Remains of a power

Modelling and optimisation

Prerequisite knowledge

- Powers beyond squares

Why do this unit?

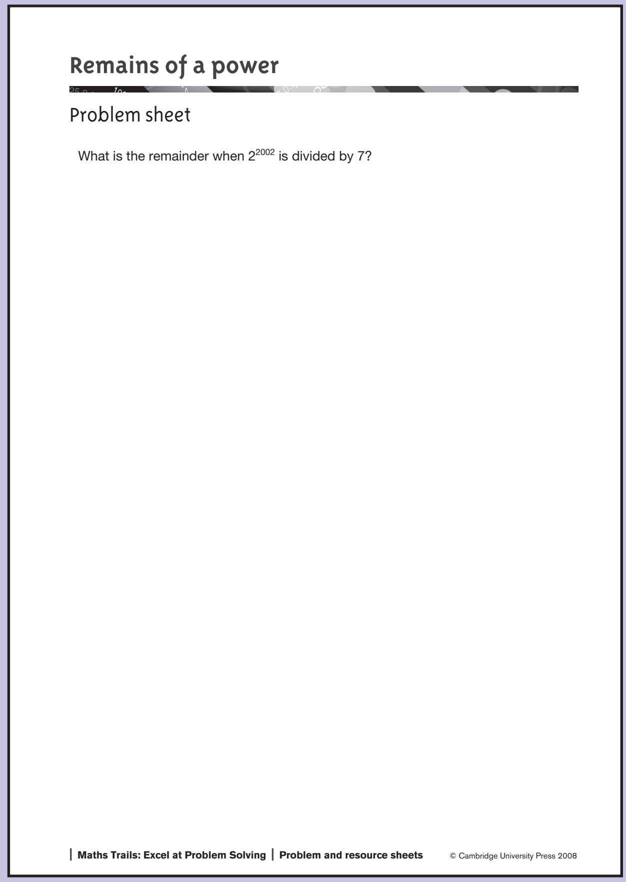
An initial attack on the problem with a calculator or spreadsheet fails. Eventually an indirect approach proves to be a way in. The spreadsheet offers an interactive environment in which to explore the extension ideas within the main problem.

Time

One or two lessons

Resources

CD-ROM: spreadsheet, problem sheet
NRICH website (optional):
www.nrich.maths.org, January 1998,
'Big powers'



Remains of a power

Problem sheet

What is the remainder when 2^{2002} is divided by 7?

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Introducing the unit

Is 2048 divisible by 7?

(The divisor 7 is used here but a smaller, easier number would do just as well.)

- How do you know you are right? [Division on a calculator produces a decimal.]
- How can you find the remainder if you are allowed to use a calculator? [Divide 2048 by 7 to produce a whole number followed by a decimal part. Subtract the whole number to leave just the decimal. Multiply the decimal by 7 to get the remainder, which is 4.]

Give pupils time to find remainders for other numbers.

A spreadsheet can do this very effectively using the MOD function (MOD gives the remainder when the first value is divided by the second). Demonstrate this to the group on a blank sheet (see 'MOD help' on the spreadsheet). Then use this sheet to check remainders for 2048 and the other numbers used by pupils.

Main part of the unit

Introduce pupils to the problem from the problem sheet.

- Will a calculator help? [No, because the number is too big.]

If needed, give pupils time to try using a calculator to see why this is not helpful.

- Will a spreadsheet help? [No, for the same reason.]

Allow time for the difficulty of the task to register with pupils. Invite general thoughts about the problem and possible approaches to it. Eventually, but not too quickly, nudge pupils towards starting simply, looking systematically at remainders for 2^1, 2^2, 2^3, 2^4, 2^5, … Invite them to produce a table of results such as:

	2^1	2^2	2^3	2^4	2^5	2^6	...
	2	4	8	16	32	64	...
Remainder	**2**	**4**	**1**	**2**	**4**	**1**	**...**

- Has that helped? How? [If the powers progress systematically we might see a pattern or might see how each answer follows from the one before.]

Show the group the sheet 'Remainder – 1' on the spreadsheet and ask pupils how that helps. [There is a pattern in the remainders: 2, 4, 1 repeats forever.]

- How do we know that this pattern does continue forever? [When a value is doubled its remainder on division by 7 is also doubled. 1 becomes 2, and 2 becomes 4. 4 doubles to 8 which has a remainder of 1 when divided by 7.]

Discuss how the sheet is constructed. Notice how MOD is used and how the up arrow (^) in a formula produces powers.

At this point ask pupils to construct their own version of the sheet, discussing the process in whatever detail is necessary. As they complete the sheet draw their attention back to the original problem.

In pairs, pupils decide on an answer which they can justify and share with the whole group. [667 cycles of 2, 4, 1 takes us to 2^{2001} so 2^{2002} divided by 7 has a remainder of 2, the next number in the sequence.]

If time allows ask pupils to pose problems of their own and to start looking for a solution to a more general case. They will report on their findings in the plenary in the form of posters. For example they might investigate:

- What happens for powers of 3, 4, 5, …? Can you see and explain any generalisations?
- What happens when you divide by numbers other than 7?

'Remainder – 2' on the spreadsheet may help.

Allow plenty of time for pupils to explore and make progress with their questions. Some pupils can be invited to notice how long a cycle is and also which numbers occur as remainders and which do not.

Remember, being able to account for the pattern is the real solution.

Plenary

Invite pupils to present interesting results and ideas from their exploration using posters or an ICT-based presentation tool.

Draw attention to the way in which the spreadsheet takes the burden of calculation and leaves us to explore the mathematics.

Discuss how a single initial question led to a deeper and wider investigation. Offer the pupils useful vocabulary by referring to that bigger field as the mathematical 'structure'.

Solution notes

When 2^{2002} is divided by 7 the remainder is 2. The sheet 'Remainder – 2' on the spreadsheet may be used to verify more general results.