NATIONAL UNIVERSITY OF SINGAPORE

Composite Boson

by

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"Everything should be made as simple as possible, but not simpler."

Albert Einstein

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Abstract

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Investigated the bosonic quality for multipartite GHZ and W entanglement of distinguishable fermions. Investigated the bosonic quality of bipartite identical fermions. Shown that under multipartite GHZ entanglement, the bosonic quality is similar to bipartite distinguishable fermions which is similar to bipartite indistinguishable fermions. Shown that bosonic quality under W entanglement decreases with more constituent fermions.

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Dedicated to Mom and Dad

Chapter 1

Introduction and History

1.1 History of Bose-Einstein Condensate

In 1924-25, Bose published a paper on quantum statistics of photons, which was later extended by Einstein to material particles. The papers theorized that when a dilute gas of weakly interacting bosons are confined in an external potential and cool below a critical temperature, then a large fraction of bosons will occupy the lowest quantum state with 0 momentum, at which quantum effects become apparent on a macroscopic scale.

In 1911, Kamerlingh Onnes discovered that the density of helium-4 liquid had a sharp maximum near 2.2K. In 1928, Wolfke and Keesom proposed that the phenomenon was due to partial phase transition of helium-4 liquid to lowest quantum state. Later, more strange properties of helium-4 liquid under phase transition was discovered. For example, there is a sudden transition to extremely low viscosity and extremely high heat conductivity. These phenomenon was later linked to BEC transition by Fritz London in 1938 when he discovered the similarity in the shape of the heat capacity curve in liquid helium with ideal Bose-Einstein gas near the transition point. However, helium-4 being a liquid rather than a gas at phase transition, has relatively strong intermolecular interactions. Therefore, Bose-Einstein equation which is a description of non-interacting bosons has to be heavily modified. In 1995, the first gaseous condensate was produced by Eric Cornell and Carl Wieman using a gas of rubidium atoms cooled to 170nK. More recent experimental development shows that Bose-Einstein condensation also applies to quasiparticles in solids. For example, a magnon in an antiferromagnet carries spin 1 and thus obeys Bose-Einstein statistics.

1.2 BCS and Fermionic Condensate

Not only a group of bosons can form BEC, fermions can form BEC also when they pair up and behave like bosons. In 1957, Bardeen, Cooper and Schrieffer proposed a theory that describes superconductivity as microscopic effect caused by the condensation of pairs of electrons into a boson-like state. According to BCS theory, electrons become correlated to each other and form Cooper pairs below certain temperature, and behaves like a BEC. 1971, helium-3 fermion was also shown to exhibit superfluidity below 0.0025K. The advent of BCS theory shows that strongly correlated fermions can exhibit bosonic-like behavior and condense to BEC under the right experimental condition. With the development of quantum information, it was proved in 2005 by Law that [3] this correlation between fermions can be described by the entanglement between fermions. And since entanglement is not restricted by the location of the fermions. It is therefore possible for two entangled fermions to be spatially separated but as whole still behave like a boson.

1.3 My Thesis

Expanding on Law's idea, we investigated the bosonic character for a system of fermions under multipartite GHZ and W entanglement. GHZ entanglement and W entanglement are different in the sense that you cannot change a GHZ entanglement into a W entanglement and vice versa by LOCC (Local Operation Classical Communication). Moreover, W entanglement is more robust than GHZ entanglement. A disturbance of one fermion in a system of fermions under GHZ, will destroy the whole entanglement structure of the system, which is different from W entanglement, whereby the remaining fermions will still be W entangled even with the disturbance of one fermion. Furthermore, I also investigated the bosonic quality of the composite system make up of two identical particles and compare its bosonic quality with distinguishable particles.

Chapter 2

Background Theory

2.1 Entanglement

Following [1], we defined quantum entanglement of bipartite system as that if the the mixed state ρ of Hilbert space $A \otimes B$ can be written as

$$\rho = \sum_{i=1} p_i \rho_{A_i} \otimes \rho_{B_i} \tag{2.1}$$

where $\rho_{A_i}(\rho_{B_i})$ is a density matrix of A(B) and $p_i > 0$ and $\sum_{i=1} p_i = 1$, then the mixed state ρ is *separable*, otherwise it is *entangled* if it is not separable.

For a state ρ of a m-partite quantum system living in a Hilbert space $H^{(1)} \otimes H^{(2)} \otimes \cdots \otimes H^{(m)}$ is *separable* if it can be written as

$$\rho = \sum_{i=1}^{n} p_i \rho_{i,1} \otimes \rho_{i,2} \otimes \cdots \otimes \rho_{i,m} \tag{2.2}$$

where each $\rho_{i,j}$ is a density matrix of $H^{(j)}$ and $p_i > 0$. Otherwise, ρ is *unseparable* or *entangled*. A multipartite entanglement is a theoretically very complex issue and it has many degrees and forms. However, it is sufficient to note that all entangled states cannot be created by LOCC.

2.1.1 Schmidt Decomposition

From [2], suppose $|\psi\rangle$ is a pure state of a composite system, $A \otimes B$. Then there exist orthonormal states $|i_A\rangle$ for system A, and orthonormal states $|i_B\rangle$ of system B such that

$$|\psi\rangle = \sum_{i=1}^{n} \lambda_i |i_A\rangle|i_B\rangle \tag{2.3}$$

where λ_i are non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$ known as *Schmidt* coefficients. The bases $|i_A\rangle$ and $|i_B\rangle$ are *Schmidt* bases for A and B respectively. The number of non-zero *Schmidt* coefficients are known as *Schmidt* number. For a bipartite system, Schmidt number denotes the amount of entanglement between system A and B. A Schmidt number of more than 1 indicates entanglement between the two systems.

2.1.2 W Entanglement

W entanglement has the property that, measurement of one party in the multipartite entanglement will not affect the entanglement structure for the remaining parties. W entangled state is defined as

$$|W\rangle = \sum_{n_1+n_2+\cdots+n_i=L} \lambda_{n_1 n_2 \ldots n_i} |n_1\rangle|n_2\rangle \ldots |n_i\rangle \tag{2.4}$$

2.1.3 GHZ Entanglement

GHZ entanglement has the property that observing the state of one party will cause all the other parties to be in the same state. GHZ state is defined as

$$|GHZ\rangle = \sum_i \lambda_i |i\rangle_1 |i\rangle_2 \ldots |i\rangle_m \tag{2.5}$$

where $|i\rangle_j$ is the state for jth particle.

2.2 Fock Space

The orthonormal basis of the Fock space consists of the vacuum state $|0\rangle$; the complete set of one-particle state $\{|\phi_\alpha\rangle : \alpha = 1, 2, 3 \dots\}$; the complete set of two-particles state; the complete set of three-particles state and so on. The formalism for Fock space is technically different for boson and fermion, and thus we will consider them separately.

2.2.1 Fermion

The creation (annihilation) operator $a_n^\dagger(a_n)$ of fermion has the following properties, where n is the quantum number of the fermion.

$$a_n^\dagger|0\rangle = |\phi_n\rangle \tag{2.6}$$

$$a_m^\dagger|\phi_n\rangle = a_m^\dagger a_n^\dagger|0\rangle = |\phi_m\phi_n\rangle = -|\phi_n\phi_m\rangle \tag{2.7}$$

$$a_n|\phi_n\rangle = |0\rangle \tag{2.8}$$

$$a_n|0\rangle = 0 \tag{2.9}$$

The vector states are defined to be antisymmetric with interchange of any two fermions. The operators also follows the following anti-commutation relations

$$\{a_n, a_m\} = \{a_n^\dagger, a_m^\dagger\} = 0 \ , \ \{a_n, a_m^\dagger\} = \delta_{nm} \ , \ \{a_n^\#, b_m^\#\} = 0 \tag{2.10}$$

(5) is the anti-commutation relation for two distinguishable fermions a and b.

