_2 = 8h^3$

  $f''(x_0) = C_{-2}(-8h^3) + C_{-1}(-h^3) + C_0(0) + C_1(h^3) + C_2(8h^3) = P''(u)$
  $= 0$

- **$\underline{P(u) = u^4}$:**

  $f_{-2} = P(-2h) = 16h^4, f_{-1} = h^4, f_0 = 0, f_1 = h^4, f_2 = 16h^4$

  $f''(x_0) = C_{-2}(16h^4) + C_{-1}(h^4) + C_0(0) + C_1(h^4) + C_2(16h^4) = P''(u)$
  $= 0$

  寫成矩陣形式，計算出係數 $C_{-2} \sim C_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & -h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -1/12 \\ 4/3 \\ -5/2 \\ 4/3 \\ -1/12 \end{bmatrix}$$

$$f''(x_0) = \frac{-f_{-2} + 16f_{-1} - 30f_0 + 16f_1 - f_2}{12h^2}$$

$$error = (x - x_{-2})(x - x_{-1})(x - x_0)(x - x_1)(x - x_2)\frac{f^5(\zeta)}{5!}$$

$$\frac{d}{dx}\left(\frac{d}{dx}error\right)|_{x=x_0} = \frac{d}{dx}\left[(x_0 - x_{-2})(x_0 - x_{-1})(x_0 - x_1)(x_0 - x_2)\frac{f^5(\zeta)}{5!}\right]$$

$$= \frac{f^5(\zeta)}{5!}[(x_0 - x_{-1})(x_0 - x_1)(x_0 - x_2) + (x_0 - x_{-2})(x_0 - x_1)(x_0 - x_2)$$

$$+ (x_0 - x_{-2})(x_0 - x_{-1})(x_0 - x_2) + (x_0 - x_{-2})(x_0 - x_{-1})(x_0 - x_1)]$$

(b) $f'''(x_0) = C_{-2}f_{-2} + C_{-1}f_{-1} + C_0f_0 + C_1f_1 + C_2f_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & -h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ -1 \\ 1/2 \end{bmatrix}$$

$$f'''(x_0) = \frac{-f_{-2} + 2f_{-1} - 2f_1 + f_2}{2h^3}$$

$$\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}error\right)\right)|_{x=x_0}$$

$$= \frac{f^5(\zeta)}{5!}[\big((x_0 - x_{-1})(x_0 - x_1) + (x_0 - x_1)(x_0 - x_2)$$

$$+ (x_0 - x_{-1})(x_0 - x_2)\big)$$

$$+ \big((x_0 - x_{-2})(x_0 - x_1) + (x_0 - x_1)(x_0 - x_2) + (x_0 - x_{-2})(x_0 - x_2)\big)$$

$$+ \big((x_0 - x_{-2})(x_0 - x_{-1}) + (x_0 - x_{-1})(x_0 - x_2) + (x_0 - x_{-2})(x_0 - x_2)\big)$$

$$+ ((x_0 - x_{-2})(x_0 - x_{-1}) + (x_0 - x_{-1})(x_0 - x_1) + (x_0 - x_{-2})(x_0 - x_1))]$$

(c) $f^{(4)}(x) = C_{-2}f_{-2} + C_{-1}f_{-1} + C_0f_0 + C_1f_1 + C_2f_2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & -h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 24 \end{bmatrix}, \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 6 \\ -4 \\ 1 \end{bmatrix}$$

$$f^{(4)}(x_0) = \frac{f_{-2} - 4f_{-1} + 6f_0 - 4f_1 + f_2}{h^4}$$

$$\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}error\right)\right)\right)|_{x=x_0}$$

$$= \frac{f^5(\zeta)}{5!}[6((x_0 - x_{-1}) + (x_0 - x_1) + (x_0 - x_{-2}) + (x_0 - x_2))]$$

3. Prove the area under any cubic between x = a and x = b is identical to the area of a parabola that matches the cubic at x = a, x = b, and x = $\frac{a+b}{2}$

3. Assume $f(x) = Ax^3 + Bx^2 + Cx + D$

$$\int_a^b f(x)\,dx = \frac{A}{4}(b^4 - a^4) + \frac{B}{3}(b^3 - a^3) + \frac{C}{2}(b^2 - a^2) + D(b-a) = \text{Area}$$

