On Krylov subspace approximations to the matrix exponential operator

Purple team

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Introduction

- Approximation of $exp(\tau A)v$ using Krylov methods, A big matrix, τ step size
- Error bounds for Arnoldi and Lanczos algorithms
- Faster convergence than $(I-\tau A)^{-1}v$ for stiff ODE problems, new integration methods
- Bound $\frac{||\tau A||^m}{m!}$, superlinear convergence for $m \gg ||\tau A||$
- Better behaviour for matrix classes: SND, skew-Hermitian, dependent on eigenvalues distribution

Matrix function definition

- $AV_m = V_m H_m + h_{m+1,m} v_m e_m^T$, $H_m = V_m^* A V_m$
- $q_{m-1}(A)v = V_m q_{m-1}(H_m) e_1 \implies (\lambda I - A)^{-1} v \approx V_m (\lambda I - H_m)^{-1} e_1$
- Numerical range $\mathcal{F}(A) = \{x^* A x : x \in \mathbb{C}^N, ||x|| = 1\}$, $\mathcal{F}(H_m) \subset \mathcal{F}(A)$
- $f(A)v = \frac{1}{i2\pi} \int_\Gamma f(\lambda) V_m (\lambda I - H_m)^{-1} e_1 d\lambda$, Γ is contour around $\mathcal{F}(A)$
- $f(A)v \approx V_m f(H_m) e_1$

Error bound lemma

- $||f(A)v - V_m f(H_m)e_1|| \leq \frac{M}{2\pi} \int_\Gamma |f(\lambda) - q_{m-1}(\lambda)||\phi(\lambda)|^{-m}|d\lambda|$
- ϕ conformal mapping from exterior of E to exterior of unit circle
- Γ piecewise smooth, f analytic and continuous to boundary
- $M = l(\partial E)/[d(\partial E)d(\Gamma)]$, $l(\partial E)$ length of boundary, $d(S)$ distance between $\mathcal{F}(A)$ and subset S, E convex closed bounded set in complex plane
- Proof idea
- $||(\lambda I - A)^{-1}v - V_m(\lambda I - H_m)^{-1}e_1|| \leq c||p_m(A)||$
- $p_m(A) = \frac{1}{2\pi i}\int_{\partial E} p_m(z)(zI - A)^{-1}dz \implies ||p_m(A)|| \leq c \max_{z \in E} |p_m(z)|$