2.2.2 Boson

The creation (annihilation) operator of boson in mode α is represented as $C_\alpha^\dagger(C_\alpha)$. The creation (annihilation) operator has the following properties.

$$C_\alpha^\dagger|0, 0, \dots, n_\alpha = 0, \dots\rangle = |\phi_\alpha\rangle = |0, 0, \dots, n_\alpha = 1, 0, \dots\rangle \tag{2.11}$$

$$C_\alpha|n_1, n_2, \dots, n_\alpha = 0, \dots\rangle = 0 \tag{2.12}$$

$$C_\alpha^\dagger|n_1, n_2, \dots, n_\alpha, \dots\rangle = (n_\alpha + 1)^{\frac{1}{2}}|n_1, n_2, \dots, n_\alpha + 1, \dots\rangle \tag{2.13}$$

$$C_\alpha|n_1, n_2, \dots, n_\alpha, \dots\rangle = n_\alpha^{\frac{1}{2}}|n_1, n_2, \dots, n_\alpha - 1, \dots\rangle \tag{2.14}$$

and follows the commutation relation

$$[C_n, C_m^\dagger] = \delta_{nm} \tag{2.15}$$

If the bosons all live in the same mode. Then we can simplify the above relations to

$$\begin{aligned} C^\dagger|0\rangle &= |1\rangle && (2.16)\\ C|0\rangle &= 0 && (2.17)\\ C^\dagger|n\rangle &= (n+1)^{\frac{1}{2}}|n+1\rangle && (2.18)\\ C|n\rangle &= n^{\frac{1}{2}}|n-1\rangle && (2.19)\\ [C, C^\dagger] &= 1 && (2.20) \end{aligned}$$

2.3 Composite Boson makes up of Two Distinguishable Fermion

The general state of a composite boson make up of two distinguishable fermions are defined as

$$|\psi\rangle = C^\dagger|0\rangle = \sum_n \lambda_n a_n^\dagger b_n^\dagger |0\rangle \tag{2.21}$$

where $a_n^\dagger$ ($b_n^\dagger$) represents the fermionic creation operator for Type A (Type B) fermions. And λ_n is the Schmidt coefficient. The commutation relation between C and $C^\dagger$ is given by

$$[C, C^\dagger] = 1 - \Delta \tag{2.22}$$

where Δ is defined as

$$\Delta = \sum_n \lambda_n^2 (a_n^\dagger a_n + b_n^\dagger b_n) \tag{2.23}$$

Which is not a strictly bosonic operator since Δ has non-zero matrix elements, depending on the states involve. The expectation values of the bosonic departure and boson number operators in $|N\rangle$ can be derived easily using (2.28), (2.31) and

(2.32)

$$\langle 1 - [C, C^\dagger] \rangle_N = 2\left(1 - \frac{\chi_{N+1}}{\chi_N}\right) \tag{2.24}$$

where χ_N is defined by the N-particle state as

$$|N\rangle = \frac{\chi_N^{-\frac{1}{2}}}{\sqrt{N!}} C^{\dagger N}|0\rangle \tag{2.25}$$

Here χ_N is the normalization constant, which is necessary since $C^\dagger$ is not a perfect bosonic operator. Furthermore, χ_N is a measure of the bosonic quality for the entire system of N composite bosons. With $\chi_N = 1$ being most bosonic and $\chi_N = 0$ being least bosonic, and any intermediate values representing sub-bosonic quality.

By multiplying (2.25) with $\langle N|$, we can express χ_N in the following form

$$\langle 0|C^N C^{\dagger N}|0\rangle = N!\chi_N \tag{2.26}$$

where χ_N is derived in [3] to be (see Appendix A for full derivation)

$$\chi_N = N! \sum_{p_1 < p_2 < \cdots < p_N} \lambda_{p_1}^2 \lambda_{p_2}^2 \ldots \lambda_{p_N}^2 \tag{2.27}$$

In order to test how good the operator C behaves as a bosonic annihilation operator, we need to examine how the operator acts on $|N\rangle$. This is defined in [3] as

$$C|N\rangle = \alpha_N \sqrt{N}|N-1\rangle + |\epsilon_N\rangle \tag{2.28}$$

where $\alpha_N = \sqrt{\frac{\chi_N}{\chi_{N-1}}}$ is a constant, and the correction term $|\epsilon_N\rangle$ is orthogonal to $|N-1\rangle$. Such a correction term is necessary because the set of $|N\rangle$ states where $N \in \{0, 1, 2, \ldots, \infty\}$ is only a subset of the entire Hilbert space associated with the constituent particles. From (2.28), we can see that it is bosonic only if

$$\alpha_N \to 1 \tag{2.29}$$

$$\langle \epsilon_N | \epsilon_N \rangle \to 0 \tag{2.30}$$

Where $\langle \epsilon_N | \epsilon_N \rangle$ is derived by [3] as

$$\langle \epsilon_N | \epsilon_N \rangle = 1 - N \frac{\chi_N}{\chi_{N-1}} + (N-1) \frac{\chi_{N+1}}{\chi_N} \tag{2.31}$$

Therefore, we can see that, the bosonic condition (2.29) and (2.30) of **one composite boson** are controlled by the ratio of normalization constants. An ideal composite boson emerges in the limit $\frac{\chi_{N\pm1}}{\chi_N} \to 1$.

2.3.1 Purity as a Bound for Bosonic Quality

To examine the bosonic nature of one composite boson further, we can look at the deviation of the resultant $N+1$ particle state from an ideal $N+1$ bosonic state $|N+1\rangle$ when we add a composite boson to an ideal N bosonic state $|N\rangle$. Following from the definition of (2.25), we have

$$C^\dagger |N\rangle = \alpha_{N+1} \sqrt{N+1} |N+1\rangle \tag{2.32}$$

where $\alpha_{N+1} = \sqrt{\frac{\chi_N}{\chi_{N-1}}}$. (2.32) shows that the bosonic quality of one composite boson is defined by the ratio $\frac{\chi_N}{\chi_{N-1}}$. It is shown by [4] that the ratio $\frac{\chi_N}{\chi_{N-1}}$ is strictly nonincreasing as N increases. Which simply means $\frac{\chi_{N+2}}{\chi_{N+1}} \leq \frac{\chi_{N+1}}{\chi_N}$ for all $N \in \{1, 2, \ldots, \infty\}$. Moreover, [4] also showed that the ratio $\frac{\chi_N}{\chi_{N-1}}$ is bounded from below and above by the purity $P(\rho) = \mathrm{Tr}\{\rho^2_{A(B)}\}$ of the reduced density matrix $\rho_{A(B)}$ of particle A(B). Since the pair is in a pure state, the purities of the two particles are guaranteed to be equal. The result was shown to be

$$1 - NP \leq \frac{\chi_{N+1}}{\chi_N} \leq 1 - P \tag{2.33}$$

The purity P has the physical meaning of quantifying the amount of entanglement between the two fermions. As P goes from 0 to 1, the amount of entanglement between the two fermions will decrease from infinity to zero. Making use of (2.21) and (2.26), and requiring $\chi_1 = 1$, we can easily show that $P = \sum_n \lambda_n^4 = 1 - \chi_2$. From which we have a bound on bosonic quality of a 2-fermion composite boson by χ_2.

$$1 - N(1 - \chi_2) \leq \frac{\chi_{N+1}}{\chi_N} \leq \chi_2 \tag{2.34}$$

Additionally, we note that $\frac{\chi_N}{\chi_{N-1}}$ is related to the condensation fraction $F = \langle N|C^\dagger C|N\rangle$ via the relation

$$F = N\frac{\chi_N}{\chi_{N-1}} \tag{2.35}$$

The value of F quantifies the number of composite bosons in the lowest quantum state in a system of N particles.

2.4 Composite Boson makes up of Two Indistinguishable Fermion

The quantifying of entanglement between indistinguishable particles is still a challenging topic, and there were several definitions of correlation and entanglement being proposed [11–13]. Here, I will follow the approach proposed by Paškauskas and You[11], that quantify quantum correlations using von Neumann entropy.

The pure state of a composite boson make up of two indistinguishable fermions can be written as

$$|\psi\rangle = \sum_{i,j=1}^{2L} \Omega_{ij} a_i^\dagger a_j^\dagger |0\rangle \tag{2.36}$$

where $\Omega_{ij} = -\Omega_{ji}$ is a $2L \times 2L$ antisymmetric matrix. By the singular value decomposition method (SVD) [13]. For any antisymmetric matrix Ω, there exists a unitary operator U such that $\Omega = UZU^\dagger$, where the matrix Z has blocks of diagonal

$$Z = \text{diag}[Z_1, Z_2, \ldots, Z_L], \;\; Z_i = \begin{bmatrix} 0 & z_i \\ -z_i & 0 \end{bmatrix} \tag{2.37}$$

This decomposition is unique and therefore it simplifies (2.36) to

$$|\psi\rangle = \sum_{k=1}^{L} 2z_k a'^\dagger_{2k-1} a'^\dagger_{2k} |0\rangle \tag{2.38}$$

whereby $a'^\dagger_k = \sum_{i=1}^{L} U_{ik} a_i^\dagger$ is the new fermionic operator, and it also satisfies all the anti-commutation relations in (2.10). The density matrix for two identical fermions is $\rho_F = |\psi\rangle\langle\psi|$. Compared to the states for two distinguishable fermions, the analog of von Neumann entropy still remains a good correlation measure for two identical fermions [11]. Unlike density matrix for distinguishable fermions, where there are two reduced density matrix one for each fermion. For identical fermions, there is only one single-particle density matrix, ρ^f(normalized to one), and is computed as

$$\rho^f_{vu} = \frac{\mathrm{Tr}(\rho_F a_u^\dagger a_v)}{\mathrm{Tr}\left(\rho_F \sum_u a_u^\dagger a_u\right)} = \frac{1}{2}\langle\psi|a_u^\dagger a_v|\psi\rangle = 2(\Omega^\dagger\Omega)_{uv} \tag{2.39}$$

The corresponding von Neumann entropy is then derived to be

$$S = -\mathrm{Tr}[\rho^f \log_2 \rho^f] = -1 - 4\sum_{k=1}^{L} |z_k|^2 \log_2\left(|z_k|^2\right) \tag{2.40}$$

The normalization condition of $|\psi\rangle$ in (2.36) requires that $4\sum_{k=1}^{L}|z_k|^2 = 1$. Under such restriction, we see that the maximum von Neumann entropy occurs when z_k is constant, that is $S_{max} = \log_2(2L)$. While for uncorrelated state, we can easily see that the minimum entropy is $S_{min} = 1$.

