Project

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Contents

1 Discrete Calculus

1.1 Differences and sums

Consider the sequence of square numbers. An elementary analysis technique is to look at successive differences between terms:

$$\begin{array}{ccccccccccc} 1 & & 4 & & 9 & & 16 & & 25 & & \cdots \\ & 3 & & 5 & & 7 & & 9 & & \cdots & \end{array}$$

Which reveals a neat property of square numbers. Generalising this method, for any sequence of terms with the x^{th} term given by $f(x)$, we can define the "difference" sequence, also called the discrete derivative of the sequence, to be

$$\Delta f(x) := f(x+1) - f(x)$$

In the case of the square numbers, we can check to see that our definition matches the example above:

$$\begin{aligned} \Delta x^2 &= (x+1)^2 - x^2 \\ &= x^2 + 2x + 1 - x^2 \\ &= 2x + 1 \end{aligned}$$

Using this definition, we can calculate discrete derivatives of elementary functions.

$$\begin{aligned} \Delta x^n &= (x+1)^n - x^n \\ &= \sum_{q=0}^{n} \binom{n}{q} x^q - x^n \\ &= \sum_{q=0}^{n-1} \binom{n}{q} x^q \end{aligned}$$

Something particularly neat happens with falling powers:

$$\begin{aligned} \Delta x^{\underline{n}} &= (x+1)^{\underline{n}} - x^{\underline{n}} \\ &= (x+1)x^{\underline{n-1}} - x^{\underline{n-1}}(x-n+1) \\ &= nx^{\underline{n-1}} \end{aligned}$$

With exponentials:

$$\begin{aligned} \Delta n^x &= n^{x+1} - n^x \\ &= (n-1)n^x \end{aligned}$$

Also, we can check to see that the difference operator is linear:

$$\begin{aligned} \Delta \big(f(x) + g(x)\big) &= \big(f(x+1) + g(x+1)\big) - \big(f(x) + g(x)\big) \\ &= \big(f(x+1) - f(x)\big) + \big(g(x+1) - g(x)\big) \\ &= \Delta f(x) + \Delta g(x) \end{aligned}$$

$$\begin{aligned} \Delta c f(x) &= cf(x+1) - cf(x) \\ &= c\big(f(x+1) - f(x)\big) \\ &= c\Delta f(x) \end{aligned}$$

Similarly, the sum operator is also linear.

1.2 Connection between sums and discrete derivatives

A remarkable result is:

$$\begin{aligned}
\sum_{x=a}^{b} \Delta f(x) &= \Delta f(a) + \Delta f(a+1) + \cdots + \Delta f(b-1) + \Delta f(b) \\
&= f(a+1) - f(a) + f(a+2) - f(a+1) + \cdots + f(b) - f(b-1) + f(b+1) - f(b) \\
&= f(b+1) - f(a)
\end{aligned}$$

And so, knowing a function's anti-discrete-derivative, or its antidifference, gives simple closed expressions for sums of that function. For a function $F(x)$ with $\Delta F(x) = f(x)$, we write

$$\sum_{x} f(x) = F(x) + c$$

for $c \in \mathbb{R}$, and

$$F(x) = \sum_{q=0}^{x-1} f(q) + F(0).$$

1.3 Quotients and products

Consider another sequence, this time factorials. An alternative analysis is to take successive quotients:

$$\begin{array}{ccccccccccc}
1 & & 2 & & 6 & & 24 & & 120 & & \cdots \\
& 2 & & 3 & & 4 & & 5 & & \cdots &
\end{array}$$

We can then define the "multiplicative difference", or the quotient of a function $f(x)$ as:

$$[Q]f(x) := \frac{f(x+1)}{f(x)}$$

Again, a connection to the product is:

$$\begin{aligned}
\prod_{x=a}^{b} [Q]f(x) &= [Q]f(a) * [Q]f(a+1) * ... * [Q]f(b-1) * [Q]f(b) \\
&= \frac{f(a+1)}{f(a)} * \frac{f(a+2)}{f(a+1)} * ... * \frac{f(b)}{f(b-1)} * \frac{f(b+1)}{f(b)} \\
&= \frac{f(b+1)}{f(a)}
\end{aligned}$$

