Core 3 Jan 06

① $\int_2^8 \frac{3}{x}\,dx = \left[3\ln x\right]_2^8 = 3\ln 8 - 3\ln 2$

$= 3(\ln 8 - \ln 2) = 3\ln \frac{8}{2} = 3\ln 4 = \ln 4^3$

$= \ln 64$

② Using $\sec^2\theta = 1 + \tan^2\theta$

$\sec^2\theta = 4\tan\theta - 2$

$\Leftrightarrow 1 + \tan^2\theta = 4\tan\theta - 2$

$\Leftrightarrow \tan^2\theta - 4\tan\theta + 3 = 0$ let $t = \tan\theta$

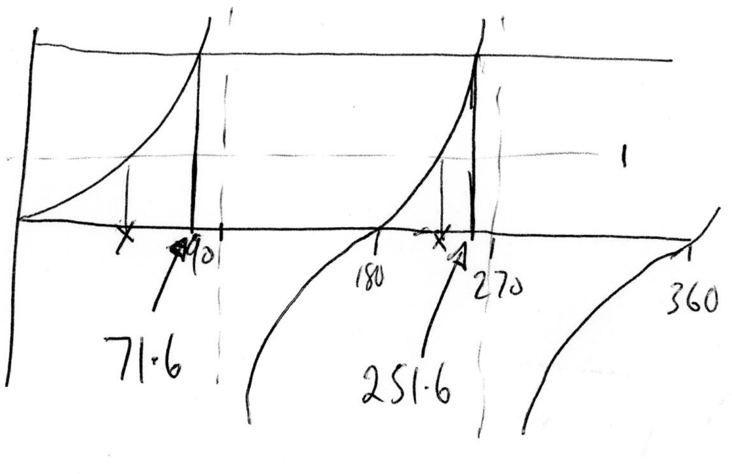
$t^2 - 4t + 3 = 0$

$(t-1)(t-3) = 0$

So $\tan\theta = 1$ or $\tan\theta = 3$

$\tan^{-1} 1 = \theta = 45°$ or $225°$

$\tan^{-1} 3 = \theta = 71.6°$ or $251.6°$



③ a) It's a product!!

$$y = \underset{u}{x^2}\,\underset{v}{(x+1)^6}$$

$u = x^2 \qquad \frac{du}{dx} = 2x$

$v = (x+1)^6 \qquad \frac{dv}{dx} = 6(x+1)^5$

this bit is really the chain rule but as it's linear you can just multiply by the coeff of x (1)

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = 6x^2(x+1)^5 + 2x(x+1)^6$$

b) $y = \dfrac{x^2+3}{x^2-3}$ $\leftarrow u$ $\leftarrow v$

QUOTIENT!!

$u = x^2+3 \qquad \frac{du}{dx} = 2x$

$v = x^2-3 \qquad \frac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \frac{2x(x^2-3) - 2x(x^2+3)}{(x^2-3)^2}$$

$$= \frac{2x^3 - 6x - 2x^3 - 6x}{(x^2-3)^2} = \frac{-12x}{(x^2-3)^2}$$

When $x=1$ $\frac{dy}{dx} = \frac{-12}{(-2)^2} = \frac{-12}{4} = -3$

④ i) $f(x) = 2 - \sqrt{x}$

As $\sqrt{x}$ is positive

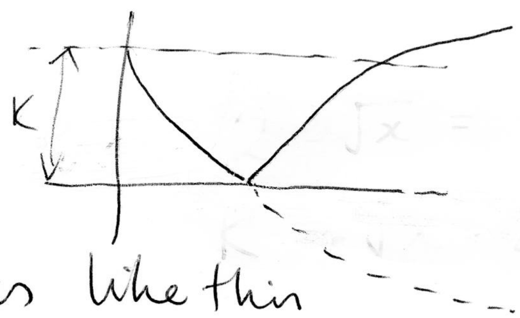
the max value of $f(x)$ is 2

So $y \leq 2$

ii) $f(4) = 2 - \sqrt{4} = 0$

$f(f(4)) = f(0) = 2$

iii) $|2 - \sqrt{x}| = K$



$y = |2 - \sqrt{x}|$ looks like this

For 2 solutions $0 < K \leq 2$

⑤ From the diagram its easier to integrate the two seperately. $y = e^{2x-1} - 1$ will give a negative area that we can 'add' on.

