111550113 謝詠晴

Numerical Methods_ Assignment 3

1. (a) Construct the divided-difference table.

x	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, \ldots x_{i+2}]$	$f[x_i, \ldots x_{i+3}]$	$f[x_i, \ldots x_{i+4}]$
-0.2	1.23	2.22	-11.883	-103.583	73.61111
0.3	2.34	-8.475	-1.525	-81.5	
0.7	-1.05	-7.56	14.775		
-0.3	6.51	-16.425			
0.1	-0.06				

(b) Interpolate for $f(0.4)$ with the first three points.

$$f(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2]$$

$$f(0.4) = 1.23 + (0.4 + 0.2) * 2.22 + (0.4 + 0.2)(0.4 - 0.3) * (-11.883)$$

$$= 1.84902$$

ANS: 1.84902

(c) Repeat (b) but use the best set of three points. Which points should be used?

Because 0.4 lies between 0.3 and 0.7, we'll choose the three points that are closest to 0.4, which are 0.1, 0.3, and 0.7.

x	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, \ldots x_{i+2}]$
0.3	2.34	-8.475	-34.125
0.7	-1.05	-1.65	
0.1	-0.06		

$$f(0.4) = 2.34 + (0.4 - 0.3) * (-8.475) + (0.4 - 0.3)(0.4 - 0.7) * (-34.125)$$

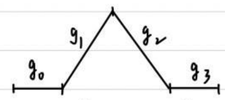
$$= 2.51625$$

ANS: 2.51625

2. Use end conditions 3 and 4 to plot the spline curve together with f(x). Which end condition gives the best fit to the function?

- For condition 3, using matrix representation HS=Y to find out S1, S2 and S3. Then substitute h and S back to the coefficient a_i to d_i. We can get four equations g_0 to g_3. The implementation is written in "q2.m".

① condition 3



	-1	-0.5	0	0.5	1
	x_0	x_1	x_2	x_3	x_4
$f(x_i)$	0	0	1	0	0
$f[x_i,x_{i+1}]$	0	2	-2	0	

$$h_0 = h_1 = \cdots = h_3 = \frac{1-(-1)}{4} = 0.5$$

$$d_i = y_i$$

$$b_i = \frac{s_i}{2}$$

$$a_i = \frac{s_{i+1}-s_i}{6h_i}$$

$$c_i = \frac{y_{i+1}-y_i}{h_i} - \frac{2h_i s_i + h_i s_{i+1}}{6}$$

$$\begin{bmatrix} 3h_0+2h_1 & h_1 & 0 \\ h_1 & 2(h_1+h_2) & h_2 \\ 0 & h_2 & 2h_2+3h_3 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = 6 \begin{bmatrix} f[x_1,x_2]-f[x_1,x_0] \\ f[x_2,x_3]-f[x_1,x_2] \\ f[x_3,x_4]-f[x_2,x_3] \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{5}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 2 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{5}{2} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 12 \\ -24 \\ 12 \end{bmatrix} \Rightarrow \begin{cases} s_1 = 8 = s_0 \\ s_2 = -16 \\ s_3 = 8 = s_n \end{cases}$$

