Trees

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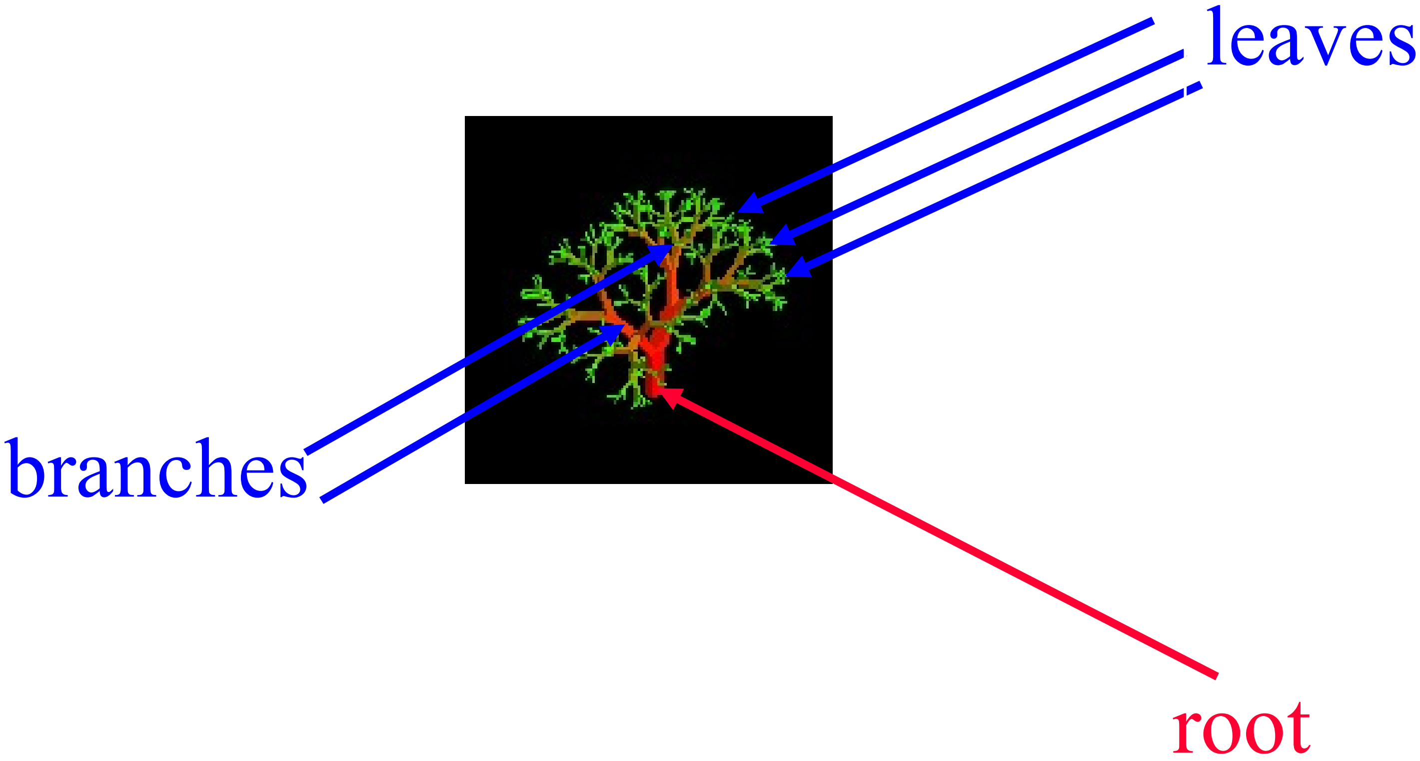
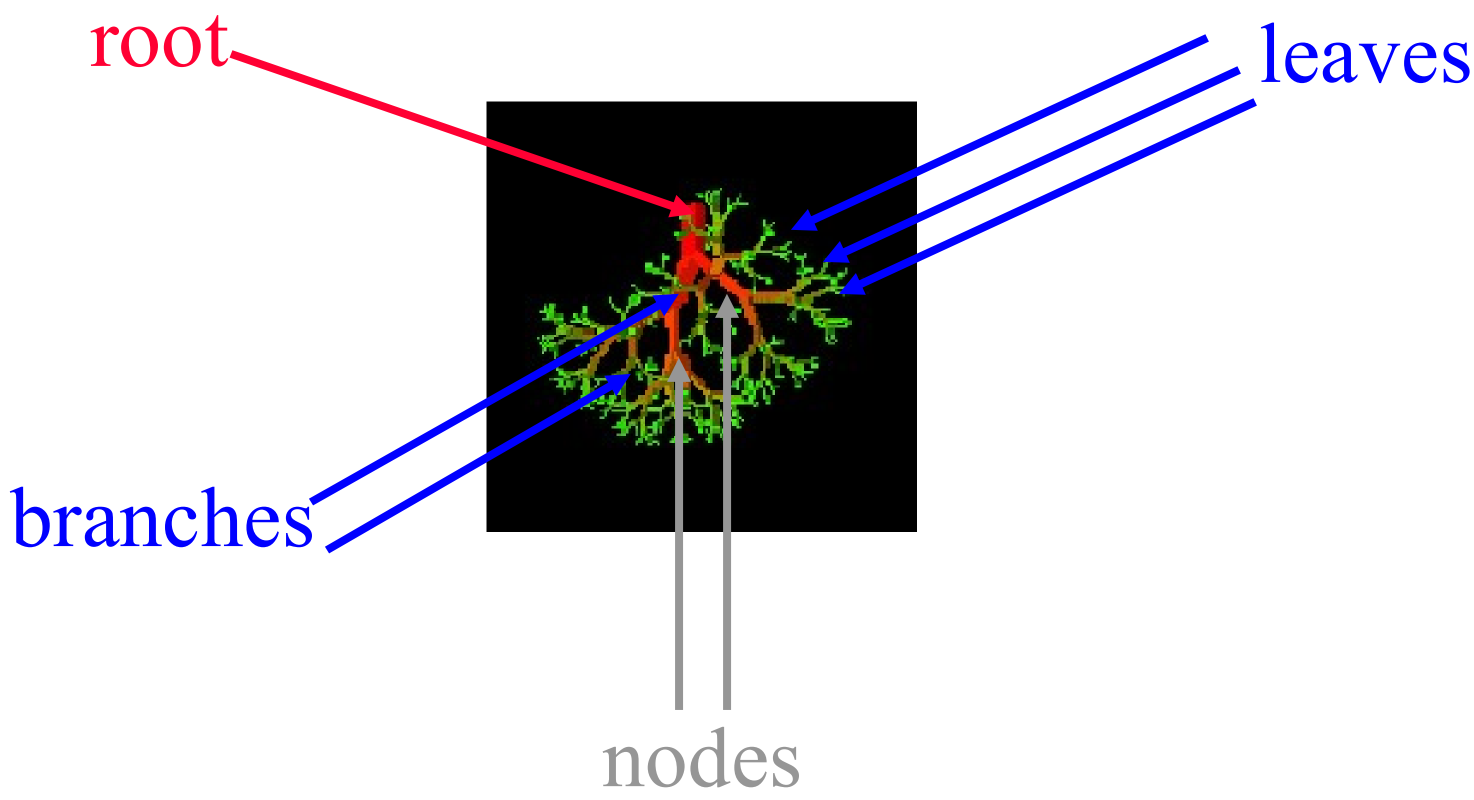
Nature Lover's View of a Tree



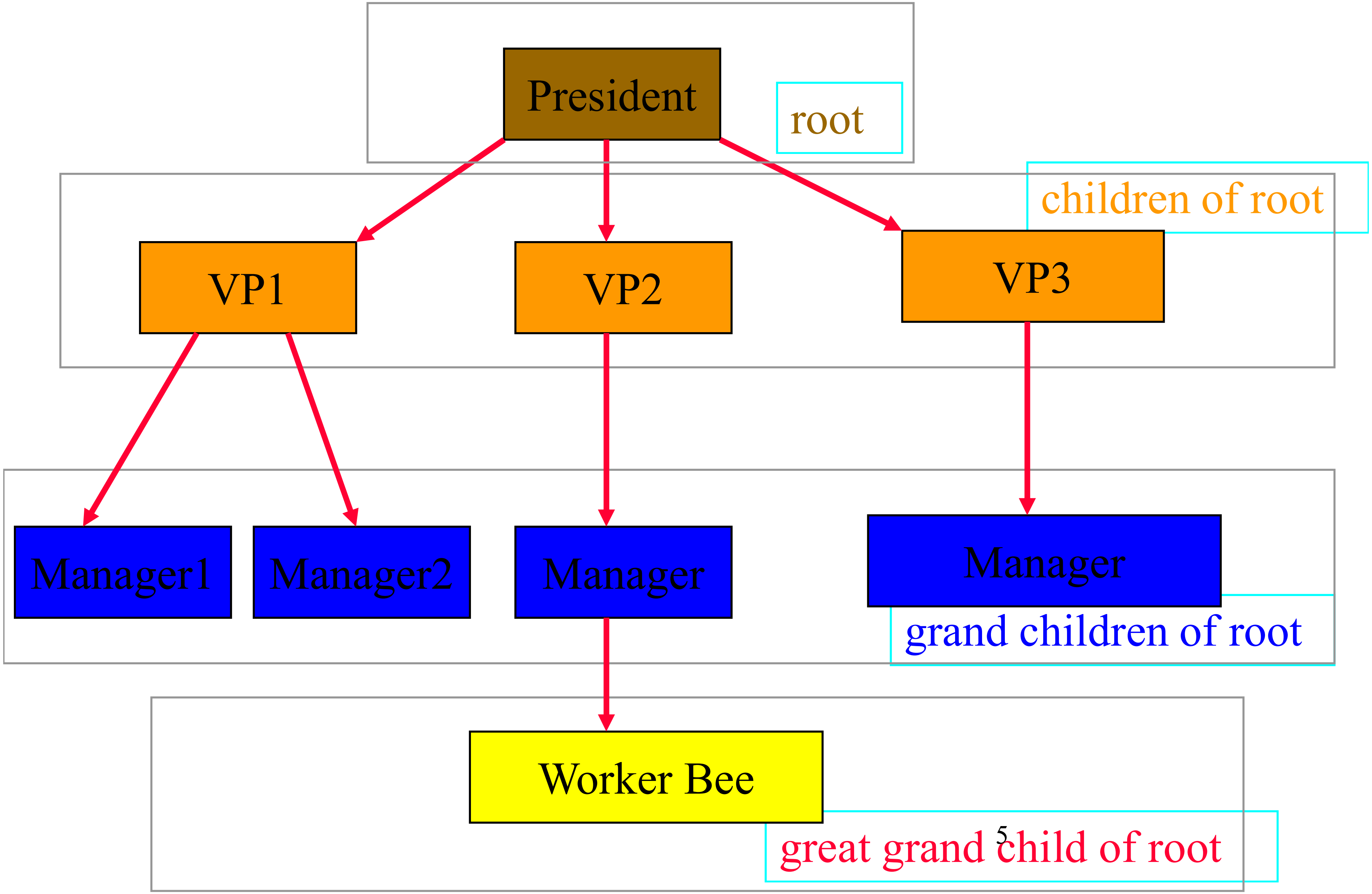
Computer Scientist's View



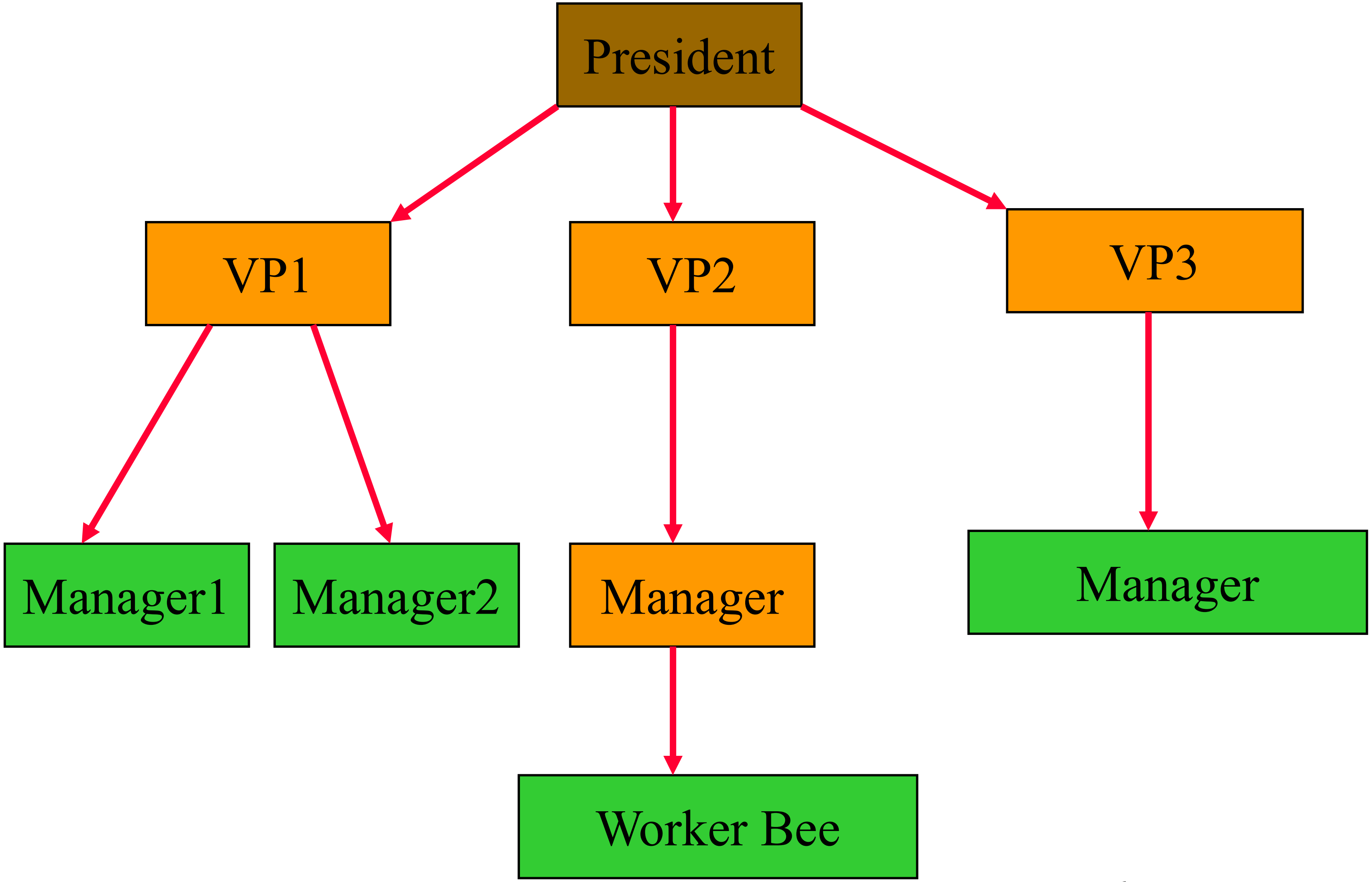
Hierarchical Data and Trees

- The element at the top of the hierarchy is the root.
- Elements next in the hierarchy are the children of the root.
- Elements next in the hierarchy are the grandchildren of the root, and so on.
- Elements that have no children are leaves.

