Pick's theorem

Deduction

Prerequisite knowledge

- Basic knowledge of properties of polygons
- Calculating areas of polygons drawn on square dotty paper (possibly by using the area of the surrounding rectangle and subtracting triangles)
- Terms including perimeter, interior, vertices

Why do this problem?

This problem offers opportunities for pupils to choose different strands for investigation, asking their own questions as well as the one(s) you pose. Many of the questions pupils are likely to ask can be picked up again in the plenary.

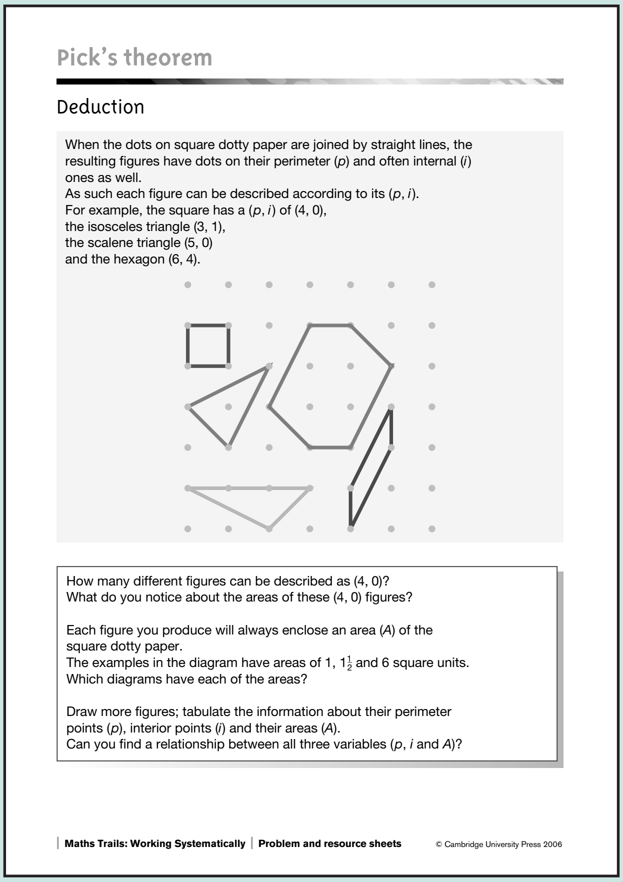
Time

At least two lessons (treated here as a continuous development)

Resources

CD-ROM: problem sheet, resource sheeet on area of polygon.

Square dotty paper, large sheets of paper and a range of other paper to create a poster of pupils' findings



Pick's theorem

Deduction

When the dots on square dotty paper are joined by straight lines, the resulting figures have dots on their perimeter (*p*) and often internal (*i*) ones as well.
As such each figure can be described according to its (*p*, *i*).
For example, the square has a (*p*, *i*) of (4, 0),
the isosceles triangle (3, 1),
the scalene triangle (5, 0)
and the hexagon (6, 4).

How many different figures can be described as (4, 0)?
What do you notice about the areas of these (4, 0) figures?

Each figure you produce will always enclose an area (*A*) of the square dotty paper.
The examples in the diagram have areas of 1, $1\frac{1}{2}$ and 6 square units.
Which diagrams have each of the areas?

Draw more figures; tabulate the information about their perimeter points (*p*), interior points (*i*) and their areas (*A*).
Can you find a relationship between all three variables (*p*, *i* and *A*)?

Maths Trails: Working Systematically | Problem and resource sheets © Cambridge University Press 2006

NRICH website (optional): www.nrich.maths.org, October 2003, 'Pick's theorem', and May 2004, 'Shearly magical'

Introducing the problem

Two key points need to be established before setting pupils off on the task:

- Have a method for obtaining the areas of shapes drawn on a dotty grid. It might be useful to spend some time on calculating areas of shapes, possibly by surrounding polygons with a rectangle and using areas of triangles.
- Make sure the context of the task and possible notation you are using are clear.

Calculating area

Using some square dotty paper, draw a polygon and surround it with a rectangle. Show by calculating areas of surrounding triangles (or rectangles) and the area of the large rectangle you can find the area of the polygon (use the resource sheet or, alternatively, an interactive tool can be found in the problem 'Shearly magical' on the NRICH website). Some of the other shapes on the problem sheet could be used to consolidate this method.

Understanding the problem

Look at the problem sheet. Show the square that is described as $(4, 0)$ and explain the notation. If the notation is confusing to pupils it is possible to simply list the two values for each shape in a table. Look at the other shapes in the diagram to check understanding.

Main activity

State the problem as on the problem sheet:

- Can you find a relationship between the three variables: points on the perimeter (p), points inside the shape (i), area of the shape (A)?

Ask the class to spend ten minutes considering how they might go about tackling the problem. The aim is not to find an answer at this stage.

After ten minutes collate some of the ideas, which might include:

- Break down the problem into an easier problem (not so many variables) by keeping one or two of the variables fixed.
- Only look at one family of shapes in terms of all shapes with the same perimeter and internal dots (p, i).
- Only work with one value of p and then systematically looking at different values of i to create a table of results, and vice-versa.
- Look at all the shapes with the same area.

Ask the groups to pick an idea, formulate a question and decide how they will go about trying to answer the question. Such questions could include:

- For a particular value for (p, i) are there particular areas?
- What shapes can we draw with the same area?
- Are there some limitations on the number of points around the perimeter for different numbers of internal points?
- What do you notice about those figures whose areas are the same?
- What ways are there of increasing the area by 1 unit?

When groups have agreed a question you might consider asking them to share their ideas with other groups and discuss methods for solving them and whether they are 'good' or 'not so good' questions or methods.

The aim is that, by the end of the sessions, pupils will have a poster which displays their findings and attempts to convince the reader that those findings are correct.

Whilst groups are working, encourage them to tabulate information about the perimeter points (p), interior points (i) and their areas (A).

Plenary

There are two things to consider in the plenary:

- a general rule
  - What have the groups noticed?
  - Which findings agree?
  - Is it possible to pull together different findings and come up with an overall rule? Has one group achieved this?
- the clarity and communication of the mathematics in the posters
  - Which presentations were clear and easy to follow?
  - Which presentations offered sound arguments?

Discuss some of the things you felt made an effective solution, which could include:

- the solution was well argued;
- the solution was well presented;
- the solution was consistent with the data;
- the solution was more than pattern spotting but tried to explain what the group discovered.

Solution notes

All (4, 0) shapes have area 1.

There are two families of shapes of the form (4, 0):

One has three points in a line, forming a triangle (the third point can be slid along a line a perpendicular distance of 1 unit away from the line joining the three points). There will never be a point in the middle because the two lines are only one unit apart.

The second family has two adjacent points on one line and two on the next. Any pair of points can be slid along their line without increasing the number of points on the perimeter – forming parallelograms of area 1 square unit.

(p, 0) areas are all of the form $\frac{1}{2}p - 1$

(p, 1) areas are all of the form $\frac{1}{2}p$

(p, 2) areas are of the form $\frac{1}{2}p + 1$

In general $A = \frac{1}{2}p + i - 1$