2.5 Addition and Subtraction as a Measure of Bosonic Quality

Expanding beyond 2 particles composite boson. [5] shows that, we can also quantify the bosonic and fermionic quality of a composite boson make up of more than 2 fermions under any type of entanglement by measuring the deviation of its occupation probability for the vacuum state after an addition and subtraction (AS) operation.

The bosonic quality of composite boson can be quantified by comparing the initial state $\rho = \sum_n p_n|n\rangle\langle n|$ with the state after addition and subtraction (AS) $\rho_{AS} = (aa^\dagger)\rho(a^\dagger a) = \sum_{n,m}\rho_{n,m}(n+1)(m+1)|n\rangle\langle m|$. The change after AS can be sufficiently measured by the probability distribution of the number of particles $\{p_0, p_1, \dots\}$, where p_n denotes the probability of detecting n particles. To develop a measure that does not depend on the nature of the particle, be it boson or fermion, we restrict our consideration only to p_0 and p_1 because they do not assume multiple occupancy. We define the measure M

$$M = p_0 - p_0^{AS} \tag{2.41}$$

where p_0^{AS} denotes the vacuum occupancy probability after AS. For bosons, the action of AS cause the occupation probability p_n to go to $p_n^{AS} = (n+1)^2 p_n$, which together with normalization, implies a decrease in p_0. The occupation probability of vacuum state p_0^{AS} after AS depends on the total probability distribution $\{p_n^{AS}\}$ after AS. Note that

$$M = p_0 - \frac{p_0}{\sum_{k+0}^{n_max} (k+1)^2 p_k} \tag{2.42}$$

where n_{max} denotes the maximum number of particles in the system. It can be easily shown that M is maximum when $p_0 = \frac{n_{max}+1}{n_{max}+2}$ and $p_{n_{max}} = 1 - p_0$. By restricting ourselves to p_0 and p_1 only, the optimal probability distribution is $p_0 = \frac{2}{3}$ and $p_1 = \frac{1}{3}$, which will give $M = \frac{1}{3}$ for the case of perfect bosons. From now on we fixed $p_0 = \frac{2}{3}$, therefore the measure is taken with respect to the state

$$\rho_{\mathcal{M}} = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1| \tag{2.43}$$

Redefining the measure as $\mathcal{M} = 3M$, we obtain

$$\mathcal{M} = 2 - 3p_0^{AS} \tag{2.44}$$

whereby ideal bosons will give $\mathcal{M} = 1$. While for ideal fermions, since the addition of one fermion to the state (2.43) will kill $|1\rangle$ and bring $|0\rangle$ to $|1\rangle$. Subsequent subtraction will then give $\rho_0^{AS} = 1$ because only $|0\rangle$ is left, this means $\mathcal{M} = -1$ for ideal fermions. Note that for distinguishable particles, AS operation will not change the state at all. Therefore, the operator for distinguishable particles will be $a_d^\dagger = \sum_{n=0}^{\infty} |n+1\rangle\langle n|$, which after AS operation will not change p_0, therefore $\mathcal{M} = 0$ for distinguishable particles. It is defined in [5] that domain $\mathcal{M} \in (-1, 0)$ represents $subfermionic$ behavior; and $\mathcal{M} \in (0, 1)$ represents *subbosonic* behavior. In addition, (2.44) shows that it's possible for $\mathcal{M} \in (1, 2)$ when $p_0^{AS} \to 0$, which is coined by [5] as *superbosonic* region.

For the case of composite boson, we can define the state of N composite bosons in Fock space as (2.25). Without lost of generality, the creation operator for the composite boson in (2.25) is not restricted to 2-fermion composite boson, it is equally valid to associate it with any composite boson make up of 2k-fermion under any form of entanglement, just that the value of χ_N for different type of composite boson will be different.

We will only be interested in states $|0\rangle$, $|1\rangle$ and $|2\rangle$ and in parameter χ_2. Note that $\chi_1 = 1$, $C^\dagger|0\rangle = |1\rangle$ and $C|1\rangle = |0\rangle$. Moreover $|\epsilon_n\rangle$ is not considered as it represents states of an unsuccessful subtraction. Consequently, the states after addition and subtraction follows

$$C^\dagger|1\rangle = \sqrt{2\chi_2}|2\rangle \tag{2.45}$$

$$C|2\rangle = \sqrt{2\chi_2}|1\rangle \tag{2.46}$$

Applying the above definition to the state (2.43), we can show that p_0^{AS} for a composite boson is $p_0^{AS} = (1 + 2\chi_2^2)$. Substituting p_0^{AS} into (2.44), we obtain $\mathcal{M}$ generally for a composite boson in terms of χ_2

$$\mathcal{M} = 2 - \frac{3}{3 - 2(1 - \chi_2^2)} \tag{2.47}$$

Since for composite boson $\chi_2 \in (0, 1)$, therefore $\mathcal{M} \in (-1, 1)$ which is in the sub-fermionic and sub-bosonic region. This result confirms the definition of measure by [5]. And assure χ_2 as an indicator of bosonic quality.

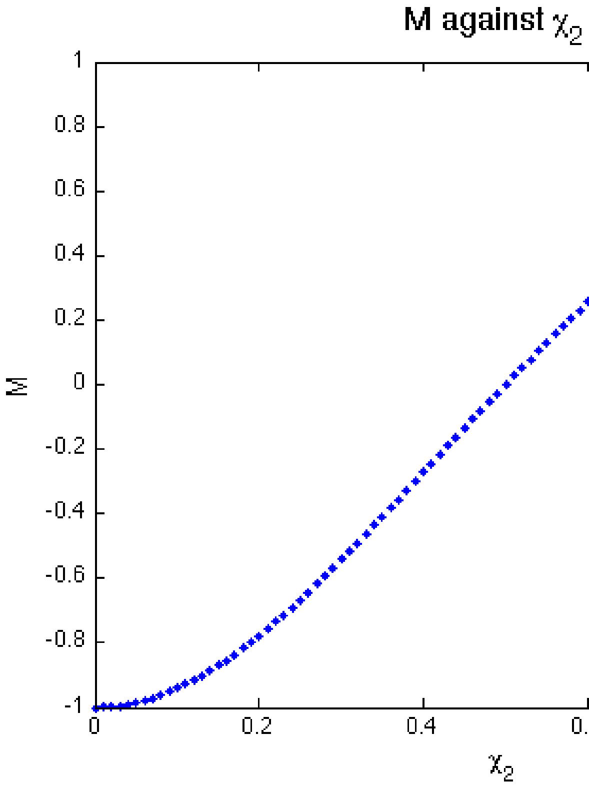

FIGURE 2.1: Plot of $\mathcal{M}$ against χ_2 for composite bosons make up of distinguishable fermions

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From Figure 2.1 above we see that the measure $\mathcal{M}$ is a monotonic increasing function of χ_2. When χ_2 is at 0, $\mathcal{M}$ is at -1, representing fully fermionic behavior. When χ_2 is at 1, $\mathcal{M}$ is at 1 also representing fully bosonic behavior. The transition between the two types of behavior occurs when $\chi_2 = 1/2$. The existence of a critical value of χ_2 for the transition between fermionic and bosonic behavior is an important result that requires further investigation.

Chapter 3

Results and Discussions

In this section I'll show that

1. *When the constituent distinguishable fermions are under GHZ entanglement, the bosonic quality of the composite system is independent of the number of its constituent fermions and is equivalent to the bosonic quality of 2-fermion system.*

2. *When the constituent distinguishable fermions are under W entanglement, the bosonic quality of the composite system decreases as the number of its constituent fermions increases.*

3. *Bosonic quality for 2 indistinguishable fermion system is the same as 2 distinguishable fermion system.*

4. *Maximum GHZ entanglement always produce better composite bosons than maximum W entanglement under the same dimension of entanglement.*

3.1 Motivation

In nature, there are many composite particles make up of fermions. An even number of fermions under certain experimental conditions behave like a boson. Recent development of quantum information had hypothesized that the bosonic quality of a composite particle is related to the entanglement of its constituent fermions. It was shown by Law [3] that in a composite system of 2 distinguishable particles, as the degree of entanglement increases, the bosonic quality of the composite boson improves as well.