And so, knowing a function's anti-multiplicative difference (alternatively called indefinite product, or antiquotient), radically simplifies products of it. For a function $F(x)$ with $[Q]F(x) = f(x)$, we write

$$\prod_{x} f(x) = cF(x)$$

for $c \in \mathbb{R}$, and

$$F(x) = F(0) \prod_{q=0}^{x-1} f(q).$$

Quotients of common functions are outlined below: Note however, that the quotient operator is not linear:

$$\begin{aligned}[Q]\big(f(x)+g(x)\big) &= \frac{f(x+1)+g(x+1)}{f(x)+g(x)} \\ &\neq \frac{f(x+1)}{f(x)} + \frac{g(x+1)}{g(x)}\end{aligned}$$

It is however, "linear" in multiplicative terms:

$$\begin{aligned}[Q]\big(f(x)*g(x)\big) &= \frac{f(x+1)*g(x+1)}{f(x)*g(x)} \\ &= \frac{f(x+1)}{f(x)}\frac{g(x+1)}{g(x)} \\ &= [Q]f(x)*[Q]g(x)\end{aligned}$$

$$\begin{aligned}[Q]\big(f(x)^c\big) &= \frac{f(x+1)^c}{f(x)^c} \\ &= \left(\frac{f(x+1)}{f(x)}\right)^c \\ &= \big([Q]f(x)\big)^c\end{aligned}$$

Similar results follow for the product operator.

1.3.1 Connections to differences and sums (useless section?)

$$\begin{aligned}[Q]c^{f(x)} &= \frac{c^{f(x+1)}}{c^{f(x)}} \\ &= c^{f(x+1)-f(x)} \\ &= c^{\Delta f(x)}\end{aligned}$$

In fact, writing a function in exponential form:

$$\begin{aligned}[Q]f(x) = [Q]\left(e^{\ln\left(f(x)\right)}\right) \\ &= \frac{e^{\ln\left(f(x+1)\right)}}{e^{\ln\left(f(x)\right)}} \\ &= e^{\ln\left(f(x+1)\right)-\ln\left(f(x)\right)} \\ &= e^{\Delta\left(\ln f(x)\right)}\end{aligned}$$

Also, note that:

$$\ln\left(\prod_{x=a}^{b} f(x)\right) = \ln\big(f(a)*f(a+1)*...*f(b)\big)$$

$$
\begin{aligned}
&= \ln\big(f(a)\big) + \ln\big(f(a+1)\big) + ... + \ln\big(f(b)\big) \\
&= \sum_{x=a}^{b} \ln\big(f(x)\big) \\
\Rightarrow \prod_{x=a}^{b} f(x) &= \exp\left(\sum_{x=a}^{b} \ln(f(x))\right)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\exp\left(\sum_{x=a}^{b} f(x)\right) &= e^{f(a)+f(a+1)+...+f(b)} \\
&= e^{f(a)} * e^{f(a+1)} * ... * e^{f(b)} \\
&= \prod_{x=a}^{b} e^{f(x)} \\
\Rightarrow \sum_{x=a}^{b} f(x) &= \ln\left(\prod_{x=a}^{b} e^{f(x)}\right)
\end{aligned}
$$

