Consecutive sums

Framing

Prerequisite knowledge

- The terminology of consecutive numbers including consecutive even and consecutive odd numbers
- The ability to use symbols to support explanations is useful but not essential

Why do this problem?

This is a problem with many avenues to pursue. The context offers opportunities for pupils to pose their own questions and this is worth encouraging, although having some ideas up your sleeve can help those pupils who are struggling with finding a direction. This problem is an example where there is a natural context for using some simple algebra to explain patterns and for simplifying expressions to see equivalence.

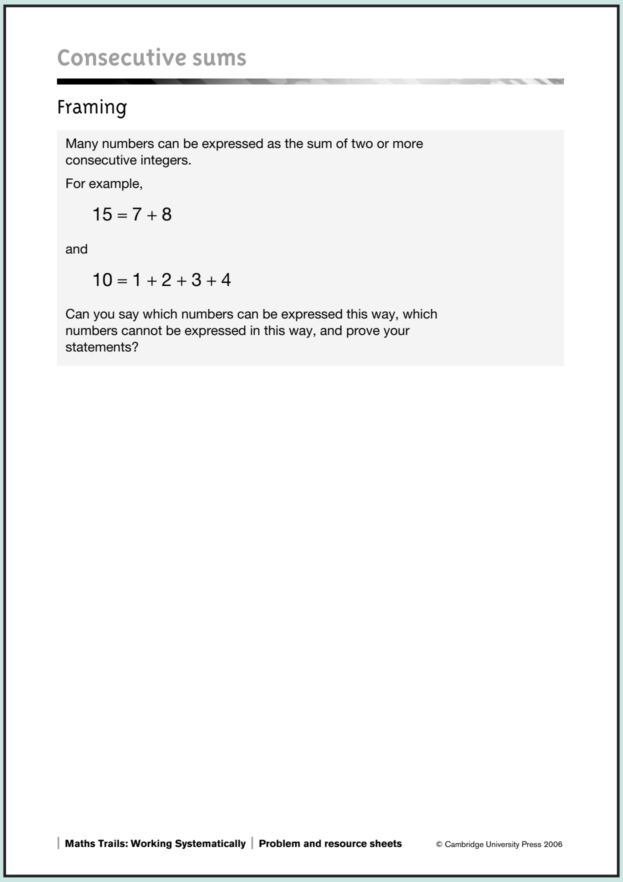
Time

Two lessons or more

Resources

CD-ROM: problem sheet

NRICH website (optional): www.nrich.maths.org, May 1997, ‘Consecutive sums’



Consecutive sums

Framing

Many numbers can be expressed as the sum of two or more consecutive integers.

For example,

$15 = 7 + 8$

and

$10 = 1 + 2 + 3 + 4$

Can you say which numbers can be expressed this way, which numbers cannot be expressed in this way, and prove your statements?

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Introducing the problem

Ask pupils some questions about consecutive numbers as an introduction, for example:

- What are consecutive numbers?
- Give four, five, six, … consecutive integers starting at 0, 3, 52, –10.
- Give four, five, six, … consecutive even numbers or odd numbers from different starting points.
- Can you find three consecutive even numbers that total 24? Can you find three consecutive odd numbers that total 24?

After reminding the class of properties of odds, evens and consecutive numbers, mental calculations involving consecutive numbers and useful shortcuts may give some clues to what follows. For example, $50 + 51 = 2 \times 50 + 1$.

Main part of the lesson

This problem has an open starting point, enabling pupils to select one aspect for development. The aim is for the pupils to produce posters in the second lesson that will pull together all their ideas.

Start with the examples given in the problem and ask the class if there are any other ways of making 15 ($7 + 8$, $4 + 5 + 6$, $1 + 2 + 3 + 4 + 5$), then 10, …

Show the group the problem. Ask pupils to work on their own for five minutes and then work with a partner.

After ten minutes stop the group and ask for ideas or anything they might have noticed.

List some questions (see the list below) that come from their first thoughts.

Before the pairs settle down to the investigation in earnest, suggest that they may wish to focus on one particular idea or question.

Stand back without intervening for at least ten minutes and note points you may wish to pursue with the class.

To prompt groups you might ask questions such as:

- What do you notice about numbers that are the sum of two consecutive numbers?
- What do you notice about numbers that are the sum of three consecutive numbers?
- Given any number, can you identify how many different ways you can write it as the sum of consecutive numbers?
- What numbers cannot be written as the sum of consecutive numbers?

Plenary

At the end of the first lesson the plenary can focus on some of the questions being tackled and point towards the production of posters in the next session. This will help pupils feel confident that there is not a 'right' answer but many answers and many problems!

Lesson 2

After recapping the ideas arising from the previous lesson, pupils should set to work producing posters which justify their findings using convincing arguments ready to share with the whole class at the end of the session. Similar prompts to those offered in the first lesson can be used and additional guidance will be needed to support the presentation of findings, hypotheses and proofs.

Solution notes

This solution refers to the case of positive integers.

Algebra has been used to keep the solution brief. There is no implication that pupils will use algebra as part of their justifications.

The number of ways of writing a number as the sum of consecutive integers is related to its oddness and its odd factors. All odd numbers can be written as the sum of two consecutive integers. For any odd number k, we know that $k-1$ and $k+1$ are even, also $(k-1)/2$ and $(k+1)/2$ are consecutive integers, so we can write $k = (k-1)/2 + (k+1)/2$.

All numbers of the form $N = pk$, where p and k are integers, and k is odd and greater than 1, can be written as the sum of consecutive integers.

For example,

$$30 = 3 \times 10 = 9 + \mathbf{10} + 11$$

$$30 = 5 \times 6 = 4 + 5 + \mathbf{6} + 7 + 8$$

$$30 = 15 \times 2 = \ldots + \mathbf{2} + \ldots$$

It is not possible to find seven numbers less than 2 but greater than 0.

If a number is a power of 2 then it is impossible to write it as a sum of consecutive integers.