# On Efficient Long-Term Extreme Response Estimation for a Moored Floating Structure


**HyeongUk Lim[1], Lance Manuel[1], Ying Min Low[2]**

[1]University of Texas at Austin, Austin, Texas, USA

[2]National University of Singapore



ASME 2018 37th International Conference on Ocean, Offshore and Arctic Engineering
Madrid, Spain • June 17-22, 2018

**Paper No. OMAE 2018-78763**

# Overview

- Background and Motivation
- Problem Definition and Example Structure
- Surrogate model for long-term extreme response using Polynomial Chaos Expansion (PCE)
- Results
- Conclusions

$K$

# Background and Motivation

- *Of interest*

  Long-term surge motion extreme response of a moored floating structure

- Uncertainty in such response extremes comes from:

  (1) variability in sea state conditions ($H_s$ and $T_p$)

  (2) variability in simulated wave train time series (given $H_s$ and $T_p$) due to random phases

- An Uncertainty Quantification Framework based on Polynomial Chaos Expansion (PCE) is developed that can treat these two sources of uncertainty – in different ways

  primary source: $(H_s, T_p)$, directly in PCE; secondary source: phase vector, $\boldsymbol{\Theta}$.

- Validation:

  → PCE "surrogate" extreme compared to Monte Carlo Simulation (MCS) "truth" systems

# Introduction

- Long-term analysis using Monte Carlo Simulation (MCS) is expensive.
  Proposal: Generalized Polynomial Chaos Expansion (PCE) framework for predicting long-term surge motion extreme response of a moored floating structure

- We consider uncertainty in:
  - environmental variables (significant wave height, $H_s$, and spectral peak period, $T_p$)
  - time-varying short-term loads (random phase vector to generate waves in each sea state).

- Validation studies involve PCE vs. MCS long-term response on the simple moored floating structure

# Uncertainty in *Environment*

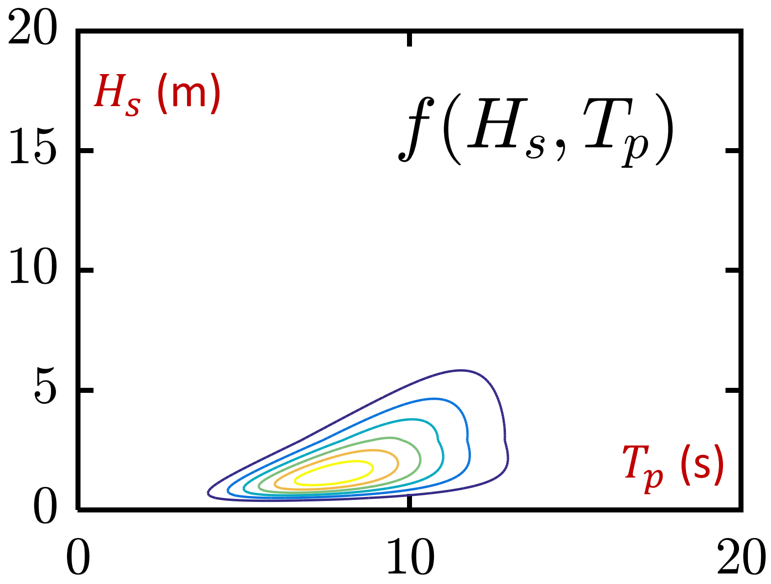
❖ Metocean Characterization for $H_s$ and $T_p$

Distribution $f(H_s, T_p)$ for a North Sea location (Haver, 2002)

$$f\left(H_s, T_p\right) = f(H_s) f(T_p|H_s)$$

$$f(H_s) = \frac{1}{\sqrt{2\pi}\xi H_s} e^{-\frac{(\ln H_s - \mu_h)^2}{2\xi^2}} \mathrm{I}(H_s \le \hat{\eta}) + \frac{\hat{\gamma}}{\rho}\left(\frac{H_s}{\rho}\right)^{\hat{\gamma}-1} e^{-\left(\frac{H_s}{\rho}\right)^{\hat{\gamma}}} \mathrm{I}(H_s > \hat{\eta})$$

$$f\left(T_p|H_s\right) = \frac{1}{\sqrt{2\pi}\sigma T_p} e^{-\frac{(\ln T_p - \mu_t)^2}{2\sigma^2}}$$




$\mu_h = 0.77, \quad \xi = 0.6565, \quad \hat{\eta} = 2.90, \quad \rho = 2.691, \quad \hat{\gamma} = 1.503$

$\mu_t(H_s) = 1.134 + 0.892 H_s^{0.225}$

$\sigma^2(H_s) = 0.005 + 0.120 e^{-0.455 H_s}$

# Uncertainty in *Response*

## Simple Moored Floating Structure

$$M\ddot{u}(t) + 2\zeta\sqrt{KM}\dot{u}(t) + Ku(t) = F_{WF}(t) + F_{LF}(t)$$

first-order (wave freq) second-order (diff freq)

$$F^{(1)}(\omega) = T^{(1)}(\omega)\eta(\omega)$$

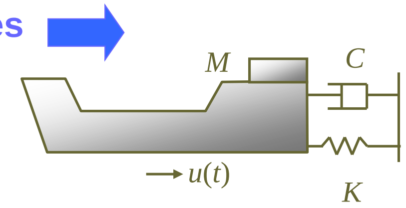
$$F^{(2)}(\omega_r, \omega_s) = T^{(2)}(\omega_r, \omega_s)\eta(\omega_r)\eta(\omega_s)$$

Irregular sea surface elevation process (wave train)

$$\eta(t) = \sum_{r=1}^{R} a_r\, e^{i(\omega_r t + \theta_r)}$$

$a_r = \sqrt{2S_\eta(\omega_r)\Delta\omega}$ ; $R$ large

$\theta_r$ = rth term in **random** phase vector




Frequency response function $(m/N)$: $H(\omega) = (-\omega^2 M + i\omega C + K)^{-1}$

$$u(t) = \sum_{r=1}^{R} H(\omega_r)T^{(1)}(\omega_r)A_r e^{i\omega_r t} + \sum_{r=1}^{R}\sum_{\substack{s=1\\ r\neq s}}^{R} H(\omega_r - \omega_s)T^{(2)}(\omega_r, \omega_s)A_r A_s^* e^{i(\omega_r - \omega_s)t}$$