$x_0 = -1 \quad x_1 = -0.5 \quad x_2 = 0 \quad x_3 = 0.5$

$d_0 = 0, \ d_1 = 0, \ d_2 = 1, \ d_3 = 0$

$b_0 = 4, \ b_1 = 4, \ b_2 = -8, \ b_3 = 4$

$a_0 = 0, \ a_1 = -8, \ a_2 = 8, \ a_3 = 0$

$c_0 = -2, \ c_1 = 2, \ c_2 = 0, \ c_3 = -2$

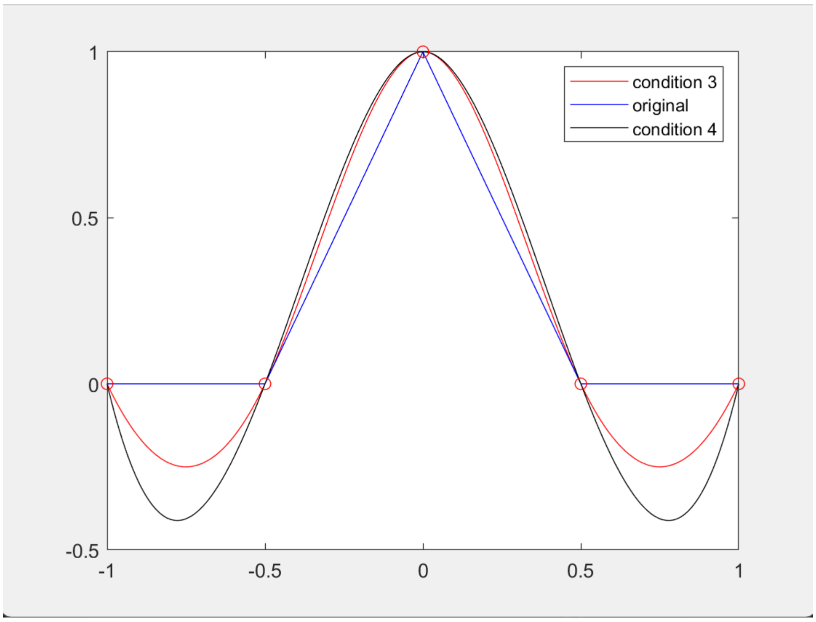
$$g_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i$$

q2.m | q3.m | q3_b.m | q4.m | q5.m | q6.m

```
x0 = linspace(-1,-0.5);
x1 = linspace(-0.5,0);
x2 = linspace(0,0.5);
x3 = linspace(0.5,1);

% condition 3
g0 = @(x) 4*(x+1).^2 - 2*(x+1);
g1 = @(x) -8*(x+0.5).^3 + 4*(x+0.5).^2 + 2*(x+0.5);
g2 = @(x) 8*x.^3 - 8*x.^2 + 1;
g3 = @(x) 4*(x-0.5).^2 - 2*(x-0.5);

plot(x0,g0(x0),'r',x1,g1(x1),'r',x2,g2(x2),'r',x3,g3(x3),'r');
hold on;

% original
x = linspace(-1, 1, 1000);
f = @(x) (x >= -1 & x < -0.5).*0 + (x >= -0.5 & x < 0.5).*(1 - abs(2*x)) + (x >= 0.5 & x <= 1).*0;
plot(x, f(x), 'b');
hold on;

% condition 4
x = linspace(-1, 1, 5);
y = f(x);
xx = linspace(-1, 1, 1000);
plot(xx,spline(x,y,xx),'k-',x,y,'ro')
hold on;
legend('condition 3','','','','original', 'condition 4');
```

- For condition 4 (not a knot), I use spline() function in MATLAB since its default end condition is "not-a-knot" condition.

- It can be seen that condition 3 fits better to the original function than condition 4.



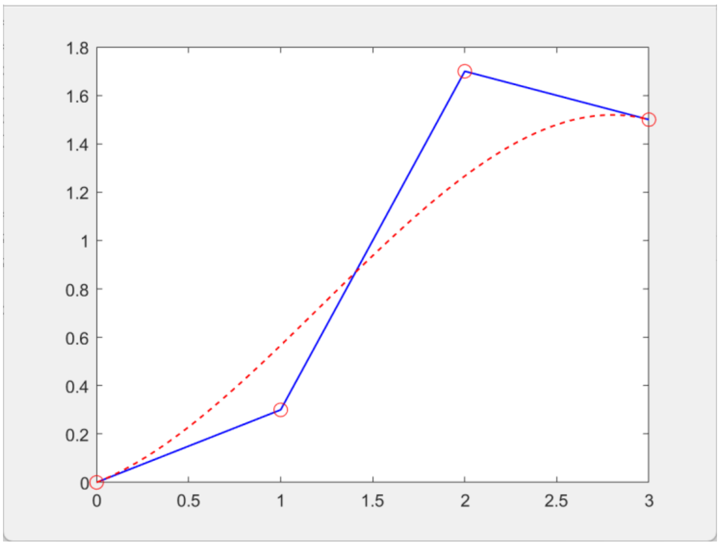
3. If these four points are connected in order by straight lines, a zigzag line is created: (0, 0), (1, 0.3), (2, 1.7), (3, 1.5).

