Core 3 - June 2006

1. $y = \sqrt{4x+1}$

$y = (4x+1)^{1/2}$

$\frac{dy}{dx} = \frac{1}{2} \times 4 (4x+1)^{-1/2} = \frac{2}{\sqrt{4x+1}}$

at $x=2$ $\quad \frac{dy}{dx} = \frac{2}{\sqrt{9}} = \frac{2}{3}$

$\frac{y-3}{x-2} = \frac{2}{3} \Longrightarrow y-3 = \frac{2}{3}(x-2)$

5 easy marks to start!!

$3y - 9 = 2x - 4$

$\underline{3y - 2x - 5 = 0}$

2. $|2x-3| < |x+1|$

Both are positive so we can square both sides

$(2x-3)^2 < (x+1)^2$

still positive so we can lose the modulus signs

$4x^2 - 12x + 9 < x^2 + 2x + 1$

$3x^2 - 14x + 8 < 0$

$(3x-2)(x-4) < 0$

To be negative one +ve one -ve

if $x < \frac{2}{3}$ $\begin{cases} (3x-2) \text{ is negative} \\ (x-4) \text{ is negative} \end{cases}$ So product is positive

if $\frac{2}{3} < x < 4$ $\begin{cases} (3x-2) \text{ is positive} \\ (x-4) \text{ is negative} \end{cases}$ So product is negative

if $x > 4$ $\begin{cases} (3x-2) \text{ is positive} \\ (x-4) \text{ is positive} \end{cases}$ So product is positive

$\therefore$ Solution is $\underline{\frac{2}{3} < x < 4}$

3. $2x^3 + 4x - 35 = 0$ one root.

i) if $x = 2$ $\quad 2 \times 2^3 + 4 \times 2 - 35 =$
$16 + 8 - 35 = -11$

if $x = 3$ $\quad 2 \times 3^3 + 4 \times 3 - 35 =$
$54 + 12 - 35 = 31$

by the sign change rule there is a root between 2 and 3.

ii) $x_{n+1} = \sqrt[3]{17.5 - 2x_n}$

Begin with any value between 2 and 3

So let $x_1 = 2.5$

$x_2 = \sqrt[3]{17.5 - 2 \times 2.5} = 2.320794$

$x_3 = \sqrt[3]{17.5 - 2 \times (ans)} = 2.342767$

$x_4 = \sqrt[3]{17.5 - 2 \times (ans)} = 2.340095$

$x_5 = \sqrt[3]{17.5 - 2 \times (ans)} = 2.340420$

So $x = 2.34$ to 2d.p.

4. $y = 5^{x-1}$

i) Take logs (ln) of both sides

$\ln y = \ln(5^{x-1})$

$\ln y = (x-1)\ln 5$

$\frac{\ln y}{\ln 5} = x - 1$

$x = 1 + \frac{\ln y}{\ln 5}$

ii) $x = 1 + \frac{\ln y}{\ln 5}$

$\ln 5$ is just a number so as $\ln y$ differentiates to $\frac{1}{y}$ then $\frac{\ln y}{\ln 5} = \frac{1}{\ln 5} \ln y$ differentiates

to $\frac{1}{\ln 5} \times \frac{1}{y} = \frac{1}{y \ln 5}$

(and the 1 differentiates to 0)

So $\frac{dx}{dy} = \frac{1}{y \ln 5}$

iii) at $x = 3$ $y = 25$

$\frac{dx}{dy} = \frac{1}{25 \ln 5}$

So $\frac{dy}{dx} = \underline{\underline{25 \ln 5}}$

5. i) $\sin 2\theta = 2 \sin\theta \cos\theta$

ii) $\sin\alpha = \frac{1}{4}$ using $\sin^2\alpha + \cos^2\alpha = 1$

$\left(\frac{1}{4}\right)^2 + \cos^2\alpha = 1$

$\frac{1}{16} + \cos^2\alpha = 1$

$\cos^2\alpha = \frac{15}{16}$

$\cos\alpha = \pm\sqrt{\frac{15}{16}}$

as α is acute $\cos\alpha = +\sqrt{\frac{15}{16}}$

Then $\sin 2\alpha = 2 \sin\alpha \cos\alpha$

$$\sin 2\alpha = 2 \times \frac{1}{4} \times \sqrt{\frac{15}{16}} = 2 \times \frac{1}{4} \times \frac{\sqrt{15}}{\sqrt{16}}$$

