Introduction

One of the first steps in problem solving is to understand what the problem is about. Building a picture in your mind is a powerful way to start. Until you have a mental picture (not necessarily an 'image'), you cannot hold enough information about the problem in order to move on. Visualising can play a key role in any journey through a problem as well as supporting us in coming to an understanding of the context. Being able to visualise is a sufficiently important problem-solving skill for us to want to provide opportunities for pupils to gain experience of it, and to practise it. This book is about creating those opportunities.

The book contains 17 problems that emphasise the value of visualising. In the 'More ideas' section we have included 16 further visualisations that can be used as stimulus material in other lessons and to give readers ideas which they can use in different contexts and develop further.

The book is aimed principally at teachers of pupils between the ages of 10 and 14. Each problem is accompanied by guidance that includes information on prerequisite knowledge, lesson outlines and solution notes. Additional teaching resources, supporting interactivities and animations, and pupil reference sheets are included on the CD-ROM. Links are given to complementary material on the NRICH website (www.nrich.maths.org). Resources on the NRICH website can be located either by using the date of publication in the text or by typing the name of the resource into the search box on the site.

All the problems and visualisations in this trail can be used in an appropriate curriculum context rather than be taught in isolation. The important thing is that there is a journey to make and that we develop problem-solving skills along the way. We hope that, after tackling some of these problems, learners will begin to use visualisation more naturally as a tool. The book will not necessarily make those who use it expert problem solvers, but it will help them unravel some of the mysteries we encounter on problem-solving journeys.

Problem solving and mathematical thinking

The purpose of this *Maths Trail* is to help give some meaning to problem solving with a particular focus on visualising. This book belongs in a series of titles that looks at particular skills needed in the problem-solving process. Other publications in the series include *Generalising* and *Working Systematically*. While this series tries to give detail to particular mathematical thinking skills, working through each book will not on its own make an expert problem solver.

We want pupils to engage in doing mathematics in a way that means more than the regurgitation of facts or the use of certain skills in particular contexts (for example, finding the area of a rectangle by remembering to multiply length by width). This book is about encouraging pupils to 'be mathematical' in the sense that they:

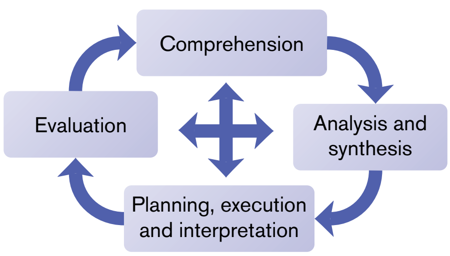
- think about and communicate ideas;
- engage in problem-solving activities;
- create and identify mathematical problems within given contexts.

Clearly, being mathematical and doing mathematics in this sense involves some mathematical content knowledge. But if mathematics is more than learning facts and practising skills, how can we support learners in 'being mathematical'? We need to offer them the opportunity to be problem solvers who can pose and explore problems. If we are to support problem solving effectively we need some guidance and structure upon which we can focus.

What is problem solving?

When we are presented with a mathematical problem, it is only a problem if we do not immediately know how to solve it. This means that something that is a problem to one person may not be a problem to another. The process of problem solving is like a journey from a state of not knowing what to do towards a destination which we hope will be a solution. The key is to have some strategies at our fingertips that will help us to get started and to identify a possible route to a solution. Our mathematical journey is often full of twists and turns, where we revisit ideas or need to step back and look for alternatives. Often a mistake or dead-end gives vital clues to the mathematics of the problem and is therefore crucial in any solution process.

To help us have a sense of direction when we are problem solving and decide what might be a good strategy to try next, it can be useful to have a model. There are many such models but the one below is a good starting point. Although the descriptions on the next page are written in a linear sequence, the diagram emphasises the reality that problem solving can be a messy process which often involves revisiting places on the journey from problem posing to problem solving.



Comprehension

This stage is about making sense of the problem by using strategies such as retelling, identifying relevant information and creating mental images. Pupils can be helped at this stage by being encouraged to re-read the problem several times and to record in some way what they understand the problem to be about (for example, by drawing a picture or making notes).

Analysis and synthesis

This stage is about 'homing in' on what the problem is asking solvers to investigate.