   (a) Using the two interior points as controls, find the cubic Bezier curve. Plot this together with the zigzag line.

```
x = [0, 1, 2, 3];
y = [0, 0.3, 1.7, 1.5];
plot(x, y, 'b-', 'LineWidth', 1);
hold on;
plot(x, y, 'ro', 'MarkerSize', 8);
hold on;


u = linspace(0, 1, 100);
bezier_x = @(u) x(1).*(1-u).^3 + x(2).*3.*u.*((1-u).^2) +  x(3).*3.*(u.^2).*(1-u) + x(4).*(u.^3);
bezier_y = @(u) y(1).*(1-u).^3 + y(2).*3.*u.*((1-u).^2) +  y(3).*3.*(u.^2).*(1-u) + y(4).*(u.^3);

plot(bezier_x(u), bezier_y(u), 'r--', 'LineWidth', 1);
```

- The implementation is written in “q3.m”. Using the equation below to define the cubic Bezier line.

- $\gamma(u) = (1-u)^3 p_0 + 3u(1-u)^2 p_1 + 3u^2(1-u)p_2 + u^3 p_3$



(b) If the second and third points (the control points) are moved, the Bezier curve will change. If these are moved vertically, where should they be located so that the Bezier curve passes through all of the original four points?

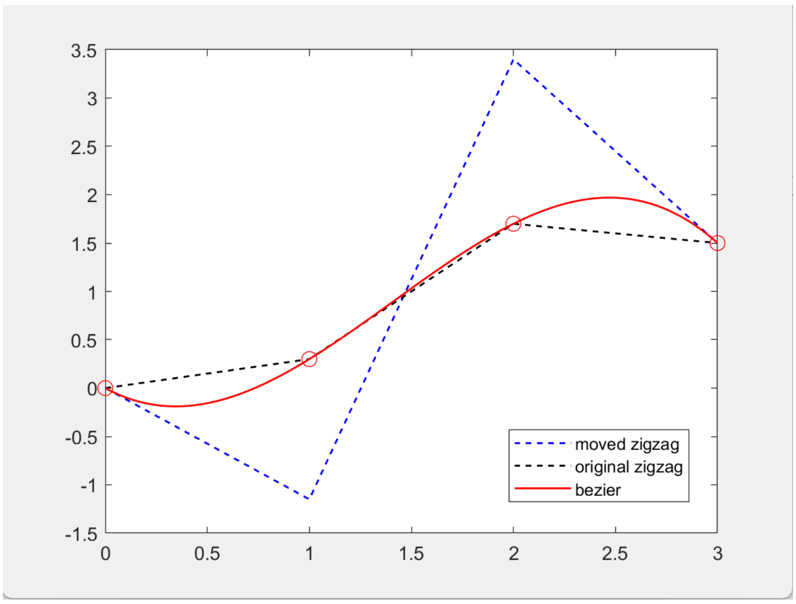
q2.m | q3.m | q3_b.m | q4.m | q5.m | q6.m

```
x = [0, 1, 2, 3];
y = [0, 0.3, 1.7, 1.5];

syms u1 u2
eq1 = x(1)*(1-u1).^3 + x(2)*3*u1.*((1-u1).^2) + x(3)*3*(u1.^2).*(1-u1) + x(4)*(u1.^3) == x(2);
eq2 = x(1)*(1-u2).^3 + x(2)*3*u2.*((1-u2).^2) + x(3)*3*(u2.^2).*(1-u2) + x(4)*(u2.^3) == x(3);

[u1,u2] = solve([eq1,eq2],[u1,u2]);
% Display the solutions
disp(['u1 = ', char(u1)]);
disp(['u2 = ', char(u2)]);

syms y2 y3
eq3 = y(1)*(1-u1).^3 + y2 *3*u1.*((1-u1).^2) + y3 *3*(u1.^2).*(1-u1) + y(4)*(u1.^3) == 0.3;
eq4 = y(1)*(1-u2).^3 + y2 *3*u2.*((1-u2).^2) + y3 *3*(u2.^2).*(1-u2) + y(4)*(u2.^3) == 1.7;

[y2,y3] = solve([eq3,eq4],[y2,y3]);
% Display the solutions
disp(['y2 = ', char(y2)]);
disp(['y3 = ', char(y3)]);

new_y = [0, y2, y3, 1.5];
u = linspace(0, 1, 100);
bezier_x = @(u) x(1)*(1-u).^3 + x(2)*3*u.*((1-u).^2) + x(3)*3*(u.^2).*(1-u) + x(4)*(u.^3);
bezier_y = @(u) new_y(1).*(1-u).^3 + new_y(2).*3.*u.*((1-u).^2) +  new_y(3).*3.*(u.^2).*(1-u) + new_y(4).*(u.^3);
% plot new zigzag line
plot(x, new_y, 'b--', 'LineWidth', 1); hold on;
% plot original zigzag line
plot(x, y ,'k--','LineWidth', 1); hold on;
plot(x, y, 'ro', 'MarkerSize', 8); hold on;
% plot bezier line
plot(bezier_x(u), bezier_y(u), 'r-', 'LineWidth', 1); hold on;
legend('moved zigzag','original zigzag','','bezier');
```

```
>> q3_b
u1 = 1/3
u2 = 2/3
y2 = -23/20
y3 = 17/5
```