$$\int_0^{1/2} (1-2x)^5 \, dx = \left[\frac{1}{6} \cdot \frac{1}{-2} (1-2x)^6\right]_0^{1/2}$$

$$= \left[\frac{-1}{12} (1-2x)^6\right]_0^{1/2} = \frac{-1}{12} \times 0 - \frac{-1}{12} \times 1$$

$$= \frac{1}{12}$$

$$\int_0^{1/2} e^{2x-1} - 1 \, .dx = \left[\frac{1}{2} e^{2x-1} + x \right]_0^{1/2}$$

$$= \left(\frac{1}{2} e^0 + 0 \right) - \left(\frac{1}{2} e^{-1} + 1 \right) = \frac{1}{2} - \frac{1}{2e} - 1$$

$$= \frac{-1}{2} - \frac{1}{2e} = \frac{-1}{2} \left(1 + \frac{1}{e} \right)$$

Total area $= \dfrac{1}{12} + \dfrac{1}{2} + \dfrac{1}{2e}$

$$= \frac{7}{12} + \frac{1}{2e}$$

(6) a) in the form $X = Ax$

$t=0 \quad X = 275$ $\qquad 275 = Ab^0 = A$

$t=10 \quad X = 440$ $\qquad 440 = 275 \times b^{10}$

$$\frac{440}{275} = b^{10}$$

$$b = \sqrt[10]{\frac{440}{275}} = \left(\frac{440}{275} \right)^{1/10}$$

When $t = 20$ $\qquad X = 275 \times \left(\left(\frac{440}{275} \right)^{1/10} \right)^{20} = 275 \times \left(\frac{440}{275} \right)^2 = 704$

b) i) $20 = 80e^{-0.02t}$

$\frac{20}{80} = e^{-0.02t}$

$\frac{20}{80} = \frac{1}{4}$ ⇒ $e^{-0.02t} = \frac{1}{4}$

take logs of both sides

$\ln e^{-0.02t} = \ln \frac{1}{4}$

$-0.02t(\ln e) = \ln \frac{1}{4}$

$\ln e = 1$

$t = \frac{\ln 1/4}{-0.02} = 69.3147$

$= 69$ (2s.f.)

ii) $Y = 80e^{-0.02t}$

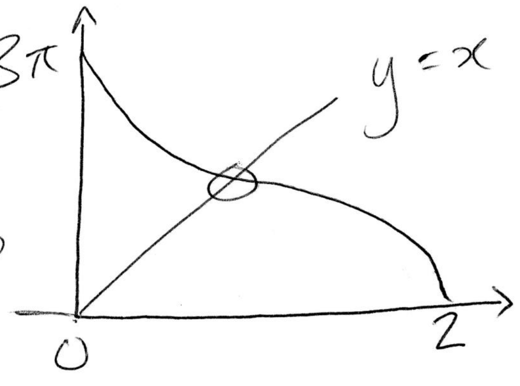
$\frac{dY}{dt} = 80 \times (-0.02) e^{-0.02t} = -1.6e^{-0.02t}$

at $t = 30$ $\frac{dY}{dt} = -1.6e^{-0.02 \times 30} = -0.88$

So the decrease rate is 0.88

(no units are given in the question for either quantity or time).

⑦ i)



(Translated 1 unit in positive x, stretch by factor of 3 in y direction).

ii) One point of intersection so one root.

iii) if $x = 1.8$ $3\cos^{-1}(x-1) - x = 0.1305$

if $x = 1.9$ $3\cos^{-1}(x-1) - x = {}^{-}0.5469$

By the sign change rule there is a root between $x = 1.8$ and 1.9.

iv) $x_1 = 2$

$x_2 = 1 + \cos\left(\frac{2}{3}\right) = 1.786$

$x_3 = 1 + \cos\left(\frac{ans}{3}\right) = 1.828$

$x_4 = 1 + \cos\left(\frac{ans}{3}\right) = 1.820$

$x_5 = 1 + \cos\left(\frac{ans}{3}\right) = 1.822$

$\therefore \alpha = 1.82$

α is a root of $3\cos^{-1}(x-1) = x$

Because $x = 1 + \cos\left(\frac{x}{3}\right) \Leftrightarrow x - 1 = \cos\left(\frac{x}{3}\right)$