- Can they represent the situation mathematically?
- What is it that they are trying to find?
- What do they think the answer might be (conjecturing and hypothesising)?
- What might they need to find out before they can get started?

Central to this stage is identifying what is unknown and what needs finding.

Planning, execution and interpretation

Once pupils have understood what the problem is about and have established what needs finding, the next stage is planning and executing a pathway to the solution. It is within this process that you might encourage pupils to think about whether they have seen something similar before and what strategies they adopted then. This will help them to identify appropriate methods and tools. Particular knowledge and skills gaps that need addressing may become evident at this stage.

During the execution phase, pupils might identify further related problems they wish to investigate. They will need to consider how they will keep track of what they have done and how they will communicate their findings. This phase of the problem-solving process ends with interpreting results and drawing conclusions.

Evaluation

Pupils can learn as much from reflecting on and evaluating what they have done as they can from the process of solving the problem itself. During this phase pupils should be expected to reflect on the effectiveness of their own approach as well as that of other people, to justify their conclusions and to assess their own learning. Evaluation may also lead to thinking about other questions that could now be investigated.

Using the model

The use of this model has a number of benefits in the classroom:

- It is a structure that can help pupils frame their problem solving and keep track of where they are in the process.
- It gives us a language which helps us to talk to pupils about what they are doing. For example:
  - 'Can you tell me what you think the problem is about?'
  - 'What are you trying to find?'
  - 'Have you seen anything like this before and what did you do then?'
  - 'Could you have solved this in a different way?'
- It offers a framework which can help structure lessons so that you can plan what pupils might do at various stages.

To support you in identifying the phases in the model, the lesson outlines indicate some key aspects of problem solving. We think it is worth drawing the attention of pupils to the model during their work on each task. Many of the activities involve engagement with all aspects of the problem-solving 'cycle'. The aspects highlighted are intended to offer a particular focus for a lesson.

What is mathematical thinking?

Within the problem-solving framework, there are many mathematical skills which pupils need to have at their fingertips. These skills involve more than numeric, geometric and algebraic manipulation. They include strategies such as:

- modelling;
- visualising;
- working systematically;
- generalising.

We would class these skills as elements of mathematical thinking that are needed to engage in the problem-solving process. This book focuses on the skills associated with *visualising*.

What is visualising?

We use some visualisation strategies almost every time we engage in problem solving. Visualisation can have a variety of purposes and involves having access to a range of skills.

Purposes of visualising

1 Modelling a situation
This is particularly useful when the situation is physically unattainable. In other words we try to see the 'unseeable', for example the inside of a 3-D object, or a very large number.

2 Stepping into a problem
Using visualisation aids understanding at the comprehension stage. Here visualisation gives us space to go more deeply into the situation, to clarify and support understanding.

3 Planning ahead
Visualising can be used throughout a problem to anticipate and to plan ahead by asking 'What will be the consequence of doing this?'.

Visualising skills

The examples in this book emphasise the purposes of visualising described above. However, they also aim to encourage the development of specific skills that underpin these purposes. The list below describes in more detail these useful visualising skills:

A Focusing on a problem or idea without talking for several minutes.

B Identifying a useful image or representation of an idea. The image might be suggested by someone else but it still has some meaning for you. This may be a representation that helps you see the larger picture and/or describe the structure of the problem.

C Scrutinising different images to identify what is the same or different. This includes:

C1 Comparing other people's representations with our own.

C2 Identifying the general and the specific in a representation and what is significant in terms of the problem at hand. For example, recognising that when two squares are put together, although there is an infinite number of possible orientations (general) you only need to consider one and to have that clearly in your mind (specific).

C3 Trying to hold more than one image in your head. For example, remembering a starting point and being able to get back again when your ideas don't work, or seeing and/or reproducing a logical sequence.

