Introduction

The emphasis of this book is on the art and techniques associated with the act of mathematical problem solving. The book contains 23 problems which benefit in some way from having a spreadsheet available, either as a ready-made environment to explore or as a tool for helping the solver create a model to use in their search for a solution. Not all problems require a spreadsheet in order to be solved. The emphasis is, rather, on how a spreadsheet could potentially help. A spreadsheet can help to reveal a pattern which it might otherwise be difficult to spot or labour-intensive to produce. In these situations the spreadsheet does not explain why the pattern is what it is (the bit of mathematics we are really interested in) – that is the task of the problem solver. However, when part of the task is the production of a spreadsheet, it is necessary for the solver to understand and represent the mathematics within the problem. This is at the heart of the experience for the learner.

This book is aimed principally at teachers of pupils between the ages of 11 and 16. Each problem is accompanied by guidance that includes information on prerequisite knowledge, lesson outlines and solution notes. Additional teaching resources, supporting spreadsheets and pupil reference sheets are included on the CD-ROM. Links are also given to complementary material on the NRICH website (www.nrich.maths.org). Resources on the NRICH website can be located either by using the date of publication in the text or by typing the name of the resource into the search box on the site.

The book is written to promote three main ideas:

- problem solving;
- using spreadsheets to help in posing or solving problems;
- learning spreadsheet skills that can make the solution of problems less labour-intensive.

The problems in this trail have been chosen because they lend themselves to being tackled with the aid of a spreadsheet – although none of them demands the use of a spreadsheet. You could therefore use this book for the problems and ignore the spreadsheet aspect of the trail altogether but clearly this would miss the point! The real challenge is for the problem solver to be able to identify the benefits for themselves. We suggest interspersing problems in this book with others that are less likely to benefit from the use of a spreadsheet and encouraging pupils to identify opportunities for the use of spreadsheets for themselves.

One issue you may have to deal with is that of pupil expectation. Many pupils expect to have to use the equipment as soon as they walk into a computer room. One way to overcome the tail of Information and Communications Technology (ICT) wagging the mathematical dog is to introduce problems in advance of the main lesson and invite pupils to think about potential routes to a solution. Remember that elegant solutions can be obtained in ways other than through the use of a spreadsheet. Having access to a computer does not mean that you have to use it! Able pupils may often find a very elegant algebraic solution and the principal aim should be to solve the problem not to use the spreadsheet. The NRICH website often includes solutions to problems that have not made use of a spreadsheet, so it is worth taking a look.

Problem solving and mathematical thinking

The purpose of this maths trail is to help give some meaning to problem solving with a particular focus on using spreadsheets. This book forms part of a series of titles which includes *Generalising*, *Working Systematically* and *Visualising*. Each book looks at particular skills that can be useful in the problem-solving process. While this collection of books promotes particular problem-solving skills, working through them is not on its own a measure of an expert problem solver.

This book is about encouraging pupils to 'be mathematical' in the sense that they:

- engage in problem-solving activities;
- think about and communicate their ideas;
- create and identify mathematical problems within given contexts.

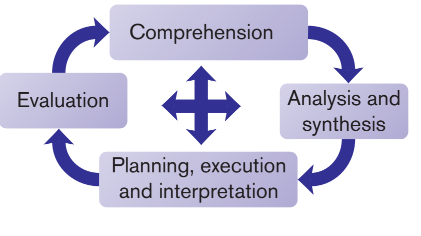
Being mathematical and doing mathematics in this sense involves some mathematical content knowledge, but mathematics is more than learning facts and practising skills and we need to support learners in 'being mathematical' in this deeper sense. We need to offer them the opportunity to be problem solvers who pose and explore their own problems. This book offers some guidance and structure upon which to focus if we are to support pupils' problem solving effectively.

What is problem solving?

When we are presented with a mathematical problem, it is only a problem if we do not immediately know how to solve it. This means that something that is a problem to one person may not be a problem to another. The process of problem solving is like a journey from a state of not knowing what to do towards a destination which we hope will be a solution. The key is to have some strategies at our fingertips that will help us to identify a possible route through to a solution. Our mathematical journey is often full of twists and turns, where we revisit ideas or need to step back and look for alternatives. Often a mistake or dead-end gives vital clues to the mathematics of the problem and is therefore crucial in any solution process.

