Real analysis - introduction

Limitation of Riemann's integral

1. Riemann integrable function space is not have one to one correspondence with $\ell^2(\mathbb{Z})$
    - By Parseval's identity, L_2 space inner product is same as $\ell^2(Z)$ space inner produc

    $$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

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    - However, it is easy to construct elements in $\ell^2(\mathbb{Z})$ that do not correspond to functions in $\mathcal{R}$ which is collection of Riemann integrable functions
    - Note that $\ell^2(\mathbb{Z})$ is complete in its norm, while $\mathcal{R}$ is not.
2. Limits of continuous functions
    - Suppose $\{f_n\}$ is a sequence of continuous functions on $[0, 1]$. We assume that $\lim_{n\to\infty} f_n(x) = f(x)$ exists for every x, and inquire as to the nature of the limiting function f.
    - If we did not suppose the convergence is uniform, it is not sufficient to say that

    $$\int_{R^d} f(x)dx = \lim_{n\to\infty} \int_{R^d} f_n(x)dx$$

3.

$\rightarrow$ There is a need for a **new integral** that can overcome the limitations of Riemann's integral which is called "Lebegue integral"

Fundamental theorem of Calculus with Lebegue integral

- What is fundamental theorem of calculus?

$$F(x) = \int_a^x f(y)dy \qquad (1)$$

$$F(b) - F(a) = \int_a^b F'(x)dx \qquad (2)$$

- When f is integrable, we want to see if F is differentiable and above relationship is holds (1)
- We want to find What conditions on F make F' exists and above condition holds (2)

Preliminaries

Point

- Let $\mathbb{R}^d$ denote the d-dimensional Euclidean space
- A point $x \in \mathbb{R}^d$ consists of a d- tuple of real numbers

  $$x = (x_1, x_2, \cdots, x_d),\ x_i \in \mathbb{R} \text{ for } 1 \le i \le d$$
- The norm of a point is denoted by $|x|$ and is defined as the standard euclidean norm
- The distance between two points x and y is $|x - y|$

Set

- The complment of a set E in $\mathbb{R}^d$ is denoted by E^c
- The reletive complement is denoted by $E - F$ where E and F are subsets of $\mathbb{R}^d$
- The distance between two sets E and F is defined by $d(E, F) = \inf\{|x - y| : x \in E \text{ and } y \in F\}$
- The open ball in $\mathbb{R}^d$ centered at x and of radius r is defined by $B_r(x) = \{y \in \mathbb{R}^d : |x - y| < r\}$
- A set $E \subset \mathbb{R}^d$ is open , if for every $x \in E$, there exists $r > 0\ s.t\ B_r(x) \subset E$
- A set E is closed if E^c is open

  * Any union of open sets is open, while the intersection of finitely many open sets is open

  * Any intersection of closed sets is closed, union of finitely many closed sets is closed.
- A set E is bounded if there is $R > 0$ such that $E < B_R(0)$
- A bounded set E is complete if it is also closed

  * Compact sets follow the Heint-Borel covering property

  : Assume E is compact, $E \in \bigcup_{\alpha \in I} O_\alpha$ and O_α is open

  Then there are finitely many open sets $O_{\alpha_1}, O_{\alpha_2}, \cdots, O_{\alpha_n}$ such that $E \in \bigcup_{j=1}^{n} O_{\alpha_j}$

  $\rightarrow$ **Any covering of a compact set contains a finite subcovering!!**

points of set

- A point $x \in \mathbb{R}^d$ is a limit point of the set E if for every $r > 0$, $(B_r(x) - \{x\}) \cap E \neq \emptyset$

  * A limit point x does not necesarily belong to the set E
- The set of all limit points of E is denoted by E'
- An isolated point of E is a point x in E such that there is $r > 0$ with $B_r(x) \cap E = \{x\}$
- The set of all interior point is called interior of E, denoted by E^o
- The closure of E, denoted by $\bar{E}$, consists of $E \cup \bar{E}$
- The boundary of E, denoted by ∂E, is the set consist of $\bar{E} - E^o$

* A set E is closed $\iff E' \subset E \iff E = \bar{E}$

- A closed set E is perfect if E does not have any isolated points.

Rectangle and Cube

- A (closed) rectangle R in $\mathbb{R}^d$ is defined by

$$R = [a_1, b_1] \times \cdots \times [a_d, b_d], \text{ where } a_j \leq b_j, 1 \leq j \leq d$$

- Side lengths of R are $b_1 - a_1, \cdots, b_d - a_d$
- The volume of R is denoted by $|R|$ and defined by $|R| = (b_1 - a_1) \times \cdots \times (b_d - a_d)$
- A (closed) cube, usually denoted by Q, is a rectangle with the same side lengths
- A union of rectangle is said to be almost disjoint , if the interiors are disjoint

Any open sets can be represented by countable union of almost disjoint closed cube

> Lemma 1.1.1
>
> Let $R, R_j\ j = 1, \cdots, N$ be rectangles such that R_j's are almost disjount and $R = \bigcup\limits_{j=1}^{n} R_j$, then $|R| = \sum_{j=1}^{n} |R_j|$
>
> Lemma 1.1.2
>
> Let $R, R_j\ j = 1, \cdots, N$ be rectangles s.t $R \subset \bigcup_{j=1}^{n} R_j$, then $|R| = \sum_{j=1}^{n} |R_j|$

- Theorem 1.1.1

  Every open set $\mathcal{O}$ of $\mathbb{R}$ can be written uniquely as a countable union of disjoint openset.

  > $proof$
  >
  > i) For each $x \in \mathcal{O}$ let I_x denote the longest open interval containing x and contained in $\mathcal{O}$
  >
  > $I_x = (a_x, b_x)$ where $a_x = \inf\{a < x : (a, x) \subset \mathcal{O}\}$ and $b_x = \sup\{x < b : (x, b) \subset \mathcal{O}\}$
  >
  > ii) if it can be written by union of disjoint set, there is a rational that can represent each set. because rationals are countable, The sets are also countable. $\rightarrow$ It is surficient to show that it can be represented by union of disjoint sets
  >
  > iii) When $I_x \cap I_y \neq \emptyset$, $I_x = I_y$ because of maximality. -> Every open set can be represented as a set of disjoint sets.

- Theorem 1.1.2 (multi-dimensional version of thm 1.1.1)

  Every open set $\mathcal{O}$ of $\mathbb{R}^d$, $d \geq 1$, can be written as a countable union of almost disjoint cubes.

  > $proof$
  >
  > i) Define set of cubes$Q^{\frac{1}{2^{k-1}}}_{n_1^k, \cdots, n_d^k}$ that fill an open set $\mathcal{O}$

> ii) because of the definition of Q, $\cup Q \subset \mathcal{O}$
>
> iii) for any point $x \in \mathcal{O}$, one can find $Q^{\frac{1}{2^{k-1}}}_{n_1^k,\cdots,n_d^k}$ that satisfies $\frac{1}{2^{k-1}} < \delta$. because $x \in B_\delta(x) \subset \mathcal{O}$
>
> $\therefore \mathcal{O} \subset \cup Q$

$\rightarrow$ **The volume of any open sets can be calculated by summations of volumes of cube!!**