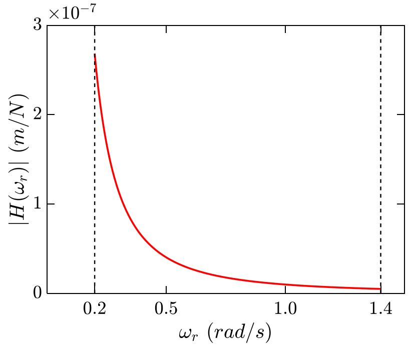
$$A_r = a_r e^{i\theta_r} \equiv a_r(H_s, T_p)e^{i\theta_r}$$

*Reference*: Low (2017)

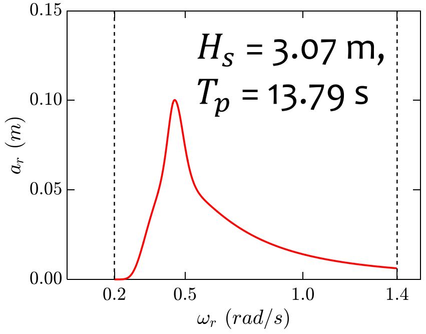
# Uncertainty in *Response*

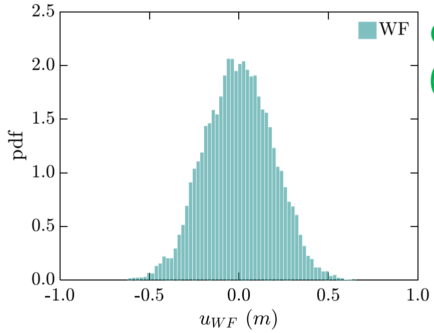
## First-Order (Wave-Frequency)

$$u_{WF}(t) = \sum_{r=1}^{R} H(\omega_r) T^{(1)}(\omega_r) A_r e^{i\omega_r t} = \sum_{r=1}^{R} H(\omega_r) T^{(1)}(\omega_r) a_r e^{i(\omega_r t + \theta_r)}$$




$T^{(1)}(\omega)$

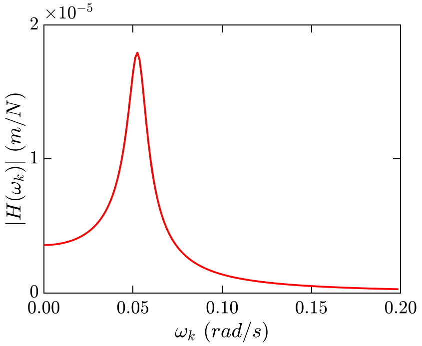


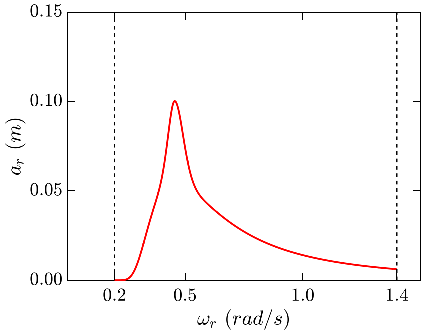

Gaussian
(0.2-1.4 rad/s)

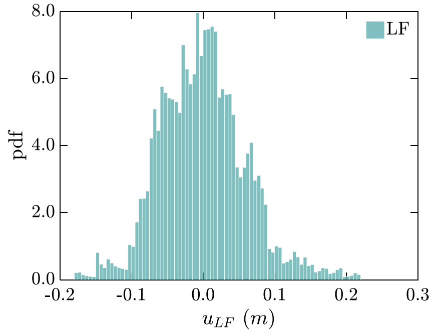
## Second-Order (Diff-Frequency)

$$u_{LF}(t) = \sum_{r=1}^{R} \sum_{\substack{s=1 \\ r \neq s}}^{R} H(\omega_r - \omega_s) T^{(2)}(\omega_r, \omega_s) A_r A_s^* e^{i(\omega_r - \omega_s)t} \cong \sum_{k=1}^{K} H(\omega_k)\, T^N(\omega_k) A_k e^{i\omega_k t}$$




$T^{(2)}(\omega_r, \omega_s)$




Non-Gaussian
(0-2 rad/s)

# Uncertainty in *Response*

Quantity of Interest (QoI) – Short-term Extreme Response, $Z$

QoI: $Z = \max\{u(t); 0 \leq t \leq T\}$, $T$: duration; 30 mins

$G_{Z|H_s,T_p}(z) = P(Z > z|H_s, T_p)$: exceedance probability of and wave-driven $z$ given $H_s$ and $T_p$

$Z$ is defined for a given pair of $H_s$ and $T_p$ (random short-term sea state parameters)
Uncertainty in $Z$ depends on random $H_s$ and $T_p$ and on $\mathbf{\Theta} = [\theta_1, \ldots, \theta_R]^T$

Useful in design: $G_Z(z) = \iint G_{Z|H_s,T_p}(z)\, f(H_s, T_p) dH_s dT_p$ → Long-term distribution of $Z$
$\mathbf{X} \equiv \{H_s, T_p, \mathbf{\Theta}\}$ → $Z(\mathbf{X})$; for $R$ ~ o($10^3$), the no. of random inputs influencing Z is too large.
Need efficient method to assess uncertainty in $Z$.

We will explore using Uncertainty Quantification
→ developing a surrogate for $Z$ using PCE (Polynomial Chaos Expansion). How?