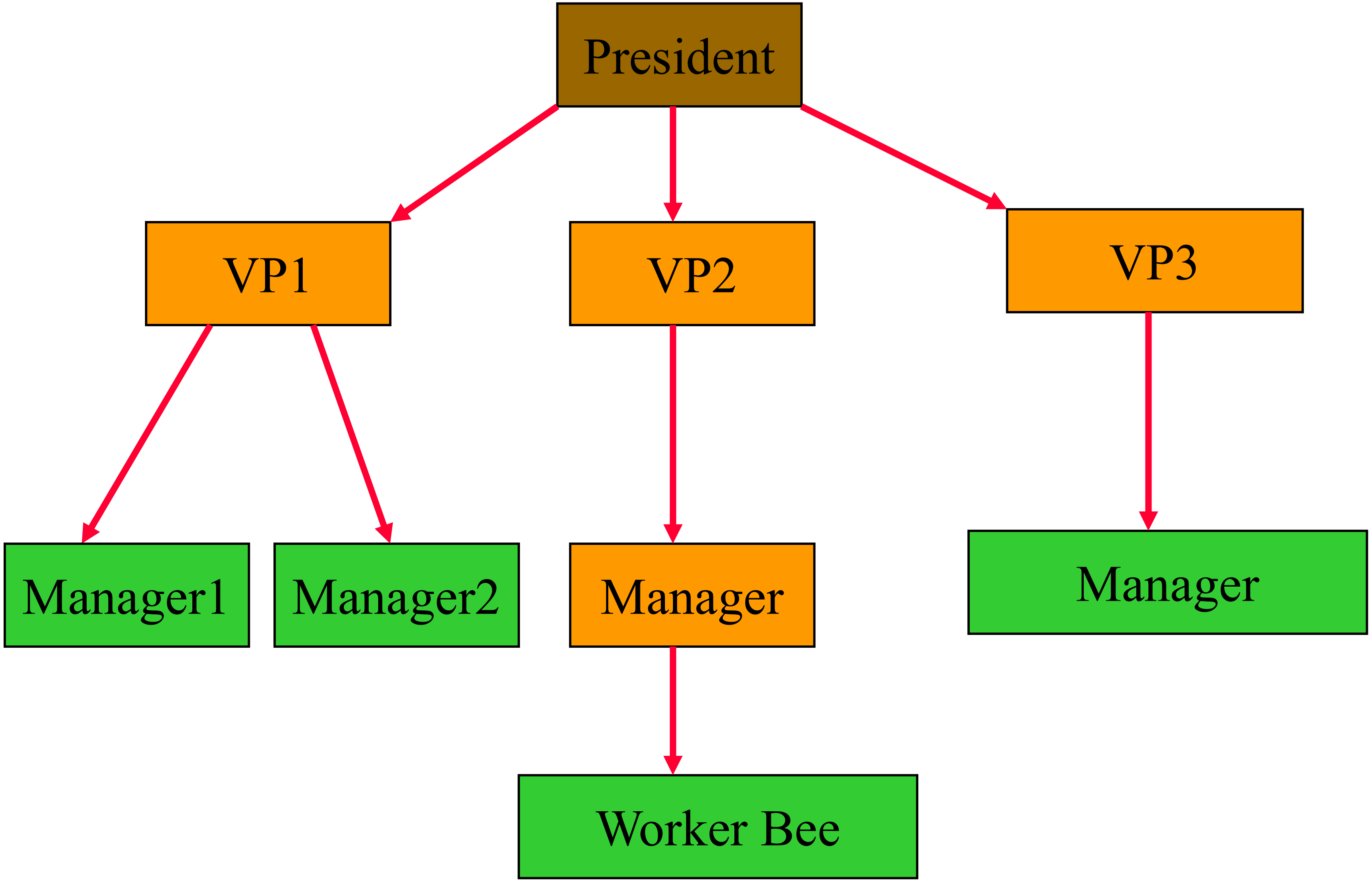
Example Tree



Leaves


President
VP1
VP2
VP3
Manager1
Manager2
Manager
Manager
Worker Bee

Parent, Grandparent, Siblings, Ancestors, Descendants



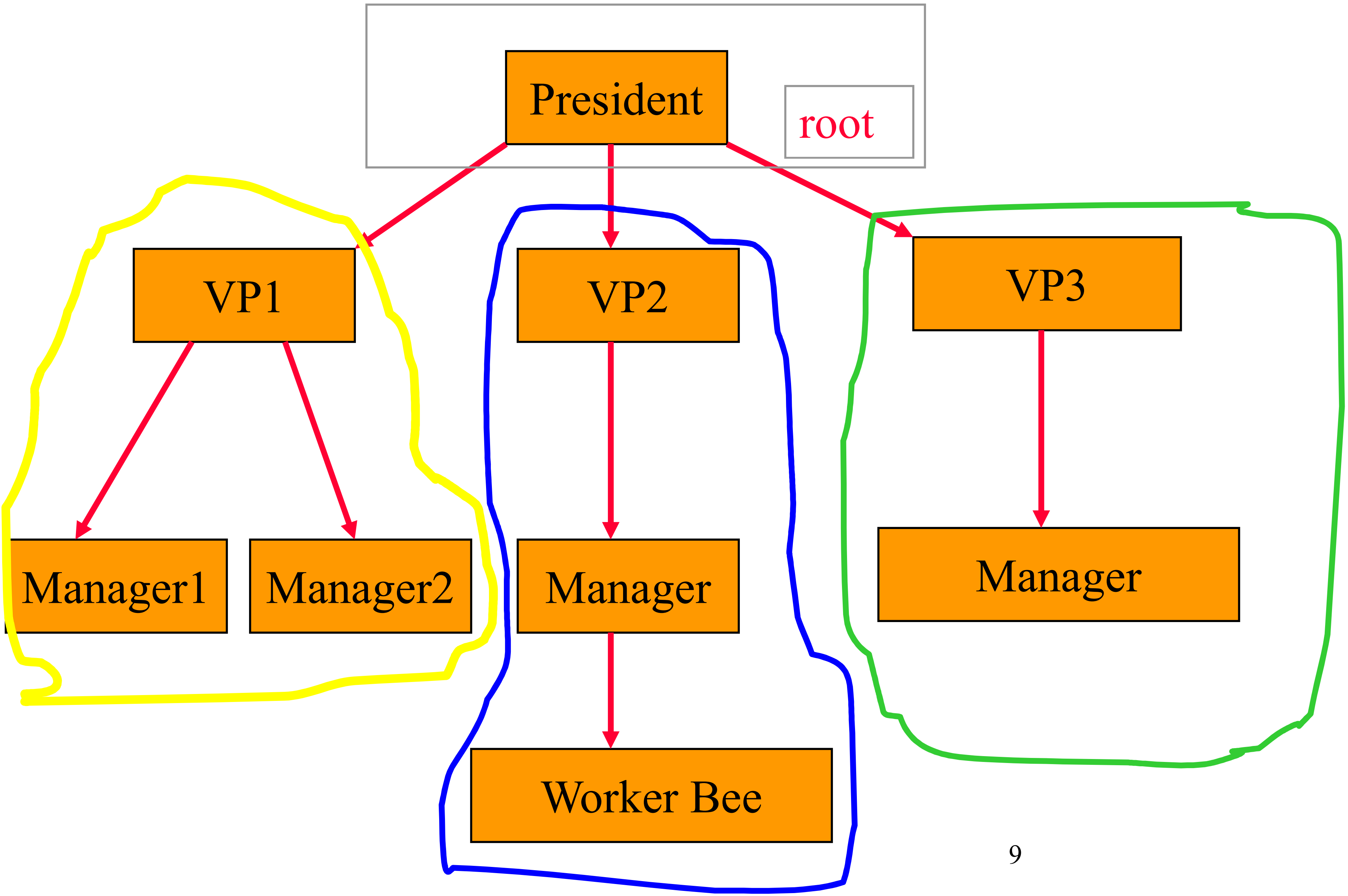
Definition of a Tree

- A *tree* is a finite set of one or more nodes such that:

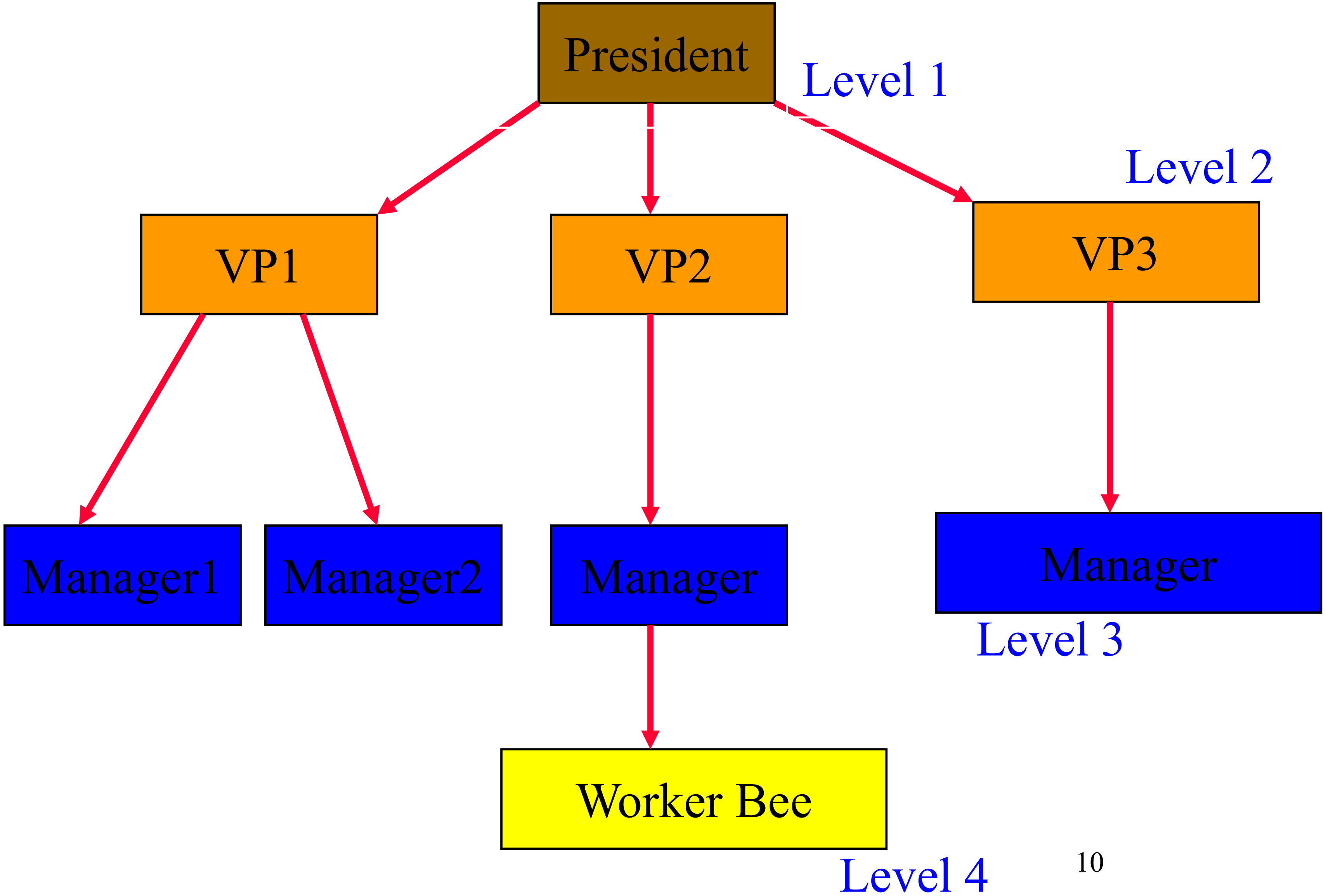
  (i) there is a specially designated node called the *root*;

  (ii) the remaining nodes are partitioned into $n \geq 0$ disjoint sets T_1, ..., T_n where each of these sets is a tree. T_1, ..., T_n are called the *subtrees* of the root.

Subtrees


President
root
VP1
VP2
VP3
Manager1
Manager2
Manager
Manager
Worker Bee

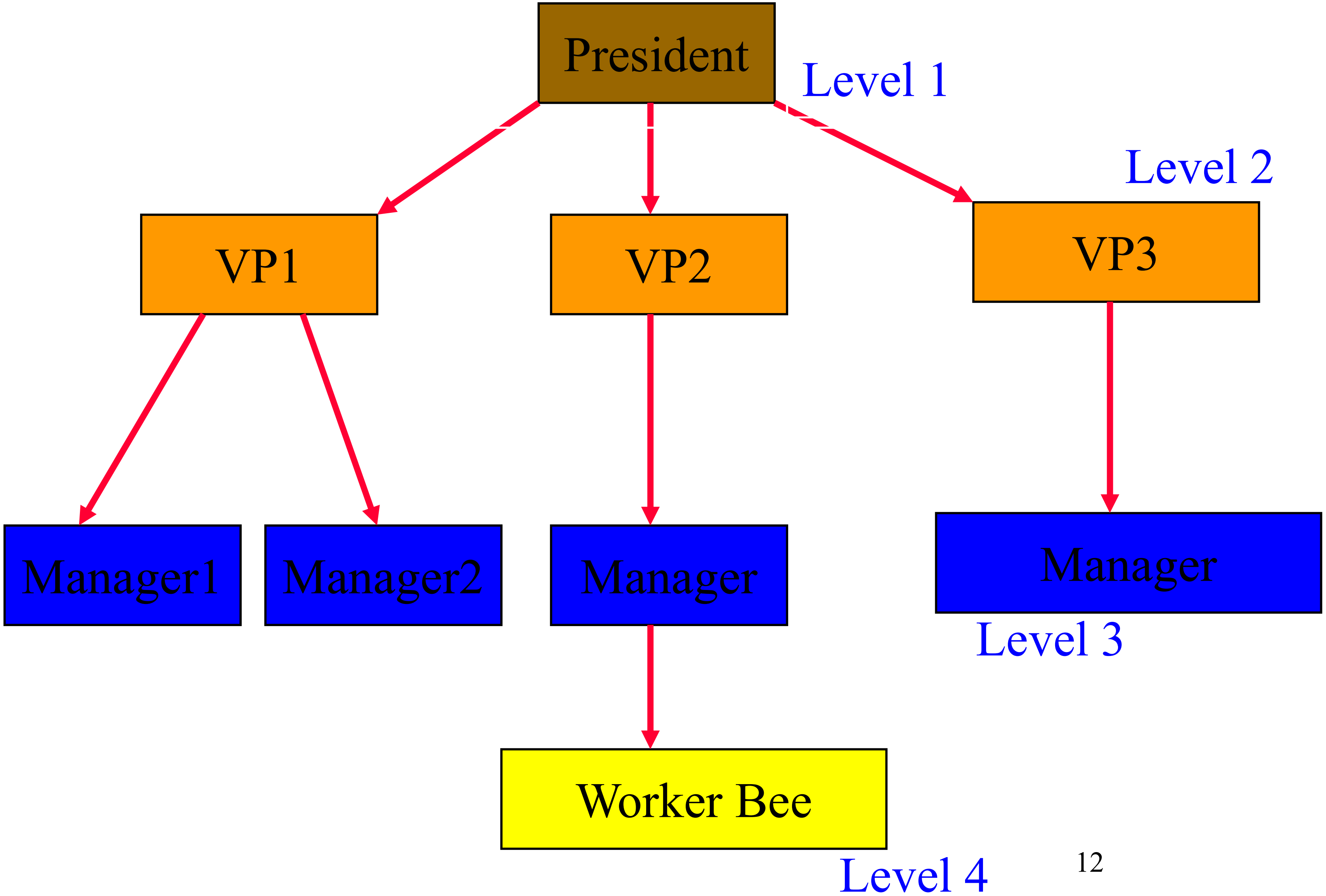
Levels


President
Level 1
Level 2
VP1
VP2
VP3
Manager1
Manager2
Manager
Manager
Level 3
Worker Bee
Level 4

