The filter and the funnel

Modelling and optimisation

Prerequisite knowledge

- Volume and surface area
- Pythagoras' theorem

Why do this unit?

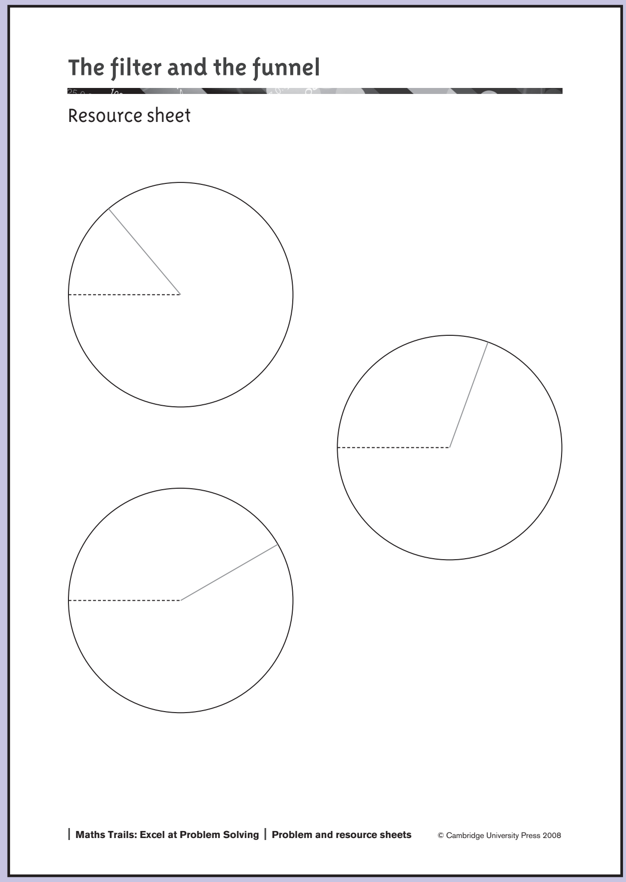
This problem reinforces the use of a spreadsheet for optimisation problems, explores the relationship between volume and surface area, and considers extreme cases.

Time

Two lessons

Resources

A plastic funnel could be a useful visual aid
CD-ROM: spreadsheet, problem sheet, resource sheet
NRICH website (optional): www.nrich.maths.org, January 2005, 'Fence it'; November 2007, 'Funnel'


The filter and the funnel

Resource sheet

| Maths Trails: Excel at Problem Solving | Problem and resource sheets © Cambridge University Press 2008

Introducing the unit

This problem is about maximising the volume of a cone.

Take a circle of diameter 10 cm, cut along a radius and show how a cone can be formed by passing one cut edge beneath the other, creating an overlap.

- As the overlap increases, the height of the cone increases. What happens to the volume? [It increases to start with and then decreases.]
- Estimate by eye when the maximum volume occurs.

The volume for a cone is one third the cylinder that it could just fit into ($\frac{1}{3}\pi r^2 h$). Measure the base radius and height for a cone made from any overlap and model calculating its volume with the pupils.

The resource sheet provides three circles with different overlaps marked. Pupils can make each of these cones (fastened with paper clips) and calculate their volumes.

- What fraction of overlap do you think would give the maximum cone volume? [using an overlap of about 60° from the original circle – the accuracy of pupils' answers will be refined later by using the spreadsheet]
- The original circle had a radius of 5 cm. If I want to make it into a cone with a base radius of 4 cm, what fraction of the circle must overlap? [One fifth must overlap. Because the radius is reducing in the ratio 5:4 the circumference must also reduce in the ratio 5:4 and so must the angle at the centre. $\frac{4}{5}$ is left so $\frac{1}{5}$ must be lost.]
- The original radius length (5 cm) has become the slope length of the cone. If the radius of the base is 4 cm can we calculate the height of that cone? [3 cm using Pythagoras' theorem]
- *Extension*: If we want to get a height of 4 cm, what overlap is needed? [Find the

radius (3 cm) using Pythagoras' theorem. The reduction ratio is 5:3 for the radius, the same for the circumference, and so the overlap must be $\frac{2}{5}$ of the original circle.]

This activity can be continued, calculating overlaps needed to create a cone with a specific radius or a specific height. This is intellectually demanding and should not be rushed; developing a good foundation of understanding ahead of the spreadsheet work is important.

Main part of the unit

From cut to cone

Show the group the sheet 'From cut to cone' on the spreadsheet and discuss the calculation in each column.

Choosing an angle of overlap determines the reduction in cone base circumference and therefore the base radius. Once the base radius is known, Pythagoras' theorem allows us to calculate the height. Using the height and the base radius, the volume can be calculated.

The maximum volume occurs with an overlap of about 66°. The graph provides a useful view of how the volume varies with overlap angle.

The funnel problem

Show the group the problem sheet and discuss how a funnel is based on a cone.

Discuss what might be meant by 'best shape'; a number of valid suggestions might be offered. Draw in the commercial constraint of using a minimum amount of plastic. [The cone with the greatest volume for its surface area will be the 'best shape'.]

Examine the sheet 'Funnel' and discuss the stages of calculation with the group. Note that the height is determined for each radius if we have a fixed volume. At this stage it would therefore be helpful if a cone shape can be found that uses the minimum surface area to contain 1 litre (1000 cm^3).

Column A holds radius values. Column B calculates the height based on a volume of 1000 cm^3 for each radius. The most useful formula for a cone's surface area is πrl (see the solution notes for an explanation), where l is the slope length. Column C calculates the slope length with that radius and height, before calculating the surface area in column D using that formula.

Pupils may wish to construct spreadsheets of their own or amend the spreadsheet 'Funnel'. Explain that they will be expected to explain and justify their findings in the plenary, so they need to think about how they will present their findings.

Plenary

Share well-executed and systematic approaches to the final problem and good justifications of any generalisations.

Solution notes

Note that the formula for the surface area of the cone is found by lying it flat to make the sector of a circle. The radius of the circle is the slope length of the cone and the fraction of the circle present is $2\pi r/2\pi l$ (or r/l as discovered earlier in the unit) – the circumference of the cone base compared with the circumference of this circle of radius l. The area is therefore $\pi l^2 \times r/l = \pi rl$.

For a volume of 1000 cm^3 the radius which gives the minimum surface area is 8.77 cm and the height is 12.99 cm.

For the minimum surface area the ratio of the height to the radius will always be √2. It does not depend on the volume of the funnel.