1.3.2 Quotients and products of common functions

$$
\begin{aligned}
[Q](x+a) &= \frac{x+a+1}{x+a} \\
&= 1 + \frac{1}{x+a}
\end{aligned}
$$

$$
\begin{aligned}
[Q]\big((x+a)!\big) &= \frac{(x+a+1)!}{(x+a)!} \\
&= x+a+1
\end{aligned}
$$

$$
\begin{aligned}
[Q](n^x) &= \frac{n^{x+1}}{n^x} \\
&= n
\end{aligned}
$$

1.4 Difference equations

1.4.1 Linear, first-order, homogeneous

Consider the difference equation

$$\Delta f(x) = \lambda f(x). \qquad (\lambda \in \mathbb{R})$$

Then, by the definition of the difference operator we can write:

$$f(x+1) = (\lambda+1)f(x) \Rightarrow \frac{f(x+1)}{f(x)} = (\lambda+1)$$

$$\Rightarrow [Q]f(x) = (\lambda + 1)$$
$$\Rightarrow f(x) = f(0)\prod_{q=0}^{x-1}(\lambda + 1)$$
$$= f(0) * (\lambda + 1)^x$$

A linear algebra approach is to find the eigenvectors of the linear transformation Δ. Suppose that the basis for our vector space is given by:

$$\{x^{\underline{0}}, x^{\underline{1}}, x^{\underline{2}}, ...\}$$

And so we can write

$$f(x) = \sum_{q=0}^{\infty} a_q x^{\underline{q}}$$

for $a_q \in \mathbb{R}$. From the difference equation, then:

$$\Delta f(x) = \lambda f(x) \Rightarrow \sum_{q=0}^{\infty} q a_q x^{\underline{q-1}} = \lambda \sum_{q=0}^{\infty} a_q x^{\underline{q}}$$

But note that the term for the $q = 0$ case of the first sum is just 0 (the difference of a constant is 0). So:

$$\sum_{q=1}^{\infty} q a_q x^{\underline{q-1}} = \lambda \sum_{q=0}^{\infty} a_q x^{\underline{q}}$$
$$\Rightarrow \sum_{q=0}^{\infty}(q+1) a_{q+1} x^{\underline{q}} = \lambda \sum_{q=0}^{\infty} a_q x^{\underline{q}}$$
$$\Rightarrow (q+1)a_{q+1} = \lambda a_q$$
$$\Rightarrow \frac{a_{q+1}}{a_q} = [Q]a_q = \frac{\lambda}{q+1}$$
$$\Rightarrow a_q = a_0 \prod_{t=0}^{q-1} \frac{\lambda}{t+1}$$
$$= a_0 \lambda^q \prod_{t=0}^{q-1}(t+1)^{-1}$$
$$= a_0 \lambda^q \left(\prod_{t=0}^{q-1}(t+1)\right)^{-1}$$
$$= a_0 \lambda^q \left(\frac{q!}{0!}\right)^{-1}$$
$$= \frac{a_0 \lambda^q}{q!}$$

Noting that

$$f(0) = \sum_{q=0}^{\infty} a_q (0)^{\underline{q}} \Rightarrow a_0 = f(0),$$

the series expression for $f(x)$ is:

$$f(x) = f(0)\sum_{q=0}^{\infty}\frac{\lambda^q}{q!}x^{\underline{q}}$$

Combining this with the previous result (meta; what to make of this?):

$$\sum_{q=0}^{\infty}\frac{\lambda^q}{q!}x^{\underline{q}} = (\lambda+1)^x$$

Note however from the binomial theorem we have:

$$(\lambda+1)^x = \sum_{q=0}^{x}\frac{x^{\underline{q}}}{q!}\lambda^q$$

And so:

$$\sum_{q=x+1}^{\infty}\frac{x^{\underline{q}}}{q!}\lambda^q = 0$$

(Wait. This is obvious. I should delete this. Crap.)

