p.11 Exercises1-2,5

$V_1 = U_1 - xU_3, V_2 = U_2, V_3 = xU_1 + U_3$

(a) prove that the vectors $V_1(p), V_2(p), V_3(p)$ are linearly independent at each point of R^3

pf)

Claim: $(c_1V_1 + c_2V_2 + c_3V_3)(p)=0 \Rightarrow c_1 = c_2 = c_3 = 0$

$\forall p \in R^3, \forall c_1, c_2, c_3 \in R$

$$(c_1V_1 + c_2V_2 + c_3V_3)(p) = c_1V_1(p) + c_2V_2(p)+c_3V_3(p)$$
$$= c_1(1,0,-p_1)_p + c_2(0,1,0)_p + c_3(p_1,0,1)_p$$
$$= (c_1, 0, -c_1p_1)_p + (0, c_2, 0)_p + (c_3p_1, 0, c_3)_p$$
$$= (c_1 + c_3p_1, c_2, -c_1p_1 + c_3)_p$$
$$= (0,0,0)_p$$

$\Rightarrow c_1+c_3p_1 = 0, c_2 = 0, -c_1p_1 + c_3 = 0$

$\Rightarrow c_1 = c_2 = c_3 = 0$

$\therefore$) linearly independent

$\because$) $V_1(p)=(U_1 - xU_3)(p)$ $V_3(p)=(xU_1 + U_3)(p)$ $V_2(p) = U_2(p)$

     $= U_1(p) - x(p)U_3(p)$ $= x(p)U_1(p) + U_3(p)$ $=(0,1,0)_p$

     $= U_1(p) - p_1U_3(p)$ $= p_1U_1(p) + U_3(p)$

     $= (1,0,-p_1)_p$ $= (p_1,0,1)_p$

$\because$) $c_1(c_1 + c_3p_1) + c_3(-c_1p_1 + c_3) = (c_1)^2 + c_1c_3p_1 - c_1c_3p_1 + (c_3)^3$
$$= (c_1)^2 + (c_3)^2$$
$$= 0$$

(b) express the vector field $xU_1 + yU_2 + zU_3$ as a linear combination of V_1, V_2, V_3

pf)

$V_1 + xV_3 = (1+x^2)U_1$

$V_2 = U_2$

$V_3 - xV_1 = (1+x^2)U_3$

$\Rightarrow U_1 = \frac{1}{1+x^2}V_1 + \frac{x}{1+x^2}V_3, U_2 = V_2, U_3 = \frac{1}{1+x^2}V_3 - \frac{x}{1+x^2}V_1$

$\Rightarrow xU_1 + yU_2 + zU_3 = \frac{x}{1+x^2}V_1 + \frac{x^2}{1+x^2}V_3 + yV_2 + \frac{z}{1+x^2}V_3 - \frac{xz}{1+x^2}V_1$

       $= \frac{x(1-z)}{1+x^2}V_1 + yV_2 + \frac{x^2+z}{1+x^2}V_3$

p.15 Exercises1-3 (3,4,5)

3.

Let $V = y^2 U_1 - xU_3$ and $f = xy, g = z^3$

$(a)V[f] = y^3$

sol)

$V[f]=y^2 U_1[xy] - xU_3[xy]$

$\quad = y^2(y) - 0$

$\quad = y^3$

$(b)V[g]=-3xz^2$

sol)

$V[g]=y^2 U_1[z^3] - xU_3[z^3]$

$\quad = 0 - x(3z^2)$

$\quad = -3xz^2$

$(c)V[fg]=y^3z^3- 3x^2yz^2$

sol)

1.

$V[fg] = y^2 U_1[xyz^3] - xU_3[xyz^3]$

$\quad = y^2(yz^3) - x(3xyz^2)$

$\quad = y^3z^3 - 3x^2yz^2$

2.

$V[f]g + fV[g]=y^3(z^3) + xy(-3xz^2)$

$\qquad\qquad = y^3z^3 - 3x^2yz^2$

∵) thm3.4.(3)

$(f)V\left[V[f]\right]=0$

sol)

$V\left[V[f]\right]=V[y^3] \quad ∵)V[f] = y^3$

$\qquad = y^2 U_1[y^3] - xU_3[y^3]$

$\qquad = 0$

$(d)fV[g] - gV[f]=-3^2yz^2 - y^3z^3$

∵)

$fV[g] = (xy)(-3xz^2), \ gV[f]=(z^3)(y^3)$

$\qquad = -3x^2yz^2 \qquad\qquad = y^3z^3$

$(e)V[f^2 + g^2]=2xy^4- 6xz^5$

sol)

$V[f^2 + g^2]=y^2 U_1[x^2y^2 + z^6]- xU_3[x^2y^2 + z^6]$

$\qquad\qquad = y^2(2xy^2) - x(6z^5)$

$\qquad\qquad = 2xy^4- 6xz^5$

4.

Prove the identity $V=\sum V[x_i]U_i$, where x_1, x_2, x_3 are the natural coordinate functions

(hint: evaluate $V = \sum v_i U_i$ on x_i)

pf)

$V[x_i]=\sum_{j=1}^{3} v_j \frac{\partial x_i}{\partial x_j}=\sum_{j=1}^{3} v_j \delta_{ij}=v_i \qquad ∵) \ \delta_{ij}=\begin{cases}1, i = j\\0, i \neq j\end{cases}$

∴) $V=\sum v_i U_i=\sum V[x_i]U_i$

5.

