Is there a theorem?

Stepping into a problem

Prerequisite knowledge

- Properties of regular shapes
- Understanding perimeter (and area for extension activities)

Why do this problem?

This problem uses loci as the context for developing a simple generalisation that can be described using basic algebraic notation.

There is also plenty of scope for extension activities.

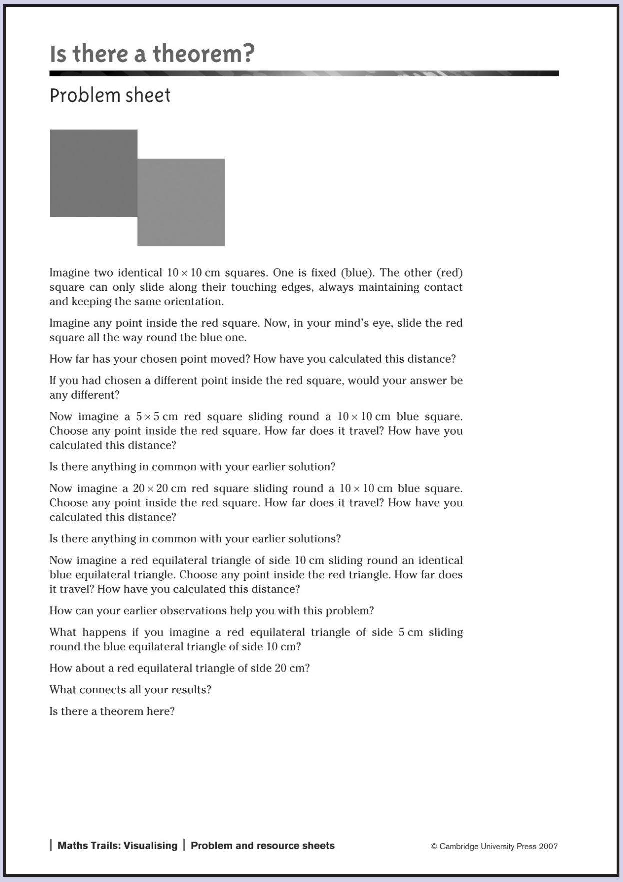
This problem also gives opportunities to pay particular attention to the analysis and synthesis phase of the problem-solving model.

Time

One or two lessons

Resources

A3 sheets
CD-ROM: problem sheet; resource sheet; interactivity
NRICH website (optional): www.nrich.maths.org, March 1997, ‘Is there a theorem?’



Is there a theorem?

Problem sheet

Imagine two identical 10 × 10 cm squares. One is fixed (blue). The other (red) square can only slide along their touching edges, always maintaining contact and keeping the same orientation.

Imagine any point inside the red square. Now, in your mind’s eye, slide the red square all the way round the blue one.

How far has your chosen point moved? How have you calculated this distance?

If you had chosen a different point inside the red square, would your answer be any different?

Now imagine a 5 × 5 cm red square sliding round a 10 × 10 cm blue square. Choose any point inside the red square. How far does it travel? How have you calculated this distance?

Is there anything in common with your earlier solution?

Now imagine a 20 × 20 cm red square sliding round a 10 × 10 cm blue square. Choose any point inside the red square. How far does it travel? How have you calculated this distance?

Is there anything in common with your earlier solutions?

Now imagine a red equilateral triangle of side 10 cm sliding round an identical blue equilateral triangle. Choose any point inside the red triangle. How far does it travel? How have you calculated this distance?

How can your earlier observations help you with this problem?

What happens if you imagine a red equilateral triangle of side 5 cm sliding round the blue equilateral triangle of side 10 cm?

How about a red equilateral triangle of side 20 cm?

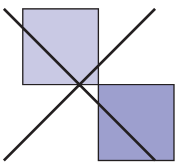
What connects all your results?

Is there a theorem here?

| Maths Trails: Visualising | Problem and resource sheets   © Cambridge University Press 2007

Introducing the problem

Ask pupils to imagine two squares: the squares are identical except that one is red and one is blue. In their mind’s eye they should put the two squares together so that the edge of the red square matches up exactly to an edge of the blue square. Clarify that the squares have to touch along the whole of an edge so ‘overhanging’ is not allowed:



Invite pupils to describe what they see, encouraging different representations. [some pupils may be visualising the squares as a tower and some as a long low block]

It might be helpful for other pupils to draw the visualisations that have been described on the board.

If pupils need support in visualising the next step, show them the interactivity.

Ask the class to start with two 10 cm squares in a long low block with red to the right of blue, and imagine a dot in the top left-hand corner of the red square.

Ask them to visualise the path of the dot as you describe the motion of the red square around the blue square.

- What shape is the path? [square]
- What is the length of the side of the square? [20 cm]
- What is the total distance travelled by the dot? [80 cm]

- Where does the path of the dot touch the blue square? [right side and bottom]
- What would happen if the dot were at the bottom left of the square? [similar 20 cm square translated, touching top and right]
- What would happen if the point were in the middle of the left-hand side? [20 cm square again but this time only touching the right-hand side and symmetrically placed around the blue square to give a horizontal line of symmetry]

Main part of the lesson

Pose some questions about what pupils have been visualising, placing some of their predictions on the board for later reference.

- Can you predict what would happen if the point was on another corner or side of the red square or placed inside the square?
- Can you predict what would happen if the sides of the red square were 5 cm, not 10 cm?
- What about sides of 20 cm?

Invite the class to work on these problems (you could hand out the problem sheet at this point). They might want to use some of the prepared squares (see the resource sheet) to help them. In order to justify their findings pupils may wish to record their work on paper rather than relying on verbal description.

Encourage pupils to develop their ideas and make conjectures, which they should record on A3 sheets before testing them out.

There will be times in the lesson when it will be appropriate to stop and discuss findings so far. You could focus on:

- Looking at why the locus is always a square of side equal to the sum of the sides of the red and blue squares; and why the total perimeter equals the sum of the perimeters of the two squares.
- Discussing the way the squares are translated for different positions of the dot.
- Asking pupils whether the generalisation will hold for other shapes, not just squares. You might wish to give them the example of equilateral triangles (see the resource sheet).

Plenary

Ask pupils to share their conjectures and their findings, perhaps by generalising their findings using words, symbols or algebraic notation if appropriate. If any pupils undertake extension work they might discuss how their findings support or contradict the generalisations stemming from the main activity.

Solution notes

For any two shapes, the first in a fixed position and the second having fixed orientation, if the second shape slides around the first and always maintains contact, then the distance travelled by any point on the second shape is the sum of the perimeters of the two shapes.