Mental frogs

Planning ahead

Prerequisite knowledge

- Experience of being systematic in an approach
- Familiarity with developing systems of recording

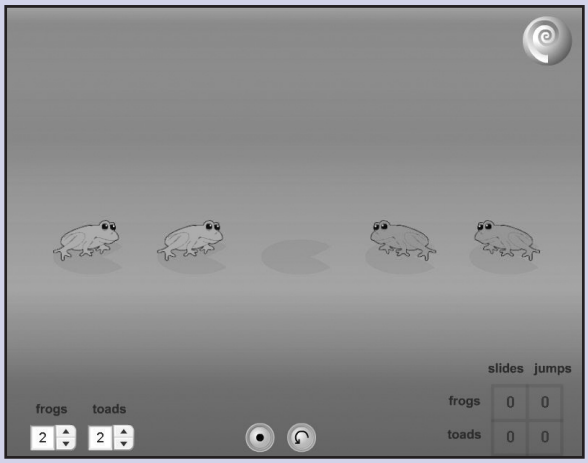
Why do this problem?

This is a challenging problem where it is useful to be able to visualise several steps ahead. Unpicking what is going on also requires an effective method of recording.

The problem also gives opportunities to pay particular attention to the analysis and synthesis, and planning, execution and interpretation phases of the problem-solving model.

Time

One lesson but with potential for two or more

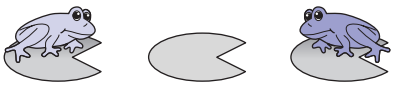


Resources

Counters in two different colours
CD-ROM: resource sheet; interactivity
NRICH website (optional):
www.nrich.maths.org, December 2000, 'Frogs'

Introducing the problem

Show the class three lily pads in a row with one frog on the left lily pad and one toad on the right lily pad (see interactivity and the resource sheet for lily pads):



Explain that a frog/toad can only hop onto an empty lily pad next to it or jump over one frog/toad onto an empty lily pad.

Demonstrate.

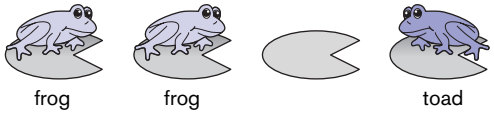
- If the frog or toad were to move, what would be possible? [the frog into the middle or the toad into the middle]

Take one of the cases – for example, the frog into the middle – and ask the class:

- What could happen next? [the frog could go back or the toad could hop over the frog]
- And then what? [you are back to the starting point or the frog hops to the opposite end]

Now ask the class to consider what would be the most efficient (takes the least number of moves) way for the frog and toad to hop if they had to swap places. The aim is to draw out from the group the fact that there are two symmetrically equivalent, efficient ways.

Now set up a situation with four lily pads in a row with two frogs on the left and one toad on the right.



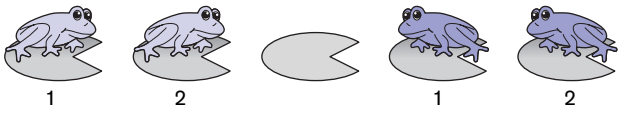
- If you were controlling the frogs and toad how would you get them to move in order to reverse positions?
- What would be your first move? Your second move?
- Is there more than one option for each move?
- Does it matter which move you choose when you do have a choice?

For example, the first move could be a frog to the right or a toad to the left. In this case either move can result in an efficient way of swapping them.

Main part of the lesson

The main part of the lesson develops the introduction by going onto five lily pads and thinking about recording the frogs' and toads' moves, leading on to other symmetrical arrangements.

Use the interactivity to show pupils five lily pads with two frogs and two toads (we have used a numbering system below to help identify each frog/toad and the moves but teachers should encourage pupils to develop their own notations):



Decide on a first move with the class and make it, for example:

Ask the class to visualise the possible next moves and their consequences. What does each move imply about the following moves?

In the example above, if frog 1 now hops right, the only option for the third move takes the frogs back to a previous position. Therefore the only efficient alternative is for toad 1 to hop over frog 2.

- If toad 1 hops over frog 2, describe the resulting arrangement of frogs and toads.
- Where is the empty lily pad?
- What could you do next?
- What would be the consequences of this next move?

Encourage pupils to visualise the consequences before they make a move.

After talking through the first two or three moves with the class, raise the issue of how they will remember what has been done. Ask for ideas. Pupils might suggest the use of different symbols for the two types of hops and possibly to indicate direction. They may also distinguish different frogs and toads as we have done here. Emphasise that the aim of recording is so that they can replicate the moves and to identify any structure that results in an efficient outcome.

Ask them to investigate in pairs the least number of moves needed, first for 2 frogs and 2 toads to swap positions and then for 3, 4, 5, ... of each. As they work, walk around the class and look for situations in which pupils are visualising consequences ('If ... then ... ') so that you can use them as examples in the plenary.

Plenary

Although the plenary could focus on the solution, this is a good opportunity to discuss the range of recording systems pupils have adopted and the role of thinking ahead.

- Would you change your mind about the system you used now you have seen some other ways?

Share with the class some of the good examples you saw of the use of visualising to think ahead. Some pupils may be able to explain how visualising helped – for example, saving time by not going down cul-de-sacs.

Lesson 2

In another lesson encourage pupils to look more deeply. Pool pupils' results on efficient solutions and encourage them to begin to conjecture about overall rules. This lesson could focus on why those rules might or might not be true.

Solution notes

For one frog and one toad, one way to swap them over is: frog to middle; toad over frog; frog to right.

The least numbers of moves for different numbers of frogs/toads are:

- for one frog and one toad: 3 moves
- for two frogs and a toad: 5 moves
- for two of each: 8 moves
- for three of each: 15 moves
- for four of each: 24 moves

In general, if n is the total number of frogs and toads (in symmetrical arrangements), the least number of moves is:

$(n/2 + 1)^2 - 1$ or $(n/2)^2 + n$ or ...

Different algebraic expressions result from different interpretations of the structure but all are equivalent.