Core 3 June 2007

1. i) Product rule

$u = x^3 \qquad v = (x+1)^5$

Product rule is

$$\frac{duv}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$\frac{du}{dx} = 3x^2 \qquad \frac{dv}{dx} = 5 \times 1(x+1)^4$

$$(x+1)^5\, 3x^2 + x^3 \times 5(x+1)^4$$

$$= 3x^2(x+1)^5 + 5x^3(x+1)^4$$

ii) $y = (3x^4+1)^{1/2} \qquad y = u^{1/2}$

Use chain rule $\quad u = 3x^4+1$

then $\quad \frac{du}{dx} = 12x^3 \qquad \frac{dy}{du} = \frac{1}{2}u^{-1/2}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 12x^3 \times \frac{1}{2}u^{-1/2}$$

$$= 6x^3(3x^4+1)^{-1/2}$$

2. Critical values $\frac{3}{4}$ from $|4x-3|$

$-\frac{1}{2}$ from $|2x+1|$

if $x < -\frac{1}{2} \qquad |4x-3|$ is $-(4x-3)$

$|2x+1|$ is $-(2x+1)$

So $\quad |4x-3| < |2x+1|$

becomes $\quad -(4x-3) < -(2x+1)$

$$-4x+3 < -2x-1$$

$$3 < 2x-1$$

$$\cancel{4}\,2 < 2x$$

$$2 < x$$

But if $x < -\frac{1}{2}$ x cannot be bigger than 2 so this does not provide an answer.

if $\quad -\frac{1}{2} < x < \frac{3}{4} \qquad |4x-3|$ is $-(4x-3)$

$|2x+1|$ is $2x+1$

So $\quad -(4x-3) < 2x+1$

$$-4x+3 < 2x+1$$

$$2 < 6x$$

$$\frac{2}{6} < x$$

$\frac{1}{3} < x$

So if $\frac{-1}{2} < x < \frac{3}{4}$ then $x > \frac{1}{3}$

Which together make $\underline{\underline{\frac{1}{3} < x < \frac{3}{4}}}$

if $x > \frac{3}{4}$ $|4x-3|$ is $4x-3$

$|2x+1|$ is $2x+1$

So $4x-3 < 2x+1$

$2x < 4$

$x <$

Which together make $\underline{\underline{\frac{3}{4} < x <}}$

So $\underline{\frac{1}{3} < x < 2}$

Alternatively as both sides of the inequality are positive you can square both sides

$|4x-3| < |2x+1|$

$(4x-3)^2 < (2x+1)^2$

$16x^2 - 24x + 9 < 4x^2 + 4x + 1$

$12x^2 - 28x + 8 < 0$

$4(3x^2 - 7x + 2) < 0$

$3x^2 - 7x + 2 < 0$

$(3x-1)(x-2) < 0$

$\underline{\underline{\frac{1}{3} < x < 2}}$

3 i) $f(169) = 3 + \sqrt{169}$

$= 3 + 13 = 16$

$ff(169) = f(16)$

$= 3 + \sqrt{16}$

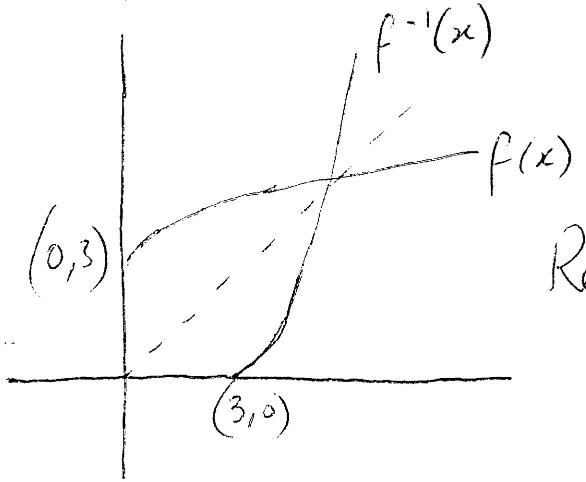
$= 3 + 4 = \underline{\underline{7}}$

ii) Let $y = 3 + \sqrt{x}$

$y - 3 = \sqrt{x}$

$(y-3)^2 = x$

$\therefore f^{-1}(x) = (x-3)^2$



Reflection in line $y = x$.

(4) $I = \int_0^{13} (2x+1)^{1/3} \, dx$

$= \left[\frac{3}{4} \times \frac{1}{2} (2x+1)^{4/3}\right]_0^{13}$

Inc power
÷ by power
÷ by coeff of ~~x~~

$= \left[\frac{3}{8} (2x+1)^{4/3}\right]_0^{13}$

$= \frac{3}{8} 27^{4/3} - \frac{3}{8} 1^{4/3}$

$= \frac{3}{8} 3^4 - \frac{3}{8} = \frac{3}{8} \times 81 - \frac{3}{8}$

$= \frac{243}{8} - \frac{3}{8} = \frac{240}{8} = 30$

ii) $y_0 = 1$

$y_1 = 14^{1/3}$

$y_2 = 27^{1/3} = 3$

Area $= \frac{1}{3} \times 6.5 \left(1 + 4 \times \sqrt[3]{14} + 3\right)$

$= 29.555$

So 29.6 to 3 sig figs.

