Analysis

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Contents

1 Curl

Theorem 1.0.0.1.

$$\nabla \times \vec{F} = 0 \Rightarrow \exists\, f : X \to \mathbb{R},\, \nabla f = \vec{F},$$

given that $\vec{F}$ is defined on X, an open, simply connected subset of $\mathbb{R}^3$.

2 Integrals

We define the Riemann-Stieltjis integral. Let μ be our non-decreasing, bounded "measure function", and f bounded over the interval $[a, b]$. Then, we define the lower and upper sums of a partition P of that interval:

$$U(f, \mu, P) := \sum_{p \in P} \sup_p(f) \Delta_p \mu \,;$$

$$L(f, \mu, P) := \sum_{p \in P} \inf_p(f) \Delta_p \mu \,.$$

Where $\Delta_p \mu = \mu(\text{endpoint}) - \mu(\text{startpoint})$. In turn we define the upper and lower Riemann-Stieltjis integrals:

$$\overline{\int_a^b} f \, d\mu := \inf_P U(f, \mu, P) \,;$$

$$\underline{\int_a^b} f \, d\mu := \sup_P L(f, \mu, P) \,.$$

(Is it okay to take the inf/sup over partitions? Aren't there "more" partitions than there are real numbers? Maybe not, by the requirement that the partition be finite?) The crucial property is that refining the partition non-strictly increases the lower sum and non-strictly decreases the upper sum. Also, we note that by the definition of inf/sup we can get arbitrarily close to the lower and upper integrals by some partition (say P_L for the lower and P_U for the upper). Then we can combine these two to get a partition which is arbitrarily close to both.

Now, we define the R-S integral as:

$$\overline{\int_a^b} f \, d\mu = \underline{\int_a^b} f \, d\mu \Rightarrow \int_a^b f \, d\mu := \overline{\int_a^b} f \, d\mu \,.$$

3 Surfaces

We may specify a surface by

$$z = \Phi(x, y).$$

Theorem 3.0.0.1.

$$\iint_{\Phi(D)} f \, dS = \iint_D f(\Phi(x, y)) \, \|\Phi_x \times \Phi_y\| \, dx \, dy$$

More generally one has the multivariable change of variables formula:

$$\int_{\Phi(D)} f(\vec{r}) \, d\vec{r} = \int_D f(\Phi(\vec{r})) \det D\Phi \, d\vec{r}$$

Proof. I don't give a proof but intuition. For proof one can see Schwarz's proof. We may just use substitution but have to take account the deformation of space The formula would be

$$\int_{\Phi(D)} Df(\vec{r}) \, d\vec{r} = \int_D Df(\Phi(\vec{r})) D\Phi \, d\vec{r}$$

as a direct generalisation from the single variable case, but if one makes Df a single variable output somehow the $D\Phi$ becomes a determinant. Haven't really figured out why. □

Definition 3.0.0.1 (Flux)**.** We define the flux as, where Σ is an oriented surface,

$$\iint_\Sigma \vec{F} \cdot d\vec{S} := \iint_\Sigma \vec{F} \cdot \hat{n} \, dS.$$

A unit normal is

$$\frac{\Phi_x \times \Phi_y}{\|\Phi_x \times \Phi_y\|}.$$

Using this, we get the formula (where $\Phi : D \to \Sigma$)

$$\iint_\Sigma \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\Phi_x \times \Phi_y)\, dx\, dy.$$

But which normal to take?

Theorem 3.0.0.2 (Green's Theorem)**.**

$$\int_D \partial_x Q - \partial_y P\, dx\, dy = \int_{\partial D} \begin{bmatrix} P \\ Q \end{bmatrix} \cdot d\vec{r}$$

Proof. We prove the result for "horizontally and/or vertically simple" regions; a region is said to be vertically simple if it can be characterised by $a < x < b$ and $g_1(x) < y < g_2(x)$. Parametrising the boundary with $\partial D = (t, g_1(t))$ and $(t, g_2(t))$ where $a < t < b$, we have $dx = dt$ and $dy = g_1'(t)$ or $g_2'(t)$. Then:

$$\begin{aligned}\int_{\partial D} \begin{bmatrix} P \\ Q \end{bmatrix} \cdot d\vec{r} = &\int_a^b P\left(\begin{bmatrix} t \\ g_1(t) \end{bmatrix}\right) dt + Q\left(\begin{bmatrix} t \\ g_1(t) \end{bmatrix}\right) g_1'(t)\, dt \\ &- \int_a^b P\left(\begin{bmatrix} t \\ g_2(t) \end{bmatrix}\right) dt + Q\left(\begin{bmatrix} t \\ g_2(t) \end{bmatrix}\right) g_2'(t)\, dt\end{aligned}$$