1.4.2 Linear, n^{th}-order, nonhomogeneous

Consider

$$\sum_{q=0}^{n} a_q\Delta^q f(x) = b. \qquad (a_q,\, b \in \mathbb{R})$$

The constant on the right can be trivially eliminated. Let $g(x) = f(x) - \frac{b}{a_0}$. As $\Delta^q\frac{b}{a_0} = 0$ for $q > 0$ (the difference of a constant is 0), we have

$$\begin{aligned}\sum_{q=0}^{n} a_q\Delta^q g(x) &= \sum_{q=0}^{n} a_q\Delta^q\left(f(x) - \frac{b}{a_0}\right) \qquad (c \in \mathbb{R})\\ &= -b + \sum_{q=0}^{n} a_q\Delta^q f(x)\\ &= 0\end{aligned}$$

And so, the problem reduces to the homogeneous case. Firstly we note the trivial solution $g(x) = 0$. Otherwise, we guess the solution to be λ^x, with $\lambda \neq 0$. (How do we know that this solution is unique? In general, the question of uniqueness to a difference/differential equation is non-trivial. For the case of linear difference equations however a proof is available.) Then, noting that

$$\Delta^q\lambda^x = (\lambda-1)^q\lambda^x,$$

we can write

$$\sum_{q=0}^{n} a_q(\lambda-1)^q\lambda^x = 0$$

$$\Rightarrow \sum_{q=0}^{n} a_q(\lambda-1)^q = 0$$

So solving linear difference equations boils down to solving a n-th order polynomial. Note that there may be more than one solution for λ, in which case each solution acts as a basis vector for the space of solutions. Unfortunately, the solution isn't this simple, as there may be repeated roots.

1.4.3 Non-linear, 1^{st}-order, nonhomogeneous

Consider

$$\Delta f(x) + h(x) * f(x) = g(x).$$

(Actually, it turns out that rephrasing the problem into a recurrence relation without the difference operator simplifies notation.) The solution is:

$$f(x) = \prod_{q=0}^{x-1} \big(1 - h(q)\big) \left(f(0) + \sum_{p=0}^{x-1} \frac{g(p)}{\prod_{q=0}^{p} \big(1 - h(q)\big)} \right)$$

In the case of constant coefficients, the expression is (somewhat) simplified; for

$$\Delta f(x) + a * f(x) = g(x).$$

we have:

$$f(x) = (1-a)^x \left(f(0) + \sum_{p=0}^{x-1} \frac{g(p)}{(1-a)^{p+1}} \right)$$

1.4.4 Proof that exponential solutions span the solution space

We may rewrite, from,

$$\sum_{q=0}^{n} a_q \Delta^q f(x) = b. \qquad (a_q,\, b \in \mathbb{R})$$

1.4.5 Conversion between operator form and recurrence form

The recurrence

$$asdasd$$

may alternative be written as

$$asd$$

How do we convert from one form to the other? We have:

$$\Delta^n f(x) = \sum_{q=0}^{n} (-1)^{n-q} \frac{n^{\underline{q}}}{q!} f(x+q)$$

This follows easily (and somewhat magically) if we define once we define the shift operator E:

$$\begin{aligned}
[E]f(x) = f(x+1) \Rightarrow \Delta f(x) &= [E-1]f(x) \\
&= f(x+1) - f(x) \\
\Rightarrow \Delta^n f(x) &= [E-1]^n f(x) \\
&= \left[\sum_{q=0}^{n} \frac{n^{\underline{q}}}{q!} (-1)^{n-q} E^q \right] f(x) \\
&= \sum_{q=0}^{n} \frac{n^{\underline{q}}}{q!} (-1)^{n-q} [E]^q f(x) \\
&= \sum_{q=0}^{n} \frac{n^{\underline{q}}}{q!} (-1)^{n-q} f(x+q)
\end{aligned}$$

For a clearer picture, we can represent this information in a matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} \begin{bmatrix} f(x) \\ f(x+1) \\ f(x+2) \\ f(x+3) \\ f(x+4) \end{bmatrix} = \begin{bmatrix} f(x) \\ \Delta f(x) \\ \Delta^2 f(x) \\ \Delta^3 f(x) \\ \Delta^4 f(x) \end{bmatrix}$$

