Core 3 June 05

1. i) $f(x) = 10 - (x+3)^2$

$f(x)$ cannot be bigger than 10 $\quad f(x) \leq 10$

ii) $ff(x) = 10 - (10 - (x+3)^2 + 3)^2$

$ff(-1) = 10 - (10 - 2^2 + 3)^2 = 10 - 9^2 = -71$

2. Either $(6x-1) = (x-1) \Rightarrow 5x = 0 \Rightarrow x = 0$

or $(6x-1) = -(x-1) \Rightarrow 6x - 1 = -x + 1$

$\Rightarrow 7x = 2 \Rightarrow x = \frac{2}{7}$

3. i) $m = 180e^{-0.017t}$ if $m = 25$

$25 = 180e^{-0.017t} \Rightarrow \frac{25}{180} = e^{-0.017t}$

$\ln\left(\frac{25}{180}\right) = -0.017t \qquad t = \frac{\ln\left(\frac{25}{180}\right)}{-0.017} = 116$

ii) $\frac{dm}{dt} = (-0.017)180e^{-0.017t}$ if $t = 55$

$\frac{dm}{dt} = 1.2$ grams per year.

4. a) $y = \frac{2}{\sqrt{x}} \qquad Vol = \int_1^5 \pi y^2 \, dx = \pi\int_1^5 \left(\frac{2}{\sqrt{x}}\right)^2 dx$

$= \pi\int_1^5 \frac{4}{x} \, dx = \pi\int_1^5 4x^{-1} \, dx = \pi\left[4\ln x\right]_1^5 = 4\pi\ln 5 - 4\pi\ln 1$

$= 4\pi\ln 5$

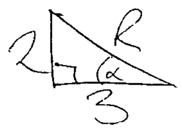
b) $\int_1^5 \sqrt{x^2+1} \, dx \approx \frac{1}{3} \times 1(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$

$= \frac{1}{3} \times 1 \times (\sqrt{2} + 4\sqrt{5} + 2\sqrt{10} + 4\sqrt{17} + \sqrt{26})$

5. i) $3\sin\theta + 2\cos\theta = R(\sin\theta + \alpha)$

$3\sin\theta + 2\cos\theta = R(\sin\theta\cos\alpha + \sin\alpha\cos\theta)$

$3 = R\cos\alpha \qquad 2 = R\sin\alpha$



$R^2 = \sqrt{2^2 + 3^2} = 13 \qquad R = \sqrt{13}$

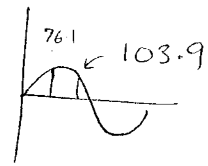
$\alpha = \tan^{-1}\left(\frac{2}{3}\right) = 33.7°$

ii) $\sqrt{13}\sin(\theta + 33.7) = \frac{7}{2}$

$\sin(\theta + 33.7) = \frac{7}{2\sqrt{13}} \qquad \sin^{-1}\left(\frac{7}{2\sqrt{13}}\right) = \theta + 33.7$

$= 76.1$ or 103.9

$\therefore \theta = 42.4°$



or $\theta = 103.9 - 33.7$

$= 70.2°$

6. a) $y = x\ln x$ use 'product rule'

$u = x \qquad v = \ln x$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{v\,du}{dx} + \frac{u\,dv}{dx} = \ln x + x \times \frac{1}{x} = \ln x + 1$

at stationary point $\ln x + 1 = 0 \qquad \ln x = -1$

$e^{\ln x} = e^{-1}$

$x = e^{-1}$

b) $y = \dfrac{4x+c}{4x-c} \begin{matrix} \leftarrow u \\ \leftarrow v \end{matrix} \Big\}$ use quotient rule.

$u = 4x+c \quad v = 4x-c$

$\dfrac{du}{dx} = 4 \quad \dfrac{dv}{dx} = 4 \quad \dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \dfrac{4(4x-c) - 4(4x+c)}{(4x-c)^2} = \dfrac{16x - 4c - 16x - 4c}{(4x-c)^2}$

$= \dfrac{-8c}{(4x-c)^2}$ which cannot equal 0 as C is nonzero.