Just looking at the P parts (why?)

$$\begin{aligned}\int_a^b \left(P\left(\begin{bmatrix} t \\ g_1(t) \end{bmatrix}\right) - P\left(\begin{bmatrix} t \\ g_2(t) \end{bmatrix}\right)\right) dt &= -\int_a^b \int_{g_1(t)}^{g_2(t)} \partial_y P\, dy\, dt \\ &= -\int_D \partial_y P\, dt.\end{aligned}$$

So $\int_{\partial D} P\, dx = -\int_D \partial_y P\, dt.$, and similarly for horizontally simple regions $\int_{\partial D} Q\, dy = \int_D \partial_x Q\, dt$. Thus for regions which are both we can combine our two expressions to get:

$$\int_D \partial_x Q - \partial_y P\, dx\, dy = \int_{\partial D} \begin{bmatrix} P \\ Q \end{bmatrix} \cdot d\vec{r}.$$

□

Though the regions for which the above theorem applies seems restrictive, we can "slice" up regions in a way such that we can apply to a broader range.

Theorem 3.0.0.3 (Divergence Theorem (2D))**.**

$$\int_C \vec{F} \cdot \hat{n}\, ds = \iint_D \nabla \cdot \vec{F}\, dx\, dy;$$

This is equivalent to Green's theorem.

Proof. Parametrise the boundary with $\gamma(t) = (x(t), y(t))$. Then the outward-pointing normal vector is

$$\hat{n} = \frac{1}{\|\gamma'(t)\|}(y'(t), -x'(t)).$$

Now using $F = (P, Q)$, one can evaluate

$$\begin{aligned}\int_C \vec{F} \cdot \hat{n}\, ds &= \int_C -Q\, dx + P\, dy \\ &= \iint_D \nabla \cdot \vec{F}\, dx\, dy.\end{aligned}$$

□

Theorem 3.0.0.4 (Stokes')**.**

$$\iint_\Sigma (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial\Sigma} \vec{F} \cdot ds$$

Proof. Notice Stokes' theorem becomes Green's if surface is in the $x - y$ plane. Anyways, let Σ be parametrised by $\Phi(x, y) = (x, y, f(x, y))$, and let $\vec{F} = (u, v, w)$. Then

$$\int_{\partial\Sigma} \vec{F} \cdot ds = \int_{\partial\Sigma} \vec{F} \cdot ds =$$

But on our surface, $z = f$ so $\frac{dz}{dt} = [Df] \begin{bmatrix} x' \\ y' \end{bmatrix} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$, giving $dz = f_x dx + f_y dy$. Substituting, using Green's, and bashing equations gives $\iint_D (\nabla \times \vec{F}) \cdot (\Phi_x \times \Phi_y)\, dx\, dy$, which gives the proof. □

Theorem 3.0.0.5 (Gauss' Divergence Theorem)**.**

$$\iiint_\Omega \nabla \cdot \vec{F}\, dV = \iint_{\partial\Omega} \vec{F} \cdot\, d\vec{S}$$

4 Curvilinear coordinates

We work in 3 dimensions because generalisations are easy. Recall that 3-dimensions means that any point in space can be uniquely specified by a 3-tuple. The system by which those tuples are interpreted is called a coordinate system. A familiar example is the standard basis, where a tuple $(a, b, c)_S$ is interpreted as $a\vec{e_1} + b\vec{e_2} + c\vec{e_3}$. Consider another coordinate system P, represented using the associated function Φ:

$$(a, b, c)_X = (\Phi(a, b, c))_S$$

A natural way to think about coordinate axes in P is to note that in standard coordinates, what we do is make two of the coordinates 0 and vary the other; or at a point, keep two of them constant and vary the other. We do the same for arbitrary coordinate systems. However, what's interesting with this is that the basis vectors are *not* constant for arbitrary coordinate systems; they change directions and magnitudes for different points in space. The natural basis vectors for a point P is:

$$\vec{h}_1 = [\partial_1 \Phi]P, \quad \vec{h}_2 = [\partial_2 \Phi]P, \quad \vec{h}_3 = [\partial_3 \Phi]P.$$

OK, I've lied. coordinate axes aren't actually what you get from keeping two coordinates fixed; coordinate axes are the the local axes that are tangent to that curved axes. By the local property of these bases, they are called *local bases*. Global bases are only possible for linear/affine coordinate systems. If the bases are mutually orthogonal, we define the Lamé coefficients

$$h_i = |\vec{h}_i|$$

and orthonormal basis

$$\vec{b}_i = \hat{h}_i.$$

A Appendix