# Surrogate Model (PCE) for Extreme Response

## Monte Carlo Simulations and PCE-based Surrogate Models

$$Z \equiv \underbrace{\boxed{Z(H_s, T_p, \boldsymbol{\Theta})}}_{\text{MCS}} \cong \underbrace{\boxed{\hat{Z}(H_s, T_p)}}_{\text{PCE}} = \sum c_i He_i(\xi(H_s, T_p))$$

$H_s$ and $T_p$ are treated explicitly as RVs

$\boldsymbol{\Theta}$ is implicitly considered in PCE coefficients, $c_i$

$G_Z(z) = \iint G_{Z|H_s,T_p}(z)\, f(H_s, T_p)\, dH_s dT_p$: unconditional exceedance probability

MCS estimate: $G_Z^{MC}(z) = \frac{1}{N_T} \sum_{k=1}^{N_T} I(Z > z | H_s^{(k)}, T_p^{(k)}, \boldsymbol{\Theta}^{(k)})$

PCE estimate: $G_{\hat{Z}}^{PCE}(z) = \frac{1}{N_T} \sum_{k=1}^{N_T} I(\hat{Z} > z | H_s^{(k)}, T_p^{(k)})$

For estimate of $G_{\hat{Z}}^{PCE}(z)$

→ need appropriate choices of polynomial order ($p$) and number of “truth system” simulations ($N_E$)

# Surrogate Model (PCE) for Extreme Response

## Polynomial Chaos Expansion (PCE)

❖ For UQ, uncertain QoI, Z, is expressed, using PCE:

$$\mathbf{X} = \{H_s, T_p\} \quad Z \equiv Z(\mathbf{X}) = \sum_{i=0}^{\infty} c_i \psi_i(\xi(\mathbf{X}))$$

$\xi$: Rosenblatt transformation

$$\Phi(q_1) = F_{H_s}(h)$$

$$\Phi(q_2) = F_{T_p|H_s}(t|h)$$

❖ A truncated PCE for $Z$ that involves polynomials up to order $p$ is:

$$Z(\boldsymbol{X}) \approx Z^{PCE}(\boldsymbol{X}) = \sum_{i=0}^{N-1} c_i \psi_i\big(\xi(\boldsymbol{X})\big), \quad N = \sum_{k=0}^{p} \binom{N_X + k - 1}{k} = \binom{N_X + p}{p}$$

$$\overline{He}_i(q_j) = (-1)^i \frac{1}{\varphi(q_j)} \frac{d^i \varphi(q_j)}{dq^i} \qquad \varphi(q_j) = \frac{1}{\sqrt{2\pi}} e^{-\frac{q_j^2}{2}}$$

❖ Multivariate Hermite polynomials are basis functions (Askey scheme)

❖ The $i^{\text{th}}$ multi-dimensional Hermite polynomial is: $He_i(\boldsymbol{Q}) = \prod_{j=1}^{N_X} \overline{He}_{\alpha_i}(Q_j)$

# Surrogate Model (PCE) for Extreme Response

## PCE Example: Building a Surrogate Model for QoI

2 variables, order-2 Hermite polynomials $\mathbf{X} = \{X_1, X_2\},\ p = 2 \Rightarrow \mathbf{Q} = \{Q_1, Q_2\}$

$$N = \sum_{k=0}^{p} \binom{N_X + k - 1}{k} = \sum_{k=0}^{2} \binom{2 + k - 1}{k} = 6$$

$$Z(\boldsymbol{X}) \approx Z^{PCE}(\boldsymbol{X}) = \sum_{i=0}^{N-1} c_i \psi_i(\xi(\boldsymbol{X})) = \sum_{i=0}^{5} c_i He_i(Q_1, Q_2) = c_0 He_0 + c_1 He_1 + c_2 He_2 + c_3 He_3 + c_4 He_4 + c_5 He_5$$

### Univariate polynomials

$$\overline{He}_i(Q_j) = (-1)^i \frac{1}{\varphi(Q_j)} \frac{d^i \varphi(Q_j)}{dq^i}$$

$$\varphi(Q_j) = \frac{1}{\sqrt{2\pi}} e^{-\frac{Q_j^2}{2}}$$

$$\overline{He}_0(Q_j) = 1$$

$$\overline{He}_1(Q_j) = Q_j$$

$$\overline{He}_2(Q_j) = Q_j^2 - 1$$

| $i$ | multi-index | |
|---|---|---|
| | $Q_1$ | $Q_2$ |
| 0 | 0 | 0 |
| 1 | 1 | 0 |
| 2 | 0 | 1 |
| 3 | 2 | 0 |
| 4 | 1 | 1 |
| 5 | 0 | 2 |

### Bivariate polynomials

$$He_0(Q_1, Q_2) = \overline{He}_0(Q_1) \cdot \overline{He}_0(Q_2) = 1$$

$$He_1(Q_1, Q_2) = \overline{He}_1(Q_1) \cdot \overline{He}_0(Q_2) = Q_1$$

$$He_2(Q_1, Q_2) = \overline{He}_0(Q_1) \cdot \overline{He}_1(Q_2) = Q_2$$

$$He_3(Q_1, Q_2) = \overline{He}_2(Q_1) \cdot \overline{He}_0(Q_2) = Q_1^2 - 1$$

$$He_4(Q_1, Q_2) = \overline{He}_1(Q_1) \cdot \overline{He}_1(Q_2) = Q_1 \cdot Q_2$$

$$He_5(Q_1, Q_2) = \overline{He}_0(Q_1) \cdot \overline{He}_2(Q_2) = Q_2^2 - 1$$

# Surrogate Model (PCE) for Extreme Response

## Polynomial Chaos Expansion (PCE) using Spectral Projection

- Spectral projection (quadrature-based integration) can be used to compute the coefficients, $c_i$:

$$c_i = \frac{E[Z(\boldsymbol{X})He_i(\boldsymbol{Q})]}{E[He_i^2(\boldsymbol{Q})]}$$

- The denominator may be derived analytically; the numerator is approximated as

$$\sum_{x_{1k}=1}^{n_q} \sum_{x_{2k}=1}^{n_q} \cdots \sum_{x_{(N_X)k}=1}^{n_q} Z\Big(\underbrace{x_{1k}, x_{2k}, \cdots, x_{(N_X)k}}_{\text{quadrature points}}\Big) He_i\big(Q(x_{1k}, x_{2k}, \cdots, x_{(N_X)k})\big)\underbrace{w_{1k} w_{2k} \cdots w_{(N_X)k}}_{\text{weights}}$$

- By averaging values of $Z$ at quadrature points, spectral projection can account for the uncertainty due to random phases in sea surface elevations (Nguyen *et al*, 2018 – for WEC extremes)