To help us have a sense of direction when we are problem solving and decide what might be a good strategy to try next, it can be useful to have a model. There are many problem-solving models but the one below is a good starting point. Although the steps are written in sequence, the diagram emphasises the reality that problem solving can be a messy process which often involves revisiting places on the journey from problem posing to problem solving.

Spreadsheets can play an important role in many of the stages of the problem-solving process.



Comprehension

This stage is about making sense of the problem by using strategies such as retelling, identifying relevant information and creating mental images. Pupils can be helped at this stage by being encouraged to re-read the problem several times, or to investigate the context by trying things out, and to record what they understand the problem to be about (for example, by drawing a picture or making notes).

Environments that encourage exploration, such as those created within a spreadsheet program, can motivate pupils to 'get stuck in' – to want to discover what is going on. Similarly, the idea of using spreadsheets as a support for solving the problem can give pupils just enough of a nudge in a useful direction to get them going. Evidence suggests that the use of ICT can be a motivation in itself.

Analysis and synthesis

This stage is about 'homing in' on what the problem is asking solvers to investigate.

- Can they represent the situation mathematically?
- What is it that they are trying to find?
- What do they think the answer might be (conjecturing and hypothesising)?
- What might they need to find out before they can get started?

Central to this stage is identifying what is unknown and what needs finding. Here pupils might think about what information they might include in a spreadsheet – giving a focus for their search. If pupils are exploring a ready-made environment the immediate feedback from any exploration can not only clarify their ideas but encourage them to pose problems of their own.

Planning, execution and interpretation

Once pupils have understood what the problem is about and have established what needs finding, the next stage is about planning and executing a pathway to the solution. It is within this process that you might encourage pupils to think about whether they have seen something similar before and what strategies they adopted then. This will help them to identify appropriate methods and tools. Particular knowledge and skills gaps that need addressing may become evident at this stage.

As pupils think about techniques that would be useful and how a spreadsheet might help, they need to be planning an approach to solving the problem. There is a complex interplay between considering how a problem might be solved and building up a model. Careful thought needs to be given to the representation of variables and controlling change. There is likely to be continual movement between planning and execution, with refinement and review forming an integral part of the solution process. Pupils may need additional encouragement to write down their findings when they are using a computer.

During the execution phase, pupils might identify further related problems they wish to investigate. They will need to consider how they will keep track of what they have done and how they will communicate their findings. This will lead on to interpreting results and drawing conclusions.

Evaluation

Pupils can learn as much from reflecting on and evaluating what they have done as they can from the process of solving the problem itself. During this phase pupils should be expected to reflect on the effectiveness of their approach as well as those of others, to justify their conclusions and assess their own learning. Evaluation may also lead to thinking about other questions that could now be investigated.

When pupils have used a spreadsheet they should be encouraged to discuss how the use of ICT helped them, whether they used the technology efficiently and how the technology opened up, or constrained, possibilities. In lesson plenaries, you may wish to introduce spreadsheet techniques that might have made a solution more efficient. These can be followed up in later lessons.

Using the model

The use of a problem-solving model has a number of benefits in the classroom.

- It is a structure that can help pupils frame their problem solving and keep track of where they are in the process.
- It gives us a language which helps us to talk to pupils about what they are doing. For example:
  - 'Can you tell me what you think the problem is about?'
  - 'What are you trying to find?'
  - 'Have you seen anything like this before and what did you do then?'
  - 'Could you have solved this in a different way?'
- It offers a framework which can help structure lessons so that you can plan what pupils might do at various stages.

We think it is very important to draw the attention of your pupils to the model during their work on each task. Many of the activities involve engagement with all aspects of the problem-solving 'cycle' but certain problems lend themselves to focusing on particular aspects that can be highlighted and made a focus for the lesson.

What is mathematical thinking?