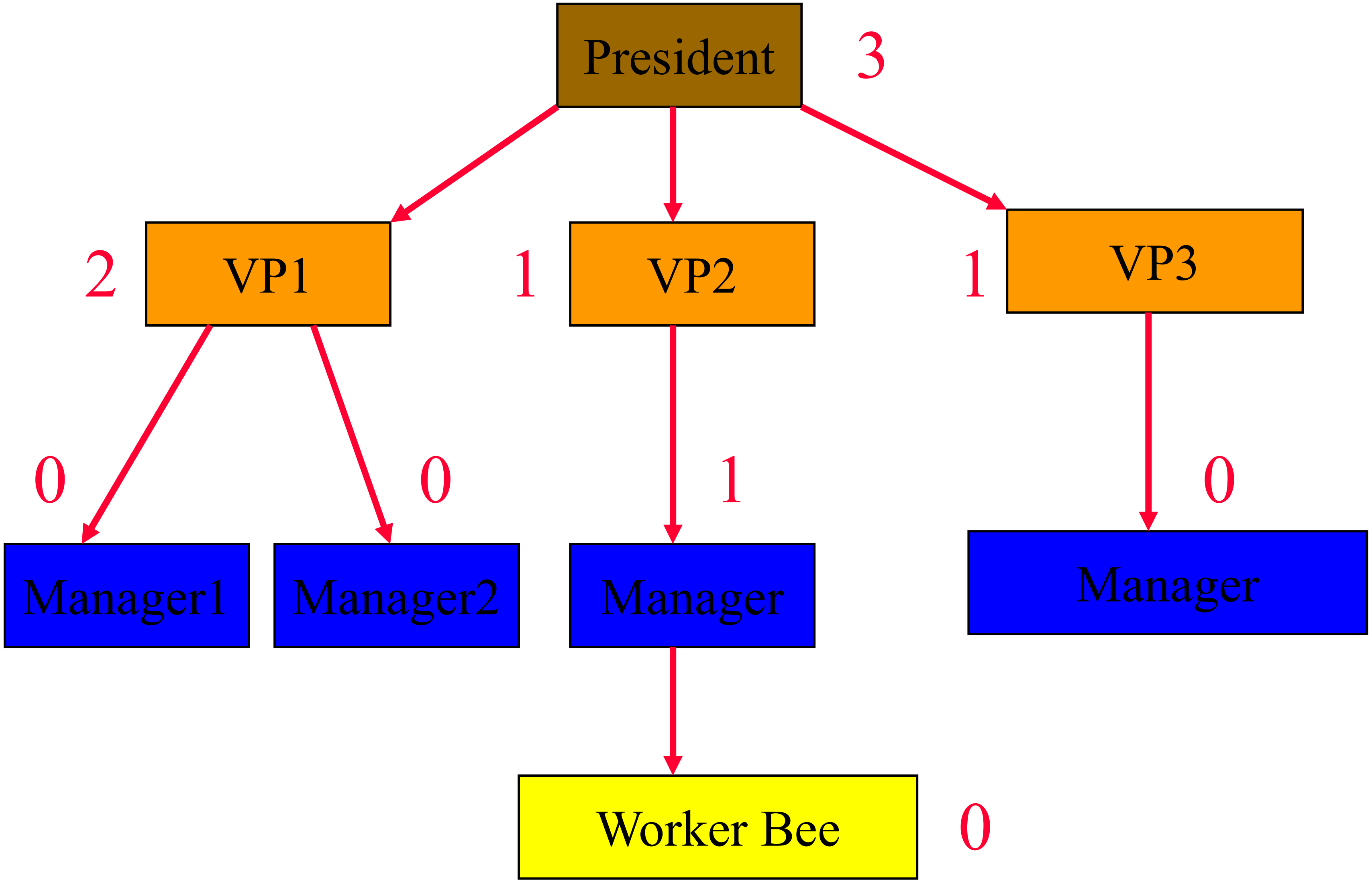
Caution

- Some texts start level numbers at 0 rather than at 1.
- Root is at level 0.
- Its children are at level 1.
- The grand children of the root are at level 2.
- And so on.
- We shall number levels with the root at level 1.

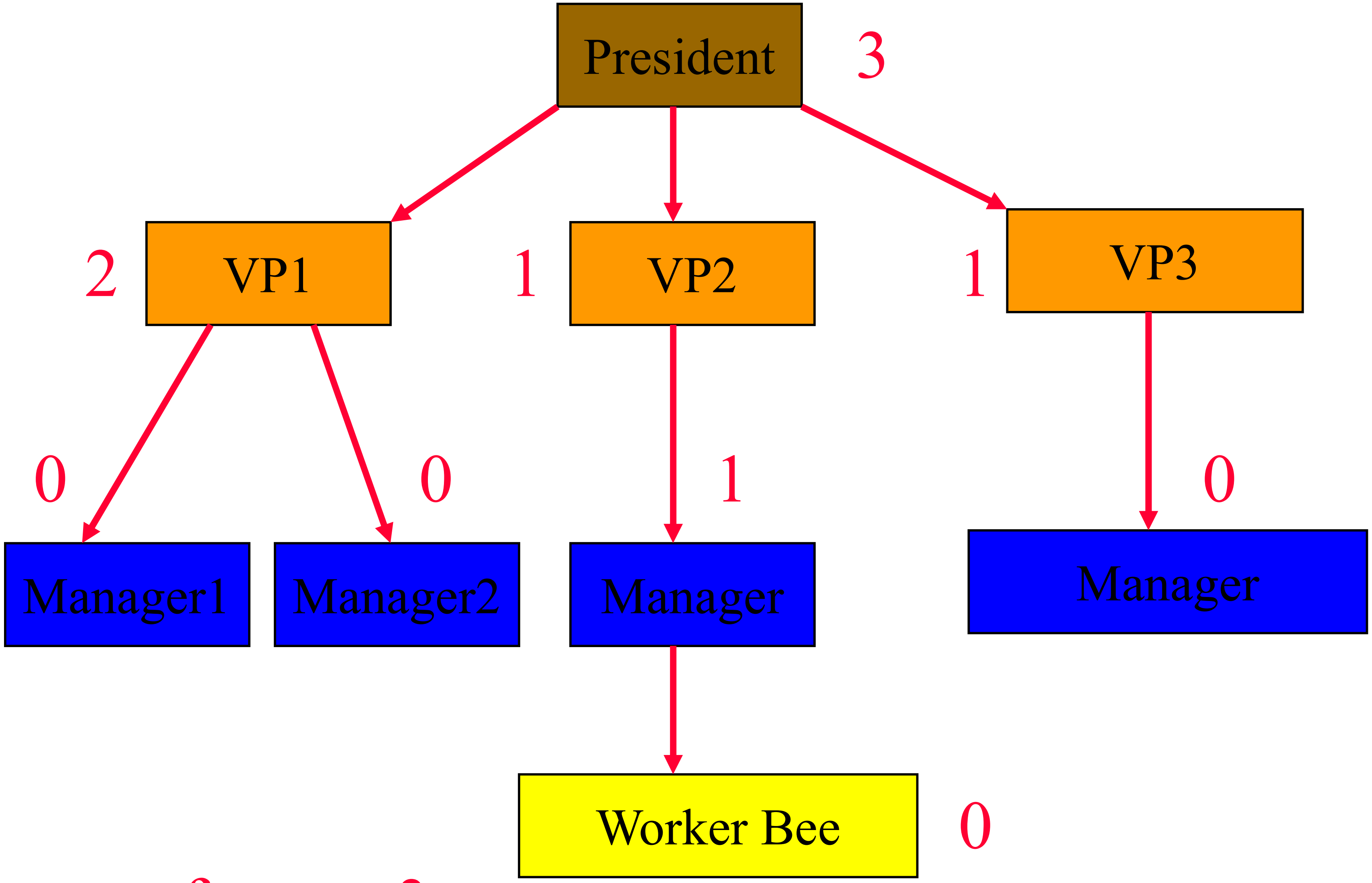
Height = Depth = Number of Levels



Node Degree = Number of Children



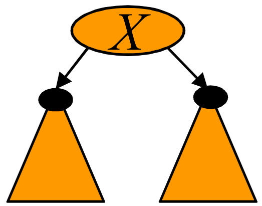
Tree Degree = Max Node Degree



Degree of tree = 3.

Terminology

- *Node*: the item of information (plus the branches to other items)
- *Degree* of a node: the number of subtrees of the node
- *Leaf* (or *terminal*) nodes: Nodes that have degree zero
  - The other nodes are referred to as *internal* (or *nonterminal*) nodes
- The roots of the subtrees of a node, *X*, are the *children* of *X*. *X* is the *parent* of its children.



- Children of the same parent are said to be *siblings*

Terminology (Cont.)

- *Ancestors* of a node: all the nodes along the path from the root to the node
- *Descendants* of a node: all the nodes that are in its subtrees
- *Level*: letting the root be at level one. If a node is at level l, then its children are at level $l + 1$
- *Height* (or *depth*) of a tree is the maximum level of any node in the tree
- *Degree of a tree*: the maximum degree of the nodes in the tree
- *Forest*: a set of disjoint trees. If we remove the root of a tree, we get a forest

Binary Tree

- Finite (possibly empty) collection of elements.
- A nonempty binary tree has a root element.
- The remaining elements (if any) are partitioned into two binary trees.
- These are called the left and right subtrees of the binary tree.

Differences Between a Tree & a Binary Tree

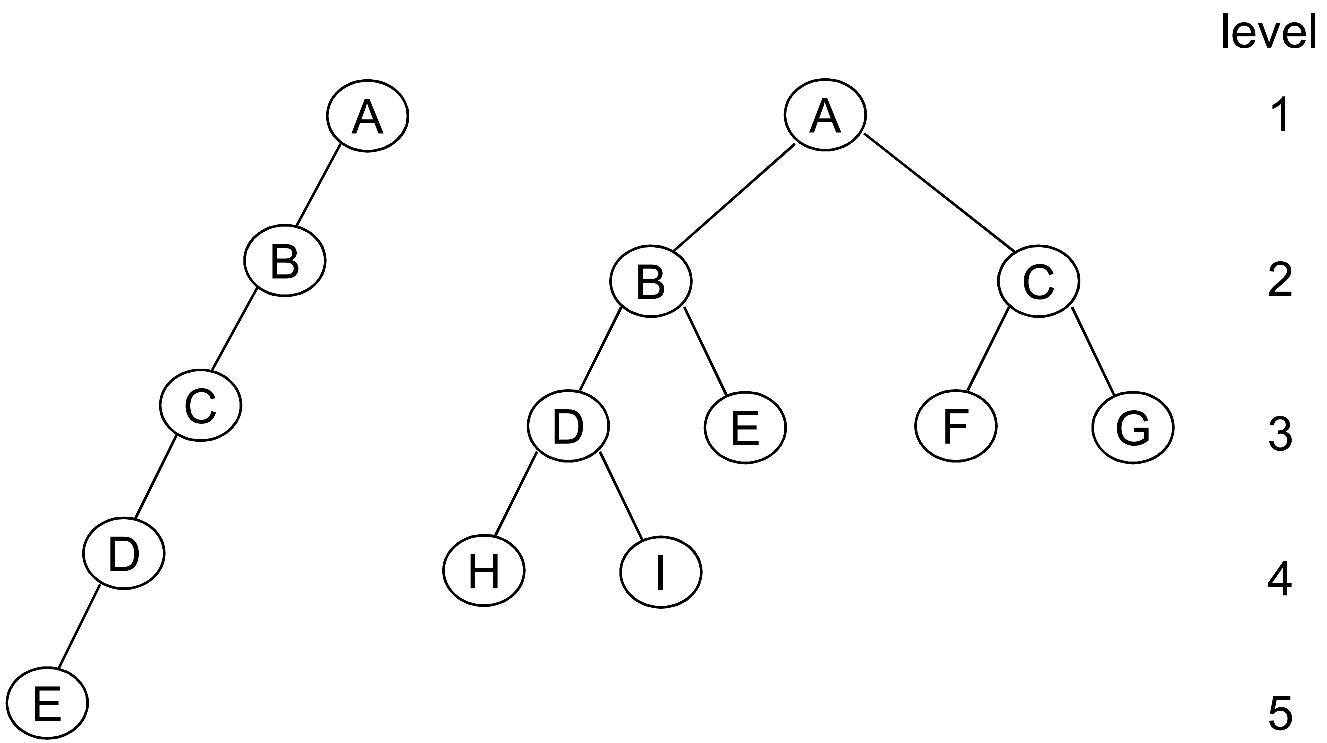
- No node in a binary tree may have a degree more than 2, whereas there is no limit on the degree of a node in a tree.
- A binary tree may be empty; a tree cannot be empty.

Differences Between a Tree & a Binary Tree (Cont.)

- The subtrees of a binary tree are ordered; those of a tree are not ordered.

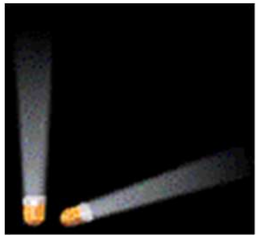
- Are different when viewed as binary trees.
- Are the same when viewed as trees.