Leaving things in operator form also helps:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} \begin{bmatrix} E^0 \\ E^1 \\ E^2 \\ E^3 \\ E^4 \end{bmatrix} = \begin{bmatrix} (E-1)^0 \\ (E-1)^1 \\ (E-1)^2 \\ (E-1)^3 \\ (E-1)^4 \end{bmatrix}$$

What might the inverse of this matrix be? Consider the expansion of $(E+1)^n$ instead; from the binomial theorem we have:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} \begin{bmatrix} E^0 \\ E^1 \\ E^2 \\ E^3 \\ E^4 \end{bmatrix} = \begin{bmatrix} (E+1)^0 \\ (E+1)^1 \\ (E+1)^2 \\ (E+1)^3 \\ (E+1)^4 \end{bmatrix}$$

Then:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} \begin{bmatrix} (E-1)^0 \\ (E-1)^1 \\ (E-1)^2 \\ (E-1)^3 \\ (E-1)^4 \end{bmatrix} = \begin{bmatrix} E^0 \\ E^1 \\ E^2 \\ E^3 \\ E^4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} \begin{bmatrix} E^0 \\ E^1 \\ E^2 \\ E^3 \\ E^4 \end{bmatrix} = \begin{bmatrix} E^0 \\ E^1 \\ E^2 \\ E^3 \\ E^4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}^{-1}$$

And so, the Using this for the n-th difference of the factorial:

$$\Delta^n x! = \sum_{q=0}^{n} \frac{n^{\underline{q}}}{q!} (-1)^{n-q} (x+q)!$$

$$= x! \sum_{q=0}^{n} \frac{n^{\underline{q}}}{q!} (-1)^{n-q} (x+q)^{\underline{q}}$$

1.5 Newton's Series

A remarkable, remarkable result is:

$$f(x) = \sum_{k=0}^{\infty} \frac{(x-a)^{\underline{k}}}{k!} \Delta^k f(a)$$

Noting that $[\Delta^n f](x_0)$ contains information about the values of $f(x)$ from $x = x_0$ to $x = x_0 + n$, one may wonder if it's possible to extract $f(x_0+n)$ from the differences of (up to the n-th order) of $f(x)$. Consider again the matrix form of the relationship between shift operations and difference operations:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} f(x) \\ \Delta f(x) \\ \Delta^2 f(x) \\ \Delta^3 f(x) \end{bmatrix} = \begin{bmatrix} f(x) \\ f(x+1) \\ f(x+2) \\ f(x+3) \end{bmatrix}$$

So

$$f(x_0 + n) = \sum_{q=0}^{n} \frac{n^{\underline{q}}}{q!} [\Delta^q f](x_0)$$

Or,

$$f(x) = f(x_0 + (x - x_0)) = \sum_{q=0}^{x-x_0} \frac{(x-x_0)^{\underline{q}}}{q!} [\Delta^q f](x_0)$$

Choosing $x_0 = 0$ simplifies the expression:

$$f(x) = \sum_{q=0}^{x} \frac{x^{\underline{q}}}{q!} [\Delta^q f](0)$$

A more intuitive explanation for this formula is discussed in _.

1.5.1 Sequence transform between top row and main diagonal

Suppose that the top row is given by $f(n)$, and the main diagonal by $g(n)$. The function $g(n)$ gives the first element of the n^{th} difference of $f(n)$, so, from our previous results:

$$\begin{aligned} g(n) &= [\Delta^n f](0) \\ &= \sum_{q=0}^{n} \frac{n^{\underline{q}}}{q!} (-1)^{n-q} f(q) \end{aligned}$$

In a matrix form:

$$\begin{bmatrix} g(0) \\ g(1) \\ g(2) \\ g(3) \\ g(4) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ f(4) \end{bmatrix}$$

