Core 4 June 2005

① $x^4 + 3x^3 + 5x^2 + 4x - 1 \equiv (x^2+x+1)(Ax^2+Bx+C) + Rx+S$

$\equiv Ax^4 + Bx^3 + Cx^2 + Ax^3 + Bx^2 + Cx + Ax^2 + Bx + C + Rx + S$

$\equiv Ax^4 + (B+A)x^3 + (C+B+A)x^2 + (C+B+R)x + (C+S)$

$A = 1$ | $A = 1$
$B + A = 3$ | $B = 2$
$C+B+A = 5$ | $C = 2$
$C+B+R = 4$ | $R = 0$
$C+S = -1$ | $S = -3$

$\therefore$ Quotient is $x^2 + 2x + 2$
Remainder is -3 (ie $0x + -3$)

② $\int_0^{\frac{1}{2}\pi} x \cos x \, dx$ (By parts!!)

$u = x$, $\frac{dv}{dx} = \cos x$

$u = x$ $\qquad \frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 1$ $\qquad v = \sin x$

$uv - \int v \frac{du}{dx} \, dx = x \sin x - \int \sin x \, dx$

$= \left[x \sin x\right]_0^{\frac{1}{2}\pi} - \left[-\cos x\right]_0^{\frac{1}{2}\pi}$

$= \left(\frac{\pi}{2} \times 1 - 0\right) - \left(-0 - -1\right)$

$= \underline{\underline{\frac{\pi}{2} - 1}}$

③ i) First find a vector to go through the 2 points $(2,-3,1)$ $(-1,-2,-4)$

$$\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} \leftarrow \text{the "gradient" vector}$$

then use either of the points given as the point on the line which defines 'where' the line is.

ie $\underset{\sim}{r} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$

or $\underset{\sim}{r} = (2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}) + t(3\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k})$

ii) Let the other line be
$\underset{\sim}{r} = (3\underset{\sim}{i} + 2\underset{\sim}{j} - 9\underset{\sim}{k}) + s(4\underset{\sim}{i} - 4\underset{\sim}{j} + 5\underset{\sim}{k})$

If they are skew then the do not cross

So try to find the point where they meet.

$$\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -9 \end{pmatrix} + s \begin{pmatrix} 4 \\ -4 \\ 5 \end{pmatrix}$$

① — $2 + 3t = 3 + 4s$
② — $-3 - t = 2 - 4s$
③ — $1 + 5t = -9 + 5s$

use ① & ② to find s & t $\qquad$ ① + ②

$-1 + 2t = 5$

$t = 3 \rightarrow s = 2$

However sub $s=2$ $t=3$ into (3) $1+15 \neq -9+10$

So no value of s & t give a common point so they are skew.

(4) i) if $x = \tan\theta$ $\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1}{(1+\tan^2\theta)^2}\,dx$

$$= \int \frac{1}{(\sec^2\theta)^2}\,dx$$

You should know this!

as $x = \tan\theta$ $\frac{dx}{d\theta} = \sec^2\theta$

as $I = \int \frac{1}{(1+x^2)^2}\,dx$ $\frac{dI}{dx} = \frac{1}{(1+x^2)^2}$

$$\frac{dI}{d\theta} = \frac{dI}{dx} \times \frac{dx}{d\theta} = \frac{1}{(\sec^2\theta)^2} \times \sec^2\theta$$

$$\frac{dI}{d\theta} = \frac{1}{\sec^2\theta}$$

$$I = \int \frac{1}{\sec^2\theta}\,d\theta = \int \cos^2\theta\,d\theta$$

ii) $\int_0^1 \frac{1}{(1+x^2)^2}\,dx$ need to change limits if we are integrating wrt θ

if $x = \tan\theta$ when $x=0$ $\theta=0$
when $x=1$ $\theta = \pi/4$

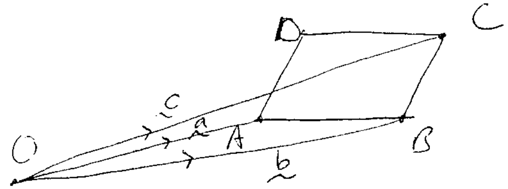
So $\int_0^{\pi/4} \cos^2\theta\,d\theta = \int_0^{\pi/4} \frac{1}{2}(1+\cos 2\theta)\,d\theta$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{4} + \frac{1}{2}\right) - (0+0)\right]$$

$$= \frac{\pi}{8} + \frac{1}{4} \quad \text{or} \quad \frac{1}{8}(\pi+2).$$

(5) i)