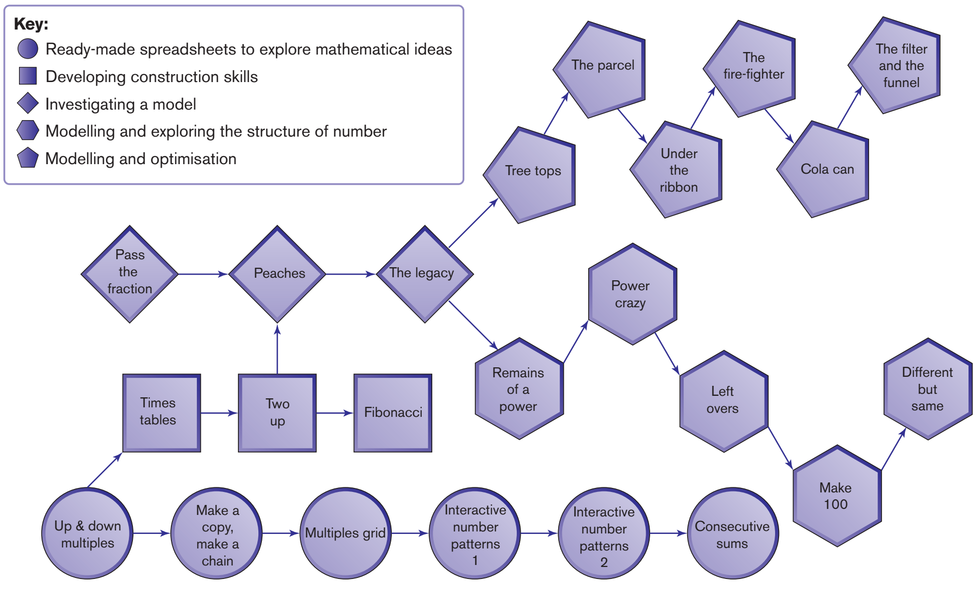
Within the problem-solving framework, there are many other mathematical skills which pupils need to have at their fingertips. These skills involve more than numeric, geometric and algebraic manipulation. They include strategies such as:

- modelling;
- visualising;
- working systematically;
- generalising.

These elements of mathematical thinking are needed to engage in the problem-solving process. The problems presented in this book encourage a number of these aspects of mathematical thinking. For example, many of the problems require pupils to understand a ready-made spreadsheet model or create a model of their own.

The 'excel at problem solving' trail

The trail structure is shown over the page and runs from left to right. The problems are in a suggested order with more difficult problems (related to mathematical content knowledge, the level of the problem solving needed or ICT skills) further along the trail.



Why use this trail?

The problems in the trail support problem posing as well as the problem-solving process. The trail attempts to offer a range of contexts which require spreadsheets to be used in different ways.

- Firstly, the spreadsheet can act as an environment where the pupil is asked to investigate what is happening and then try to explain it. A simple example of this can be found in 'Interactive number patterns 1'. Here, the environment needs to be explored and the underpinning mathematics explained. These environments may be used as a stimulus for class discussion or for individuals or small groups to act upon.
- Secondly, the spreadsheet may act as a labour-saving device, enabling the user to generate a large amount of information out of which it is possible to spot patterns. For example, in 'Make 100' the spreadsheet does not explain the pattern but helps the solver to generate enough examples for a pattern to emerge. The challenge is to explain the pattern.
- Thirdly, the spreadsheet can be used to achieve rapid repeated calculation. This allows the solver to focus on the context, such as searching for an optimal solution, rather than the calculation itself. There are two types of problem of this general kind.
  - In the first type the model is already created for the user, as, for example, in 'Remains of a power'. In this case pupils move towards an understanding of what the model is doing. We describe this process of examining the structure and making sense of the formulae used as 'deconstruction'.
  - In the second type pupils create a model for themselves, as in, for example, 'Tree tops'. Here they unpick the underlying mathematics in order to understand what is needed and construct a model for themselves.

In engaging with the problems in this book pupils might:

- investigate a pre-programmed model to identify patterns and relationships and its underpinning rules;
- investigate the consequences of altering the values of variables in a model;
- design and construct models of their own;
- evaluate models and share ideas.

How to use the trail

This trail is an organised set of curriculum resources, including teacher notes and hints for pupils, designed to develop purposeful use of a spreadsheet. The trail contains 23 problems and a guide to the Excel techniques used throughout the book. The trail has three main pathways:

- ready-made spreadsheets to explore mathematical ideas;
- mathematical tasks that can be used to build confidence with using spreadsheets;
- problems which explore mathematical structure and optimisation.

We have divided this last type of resource into three main categories. Firstly, there are problems where learners are invited to explore a ready-made model and secondly, there are problems where the model gives opportunities to explore the structure of number. Finally, we have included problems that involve using a model to aid optimisation. We suggest that the problems offered of the first type are the most accessible and therefore might be attempted before moving on to exploring the structure of number and/or optimisation.

