More sequences and series

Generalising and creating formulae

Prerequisite knowledge

- Pupils may benefit from tackling the problem ‘Sequences and series’ before this
- Square numbers
- Consecutive odd numbers

Why do this problem?

It encourages the exploration of the properties of numbers and their inter-relatedness with other sets of numbers. It relates numerical ideas to spatial representation and vice versa.

Time

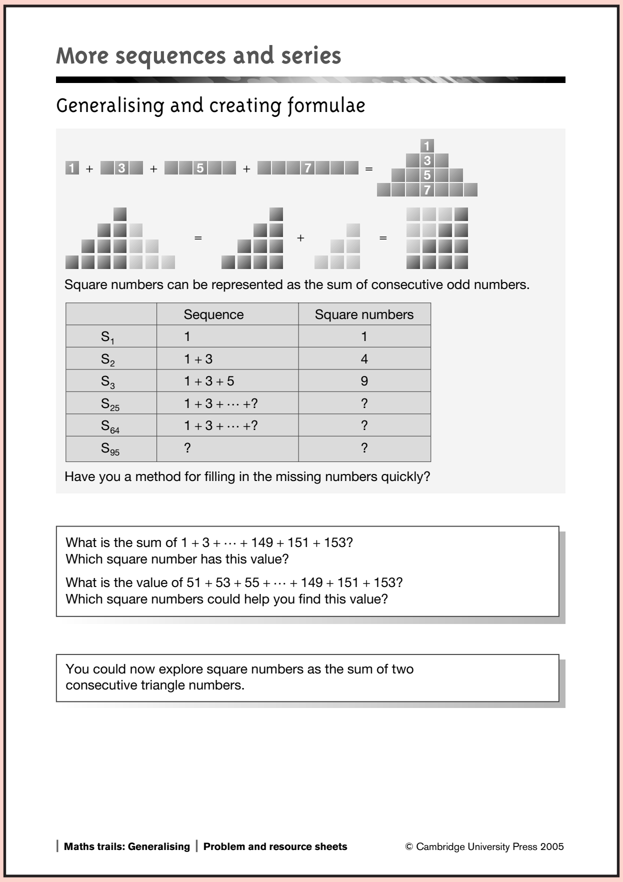
One lesson

Resources

CD-ROM: pupil worksheet; OHT of grid

NRICH website (optional): www.nrich.maths.org, May 2005, ‘More sequences and series’

Counters and an OHT with an appropriately scaled grid or interlocking cubes would be useful for demonstration purposes and may support pupils in their visualisations. An interactive tool is available on the NRICH site which allows a choice of consecutive odd numbers.



More sequences and series

Generalising and creating formulae

Square numbers can be represented as the sum of consecutive odd numbers.

	Sequence	Square numbers
S_1	1	1
S_2	1 + 3	4
S_3	1 + 3 + 5	9
S_{25}	1 + 3 + ⋯ +?	?
S_{64}	1 + 3 + ⋯ +?	?
S_{95}	?	?

Have you a method for filling in the missing numbers quickly?

What is the sum of 1 + 3 + ⋯ + 149 + 151 + 153?
Which square number has this value?

What is the value of 51 + 53 + 55 + ⋯ + 149 + 151 + 153?
Which square numbers could help you find this value?

You could now explore square numbers as the sum of two consecutive triangle numbers.

Maths trails: Generalising | Problem and resource sheets © Cambridge University Press 2005

Introducing the problem

Introduce triangle numbers represented as staircases and ask pupils to use this imagery to work out mentally each of the next triangle numbers, perhaps coming to the OHP and creating them with OHP counters or by colouring a grid.

Make a list of the triangle numbers and, without indicating that they are consecutive, take staircase representations of two consecutive triangle numbers, putting them together to make a double staircase.

- What do you notice about this staircase?

Pupils might notice that it is made from odd numbers or even that the two triangle numbers were consecutive. They might notice that the staircase is symmetrical and that, depending on the height of the staircase, the total number of blocks making the staircase is odd or even. If this latter point is raised a discussion of when the total is odd or even may arise. Someone may even notice that the total is a square number.

Of particular interest in the rest of the lesson is the choice of consecutive triangle numbers and their link to the square numbers, so this point can be noted and used in setting the scene for the main part of the lesson.

Main part of the lesson

First step

Ask pupils to work in small groups to:

- create staircases of the triangle numbers and to investigate which pairs of triangle numbers produce double staircases (consecutive ones);

- describe the widths of each tier of the staircases (each layer has an odd number, starting with 1 at the top and then including every odd number up to the odd number represented by the bottom tier).

Can they explain why their findings are always true?

It may be the case that you are asking pupils to confirm their hypothesis from the introduction to the lesson.

Second step – reversing the process

The aim now is to take the notion that any double staircase can be made into a square and that every square number is the sum of all the odd numbers up to $2s + 1$, where s is the side of the square.

Display a double staircase made with cubes or counters and show how it can be broken into two pieces to form a square.

- Is it always the case that a staircase of consecutive odd numbers can be cut once to create a square?
- What is the relationship between the size of the staircase and the size of the square? Can you use this rule to work out the sums of consecutive odd numbers?

Plenary

Encourage pupils to describe a strategy for adding a set of consecutive odd numbers not starting at zero. It is not necessary for them to be able to describe this strategy algebraically. Asking them to sum some consecutive odd numbers together modelling what they are doing with diagrams of staircases may help to consolidate their findings.

Solution notes

The sum of the first n odd numbers is a square number given by $[\frac{1}{2}(n+1)]^2$.

$S_{25} = 625$

$S_{64} = 4096$

$S_{95} = 9025$

$1 + 3 + \cdots + 151 + 153 = S_{77} = 5929$

$51 + 53 + \cdots + 151 + 153 = S_{77} - S_{25} = 5304$