Diminishing returns

Planning ahead

Prerequisite knowledge

- Halving in a spatial context

Why do this problem?

The problem helps to extend pupils' familiarity with fractions but has the potential to develop into work on limits. The concept of limits is intriguing and can stimulate discussion as well as the idea that something that goes on forever can in fact have boundaries. Limits are often taught in abstract terms, but these concrete examples are a good introduction to what is not such a difficult concept. At the deepest level some explanation of how to find exact limits is given as an extension.

The problem also gives opportunities to pay particular attention to the evaluation phase of the problem-solving model.

Time

Two lessons

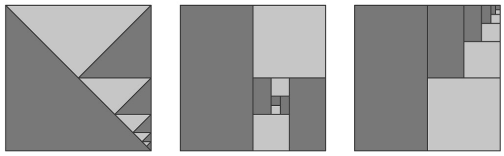
Resources

Paper in two different colours; scissors
CD-ROM: problem sheet; animation
NRICH website (optional): www.nrich.maths.org, September 1997, 'Zooming in on the squares'

Diminishing returns

Problem sheet

The challenge is to recreate one of these three patterns with a partner.



Before you get started

You need two sheets of differently coloured paper (here we chose purple and pink). By folding and cutting, make two squares of the same size.

Making a pattern

Decide which colour square you are not going to cut up. It will form the base of your pattern. The other square you will cut and paste onto the base.

Decide which pattern (from the three above) you are going to recreate with your partner. Work together to make one copy.

On each part of the pattern (including the bits of the base showing through) write the fraction of the whole pattern it represents.

Make a note of each of the fractions as you go along.

Can you estimate what fraction of the whole square each colour covers?

Using the fractions that you have noted down, can you write down a sum that represents your estimate?

| Maths Trails: Visualising | Problem and resource sheets

Introducing the problem

Before playing the animation, tell pupils you will ask them to describe what they saw – including how they might recreate the 'final' pattern. Play the animation through twice without pausing.

Discuss the sequence they have seen by asking:

- How did it start?
- What happened first?
- Then what happened?
- What colours were used?
- How were the colours significant?

Replay the animation, pausing to discuss and clarify points that have arisen in the earlier discussion, drawing out the system to the developing pattern, the fractions shaded, and so on.

Main part of the lesson

The aim of this activity (which may take two lessons) is for pupils to recreate the animation they have just seen (or a similar one – see the problem sheet) using coloured paper.

Afterwards, talk to pupils about how they recreated the patterns, drawing out issues such as:

- how pupils visualised what happened and how it helped them to plan what they would do;
- continued halving;

- the fact that even though the patterns look different, they use the same fractions;
- the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$;
- the sum of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ is 1;
- the recognition that $\frac{1}{2}, \frac{1}{8}, \ldots$ will be in one colour and $\frac{1}{4}, \frac{1}{16}, \ldots$ will be in the other colour;
- pupils' estimations of the fraction of the original square covered by each colour.

Some pupils may be able to make the connection between the sums of the series that represent each colour and their estimations of the limits which should sum to 1.

Plenary

Ask the class to think about what happens to the fraction if you add a large green square to the bottom edge of the animation. Can they extend this idea to the triangles in square pattern?

Similar animations using triangles can be found in the 'More ideas' section, under 'Triangles, triangles, triangles, … '.

Solution notes

Here are some ideas for explanations for more able pupils:

You can think of each pattern as made up from: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ or as two halves, that is, $\frac{1}{2} + (\frac{1}{4} + \frac{1}{8} + \ldots)$

Equating these:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \frac{1}{2} + \frac{1}{2}(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots)$$

Rearranging:

$$1(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots) - \frac{1}{2}(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots) = \frac{1}{2}$$

Therefore $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ must equal 1.

Looking at the pink areas only on the problem sheet, the total area of the pink,

$$P = \frac{1}{4} + \frac{1}{16} + \ldots$$
$$P = \frac{1}{4}(1 + \frac{1}{4} + \frac{1}{16} + \ldots)$$
$$\therefore\ P = \frac{1}{4}(1 + P)$$
$$\therefore\ 4P = 1 + P$$
$$P = \frac{1}{3}.$$