Cyclotomy in Radicals

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Contents

1 Vandermonde's Method

1.1 Theory

We wish to express the n^{th} roots of unity in radicals with index lower than n. The following method is due to Vandermonde and Gauss.

We consider a primitive p^{th} root ζ where p is prime (results for composite numbers follow from the prime numbers, as we will show later). We begin by considering the vector space $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$. A natural basis is $\{\zeta^i \mid i = 1, 2, \ldots, p-1\}$. However, we will instead work with $\{\zeta^i \mid i = g^1, g^2, \ldots, g^{p-1}\}$ where g is a primitive root modulo p. Although this is simply a re-ordering of the previous basis, it is helpful as automorphisms in $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ will simply "shift" the coefficients of a vector.

To illustrate this, consider the automorphism ϕ in the Galois group which maps ζ to ζ^g. To notate vectors we will use $[C_1, C_2, \ldots, C_n] = C_1\zeta^{g^1} + C_2\zeta^{g^2} + \cdots + C_n\zeta^{g^n}$, where $n = p-1$ is the dimension of the vector space. Noting that ϕ takes ζ^{g^i} to $\zeta^{g^{i+1}}$, we see that $\phi([C_1, C_2, \ldots, C_n]) = [C_n, C_1, C_2, \ldots, C_{n-1}]$. Ultimately, we want an expression for $[0, \ldots, 0, 1]$ (or any other vector with only one non-zero entry) by adding/multiplying/radicalising vectors whose values we already know. Vandermonde's method uses addition, which our current basis is optimised for.

A natural way to abuse the coefficient-shifting property of ϕ is by considering the vector $[\,\omega^1, \omega^2, \ldots, \omega^n\,]$ where ω is a primitive n^{th} root of unity.

$$\begin{aligned}\phi\left([\,\omega^1, \omega^2, \ldots, \omega^n\,]\right) &= [\,\omega^n, \omega^1, \ldots, \omega^{n-1}\,]\\ &= \omega^{-1}[\,\omega^1, \omega^2, \ldots, \omega^n\,]\end{aligned}$$

The genius step is to consider such vectors for all n^{th} roots of unity and notice that adding them up gives the desired $[0, \ldots, 0, 1]$:

$$\begin{array}{ccccccccccc} & (\omega^1)^1\zeta^1 & + & (\omega^1)^2\zeta^2 & + & \cdots & + & (\omega^1)^n\zeta^n & =: & V_1 \\ + & & & & & & & & & \\ & (\omega^2)^1\zeta^1 & + & (\omega^2)^2\zeta^2 & + & \cdots & + & (\omega^2)^n\zeta^n & =: & V_2 \\ + & & & & & & & & & \\ & & & & & \vdots & & & & \\ + & & & & & & & & & \\ & (\omega^n)^1\zeta^1 & + & (\omega^n)^2\zeta^2 & + & \cdots & + & (\omega^n)^n\zeta^n & =: & V_n \\ = & & & & & & & & & \\ & 0 & + & 0 & + & \cdots & + & n\zeta^n & = & n\zeta^n \end{array} \tag{1}$$

Each of the first $n-1$ columns add to 0 because ω^i is a root of unity whose order divides n, and the columns cycles over powers of ω^i from 1 to n. The column is non-zero only when $\omega^i = 1$.

Furthermore, each of the rows V_i are expressible in radicals. Noting that

$$\begin{aligned}\phi\left(V_i\right) &= \phi\left([\,(\omega^i)^1, (\omega^i)^2, \ldots, (\omega^i)^n\,]\right)\\ &= [\,(\omega^i)^n, (\omega^i)^1, \ldots, (\omega^i)^{n-1}\,]\\ &= [\,\omega^1, \omega^2, \ldots, \omega^n\,],\end{aligned}$$

it follows that

$$\begin{aligned}\phi\left(V_i^n\right) &= \phi\left(V_i\right)^n && \text{(automorphisms preserve products)}\\ &= \omega^{-in}V_i^n\\ &= V_i^n.\end{aligned}$$