Sometimes we place an emphasis on deconstruction of existing spreadsheets and at other times the emphasis is on pupils constructing their own spreadsheet models. In any activity where we suggest the construction of a spreadsheet by pupils you may feel it is more appropriate for pupils to work on understanding the structure of a pre-prepared spreadsheet. We therefore have included ready-made spreadsheets for all the problems.

You might wish to use the trail as a 'course' for pupils over a short or long period of time, with them working as a whole class, in small groups or individually. The trail indicates an ordering of the materials to support each pupil's developing skills over time. However, it is also possible to dip in and out of the materials at appropriate points in a scheme of work. The timings indicated in the teacher notes for each unit are indicative as the intention is to encourage extension and pupil investigation beyond what is made explicit. The trail is not intended to be a strait jacket.

The problems offer opportunities for pupils of a wide age and ability range, and do not imply a particular view of classroom organisation. However, there is an underlying message concerning classroom practice and the learning of mathematics as a collaborative experience, valuing the journey through a problem rather than just the answer. While it is not necessary for pupils to work in groups, there is an expectation that pupils will be given opportunities to talk about their mathematical experiences *en route* as well as at the 'conclusion' of their investigations.

Questions to consider when using spreadsheets

The following questions may serve as a useful focus for reflection on the use of spreadsheets.

- What aspect of mathematics is the work addressing? Is it doing so successfully?
- What prerequisite skills are needed before starting the task (mathematics and ICT)?
- Is any specific vocabulary needed in order to be able to talk effectively about the activity, the model or the outcomes?
- How is the spreadsheet work supporting the mathematics being covered?
- Is using the spreadsheet enabling you to do things you would not have been able to do better (or as well) another way?

Ideas for managing sessions

There are no step-by-step rules that will always lead to a solution in any of the problems we have used in this book. The aim is to encourage pupils to share ideas and different approaches and learn to value those differences while identifying elegant and efficient solutions whenever appropriate. After working on this trail, pupils should feel able to recognise where the use of a spreadsheet might be helpful and feel empowered to use the ICT skills they have learnt appropriately and more effectively when problem solving in the future.

The lesson notes included in this book are intended purely as a guide. As indicated in what follows, there are as many approaches to teaching as there are pupils in a class! All we can do is offer some ideas without any intention of being prescriptive.

Working with the whole class

The following might be a typical whole-class approach to a problem.

Introduce the problem

Building on the introductory activity, introduce the problem and, where it is helpful, start to model an approach or share your thoughts on what mathematics underpins a situation. Then ask the pupils to work entirely on their own for five minutes – giving them time to experiment and to familiarise themselves with the context. At this point it is worth emphasising that pupils are not expected to be working neatly towards a solution, either in terms of what they might do with an existing spreadsheet or in terms of what a spreadsheet they are going to create for themselves might look like. They are simply finding out what the problem might be about. After a short time you will stop them and ask them to work with a partner in order to share what they have discovered or thought about. The aim of the time spent working in small groups and pairs is to share initial ideas which can then be discussed as a whole class. This part of the lesson corresponds to the 'comprehension' phase of the problem-solving process.

Sharing and moving on

Stopping pupils after a short period of time to share findings and ideas of what the problem is about offers opportunities to 'get started' for those who have not found a way into the problem and the chance to refine and develop ideas as a community. It is not enough to 'know how to do it' but to leave room for new ideas and questions to be discussed. This is about valuing the journey, including the cul-de-sacs we may encounter or the different routes we may take. During this time pupils are beginning to identify potentially promising approaches to tackling the problem.

Planning, execution and interpretation

So far we have encouraged our pupils to:

Think – Pair – Share

After these early discussions pupils need further time to investigate the problems and consider possible routes to their solutions, create spreadsheet models of their own and/or consider further problems that might arise. Sharing and iteration of discussions will help to give all a sense of owning the mathematics and ensure that, as far as possible, as many different approaches to the problem are considered, not simply 'the answer'.

Evaluation

During this last phase pupils discuss their findings, convincing themselves, and their classmates, that any findings they have or conjectures that they wish to put forward are reasonable. Time considering different solutions and their 'efficiency' or 'accessibility' are invaluable in opening up the mathematics and helping pupils value different approaches. Discussing the value that the use of spreadsheets has brought to the problem is an essential element of any lesson. Then, of course, it is worth considering all the other problems or ideas that the context has stimulated pupils to consider, and the process can start again!