# Surrogate Model (PCE) for Extreme Response

## Polynomial Chaos Expansion (PCE) using Linear Regression

$N_E$ response valuations conducted account for random $H_s, T_p$ values and variability from $\boldsymbol{\Theta}$

$$\boldsymbol{c} = \arg\min_{c_i} \sum_{m=1}^{N_E} \left[ Z(\boldsymbol{X}^{(m)}) - \sum_{i=0}^{N-1} c_i He_i(\xi(\boldsymbol{X}^{(m)})) \right]^2$$

$$\boldsymbol{Hc} = \boldsymbol{Z} \quad \rightarrow \quad \begin{bmatrix} He_0(\xi(\boldsymbol{X}^{(1)})) & \cdots & He_{N-1}(\xi(\boldsymbol{X}^{(1)})) \\ \vdots & \ddots & \vdots \\ He_0(\xi(\boldsymbol{X}^{(N_E)})) & \cdots & He_{N-1}(\xi(\boldsymbol{X}^{(N_E)})) \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_{N-1} \end{bmatrix} = \begin{bmatrix} Z(\boldsymbol{X}^{(1)}) \\ \vdots \\ Z(\boldsymbol{X}^{(N_E)}) \end{bmatrix}$$

$\boldsymbol{H}: N_E \times N$ $\quad \boldsymbol{c}: N \times 1$ $\quad \boldsymbol{Z}: N_E \times 1$

Least squares solution: $\boldsymbol{c} = (\boldsymbol{H}^T \boldsymbol{H})^{-1} \boldsymbol{H}^T \boldsymbol{Z}$

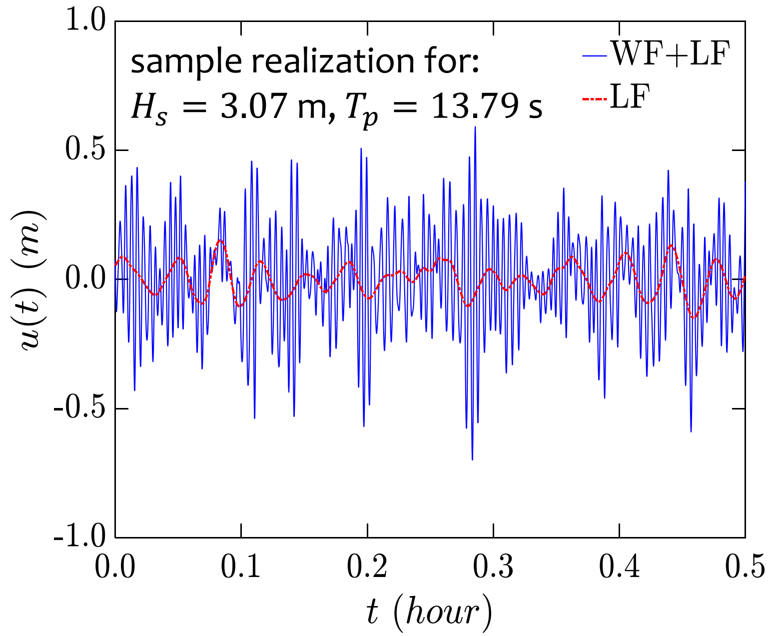
$$\boxed{Z^{PCE}(H_s, T_p) = \sum c_i He_i(\xi(H_s, T_p))}$$

# Training Data for PCE Surrogate Development

## Response Stochastic Simulations

$k$th sample of $(H_s, T_p, \mathbf{\Theta}) \rightarrow$ $\boxed{A_r = a_r e^{i\theta_r}, \qquad a_r = \sqrt{2S_\eta(\omega_r)\Delta\omega}}$

$$u^{(k)}(t) = \sum_{r=1}^{R} H(\omega_r)T^{(1)}(\omega_r)A_r e^{i\omega_r t} + \sum_{r=1}^{R}\sum_{\substack{s=1\\ r\neq s}}^{R} H(\omega_r-\omega_s)T^{(2)}(\omega_r,\omega_s)A_r A_s^* e^{i(\omega_r-\omega_s)t}$$




$$Z^{(k)} = \max\{u^{(k)}(t); 0 \le t \le T\}$$

involving randomness of $H_s$ and $T_p$, $\mathbf{\Theta} = [\theta_1, \ldots, \theta_R]^T$

- Using Inverse Fast Fourier Transform, realizations of 30-min time series are done quite efficiently

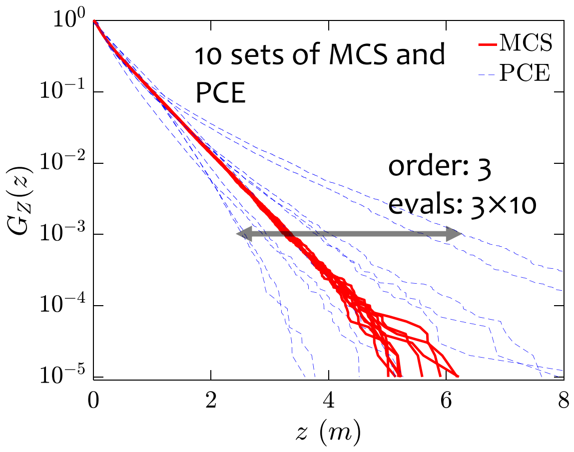
- Relative importance of WF and LF components to the response is evident

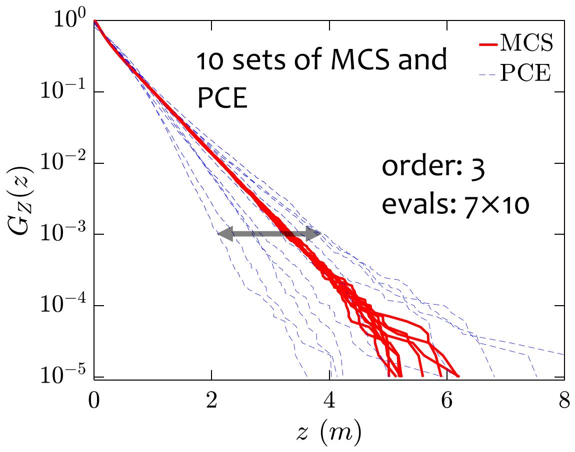
# Results: Long-Term Extreme Response

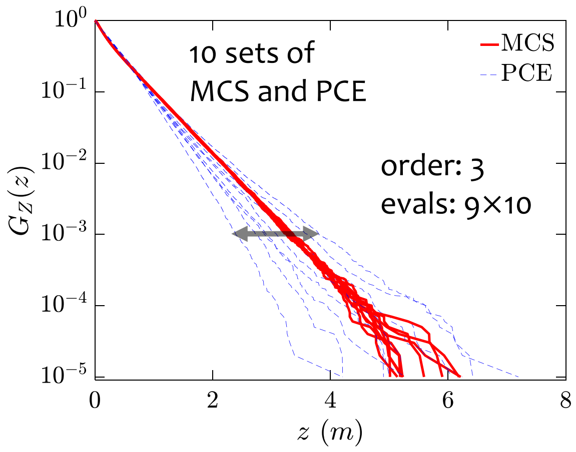
## Parameters for PCE-based Surrogate Models