So V_i^n is invariant under ϕ. Given $V_i^n = [C_1, C_2, \ldots, C_n]$ we have $[C_1, C_2, \ldots, C_n] = [C_n, C_1, \ldots, C_n{-}1] = \cdots = [C_n - 1, C_n, \ldots, C_n - 2]$, so $C_1 = C_2 = \cdots = C_n$. $[1, 1, \ldots, 1] = -1$ so $V_i^n = -C_1$. Theoretically we've established now that expressing ζ in radicals is possible, but calculating C_1 in practicality is non-trivial. (write notes on this!) Suppose we want to find the coefficient of ζ^1 in $[a_1, a_2, \ldots, a_n][b_1, b_2, \ldots, b_n]$. We want to determine entries A_i and B_j such that

$$\zeta^{g^i}\zeta^{g^j} = \zeta^1 \Rightarrow g^i + g^j \equiv 1 \pmod p$$

For example, take $p = 11$ and $g = 2$.

g^i	g^j	i	j
2	-1	1	5
3	-2	8	6
4	-3	2	3
5	-4	4	7
6	-5	9	9

The problem of determining which coefficients are involved boils down to calculating discrete logarithms, for which no efficient solution is known.

From (1) one may write:

$$\zeta^{-1} = \frac{\sqrt[n]{V_1} + \sqrt[n]{V_2} + \cdots + \sqrt[n]{V_n}}{n}.$$

However, this equation is not necessarily correct. While we know V_i^n, there are n possible values for V_i. For example, from $x^4 = 1$ we are unsure wheter $x = \pm 1$ or $x = \pm i$. A way to mitigate this problem is by noting that the correct form of V_i can be determined from the value of V_1. $\phi(V_i V_1^{n-i}) = \omega^{-i} V_i \omega^{i-n} V_1^{n-i} = V_i V_1^{n-i}$, so $V_i V_1^{n-i}$ may be determined using the same method for determining V_i^n. Thus, we can instead write:

$$\zeta = \frac{1}{n}\left(V_1 + \frac{V_2 V_1^{n-2}}{V_1^{n-2}} + \frac{V_3 V_1^{n-3}}{V_1^{n-3}} + \cdots + \frac{V_n V_1^{n-n}}{V_1^{n-n}}\right).$$

But which root of V_1^n should we choose? It turns out that choosing any will give us some p^{th} root ζ^j of unity. Suppose we mistake V_1 for $\omega^j V_1$. Then, we will mistake V_i for

$$\frac{V_i V_1^{n-i}}{(\omega^j V_1)^{n-i}} = \omega^{ij} V_i.$$

Using the diagram from before, this has the effect of shifting all coefficients by j to the left, ultimately giving us $zeta^{-j}$ instead of $zeta^{-1}$:

$$\begin{array}{ccccccccccccc}
 & (\omega^1)^{1+j}\zeta^1 & + & (\omega^1)^{2+j}\zeta^2 & + & \cdots & + & (\omega^1)^0\zeta^{-j} & + & \cdots & + & (\omega^1)^{n+j}\zeta^n & =: & \omega^j V_1 \\
+ & & & & & & & & & & & & & \\
 & (\omega^2)^1\zeta^1 & + & (\omega^2)^2\zeta^2 & + & \cdots & + & (\omega^2)^0\zeta^{-j} & + & \cdots & + & (\omega^2)^n\zeta^n & =: & \omega^{2j} V_2 \\
+ & & & & & & & & & & & & & \\
 & & & & & \vdots & & & & \vdots & & & & \\
+ & & & & & & & & & & & & & \\
 & (\omega^n)^1\zeta^1 & + & (\omega^n)^2\zeta^2 & + & \cdots & + & (\omega^n)^0\zeta^{-j} & + & \cdots & + & (\omega^n)^n\zeta^n & =: & \omega^{nj} V_n \\
= & & & & & & & & & & & & & \\
 & 0 & + & 0 & + & \cdots & + & n\zeta^{-j} & + & \cdots & + & 0 & = & n\zeta^{-j}
\end{array}$$

