Take three from five

Framing

Prerequisite knowledge

- Knowledge of properties of odd and even numbers
- Algebra is useful but not essential

Why do this problem?

The problem extends the idea of divisibility tests for identifying factors to finding remainders. It gives the opportunity to spend time extending the idea that all integers can be partitioned into odds and evens, to the partitions relating to division by 3 (numbers can only have remainders of 0, 1 or 2 when divided by 3). This is a powerful idea. The problem can be used to discuss representations of numbers in modulo 3, including graphic and algebraic forms.

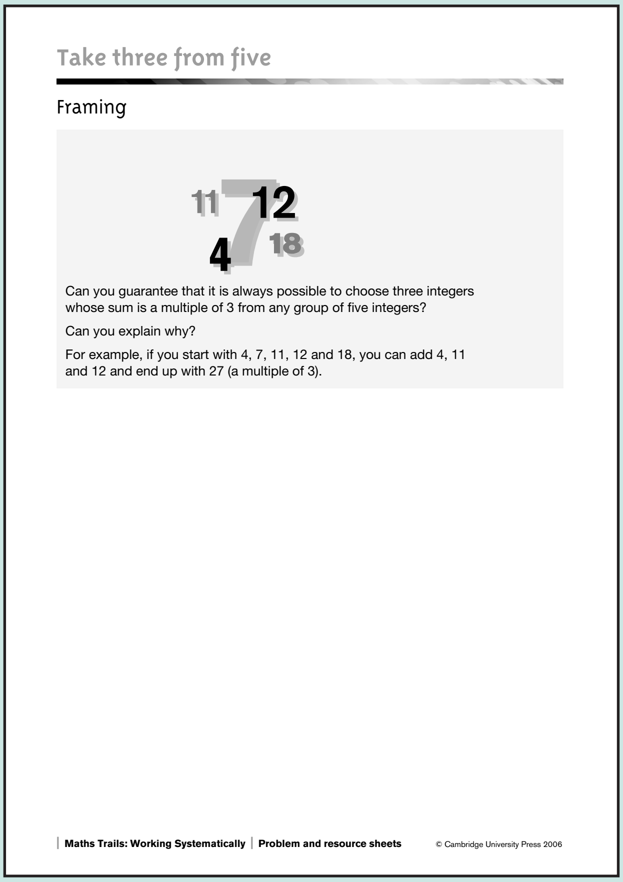
Time

One lesson

Resources

CD-ROM: problem sheet

NRICH website (optional): www.nrich.maths.org, October 2003, 'Take three from five'


Take three from five

Framing

Can you guarantee that it is always possible to choose three integers whose sum is a multiple of 3 from any group of five integers?

Can you explain why?

For example, if you start with 4, 7, 11, 12 and 18, you can add 4, 11 and 12 and end up with 27 (a multiple of 3).

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Introducing the problem

A short introduction classifying all numbers as either odd or even, including working on some very large numbers, is a good starting point.

Ask for three integers and write them on the board. Identify two whose sum is an even number. Repeat this process.

Ask the pupils if it is always possible to find two numbers out of three that add together to make an even number. In order to explain why this is the case, discuss the fact that all numbers are either odd or even and therefore the three numbers can only be:

EEE, OOO, EOO or OEE

This means that it is always possible to find two out of three numbers whose sum is divisible by 2.

Main part of the lesson

Ask the whole group to offer any five numbers and demonstrate that you can identify three that can be added together to make a multiple of 3.

Repeat once or twice.

Ask the group to see if they can find five numbers where this is not the case. Working in pairs pupils identify and test sets of numbers. If anyone thinks they have a set of five they can write them on the board.

After a short time stop and review: it always seems to be possible, but why?

Encourage the group to give some sets of five very large numbers. To help you pick three you might find it useful to employ divisibility rules to identify those with remainder 0, 1 and 2.

A discussion on how you can find three of the five numbers whose sum is divisible by 3 so quickly might be fruitful: you do not have to add all the digits because you can eliminate any whose sum is divisible by 3.

For example, for 762519782.

Immediately cross out the 6 and 9. $7\cancel{6}251\cancel{9}782$

Then $7 + 8$ is divisible by 3, so cross these digits out. $7\cancel{6}251\cancel{9}\cancel{7}\cancel{8}2$

Similarly 7 and 2, 1 and 2. $\cancel{7}\cancel{6}\cancel{2}5\cancel{1}\cancel{9}\cancel{7}\cancel{8}\cancel{2}$

When 5 is divided by 3 the remainder is 2, so the number 762519782 when divided by 3 has a remainder 2.

Ask the group to use the idea in the introductory activity, but apply it to numbers that are divisible by 3 and those that are not. Pupils could work in pairs to look further at the problem.

During the time pupils are working it may be worth stopping to discuss key ideas. Alternatively, ask pupils to write notes of things they have observed on the board, so that groups who are making slower progress can pick up on meaningful prompts.

Plenary

There are two key points:

- numbers have a remainder of 0 or 1 or 2 when divided by 3;
- looking at possible combinations of three from five, it is always possible to find three whose sum is divisible by 3.

Possible extensions include:

- If I remove the requirement that I chose exactly three numbers (so I could take one, two or three numbers) is it still necessary to have five numbers?
- What happens if I want to ensure the sum is divisible by 4?

Solution notes

Either a number is a multiple of 3, or leaves a remainder of 1 or 2 when divided by 3.

These numbers can be written in the forms:

Type 1	$3a$
Type 2	$3b + 1$
Type 3	$3c + 2$

Three type 1 numbers sum to $9a = 3(3a)$, which is divisible by 3.

Three type 2 numbers sum to $9b + 3 = 3(3b + 1)$, which is divisible by 3.

Three type 3 numbers sum to $9c + 6 = 3(3c + 2)$, which is divisible by 3.

Type 1 + type 2 + type 3 $= 3a + 3b + 3c + 3 = 3(a + b + c + 1)$, which is divisible by 3.

Any group of five numbers will either include three numbers of the same type, or at least one of each type of number. Therefore it is always possible to choose three numbers whose sum is divisible by 3.