Consecutive sums

Ready-made spreadsheets to explore mathematical ideas

Prerequisite knowledge

- Simplification of algebraic expressions

Why do this unit?

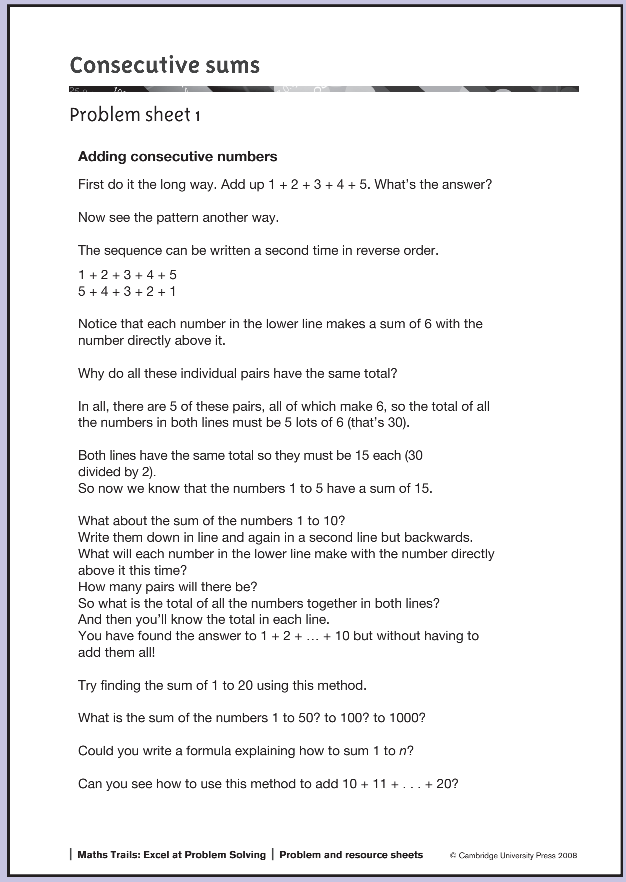
The sum of consecutive integers occurs frequently in mathematics. This activity uses an interactive spreadsheet to assist visualisation, leading to a key formula, and tests conjectures about sums of consecutive numbers.

Time

Two lessons

Resources

CD-ROM: spreadsheet, problem sheets 1 and 2
NRICH website (optional): www.nrich.maths.org, May 2004, 'Sequences and series'; December 2001, 'Proof sorter – sum of an AP'; May 1997, 'Consecutive sums'



Consecutive sums

Problem sheet 1

Adding consecutive numbers

First do it the long way. Add up 1 + 2 + 3 + 4 + 5. What's the answer?

Now see the pattern another way.

The sequence can be written a second time in reverse order.

1 + 2 + 3 + 4 + 5
5 + 4 + 3 + 2 + 1

Notice that each number in the lower line makes a sum of 6 with the number directly above it.

Why do all these individual pairs have the same total?

In all, there are 5 of these pairs, all of which make 6, so the total of all the numbers in both lines must be 5 lots of 6 (that's 30).

Both lines have the same total so they must be 15 each (30 divided by 2).
So now we know that the numbers 1 to 5 have a sum of 15.

What about the sum of the numbers 1 to 10?
Write them down in line and again in a second line but backwards.
What will each number in the lower line make with the number directly above it this time?
How many pairs will there be?
So what is the total of all the numbers together in both lines?
And then you'll know the total in each line.
You have found the answer to 1 + 2 + … + 10 but without having to add them all!

Try finding the sum of 1 to 20 using this method.

What is the sum of the numbers 1 to 50? to 100? to 1000?

Could you write a formula explaining how to sum 1 to *n*?

Can you see how to use this method to add 10 + 11 + . . . + 20?

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Introducing the unit

The introductory activity aims to give pupils an efficient means of adding a long sequence of consecutive numbers.

Open 'Quick adding' on the spreadsheet and explain to pupils that this sheet shows the calculation $1 + 2 + 3 + \ldots + n$. Tell pupils that they are going to find a quick way of doing the calculation, even when the numbers are large and there are lots of them.

Use the spinner buttons by the top table to show the results for $n = 6$, 7, 8, 9 and 10. Adding the consecutive numbers 1 to 6 gives a total of 21, 1 to 7 gives 28 and so on.