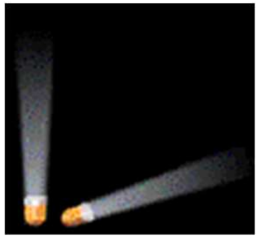
Examples

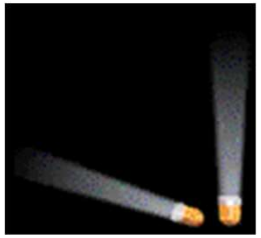


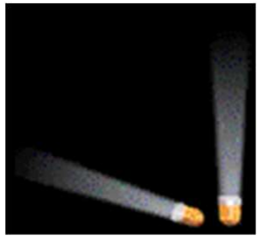
(a) Skewed binary tree (b) Complete binary tree

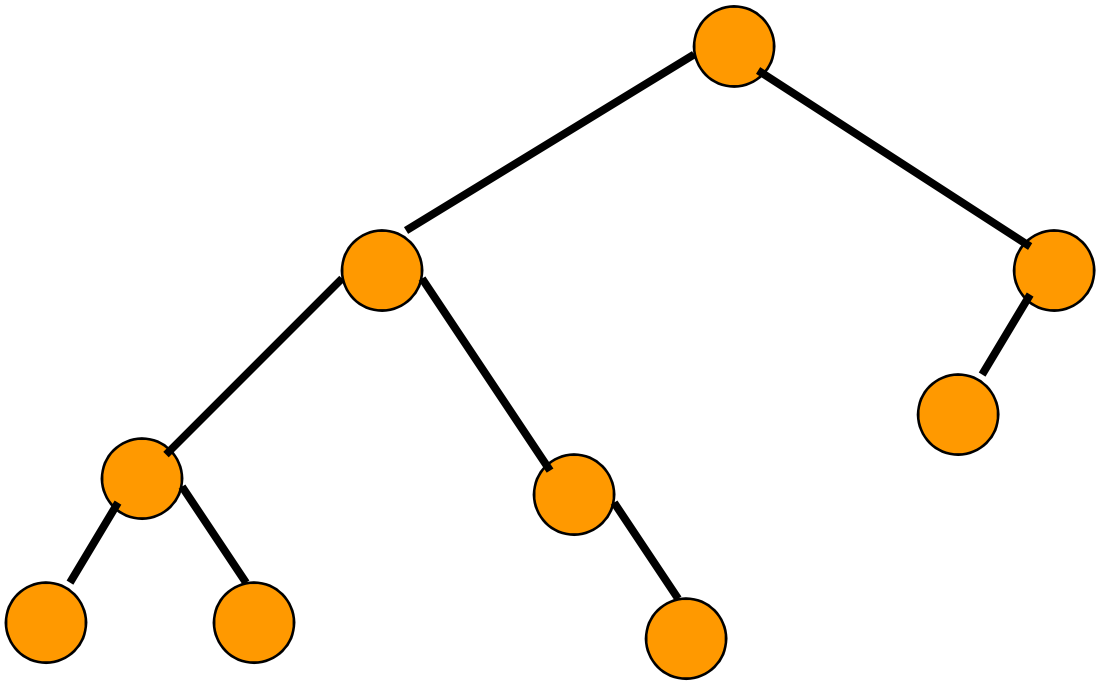
Binary Tree Properties & Representation





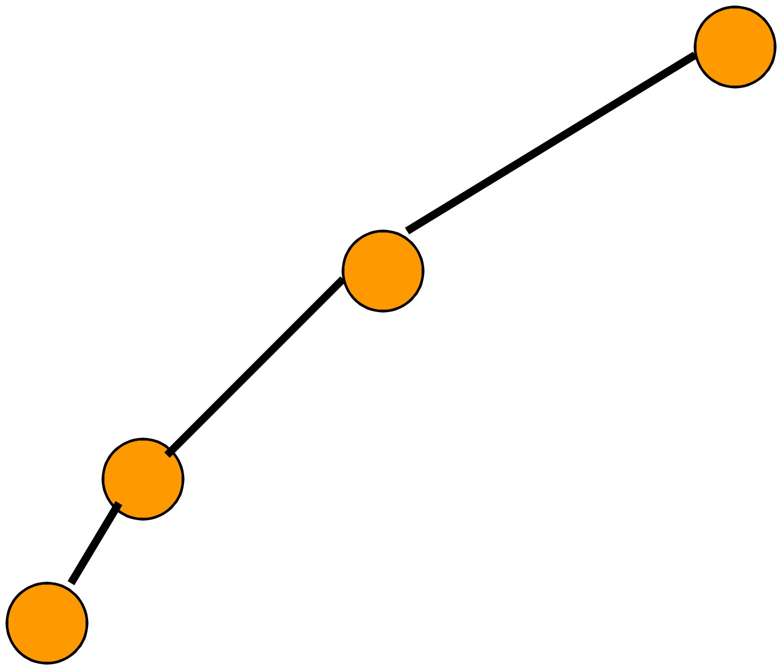






Minimum Number of Nodes

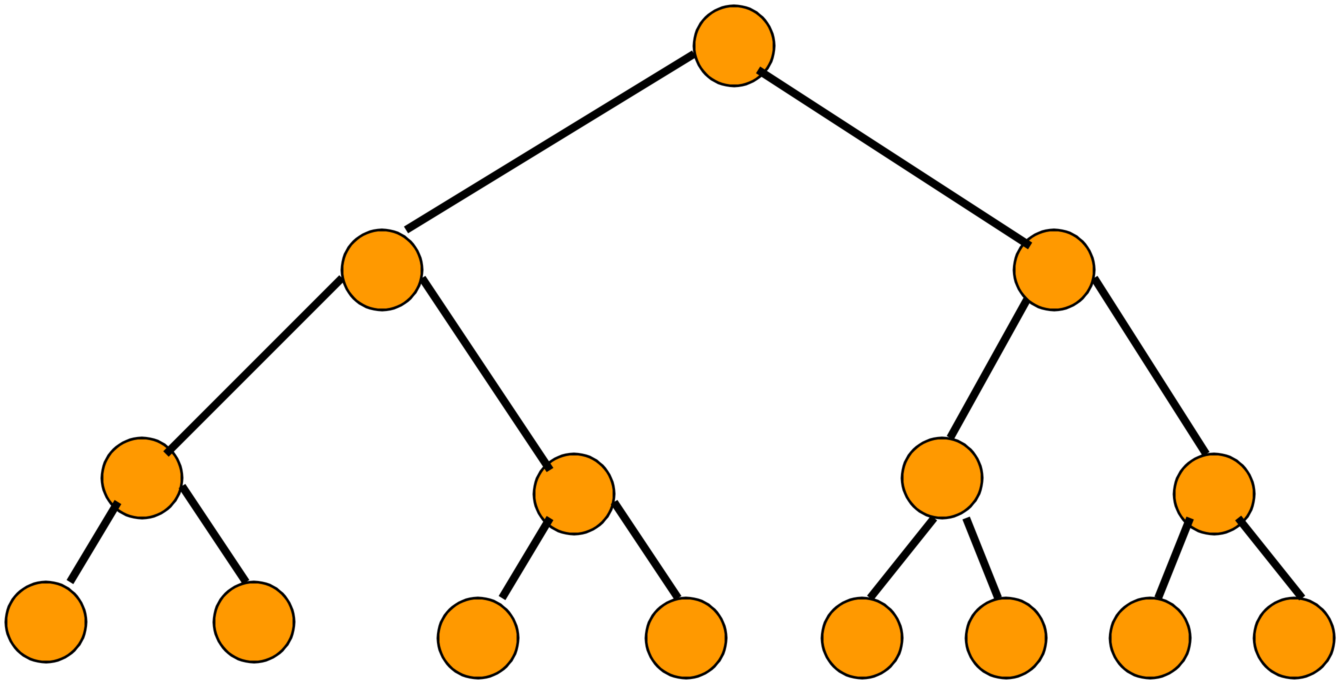
- Minimum number of nodes in a binary tree whose height is *h*.
- At least one node at each of first *h* levels.



minimum number of nodes is *h*

Maximum Number of Nodes

- All possible nodes at first *h* levels are present.



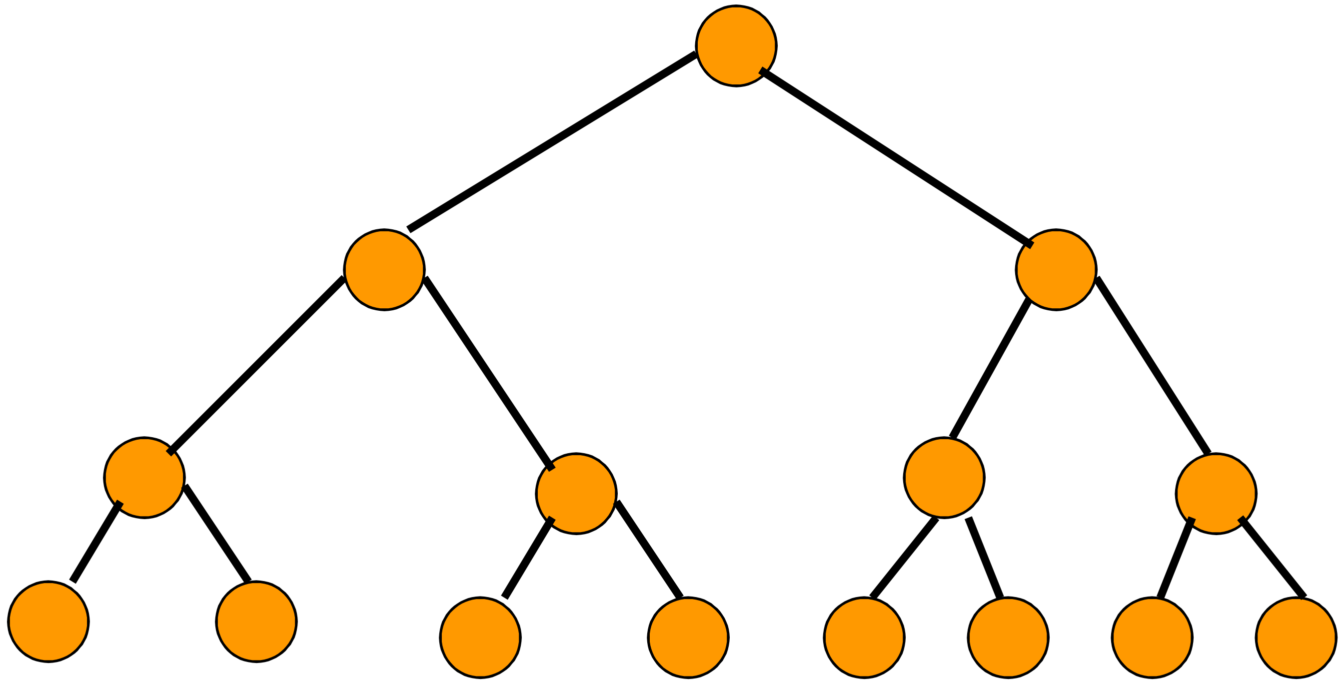
Maximum number of nodes

$= 1 + 2 + 4 + 8 + \ldots + 2^{h-1}$

$= 2^h - 1$

Full Binary Tree

- A full binary tree of a given height h has $2^h - 1$ nodes.



Height 4 full binary tree has 15 nodes.

Number of Nodes & Height

- Let n be the number of nodes in a binary tree whose height is h.
- $h \le n \le 2^h - 1$
- $\log_2(n+1) \le h \le n$
- skewed tree: $h = n$
- full binary tree: $h = \log_2(n+1)$

Properties of Binary Trees

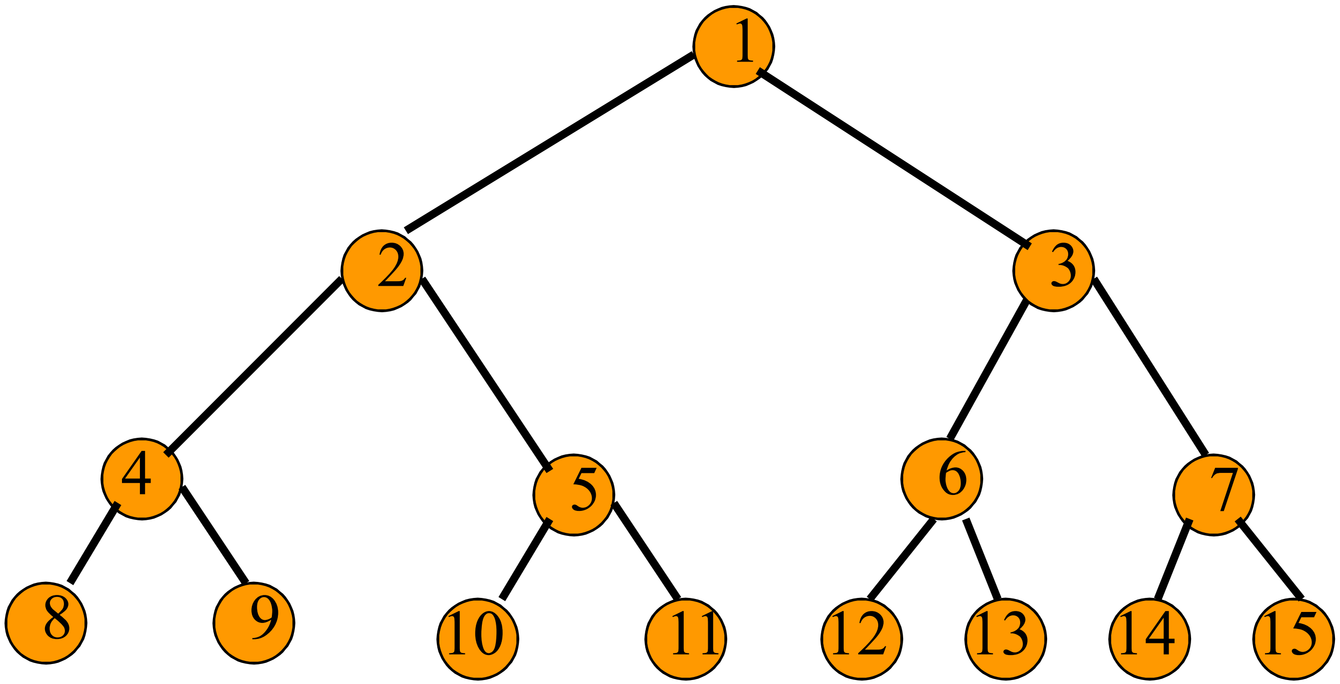
- The maximum number of nodes on level i of a binary tree is 2^{i-1}, $i \geq 1$
  - The proof is by induction on i

- The maximum number of nodes in a binary tree of depth k is $2^k - 1$

$$\sum_{i=1}^{k} 2^{i-1} = 2^k - 1$$

Numbering Nodes in a Full Binary Tree

- Number the nodes 1 through $2^h - 1$.
- Number by levels from top to bottom.
- Within a level number from left to right.