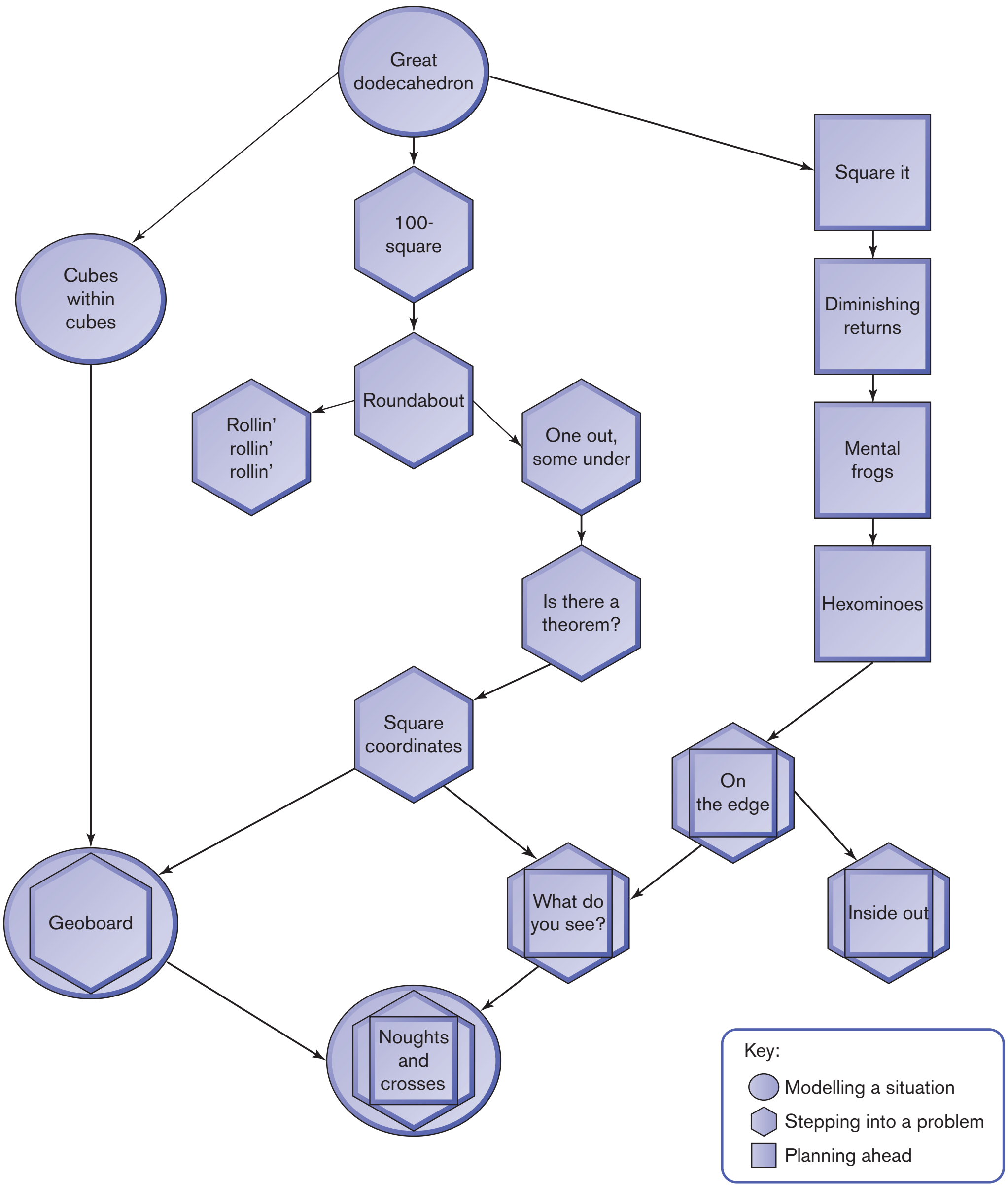
D Making connections. For example, remembering the processes or underpinning structures rather than individual images.

E Describing a representation (your own or someone else's) to an audience. For example, to explain or clarify thinking or share interpretations.

The visualising trail

The trail structure is shown below and runs from top to bottom. It includes problems focusing on the three purposes of visualising and covers the range of visualising skills described earlier.

In general, a problem which is further down the trail and coded with the same shape requires higher efficiency in that aspect of visualising or involves some aspect of mathematics associated with a higher level of maturity.



Why use this trail?

