Cubes within cubes

Modelling a situation

Prerequisite knowledge

- Properties of cubes

Why do this problem?

This problem offers opportunities for pupils to present their thought processes, rather than just their findings, in the form of a storyboard. Storyboards enable pupils to tell the story by illustrating or making notes of each significant stage in the process, rather like a cartoon strip.

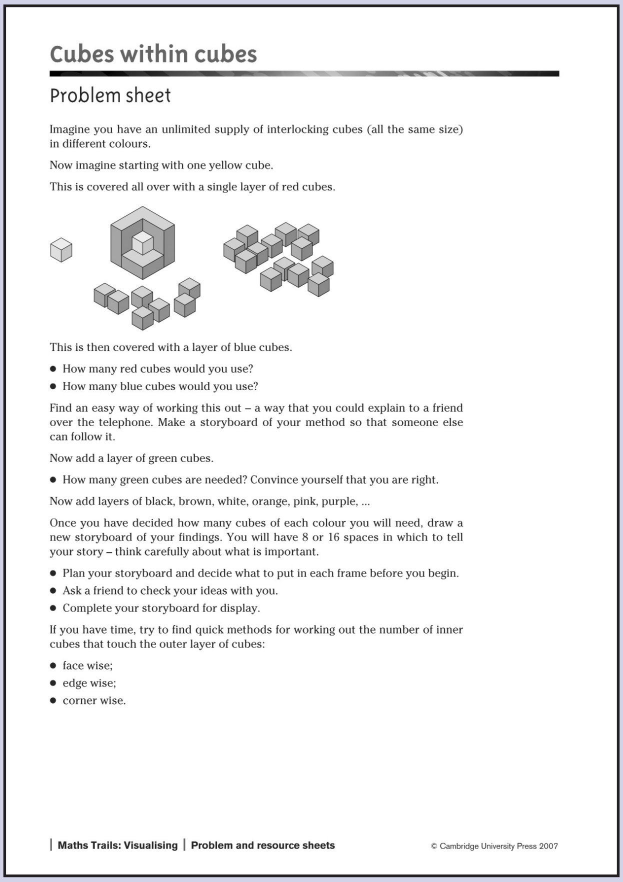
The geometric context lends itself to some nice algebra with plenty of room to extend the problem to consider the positions of different small cubes relative to the 'inner' cube.

The level at which pupils use algebraic notation can vary greatly. Descriptive 'proofs' are perfectly adequate for younger or lower-attaining pupils. There are opportunities to investigate the equivalence of algebraic expressions arising out of the problem. ICT could be used to present findings.

This problem also gives opportunities to pay attention to the planning, execution and interpretation, and evaluation phases of the problem-solving model.

Time

One lesson


Cubes within cubes

Problem sheet

Imagine you have an unlimited supply of interlocking cubes (all the same size) in different colours.

Now imagine starting with one yellow cube.

This is covered all over with a single layer of red cubes.

This is then covered with a layer of blue cubes.

- How many red cubes would you use?
- How many blue cubes would you use?

Find an easy way of working this out – a way that you could explain to a friend over the telephone. Make a storyboard of your method so that someone else can follow it.

Now add a layer of green cubes.

- How many green cubes are needed? Convince yourself that you are right.

Now add layers of black, brown, white, orange, pink, purple, ...

Once you have decided how many cubes of each colour you will need, draw a new storyboard of your findings. You will have 8 or 16 spaces in which to tell your story – think carefully about what is important.

- Plan your storyboard and decide what to put in each frame before you begin.
- Ask a friend to check your ideas with you.
- Complete your storyboard for display.

If you have time, try to find quick methods for working out the number of inner cubes that touch the outer layer of cubes:

- face wise;
- edge wise;
- corner wise.

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Resources

CD-ROM: problem sheet; resource sheets 1 and 2
NRICH website (optional): www.nrich.maths.org, February 2003, 'Cubes within cubes'

Introducing the problem

Ask pupils to imagine they have an unlimited supply of interlocking cubes (all the same size) in different colours.