$$= 2 \times \frac{1}{4} \times \frac{\sqrt{15}}{4} = \frac{1}{8}\sqrt{15}$$

iii) $0^\circ < \beta < 90^\circ$

$5 \sin 2\beta \sec\beta = 3$

$5 \times 2\sin\beta\cos\beta \frac{1}{\cos\beta} = 3$

$\sin\beta = 0.3$

$\sin^{-1} 0.3 = \beta = 17.5^\circ$

6. i) $f(x) = 2 - x^2 \quad x \leq 0$

$ff(-3)$

$f(-3) = 2 - (-3)^2 = 2 - 9 = -7$

$f(-7) = 2 - (-7)^2 = 2 - 49 = -47$

ii) let $y = 2 - x^2$

$x^2 = 2 - y$

$x = \pm\sqrt{2 - y}$

so $f^{-1}(x) = \pm\sqrt{2 - x}$

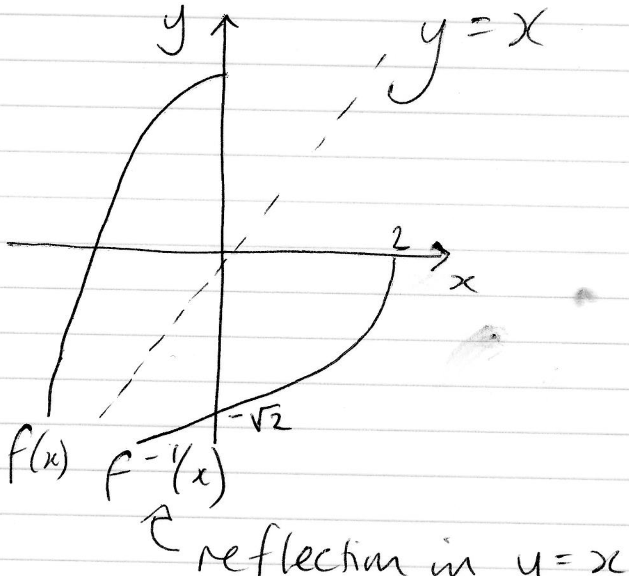
but the doman for $f(x)$ is $x \leq 0$ ie negatives

so the range for $f^{-1}(x)$ must be the same

(negative) so $f^{-1}(x) \leq 0$

so $f^{-1}(x) = -\sqrt{2 - x}$

iii)



if $x = 0$ $y = -\sqrt{2}$

if $y = 0$ $x = 2$

7. a) $\displaystyle\int_1^2 \frac{2}{(4x-1)^2} \cdot dx = \int_1^2 2(4x-1)^{-2} \cdot dx$

$= \left[2 \times \frac{1}{4} \times \frac{1}{-1} (4x-1)^{-1}\right]_1^2$

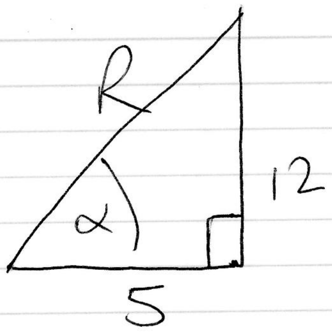
÷ by coeff of x   ÷ new power

$= \left[-\frac{1}{2}(4x-1)^{-1}\right]_1^2$

$= \left(\frac{-1}{2} \times \frac{1}{7}\right) - \left(\frac{-1}{2} \times \frac{1}{3}\right)$

$= \frac{-1}{14} + \frac{1}{6}$

$\frac{-3}{42} + \frac{7}{42} = \frac{4}{42} = \frac{2}{21}$



$$\tan\alpha = \frac{12}{5} \qquad \alpha = \tan^{-1}\left(\frac{12}{5}\right) = 67.4°$$

$$R^2 = 5^2 + 12^2 = 25 + 144 = 169$$

$$R = 13.$$

$$13\cos(x - 67.4)$$

ii) To transform $y = \cos x$ into $13\cos(x - 67.4)$
a translation by $67.4°$ in the positive
x direction.

and a stretch by factor 13 in the y direction.