$\cos^{-1}(x-1) = \frac{x}{3} \Leftrightarrow 3\cos^{-1}(x-1) = x$

8 i) $y = \ln(5-x^2)$

$\frac{dy}{dx} = ?$ chain rule - a function of a function

Let $u = 5 - x^2$ $\qquad$ $y = \ln u$

$\frac{du}{dx} = -2x$ $\qquad$ $\frac{dy}{du} = \frac{1}{u}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -2x\frac{1}{5-x^2}$ or $-2x(5-x^2)^{-1}$

at $x=2$ (P) then gradient is $-2 \times 2 \times 1$
$= -4$

$\frac{y-0}{x-2} = -4$ $\qquad$ $y = -4x + 8$

ii) $x_0 = 0$ $\qquad$ $y_0 = \ln 5$ $\qquad$ $h = \frac{1}{2}$

$x_1 = 0.5$ $\qquad$ $y_1 = \ln 4.75$

$x_2 = 1$ $\qquad$ $y_2 = \ln 4$

$x_3 = 1.5$ $\qquad$ $y_3 = \ln 2.75$

$x_4 = 2$ $\qquad$ $y_4 = \ln 1$

Simpson's rule gives $\frac{1}{3} \times \frac{1}{2}\left(\ln 5 + 4\ln 4.75 + 2\ln 4 + 4\ln 2.75 + \ln 1\right)$

$= 2.44$

iii) OPQ is a triangle Q has coordinates (0, 8)

So area of triangle is $\frac{1}{2} \times 2 \times 8 = 8$

$8 - 2.44 = 5.56$

⑨ i) $\sin 3\theta = \sin(\underset{A}{2\theta} + \underset{B}{\theta})$

$=$

$\sin 2\theta \cos\theta + \sin\theta \cos 2\theta$

$= 2\sin\theta\cos\theta\cos\theta + \sin(\cos^2\theta - \sin^2\theta)$

$= 2\sin\cos^2\theta + \sin\cos^2\theta - \sin^3\theta$

$= 3\sin\cos^2\theta - \sin^3\theta$

$= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta$

$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

ii) $9\sin\left(\frac{10}{3}\alpha\right) - 12\sin^3\left(\frac{10}{3}\alpha\right)$ looks a bit like this

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$3\sin 3\theta = 9\sin\theta - 12\sin^3\theta$

where $\theta = \frac{10}{3}\alpha$

So $9\sin\left(\frac{10}{3}\alpha\right) - 12\sin^3\left(\frac{10}{3}\alpha\right) = 3\sin\left(3\times\frac{10}{3}\alpha\right)$

$= 3\sin(10\alpha)$ So max value is $\underline{\underline{3}}$

occurs when $\sin 10\alpha = 1$ ie $10\alpha = 90$

$\underline{\underline{\alpha = 9}}$

iii) $3\sin 6\beta \operatorname{cosec} 2\beta = 4$

$= 3\sin 6\beta \dfrac{1}{\sin 2\beta} = 4$

thin of $3\sin 6\beta$ as $3\sin 3\theta$ where $\theta = 2\beta$

so

$$3\sin 3\theta = 9\sin\theta - 12\sin^3\theta$$

$$3\sin 6\beta = 9\sin 2\beta - 12\sin^3 2\beta$$

$$\therefore\ 3\sin 6\beta \frac{1}{\sin 2\beta} = \frac{9\sin 2\beta - 12\sin^3 2\beta}{\sin 2\beta} = 4$$

$$= 9 - 12\sin^2 2\beta = 4$$

$$12\sin^2 2\beta = 5$$

$$\sin 2\beta = \pm\sqrt{\frac{5}{12}}$$

as $0 < \beta < 90$ $\quad 0 < 2\beta < 180$

so $\sin 2\beta = +\sqrt{\frac{5}{12}}$ only

$$\sin^{-1}\left(\sqrt{\frac{5}{12}}\right) = 40.2 \ (\text{or } 139.8) = 2\beta$$

So $\beta = 20.1°$ or $69.9°$