We often present a problem to pupils and then jump straight into thinking about how to solve it without giving them time to create a picture of what is going on in their own minds. One thing the lessons in this book encourage is giving pupils that time and some strategies for 'getting to know' a problem by using their visualising skills. It can be frustrating when pupils jump into problems without thinking ahead and considering 'what would happen if' because as a result they often get nowhere quickly. Visualising is a strategy we need to encourage pupils to employ throughout problem solving. Mental images can also act as a focus for discussion and sharing.

The table below lists the problems in the trail and gives suggestions for the major visualisation foci of each one. The numbers and letters refer to the purposes of visualising and the visualising skills listed in the section 'What is visualising?'. Clearly a slightly different emphasis during teaching can significantly change the visualisation focus.

	Purposes			**Skills**						
	1	**2**	**3**	**A**	**B**	**C1**	**C2**	**C3**	**D**	**E**
Great dodecahedron	X									X
Cubes within cubes	X			X		X			X	X
100-square		X		X			X		X	
Roundabout		X		X		X		X		X
Rollin' rollin' rollin'		X		X		X		X		X
One out, some under		X		X					X	
Is there a theorem?		X				X		X		X
Square coordinates		X						X		
Square it			X				X			
Diminishing returns			X						X	X
Mental frogs			X		X			X		
Hexominoes			X	X				X	X	X
On the edge		X	X				X		X	
Inside out		X	X				X		X	
What do you see?		X	X	X			X	X	X	X
Geoboard		X	X		X	X	X			X
Noughts and crosses	X	X	X							X

How to use this trail

This trail is an organised set of curriculum resources, including teacher notes and pupil hints, designed to develop pupils' mathematical thinking and specifically focusing on the skill of visualising.

The book has two main elements:

- Whole lessons where visualising forms a major learning objective.
- 'More ideas', which can be used as lesson starters at any time. We would recommend that these are chosen to reflect the lesson content as a whole. For example, selecting an idea based on sequences of numbers if the lesson is on some aspect of number work related to sequences.

You might wish to use the trail as a 'course' for pupils over a short or long period of time, with them working as a whole class, in small groups or individually. The trail indicates an order of materials to support each pupil's developing skills over time. However, it is also possible to dip in and out of the materials at appropriate points in a scheme of work. The visualisations in the 'More ideas' section are intended to encourage regular reminders about visualising in 'normal' lessons. The trail has a simple ordering of problems and is intended to illustrate aspects of visualising arising from different contexts, including number and space and shape. It is not intended to be a straitjacket. The timings indicated in the teacher notes for each problem are indicative as the intention is to encourage extension and pupil investigation beyond what is made explicit.

The problems offer opportunities for pupils of a wide age and ability range, and do not imply a particular view of classroom organisation. However, there is an underlying message concerning classroom practice and the learning of mathematics as a collaborative experience, valuing the journey through a problem rather than the answer. While it is not necessary for pupils to work in groups, there is an expectation that pupils will be given opportunities to talk about their mathematical experiences *en route* as well as at the 'conclusion' of their investigations.

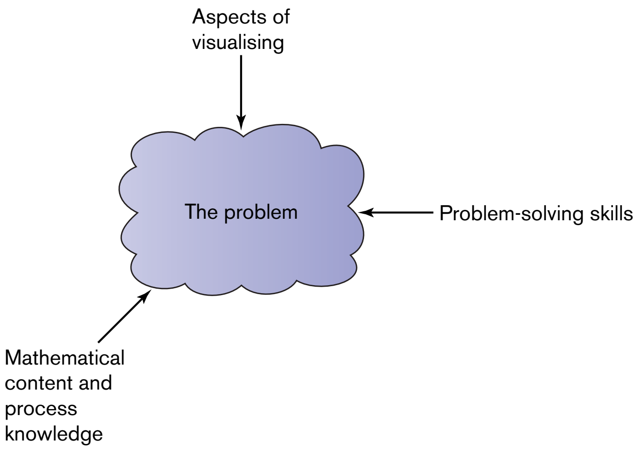
Although the trail map indicates a very linear structure, reflecting progressively more-challenging contexts for visualising, there is considerable flexibility available.

We hope that many of the visualisations will stimulate ideas of your own.