With this motivation, we expand Law's result from 2-fermion system to arbitrary 2k-fermion system under W and GHZ entanglement. In addition, we also investigate the bosonic quality for indistinguishable 2-fermion system.

3.2 Bosonic Quality of Distinguishable Fermions under GHZ Entanglement

If we assume maximum entanglement in (2.21), which means $\lambda_n = \lambda$ is constant, and if we truncate the summation at L, therefore we only consider $C^\dagger_{max} = \sum_{n=1}^{L} \lambda a^\dagger_n b^\dagger_n$. By the normalization condition where $\langle 0|C_{max} C^\dagger_{max}|0\rangle = 1$ gives $\lambda = \frac{1}{\sqrt{L}}$. Note that the consideration of finite L is natural if we deal with the excitations confined in a finite condensed matter region. Moreover, for particle in free space, due to the finiteness of the resources such as energy, it is natural to assume finite sums over λ_n. It was shown in [5] that under maximum entanglement for two distinguishable fermion system.

$$\langle 0|C^N_{max} C^{\dagger N}_{max}|0\rangle = N!\mathcal{M}_N \tag{3.1}$$

in which $\mathcal{M}_N$ is given as

$$\mathcal{M}_N = \frac{L!}{L^N(L-N)!} \tag{3.2}$$

$\mathcal{M}_N$ falls in the range $0 \leq \mathcal{M}_N \leq 1$, with 0 being fully fermionic and 1 being fully bosonic. Furthermore, $\mathcal{M}_N(L)$ is a strictly increasing function of dimension L, whereby a larger L indicates more entanglement. It can be shown that as entanglement goes to infinity, i.e. $\lim_{L\to\infty} \mathcal{M}_N(L) = 1$, the bosonic quality of the system of composite bosons approach ideal bosons.

Bosonic quality of 2k fermions under GHZ entanglement. It can be shown that generally for 2k fermions under GHZ entanglement, its bosonic quality is same as 2 fermion composite boson under the same distribution of λ_n. We define the creation operator for GHZ composite boson make up of 2k distinguishable fermions as

$$C^{\dagger}_{2k,GHZ} = \sum_{n=1}^{L} \lambda_n a_n^{(1)\dagger} a_n^{(2)\dagger} \dots a_n^{(2k)\dagger} \tag{3.3}$$

where λ_n is the GHZ coefficient and generally it can be complex, $a_n^{(k)\dagger}$ is the fermionic creation operator for Type k fermion. Similar to (2.26), we can define $\chi^{(N)}_{2k,GHZ}$ as

$$\langle 0 | C^N_{2k,GHZ} C^{\dagger N}_{2k,GHZ} | 0 \rangle = N! \chi^{(N)}_{2k,GHZ} \tag{3.4}$$

Where $\chi^{(N)}_{2k,GHZ}$ indicates the collective bosonic quality of N composite bosons each make up of 2k fermions under GHZ entanglement. And we show that (see Appendix B)

$$\chi^{(N)}_{2k,GHZ} = N! \sum_{r_1 < r_2 < \dots < r_N} |\lambda_{r_1}|^2 |\lambda_{r_2}|^2 \dots |\lambda_{r_N}|^2 \tag{3.5}$$

Which has exactly the same form as χ_N in (2.27). This means that if the entanglement structure of the GHZ state is the same as the two-distinguishable-fermion composite boson state, then their bosonic quality will be the same also.

From above we can also imply that under maximum GHZ entanglement, whereby $\lambda_n = \lambda$ is constant, we have

$$\mathcal{M}^{(N)}_{2k,GHZ} = \mathcal{M}_N \tag{3.6}$$

where $\mathcal{M}^{(N)}_{2k,GHZ}$ is defined similarly as (3.1) for the maximally entangled GHZ creation and annihilation operator.

The commutation relation for $[C_{2k,GHZ}, C^{\dagger}_{2k,GHZ}]$ is derived as

$$[C_{2k,GHZ}, C^{\dagger}_{2k,GHZ}] = 1 - \Delta$$

where Δ is derived as

$$\begin{aligned} \Delta = \sum_n \lambda_n^2 [&S_n(2k-1) - S_n(2k-2) \\ &+ \dots + (-1)^{2k-r+1} S_n(2k-r) + \dots + S_n(1)] \end{aligned} \tag{3.7}$$

and $S_n(m)$ represents the sum of all the possible combinations of m operators taken from the set $\{A_n^{(1)}, A_n^{(2)}, \ldots, A_n^{(2k)}\}$, such that for each combination $A_n^{(i_1)} A_n^{(i_2)} \ldots A_n^{(i_m)}$, $i_1 < i_2 < \cdots < i_m$, and $A_n^{(r)} = a_n^{(r)\dagger} a_n^{(r)}$. It can be seen that the matrix elements of Δ is obviously not equal to 0, and it represents the bosonic departure of the operators from perfect bosonic operators.

Since $\chi_N = \chi_{2k,GHZ}^{(N)}$, we can derived similarly as (2.24) the expectation values of the bosonic departure and bosonic operators in $|N\rangle$ as

$$\langle 1 - [C_{2k,GHZ}, C_{2k,GHZ}^{\dagger}] \rangle_N = 2\left(1 - \frac{\chi_{2k,GHZ}^{(N+1)}}{\chi_{2k,GHZ}^{(N)}}\right) \tag{3.8}$$

Where in this case $\frac{\chi_{2k,GHZ}^{(N+1)}}{\chi_{2k,GHZ}^{(N)}}$ represents the bosonic quality of the creation(annihilation) operator in adding(subtracting) one composite boson from the system.

Since $\chi_{2k,GHZ}^{(N)} \leq \mathcal{M}_{2k,GHZ}$ under all possible distribution of λ_n. Therefore, we can set an upperbound for the bosonic quality of a GHZ composite boson using (2.34)

$$\frac{\chi_{2k,GHZ}^{(N+1)}}{\chi_{2k,GHZ}^{(N)}} \leq \mathcal{M}_{2k,GHZ}^{(2)} \tag{3.9}$$

The result under this section shows that for a GHZ state that consists of any even number of fermions more than 2, and with the same distribution of the GHZ coefficients λ_n as the 2 indistinguishable fermion state, then they will have the same bosonic character.

3.3 Bosonic Quality of Distinguishable Fermions under W Entanglement

For composite boson make up of 2k fermions under W entanglement. The creation operator can be defined generally as

$$C_{2k,W}^{\dagger} = \sum_{n_1+n_2+\cdots+n_{2k}=L} \lambda_{n_1 n_2 \ldots n_{2k}} a_{n_1}^{(1)\dagger} a_{n_2}^{(2)\dagger} \ldots a_{n_{2k}}^{(2k)\dagger} \tag{3.10}$$

where $n_1, n_2, \ldots, n_{2k} \in \{0, 1, 2, \ldots, L\}$, and similar to (3.4) $\chi^{(N)}_{2k,W}$ is defined as

$$\langle 0|C^N_{2k,W} C^{\dagger N}_{2k,W}|0\rangle = N!\chi^{(N)}_{2k,W} \tag{3.11}$$

Due to the complexity of the problem for W entanglement. The explicit form of $\chi^{(N)}_{2k,W}$ will not be provided here and it can be a problem for future research. Here, I will provide the result for $\mathcal{M}^{(2)}_{2k,W}$ defined as

$$\langle 0|C^2_{2k,W_{max}} C^{\dagger^2}_{2k,W_{max}}|0\rangle = 2\mathcal{M}^{(2)}_{2k,W}(L) \tag{3.12}$$

Note that $\mathcal{M}^{(2)}_{2k,W}$ is sufficient to analyze the bosonic quality of a composite boson, since by (2.47), the measure $\mathcal{M}$ for bosonic quality only depends on $\chi^{(2)}_{2k,W}$ which is bounded by $\mathcal{M}^{(2)}_{2k,W}$, i.e $\chi^{(2)}_{2k,W} \leq \mathcal{M}^{(2)}_{2k,W}$. To find $\mathcal{M}^{(2)}_{2k,W}$, we define $\lambda_{n_1 n_2 \ldots n_{2k}} = \lambda$ to be constant. Normalization of $\langle 0|C_{2k,W} C^\dagger_{2k,W}|0\rangle = 1$ yields $\sum_{n_1+n_2+\cdots+n_{2k}=L} |\lambda_{n_1 n_2 \ldots n_{2k}}|^2 = 1$, which gives $\lambda^2 = \binom{L+2k-1}{2k-1}^{-1}$.

$$\langle 0|C^2_{2k,W_{max}} C^{\dagger^2}_{2k,W_{max}}|0\rangle = 2\mathcal{M}^{(2)}_{2k,W}(L) \tag{3.13}$$

whereby $\mathcal{M}^{(2)}_{2k,W}(L)$ is derived in Appendix C to give

$$\mathcal{M}^{(2)}_{2k,W}(L) = 1 + \frac{\lambda^4}{2} \sum_{s=1}^{2k-1} \sum_{r=0}^{L} \left\{ (-1)^s \binom{2k}{s} \left[\binom{L+2k-1-r-s}{2k-1-s} \binom{r+s-1}{s-1} \right]^2 \right\} \tag{3.14}$$

$\mathcal{M}^{(2)}_{2k,W}$ characterize the bosonic quality of a composite boson make up of 2k fermions under maximum W entanglement. $\mathcal{M}^{(2)}_{2k,W}$ takes the range $0 \leq \mathcal{M}^{(2)}_{2k,W} \leq 1$, in which 0 being fully fermionic and 1 being fully bosonic. Moreover, $\mathcal{M}^{(2)}_{2k,W}(L)$ is a strictly increasing function of L, where a larger L means more entanglement. As L goes to infinity, $\mathcal{M}^{(2)}_{2k,W}(L) \to 1$, this means that under infinite W entanglement, the composite boson behaves like a real boson.