Or:

$$\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \\ f(4) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ g(2) \\ g(3) \\ g(4) \end{bmatrix}$$

Suppose we are concerned with the $(k+1)^{\text{th}}$ diagonal, given by $g_k(n)$ (so the first diagonal is $g_0(n)$). The function $g_k(n)$ gives the $(k+1)^{\text{th}}$ element of the n^{th} difference of $f(n)$, so, from our previous results:

$$\begin{aligned} g_k(n) &= [\Delta^n f](k) \\ &= \sum_{q=0}^{n} \frac{n^{\underline{q}}}{q!}(-1)^{n-q} f(k+q) \end{aligned}$$

In a matrix form:

$$\begin{bmatrix} g_k(0) \\ g_k(1) \\ g_k(2) \\ g_k(3) \\ g_k(4) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} \begin{bmatrix} f(k) \\ f(k+1) \\ f(k+2) \\ f(k+3) \\ f(k+4) \end{bmatrix}$$

Or:

$$\begin{bmatrix} f(k) \\ f(k+1) \\ f(k+2) \\ f(k+3) \\ f(k+4) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} \begin{bmatrix} g_k(0) \\ g_k(1) \\ g_k(2) \\ g_k(3) \\ g_k(4) \end{bmatrix}$$

1.5.2 Stirling

To express powers in terms of falling powers, we can use Newton's series. For $f(x) = x^n$,

$$\begin{aligned} [\Delta^k f](0) &= \sum_{q=0}^{k} \frac{k^{\underline{q}}}{q!}(-1)^{k-q} f(q) \\ &= \sum_{q=0}^{k} \frac{k^{\underline{q}}}{q!}(-1)^{k-q} q^n \end{aligned}$$

And so,

$$x^n = \sum_{q=0}^{x} \left(\frac{x^{\underline{q}}}{q!} \sum_{p=0}^{q} \frac{q^{\underline{p}}}{p!}(-1)^{k-p} p^n \right)$$

The coefficient of the falling power here is defined as the Stirling numbers of the second kind, denoted and defined as:

$$S(n,q) := \frac{1}{q!} \sum_{p=0}^{q} \frac{q^{\underline{p}}}{p!}(-1)^{k-p} p^n$$

So

$$x^n = \sum_{q=0}^{x} S(n,q)\, x^{\underline{q}}$$

Some values of the Stirling numbers are given below in matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & 7 & 6 & 1 \end{bmatrix} \begin{bmatrix} x^{\underline{0}} \\ x^{\underline{1}} \\ x^{\underline{2}} \\ x^{\underline{3}} \\ x^{\underline{4}} \end{bmatrix} = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix}$$

A natural extension to this is the inverse matrix; how does one construct falling powers from normal powers? The coefficients of the inverse matrix are given by the Stirling numbers of the first kind:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & -3 & 1 & 0 \\ 0 & -6 & 11 & -6 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} = \begin{bmatrix} x^{\underline{0}} \\ x^{\underline{1}} \\ x^{\underline{2}} \\ x^{\underline{3}} \\ x^{\underline{4}} \end{bmatrix}$$

It turns out that the explicit formula for Stirling numbers of the first kind is not nearly as simple as the one for the second kind...

1.5.3 Polynomial extrapolation

Suppose we want to continue the sequence:

$$2 \qquad 3 \qquad 5 \qquad 7 \qquad 11.$$

Successively applying difference operators give:

$$\begin{array}{ccccccccc} 2 & & 3 & & 5 & & 7 & & 11 \\ & 1 & & 2 & & 2 & & 4 & \\ & & 1 & & 0 & & 2 & & \\ & & & -1 & & 2 & & & \\ & & & & 3 & & & & \end{array}$$