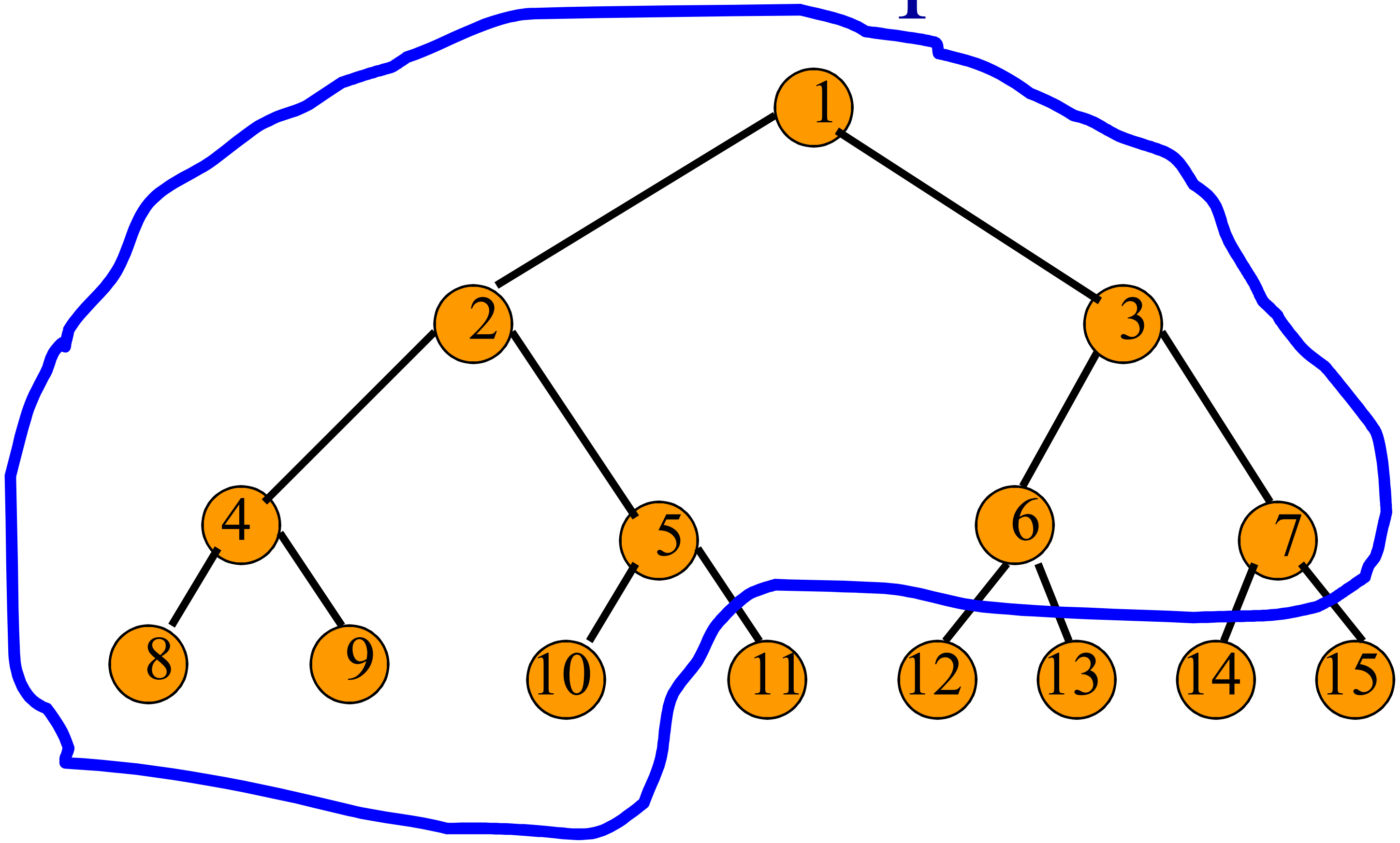
Complete Binary Tree

- Start with a full binary tree that has at least n nodes.
- Number the nodes as described earlier.
- The binary tree defined by the nodes numbered 1 through n is the unique n node complete binary tree.

Complete Binary Tree (Cont.)

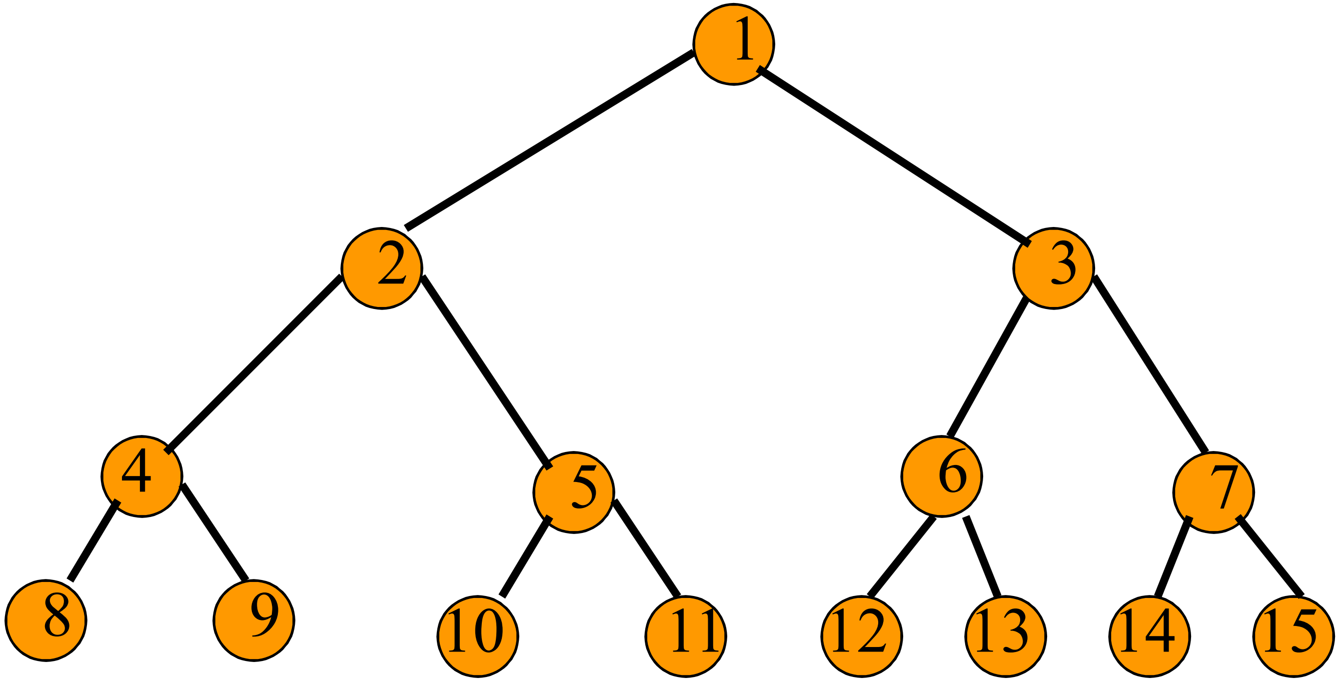
- Definition
  - A binary tree with n nodes and of depth h is *complete* iff its nodes correspond to the nodes which renumbered one to n in the full binary tree of depth h.
- Note
  - $h = \lceil \log_2(n+1) \rceil$

Example



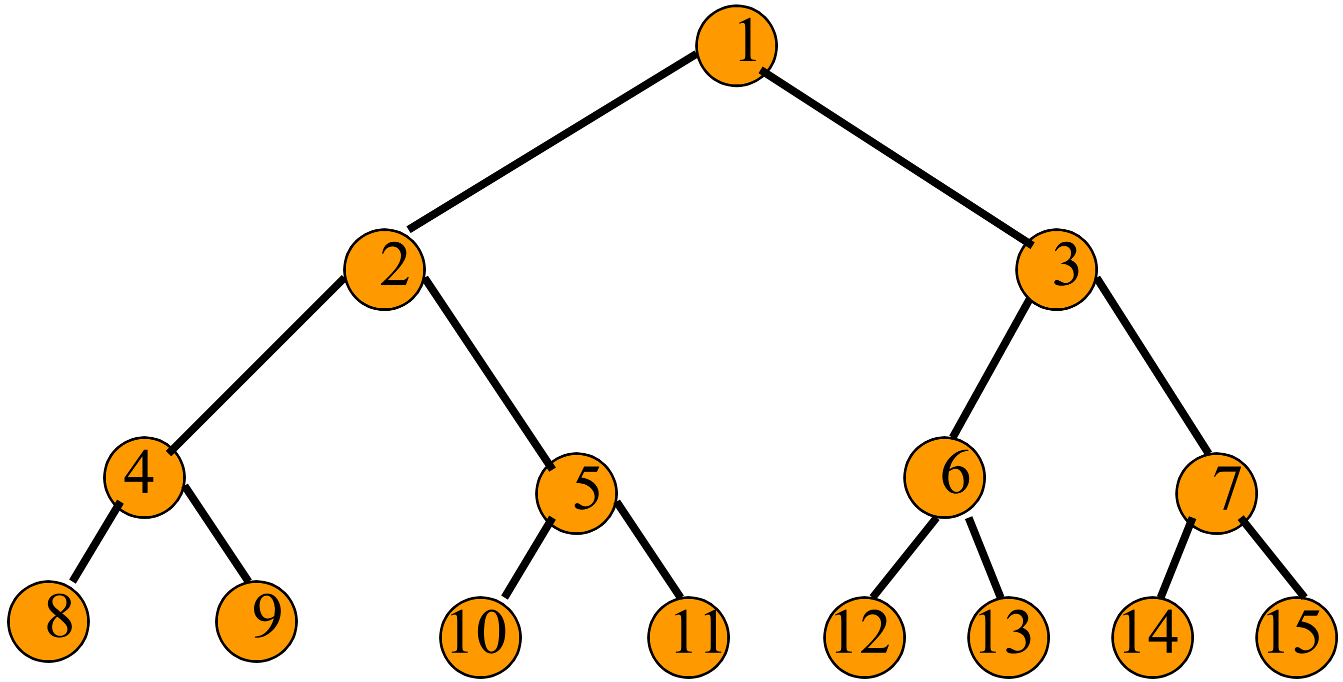
- Complete binary tree with 10 nodes.

Node Number Properties



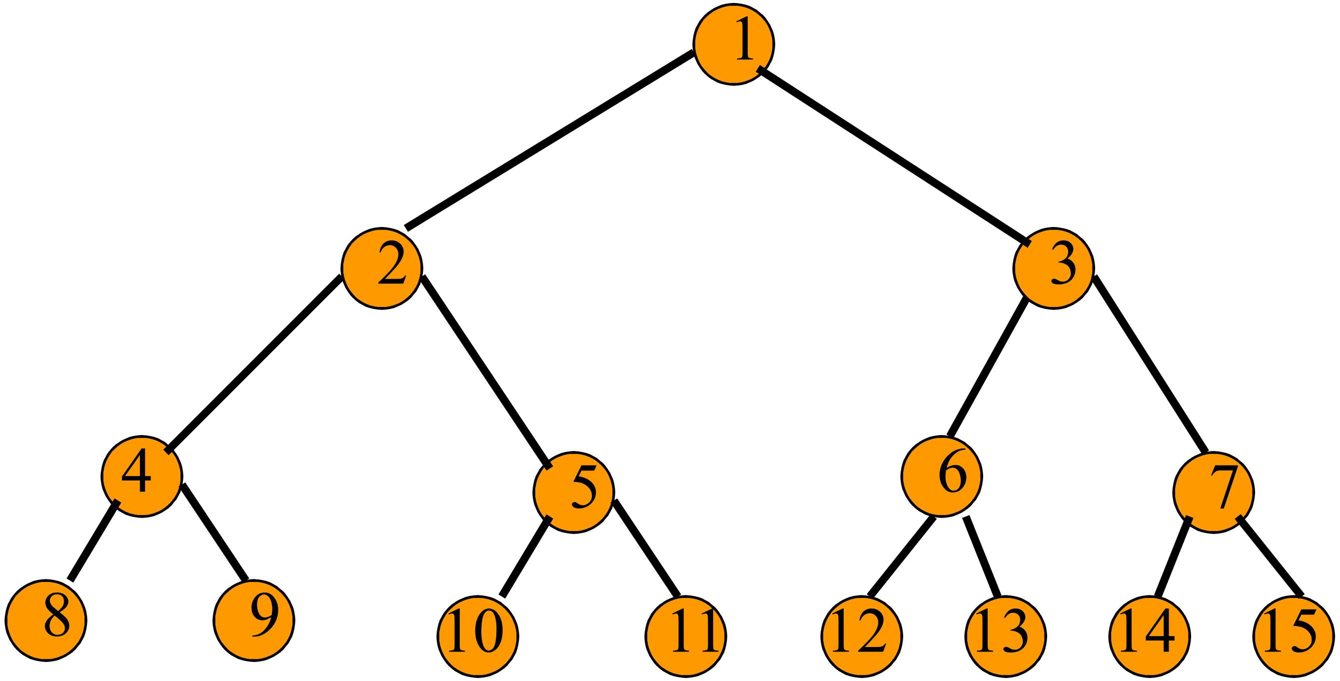
- Parent of node i is node i / 2, unless i = 1.
- Node 1 is the root and has no parent.

Node Number Properties (Cont.)



- Left child of node i is node $2i$, unless $2i > n$, where n is the number of nodes.
- If $2i > n$, node i has no left child.

Node Number Properties (Cont.)