<u>ANS: (1, -23/20) and (2,17/5)</u>

- The implementation is written in 'q3_b.m'.
- Since they move vertically, the x coordinates are fixed. Therefore, we first find u such that bezier_x is equal to their x coordinates. Then substitute u back to the equations and find new y coordinates that satisfied "bezier_y = original y coordinates" so that the Bezier curve can passes through all the original four points.



4. The function whose values are tabulated below is $z = x + e^y$. Construct the B-spline surface from the rectangular array of 16 points nearest to (2.8, 0.54) and find $z(2.8, 0.54)$.

   Choose 16 points that is nearest to (2.8, 0.54).

x\y	0.2	0.4	0.5	0.7
1.3	2.521	2.792	2.949	3.314
2.5	3.721	3.992	4.149	4.514
3.1	4.321	4.592	4.749	5.114
4.7	5.921	6.192	6.349	6.714

```
q3.m × | q2.m × | q3_b.m × | q4.m × | q5.m × | q6.m × | +

X = [1.3 1.3 1.3 1.3;2.5 2.5 2.5 2.5;3.1 3.1 3.1 3.1; 4.7 4.7 4.7 4.7];
Y = [0.2 0.2 0.2 0.2;0.4 0.4 0.4 0.4;0.5 0.5 0.5 0.5; 0.7 0.7 0.7 0.7];
Z = [2.521 2.792 2.949 3.314; 3.721 3.992 4.149 4.514; 4.321 4.592 4.749 5.114; 5.921 6.192 6.349 6.714];
M = [-1 3 -3 1; 3 -6 3 0; -3 0 3 0; 1 4 1 0];
U = @(u)[u.^3 u.^2 u 1];
V = @(v)[v.^3; v.^2 ;v ;1];
% x(u,v) = 1/36 uMXM'v'

x_eq = @(u, v) (1/36) * U(u) * M * X * M' * V(v);
y_eq = @(u, v) (1/36) * U(u) * M * Y * M' * V(v);
z_eq = @(u, v) (1/36) * U(u) * M * Z * M' * V(v);

x_val = 2.8;
y_val = 0.54;
u_val = (x_val-1.3)/(4.7-1.3);
v_val = (y_val-0.2)/(0.7-0.2);
result = z_eq(u_val, v_val);
disp(result);
```

$$z(u,v) = \frac{1}{36}[u^3, u^2, u, 1]MZM^T[v^3, v^2, v, 1]^T$$

- The implementation is written in ‘q4.m’.
- Since u and v range between 0 and 1, we need to transform (2.8, 0.54) to fit the range first, which results in u_val and v_val.

```
>> q4
    4.3706
```

ANS: $z(2.8, 0.54) = 4.3706$

5. (a) Develop the normal equations to fit the (x, y) data to a plane.