Estimation of $\hat{Z}(H_s, T_p)$ depends on $p$ (order of polynomial) and $N_E$ (no. of evaluations)

$$\#(\boldsymbol{c}) = \binom{p + \#(H_s, T_p)}{p} = \binom{3+2}{3} = 10, \qquad \boldsymbol{c} = (\boldsymbol{H}^T\boldsymbol{H})^{-1}\boldsymbol{H}^T\boldsymbol{Z}$$



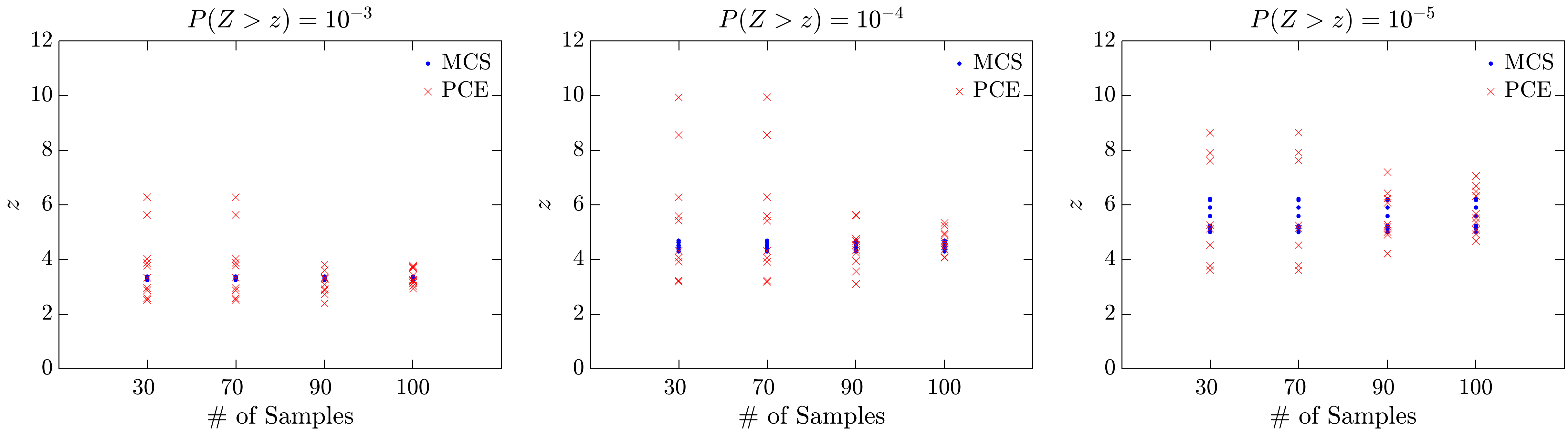




For $G_Z(z) = 10^{-3}$ levels: given $p = 3$, uncertainty in extremes reduced as $N_E$ increases

# Results: Long-Term Extreme Response

## Parameters for PCE-based Surrogate Models

10 sets of MCS and PCE; order-3 polynomials




100 samples for estimation of ***c*** yield comparable uncertainty for PCE vs. MCS

# Results: Long-Term Extreme Response

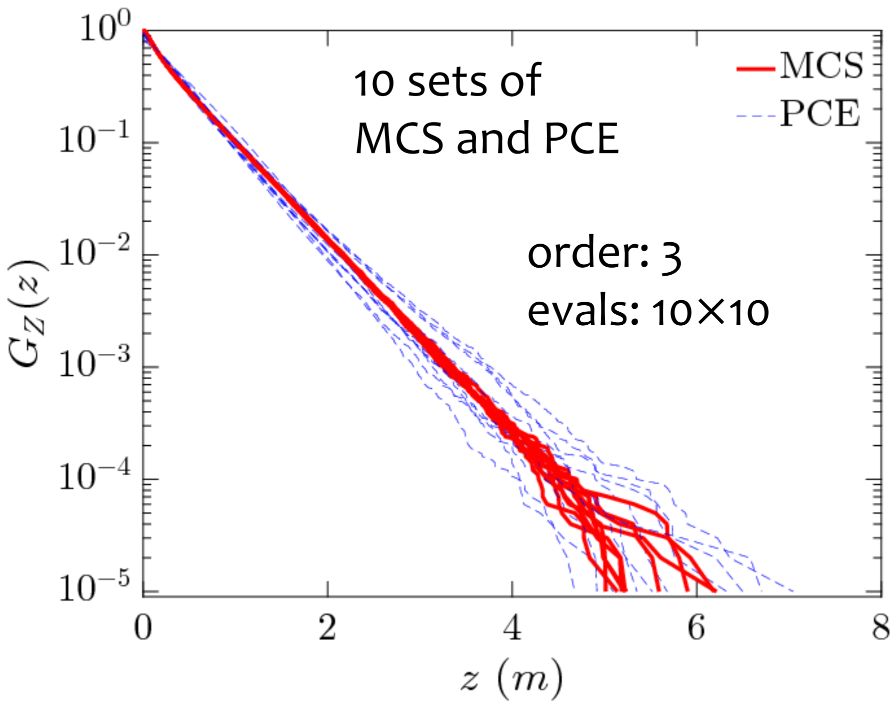
## Exceedance Probabilities of Extreme Surge Motion

A rule of thumb: $p \geq 3$ and $N_E \leq 3N$ to achieve satisfactory accuracy for $G_Z(z) \leq 10^{-4}$ (Sudret, 2007; Blatman, 2010)