Working with individual pupils

Trails can also be a useful tool for teachers to use with individual pupils who need the challenge of problem-solving activities that have the potential to incorporate the use of ICT. Pupils who quickly grasp the mechanical aspects of mathematics and/or ICT but find it difficult to work on open-ended tasks can be encouraged to tackle some of the problems in the trail. The sense of direction and purpose in the trail, when shared with the pupil(s), can give them the opportunity to build up a repertoire of approaches to such problems, particularly in emphasising where there are opportunities to use spreadsheets, and give them more confidence when confronted with similar activities in the future.

Working with groups

Groups of pupils can often find their own way into problems simply by being given the context and being asked to:

- identify what the question is about;
- consider strategies for solution;
- plan what they are going to do;
- execute their plans.

One important aim might be to encourage pupils to communicate their findings to others in the class or their teacher, describing the problems they had as well as how they chose their methods and executed their solutions. This communication does not have to take the form of written output but could be verbal or in the form of a poster or presentation.

Finally encourage the group to answer evaluative questions such as:

- Could we have done this more efficiently?
- What have we learnt that is new?
- Have we met anything like this before? Were we able to make connections?
- What additional questions did we come up with and answer while we were working on the problem?
- Are there some questions still to be answered?
- How did the spreadsheet help?

Many further examples of problems are available on the NRICH website (www.nrich.maths.org) if you wish to extend any of the work in a particular area of mathematics or simply to reinforce ideas and skills. Happy trailing!

Assessment

Prerequisite knowledge

Three aspects to pupils' prerequisite knowledge need to be considered.

Firstly, we need to consider pupils' abilities to tackle problem-solving situations independently. This trail emphasises the importance of applying problem-solving and mathematical thinking skills and, while it continues to employ and develop those skills, complementing the work in other books in this series, it does not assume any particular proficiency in or familiarity with problem solving. As the main purpose of the trail is to support, develop and extend these skills, it means that, as a teacher, the amount of scaffolding and support you will be offering will vary considerably according to pupils' prior experiences of problem solving and their individual needs.

Secondly, each problem depends upon knowledge of particular aspects of mathematical curriculum content.

Lastly, a problem may assume familiarity with a particular spreadsheet skill.

Assessment for learning

The notes and other documentation for each problem aim to support formative and summative assessment opportunities. Outline solutions to problems are included to give some guidelines on expected outcomes. Edited pupil solutions can be found to many of the problems on the NRICH website at www.nrich.maths.org. The advantage of looking at solutions on the website is that they can give you an idea of the variety you can expect from your pupils.

In some cases, it might be best not to highlight the content objectives at the beginning of the lesson as doing this could close down possibilities for pupils to think independently. We therefore suggest that the lesson objectives shared with the pupils should not 'reveal' obvious routes to a solution.

Listening and questioning are important tools in the process of formative assessment. To support this:

- all problems have suggested prompts for teachers and mentors to use;
- pupils are encouraged to hypothesise and share ideas with fellow pupils, arguing their case – these are ideal opportunities to listen;
- whole-group discussions during the lesson can be used to reveal pupils' understanding, misconceptions or lack of awareness of, or confidence in, the necessary mathematical knowledge;
- peer assessment can often shed valuable light on the understanding of the assessor as well as the assessed;
- reviewing and reflecting on the lesson outcomes with the pupils can help the teacher make judgements and also be used by pupils as an opportunity for self or peer assessment.

As highlighted earlier, much of the work and learning is about the journey through each problem. It is not necessary for pupils to have well-rounded, written solutions for sound assessment judgements to be made. While feedback through marking is sometimes appropriate, oral and continuous feedback throughout the problem-solving process is just as valuable.

'To be effective feedback should cause thinking to take place'

Assessment for learning in everyday lessons (DfES, 2004)

Assessing learning over the whole trail

In any problem there is the potential for four main outcomes:

- the introduction, development or consolidation of an aspect of mathematics;
- application and practice of problem-solving skills;
- practice and extension of capability with spreadsheets;
- application of mathematical understanding and technology to the solution of a problem.