In addition, as the number of constituent fermions under maximum W entanglement in the composite system increases, the bosonic quality of the composite system decreases, which is unlike GHZ entanglement, whereby the number of constituent fermions does not affect the bosonic quality of the composite system. Our result shows that

$$\mathcal{M}^{(2)}_{4,W} > \mathcal{M}^{(2)}_{6,W} > \mathcal{M}^{(2)}_{8,W} > \ldots \tag{3.15}$$

Using the bosonic measure defined in (2.47), we expressed the bosonic measure $\mathcal{M}$ in terms of entanglement defined by the parameter L.

$$\mathcal{M} = 2 - \frac{3}{3 - 2\left[1 - \mathcal{M}_{2k,W}^{(2)}{}^{2}(L)\right]} \tag{3.16}$$

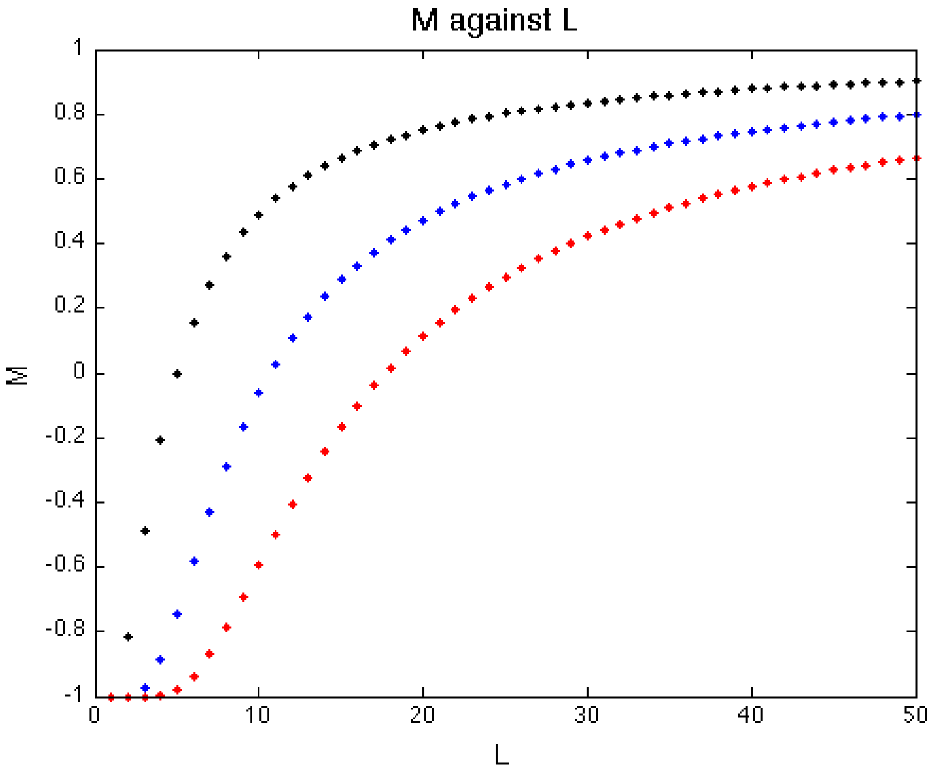

FIGURE 3.1: Plot of measure $\mathcal{M}$ against entanglement parameter L for composite bosons make up of distinguishable fermions under W entanglement. In which black dots represent 4-fermion composite boson, blue dots represent 6-fermion composite boson and red-dots represent 8-fermion composite boson.

From the above figure we can see that $\mathcal{M}$ increases monotonically with L, which means that bosonic quality improves as entanglement increases. In addition, unlike GHZ composite boson, whose bosonic quality is independent of the number of its constituent fermions, W composite boson make up of different number of fermions exhibits different bosonic quality under the same distribution of the coefficient λ.

As discussed under Section 2.4, as the measure $\mathcal{M}$ goes from -1 to 0 to 1, it represents the transition of a composite particle from fully fermionic to distinguishable to fully bosonic. It is especially interesting to note that, there is a transition point from fermionic to bosonic when the curve crosses the $\mathcal{M} = 0$ line. As seen from the curve above, that transition point occurs at different L for composite boson makes up of different number of fermions. Generally under maximum W entanglement, as shown in Figure 3.1, the transition point occurs at a larger L for larger composite boson. This implies that there is a critical entanglement point whereby if we increase our entanglement beyond that point, we will be able to produce a composite particle that exhibit bosonic behaviour, below which, we will have a composite particle behaving like its constituent fermions.

3.4 Bosonic Quality of Two Indistinguishable Fermions

From (2.36), we have shown that the composite particle state of 2 identical fermions can be defined as $|\psi\rangle = \sum_{k=1}^{L} 2z_k a_{2k-1}^{'\dagger} a_{2k}^{'\dagger}|0\rangle$, from which the creation operator will be defined as

$$C_{2,id}^{\dagger} = \sum_{k=1}^{L} \lambda_k a_{2k-1}^{'\dagger} a_{2k}^{'\dagger} \tag{3.17}$$

whereby $\lambda_k = 2z_k$. If we define $N!\chi_{2,id}^{(N)} = \langle 0|C_{2,id}^{N} C_{2,id}^{\dagger N}|0\rangle$. We show that (see Appendix D)

$$\chi_{2,id}^{(N)} = N! \sum_{m_1<m_2<\cdots<m_N} |\lambda_{m_1}|^2 |\lambda_{m_2}|^2 \ldots |\lambda_{m_N}|^2 \tag{3.18}$$

Which has the same expression as χ_N and $\chi_{2k,GHZ}^{(N)}$. Specifically, under the same entanglement structure

$$\chi_{2,id}^{(N)} = \chi_N = \chi_{2k,GHZ}^{(N)} \tag{3.19}$$

This result is quite striking because it means that the bosonic quality of a 2-fermion composite boson is independent of its constituent fermions, be it distinguishable or indistinguishable as long as the distribution of λ_n is the same. Example of such case, is when both are under maximum entanglement, in which their coefficieints are constant, and as such, we will not be able to identify the content of the composite boson by simply looking at his bosonic behavior as a whole.

Experimentally, it was verified for both cases to exhibit bosonic behavior. For example, the Cooper pair is make up of two identical electrons which exhibits bosonic behavior. And Helium-4 condensate is a composite boson that make up of distinguishable fermions. Moreover, it will be interesting to compare the bosonic quality of 2 fermions experimentally when they are prepared in state (2.38) and (2.21) for indistinguishable and distinguishable particles respectively.

3.5 Comparison of Bosonic Quality for Different Entanglements

The bosonic quality of composite boson make up of 2k distinguishable fermions under GHZ entanglement is the same as composite boson make up of 2 distinguishable fermions. Moreover, it is also the same as 2 indistinguishable fermions composite system.

We have shown that under maximum entanglement, the bosonic quality of the composite system under the same parameter L follows

$$\mathcal{M}_2 = \mathcal{M}^{(2)}_{2,id} = \mathcal{M}^{(2)}_{2k,GHZ} > \mathcal{M}^{(2)}_{4,W} > \mathcal{M}^{(2)}_{6,W} > \mathcal{M}^{(2)}_{8,W} > \dots \tag{3.20}$$

Moreover, although not shown in this paper, the relation can be postulated to hold for N composite bosons.

$$\mathcal{M}_N = \mathcal{M}^{(N)}_{2,id} = \mathcal{M}^{(N)}_{2k,GHZ} \overset{?}{>} \mathcal{M}^{(N)}_{4,W} \overset{?}{>} \mathcal{M}^{(N)}_{6,W} \overset{?}{>} \mathcal{M}^{(N)}_{8,W} \overset{?}{>} \dots \tag{3.21}$$

Where $\overset{?}{>}$ requires future prove.

The above result simply says that under maximum entanglement, the bosonic quality of 2-distinguishable, 2-indistinguishable, and 2k GHZ indistinguishable state is the same, and all of them are more bosonic than W state under the same parameter L.