5 i) $m = 240e^{-0.04t}$

at $t = 0$ $\quad m = 240$

So find t

$120 = 240e^{-0.04t}$

$0.5 = e^{-0.04t}$

$\ln 0.5 = -0.04t \ln e$ $\quad 1$

$\dfrac{\ln 0.5}{-0.04} = t = 17.3$ years

ii) $\dfrac{dM}{dt} = 240 \times {}^{-}0.04e^{-0.04t}$

$= {}^{-}9.6e^{-0.04t}$

Equate to ${}^{-}2.1$ ← 'rate of change' is -ve

$-2.1 = -9.6e^{-0.04t}$

$\dfrac{2.1}{9.6} = e^{-0.04t}$

$\ln\left(\dfrac{2.1}{9.6}\right) = -0.04t$

$t = \dfrac{\ln\left(\frac{2.1}{9.6}\right)}{-0.04} = 37.9956\ldots$

$= 38$ years

6 i) $\displaystyle\int_0^a \left(6e^{2x} + x\right).dx = 42$

$\left[3e^{2x} + \dfrac{x^2}{2}\right]_0^a = 42$

$\left(3e^{2a} + \dfrac{a^2}{2}\right) - \left(3 + 0\right) = 42$

$3e^{2a} + \dfrac{a^2}{2} - 3 = 42$

$3e^{2a} + \dfrac{a^2}{2} = 45$

$6e^{2a} + a^2 = 90$

$e^{2a} + \dfrac{a^2}{6} = 15$

$$e^{2a} = 15 - \frac{a^2}{6}$$

$$\ln\left(e^{2a}\right) = \ln\left(15 - \frac{a^2}{6}\right)$$

$$2a = \ln\left(15 - \frac{a^2}{6}\right)$$

$$a = \frac{1}{2}\ln\left(15 - \frac{a^2}{6}\right)$$

ii)

$$a = \frac{1}{2}\ln\left(15 - \frac{1}{6}\right) = 1.348438451$$

$$a = \frac{1}{2}\ln\left(15 - \frac{(ans)^2}{6}\right) = 1.3438\ldots$$

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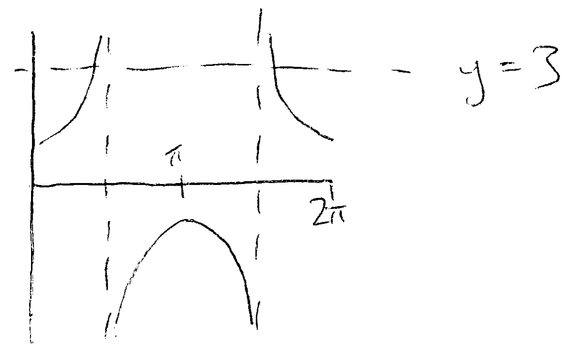
$$\underline{\underline{1.344}}$$

⑦ i)



ii) $\sec x = 3$

$$\frac{1}{\cos x} = 3 \Rightarrow \cos x = \frac{1}{3}$$

$$\cos^{-1}\left(\frac{1}{3}\right) = x = \underline{1.23 \text{ rads}}$$

$$\text{or } 2\pi - 1.23 \; \cancel{\text{or}} \; \underline{5.05 \text{ rads}}$$

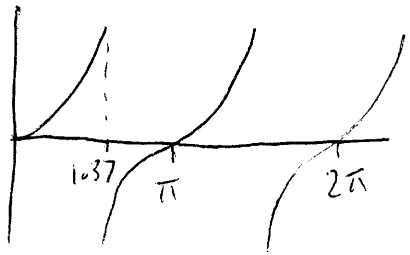
iii) $\sec\theta = 5\,\text{cosec}\,\theta$

$$\frac{1}{\cos\theta} = \frac{5}{\sin\theta}$$

$$\frac{\sin\theta}{\cos\theta} = 5$$

$$\tan\theta = 5$$

$$\tan^{-1}(5) = \theta = \underline{1.37 \text{ rads}}$$



$\underline{\underline{\omega}}$ $\quad \pi + 1.37 = \underline{4.51 \text{ rads}}$

(8) i) $y = \dfrac{4\ln x - 3}{4\ln x + 3}$ $\begin{matrix} \leftarrow u \\ \leftarrow v \end{matrix}$ Quotient Rule

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} \quad u\frac{dv}{dx}}{v^2}$$

