Convex Optimization - Differentiable problem

Dual problem

- Primal problem

$$\text{minimize } f(x)$$
$$\text{subject to } g_i(x) \leq 0, \quad i = 1, \cdots, m$$

$$\text{where } f, g_i \text{are comvex functions on } \mathbb{R}^n$$

- Domain : $\mathcal{D} = \textbf{dom } f \cap (\cap_{i=1}^{m} \textbf{dom } g_i)$
- Feasible : a point $x \in \mathcal{D}$ satisfies the constraints $g_i(x) \leq 0$ for $i = 1, \cdots, m$
- Lagrangian L :

$$L(x, \alpha) = f(x) + \sum_{i=1}^{m} \alpha_i g_i(x) \text{ where } \alpha_i \geq 0, \forall i$$

- (Lagrange) Dual function h

$$h(\alpha) = \inf_x \left\{ f(x) + \sum_{i=1}^{m} \alpha_i g_i(x) \right\}$$

- $h(\alpha) \leq p^\star \text{where } p^\star$ is optimal value of primal problem

$$\begin{aligned} p^\star &= \inf_{x : g_i(x) \leq 0} f(x) \\ &\geq \inf_{x : g_i(x) \leq 0} \left\{ f(x) + \sum_{i=1}^{m} \alpha_i g_i(x) \right\} \\ &\geq \inf_x \left\{ f(x) + \sum_{i=1}^{m} \alpha_i g_i(x) \right\} = h(\alpha), \forall \alpha \geq 0 \end{aligned}$$

- Lagrange Dual problem

$$\text{maximize } h(\alpha)$$
$$\text{subject to } \alpha \geq 0$$

- Weak Duality

$$d^\star \leq p^\star$$
$$\text{where } d^\star \text{is optimal value of the dual problem}$$

- Strong Duality

$$d^\star = p^\star$$
$$\text{When Slater's condition is satisfied}$$

- Slater's condition

$$\exists x \in \text{relint} \mathcal{D} \text{ s.t } g_i(x) < 0, \forall i$$

- refined Slater's condition

$$\exists x \in \text{relint}\mathcal{D}\text{s.t}g_i(x) \leq 0, \forall i, \text{for affine } g_i$$

- Complementary slackness : When Strong Duality holds, $x^\star$ and $\alpha^\star$ are primal and dual optimal point

$$\begin{aligned} f(x^\star) &= h(\alpha^\star) \ \ (\because \text{strong duality}) \\ &= \inf_x \left\{ f(x) + \sum_{i=1}^m \alpha_i^\star g_i(x) \right\} \\ &\leq f(x^\star) + \sum_{i=1}^m \alpha_i^\star g_i(x^\star) \\ &\leq f(x^\star) \end{aligned}$$

$$\therefore \sum_{i=1}^m \alpha_i^\star g_i(x^\star) = 0 \rightarrow \alpha_i^\star g_i(x^\star) = 0, \forall i [\text{Complementary Slackness}]$$

  This means that ith optimal Lagrange multiplier is zero unless the ith constraint is active at the optimum

KKT optimality conditions (Karash - Kuhn - Tucker)

- Assumption : f, g_i are convex and differentiable
- $x^\star$and $\alpha^\star$ are any primal and dual optimal points with zero duality gap

  $\iff$ Following KKT conditions are satisfied

$$\begin{aligned} \text{i)}&g_i(x^\star) \leq 0, \forall i (\text{feasibility}) \\ \text{ii)}&\alpha_i^\star \geq 0, \forall i (\text{Lagrange multiplier}) \\ \text{iii)}&\alpha_i^\star g_i(x^\star) = 0, \forall i (\text{Complementary Slackness}) \\ \text{iv)}&\nabla f(x^\star) + \sum_{i=1}^m \alpha_i^\star \nabla g_i(x^\star) = 0 (\text{First derivative have to be 0}) \end{aligned}$$