Analysis

Edwin Park

2023

Contents

1 Taylor Polynomials

$$\begin{aligned}
f(\vec{x}) &= f(\vec{x_0} + (\vec{x} - \vec{x_0})) \\
&\approx f(\vec{x_0}) + D_f(\vec{x_0})(\vec{x} - \vec{x_0}) + \frac{1}{2}(\vec{x} - \vec{x_0})^\top H_f(\vec{x_0})(\vec{x} - \vec{x_0}) \\
&= f(\vec{x_0}) + \begin{bmatrix} f_x(\vec{x_0}) & f_y(\vec{x_0}) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x - x_0 & y - y_0 \end{bmatrix} \begin{bmatrix} f_{xx}(\vec{x_0}) & f_{xy}(\vec{x_0}) \\ f_{yx}(\vec{x_0}) & f_{yy}(\vec{x_0}) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}
\end{aligned}$$

Say we wanted to find an upper bound for the error of a first-order approximation at $\vec{x}$. The error then is equal to the quadratic term evaluated at some point in between the central point, $\vec{x_0}$, and the point of interest, $\vec{x}$. In particular, it is, for some $t \in [0, 1]$,

$$\frac{1}{2} \left[D^2_{\vec{x} - \vec{x_0}} f \right] (t\vec{x} + (1-t)\vec{x_0}) = \frac{1}{2} \begin{bmatrix} x - x_0 & y - y_0 \end{bmatrix} \begin{bmatrix} f_{xx}(t\vec{x} + (1-t)\vec{x_0}) & f_{xy}(t\vec{x} + (1-t)\vec{x_0}) \\ f_{yx}(t\vec{x} + (1-t)\vec{x_0}) & f_{yy}(t\vec{x} + (1-t)\vec{x_0}) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}.$$