- The implementation is written in ‘q5.m’.

```
q2.m × | q3.m × | q3_b.m × | q4.m × | q5.m × | q6.m × | +

x = [0.40 1.2 3.4 4.1 5.7 7.2 9.3];
y = [0.70 2.1 4.0 4.9 6.3 8.1 8.9];
z = [0.031 0.933 3.058 3.349 4.870 5.757 8.921];

n = length(x');
A = [x' y' ones(n,1)];
b = z';

% A^TAa = A^Tb, a is the coefficient that we want
coef = (A'*A) \ (A'*b);
```

- Using the equation: $A^T Aa = A^T b$, where A and b are the matrices displayed in the screenshot below, we can find the coefficients a, b, and c.

```
>> q5
A
    0.4000    0.7000    1.0000
    1.2000    2.1000    1.0000
    3.4000    4.0000    1.0000
    4.1000    4.9000    1.0000
    5.7000    6.3000    1.0000
    7.2000    8.1000    1.0000
    9.3000    8.9000    1.0000

b
    0.0310
    0.9330
    3.0580
    3.3490
    4.8700
    5.7570
    8.9210

A^TAa = A^Tb, a is the coefficient that we want
```

```
A^TA:
  200.7900  213.4900   31.3000
  213.4900  229.4200   35.0000
   31.3000   35.0000    7.0000

A^Tb:
  177.4348
  187.3327
   26.9190

a:
    1.5961   -0.7024    0.2207
```

(b) Use these equations to fit z = ax + by + c.

ANS: $z = 1.5961x - 0.7024y + 0.2207$

(c) What is the sum of the squares of the deviations of the points from the plane?

$$\text{sum of the squares of the deviations} = \sum (Aa - b)^2$$

```
% Aa = b
x = A * coef;
r = A * coef - b;
e = r.^2;
disp(['sum of the squares of the deviations: ' num2str(sum(e))])
```

```
sum of the squares of the deviations: 0.3194
```

ANS: 0.3194

6. Find the first few terms of the Chebyshev series for cos(x) by rewriting the Maclaurin series in terms of the T(x)'s and collecting terms. Compare the error of both the Chebyshev series and the power series after truncating each to the fourth degree.

6.

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$T_{n+1}(x) = 2x T_n(x) - T_{n-1}(x)$, $T_0(x) = 1$, $T_1(x) = x$

① $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

② $1 = T_0$, $x^2 = \frac{1}{2}(T_0 + T_2)$, $x^4 = \frac{1}{8}(3T_0 + 4T_2 + T_4)$

$$x^6 = \frac{1}{32}T_6 + \frac{3}{2}\cdot\frac{1}{8}(3T_0 + 4T_2 + T_4) - \frac{9}{16}\cdot\frac{1}{2}(T_0 + T_2) + \frac{1}{32}T_0$$

$$= \frac{1}{32}T_6 + \frac{5}{16}T_0 + \frac{15}{32}T_2 + \frac{3}{16}T_4$$

$$\cos x = T_0 - \frac{1}{2}\cdot\frac{1}{2}(T_0 + T_2) + \frac{1}{24}\cdot\frac{1}{8}(3T_0 + 4T_2 + T_4) - \frac{1}{720}\left(\frac{5}{16}T_0 + \frac{15}{32}T_2 + \frac{3}{16}T_4 + \frac{1}{32}T_6\right)$$

T_0 : 1

T_1 : x

T_2 : $2x^2 - 1$

T_3 : $4x^3 - 3x$

T_4 : $8x^4 - 8x^2 + 1$

T_5 : $16x^5 - 20x^3 + 5x$

T_6 : $32x^6 - 48x^4 + 18x^2 - 1 = T_6$

$$\cos x = T_0 - \frac{1}{2}\cdot\frac{1}{2}(T_0+T_2) + \frac{1}{24}\cdot\frac{1}{8}(3T_0+4T_2+T_4)$$
$$-\frac{1}{720}\left(\frac{5}{16}T_0+\frac{15}{32}T_2+\frac{3}{16}T_4+\frac{1}{32}T_6\right)$$

$$= \left(1-\frac{1}{4}+\frac{1}{64}-\frac{1}{720}\cdot\frac{5}{16}\right)+\left(\frac{-1}{4}+\frac{1}{48}-\frac{1}{720}\cdot\frac{15}{32}\right)(2x^2-1)$$
$$+\left(\frac{1}{192}-\frac{1}{720}\cdot\frac{3}{16}\right)(8x^4-8x^2+1)-\frac{1}{720}\cdot\frac{1}{32}(32x^6-48x^4+18x^2-1)$$

$$= 0.7652 + (-0.2298)(2x^2-1) + 0.0049(8x^4-8x^2+1)$$
$$-0.000043(32x^6-48x^4+18x^2-1)$$