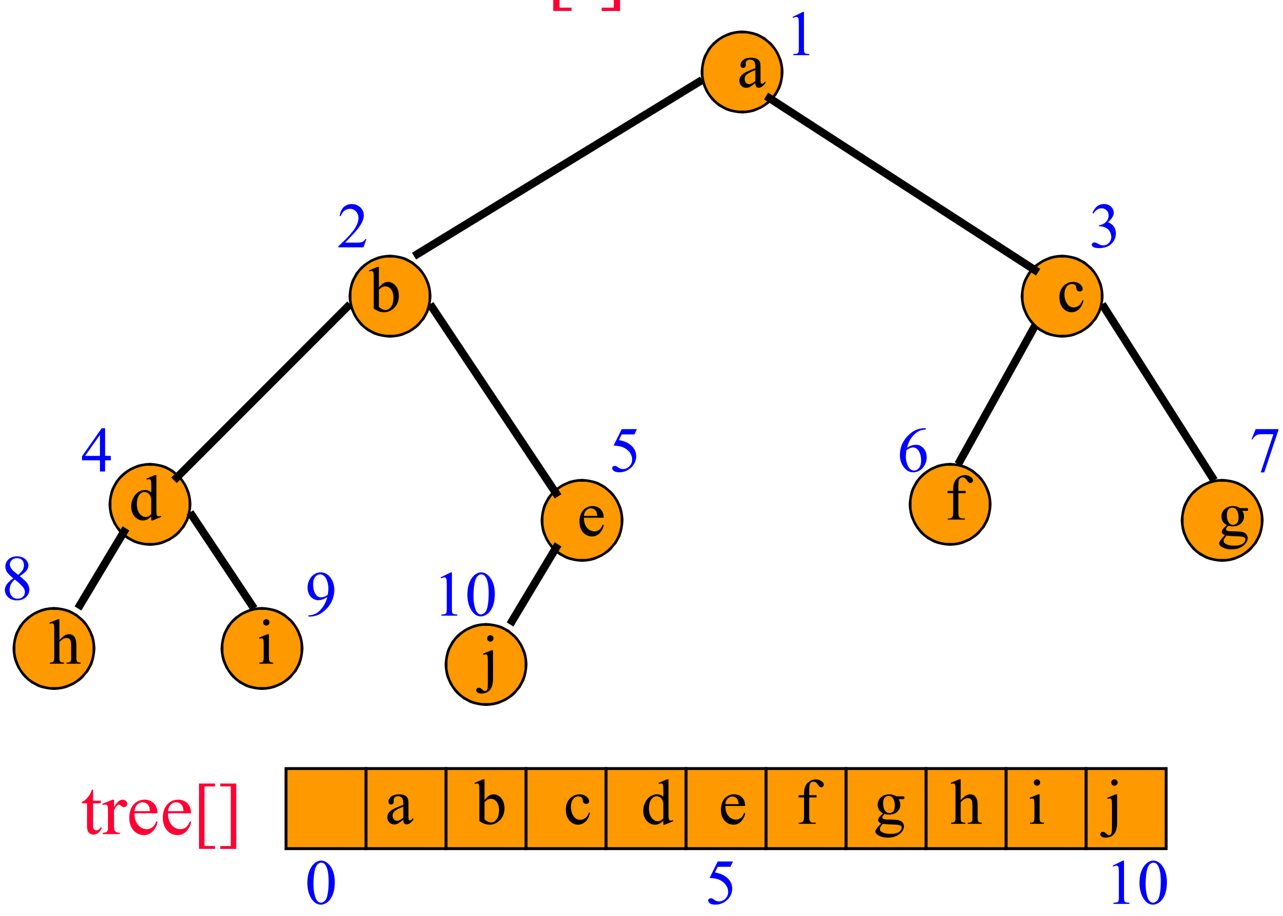
- Right child of node i is node $2i+1$, unless $2i+1 > n$, where n is the number of nodes.
- If $2i+1 > n$, node i has no right child.

Binary Tree Representation

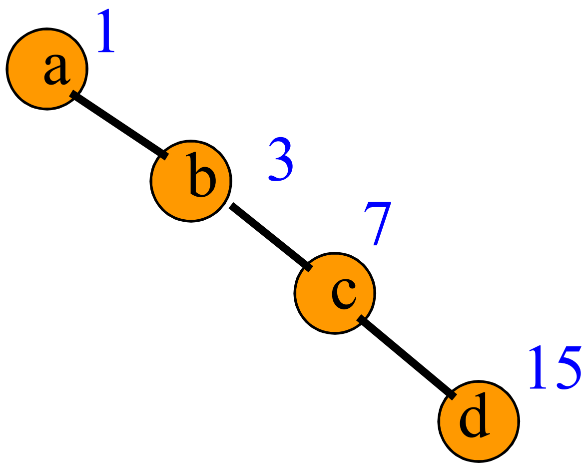
- Array representation.
- Linked representation.

Array Representation

- Number the nodes using the numbering scheme for a full binary tree. The node that is numbered *i* is stored in tree[*i*].



Right-Skewed Binary Tree



tree[]		a	-	b	-	-	-	c	-	-	-	-	-	-	-	d
	0					5					10					15

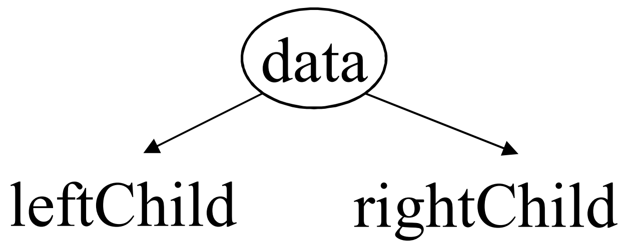
Disadvantages of Array Representation

- While the array representation appears to be good for complete binary trees, it is wasteful for many other binary trees

Linked Representation

- Each binary tree node is represented as an object whose data type is TreeNode.
- The space required by an n node binary tree is *n* * (space required by one node)

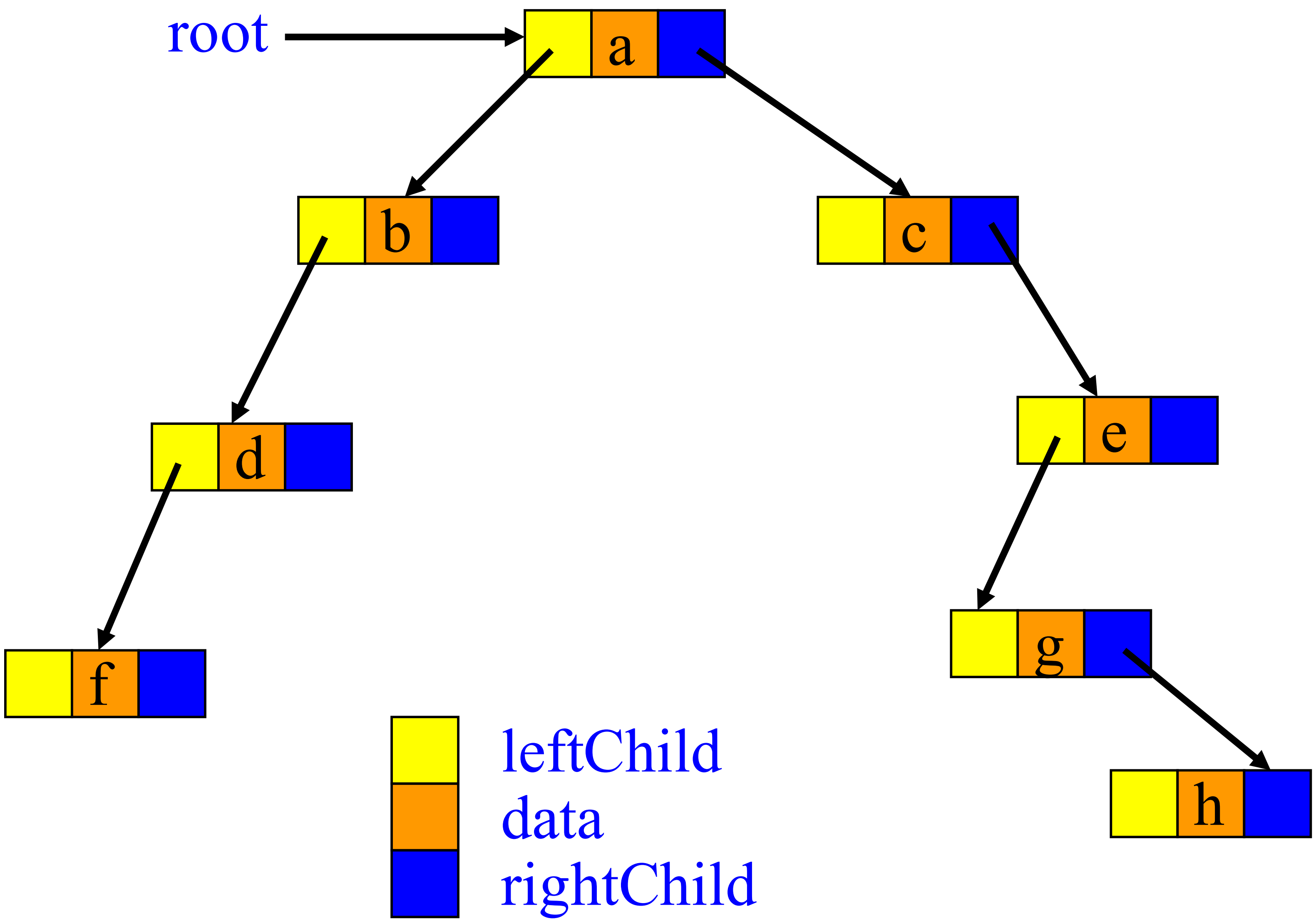
left child	data	right child



TreeNode Class

```
template <class T>
class TreeNode
{
   T data;
   TreeNode<T> *leftChild,
               *rightChild;
   TreeNode()
      {leftChild = rightChild = NULL;}
   // other constructors come here
};
```

Linked Representation Example



Binary Tree Traversal

- Arithmetic Expressions
- Inorder Traversal
- Preorder Traversal
- Postorder Traversal
- Level-order Traversal

Arithmetic Expressions

- (a + b) * (c + d) + e – f/g*h + 3.25
- Expressions comprise three kinds of entities.
  - Operators (+, -, /, *).
  - Operands (a, b, c, d, e, f, g, h, 3.25, (a + b), (c + d), etc.).
  - Delimiters ((,)).

Operator Degree

- Number of operands that the operator requires.
- Binary operator requires two operands.
  - a + b
  - c / d
  - e - f
- Unary operator requires one operand.
  - + g
  - - h

Infix Form

- Normal way to write an expression.
- Binary operators come in between their left and right operands.
  - a * b
  - a + b * c
  - a * b / c
  - (a + b) * (c + d) + e – f/g*h + 3.25

Operator Priorities

- How do you figure out the operands of an operator?
  - a + b * c
  - a * b + c / d
- This is done by assigning operator priorities.
  - priority(*) = priority(/) > priority(+) = priority(-)
- When an operand lies between two operators, the operand associates with the operator that has higher priority.

Tie Breaker

- When an operand lies between two operators that have the same priority, the operand associates with the operator on the left.
  - a + b - c
  - a * b / c / d

Delimiters

- Subexpression within delimiters is treated as a single operand, independent from the remainder of the expression.
  - (a + b) * (c – d) / (e – f)

Infix Expression Is Hard to Parse

- Need operator priorities, tie breaker, and delimiters.
- This makes computer evaluation more difficult than is necessary.
- Postfix and prefix expression forms do not rely on operator priorities, a tie breaker, or delimiters.
- So it is easier for a computer to evaluate expressions that are in these forms.

Postfix Form

- The postfix form of a variable or constant is the same as its infix form.
  - a, b, 3.25
- The relative order of operands is the same in infix and postfix forms.
- Operators come immediately after the postfix form of their operands.
  - Infix = a + b
  - Postfix = ab+

Postfix Examples

- Infix = a + b * c
  - Postfix = a b c * +
- Infix = a * b + c
  - Postfix = a b * c +
- Infix = (a + b) * (c – d) / (e + f)
  - Postfix = a b + c d - * e f + /

Unary Operators

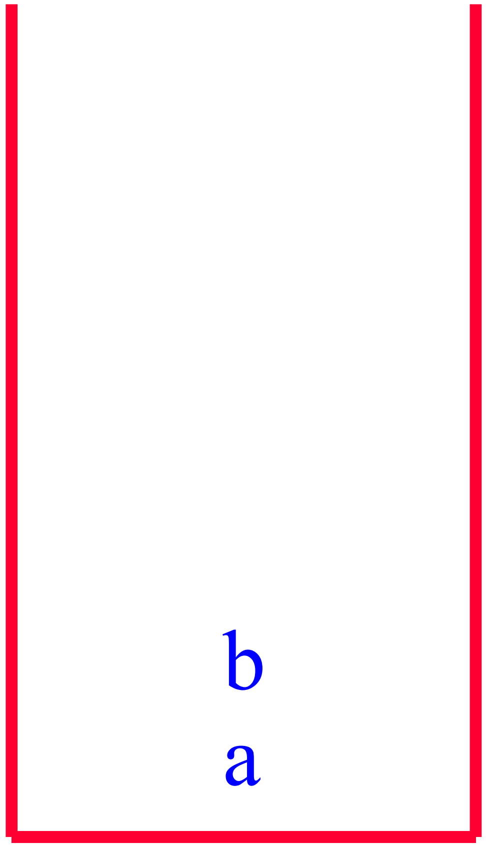
- Replace with new symbols.
  - + a => a @
  - + a + b => a @ b +
  - - a => a ?
  - - a-b => a ? b -

Postfix Evaluation

- Scan postfix expression from left to right pushing operands on to a stack.
- When an operator is encountered, pop as many operands as this operator needs; evaluate the operator; push the result on to the stack.
- This works because, in postfix, operators come immediately after their operands.

Postfix Evaluation

- (a + b) * (c – d) / (e + f)
- a b + c d - * e f + /
- a b + c d - * e f + /
- a b + c d - * e f + /
- a b + c d - * e f + /



stack

Postfix Evaluation

- (a + b) * (c – d) / (e + f)
- a b + c d - * e f + /
- a b + c d - * e f + /
- a b + c d - * e f + /
- a b + c d - * e f + /
- a b + c d - * e f + /
- a b + c d - * e f + /
- a b + c d - * e f + /

d

c

(a + b)

stack

Postfix Evaluation

- $(a + b) * (c - d) / (e + f)$
- a b + c d - * e f + /
- a b + c d - * e f + /

(c – d)
(a + b)

stack

Postfix Evaluation

- (a + b) * (c – d) / (e + f)
- a b + c d - * e f + /
- a b + c d - * e f + /
- a b + c d - * e f + /
- a b + c d - * e f + /
- a b + c d - * e f + /

stack
f
e
(a + b)*(c – d)

Postfix Evaluation

- (a + b) * (c – d) / (e + f)
- a b + c d - * e f + /
- a b + c d - * e f + /
- a b + c d - * e f + /
- a b + c d - * e f + /
- a b + c d - * e f + /
- a b + c d - * e f + /

(e + f)

(a + b)*(c – d)

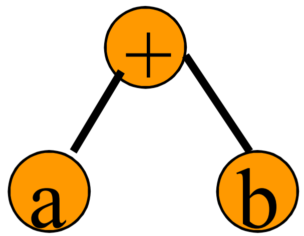
stack

Prefix Form

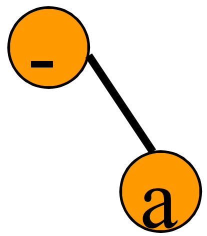
- The prefix form of a variable or constant is the same as its infix form.
  - a, b, 3.25
- The relative order of operands is the same in infix and prefix forms.
- Operators come immediately before the prefix form of their operands.
  - Infix = a + b
  - Postfix = ab+
  - Prefix = +ab