Ideas for managing sessions

Each lesson in this book has three distinct foci, each supporting and complementing the others:

- mathematical content and process knowledge;
- problem-solving skills;
- aspects of visualising (being used both as a tool to support the learning and as a skill to be developed in its own right).



There are no step-by-step rules that will always lead to a solution in any of the problems used in this book. The aim is to encourage pupils to share ideas and different approaches and to learn to value those differences while identifying elegant and efficient solutions whenever appropriate. After working on this trail, pupils will be able to recognise where visualising might be helpful and feel empowered to use visualising more effectively when problem solving in the future.

There are as many approaches to teaching as there are pupils in a class so these lesson notes are intended purely as a guide. All we can do is seed some ideas without any intention of being prescriptive.

Working with the whole class

The following might be a typical whole-class approach to a problem.

Introduce the problem

Building on the introductory activity, introduce the problem and, where it is helpful, start to model an approach or share your visualisation. Then ask pupils to work entirely on their own for five minutes – giving them time to play, and to familiarise and visualise the context for themselves. At this point it is worth emphasising that they are not expected to be working neatly towards a solution but simply finding out what the problem might be about. Tell them that after a short time you will stop them and ask them to work with a partner for a further five minutes in order to share what they have discovered. The aim of the time spent working in pairs and small groups is to share initial ideas that can later be discussed as a whole class. This part of the lesson corresponds to the 'comprehension' phase of the problem-solving process.

Sharing and moving on

Asking pairs or groups to share findings and ideas of what the problem is about offers opportunities for those who have not found a way into the problem to 'get started' and gives the class the chance to refine and develop ideas as a community. Here the important thing is not just 'knowing how to do it' but leaving room for new ideas and questions to be discussed. This is about valuing the journey, including the cul-de-sacs we may encounter or the different routes we may take. During this time pupils can begin to identify potentially promising approaches to tackling the problem.

Planning, execution and interpretation

So far we have encouraged pupils to:

Think – Pair – Share

After these early discussions pupils need further time to investigate the problem and consider possible routes to their solutions and/or consider further problems that might arise. Sharing and iteration of discussions will help to give all a sense of owning the mathematics and ensure that, as far as possible, as many different approaches to the problem are considered, not simply 'the answer'. In this phase, pupils will execute their plans and interpret their findings.

Evaluation

During this last phase pupils discuss their findings, convincing themselves and their classmates that any findings they have or conjectures they wish to put forward are reasonable. Time considering different solutions and their 'efficiency' or 'accessibility' is invaluable in opening up the mathematics and helping pupils value different approaches. Then, of course, it is worth talking about all the other problems or ideas that the context has stimulated pupils to consider and the process can start again!

Working with individual pupils

The trail can also be a useful tool for teachers to use with individual pupils who need the challenge of problem-solving activities. Pupils who quickly grasp the mechanical aspects of mathematics but find it difficult to work on open-ended tasks can be encouraged to tackle some of the problems in the trail. The sense of direction and purpose of the trail, when shared with the pupil(s), particularly in emphasising where there are opportunities to visualise, can enable them to build up a repertoire of approaches to such problems and give them more confidence when confronted with similar activities in the future.

Working with groups

Groups of pupils can often find their own way into problems simply by being given the context and being asked to:

- identify what the question is about;
- consider strategies for solution;
- plan what they are going to do;
- execute their plans and interpret their results.

One important aim might be to encourage pupils to communicate their findings to others in the class or to their teacher, describing the difficulties they had as well as how they chose their approach and executed their solutions. This communication does not have to take the form of written output but could be verbal or in the form of a poster or presentation.

Finally, encourage the group to answer evaluative questions such as:

- Could we have done this more efficiently?
- What have we learnt that is new?
- Have we met anything like this before and were we able to make connections?
- What additional questions did we come up with and answer while we were working on the problem?
- Are there some questions still to be answered?

Many further examples of problems are available on the NRICH website (www.nrich.maths.org) if you wish to extend any of the work in a particular area of mathematics, or simply to reinforce ideas and skills.

Assessment

Prerequisite knowledge

Two aspects to pupils' prerequisite knowledge need to be considered.