7. i) $\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1-\cos^2 x)$
$= 2\cos^2 x - 1$

ii) $\dfrac{4\cos 2x}{1+\cos 2x} = \dfrac{4(2\cos^2 x - 1)}{1 + 2\cos^2 x - 1} = \dfrac{8\cos^2 x - 4}{2\cos^2 x}$

$= \dfrac{8\cos^2 x}{2\cos^2 x} - \dfrac{4}{2\cos^2 x} = 4 - \dfrac{2}{\cos^2 x} = 4 - 2\sec^2 x$

iii) $\dfrac{4\cos 2x}{1+\cos 2x} = 3\tan x - 7$

$4 - 2\sec^2 x = 3\tan x - 7$

use $\sec^2 x = 1 + \tan^2 x$

$4 - 2(1+\tan^2 x) = 3\tan x - 7$

$4 - 2 - 2\tan^2 x = 3\tan x - 7$

$2 = 2\tan^2 x + 3\tan x - 7$

Which factorises to $(2\tan x - 3)(\tan x + 3) = 0$

$\therefore \tan x = \dfrac{3}{2}$ or -3

use $\tan^{-1}\left(\frac{3}{2}\right)$ and $\tan^{-1}(-3)$

and graphs to get $0.983, 4.12, 1.89, 5.03$
$(0.313\pi, 1.31\pi, 0.602\pi, 1.6\pi)$

8. i) $e^{1/5 x} = \sqrt[3]{3x+8}$

$e^{1/5 x} - \sqrt[3]{3x+8} = 0$

if $x = 5.2 \quad e^{\frac{5.2}{5}} - \sqrt[3]{3 \times 5.2 + 8} = -0.04$

if $x = 5.3 \quad e^{5.3/5} - \sqrt[3]{3 \times 5.3 + 8} = 0.006$

sign changes from 5.2 to 5.3

ii) $e^{1/5 x} = \sqrt[3]{3x+8} \Rightarrow e^{1/5 x} = (3x+8)^{1/3}$

$\ln e^{1/5 x} = \ln(3x+8)^{1/3}$

$\dfrac{1}{5}x = \dfrac{1}{3}(3x+8) \Rightarrow x = \dfrac{5}{3}(3x+8)$

iii) As $x = \dfrac{5}{3}(3x+8)$

Starting with 5.2 gives $\rightarrow 5.2687 \rightarrow 5.2832 \rightarrow 5.2863 \rightarrow 5.2869$

starting with 5.3 gives $\rightarrow 5.2898 \rightarrow 5.2877 \rightarrow 5.2872 \rightarrow 5.2871$

$\therefore$ to 2dp 5.29

iv) $\int_0^{5.29} (3x+8)^{1/3} \, dx - \int_0^{5.29} e^{1/5x} \, dx$

$= \left[\frac{3}{4} \times \frac{1}{3} (3x+8)^{4/3}\right]_0^{5.29} - \left[5e^{1/5x}\right]_0^{5.29}$

$= \underline{\underline{3.78}}$

9. i) Translate by 7 units in negative x.
Stretch by factor of $\frac{1}{m}$ in x direction
Translate 4 units in negative x.

ii) Each y value is the image of a unique x value.

$y = \sqrt{mx+7} - 4$

$y + 4 = \sqrt{mx+7}$

$(y+4)^2 = mx+7$

$x = \frac{(y+4)^2 - 7}{m}$ $\quad \therefore \; f^{-1}(x) = \frac{(x+4)^2 - 7}{m}$

iii) The function and its inverse are reflected in $y = x$ so if they meet it is on the line $y = x$
for this to happen $f(x) = x$ (or $f^{-1}(x) = x$)

if $x = \sqrt{mx+7} - 4$ then $x + 4 = \sqrt{mx+7}$
so $(x+4)^2 = mx+7$
$x^2 + 8x + 16 = mx + 7 \implies x^2 + (8-m)x + 9 = 0$

For this to have real solutions discriminant must be ≥ 0

$b^2 - 4ac \geq 0$

$(8-m)^2 - 4 \times 9 \geq 0$

$64 - 16m + m^2 - 36 \geq 0$

$m^2 - 16m + 28 \geq 0$

$(m-14)(m-2) \geq 0$

So equation has roots if $m \geq 14$ or $m \leq 2$

So as they do ~~not~~ meet (on line $y = x$) then the equation has no solution

$\therefore \; \underline{\underline{2 < m < 14}}$