As a result of having ICT and mathematical outcomes attached to any task there is a need to find the balance which does justice to them both. Associated with this balance is the way in which ICT is integrated into the teaching and learning. Do ICT skills need to be learnt separately, in advance of their application, or can they be successfully integrated into the mathematics lesson without losing the focus on mathematics? It is worth reiterating that all the activities in the book place the emphasis firmly on the mathematics. Some additional time may be needed to support the ICT needs of your pupils, or help your pupils support each other in securing the particular techniques they might need when creating a spreadsheet or modifying an existing spreadsheet. Often this ‘additional’ time (what we call the ‘ICT overhead’) should not be very large in any single unit. If pupils use spreadsheets regularly, they will build up expertise and any new technique will only be a small step on from where they already are. In addition, ICT lends itself to peer support. Our experience in the classroom indicates that some pupils already have the skills needed or can pick them up very quickly and help others in the class. It is not necessary for the teacher to have to teach every child every new technique individually.

Resources

The resources for every problem include a range of spreadsheets in a single Microsoft Excel workbook file, along with supporting problem and resource sheets. The content of each Excel file reflects the nature of the activity and, as a result, their content varies. Each element of the resource is referred to within the book's text. The nature of the content of each spreadsheet is very dependent on the unit it supports.

- Units where pupils will be using the spreadsheet as an interactive environment may contain more than one sheet, each with slightly different versions of the environment. These pre-configured conversions enable the teacher and pupils to focus on and develop particular aspects of the problem.
- Units where pupils are expected to produce spreadsheet models of their own will have Excel workbooks with sheets that contain examples of what to expect. These sheets can be used for demonstration purposes or for teacher reference.
- Units where pupils may be expected to explore an existing spreadsheet and amend it in the course of their work will contain Excel workbooks with partially completed sheets that can be used for demonstrating particular points, as well as completed examples of possible solutions.
- Where a new Excel technique might be used the Excel workbook contains a tutorial sheet.
- Solutions are often contained within the example spreadsheet and where this is the case they are referred to in the text.

Excel workbooks normally contain more than one spreadsheet – accessible via the tabs at the bottom of each sheet. Tabs have been colour-coded to help you find your way around:

- blue for sheets which might form an integral part of the lesson;
- green for supplementary sheets;
- red for example solution sheets;
- yellow for techniques.

The spreadsheet techniques are also included in the final section, 'Some Excel techniques explained'.

Excel techniques and use of the CD-ROM material

The use of ICT normally involves a tension between the learning of skills and the application of those skills. Many pupils are taught ICT skills in isolation. This book promotes a view that the ICT is there to support and enhance the mathematics. The need to develop and practise skills should arise naturally from the needs of a problem. The skills that might be required are part of a single learning experience for the pupil. Through engaging in the tasks in this trail pupils will:

- learn about spreadsheets through problem solving;
- use spreadsheets to support the problem-solving process.

Some skills are usually assumed. Pupils must be able to:

- input formulae;
- copy and paste;
- save different versions of the same sheet.

Other important techniques are described within the units where they occur, with technical instructions in the section 'Some Excel techniques explained' at the end of the book. In summary these are:

- **Using a graph.** Optimisation problems (for example 'The filter and the funnel' or 'Cola can') can be managed without a graph by examining the values in cells, but adding the graph offers a view of how the function behaves generally and can enrich our understanding of the problem and its solution.
- **Relative and absolute references.** This option enables a single cell value to control many other cells without losing the ability to swiftly replicate formulae using **Fill down** and similar techniques. In 'Make 100' and 'Different but same' this choice between relative and absolute references enables a complete table of results to be produced from a single formula.
- **Conditional formatting** can be enormously useful because it makes conspicuous the values that are of particular interest. These values could be located by simple cell-by-cell inspection, but colour change via conditional formatting not only picks these values out but often draws attention to pattern as, for example, in 'Power crazy'.

Finally there are two functions in Excel that are well worth knowing about from a problem-solving point of view. The **MOD** function produces the remainder and is used, for example, in 'Remains of a power'. The **INT** function separates the integer and decimal parts of a number and is used to powerful effect to isolate the digits within a value.

Computer screen specifications vary. The display size for any spreadsheet can be adjusted using **Zoom** on the **View** menu.

Some further reading

Burton, L. (1984) *Thinking Things Through*. Oxford: Basil Blackwell.
Mason, J. with Burton, L. and Stacey, K. (1982) *Thinking Mathematically*. Wokingham: Addison-Wesley.
Piggott, J. (2005) 'An investigation into the nature of mathematical enrichment: a case study of implementation'. Ph.D. dissertation, Institute of Education, London.
Polya, G. (1945) *How to Solve It*. London: Penguin.