$u = 4\ln x - 3 \qquad v = 4\ln x + 3$

$\dfrac{du}{dx} = \dfrac{4}{x} \qquad \dfrac{dv}{dx} = \dfrac{4}{x}$

$$\frac{dy}{dx} = \frac{(4\ln x + 3)\frac{4}{x} - (4\ln x - 3)\frac{4}{x}}{(4\ln x + 3)^2}$$

$$= \frac{(16\ln x - 12) - (16\ln x - 12)}{x(4\ln x + 3)^2}$$

$$= \frac{24}{x(4\ln x + 3)^2}$$

ii) Crosses x-axis if $y = 0$

$\therefore \quad 4\ln x - 3 = 0$

$\ln x = \dfrac{3}{4}$

$e^{\ln x} = e^{3/4}$

$x = e^{3/4}$

at $x = e^{3/4}$

$$\frac{dy}{dx} = \frac{24}{e^{3/4}(4\ln e^{3/4} + 3)^2}$$

$$= \frac{24}{e^{3/4}(4 \times \frac{3}{4} + 3)^2}$$

$$= \frac{24}{e^{3/4} \times 6^2} = \frac{2}{3e^{3/4}}$$

$$= \underline{\tfrac{2}{3}e^{-3/4}}$$

iii) $\int_1^e \pi y^2 \, dx$

$\int_1^e \pi \left(\frac{2}{x^{1/2}(4\ln x+3)}\right)^2 . dx$

$= \int_1^e \pi \frac{4}{x(4\ln x+3)^2} . dx$ make it look like part (i)

$= \int_1^e \frac{\pi}{6} \times \frac{24}{x(4\ln x+3)^2} . dx$

$= \frac{\pi}{6}\int_1^e \frac{24}{x(4\ln x+3)^2} . dx$

$= \frac{\pi}{6}\left[\frac{4\ln x - 3}{4\ln x+3}\right]_1^e$

$= \frac{\pi}{6}\left[\left(\frac{4\ln e - 3}{4\ln e+3}\right) - \left(\frac{4\ln 1 - 3}{4\ln 1 + 3}\right)\right]$

$= \frac{\pi}{6}\left[\frac{1}{7} - \frac{-3}{3}\right] = \frac{\pi}{6} \times \frac{8}{7}$

$= \frac{8\pi}{42} = \frac{4\pi}{21}$

9 $\tan(\theta+60)\tan(\theta-60)$

$\tan 60 = \sqrt{3}$

$\tan(\theta+60) = \frac{\tan\theta + \tan 60}{1-\tan\theta\tan 60}$

$= \frac{\tan\theta + \sqrt{3}}{1-\sqrt{3}\tan\theta}$

$\tan(\theta-60) = \frac{\tan\theta - \tan 60}{1+\tan\theta\tan 60}$

$= \frac{\tan\theta - \sqrt{3}}{1+\sqrt{3}\tan\theta}$

So $\tan(\theta+60)\tan(\theta-60)$

$= \left(\frac{\tan\theta+\sqrt{3}}{1-\sqrt{3}\tan\theta}\right)\left(\frac{\tan\theta-\sqrt{3}}{1+\sqrt{3}\tan\theta}\right)$

$$= \frac{\tan^2\theta - 3}{1 - 3\tan^2\theta}$$

ii) $\tan(\theta+60)\tan(\theta-60) = 4\sec^2\theta - 3$

$$\frac{\tan^2\theta - 3}{1 - 3\tan^2\theta} = 4\sec^2\theta - 3$$

$$(\tan^2\theta - 3) = (4\sec^2\theta - 3)(1 - 3\tan^2\theta)$$

$$(\tan^2\theta - 3) = (4(1+\tan^2\theta) - 3)(1 - 3\tan^2\theta)$$

$$(\tan^2\theta - 3) = (4 + 4\tan^2\theta - 3)(1 - 3\tan^2\theta)$$

$$(\tan^2\theta - 3) = (1 + 4\tan^2\theta)(1 - 3\tan^2\theta)$$

$$(\tan^2\theta - 3) = -12\tan^4\theta + \tan^2\theta + 1$$

$$12\tan^4\theta - 4 = 0$$

$$12\tan^4\theta = 4 \qquad \tan^4\theta = \frac{1}{3}$$

$$\tan\theta = \pm\frac{1}{\sqrt[4]{3}}$$

$$\tan^{-1}\left(\frac{1}{\sqrt[4]{3}}\right) = \underline{37.2} \quad (1\text{ d.p.})$$

$$\tan^{-1}\left(\frac{-1}{\sqrt[4]{3}}\right) = -37.23$$

But $0 < \theta < 180$

So $-37.23 + 180 = \underline{142.8}$ (1dp)

(iii) $\tan(\theta+60)\tan(\theta-60) = k^2$

$$\frac{\tan^2\theta - 3}{1 - 3\tan^2\theta} = k^2$$

$$\tan^2\theta - 3 = k^2 - 3k^2\tan^2\theta$$

$$\tan^2\theta + 3k^2\tan^2\theta = k^2 + 3$$

$$\tan^2\theta(1 + 3k^2) = k^2 + 3$$

$$\tan^2\theta = \frac{k^2+3}{1+3k^2}$$

$\therefore$ $\tan^2\theta$ = Positive for all values of k

$$\therefore \tan\theta = \pm\sqrt{\frac{k^2+3}{1+3k^2}}$$

So there are 2 values $0 < \theta < 180$