But $\hat{Z}(H_s, T_p)$ does not explicitly account for uncertainty introduced by $\boldsymbol{\Theta}$
→ larger $N_E$ is needed; $N_E = 10N$

A proper choice of the polynomial order ($p$): 3
The number of simulations ($N_E$) of the truth system:

$$10 \times N = 100;\ N = \binom{2+p}{2} = \binom{5}{2} = 10$$



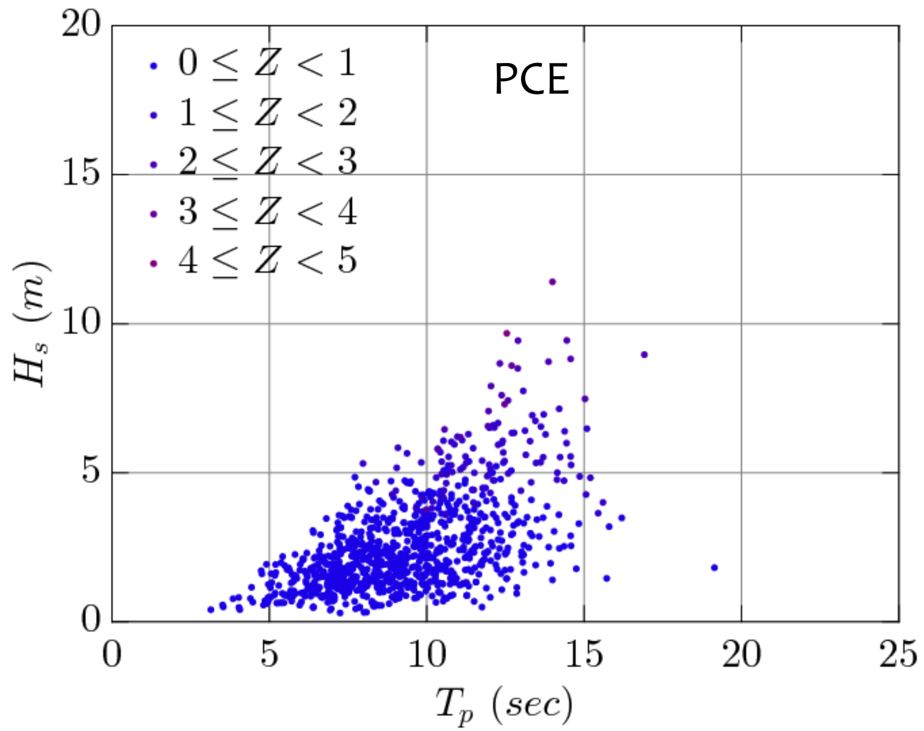

Slightly greater variability than MCS but no bias in the PCE predictions

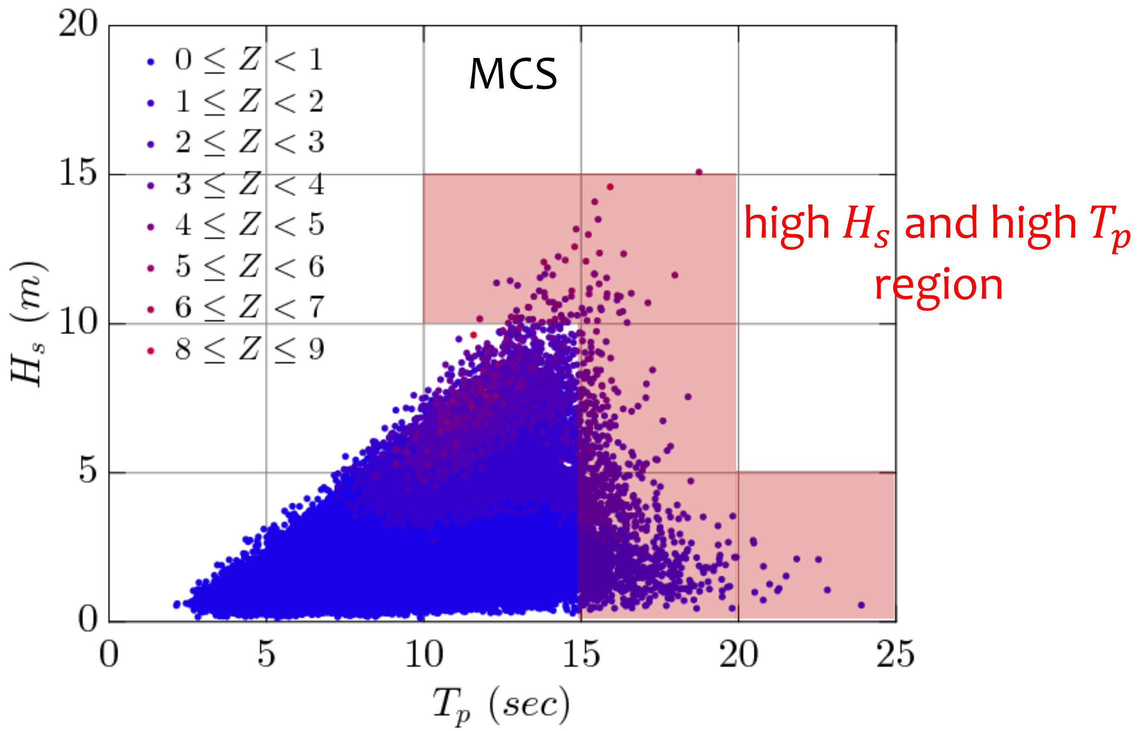
# Results: Long-Term Extreme Response

## Drawn Samples for PCE and MCS

Recall that $G_Z(z) = \iint G_{Z|H_s,T_p}(z)\, f(H_s, T_p)\, dH_s dT_p$

Inaccuracy in PCE $G_Z$ prediction may result from deficient approximations of $(Z|H_s, T_p)$




