Analysis

Edwin Park

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Contents

1 Curl

Theorem 1.0.0.1.

$$\nabla \times \vec{F} = 0 \Rightarrow \exists\, f : X \to \mathbb{R},\, \nabla f = \vec{F},$$

given that $\vec{F}$ is defined on X, an open, simply connected subset of $\mathbb{R}^3$.

2 Integrals

We define the Riemann-Stieltjis integral. Let μ be our non-decreasing, bounded "measure function", and f bounded over the interval $[a, b]$. Then, we define the lower and upper sums of a partition P of that interval:

$$U(f, \mu, P) := \sum_{p \in P} \sup_p(f) \Delta_p \mu \,;$$

$$L(f, \mu, P) := \sum_{p \in P} \inf_p(f) \Delta_p \mu \,.$$

Where $\Delta_p \mu = \mu(\text{endpoint}) - \mu(\text{startpoint})$. In turn we define the upper and lower Riemann-Stieltjis integrals:

$$\overline{\int_a^b} f \, d\mu := \inf_P U(f, \mu, P) \,;$$

$$\underline{\int_a^b} f \, d\mu := \sup_P L(f, \mu, P) \,.$$

(Is it okay to take the inf/sup over partitions? Aren't there "more" partitions than there are real numbers? Maybe not, by the requirement that the partition be finite?) The crucial property is that refining the partition non-strictly increases the lower sum and non-strictly decreases the upper sum. Also, we note that by the definition of inf/sup we can get arbitrarily close to the lower and upper integrals by some partition (say P_L for the lower and P_U for the upper). Then we can combine these two to get a partition which is arbitrarily close to both.

Now, we define the R-S integral as:

$$\overline{\int_a^b} f \, d\mu = \underline{\int_a^b} f \, d\mu \Rightarrow \int_a^b f \, d\mu := \overline{\int_a^b} f \, d\mu \,.$$

3 Surfaces

We may specify a surface by

$$z = \Phi(x, y).$$

Theorem 3.0.0.1.

$$\iint_{\Phi(D)} f \, dS = \iint_D f(\Phi(x, y)) \, \|\Phi_x \times \Phi_y\| \, dx \, dy$$

Definition 3.0.0.1 (Flux)**.** We define the flux as, where Σ is an oriented surface,

$$\iint_\Sigma \vec{F} \cdot d\vec{S} := \iint_\Sigma \vec{F} \cdot \hat{n} \, dS.$$

A unit normal is

$$\frac{\Phi_x \times \Phi_y}{\|\Phi_x \times \Phi_y\|}.$$

Using this, we get the formula (where $\Phi : D \to \Sigma$)

$$\iint_\Sigma \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\Phi_x \times \Phi_y) \, dx \, dy.$$

But which normal to take?

Theorem 3.0.0.2 (Green's Theorem)**.**

$$\int_D \partial_x Q - \partial_y P \, dx \, dy = \int_{\partial D} \begin{bmatrix} P \\ Q \end{bmatrix} \cdot d\vec{r}$$

Proof. We prove the result for "horizontally and/or vertically simple" regions; a region is said to be vertically simple if it can be characterised by $a < x < b$ and $g_1(x) < y < g_2(x)$. Parametrising the boundary with $\partial D = (t, g_1(t))$ and $(t, g_2(t))$ where $a < t < b$, we have $dx = dt$ and $dy = g_1'(t)$ or $g_2'(t)$. Then:

$$\begin{aligned} \int_{\partial D} \begin{bmatrix} P \\ Q \end{bmatrix} \cdot d\vec{r} &= \int_a^b P\left(\begin{bmatrix} t \\ g_1(t) \end{bmatrix}\right) dt + Q\left(\begin{bmatrix} t \\ g_1(t) \end{bmatrix}\right) g_1'(t) \, dt \\ &- \int_a^b P\left(\begin{bmatrix} t \\ g_2(t) \end{bmatrix}\right) dt + Q\left(\begin{bmatrix} t \\ g_2(t) \end{bmatrix}\right) g_2'(t) \, dt \end{aligned}$$

Just looking at the P parts (why?)

$$\begin{aligned} \int_a^b \left(P\left(\begin{bmatrix} t \\ g_1(t) \end{bmatrix}\right) - P\left(\begin{bmatrix} t \\ g_2(t) \end{bmatrix}\right)\right) dt &= -\int_a^b \int_{g_1(t)}^{g_2(t)} \partial_y P \, dy \, dt \\ &= -\int_D \partial_y P \, dt. \end{aligned}$$

So $\int_{\partial D} P \, dx = -\int_D \partial_y P \, dt.$, and similarly for horizontally simple regions $\int_{\partial D} Q \, dy = \int_D \partial_x Q \, dt$. Thus for regions which are both we can combine our two expressions to get:

$$\int_D \partial_x Q - \partial_y P \, dx \, dy = \int_{\partial D} \begin{bmatrix} P \\ Q \end{bmatrix} \cdot d\vec{r}.$$

□

Though the regions for which the above theorem applies seems restrictive, we can "slice" up regions in a way such that we can apply to a broader range.

Theorem 3.0.0.3 (Divergence Theorem (2D))**.**

$$\int_C \vec{F} \cdot \hat{n} \, ds = \iint_D \nabla \cdot \vec{F} \, dx \, dy;$$

This is equivalent to Green's theorem.

Proof. Parametrise the boundary with $\gamma(t) = (x(t), y(t))$. Then the outward-pointing normal vector is

$$\hat{n} = \frac{1}{\|\gamma'(t)\|}(y'(t), -x'(t)).$$

Now using $F = (P, Q)$, one can evaluate

$$\begin{aligned} \int_C \vec{F} \cdot \hat{n} \, ds &= \int_C -Q \, dx + P \, dy \\ &= \iint_D \nabla \cdot \vec{F} \, dx \, dy. \end{aligned}$$

□

Theorem 3.0.0.4 (Stokes')**.**

$$\iint_\Sigma (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial \Sigma} \vec{F} \cdot ds$$

Proof. Notice Stokes' theorem becomes Green's if surface is in the $x - y$ plane. Anyways, let Σ be parametrised by $\Phi(x, y) = (x, y, f(x, y))$, and let $\vec{F} = (u, v, w)$. Then

$$\int_{\partial\Sigma} \vec{F} \cdot ds = \int_{\partial\Sigma} \vec{F} \cdot ds =$$

But on our surface, $z = f$ so $\frac{dz}{dt} = [Df] \begin{bmatrix} x' \\ y' \end{bmatrix} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}$, giving $dz = f_x dx + f_y dy$. Substituting, using Green's, and bashing equations gives $\iint_D (\nabla \times \vec{F}) \cdot (\Phi_x \times \Phi_y)\, dx\, dy$, which gives the proof. □

Theorem 3.0.0.5 (Gauss' Divergence Theorem)**.**

$$\iiint_\Omega \nabla \cdot \vec{F}\, dV = \iint_{\partial\Omega} \vec{F} \cdot d\vec{S}$$

A Appendix