Position vector of D is $\overrightarrow{OD}$

$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD}$ $\overrightarrow{OA} = \underset{\sim}{a}$

$\overrightarrow{AD} = \overrightarrow{BC} = \underset{\sim}{c} - \underset{\sim}{b}$

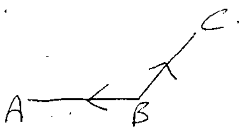
so $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \underset{\sim}{a} + \underset{\sim}{c} - \underset{\sim}{b}$

as $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$
$\underset{\sim}{b} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$
$\underset{\sim}{c} = \underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$

$$\underset{\sim}{a} + \underset{\sim}{c} - \underset{\sim}{b} = (2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}) + (\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}) - (3\underset{\sim}{i} - 2\underset{\sim}{j})$$

$$= 0\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k} = 2\underset{\sim}{j} + \underset{\sim}{k}$$

ii) To work out $\angle ABC$ do the dot product of $\vec{BA}$ and $\vec{BC}$.



$$\vec{BA} = \underset{\sim}{a} - \underset{\sim}{b} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k}$$
$$\vec{BC} = \underset{\sim}{c} - \underset{\sim}{b} = -2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$$
$$\vec{BA} \cdot \vec{BC} = (-\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k}) \cdot (-2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k})$$
$$= -1 \times -2 + 3 \times 1 + 3 \times -2 = -1$$
$$\vec{BA} \cdot \vec{BC} = |\vec{BA}||\vec{BC}|\cos\theta = \sqrt{1+9+9}\sqrt{4+1+4}\cos\theta$$
$$\therefore \quad -1 = \sqrt{19}\sqrt{9}\cos\theta$$
$$\frac{-1}{3\sqrt{19}} = \cos\theta$$
$$\cos^{-1}\left(\frac{-1}{3\sqrt{19}}\right) = 94.39^\circ$$

(6) i) $xy^2 = 2x + 3y$

product rule

$$x2y\frac{dy}{dx} + y^2 = 2 + 3\frac{dy}{dx}$$
$$2xy\frac{dy}{dx} - 3\frac{dy}{dx} = 2 - y^2$$
$$\frac{dy}{dx}(2xy - 3) = 2 - y^2$$
$$\frac{dy}{dx} = \frac{2 - y^2}{2xy - 3}$$

ii) If tangent parallel to y axis then

$$2xy - 3 = 0$$

Obviously it must be a point on the curve so $xy^2 = 2x + 3y$ $\therefore$ Solve Simultaneously

So $2xy = 3$
$$y = \frac{3}{2x}$$
$$x\left(\frac{3}{2x}\right)^2 = 2x + 3\left(\frac{3}{2x}\right)$$
$$\frac{9}{4x} = 2x + \frac{9}{2x}$$
$$9 = 8x^2 + 18$$
$$8x^2 = -9$$
$$x^2 = -\frac{9}{8}$$

but x^2 cannot equal $-\frac{9}{8}$

(7) $x = t^2 \qquad y = \frac{1}{t}$

i) $$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-1}{t^2} \div 2t$$
$$= \frac{-1}{2t^3}$$

ii) if $x = 4 \qquad y = -1/2$
$$t^2 = 4 \qquad \frac{1}{t} = -1/2$$
$\therefore$ $t = -2$ (not $+2$)

7 $\therefore \dfrac{dy}{dx} = \dfrac{1}{16}$. $\dfrac{y - -\frac{1}{2}}{x-4} = \frac{1}{16}$

$y + \frac{1}{2} = \frac{1}{16}(x-4)$

$16y + 8 = x \quad 4$

$\underline{\underline{12 = x - 16y}}$

iii) solve simultaneously $x - 16y = 12$
$x = t^2$
$y = 1/t$

So $t^2 - \dfrac{16}{t} = 12$

$t^3 - 16 = 12t \Rightarrow t^3 - 12t - 16 = 0$

We know $t=-2$ is a solution so $t+2$ is a factor

$t^3 - 12t - 16 \equiv (t+2)(At^2 + Bt + C)$

$\equiv At^3 + Bt^2 + Ct + 2At^2 + 2Bt + 2C$

$A = 1$
$B + 2A = 0$
$C + 2B = -12$
$2C = -16$

$A = 1$
$B = -2$
$C = -8$

$(t+2)(t^2 - 2t - 8)$

$(t+2)(t+2)(t-4)$

$\therefore$ only other solution is $\underline{t = 4}$

8 i) $\dfrac{3x+4}{(1+x)(2+x)^2} = \dfrac{A}{1+x} + \dfrac{B}{2+x} + \dfrac{C}{(2+x)^2}$

