Exterior measure

- Definition 1.2.1

  Let E be any subset of $\mathbb{R}^d$. The exterior measure(= outer measure) of E is defined by

  $$m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|$$

  where the infimum is taken over all countable covering $E \subset \bigcup_{j \in \mathbb{N}} Q_j$ by closed cube

- Remark 1.2.1

  1. It is not sufficient to allow finite sums in the def of $m_*(E)$
  2. The covering by cubes in the def of $m_*(E)$ can be replaced by coverings of rectangles or balls
  3. The exterior measure of a point is zero
  4. The exterior measure of a closed cube is equal to its volume $m_*(Q) = |Q|$
  5. The exterior measure of a rectangle is equal to is volume
  6. The exterior measure of $\mathbb{R}^d$ is infinite

  $\rightarrow$ by rmk 3. and 6. m_* is a mapping function $\{E \subset \mathbb{R}^d\} \rightarrow [0, \infty]$

- Theorem 1.2.1 (properties of exterior measure)

  1. for every $\epsilon > 0,$ there exists a covering $E \subset \bigcup_{j \in \mathbb{N}} Q_j$ s.t $\sum_j |Q_j| \leq m_*(E) + \epsilon$ (by def of infimum)
  2. (Monotonicity) if $E_1 \subset E_2$, then $m_*(E_1) \leq m_*(E_2)$

     > pf
     >
     > i) Let $\{Q_{ij}\}$ be a coverings of E_i. $E_i \subset \{Q_{ij}\}$
     >
     > ii) $E_1 \sub E_2 \rightarrow $ $\{Q_{2j}\} \subset \{Q_{1j}\}$
     >
     > iii) $m_*(E_1) \leq m_*(E_2)$

  3. (countable sub-additivity) if $E \subset \bigcup_{j=1}^{N} E_j$, then $m_*(E_1) \leq m_*(E_2)$

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     >
     > i) When $m_*(E_j) = \infty$ for some j, the inequality holds
     >
     > ii) Assume that $m_*(E_j) < \infty, j \in \mathbb{N}$
     >
     > iii) $\sum_k |Q_{j,k}| \leq m_*(E_j) + \frac{\epsilon}{2^j}$ for covering $\{Q_{j,k}\}$ of E_j $(\because \text{Thm 1.2.1-1})$
     >
     > iv) $E \subset \cup_j E_j \subset \cup_j \cup_k Q_{j,k}$
     >
     > v) $m_*(E) \leq \sum_j \sum_k |Q_{j,k}| \leq \sum_j (m_*(E_j) + \frac{\epsilon}{2^j}) = \sum_j m_*(E_j) + \epsilon$

  4. Let $E \subset \mathbb{R}^d$, $m_*(E) = \inf m_*(\mathcal{O})$ where the infimum is taken over all open sets $\mathcal{O} \supset E$

     > pf
     >
     > i) $m_*(E) \leq m_*(\mathcal{O})$ by monotonicity

> ii) We can choose closed covering $\sum_j |Q_j| \le m_*(E) + \frac{\epsilon}{2}$ by def of exterior measure
>
> iii) we can choose open cube $|Q_j^o| \le |Q_j| + \frac{\epsilon}{2^{j+1}}$ $\because$ volume of open set and closed set is same
>
> iv) Set $\mathcal{O} = \bigcup_j Q_j^o$ then $m_*(\mathcal{O}) \le \sum_j |Q_j^o| \le \sum_j |Q_j| + \frac{\epsilon}{2} \le m_*(E) + \epsilon$

5. If $E = E_1 \cup E_2$, and $d(E_1, E_2) > 0$, then $m_*(E) = m_*(E_1) + m_*(E_2)$

> i) Let $d(E_1, E_2) = \delta > 0$
>
> ii) Then we can find covers of each E_j that has diameter less than δ -> It can be divided into which of the two sets are included. We can index it by $J_1 \text{ and } J_2$
>
> iii) $m_*(E_1) + m_*(E_2) \le \sum_{J_1} |Q_j| + \sum_{J_2} |Q_j| = \sum_j |Q_j| \le m_*(E) + \epsilon$ ($\ge$)
>
> iv) $m_*(E) \le m_*(E_1) + m_*(E_2)$ by sub-additivity ($\le$)

6. If a set E is countable union of almost disjoint cubes $E = \bigcup Q_j$, then $m_*(E) = \sum |Q_j|$

> i) ($\le$) holds because of sub-additivity
>
> ii) Define $|Q_j| \le |\tilde{Q}_j| + \frac{\epsilon}{2_j}$ (a smaller cube contained in Q_j). and mutually disjoint
>
> iii) $m_*(E) \ge m_*(\bigcup_j \tilde{Q}_j) = \sum_j |\tilde{Q}_j| \ge \sum_j Q_j - \epsilon$ ($\because item(5)$) $\rightarrow (\ge)$ holds