Binary Tree Form

- a + b

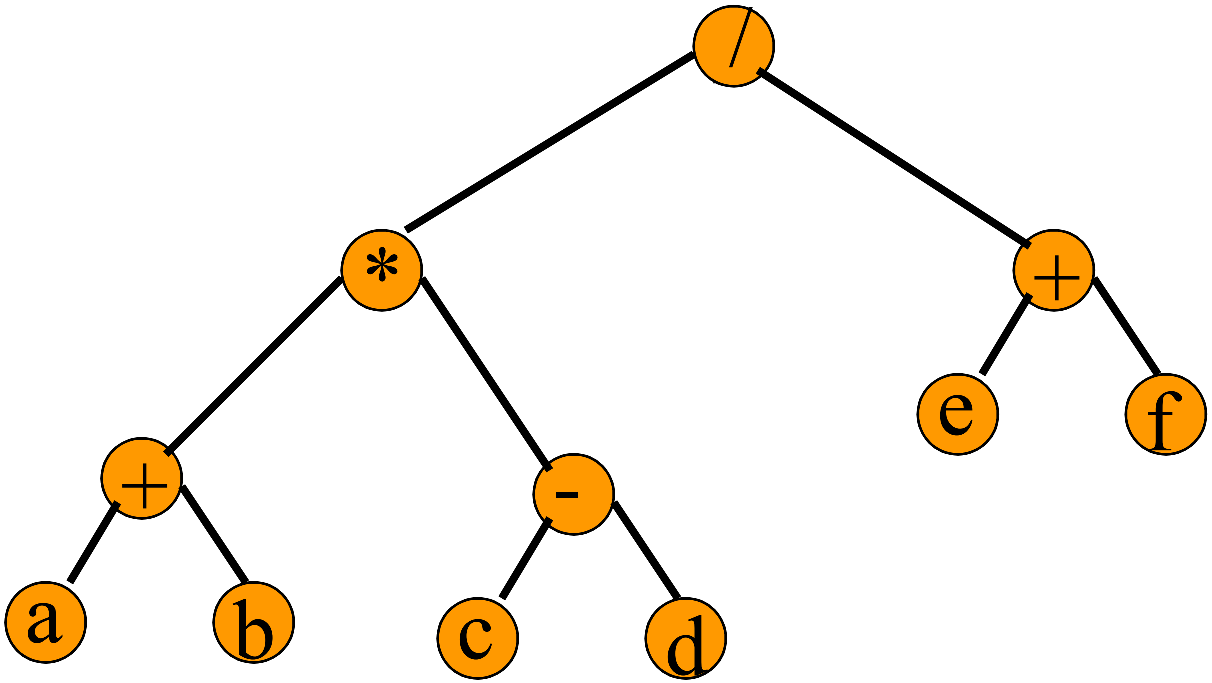


- \- a



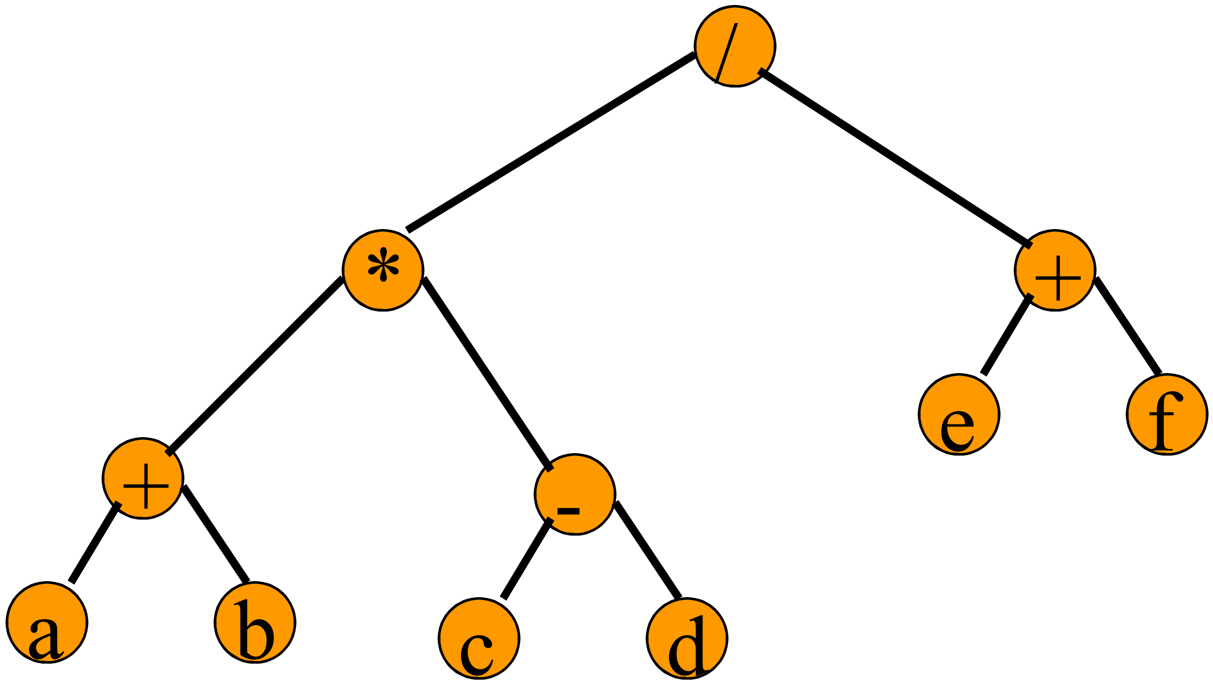
Binary Tree Form

- (a + b) * (c – d) / (e + f)



Merits of Binary Tree Form

- Left and right operands are easy to visualize.
- Simple recursive evaluation of expression.



Binary Tree Traversal

- Many binary tree operations are done by performing a traversal of the binary tree.
- In a traversal, each element of the binary tree is visited exactly once.
- During the visit of an element, all action (make a clone, display, evaluate the operator, etc.) with respect to this element is taken.

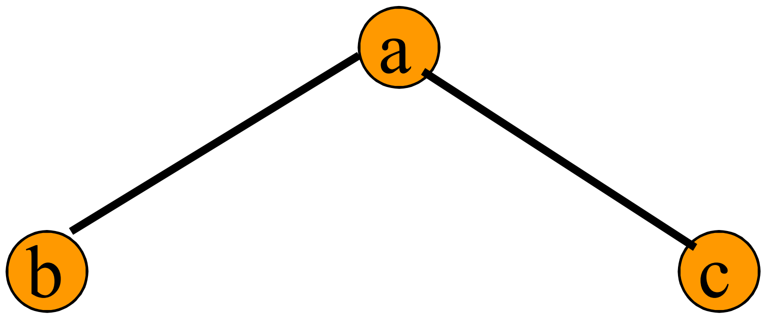
Binary Tree Traversal Methods

- Preorder
- Inorder
- Postorder
- Level order

Preorder Traversal

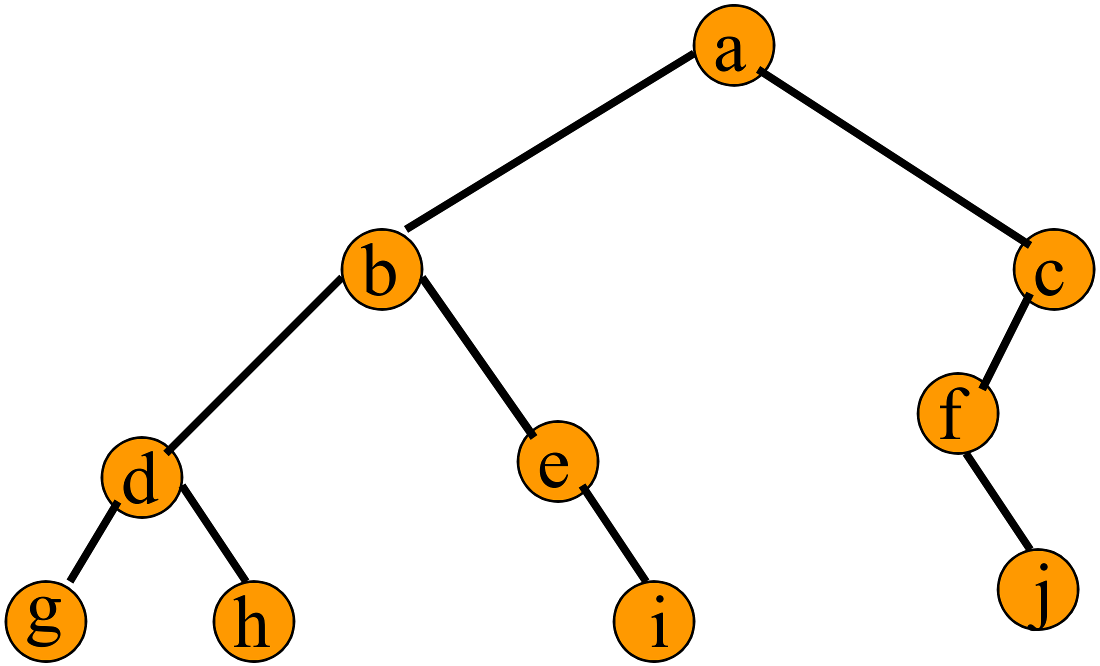
```
template <class T>
void PreOrder(TreeNode<T> *t)
{
   if (t != NULL)
   {
      Visit(t);
      PreOrder(t->leftChild);
      PreOrder(t->rightChild);
   }
}
```

Preorder Example (Visit = print)



a b c

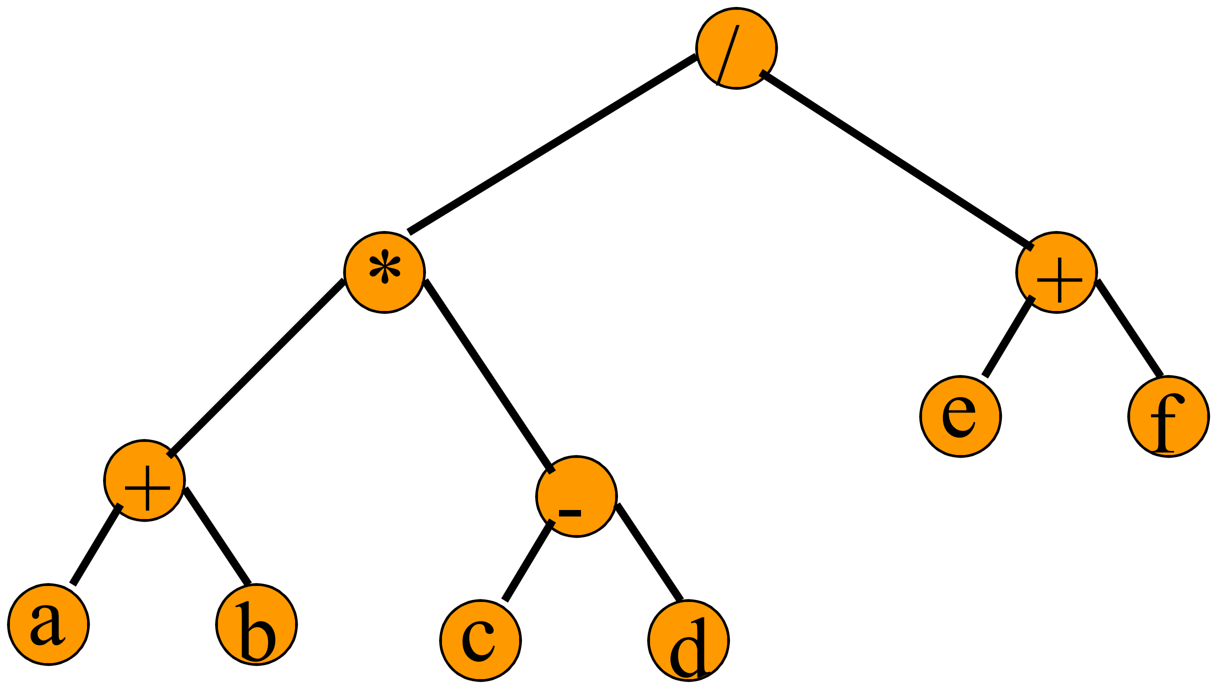
Preorder Example (Visit = print)




```
template <class T>
void PreOrder(TreeNode<T> *t)
{
    if (t != NULL)
    {
        Visit(t);
        PreOrder(t->leftChild);
        PreOrder(t->rightChild);
    }
}
```

a b d g h e i c f j

Preorder of Expression Tree



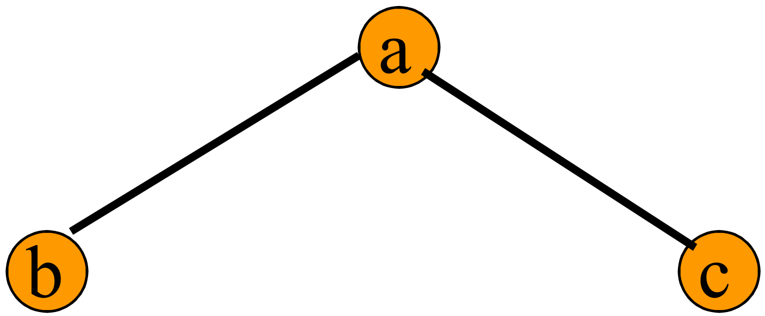
/ * + a b - c d + e f

Gives prefix form of expression!

