American billions

Interpretation

Prerequisite knowledge

- Divisibility rules

Why do this problem?

This starts pupils thinking about divisibility rules and requires a very systematic and organised approach in order to get to the (unique) solution efficiently. The starting points are accessible but the problem gets more difficult before getting easier!

It is not coincidence that this lovely problem is at the end of the trail. It is easy to get started but feels rather ‘out of control’ in the middle as you begin to list possibilities.

Time

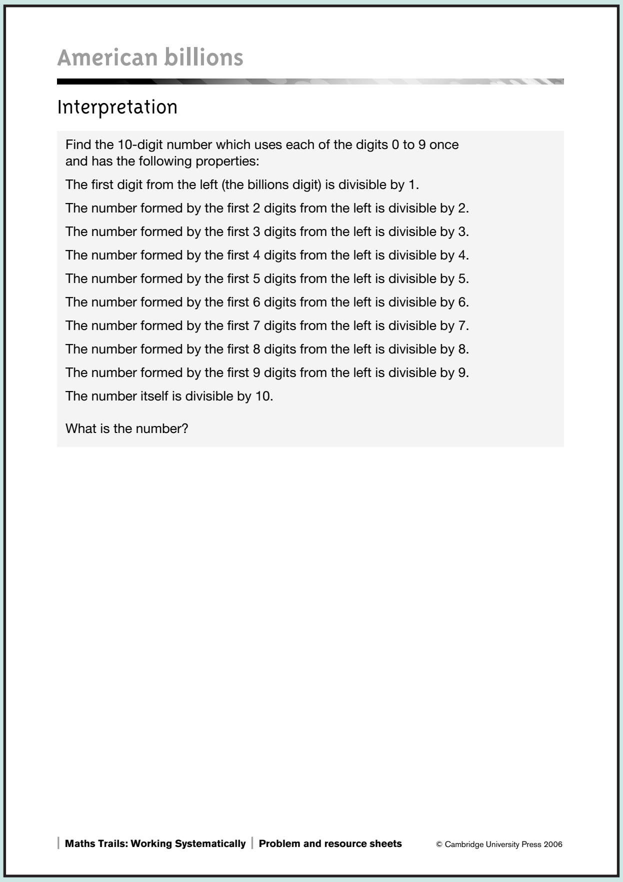
One or two lessons, or a project to return to over a longer period of time

Resources

CD-ROM: problem sheet

NRICH website (optional): www.nrich.maths.org, September 2001, ‘American billions’

Digit cards and squared paper to help record possibilities in an organised way might be helpful.



American billions

Interpretation

Find the 10-digit number which uses each of the digits 0 to 9 once and has the following properties:

The first digit from the left (the billions digit) is divisible by 1.

The number formed by the first 2 digits from the left is divisible by 2.

The number formed by the first 3 digits from the left is divisible by 3.

The number formed by the first 4 digits from the left is divisible by 4.

The number formed by the first 5 digits from the left is divisible by 5.

The number formed by the first 6 digits from the left is divisible by 6.

The number formed by the first 7 digits from the left is divisible by 7.

The number formed by the first 8 digits from the left is divisible by 8.

The number formed by the first 9 digits from the left is divisible by 9.

The number itself is divisible by 10.

What is the number?

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Introducing the problem

Write some numbers on the board and ask pupils to identify:

- all those that are divisible by 2, 3, 4, 5, etc.
- which factors are easy to identify and which are harder.

Discuss shortcuts for testing whether a number is divisible by 3 or 9. For example, 1836981 is easily recognised as being divisible by 9 without adding all the digits – by grouping it as:

18 36 9 81

Each of the numbers in this partition is divisible by 9 so the whole number is divisible by 9. What does this tell you about a number made up from all the digits from 0 to 9?

Main part of the lesson

Show the problem to the group and ask them to work on their own for ten minutes – noting anything they think might help them to start to solve the problem.

After ten minutes ask for ideas, with reasons.

It is often the case that pupils recognise the number will end in a zero and that the fifth digit will have to be a 5.

Discuss the positioning of odd and even numbers in the sequence. This may prove to be useful in reducing possible combinations early on.

The next step is to encourage recording possibilities in a systematic way. Give the group time to work on ways of organising possibilities. There is a good chance that support will be needed and this should reflect the likely approaches the groups adopt to the solution. The two methods outlined below may offer some guidance but are not exhaustive. Each still needs pupils to list possibilities.

Guidance for method 1

The class has already established:

xxxx5xxxx0

Since an odd number is not divisible by an even number the pattern must go odd, even, odd, even, ...

Since the number formed by the first four digits must be divisible by 4 and multiples of 100 are divisible by 4, the digits in the first two positions will not affect whether the number formed by the first four digits is a multiple of 4.

The third digit can only be 1, 3, 7 or 9 (odd numbers but not 5) and this means that the only possibilities for the fourth digit are 2 and 6 (12 or 16 or 32 or 36 or ...)

xxx25xxxx0

xxx65xxxx0

Since the number formed by the first eight digits must be divisible by 8, the sixth digit is even. Because multiples of an even number of hundreds are divisible by 8, only the seventh and eighth digits need to be considered. If the number formed by these last two digits is divisible by 8, the number formed by the first eight digits is divisible by 8.

This means that you are well on the way but there are still lists of possibilities to consider.

Guidance for method 2

Having established the position of the 0 and 5 and that the numbers alternate odd–even, this method starts from the left and works along the number considering all possibilities.

For example the first two digits could be:

12, 14, 16, 18, 32, 34, 36, 38, 72, 74, 76, 78, 92, 94, 96, 98

The first three digits need to be divisible by three so this list grows a little:

123, 129, 147, ... work, but 127, ... do not

Continuing in this way the lists get longer before – almost like magic – it reduces to the unique solution. Of course a table would be much better than a list like the one above as a way of recording!

Plenary

The need to work systematically in recording possibilities is essential to this problem and its value needs to be emphasised in the plenary. Recap the divisibility rules that were useful at the different stages. If time allows, the production of some wall charts explaining the process and divisibility tests would be both a useful classroom resource and assessment tool.

Solution notes

The 10-digit number is:

3816547290

since
3 is divisible by 1,
38 is divisible by 2,
381 is divisible by 3,
3816 is divisible by 4,
38165 is divisible by 5,
381654 is divisible by 6,
3816547 is divisible by 7,
38165472 is divisible by 8,
381654729 is divisible by 9,
3816547290 is divisible by 10.