- Can you see a connection between the last number in the sequence and the total (sum) we arrive at? [Each time the total increases by the value of the last number – 6 makes 21, 7 makes 28, 8 makes 36, 9 makes 45 and 10 makes 55.]
- What do you notice about 21, 28, 36, 45 and 55? [They are triangle numbers.]

Set n to 6 and point out the arrangement of numbers in the sheet. 1 to 6 is above 6 to 1. Draw attention to the sum of each column, 7, as the upper number is added to the number below it.

- Why is this total the same all the way along? [Values ascend in the upper line and descend in the lower line by the same amount.]
- What is the sum of each pair? [One more than the largest number, $n + 1$.] How many columns are there? [n]
- So what is the total of both rows all together? [$n(n + 1)$]
- And what must the total for one line be? [$\frac{1}{2}n(n + 1)$]

Ask pupils to check this for enough specific n values until they have a strong understanding of the result and can justify it and use it.

We have looked at consecutive numbers which start at 1. What happens if we do not start at 1? Problem sheet 1 may be helpful at this stage. Choose a few examples to clarify what is being discussed and invite pupils to suggest methods for finding the sum. Some pupils may suggest a method similar to the method when starting at 1 (reverse and add in pairs, multiply and halve). Other pupils may suggest a subtraction method (the sum of 16 to 100 is equal to the sum of 1 to 100 minus the sum 1 to 15).

Main part of the unit

Show the problem 'Consecutive sums' on problem sheet 2. Explain that the class is going to work in pairs to find sums using consecutive numbers that need not start from 1.

As pupils work in groups on the problem some questions to stimulate discussions might include:

- Do you notice which numbers can be formed from the sum of two consecutive numbers? [All the odd numbers – because with two consecutive numbers one must be odd and one even.]
- Why can't you make an even number from two consecutive numbers? [As above.]
- Can you think of a quick way of finding the two consecutive numbers? [Roughly half the odd number $\frac{1}{2}(n+1)$.]
- What numbers can be represented by three consecutive numbers?
- What about four or five consecutive numbers? [Related to multiples.]
- Is there a method for producing more using the answers you already have? [There are many methods but an example might be: 'If I have $15=4+5+6$ then I can go up from 15 in steps of 3 by adding one to each number like this: $18=5+6+7$' – this justifies the multiples suggested in the question above.]
- Can you predict what consecutive sums are possible given any number? [Yes, by looking at its factors.]

Investigate sums that are not possible.

- Are there any numbers you haven't been able to find consecutive sums for? [Pupils may offer quite a few but through discussion, reduce this to powers of 2.]

Plenary

'Sums' and 'Sums zoom 20%' on the spreadsheet automatically calculate sums of consecutive numbers and are available to support discussions in the plenary.

There will be lots to share from the main activity but you may wish to focus on one idea that has been noted but not justified. For example, prove that pure powers of 2 can never be made as the sum of consecutive numbers.

Follow this by asking pupils to justify the conclusion to each other and to produce a proof in their own words on paper.

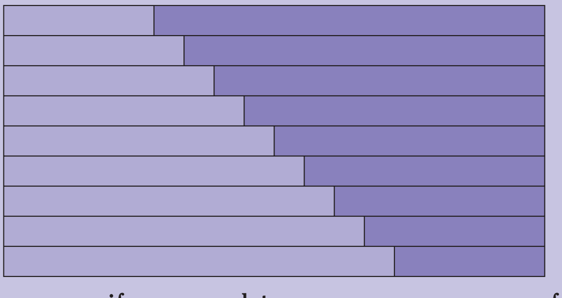
Solution notes

Plenary

Visually: This particular image happens to have nine strips but it is offered as the general case, so the number of strips (number of consecutive numbers) could be odd or even.

If the number of strips is odd, the longest and shortest strips will either both be even or both odd. This means their sum (the width of the rectangle) must be even. In either case the rectangle in the image has its height odd and its width even.

If instead the number of strips is even, the longest and shortest strips will be odd and even (or even and odd) respectively – so their sum must be odd. In this case the rectangle in the image would have its height even and its width odd.



However, if we need to arrange a power of 2 into a rectangle, both dimensions must be even, and this cannot be done with an odd or an even number of strips, so cannot be done at all.

Algebraically: $\frac{1}{2}n(n+1)$ always contains an odd factor because either n or $n + 1$ must be odd, and the other one even. The even factor can be divided by the 2 but the overall result must always contain an odd factor, and pure powers of 2 will only have even factors, so they will never match the sum for a set of consecutive numbers.