Pair products

Generalising from number

Prerequisite knowledge

- The idea of consecutive numbers and how they can be generated from the first number
- Expansion of binomial brackets enables the underlying structure to be represented

Why do this problem?

This is a chance to make use of some standard algebraic manipulation to explain a numerical generalisation.

Time

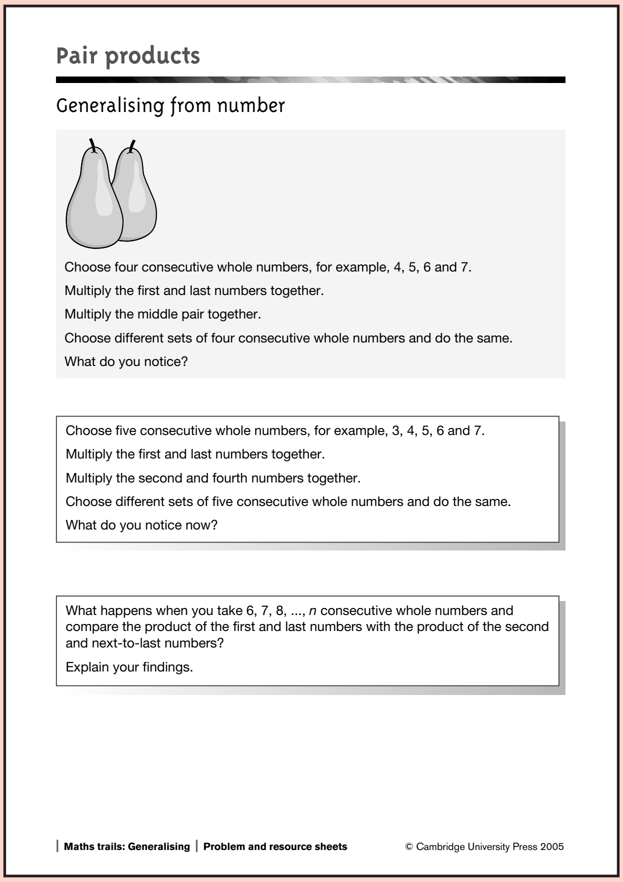
One lesson

Resources

CD-ROM: pupil worksheet

NRICH website (optional): www.nrich.maths.org, May 2004, 'Pair products'

Calculators should be available so that pupils can try a range of number groups without getting too bogged down with the arithmetic.



Pair products

Generalising from number

Choose four consecutive whole numbers, for example, 4, 5, 6 and 7.
Multiply the first and last numbers together.
Multiply the middle pair together.
Choose different sets of four consecutive whole numbers and do the same.
What do you notice?

Choose five consecutive whole numbers, for example, 3, 4, 5, 6 and 7.
Multiply the first and last numbers together.
Multiply the second and fourth numbers together.
Choose different sets of five consecutive whole numbers and do the same.
What do you notice now?

What happens when you take 6, 7, 8, ..., *n* consecutive whole numbers and compare the product of the first and last numbers with the product of the second and next-to-last numbers?
Explain your findings.

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Introducing the problem

Use a short introduction on consecutive numbers, involving some mental arithmetic:

- Give the next five consecutive numbers after 9999 ...

Move on to the first example on the problem sheet: using 4, 5, 6 and 7, calculate the product of the inner and outer number pairs. Ask for any observations from pupils.

It is difficult to generalise from just one result so suggest pupils try some more sets of four consecutive numbers. At this stage encourage pupils to work individually – this may help all of them to understand the requirements of the problem.

Main part of the lesson

After 5 minutes stop the class to spend some time collating findings on the board without immediately discussing what pupils noticed. This will give pupils who have not had time to do more than one or two examples, or who have not spotted a pattern, to be able to see sufficient data to identify a pattern for themselves.

Ask the class for observations and make a note – this might include a range of ideas. For example:

- 'The products are always even.'
  Is this always the case? Why?
- 'The difference in the two products is always 2.'
  Is this always the case? Why?

Encourage pupils to work on explaining or refuting the theories they have just developed. It may be worth talking about using symbols to express consecutive numbers – this is especially necessary for the second observation (the difference in the two products always being 2) but pupils may be able to explain the first in terms of the sequence of odd and even numbers.

After a reasonable amount of time, working in small groups, pupils may be ready to give a convincing argument for one or more of the rules and why they do or do not work. These findings will need some pulling together and feedback, including some help with representation and confirmation of basic principles.

The last part of the main activity will be extending the ideas as on the problem sheet, with the potential to go even further by considering how the product of the outside pair compares with the product of other pairs. Pupils could work towards displaying one finding with an explanation of why it is or is not true.

Plenary

A pulling together of pupils' findings can be achieved in many ways including:

- pinning the posters up for display;
- choosing one or two groups to share their findings with the rest of the class.

Solution notes

For four consecutive numbers the product of the first and last numbers is always 2 less than the product of the middle two numbers.

Explanation: Suppose the first number is x. Then the second number is $x + 1$, the third is $x + 2$, and the fourth is $x + 3$.

The product of the first and fourth numbers is: $x(x + 3) = x^2 + 3x$.

The product of the second and third numbers is: $(x + 1)(x + 2) = x^2 + 3x + 2$.

So $(x + 1)(x + 2) = x(x + 3) + 2$ for any chosen value of x.

With five consecutive whole numbers the product of the first and last numbers is always 3 less than the product of the second and fourth numbers.

Explanation: Using the same symbols as above, the product of the first and last numbers is: $x(x + 4) = x^2 + 4x$.

The product of the second and fourth numbers is: $(x + 1)(x + 3) = x^2 + 4x + 3$.

So $(x + 1)(x + 3) = x(x + 4) + 3$ for any chosen value of x.

With *n* consecutive whole numbers the product of the first and last numbers is always $n - 2$ less than the product of the second and penultimate numbers.

Explanation: Using the same symbols as above, the last number (the nth number) will be $x + n - 1$. Thus the penultimate (next-to-last) number will be $x + n - 2$.

The product of the first and last numbers is: $x(x + n - 1) = x^2 + nx - x$.

The product of the second and the penultimate numbers is:

$$\begin{aligned}(x + 1)(x + n - 2) &= x^2 + nx - 2x + x + n - 2\\ &= x^2 + nx - x + n - 2.\end{aligned}$$

So $(x + 1)(x + n - 2) = x(x + n - 1) + n - 2$; that is, the product of the second and penultimate numbers will always exceed the product of the first and last numbers by exactly $n - 2$.