Below is the plot of $\mathcal{M}$ against L for 2 composite boson system.

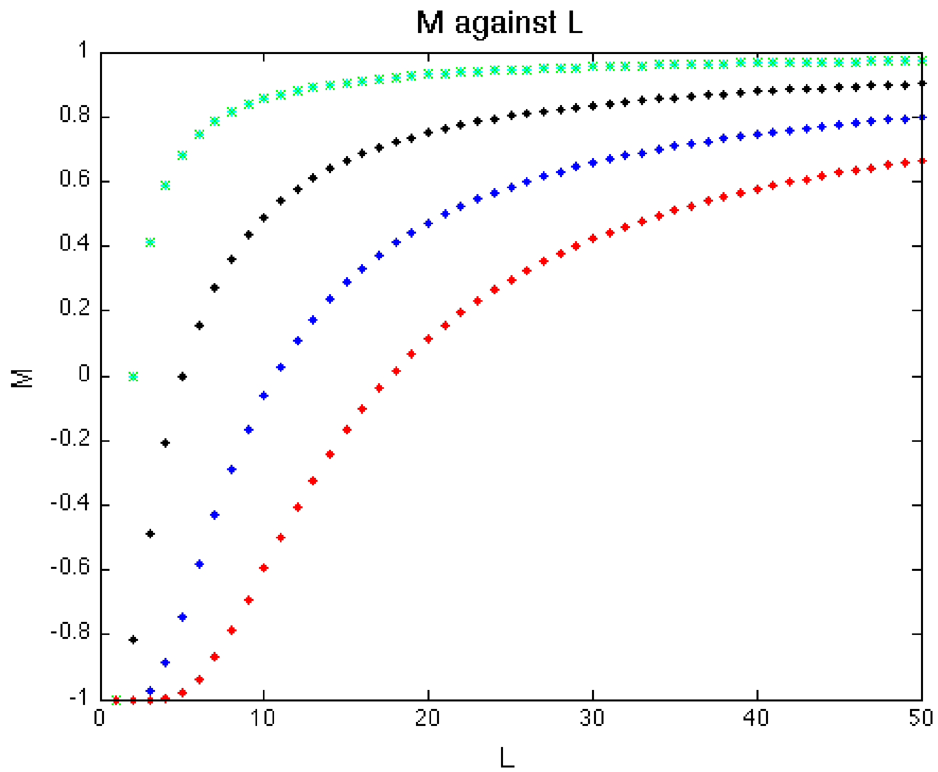

FIGURE 3.2: It should be noted that while L for GHZ entanglement represents L dimension, L for W entanglement represents $\binom{L+2k-1}{2k-1}$ dimensions, where 2k is the number of fermions. Green dots - GHZ, black dots - 4 fermion W, blue dots - 6 fermion W, red dots - 8 fermion W.

As mention in the previous section, there is a transition from fermionic to bosonic when the curve cross the the $\mathcal{M} = 0$ line. For any two particle state, the lowest order of entanglement is $L = 2$. From the above Figure, we see that $\mathcal{M}(2) = 0$, which means a two particle state or any multipartite GHZ state under maximum entanglement reach the transition point with just $L = 2$. This implies that these states are highly bosonic, and in fact more bosonic than W state.

3.6 Future Research

The goal for the future research is to translate the pure mathematical derivation for bosonic quality into more realistic description of real atoms and molecules. And find an experimental implementation of the above results. In addition, we would also like to seek a prove for the postulation (3.21) although the postulation is most likely to hold true. Furthermore, we would also like to extend the result of bosonic quality of bipartite indistinguishable fermions to multipartite indistinguishable fermions.

Chapter 4

Conclusion

In the thesis, we had made some interesting discoveries with respect to bosonic nature of composite bosons. When BEC was proposed about 85 years ago, and BCS about 50 years ago, it is not directly clear that why a group of strongly correlated fermions can exhibit bosonic behavior, later with the advent of quantum information, it seems that such correlation maybe describable by quantum entanglement. Entanglement may not be the only reason for fermions to behave like bosons, but it provides a convincing reason to do so. Since entanglement is not confined by space, you can prepare two spatially separated fermions under entanglement, and together they can behave like a boson. Bringing entanglement into the description of correlations between fermions is something new and highly interesting.

In this thesis, I had shown a few interesting results using the formulation of second quantization. I had shown that if we can entangle 4 or more even number of distinguishable fermions in GHZ state, then irregardless of the number of fermions, they will all behave like 2 indistinguishable fermions under the same entanglement. Furthermore, I had also shown that the bosonic quality of two indistinguishable fermions is exactly the same as two distinguishable fermions. Since quantification of entanglement between indistinguishable fermions is still not fully understood. This result may therefore throw some light into our understanding of entanglements between indistinguishable fermions.

In addition, I had also shown that under maximum W entanglement, there is a particular amount of entanglement that can transit the composite particle from fermionic to bosonic. And the amount of entanglement required for transition is different for W composite particle make up of different number of fermions. Generally for a W composite particle, the transition point occurs at higher dimension as the number of its constituent fermions increases, while for a GHZ composite particle, the transition into bosonic state can occur with a dimension of only 2. Since bosonic quality of GHZ composite particle is the same as any 2-fermion composite particle. This means that any 2 fermion composite particle can exhibit bosonic behavior with just 2 dimension of entanglement, which is the lowest order of entanglement. This means that GHZ entanglement are very strong in producing composite bosons. However GHZ has its disadvantage, being extremely fragile to interactions, it is very difficult to maintain a system in the GHZ state for a long time, since interaction with the environment is a common event in the lab. Therefore there is give and take for the type of composite boson that we want to produce. W being more robust but requires stronger entanglement. GHZ being more bosonic are more fragile.

Appendix A

Derivation of χ_N

For $C^\dagger = \sum_{n=1}^{L} \lambda_n a_n^\dagger b_n^\dagger$ then

$$
\begin{aligned}
&\langle 0|C^N C^{\dagger N}|0\rangle \\
=&\langle 0| \sum_{\substack{s_1,s_2,\ldots,s_N \\ r_1,r_2,\ldots,r_N}} \lambda_{s_1}\lambda_{s_2}\ldots\lambda_{s_N}\lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_N} b_{s_N}a_{s_N}b_{s_{N-1}}a_{s_{N-1}}\ldots b_{s_1}a_{s_1}a^\dagger_{r_1}b^\dagger_{r_1}a^\dagger_{r_2}b^\dagger_{r_2}\ldots a^\dagger_{r_N}b^\dagger_{r_N}|0\rangle \\
=&\langle 0| \sum_{\substack{s_1,s_2,\ldots,s_N \\ r_1,r_2,\ldots,r_N}} \lambda_{s_1}\lambda_{s_2}\ldots\lambda_{s_N}\lambda_{r_1}\lambda_{r_2}\ldots\lambda_{r_N} a_{s_N}a_{s_{N-1}}\ldots a_{s_1}a^\dagger_{r_1}a^\dagger_{r_2}\ldots a^\dagger_{r_N} \\
&\quad b_{s_N}b_{s_{N-1}}\ldots b_{s_1}b^\dagger_{r_1}b^\dagger_{r_2}\ldots b^\dagger_{r_N}|0\rangle \\
=&N!\langle 0| \sum_{r_1,r_2,\ldots,r_N} \lambda^2_{r_1}\lambda^2_{r_2}\ldots\lambda^2_{r_N} a_{r_N}a_{r_{N-1}}\ldots a_{r_1}a^\dagger_{r_1}a^\dagger_{r_2}\ldots a^\dagger_{r_N}|0\rangle \\
=&N!\langle 0| \sum_{r_1<r_2<\cdots<r_N} \lambda^2_{r_1}\lambda^2_{r_2}\ldots\lambda^2_{r_N} a_{r_N}a_{r_{N-1}}\ldots a_{r_1}a^\dagger_{r_1}a^\dagger_{r_2}\ldots a^\dagger_{r_N}|0\rangle \\
=&(N!)^2 \sum_{r_1<r_2<\cdots<r_N} \lambda^2_{r_1}\lambda^2_{r_2}\ldots\lambda^2_{r_N}
\end{aligned}
$$

where $s_1, s_2, \ldots, s_N, r_1, r_2, \ldots, r_N \in \{1, 2, \ldots, L\}$

Therefore χ_N is equal to

$$\chi_N = N! \sum_{r_1<r_2<\cdots<r_N} \lambda^2_{r_1}\lambda^2_{r_2}\ldots\lambda^2_{r_N}$$