1.2 3rd Roots of Unity

We have $\omega = -1$, $p = 3$ and $n = 2$. A (and also the only) primitive root for 3 is 2, so $g = 2$. Noting $\zeta^3 = 1$, we have

$$\begin{aligned} V_1 &= (-1)\zeta^{2^2} + (1)\zeta^{2^1} \\ &= \zeta^2 - \zeta \\ \Rightarrow V_1^2 &= \zeta^2 - 2\zeta^3 + \zeta^4 \\ &= \zeta^2 + \zeta - 2 \\ &= [3, 3] \end{aligned}$$

So $V_1 = \pm\sqrt{-3}$. Take $V_1 = \sqrt{-3}$.

$$\begin{aligned} V_2 V_1^{2-2} &= (\zeta + \zeta^2) \\ &= -1 \end{aligned}$$

So ultimately,

$$\begin{aligned} \zeta^{-1} &= \frac{1}{n}\left(V_1 + \frac{V_2 V_1^{n-2}}{V_1^{n-2}}\right) \\ &= \frac{1}{2}\left(\sqrt{-3} - 1\right) \\ &= -\frac{1}{2} + \frac{\sqrt{3}}{2} i \end{aligned}$$

If we chose $V_1 = -\sqrt{-3}$ then we would have gotten $\zeta^{-1} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$.

1.3 5^{th} Roots of Unity

From here on we omit the details of the calculation of products of V_i and only show relevant results. Calculating V_i can be done with a computer algebra system.
For ease of notation, we let q be the number minimally satisfying $\eta^q \sqrt[k]{V_i^k} = V_i$ where $k = \frac{n}{(i,n)}$ and η is the "minimal" k^{th} primitive root (meta; why not others? what if we try the others?), and the radical evaluates to the root with minimal argument (from 0 to 2π), and q' satisfy the same equation but with the radical evaluating to the "principal" value (meta; the default implementation in mathematica is problematic as it uses the range -pi to pi; unless this is remedied q' is more or less useless).

$g = 2$:

i	$V_i V_1^{n-i}$	V_i^k	k	q
1	$-15+20i$	$-15+20i$	4	1
2	$-5-10i$	5	2	1
3	-5	$-15-20i$	4	0
4	-1	-1	1	0

$g = 3$:

i	$V_i V_1^{n-i}$	V_i^k	k	q
1	$-15-20i$	$-15-20i$	4	0
2	$-5+10i$	5	2	1
3	-5	$-15+20i$	4	1
4	-1	-1	1	0

1.4 7^{th} Roots of Unity

$g = 3$:

i	$V_i V_1^{n-i}$	V_i^k	k	q
1	$\frac{1}{2}(-7)\left(71 + 39i\sqrt{3}\right)$	$\frac{1}{2}(-7)\left(71 + 39i\sqrt{3}\right)$	6	2
2	$\frac{1}{2}(-7)\left(17 + 19i\sqrt{3}\right)$	$\frac{1}{2}(-7)\left(-1 + 3i\sqrt{3}\right)$	3	2
3	$\frac{1}{2}(-7)\left(13 + 3i\sqrt{3}\right)$	-7	2	0
4	$\frac{1}{2}(-7)i\left(\sqrt{3} + 5i\right)$	$\frac{1}{2}(-7)\left(-1 - 3i\sqrt{3}\right)$	3	0
5	-7	$\frac{1}{2}(-7)\left(71 - 39i\sqrt{3}\right)$	6	0
6	-1	-1	1	0

$g = 5$:

i	$V_i V_1^{n-i}$	V_i^k	k	q
1	$\frac{1}{2}(-7)\left(71 - 39i\sqrt{3}\right)$	$\frac{1}{2}(-7)\left(71 - 39i\sqrt{3}\right)$	6	0
2	$\frac{1}{2}(-7)\left(17 - 19i\sqrt{3}\right)$	$\frac{1}{2}(-7)\left(-1 - 3i\sqrt{3}\right)$	3	0
3	$\frac{1}{2}(-7)\left(13 - 3i\sqrt{3}\right)$	-7	2	0
4	$\frac{7}{2}i\left(\sqrt{3} - 5i\right)$	$\frac{1}{2}(-7)\left(-1 + 3i\sqrt{3}\right)$	3	2
5	-7	$\frac{1}{2}(-7)\left(71 + 39i\sqrt{3}\right)$	6	2
6	-1	-1	1	0