# of samples: 10 × 100 (PCE) vs 100,000 (MCS)

# Results: Long-Term Extreme Response

## PCE Short-term Extreme Response Prediction Accuracy

Specific sea states $H_s$ (m)-$T_p$ (s): 0.73-7.14; 2.16-8.98; 5.58-11.56

$H_s$: 0.1, 0.5, 0.9 non-exceedance marginal quantiles; e.g., $P(H_s < 0.73) = 0.1$
$T_p$: conditional median values for associated $H_s$ values; e.g., $F_{T_p|H_s=0.73}(7.14) = 0.5$

by 100 realizations of random phases

$$\mathrm{E}[Z(H_s, T_p, \boldsymbol{\Theta})]$$

$$Z = \max\{u(t); 0 \le t \le 30\ mins\}$$

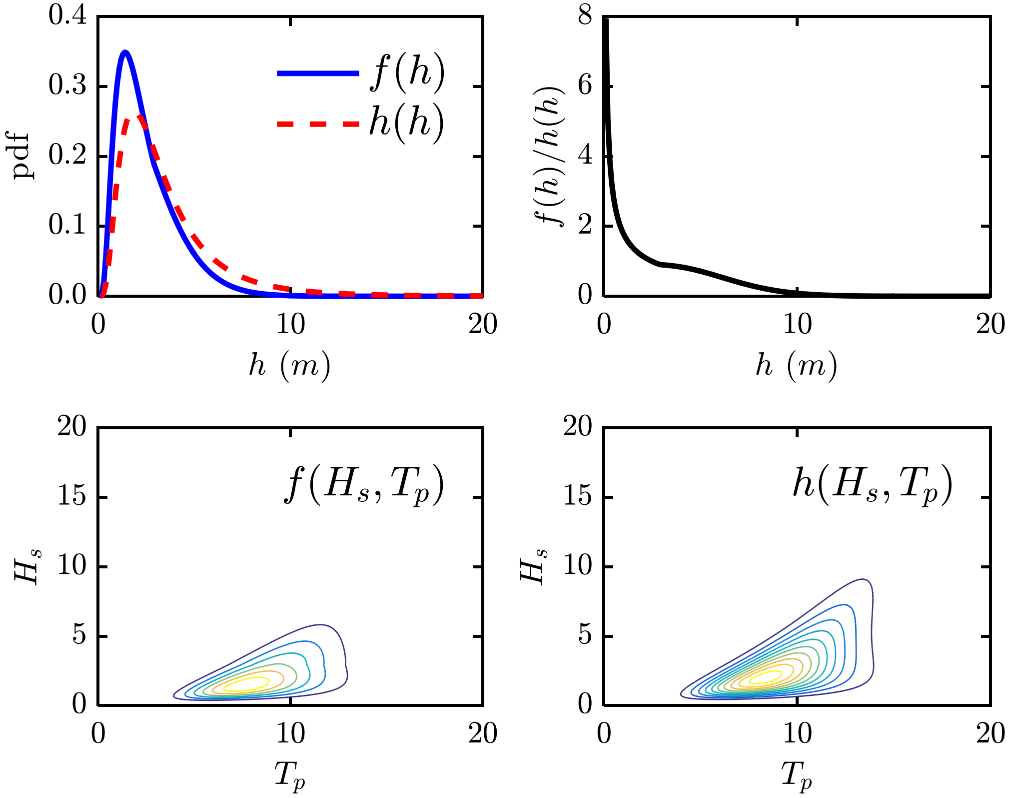
$$\hat{Z}(H_s, T_p)$$

| $H_s$ (m) | $T_p$ (s) | Truth System | PCE1 | PCE2 | PCE3 | PCE4 | PCE5 | PCE6 | PCE7 | PCE8 | PCE9 | PCE10 | Mean |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.73 | 7.14 | **0.034** | 0.041 | 0.036 | 0.011 | -0.012 | 0.076 | 0.079 | 0.047 | -0.072 | 0.026 | 0.025 | **0.026** |
| 2.16 | 8.98 | **0.250** | 0.232 | 0.266 | 0.285 | 0.271 | 0.238 | 0.234 | 0.244 | 0.294 | 0.282 | 0.262 | **0.261** |
| 5.58 | 11.56 | **1.340** | 1.220 | 1.216 | 1.317 | 1.419 | 1.215 | 1.246 | 1.347 | 1.495 | 1.264 | 1.374 | **1.311** |

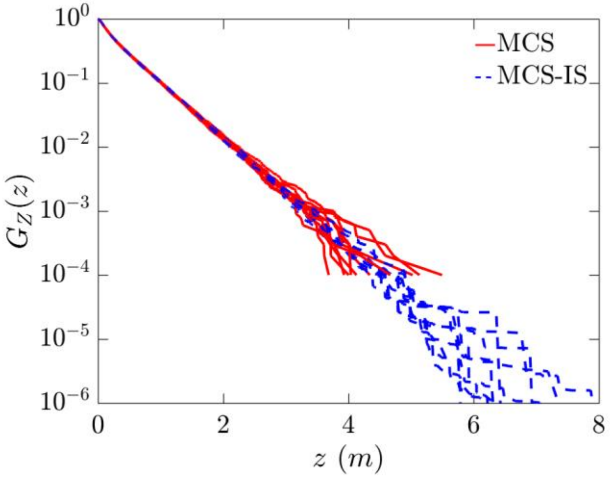
## Monte Carlo Simulations with Importance Sampling

Importance sampling can be used to reduce sampling variability resulted from long-term uncertainty

MCS-IS estimate: $G_Z^{MC-IS}(z) = \frac{1}{N_T}\sum_{k=1}^{N_T} I(Z > z|H_s^{(k)}, T_p^{(k)}) \times \frac{f(h^{(k)},t^{(k)})}{h(h^{(k)},t^{(k)})} = \frac{1}{N_T}\sum_{k=1}^{N_T} I(Z > z|H_s^{(k)}, T_p^{(k)}) \times \frac{f(h^{(k)})}{h(h^{(k)})}$




- PDF $h$ is chosen to result in more higher $H_s$ samples
- → $\mu_{H_s}^f = 2.54$ m; $\mu_{H_s}^h = \mu_{H_s}^f + 1 = 3.54$ m




\# of MCS samples: 10,000
\# of MCS-IS samples: 10,000

Because of the weights ($\frac{f}{h} < 1$ at high $H_s$), $G_Z$ under $10^{-4}$ can be estimated, even with $10^4$ samples