Appendix B

Derivation of $\chi^{(N)}_{2k,GHZ}$

For $C^{\dagger}_{2k,GHZ} = \sum_{n=1}^{L} \lambda_n a_n^{(1)\dagger} a_n^{(2)\dagger} \ldots a_n^{(2k)\dagger}$ then

$$
\begin{aligned}
&\langle 0| C^{N}_{2k,GHZ} C^{\dagger N}_{2k,GHZ} |0\rangle \\
=&\langle 0| \sum \lambda_{r_N}\lambda_{r_{N-1}}\ldots\lambda_{r_1}\lambda_{s_1}\lambda_{s_2}\ldots\lambda_{s_N}(a^{(1)}_{r_N}a^{(1)}_{r_{N-1}}\ldots a^{(1)}_{r_1}a^{(1)\dagger}_{s_1}a^{(1)\dagger}_{s_2}\ldots a^{(1)\dagger}_{s_N}) \\
&(a^{(2)}_{r_N}a^{(2)}_{r_{N-1}}\ldots a^{(2)}_{r_1}a^{(2)\dagger}_{s_1}a^{(2)\dagger}_{s_2}\ldots a^{(2)\dagger}_{s_N})\ldots(a^{(2k)}_{r_N}a^{(2k)}_{r_{N-1}}\ldots a^{(2k)}_{r_1}a^{(2k)\dagger}_{s_1}a^{(2k)\dagger}_{s_2}\ldots a^{(2k)\dagger}_{s_N})|0\rangle \\
=&N!\langle 0| \sum |\lambda_{r_1}|^2|\lambda_{r_2}|^2\ldots|\lambda_{r_N}|^2(a^{(1)}_{r_N}a^{(1)}_{r_{N-1}}\ldots a^{(1)}_{r_1}a^{(1)\dagger}_{r_1}a^{(1)\dagger}_{r_2}\ldots a^{(1)\dagger}_{r_N}) \\
&(a^{(2)}_{r_N}a^{(2)}_{r_{N-1}}\ldots a^{(2)}_{r_1}a^{(2)\dagger}_{r_1}a^{(2)\dagger}_{r_2}\ldots a^{(2)\dagger}_{r_N})\ldots(a^{(2k-1)}_{r_N}a^{(2k-1)}_{r_{N-1}}\ldots a^{(2k-1)}_{r_1}a^{(2k-1)\dagger}_{r_1}a^{(2k-1)\dagger}_{r_2}\ldots a^{(2k-1)\dagger}_{r_N})|0\rangle \\
=&N!\langle 0| \sum |\lambda_{r_1}|^2|\lambda_{r_2}|^2\ldots|\lambda_{r_N}|^2(a^{(1)}_{r_N}a^{(1)}_{r_{N-1}}\ldots a^{(1)}_{r_1}a^{(1)\dagger}_{r_1}a^{(1)\dagger}_{r_2}\ldots a^{(1)\dagger}_{r_N}) \\
&(a^{(2)}_{r_N}a^{(2)}_{r_{N-1}}\ldots a^{(2)}_{r_1}a^{(2)\dagger}_{r_1}a^{(2)\dagger}_{r_2}\ldots a^{(2)\dagger}_{r_N})\ldots(a^{(2k-2)}_{r_N}a^{(2k-2)}_{r_{N-1}}\ldots a^{(2k-2)}_{r_1}a^{(2k-2)\dagger}_{r_1}a^{(2k-2)\dagger}_{r_2}\ldots a^{(2k-2)\dagger}_{r_N}) \\
&[\delta_{r_1r_1}\delta_{r_2r_2}\ldots\delta_{r_Nr_N} + (\text{sum of impure delta sets})]\,|0\rangle \\
=&N!\langle 0| \sum |\lambda_{r_1}|^2|\lambda_{r_2}|^2\ldots|\lambda_{r_N}|^2(a^{(1)}_{r_N}a^{(1)}_{r_{N-1}}\ldots a^{(1)}_{r_1}a^{(1)\dagger}_{r_1}a^{(1)\dagger}_{r_2}\ldots a^{(1)\dagger}_{r_N}) \\
&(a^{(2)}_{r_N}a^{(2)}_{r_{N-1}}\ldots a^{(2)}_{r_1}a^{(2)\dagger}_{r_1}a^{(2)\dagger}_{r_2}\ldots a^{(2)\dagger}_{r_N})\ldots(a^{(2k-2)}_{r_N}a^{(2k-2)}_{r_{N-1}}\ldots a^{(2k-2)}_{r_1}a^{(2k-2)\dagger}_{r_1}a^{(2k-2)\dagger}_{r_2}\ldots a^{(2k-2)\dagger}_{r_N})|0\rangle \\
=&N!\langle 0| \sum |\lambda_{r_1}|^2|\lambda_{r_2}|^2\ldots|\lambda_{r_N}|^2(a^{(1)}_{r_N}a^{(1)}_{r_{N-1}}\ldots a^{(1)}_{r_1}a^{(1)\dagger}_{r_1}a^{(1)\dagger}_{r_2}\ldots a^{(1)\dagger}_{r_N})|0\rangle \\
=&(N!)^2 \sum_{r_1<r_2<\cdots<r_N} |\lambda_{r_1}|^2|\lambda_{r_2}|^2\ldots|\lambda_{r_N}|^2
\end{aligned}
$$

where $s_1, s_2, \ldots, s_N, r_1, r_2, \ldots, r_N \in \{1, 2, \ldots, L\}$

Which means

$$\chi^{(N)}_{2k,GHZ} = N! \sum_{r_1<r_2<\cdots<r_N} |\lambda_{r_1}|^2|\lambda_{r_2}|^2\ldots|\lambda_{r_N}|^2$$

and (sum of impure delta sets) will relabel at least two of the fermionic creation operator from the set $a^{(2k-2)\dagger}_{r_1} a^{(2k-2)\dagger}_{r_2} \dots a^{(2k-2)\dagger}_{r_N}$ with the same quantum number. Since by Pauli Exclusion Principle, we cannot add 2 fermions with the same quantum number to the same system. Therefore

$$(a^{(1)}_{r_N} a^{(1)}_{r_{N-1}} \dots a^{(1)}_{r_1} a^{(1)\dagger}_{r_1} a^{(1)\dagger}_{r_2} \dots a^{(1)\dagger}_{r_N})(a^{(2)}_{r_N} a^{(2)}_{r_{N-1}} \dots a^{(2)}_{r_1} a^{(2)\dagger}_{r_1} a^{(2)\dagger}_{r_2} \dots a^{(2)\dagger}_{r_N})$$
$$\dots (a^{(2k-2)}_{r_N} a^{(2k-2)}_{r_{N-1}} \dots a^{(2k-2)}_{r_1} a^{(2k-2)\dagger}_{r_1} a^{(2k-2)\dagger}_{r_2} \dots a^{(2k-2)\dagger}_{r_N})(\text{sum of impure delta sets})|0\rangle = 0$$

By doing so, we can collapse the number of indistinguishable fermions by one. Repeating the same process as above. We can ultimately collapse the number of creation operators to that of two indistinguishable fermions. Therefore yielding $\chi^{(N)}_{2k,GHZ} = \chi_N$.

Appendix C

Derivation of $\mathcal{M}^{(2)}_{2k,W}$

The approach of finding $\mathcal{M}^{(2)}_{2k,W}$ starts by finding $\mathcal{M}^{(2)}_{4,W}$ first, and through the pattern exhibit in the result for $\mathcal{M}^{(2)}_{4,W}$, $\mathcal{M}^{(2)}_{2k,W}$ becomes clear. First derive the explicit form for $\chi^{(2)}_{4,W}$. $C^\dagger_{4,W}$ is defined as

$$C^\dagger_{4,W} = \sum_{n_1+n_2+n_3+n_4=L} \lambda_{n_1n_2n_3n_4} a^\dagger_{n_1} b^\dagger_{n_2} c^\dagger_{n_3} d^\dagger_{n_4}$$

where $n_1, n_2, n_3, n_4 \in \{0, 1, 2, \ldots, L\}$. To save notation, I define $n_1+n_2+n_3+n_4 = \tilde{n}$ and $\lambda_{n_1n_2n_3n_4} = \lambda_{\ddot{n}}$