1.5 11^{th} Roots of Unity

$g = 2$:

i	$V_iV_1^{n-i}$	V_i^k	k	q
1	$\frac{1}{4}(-11)i\left(-25\sqrt{3905050-1323310\sqrt{5}}+i\left(12475\sqrt{5}+30341\right)\right)$	$\frac{1}{4}(-11)i\left(-25\sqrt{3905050-1323310\sqrt{5}}+i\left(12475\sqrt{5}+30341\right)\right)$	10	2
2	$-11\left(\frac{5627\sqrt{5}}{4}+\frac{1}{4}i\sqrt{33331490\sqrt{5}+74804650}+\frac{7773}{4}\right)$	$-11\left(\frac{25\sqrt{5}}{4}+\frac{5}{2}i\sqrt{\frac{1}{2}\left(205-89\sqrt{5}\right)}+\frac{89}{4}\right)$	5	0
3	$-11\left(-180\sqrt{5}+6i\sqrt{9190\sqrt{5}+21250}+919\right)$	$\frac{1}{8}(-11)i\left(33322i+2410i\sqrt{5}-50\sqrt{3905050-1705682\sqrt{5}}\right)$	10	6
4	$-11\left(\frac{435\sqrt{5}}{4}-\frac{1}{4}i\sqrt{173662\sqrt{5}+388330}-\frac{1093}{4}\right)$	$-11\left(-\frac{25\sqrt{5}}{4}+\frac{5}{2}i\sqrt{\frac{1}{2}\left(89\sqrt{5}+205\right)}+\frac{89}{4}\right)$	5	0
5	$-11\left(-\frac{175\sqrt{5}}{4}-\frac{25}{4}i\sqrt{50-10\sqrt{5}}+\frac{69}{4}\right)$	-11	2	0
6	$-11\left(-\frac{35\sqrt{5}}{4}-\frac{1}{4}i\sqrt{5822\sqrt{5}+13130}+\frac{5}{4}\right)$	$-11\left(-\frac{25\sqrt{5}}{4}-\frac{5}{2}i\sqrt{\frac{1}{2}\left(89\sqrt{5}+205\right)}+\frac{89}{4}\right)$	5	4
7	$-11\left(-\frac{13\sqrt{5}}{4}+\frac{5}{4}i\sqrt{2\sqrt{5}+10}-\frac{35}{4}\right)$	$-11\left(\frac{1205\sqrt{5}}{4}-\frac{25}{4}i\sqrt{3905050-1705682\sqrt{5}}-\frac{41611}{4}\right)$	10	8
8	$-11\left(-\frac{3\sqrt{5}}{4}+\frac{1}{4}i\sqrt{50-10\sqrt{5}}+\frac{9}{4}\right)$	$-11\left(\frac{25\sqrt{5}}{4}-\frac{5}{2}i\sqrt{\frac{1}{2}\left(205-89\sqrt{5}\right)}+\frac{89}{4}\right)$	5	4
9	-11	$-11\left(\frac{12475\sqrt{5}}{4}-\frac{25}{4}i\sqrt{3905050-1323310\sqrt{5}}-\frac{27931}{4}\right)$	10	2
10	-1	-1	1	0

$g = 6$:

i	$V_i V_1^{n-i}$	V_i^k	k	q
1	$\frac{1}{4}(-11)\left(12475\sqrt{5}-25i\sqrt{3905050-1323310\sqrt{5}}-27931\right)$	$\frac{1}{4}(-11)\left(12475\sqrt{5}-25i\sqrt{3905050-1323310\sqrt{5}}-27931\right)$	10	2
2	$\frac{1}{4}(-11)\left(-5627\sqrt{5}+i\left(768\sqrt{5}+1495\right)\sqrt{2\left(\sqrt{5}+5\right)}+8691\right)$	$\frac{1}{4}(-11)\left(25\sqrt{5}-5i\sqrt{410-178\sqrt{5}}+89\right)$	5	4
3	$-11\left(-180\sqrt{5}-6i\sqrt{9190\sqrt{5}+21250}+919\right)$	$\frac{1}{4}(-11)\left(1205\sqrt{5}-25i\sqrt{3905050-1705682\sqrt{5}}-41611\right)$	10	8
4	$\frac{1}{4}(-11)\left(-5\left(87\sqrt{5}+79\right)-i\sqrt{173662\sqrt{5}+388330}\right)$	$\frac{1}{4}(-11)\left(-25\sqrt{5}-5i\sqrt{178\sqrt{5}+410}+89\right)$	5	4
5	$\frac{1}{4}(-11)\left(175\sqrt{5}-25i\sqrt{50-10\sqrt{5}}+299\right)$	-11	2	0
6	$\frac{1}{4}(-11)\left(35\sqrt{5}-i\sqrt{5822\sqrt{5}+13130}-37\right)$	$\frac{1}{4}(-11)\left(-25\sqrt{5}+5i\sqrt{178\sqrt{5}+410}+89\right)$	5	0
7	$\frac{1}{4}(-11)\left(13\left(\sqrt{5}-1\right)+5i\sqrt{2\left(\sqrt{5}+5\right)}\right)$	$\frac{1}{4}(-11)\left(-1205\sqrt{5}-25i\sqrt{3905050-1705682\sqrt{5}}-16661\right)$	10	6
8	$\frac{1}{4}(-11)\left(3\sqrt{5}+i\sqrt{50-10\sqrt{5}}+7\right)$	$\frac{1}{4}(-11)\left(25\sqrt{5}+5i\sqrt{410-178\sqrt{5}}+89\right)$	5	0
9	-11	$\frac{1}{4}(-11)\left(-12475\sqrt{5}-25i\sqrt{3905050-1323310\sqrt{5}}-30341\right)$	10	2
10	-1	-1	1	0

$g = 7$:

i	$V_i V_1^{n-i}$	V_i^k	k	q
1	$\frac{1}{4}(-11)\left(-1205\sqrt{5}-25i\sqrt{3905050-1705682\sqrt{5}}-16661\right)$	$\frac{1}{4}(-11)\left(-1205\sqrt{5}-25i\sqrt{3905050-1705682\sqrt{5}}-16661\right)$	10	6
2	$\frac{1}{4}(-11)\left(-459\sqrt{5}-i\sqrt{22283998\sqrt{5}+74804650}+2605\right)$	$\frac{1}{4}(-11)\left(-25\sqrt{5}-5i\sqrt{178\sqrt{5}+410}+89\right)$	5	4
3	$-11\left(180\sqrt{5}-6i\sqrt{21250-9190\sqrt{5}}+919\right)$	$\frac{1}{4}(-11)\left(12475\sqrt{5}-25i\sqrt{3905050-1323310\sqrt{5}}-27931\right)$	10	2
4	$\frac{1}{4}(-11)\left(-349\sqrt{5}+i\sqrt{103202\sqrt{5}+388330}-1179\right)$	$\frac{1}{4}(-11)\left(25\sqrt{5}+5i\sqrt{410-178\sqrt{5}}+89\right)$	5	0
5	$\frac{1}{4}(-11)\left(-115\sqrt{5}+25i\sqrt{50-22\sqrt{5}}+359\right)$	-11	2	0
6	$\frac{1}{4}(-11)\left(21\sqrt{5}+i\sqrt{2882\sqrt{5}+13130}+19\right)$	$\frac{1}{4}(-11)\left(25\sqrt{5}-5i\sqrt{410-178\sqrt{5}}+89\right)$	5	4
7	$\frac{1}{4}(-11)\left(-5\sqrt{5}-i\sqrt{1450-190\sqrt{5}}-19\right)$	$\frac{1}{4}(-11)\left(-12475\sqrt{5}-25i\sqrt{3905050-1323310\sqrt{5}}-30341\right)$	10	2
8	$\frac{1}{4}(-11)\left(\sqrt{5}-i\sqrt{50-22\sqrt{5}}+11\right)$	$\frac{1}{4}(-11)\left(-25\sqrt{5}+5i\sqrt{178\sqrt{5}+410}+89\right)$	5	0
9	-11	$\frac{1}{4}(-11)\left(1205\sqrt{5}-25i\sqrt{3905050-1705682\sqrt{5}}-41611\right)$	10	8
10	-1	-1	1	0