$\times(1+x)$ $\dfrac{3x+4}{(2+x)^2} = A + \dfrac{B(1+x)}{2+x} + \dfrac{C(1+x)}{(2+x)^2}$

Let $x=-1$ $\dfrac{1}{1} = \underline{\underline{A = 1}}$

$\times(2+x)^2$ $\dfrac{3x+4}{1+x} = \dfrac{A(2+x)^2}{1+x} + B(2+x) + C$

let $x = -2$

$\dfrac{-2}{-1} = \underline{\underline{C = 2}}$

$\dfrac{3x+4}{(1+x)(2+x)^2} \equiv \dfrac{1}{1+x} + \dfrac{B}{2+x} + \dfrac{2}{(2+x)^2}$

Let $x=0$ $\dfrac{4}{1 \times 2^2} = \dfrac{1}{1} + \dfrac{B}{2} + \dfrac{2}{4}$

$1 = 1 + \dfrac{B}{2} + \dfrac{1}{2}$

$\underline{\underline{B = -}}$

ii) $(1+x)^{-1} - (2+x)^{-1} + 2(2+x)^{-2}$

$\left(1 + -1x + \frac{-2\times -1}{2\times 1}x^2\right) - (2^{-1})\left(1+\frac{x}{2}\right)^{-1} + 2\times(2^{-2})\left(1+\frac{x}{2}\right)^{-2}$

$(1 - x + x^2) - \frac{1}{2}\left(1 - \frac{x}{2} + \frac{-1\times -2}{2\times 1}\left(\frac{x}{2}\right)^2\right) + \frac{2}{4}\left(1 - 2\frac{x}{2} + \frac{-2\times -3}{2\times 1}\left(\frac{x}{2}\right)^2\right.$

$1 - x + x^2 - \frac{1}{2} + \frac{x}{4} - \frac{x}{8} + \frac{1}{2} - \frac{x}{2} + \frac{3}{8}x^2$

$= 1 - \frac{5}{4}x + \frac{5}{4}x^2$

iii) $|x| < 1$ $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$

From (A)   From (B) & (C)

$\therefore |x| < 1$

9 i) $\theta - 20$ is the difference between Room temp. and temp. of liquid

K is constant of proportionality (−) to indicate temp. falling

$\frac{d\theta}{dt}$ is rate of change of temp w.r.t temp

$\therefore \frac{d\theta}{dt} = -K(\theta - 20)$

ii) $\frac{dt}{d\theta} = \frac{-1}{K}\frac{1}{\theta - 20} \Rightarrow t = \frac{-1}{K}\ln|\theta - 20| + c$

$-Kt = \ln|\theta - 20| + c$

$\ln|\theta - 20| = -Kt + c$

2 boundary conditions (1) $t = 0 \quad \theta = 100$
$t = 5 \quad \theta = 68$

~~$t = -K\ln|\theta - 20| + c$~~

~~$0 = -$~~

$\ln 80 = c$

$\ln 48 = -5K + \ln 80$

$5K = \ln 80 - \ln 48$

$5K = \ln\frac{80}{48}$

$5K = \ln 5/3$

$K = \frac{1}{5}\ln\frac{5}{3}$

$\therefore \ln|\theta - 20| = -\left(\frac{1}{5}\ln 5/3\right)t + \ln 80$

$e^{\ln|\theta - 20|} = e^{-\left(\frac{1}{5}\ln 5/3\right)t + \ln 80}$

$= e^{\ln|\theta - 20|} = e^{-\left(1/5 \ln 5/3\right)t}e^{\ln 80}$

$\theta - 20 = e^{-\left(1/5 \ln 5/3\right)t}80$

$\theta = 80e^{-\left(1/5 \ln 5/3\right)t} + 20$

iii) By another 32°C $t = 68 - 32 = 36$

if $\theta = 36$ $36 = 80e^{-\left(1/5 \ln 5/3\right)t} + 20$

$\frac{16}{80} = e^{-\left(1/5 \ln 5/3\right)t}$

$\ln\left(\frac{1}{5}\right) = \frac{-1}{5}\ln 5/3\, t$

$t = \frac{-5\ln\frac{1}{5}}{\ln 5/3} = \underline{\underline{15.75}}