Invite them to imagine starting with one yellow cube. This is surrounded by a layer of red cubes which forms an outer cube.

- Describe what you see.
- How many red cubes touch the yellow cube face to face? [6]
- Which red cubes touch the edges of the yellow cube? [the ones in the middle of each edge of the large red cube]
- How many red cubes touch the yellow cube in this way? (Be careful!) [12]
- How do the red cubes in the corners of the large red cube touch the yellow cube? [only at the corners]
- How many touch in this way? [8]
- How many red cubes are there altogether? [26]
- How many red and yellow cubes are there? [27] Can you explain why? [3^3]

Main part of the lesson

Now ask the class to imagine that the red cube is covered with a layer of blue cubes.

Ask pairs of pupils to work out how many blue cubes are needed and to make a note of how they did this. (Groups who complete this task quickly might like to think about the numbers of blue cubes that touch the red cubes in different ways.)

After a few minutes you might wish to stop the groups to discuss the different methods for finding out the number of blue cubes and to introduce the idea of how to complete a storyboard of their method so that someone else can follow it.

You might wish to create a storyboard of your own that describes a process from the previous lesson or you might start to create a storyboard with the class based on the introductory session.

Ask the groups to work on cubes of different layers as described on the problem sheet. The aim is to produce a storyboard of part or all of their findings. You might wish to give them the choice of 8 or 16 spaces (resource sheets 1 and 2) in which to tell their story.

After about 15 minutes, remind groups to start planning their storyboards.

Pupils may wish to use ICT to tell their story, perhaps using presentation software.

Plenary

Select for discussion one or two storyboards that draw out key features of different convincing arguments. The storyboards are a useful assessment tool and could form an excellent display.

If pupils have confidence in algebraic representation you might like to demonstrate the equivalence of different methods.

You could encourage pupils to finish or refine their storyboards for homework.

Solution notes

The general cube in these examples has side length of n.

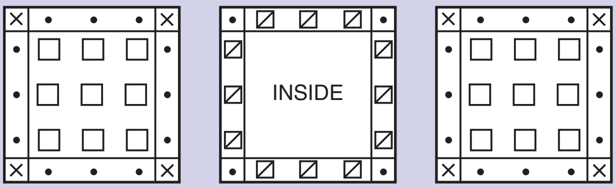
Example 1

The number of cubes on the outside equals the volume of the whole cube minus the volume of the cube formed by the next layer down. That is:

$$n^3 - (n-2)^3 = 6n^2 - 12n + 8$$

So in the case of the red layer ($n = 3$), the total number of cubes $= 3^3 - 1^3 = 26$.

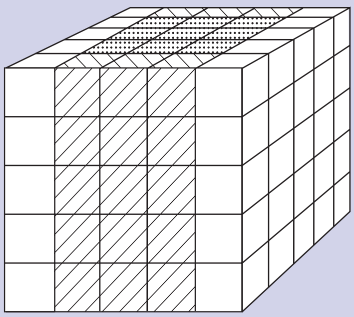
Example 2



$$\begin{array}{ll} \text{corners of cube} = 8 & \\ \left.\begin{array}{l}\text{middle of top and bottom faces} = 2(n-2)^2 \\ \text{middle of ‘inbetween’ faces} = 4(n-2)^2\end{array}\right\} & 6(n-2)^2 \\ \text{edges of cube} = 12(n-2) & \end{array}$$

$$\begin{aligned} \text{Total} &= 6(n-2)^2 + 12(n-2) + 8 \\ &= 6n^2 - 12n + 8 \end{aligned}$$

Example 3



‘left’ and ‘right’ sides $= 2n^2$

‘front’ and ‘back’ sides $= 2n(n-2)$

‘top’ and ‘bottom’ sides $= 2(n-2)^2$

$$\begin{aligned} \text{Total} &= 2n^2 + 2n(n-2) + 2(n-2)^2 \\ &= 6n^2 - 12n + 8 \end{aligned}$$