$g = 8$:

i	$V_i V_1^{n-i}$	V_i^k	k	q
1	$\frac{1}{4}(-11)\left(1205\sqrt{5} - 25i\sqrt{3905050 - 1705682\sqrt{5}} - 41611\right)$	$\frac{1}{4}(-11)\left(1205\sqrt{5} - 25i\sqrt{3905050 - 1705682\sqrt{5}} - 41611\right)$	10	8
2	$\frac{1}{4}(-11)\left(459\sqrt{5} - i\sqrt{22283998\sqrt{5} + 74804650} + 13859\right)$	$\frac{1}{4}(-11)\left(-25\sqrt{5} + 5i\sqrt{178\sqrt{5} + 410} + 89\right)$	5	0
3	$-11\left(180\sqrt{5} + 6i\sqrt{21250 - 9190\sqrt{5}} + 919\right)$	$\frac{1}{4}(-11)\left(-12475\sqrt{5} - 25i\sqrt{3905050 - 1323310\sqrt{5}} - 30341\right)$	10	2
4	$\frac{1}{4}(-11)\left(349\sqrt{5} + i\sqrt{103202\sqrt{5} + 388330} - 309\right)$	$\frac{1}{4}(-11)\left(25\sqrt{5} - 5i\sqrt{410 - 178\sqrt{5}} + 89\right)$	5	4
5	$\frac{1}{4}(-11)\left(115\sqrt{5} + 25i\sqrt{50 - 22\sqrt{5}} + 9\right)$	-11	2	0
6	$\frac{1}{4}(-11)i\left(\sqrt{2882\sqrt{5} + 13130} + 3i\left(7\sqrt{5} + 17\right)\right)$	$\frac{1}{4}(-11)\left(25\sqrt{5} + 5i\sqrt{410 - 178\sqrt{5}} + 89\right)$	5	0
7	$\frac{1}{4}(-11)i\left(19i + 5i\sqrt{5} + \sqrt{1450 - 190\sqrt{5}}\right)$	$\frac{1}{4}(-11)\left(12475\sqrt{5} - 25i\sqrt{3905050 - 1323310\sqrt{5}} - 27931\right)$	10	2
8	$\frac{1}{4}(-11)\left(-\sqrt{5} - i\sqrt{50 - 22\sqrt{5}} + 5\right)$	$\frac{1}{4}(-11)\left(-25\sqrt{5} - 5i\sqrt{178\sqrt{5} + 410} + 89\right)$	5	4
9	-11	$\frac{1}{4}(-11)\left(-1205\sqrt{5} - 25i\sqrt{3905050 - 1705682\sqrt{5}} - 16661\right)$	10	6
10	-1	-1	1	0

1.6 13^{th} Roots of Unity

$g = 2$:

i	$V_i V_1^{n-i}$	V_i^k	k	q
1	$-13\left(-6210\sqrt{3}-\frac{3}{2}i\left(10175\sqrt{3}-93012\right)-\frac{685795}{2}\right)$	$-13\left(-6210\sqrt{3}-\frac{3}{2}i\left(10175\sqrt{3}-93012\right)-\frac{685795}{2}\right)$	12	4
2	$-13\left(10736\sqrt{3}+i\left(-52536\sqrt{3}-8906\right)-43581\right)$	$-13\left(\frac{337}{2}+\frac{15i\sqrt{3}}{2}\right)$	6	0
3	$-13\left(-10746\sqrt{3}+i\left(-2196\sqrt{3}-20895\right)+4270\right)$	$65-156i$	4	1
4	$-13\left(2160\sqrt{3}+i\left(4200-2142\sqrt{3}\right)+4165\right)$	$-13\left(\frac{5}{2}+\frac{3i\sqrt{3}}{2}\right)$	3	0
5	$-13\left(-900\sqrt{3}+\frac{15}{2}i\left(119\sqrt{3}-8\right)-\frac{119}{2}\right)$	$-13\left(6210\sqrt{3}+\frac{3}{2}i\left(10175\sqrt{3}+93012\right)-\frac{685795}{2}\right)$	12	11
6	$-13\left(345\sqrt{3}-\frac{1}{2}i\left(135\sqrt{3}-46\right)+\frac{9}{2}\right)$	13	2	0
7	$-13\left(90\sqrt{3}+\frac{3}{2}i\left(25\sqrt{3}+4\right)-\frac{5}{2}\right)$	$-13\left(6210\sqrt{3}-\frac{3}{2}i\left(10175\sqrt{3}+93012\right)-\frac{685795}{2}\right)$	12	6
8	$-13\left(\frac{15}{2}i\left(\sqrt{3}-4\right)+18\sqrt{3}+\frac{25}{2}\right)$	$-13\left(\frac{5}{2}-\frac{3i\sqrt{3}}{2}\right)$	3	2
9	$-13\left(\frac{9\sqrt{3}}{2}-\frac{3}{2}i\left(2\sqrt{3}+5\right)-5\right)$	$65+156i$	4	0
10	$-13\left(\sqrt{3}+\frac{1}{2}i\left(3\sqrt{3}+2\right)-\frac{3}{2}\right)$	$-13\left(\frac{337}{2}-\frac{15i\sqrt{3}}{2}\right)$	6	5
11	-13	$-13\left(-6210\sqrt{3}+\frac{3}{2}i\left(10175\sqrt{3}-93012\right)-\frac{685795}{2}\right)$	12	1
12	-1	-1	1	0

1.7 17^{th} Roots of Unity

$g = 3$:

i	$V_i V_1^{n-i}$	V_i^k	k	q
1	$-17\left((332126703+180803848i)+(28762800-89862184i)\sqrt{2}\right)$	$-17\left((332126703+180803848i)+(28762800-89862184i)\sqrt{2}\right)$	16	2
2	$-17\left((-26561768+39707967i)+(43752888-94790412i)\sqrt{2}\right)$	$(136-255i)\left(24\sqrt{2}+287i\right)$	8	7
3	$-17\left((17989751-7762652i)+(3741959-14838356i)\sqrt{2}\right)$	$-17\left((332126703-180803848i)+(138288240-114771640i)\sqrt{2}\right)$	16	4
4	$-17\left((-2122627+2885880i)-(2101116+2581618i)\sqrt{2}\right)$	$-255-136i$	4	0
5	$-17\left((478225+539044i)-(319831-225440i)\sqrt{2}\right)$	$-17\left((332126703+180803848i)+(138288240+9649528i)\sqrt{2}\right)$	16	12
6	$-17\left((-128548+25357i)-(13434-265548i)\sqrt{2}\right)$	$(-136-255i)\left(24\sqrt{2}+287i\right)$	8	1
7	$-17\left((-17727-1716i)-(94840-23386i)\sqrt{2}\right)$	$-17\left((332126703-180803848i)+(28762800-77188856i)\sqrt{2}\right)$	16	8
8	$-17\left((-3103-644i)-(8936+4908i)\sqrt{2}\right)$	17	2	0
9	$-17\left((-1393-140i)+(668-1438i)\sqrt{2}\right)$	$-17\left((332126703+180803848i)-(33166128-15838968i)\sqrt{2}\right)$	16	15
10	$-17\left((532+371i)+(654+120i)\sqrt{2}\right)$	$(-136+255i)\left(24\sqrt{2}-287i\right)$	8	6
11	$-17\left((175-100i)-(117-132i)\sqrt{2}\right)$	$-17\left((332126703-180803848i)-(133884912-40748424i)\sqrt{2}\right)$	16	11
12	$-17\left((-3+32i)+(12+14i)\sqrt{2}\right)$	$-255+136i$	4	3
13	$-17\left((9+12i)-(7-4i)\sqrt{2}\right)$	$-17\left((332126703+180803848i)-(133884912-64373688i)\sqrt{2}\right)$	16	3
14	$-17\sqrt{8\sqrt{2}-4i\sqrt{\sqrt{2}+10}-1}$	$(136+255i)\left(24\sqrt{2}-287i\right)$	8	0
15	-17	$-17\left((332126703-180803848i)-(116402128-92885528i)\sqrt{2}\right)$	16	5
16	-1	-1	1	0

2 Galois Theory