Inorder Traversal

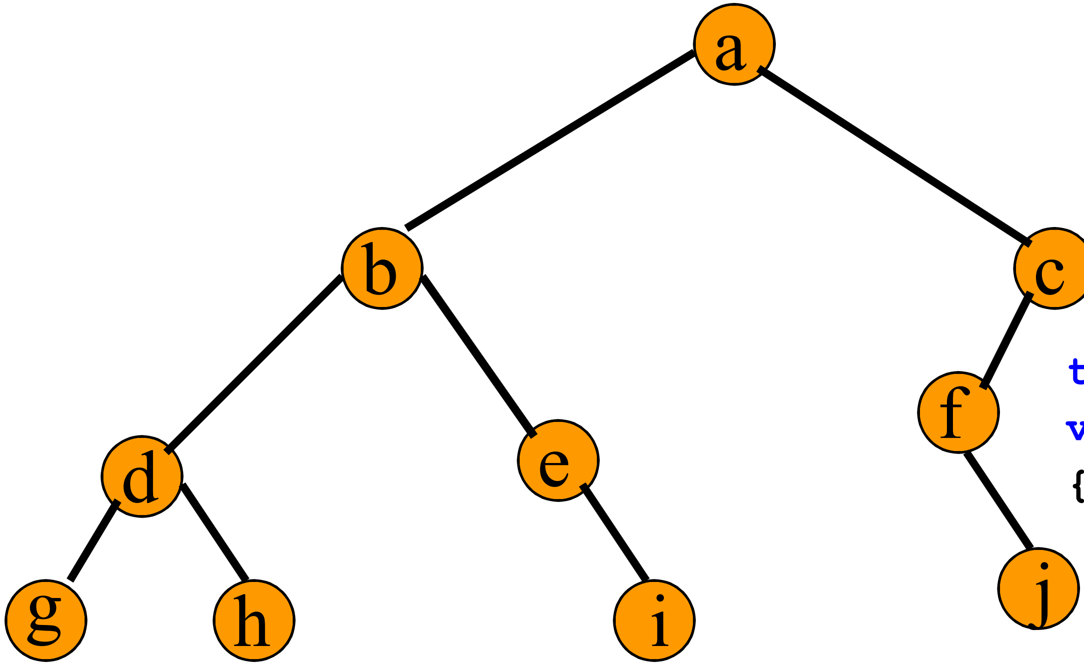
```
template <class T>
void InOrder(TreeNode<T> *t)
{
    if (t != NULL)
    {
        InOrder(t->leftChild);
        Visit(t);
        InOrder(t->rightChild);
    }
}
```

Inorder Example (Visit = print)



b a c

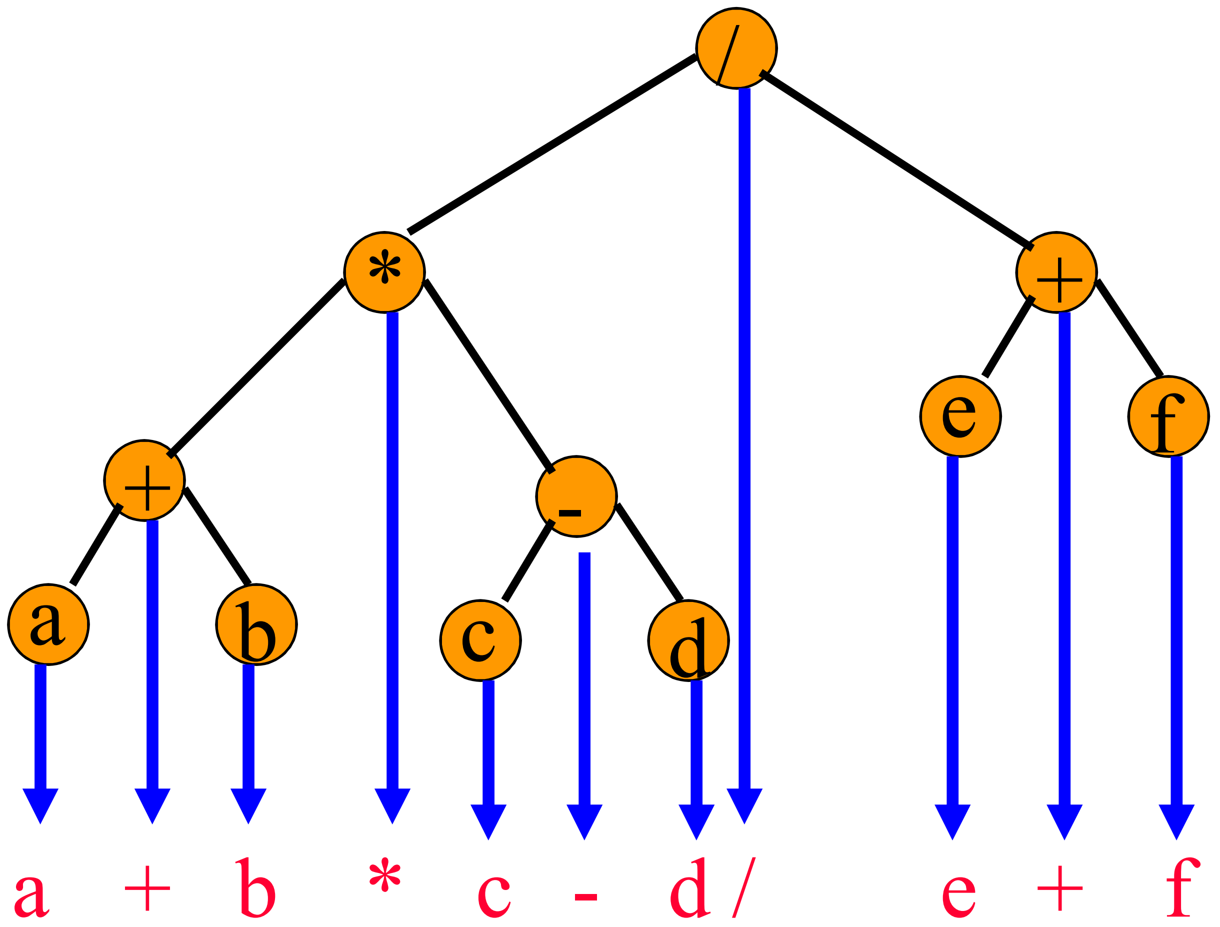
Inorder Example (Visit = print)




```
template <class T>
void InOrder(TreeNode<T> *t)
{
    if (t != NULL)
    {
        InOrder(t->leftChild);
        Visit(t);
        InOrder(t->rightChild);
    }
}
```

g d h b e i a f j c

Inorder of Expression Tree

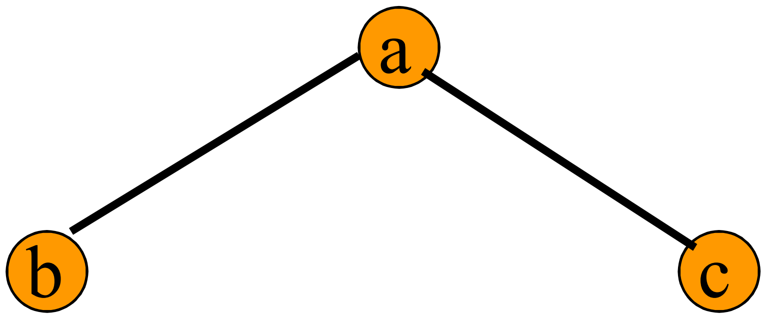


Gives infix form of expression (sans parentheses)!

Postorder Traversal

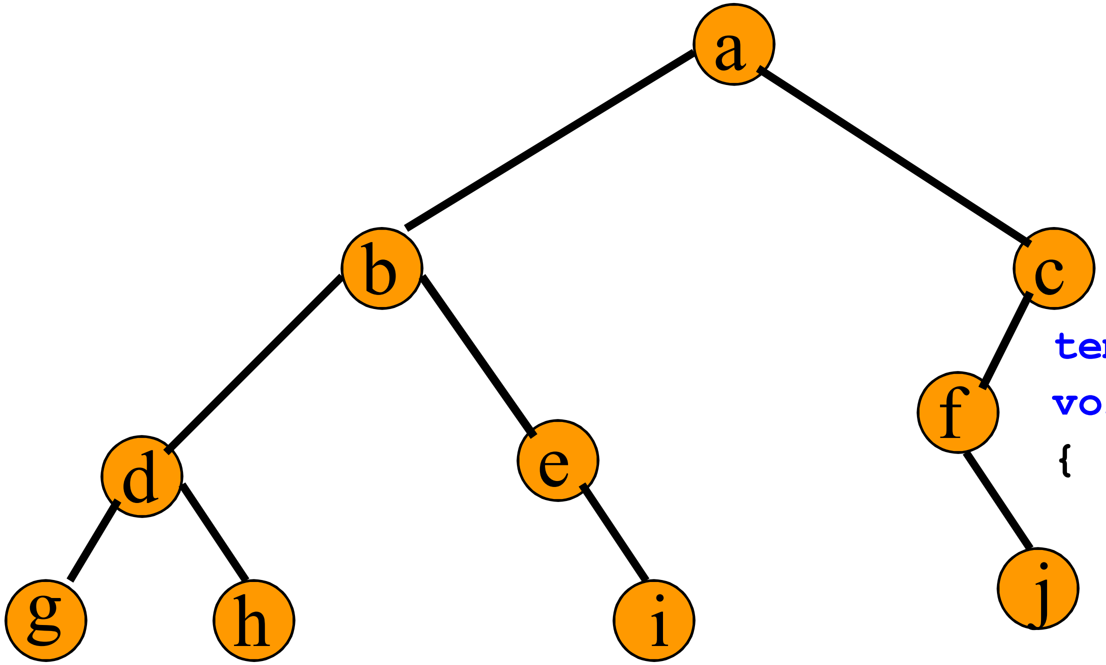
```
template <class T>
void PostOrder(TreeNode<T> *t)
{
    if (t != NULL)
    {
        PostOrder(t->leftChild);
        PostOrder(t->rightChild);
        Visit(t);
    }
}
```

Postorder Example (Visit = print)



b c a

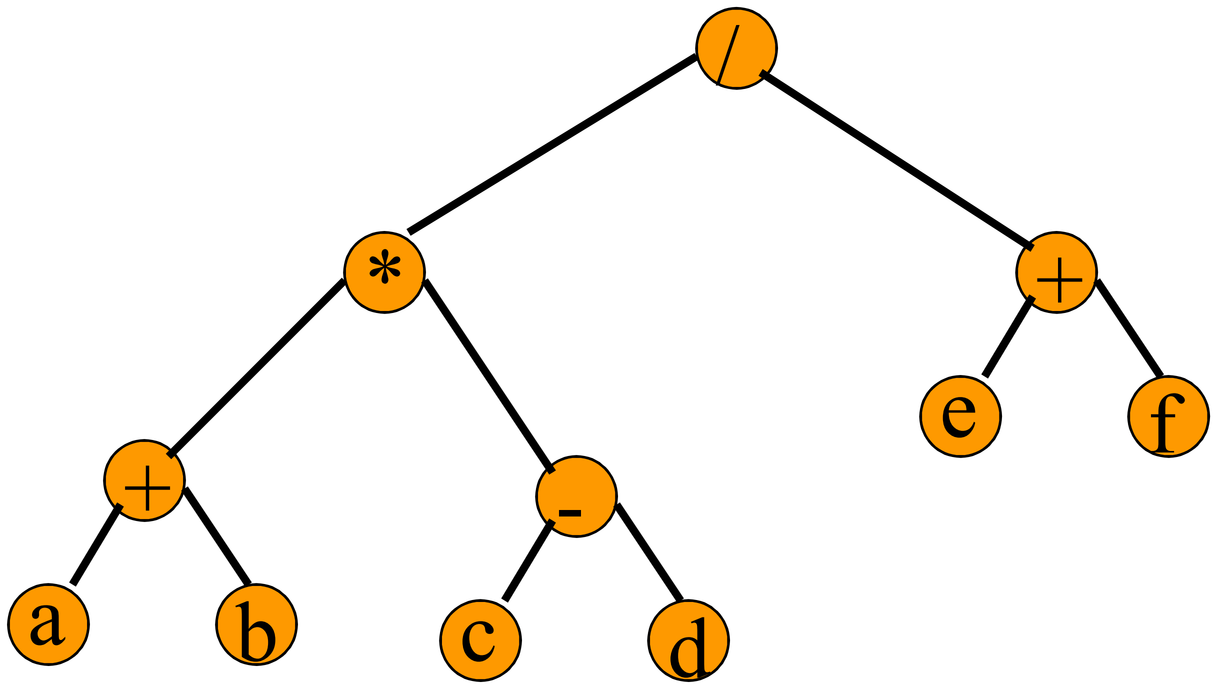
Postorder Example (Visit = print)




```
template <class T>
void PostOrder(TreeNode<T> *t)
{
    if (t != NULL)
    {
        PostOrder(t->leftChild);
        PostOrder(t->rightChild);
        Visit(t);
    }
}
```

g h d i e b j f c a

Postorder of Expression Tree



a b + c d - * e f + /

Gives postfix form of expression!

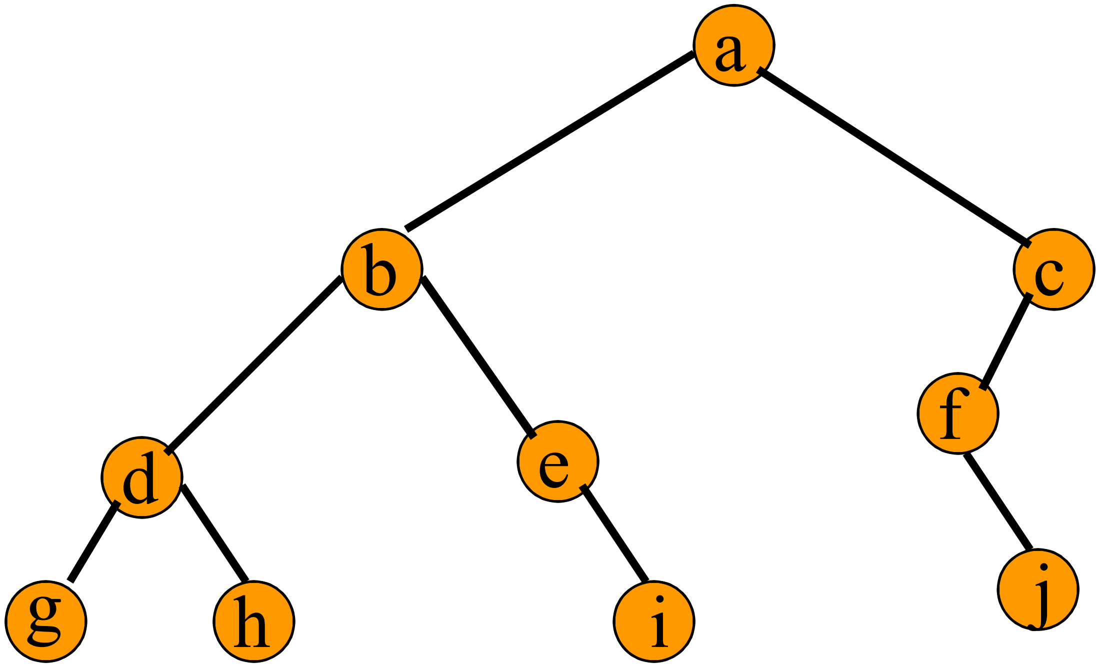
Level Order

- Visit the root first, then the root's left child, followed by the root's right child
- Continue in this manner, visiting the nodes at each new level from the leftmost node to the rightmost node

Level Order (Cont.)

```
template <class T>
void Tree<T>::LevelOrder()
{
  Queue<TreeNode<T>*> q;
  TreeNode<T> * currentNode = root;
  while(currentNode){
    Visit(currentNode);
    if(currentNode->leftChild) q.Push(currentNode->leftChild);
    if(currentNode->rightChild) q.Push(currentNode->rightChild);
    if(q.IsEmpty()) return;
    currentNode = q.Front();
    q.Pop();
  }
}
```

Level-Order Example (Visit = print)



a b c d e f g h i j

```
template <class T>
void Tree<T>::LevelOrder()
{
  Queue<TreeNode<T>*> q;
  TreeNode<T> * currentNode = root;
  while(currentNode){
    Visit(currentNode);
    if(currentNode->leftChild)
      q.Push(currentNode->leftChild);
    if(currentNode->rightChild)
      q.Push(currentNode->rightChild);
    if(q.IsEmpty()) return;
    currentNode = q.Front();
    q.Pop();
  }
}
```

Homework #1

- Lemma 5.3: For any nonempty binary tree, *T*, if n_0 is the number of leaf nodes and n_2 the number of nodes of degree 2, then $n_0 = n_2 + 1$
  - Proof: Homework #1-1

- Homework #1-2
  - Implement and test Programs 5.1 – 5.7 in the textbook

- 유캠퍼스에 보고서로 제출 (9월 10일 자정까지)