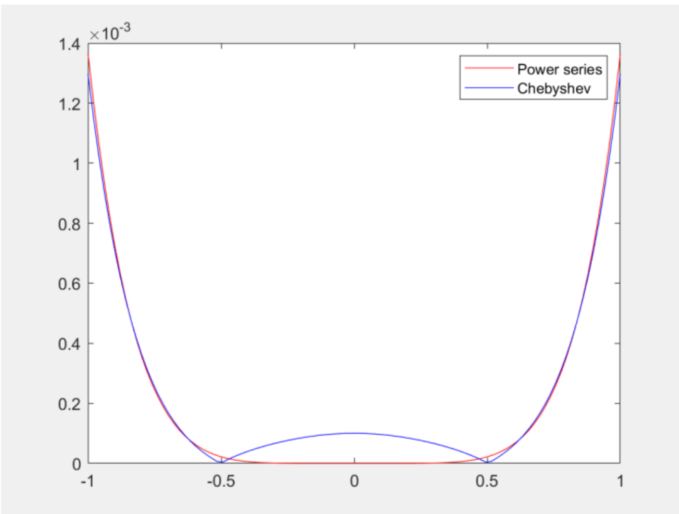
$$\fallingdotseq 0.9999 - 0.4996x^2 + 0.0413x^4$$

- Use MATLAB to plot the error which is written in ‘q6.m’.

- Power series has smaller error.

q3.m | q2.m | q3_b.m | q4.m | q5.m | q6.m

```

x = linspace(-2, 1, 100);
f1 = @(x) 1 - (x.^2)/2 + (x.^4)/24;
f2 = @(x) 0.9999 - 0.4996*(x.^2) + 0.0413*(x.^4);
f3 = @(x) cos(x);
y1 = f1(x);
y2 = f2(x);
y3 = f3(x);
error_f1 = abs(f3(x) - f1(x));
error_f2 = abs(f3(x) - f2(x));
plot(x,error_f1,'r');
hold on;
plot(x,error_f2,'b');
hold on;
legend('Power series','Chebyshev');
```



7. Find the Fourier coefficients for $f(x) = x^2 - 1$ if it is periodic and one period extends from x = −1 to x = 2.

7. 已知 $f(x) = x^2 - 1$ 是 even function , $[-1, 2]$, $L = \frac{3}{2}$

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{2n\pi x}{P}\right) + B_n \sin\left(\frac{2n\pi x}{P}\right) \right]$$

$$A_0 = \frac{1}{L} \int_{-1}^{2} f(x)\,dx = \frac{2}{3} \int_{-1}^{2} f(x)\,dx$$

$$= \frac{2}{3} \int_{-1}^{2} (x^2 - 1)\,dx = 0$$

$$B_n = \frac{1}{L} \int_{-1}^{2} f(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{3} \int_{-1}^{2} (x^2 - 1) \sin\left(\frac{2n\pi x}{3}\right) dx$$

$$A_n = \frac{1}{L} \int_{-1}^{2} f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{2}{3} \int_{-1}^{2} (x^2 - 1) \cos\left(\frac{2n\pi x}{3}\right) dx$$

$$\therefore\ f(x) = \sum_{n=1}^{\infty} \left[\frac{2}{3} \int_{-1}^{2} (x^2 - 1) \cos\left(\frac{2n\pi x}{3}\right) dx \cdot \cos\left(\frac{2n\pi x}{3}\right) + \frac{2}{3} \int_{-1}^{2} (x^2 - 1) \sin\left(\frac{2n\pi x}{3}\right) dx \cdot \sin\left(\frac{2n\pi x}{3}\right) \right]$$

#