No pattern is discernable, but we can assume that the bottom-most sequence will always be 3:

$$\begin{array}{ccccccccccc} 2 & & 3 & & 5 & & 7 & & 11 & & 22 \\ & 1 & & 2 & & 2 & & 4 & & 11 & \\ & & 1 & & 0 & & 2 & & 7 & & \\ & & & -1 & & 2 & & 5 & & & \\ & & & & 3 & & 3 & & & & \end{array}$$

This gives 22 as the next element. What is the equation for the n-th element? Assume that the top row, $f(n)$ is given by a polynomial in the falling powers:

$$f(n) = a_0 + a_1 n^{\underline{1}} + a_2 n^{\underline{2}} + \cdots$$

Then, $f(0)$, the first element of the top row (0-indexed), is equal to a_0, as all terms with a factor of n will disappear. So $a_0 = 2$. Now, consider the second row, the difference of the top row. It is given by:

$$\Delta f(n) = a_1 + 2a_2 n^{\underline{1}} + 3a_3 n^{\underline{2}} + \cdots$$

Then, $[\Delta f](0)$, the first element of the second row is equal to $a_1 - a_1 = 1$. Now, consider the third row, the difference of the second row. It is given by:

$$\Delta^2 f(n) = 2a_2 + 6a_3 n^{\underline{1}} + 12a_4 n^{\underline{2}} + \cdots$$

Then, $[\Delta^2 f](0)$, the first element of the third row is equal to $2a_2 - a_2 = \frac{1}{2}$. And so,

$$a_n = \frac{1}{n!}\big(\text{first element of (n+1)-th row}\big)$$

The $n+1$ is just an artifact of 0-indexing and the English language. Looking back at the first elements of the rows in our sequence before:

$$\begin{array}{ccccccccccc}
2 & & 3 & & 5 & & 7 & & 11 & & 22 \\
& 1 & & 2 & & 2 & & 4 & & 11 & \\
& & 1 & & 0 & & 2 & & 7 & & \\
& & & -1 & & 2 & & 5 & & & \\
& & & & 3 & & 3 & & & &
\end{array}$$

We see that $a_0 = 2$, $a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = -\frac{1}{6}$ and $a_4 = \frac{1}{8}$. Coefficients after a_5 must all be 0, as we assumed a row of 3's in the 5^{th} row. And so, the top row is given by:

$$f(n) = 2 + n^{\underline{1}} + \frac{1}{2}n^{\underline{2}} - \frac{1}{6}n^{\underline{3}} + \frac{1}{8}n^{\underline{4}}$$

Using a change of basis matrix (transpose of Stirling matrix of first kind from before) to convert this into normal powers give:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & -6 \\ 0 & 0 & 1 & -3 & 11 \\ 0 & 0 & 0 & 1 & -6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ \frac{1}{2} \\ -\frac{1}{6} \\ \frac{1}{8} \end{bmatrix} = \begin{bmatrix} 2 \\ -\frac{7}{12} \\ \frac{19}{8} \\ -\frac{11}{12} \\ \frac{1}{8} \end{bmatrix}$$

1.6 Conversion to differential operator

$$\frac{d^n}{dx^n} f(x) = n! \sum_{q=n}^{\infty} \frac{s(q,n)}{q!} \Delta^q f(x)$$

1.7 Applications

1.7.1 Solution to functional equations

Simple functional equations containing terms $f^n(x)$ may be solved using recurrence/difference equations: Consider the problem of finding $f(x)$ such that

$$f(f(x)) = 6f(x) - x$$

Let $g(0) = x$ and $g(n+1) = f\left(g(n)\right)$. Then:

$$g(2) = 6g(1) - g(0)$$

Which, using the characteristic polynomial, gives

1.7.2 Runtime of recursive functions

Suppose that the runtime of a program is given by

$$\begin{aligned} T(n) &= T\left(\frac{n}{a}\right) + f(n) \\ \Rightarrow T(n) - T\left(\frac{n}{a}\right) &= f(n). \end{aligned}$$