$$\begin{aligned}
2\chi^{(2)}_{4,W} =& \langle 0|C^2_{4,W} C^{\dagger 2}_{4,W}|0\rangle \\
=& \langle 0| \sum_{\tilde{n},\tilde{m},\tilde{r},\tilde{s}=L} \lambda_{\ddot{n}}\lambda_{\ddot{m}}\lambda^*_{\ddot{r}}\lambda^*_{\ddot{s}} a_{m_1}a_{n_1}a^\dagger_{r_1}a^\dagger_{s_1} b_{m_2}b_{n_2}b^\dagger_{r_2}b^\dagger_{s_2} c_{m_3}c_{n_3}c^\dagger_{r_3}c^\dagger_{s_3} d_{m_4}d_{n_4}d^\dagger_{r_4}d^\dagger_{s_4}|0\rangle \\
=& \langle 0| \sum_{\tilde{n},\tilde{m},\tilde{r},\tilde{s}=L} \lambda_{\ddot{n}}\lambda_{\ddot{m}}\lambda^*_{\ddot{r}}\lambda^*_{\ddot{s}} (\delta_{n_1r_1}\delta_{m_1s_1} - \delta_{m_1r_1}\delta_{n_1s_1})(\delta_{n_2r_2}\delta_{m_2s_2} - \delta_{m_2r_2}\delta_{n_2s_2}) \\
& (\delta_{n_3r_3}\delta_{m_3s_3} - \delta_{m_3r_3}\delta_{n_3s_3})(\delta_{n_4r_4}\delta_{m_4s_4} - \delta_{m_4r_4}\delta_{n_4s_4})|0\rangle \\
=& \sum_{\tilde{r},\tilde{s}=L} \big[\lambda_{r_1r_2r_3r_4}\lambda_{s_1s_2s_3s_4} + \lambda_{s_1s_2s_3s_4}\lambda_{r_1r_2r_3r_4} \\
& + (-\lambda_{s_1r_2r_3r_4}\lambda_{r_1s_2s_3s_4} - \lambda_{r_1s_2r_3r_4}\lambda_{s_1r_2s_3s_4} - \lambda_{r_1r_2s_3r_4}\lambda_{s_1s_2r_3s_4} - \lambda_{r_1r_2r_3s_4}\lambda_{s_1s_2s_3r_4}) \\
& + (\lambda_{s_1s_2r_3r_4}\lambda_{r_1r_2s_3s_4} + \lambda_{s_1r_2s_3r_4}\lambda_{r_1s_2r_3s_4} + \lambda_{s_1r_2r_3s_4}\lambda_{r_1s_2s_3r_4} \\
& + \lambda_{r_1s_2s_3r_4}\lambda_{s_1r_2r_3s_4} + \lambda_{r_1s_2r_3s_4}\lambda_{s_1r_2s_3r_4} + \lambda_{r_1r_2s_3s_4}\lambda_{s_1s_2r_3r_4}) \\
& + (-\lambda_{s_1s_2s_3r_4}\lambda_{r_1r_2r_3s_4} - \lambda_{s_1s_2r_3s_4}\lambda_{r_1r_2s_3r_4} - \lambda_{s_1r_2s_3s_4}\lambda_{r_1s_2r_3r_4} - \lambda_{r_1s_2s_3s_4}\lambda_{s_1r_2r_3r_4})\big]\lambda^*_{\ddot{r}}\lambda^*_{\ddot{s}}
\end{aligned}$$

If we assume $\lambda_{\ddot{n}} = \lambda$ is constant. By normalization condition $\langle 0|C_{4,W}C^{\dagger}_{4,W}|0\rangle = 1$, we get $\sum_{\tilde{n}=L} |\lambda_{\ddot{n}}|^2 = 1$, which gives $\lambda = \binom{L+3}{3}$. Evaluating of $2\mathcal{M}^{(2)}_{4,W} = \langle 0|C^2_{4,W_{max}}C^{\dagger 2}_{4,W_{max}}|0\rangle$ gives

$$
\begin{aligned}
2\mathcal{M}^{(2)}_{4,W} = &\sum_{\tilde{r},\tilde{s}=L} \Big[2\lambda_{r_1r_2r_3r_4}\lambda_{s_1s_2s_3s_4} - \binom{4}{1}\lambda_{s_1r_2r_3r_4}\lambda_{r_1s_2s_3s_4} \\
&+ \binom{4}{2}\lambda_{s_1s_2r_3r_4}\lambda_{r_1r_2s_3s_4} - \binom{4}{3}\lambda_{s_1s_2s_3r_4}\lambda_{r_1r_2r_3s_4}\Big]\lambda^*_{\ddot{r}}\lambda^*_{\ddot{s}} \\
=&2 + \lambda^4\Bigg\{ - \binom{4}{1}\left[\left(\binom{L+2}{2}\binom{0}{0}\right)^2 + \left(\binom{L+1}{2}\binom{1}{0}\right)^2 + ... + \left(\binom{2}{2}\binom{L}{0}\right)^2\right] \\
&+ \binom{4}{2}\left[\left(\binom{L+1}{1}\binom{1}{1}\right)^2 + \left(\binom{L}{1}\binom{2}{1}\right)^2 + ... + \left(\binom{1}{1}\binom{L+1}{1}\right)^2\right] \\
&- \binom{4}{3}\left[\left(\binom{L}{0}\binom{2}{2}\right)^2 + \left(\binom{L-1}{0}\binom{3}{2}\right)^2 + ... + \left(\binom{0}{0}\binom{L+2}{2}\right)^2\right]\Bigg\}
\end{aligned}
$$

From the result of $\mathcal{M}^{(2)}_{4,W}$, $\mathcal{M}^{(2)}_{2k,W}$ becomes clear

$$
\begin{aligned}
2\mathcal{M}^{(2)}_{2k,W} =&2 + \lambda^4\Bigg\{(-1)\binom{2k}{1}\Bigg[\left(\binom{L+2k-2}{2k-2}\binom{0}{0}\right)^2 \\
&+ \left(\binom{L+2k-3}{2k-2}\binom{1}{0}\right)^2 + \cdots + \left(\binom{2k-2}{2k-2}\binom{L}{0}\right)^2\Bigg] \\
&+ (-1)^2\binom{2k}{2}\Bigg[\left(\binom{L+2k-3}{2k-3}\binom{1}{1}\right)^2 \\
&+ \left(\binom{L+2k-4}{2k-3}\binom{2}{1}\right)^2 + ... + \left(\binom{2k-3}{2k-3}\binom{L+1}{1}\right)^2\Bigg] \\
&+ (-1)^3\binom{2k}{3}\Bigg[\left(\binom{L+2k-4}{2k-4}\binom{2}{2}\right)^2 \\
&+ \left(\binom{L+2k-5}{2k-4}\binom{3}{2}\right)^2 + ... + \left(\binom{2k-4}{2k-4}\binom{L+2}{2}\right)^2\Bigg] \\
&+ \\
&\cdots \\
&+ (-1)^{2k-1}\binom{2k}{2k-1}\Bigg[\left(\binom{L}{0}\binom{2k-2}{2k-2}\right)^2 \\
&+ \left(\binom{L-1}{0}\binom{2k-1}{2k-2}\right)^2 + ... + \left(\binom{0}{0}\binom{L+2k-2}{2k-2}\right)^2\Bigg]\Bigg\}
\end{aligned}
$$

Therefore we have

$$\mathcal{M}^{(2)}_{2k,W}(L) = 1 + \frac{\lambda^4}{2}\sum_{s=1}^{2k-1}\sum_{r=0}^{L}\left\{(-1)^s\binom{2k}{s}\left[\binom{L+2k-1-r-s}{2k-1-s}\binom{r+s-1}{s-1}\right]^2\right\}$$

Appendix D

Derivation of $\chi_{2,id}^{(N)}$

For $C_{2,id}^{\dagger} = \sum_{k=1}^{L} \lambda_k a_{2k-1}^{\dagger} a_{2k}^{\dagger}$, we firstly find

$$
\begin{aligned}
&C_{2,id}^{\dagger N}|0\rangle \\
&= \sum_{m_1,m_2,\ldots,m_N} \lambda_{m_1}\lambda_{m_2}\ldots\lambda_{m_N} a_{2m_1-1}^{\dagger}a_{2m_1}^{\dagger}a_{2m_2-1}^{\dagger}a_{2m_2}^{\dagger}\ldots a_{2m_N-1}^{\dagger}a_{2m_N}^{\dagger}|0\rangle \\
&= N! \sum_{m_1<m_2<\cdots<m_N} \lambda_{m_1}\lambda_{m_2}\ldots\lambda_{m_N} a_{2m_1-1}^{\dagger}a_{2m_1}^{\dagger}a_{2m_2-1}^{\dagger}a_{2m_2}^{\dagger}\ldots a_{2m_N-1}^{\dagger}a_{2m_N}^{\dagger}|0\rangle
\end{aligned}
$$

where $m_1, m_2, \ldots, m_N \in \{1, 2, \ldots, L\}$

Since the set of fermionic states $a_{m_1}^{\dagger}a_{m_2}^{\dagger}\ldots a_{m_N}^{\dagger}|0\rangle$ whereby $m_1 < m_2 < \cdots < m_N$ are orthogonal. Therefore

$$
\begin{aligned}
N!\chi_{2,id}^{(N)} =& \langle 0|C_{2,id}^{N} C_{2,id}^{\dagger N}|0\rangle \\
=&(N!)^2 \sum_{m_1<m_2<\cdots<m_N} |\lambda_{m_1}|^2|\lambda_{m_2}|^2\ldots|\lambda_{m_N}|^2
\end{aligned}
$$

And so we have

$$
\chi_{2,id}^{(N)} = N! \sum_{m_1<m_2<\cdots<m_N} |\lambda_{m_1}|^2|\lambda_{m_2}|^2\ldots|\lambda_{m_N}|^2
$$

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