Firstly, we need to consider pupils' abilities to tackle problem-solving situations independently. This trail emphasises the importance of applying problem-solving and mathematical-thinking skills (the emphasis here is on visualising), but it does not assume any particular level of efficiency in problem solving nor any familiarity with it. As the main purpose of the trail is to support, develop and extend these skills, it means that, as a teacher, the amount of scaffolding and support you offer will vary considerably according to pupils' prior experiences of problem solving and their individual needs.

Secondly, each problem depends upon knowledge of particular aspects of mathematical curriculum content.

Assessment for learning

The notes and other documentation for each problem aim to support opportunities for formative and summative assessment. Outline solutions to the problems are included to give some guidelines on expected outcomes. Where applicable, edited solutions by pupils can be found on the NRICH website at www.nrich.maths.org. The advantage of looking at solutions on the website is that they can give you an idea of the variety you can expect from your own pupils.

The key outcome for all of the activities is to develop pupils' skills in visualising within a range of contexts. It might be best not to highlight the content objectives at the beginning of the lesson as this could close down possibilities for pupils to think independently. We therefore offer a cautionary note that lesson objectives shared with the pupils should not reveal obvious routes to a solution.

Listening and questioning are important tools in the process of formative assessment. To support this:

- all problems have suggested prompts for teachers and mentors to use;
- pupils are encouraged to hypothesise and share ideas with fellow pupils, arguing their case – these are ideal opportunities to listen;
- whole-group discussions during the lesson can be used to reveal pupils' understanding, misconceptions or lack of awareness of the necessary mathematical knowledge;
- the use of peer assessment is encouraged as this can often shed valuable light on the understanding of the assessor as well as the assessed;
- reviewing and reflecting on the lesson outcomes with pupils are suggested as ways of helping the teacher make judgements and allowing pupils an opportunity for self or peer assessment.

A pupil self-assessment sheet is included on the CD-ROM and can be made available to pupils at the end of each problem.

As highlighted earlier, much of the work and learning is about the journey through each problem. It is not necessary for pupils to have well-rounded, written solutions for sound assessment judgements to be made. While feedback through marking is sometimes appropriate, oral and continuous feedback throughout the problem-solving process is just as valuable.

'To be effective feedback should cause thinking to take place'
(*Assessment for learning in everyday lessons*, DfES, 2004)

Assessing learning over the whole trail

This section considers the possible outcomes of completing the whole trail, or at least a significant part of it, for assessment purposes.

Pupils and teachers have the opportunity to assess what learning has taken place after each of the problems (see the pupil assessment sheet on the CD-ROM). Here are some concluding thoughts to help with the assessment process over the whole of the trail.

Pupils should look back over their work and think about what they have been doing so that they can identify:

- ways in which they have used visualisation;
- mathematical facts they have used;
- the range of mathematical skills they have employed;
- some thinking and problem-solving skills they have developed;
- instances where they were able to be independent and find things out for themselves;
- places where they have compared ideas or methods and evaluated their choice;
- times when they have asked themselves 'What if … ?' or 'What if not … ?' questions;
- other relevant questions to ask themselves and others;
- things that have fired their curiosity and got them to ask more questions;
- where they have persevered and not been frightened by 'complicated work';
- places where they have found use of symbols helpful.

Resources

The book uses animations and interactivities in a number of places. These are written in Macromedia Flash (version 8). The player can be downloaded free by following the links in the Help section of the NRICH website (http://nrich.maths.org/public/viewer.php?obj_id=4779). Where a problem would benefit from slightly different layouts depending on whether it is used as a whole-class teaching resource or as a pupil reference sheet, we have produced separate supporting material for this purpose. The animations, interactivities, problem sheets, resource sheets and other electronic-format materials are included on the CD-ROM.

Some further reading

Burton, L. (1984) *Thinking Things Through*. Oxford: Basil Blackwell.

Mason, J. with Burton, L. and Stacey, K. (1982) *Thinking Mathematically*. Wokingham: Addison-Wesley.

Piggott, J. (2005) 'An investigation into the nature of mathematical enrichment: a case study of implementation'. Ph.D. dissertation, Institute of Education, London.

Polya, G. (1945) *How to Solve It*. London: Penguin.