Analysis

Edwin Park

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Contents

1 Taylor Polynomials

$$
\begin{aligned}
f(\vec{x}) &= f(\vec{x_0} + (\vec{x} - \vec{x_0})) \\
&\approx f(\vec{x_0}) + D_f(\vec{x_0})(\vec{x} - \vec{x_0}) + \frac{1}{2}(\vec{x} - \vec{x_0})^\top H_f(\vec{x_0})(\vec{x} - \vec{x_0}) \\
&= f(\vec{x_0}) + \begin{bmatrix} f_x(\vec{x_0}) & f_y(\vec{x_0}) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x - x_0 & y - y_0 \end{bmatrix} \begin{bmatrix} f_{xx}(\vec{x_0}) & f_{xy}(\vec{x_0}) \\ f_{yx}(\vec{x_0}) & f_{yy}(\vec{x_0}) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}
\end{aligned}
$$

Say we wanted to find an upper bound for the error of a first-order approximation at $\vec{x}$. The error then is equal to the quadratic term evaluated at some point in between the central point, $\vec{x_0}$, and the point of interest, $\vec{x}$. In particular, it is, for some $t \in [0, 1]$,

$$
\frac{1}{2} \left[D^2_{\vec{x} - \vec{x_0}} f \right] (t\vec{x} + (1-t)\vec{x_0}) = \frac{1}{2} \begin{bmatrix} x - x_0 & y - y_0 \end{bmatrix} \begin{bmatrix} f_{xx}(t\vec{x} + (1-t)\vec{x_0}) & f_{xy}(t\vec{x} + (1-t)\vec{x_0}) \\ f_{yx}(t\vec{x} + (1-t)\vec{x_0}) & f_{yy}(t\vec{x} + (1-t)\vec{x_0}) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}.
$$

2 Integral Theorems

We may specify a surface by

$$
(x, y, z) = \Phi(a, b).
$$

Theorem 2.0.0.1.

$$
\iint_{\Phi(D)} f \, dS = \iint_D f(\Phi(a, b)) \, \|\Phi_a \times \Phi_b\| \, da \, db
$$

More generally one has the multivariable change of variables formula:

$$
\int_{\Phi(D)} f(\vec{r}) \, d\vec{r} = \int_D f(\Phi(\vec{r})) \det D\Phi \, d\vec{r}.
$$

Theorem 2.0.0.2 (Green's Theorem)**.**

$$
\int_D \partial_x Y - \partial_y X \, dx \, dy = \int_{\partial D} \begin{bmatrix} X \\ Y \end{bmatrix} \cdot d\vec{r}
$$

Theorem 2.0.0.3 (Divergence Theorem (2D))**.**

$$
\int_C \vec{F} \cdot \hat{n} \, ds = \iint_D \nabla \cdot \vec{F} \, dx \, dy;
$$

This is equivalent to Green's theorem.

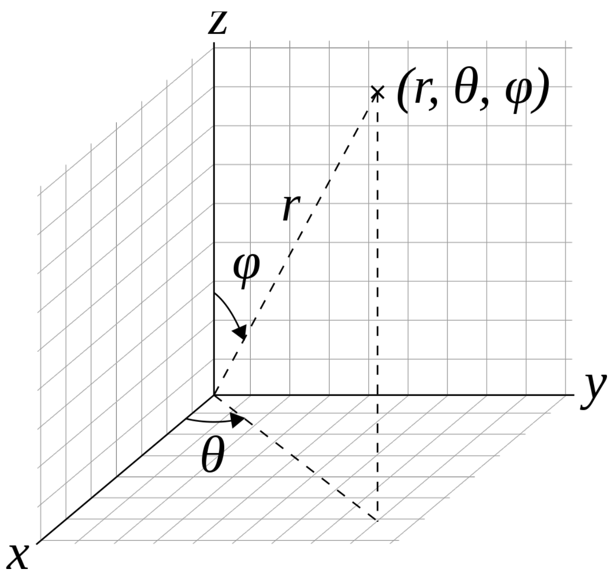
Theorem 2.0.0.4 (Stokes')**.**

$$
\iint_\Sigma (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial\Sigma} \vec{F} \cdot ds
$$

Theorem 2.0.0.5 (Gauss' Divergence Theorem)**.**

$$
\iiint_\Omega \nabla \cdot \vec{F} \, dV = \iint_{\partial\Omega} \vec{F} \cdot d\vec{S}
$$

3 Spherical



$$
\begin{aligned}
x &= r\sin(\phi)\cos(\theta) \\
y &= r\sin(\phi)\sin(\theta) \\
z &= r\cos(\phi) \\
\det D\Phi &= r^2\sin(\phi) \\
\theta &\in [0, 2\pi) \\
\phi &\in [0, \pi)
\end{aligned}
$$