This is common for recursive algorithms which solve sub-problems smaller by a factor a. The solution is to introduce a new function, $Q(n)$ such that

$$Q\left(\log_a(n) + 1\right) = T(n).$$

This gives:

$$\begin{aligned} T(n) - T\left(\frac{n}{a}\right) &= Q\left(\log_a(n) + 1\right) - Q\left(\log_a\left(\frac{n}{a}\right) + 1\right) \\ &= Q\left(\log_a(n) + 1\right) - Q\left(\log_a(n)\right) \\ &= Q(x+1) - Q(x) \qquad (x = \log_a(n)) \\ &= \Delta Q(x) \end{aligned}$$

And so (changing variables to t for clarity):

$$\begin{aligned} \Delta Q(t) = f(e^t) \Rightarrow Q(t) &= \sum_{q=1}^{t-1} f(e^q) + Q(1) \\ \Rightarrow T(t) &= \sum_{q=1}^{\log_a(t)} f(e^q) + T(1) \end{aligned}$$

Applying this formula to

$$T(n) = T\left(\frac{n}{2}\right) + 1,\, T(1) = 1$$

gives

$$\begin{aligned} T(n) &= \sum_{q=1}^{\log_a(n)} 1 + 1 \\ &= \log_a(n) + 1 \end{aligned}$$

1.7.3 Master Theorem

The runtime for divide-and-conquer algorithms are given in general by:

$$\begin{aligned} T(n) &= aT\left(\frac{n}{b}\right) + f(n) \\ \Rightarrow T(n) - T\left(\frac{n}{b}\right) &= (a-1)T\left(\frac{n}{b}\right) + f(n). \end{aligned}$$

Again using $x = \log_b(n)$ and $Q(x) = T(n)$:

$$\begin{aligned} T(n) &= aT\left(\frac{n}{b}\right) + f(n) \\ \Rightarrow T(n) - T\left(\frac{n}{b}\right) &= (a-1)T\left(\frac{n}{b}\right) + f(n). \end{aligned}$$

While similar

2 Calculus

Define exp
show ddx lnx =x-1
show ddx x to the a

2.1 Taylor Series

(but when is this true?) Assume that:

$$f(x) = \sum_{q=0}^{\infty} a_q (x - x_0)^q$$

Then

$$\begin{aligned} \frac{d^n}{dx^n} f(x) &= \frac{d^n}{dx^n} \sum_{q=0}^{\infty} a_q (x - x_0)^q \\ &= \sum_{q=0}^{\infty} \frac{d^n}{dx^n} a_q (x - x_0)^q \\ &= \sum_{q=0}^{\infty} q^{\underline{n}} a_q (x - x_0)^{q-n} \end{aligned}$$

Consider the case when $x = x_0$. All terms in the sum will reduce to 0 except when $q = n$:

$$\frac{d^n}{dx^n} f(x_0) = n^{\underline{n}} a_n \Rightarrow a_n = \frac{f^{(n)}(x_0)}{n!}$$

And ultimately:

$$f(x) = \sum_{q=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^q$$

2.1.1 Binomial

From Taylor's we have:

$$f(x) = \sum_{q=0}^{\infty} \frac{f^{(q)}(a)}{q!}(x-a)^q$$

But

$$\frac{d^q}{dx^q}x^r = r^{\underline{q}}x^{r-q}$$

And so:

$$x^r = \sum_{q=0}^{\infty} \frac{r^{\underline{q}}}{q!}a^{r-q}(x-a)^q \tag{1}$$

Making the substitution $x = (a+b)$ gives:

$$(a+b)^r = \sum_{q=0}^{\infty} \frac{r^{\underline{q}}}{q!}a^{r-q}(b)^q$$

Consider $(1+x)^{\frac{1}{2}}$. From (1) and choosing $a = 1$ we have:

$$(1+x)^{\frac{1}{2}} = \sum_{q=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{\underline{q}}}{q!}x^q$$