iii) $5\cos x + 12\sin x = 2$

$$13\cos(x - 67.4) = 2$$

$$\cos(x - 67.4) = 2/13$$

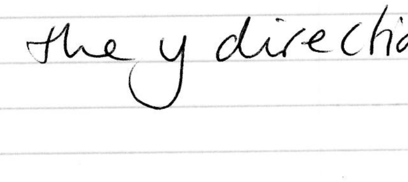
let $\phi = x - 67.4$

$$0 < x < 360$$

$$-67.4 < \phi < 292.6$$

$$\cos^{-1}\left(\frac{2}{13}\right) = 81.15°$$

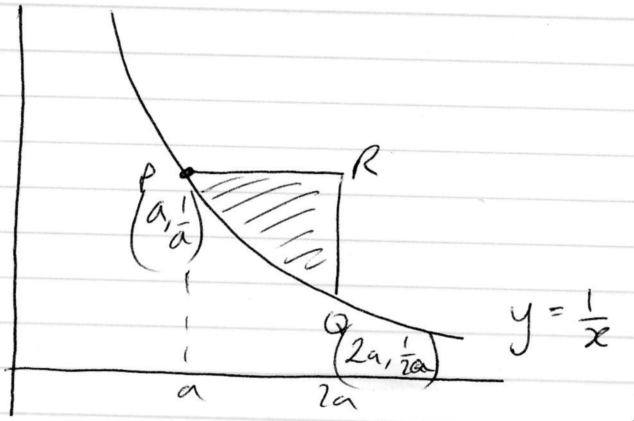
-81.15 is outside the range



other solution $360 - 81.15$

$$= 278.85$$

b)



Coordinates of R must be $\left(2a, \frac{1}{a}\right)$

Area of rectangle $(2a - a) \times \left(\frac{1}{a} - 0\right)$

$$a \times \frac{1}{a} = 1$$

$$\int_a^{2a} \frac{1}{x} \cdot dx = \Big[\ln x\Big]_a^{2a}$$

$$= \ln 2a - \ln a$$

$$= \ln \frac{2a}{a} = \ln 2$$

So shaded area is $1 - \ln 2$

$$= \ln e - \ln 2 = \ln\left(\frac{e}{2}\right) = \underline{\underline{\ln\left(\tfrac{1}{2}e\right)}}$$

8. i) $5\cos x + 12\sin x = R\cos(x - \alpha)$

$5\cos x + 12\sin x = R(\cos x\cos\alpha + \sin x\sin\alpha)$

$5\cos x = R\cos\alpha\cos x$

$5 = R\cos\alpha \qquad \cos\alpha = \frac{5}{R}$

$12\sin x = R\sin\alpha\sin x$

$12 = R\sin\alpha \qquad \sin\alpha = \frac{12}{R}$

$\phi = 81.15$ or 278.85

$\phi = x - 67.4$

$x = \phi + 67.4$

$x = 81.15 + 67.4 = \underline{148.55}$

and $x = 278.85 + 67.4 = \underline{346.25}$

9. $y = 2\ln(x-1)$

Rotate around the y-axis it's even in <u>bold.</u>

$$\int_0^p \pi x^2 \, dy$$

So we need to rearrange the equation of the curve.

$$y = 2\ln(x-1)$$

$$\frac{y}{2} = \ln(x-1)$$

$$e^{y/2} = x-1$$

$$e^{y/2} + 1 = x$$

$$\int_0^p \pi\left(e^{y/2}+1\right)^2 dy = \int_0^p \pi\left(\left(e^{y/2}\right)^2 + 2e^{y/2} + 1\right) dy$$

$$= \int_0^p \pi\left(e^y + 2e^{y/2} + 1\right) dy$$

$$= \pi\left[e^y + 2 \times 2e^{y/2} + y\right]_0^p$$

$\div$ by $\frac{1}{2}$ (coeff ~~of~~ of y)

$$= \left[\left(e^{p} + 4e^{p/2} + p\right) - \left(e^{0} + 4e^{0} + 0\right)\right]\pi$$

$$V = \pi\left(e^{p} + 4e^{p/2} + p - 5\right)$$

ii) $\frac{dp}{dt} = 0.2$

$$\frac{dV}{dp} = \pi\left(e^{p} + 4 \times \tfrac{1}{2} e^{p/2} + 1\right)$$

$$\frac{dV}{dt} = \frac{dV}{dp} \times \frac{dp}{dt} = 0.2\pi\left(e^{p} + 2e^{p/2} + 1\right)$$

at $p = 4$ $\quad \frac{dV}{dt} = 0.2\pi\left(e^{4} + 2e^{2} + 1\right)$

$$\frac{dV}{dt} = \underline{44.2 \text{ cm}^3 \text{min}^{-1}}$$