Using Simpson's $\frac{1}{3}$ Rule,

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a+b)^2 = a^2 + 2ab + b^2$

$$\int_a^b f(x)\,dx = \frac{h}{3}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

$$= \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

$$= \frac{b-a}{6}\left[(Aa^3 + Ba^2 + Ca + D) + 4\left[A\left(\frac{a+b}{2}\right)^3 + B\left(\frac{a+b}{2}\right)^2 + C\left(\frac{a+b}{2}\right) + D\right] + (Ab^3 + Bb^2 + Cb + D)\right]$$

$$= \frac{b-a}{6}\left[\left(A + \frac{4}{8}A\right)(a^3 + b^3) + (B+B)(a^2+b^2) + (C+2C)(a+b) + 6D + \frac{4}{8}A(3a^2b + 3ab^2) + \frac{4}{4}B(2ab)\right]$$

$a^3b,\ ab^3,\ a^2b,\ ab^2$ 項會被抵消

$$= \frac{A}{4}(b^4 - a^4) + \frac{B}{3}(b^3 - a^3) + \frac{C}{2}(b^2 - a^2) + D(b-a)$$

$$= \text{exact area of } f(x) \text{ between } a \text{ and } b \ \#$$

4. Compute the integral of $f(x) = \frac{\sin(x)}{x}$ between $x = 0$ and $x = 1$ using Simpson's 1/3 rule with h = 0.5 and then with h = 0.25.

4.

① $h = 0.5$, interval $[0, 1]$, $f(x) = \frac{\sin x}{x}$

$x_0 = 0,\ x_1 = 0.5,\ x_2 = 1$

$f(x_0) = 1,\ f(x_1) = \frac{\sin(0.5)}{0.5},\ f(x_2) = \frac{\sin 1}{1}$

$$\int_0^1 f(x)\,dx \approx \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$= \frac{1}{6}\left[1 + 8\sin(0.5) + \sin(1)\right]$$

② $h = 0.25$, $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$

$$\int_0^1 f(x)\,dx \approx \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$= \frac{0.25}{3}\left[1 + 4\frac{\sin(0.25)}{0.25} + 2\frac{\sin(0.5)}{0.5} + 4\frac{\sin(0.75)}{0.75} + \sin(1)\right]$$

q4.m

```
% h = 0.5
f = 1/6 * (1 + 8*sin(0.5) + sin(1));
fprintf('h_0.5 = %.10f\n', f);

% h = 0.25
f2 = 0.25/3 * (1 + 16*sin(0.25) + 4*sin(0.5)  + (4/0.75)*sin(0.75) + sin(1));
fprintf('h_0.25 = %.10f\n', f2);

% extrapolation
extra = f2 + (f2-f)/15;
fprintf('extrapolation = %.10f\n', extra);

% true value
f = @(x) sin(x) ./ x;
true_value = integral(f, 0, 1);
fprintf('true_value = %.10f\n', true_value);
```

```
>> q4
h_0.5 = 0.9461458823
h_0.25 = 0.9460869340
extrapolation = 0.9460830041
true_value = 0.9460830704
```

$$error = \frac{-1}{90} h^5 f^{(4)}(\xi)$$

The order of the error after the extrapolation is $O(h^5)$

And the result of choosing h as 0.25 is closer to the true answer.

5. (a) Using the trapezoidal rule in both directions.

$$\int_a^b f(x)dx \approx \frac{h}{2}(f_0 + 2f_1 + \cdots + 2f_{n-1} + f_n)$$

q5_a.m | q5_b.m | q5_c.m

```
fx = @(x) exp(x);
fy = @(y) sin(2 * y);
x_min = -0.2; x_max = 1.4;
y_min = 0.4; y_max = 2.6;

h = 0.1;
x = x_min:h:x_max;
y = y_min:h:y_max;

int_x = 0; int_y = 0;

for i = 1:length(x)
    if i == 1 || i == length(x)
        int_x = int_x + (h/2) * (fx(x(i)));
    else
        int_x = int_x + (h/2) * (2*fx(x(i)));
    end
end
for i = 1:length(y)
    if i == 1 || i == length(y)
        int_y = int_y + (h/2) * (fy(y(i)));
    else
        int_y = int_y + (h/2) * (2*fy(y(i)));
    end
end
fprintf('integral_value = %.10f\n', int_x*int_y);
```

```
>> q5_a
integral_value = 0.3683399551
```

ANS: 0.36834

(b) Using Simpson's 1/3 rule in both directions.