# Results: Long-Term Extreme Response

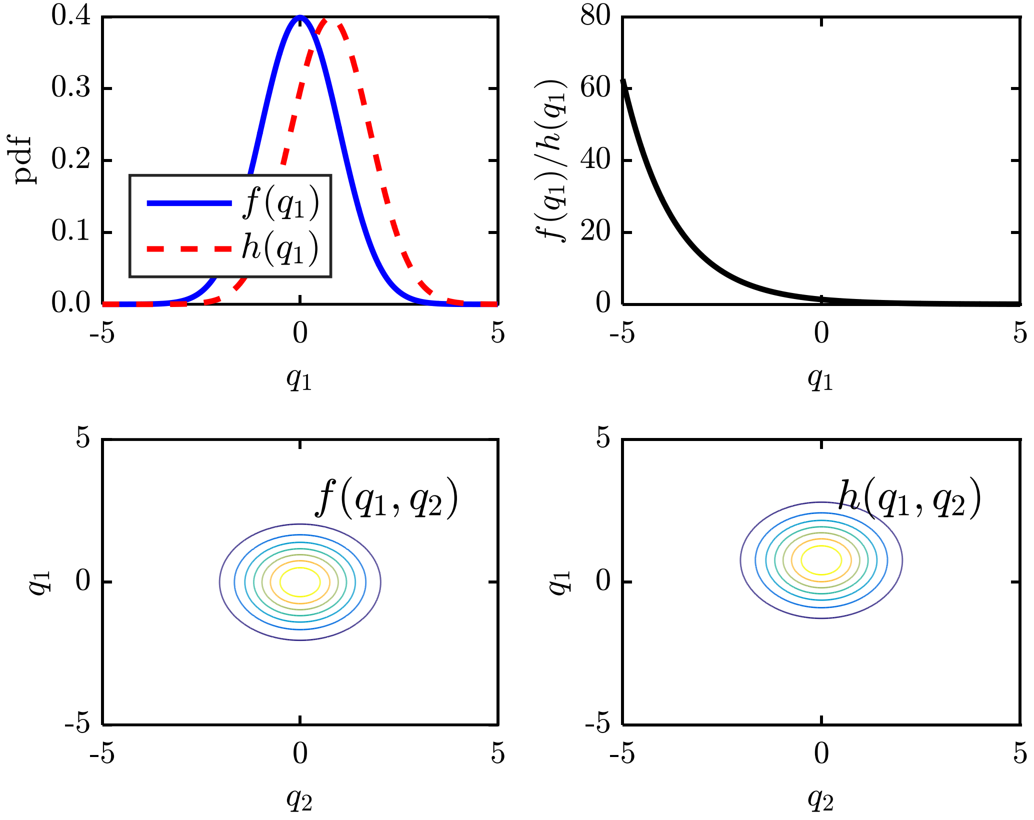
## PCE with Importance Sampling

Importance sampling can also be used in PCE

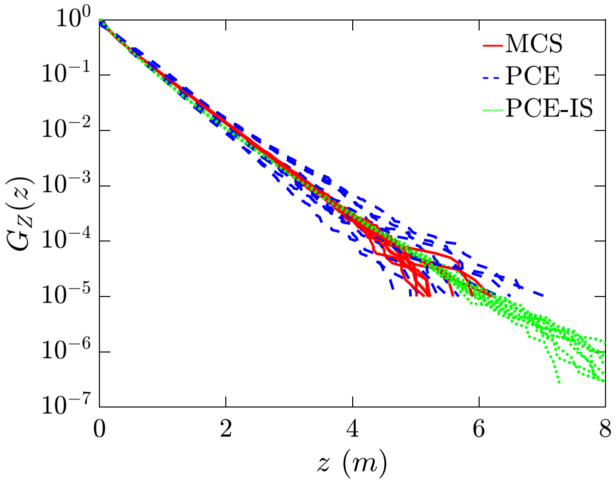
PCE-IS estimate: $G_{\hat{Z}}^{PCE}(z) = \frac{1}{N_T}\sum_{k=1}^{N_T} I(\hat{Z} > z|H_s^{(k)}, T_p^{(k)}) \times \frac{f(q_1{}^{(k)}, q_2{}^{(k)})}{h(q_1{}^{(k)}, q_2{}^{(k)})} = \frac{1}{N_T}\sum_{k=1}^{N_T} I(\hat{Z} > z|H_s^{(k)}, T_p^{(k)}) \times \frac{f(q_1{}^{(k)})}{h(q_1{}^{(k)})}$

$\Phi(q_1) = F_{H_s}(h)$
$\Phi(q_2) = F_{T_p|H_s}(t|h)$

Note that $q_1$ and $q_2$ are independent standard normal variables ($f$)




- PDF $h$ is chosen to obtain more samples in higher $Q_1$ ranges
- Ex) $\mu_{Q_1}^f = 0; \mu_{Q_1}^h = \mu_{Q_1}^f + 0.77;\ F_{H_s}^{-1}(\Phi(0.77)) = 3.54$ (m)




# of MCS samples: 100,000
# of PCE samples: 100,000
# of PCE-IS samples: 100,000

There is no complexity in selecting importance sampling density ($h$)

Because of the weights ($\frac{f}{h} < 1$ at high $Q_1(H_s)$), $G_Z$ under $10^{-5}$ can be estimated, even with $10^5$ samples

# Conclusions

- Prediction of long-term extreme response of a simple moored floating offshore structure was carried out using a Polynomial Chaos Expansion (PCE) surrogate

- The moored floating structure subjected to first-order and second-order wave loading was considered in numerical studies

- Uncertainty in the response extremes was treated by explicit inclusion of $H_s$ and $T_p$ in PCE; uncertainty resulting from random phases in the wave train was accounted for by using multiple suites of data in linear regression, used for PCE coefficient estimation

- PCE surrogates yielded accurate prediction when compared against the truth system MCS and with significant saving; importance sampling leads to unbiased prediction with even better control (reduced confidence intervals on long-term predictions)

# Acknowledgment

- Discussions with Mr. Phong Nguyen (PhD Candidate)

  Lead author of closely related work,

  *On the Development of an Efficient Surrogate Model for Predicting Long-term Extreme Loads on a Wave Energy Converter*, OMAE2018-78766, Thursday, 15:30 – Session 9-4-1


*gracias!*