$$\int_a^b f(x)dx \approx \frac{h}{3}(f_0 + 4f_1 + f_2 \ldots + f_{n-2} + 4f_{n-1} + f_n)$$

q5_a.m | q5_b.m | q5_c.m

```
fx = @(x) exp(x);
fy = @(y) sin(2 * y);
x_min = -0.2; x_max = 1.4;
y_min = 0.4; y_max = 2.6;

h = 0.1;
x = x_min:h:x_max;
y = y_min:h:y_max;

int_x = 0; int_y = 0;

for i = 1:2:length(x)-2
    int_x = int_x + h/3*(fx(x(i)) + 4*fx(x(i+1)) + fx(x(i+2)));
end

for i = 1:2:length(y)-2
    int_y = int_y + h/3*(fy(y(i)) + 4*fy(y(i+1)) + fy(y(i+2)));
end

fprintf('integral_value = %.10f\n', int_x*int_y);
```

```
>> q5_b
integral_value = 0.3692685195
```

ANS: 0.36927

(c) Using Gaussian quadrature, three-term formulas in both directions.

- For $\int_{-0.2}^{1.4} e^x dx$, change to variable t for limits [-1 1]

$$x = \frac{(1.4 + 0.2)t + 1.4 - 0.2}{2} = 0.8t + 0.6$$

$$t_1 = -0.7746, t_2 = 0, t_3 = 0.7746$$

$$w_1 = 0.5555, w_2 = 0.8888, w_3 = 0.5555$$

$$\int_{-0.2}^{1.4} e^x dx = 0.8 \int_{-1}^{1} e^{0.8t+0.6} dt = 0.8[w_1 f(t_1) + w_2 f(t_2) + w_3 f(t_3)]]$$

- For $\int_{0.4}^{2.6} \sin(2y) dy$, change to variable t for limits [-1 1]

$$y = \frac{(2.6 - 0.4)t + 2.6 + 0.4}{2} = 1.1t + 1.5$$

$$t_1 = -0.7746, t_2 = 0, t_3 = 0.7746$$

$$w_1 = 0.5555, w_2 = 0.8888, w_3 = 0.5555$$

$$\int_{0.4}^{2.6} \sin(2y) dy = 1.1 \int_{-1}^{1} \sin(2.2t + 3) dt = 1.1[w_1 f(t_1) + w_2 f(t_2) + w_3 f(t_3)]]$$

q5_c.m | q6.m | +

```
fx = @(t) exp(0.8*t+0.6);
fy = @(t) sin(2.2*t+3);

t1=-sqrt(3/5); t2=0; t3=sqrt(3/5);
w1=5/9; w2=8/9; w3=5/9;

int_x= 0.8*(w1*fx(t1) + w2*fx(t2) + w3*fx(t3));
int_y = 1.1*(w1*fy(t1) + w2*fy(t2) + w3*fy(t3));

fprintf('integral_value = %.10f\n', int_x*int_y);
```

```
>> q5_c
integral_value = 0.3723777181
```

<u>ANS: 0.37238</u>

- Analytical solution: 0.36927

q5_analytical.m | +

```
func = @(x,y) exp(x).*sin(2.*y);
q = integral2(func,-0.2,1.4,0.4,2.6);
fprintf('analytical_value = %.10f\n', q);
```

```
>> q5_analytical
analytical_value = 0.3692650166
```

Using Simpson's 1/3 rule's answer is closest to analytical solution.

6. Please use Monte Carlo Integration to compute the double integral of $f(x, y) = (x-1)^2 + \frac{y^2}{16}$ where $R = [-2,3] \times [-1,2]$

$$\int_{-2}^{3}\int_{-1}^{2}(x-1)^2 + \frac{y^2}{16}dydx \approx (2+1)(3+2)\frac{1}{N}\sum_{i=1}^{N} f(x_i, y_i)$$

Choosing N as 10000 and generating 10000 random numbers (x_i, y_i).

q5_c.m | q6.m

```
f = @(x, y) (x - 1).^2 + (y.^2 / 16);
x_min = -2; x_max = 3; y_min = -1; y_max = 2;
N = 10000;

% Generate random points
x_rand = x_min + (x_max - x_min) * rand(N, 1);
y_rand = y_min + (y_max - y_min) * rand(N, 1);

f_rand = f(x_rand, y_rand);

area = (x_max - x_min) * (y_max - y_min);
integral_value = area * mean(f_rand);

fprintf('integral_value: %.10f\n', integral_value);
```

```
>> q6
integral_value: 35.9471026227
```

ANS: 35.94710