If $V[f] = W[f]$ for every function f on R^3, prove that V=W

pf)

$V = \sum V[x_i]U_i = \sum W[x_i]U_i = W$

$\therefore) V = W$

$\because)$

$\forall p \in R^3$

$V[f](p) = V(p)[f]$

$\qquad = \sum v_i(p)\dfrac{\partial f}{\partial x_i}(p)$

$\qquad = \sum \left(v_i\dfrac{\partial f}{\partial x_i}\right)(p)$

$\qquad = \left(\sum v_i\dfrac{\partial f}{\partial x_i}\right)(p)$

$\Rightarrow V[f] = \left(\sum v_i\dfrac{\partial f}{\partial x_i}\right) \qquad \because) U_i[f] = \dfrac{\partial f}{\partial x_i}$

$\Rightarrow V = \sum v_i U_i$

$\left(\sum w_i\dfrac{\partial f}{\partial x_i}\right) = W[f] = V[f] = \left(\sum v_i\dfrac{\partial f}{\partial x_i}\right)$

$\Rightarrow \left(\sum w_i\dfrac{\partial f}{\partial x_i}\right) = \left(\sum v_i\dfrac{\partial f}{\partial x_i}\right)$

$\Rightarrow \sum w_i U_i = W = V = \sum v_i U_i$

2. Let $\phi = \frac{dx}{y}$ and $\psi = zdy$

Check the Leibnizian formula (3) of Thm6.4 in this case by computing each term separately

pf)

(i) $d(\phi \wedge \psi) = d\left(\frac{z}{y} dxdy\right) = \frac{\partial\left(\frac{z}{y}\right)}{\partial z} dzdxdy = \frac{1}{y} dzdxdy = \frac{1}{y} dxdydz$

(ii) $d\phi \wedge \psi - \phi \wedge d\psi = (\frac{d\,(dx)}{y} \wedge zdy\,) - (\frac{dx}{y} \wedge d(zdy))$

$\qquad\qquad\qquad = 0 - \frac{1}{y} \wedge dx \wedge dz \wedge dy$

$\qquad\qquad\qquad = \frac{1}{y} dxdydz$

by (i) and (ii)

$\therefore)\ \ d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi$

3. For any function f show that $d(df) = 0$

Deduce that $d(f\,dg) = df \wedge dg$

pf)

(1) $df = \sum_\square \frac{\partial f}{\partial x_i} dx_i$

$d(df) = d(\sum_\square \frac{\partial f}{\partial x_i} dx_i\ ) = \sum_{i,j} \frac{\partial^2 f}{\partial x_i\, \partial x_j} dx_j dx_i = \sum_{i<j} \left(\frac{\partial^2 f}{\partial x_j\, \partial x_i} - \frac{\partial^2 f}{\partial x_i\, \partial x_j} \right) dx_i dx_j = 0$

$\qquad =$

(2) $d(f\,dg) = df \wedge dg + f\,d(dg) = df \wedge dg \quad (\because (1)\ d(dg) = 0)$

4. Simplify the following forms

(a) $d(f\,dg + g\,df)$

Sol)

$d(f\,dg) + d(g\,df) = df \wedge dg + dg \wedge df \qquad (\because \text{e.x } 3)$

$\qquad\qquad\qquad = df \wedge dg - df \wedge dg$

$\qquad\qquad\qquad = 0$

(b) $d\left((f - g)(df + dg) \right)$

Sol)

$d(f\,df + f\,dg - g\,df - g\,dg) = d(f\,df) + d(f\,dg) - d(g\,df) - d(g\,dg)$

$\qquad\qquad\qquad\qquad = 0 + df \wedge dg - dg \wedge df - 0 \qquad (\because \text{e.x } 3)$

$\qquad\qquad\qquad\qquad = df \wedge dg + df \wedge dg$

$\qquad\qquad\qquad\qquad = 2(df \wedge dg)$

$(c) d(f\,dg \wedge g\,df)$

Sol)

$d(f\,dg \wedge g\,df) = (d(f\,dg) \wedge g\,df) - (f\,dg \wedge d(g\,df))$

$\qquad\qquad = ((df \wedge dg) \wedge g\,df) - (f\,dg \wedge (dg \wedge df))$ ($\because$ e.x 3, thm6.4)

$\qquad\qquad = 0$

$(d) d(gf\,df) + d(f\,dg)$

Sol)

$d(gf\,df) = d(gf) \wedge df = (g\,df + f\,dg) \wedge df = f\,dg \wedge df = -f\,df \wedge dg$

By e.x.3 $d(f\,dg) = df \wedge dg$

Thus $d(gf\,df) + d(f\,dg) = -f\,df \wedge dg + df \wedge dg$

p.40 Exercises1-7 (4,5,6)

4. $F(u,v) = (u^2 - v^2, 2uv)$

Find a formula for the Jacobian matrix of F at all points,

and decuce that F_{*p} is a linear isomorphism at every point of R^2 except the origin

Pf)

 Let Jacobian matrix of $F := JF$

$$JF = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix}$$

$F_{*p} : T_p(R^2 - \{0\}) \to T_{F(p)}(R^2 - \{0\})$ at $p = (u,v)$ is linear transformation and JF at $p = (u,v)$ has rank=2

Therefore F_{*p} is one-to-one

Thus F_{*p} is linear isomorphism

($\because$ linear transformation+one-to-one $\Rightarrow$ linear isomorphism)

5.

If $F: R^n \to R^m$ is a linear transformation , prove that $F_*(v_p) = F(v)_{F(p)}$

Pf)

Suppose $F = (f_1, f_2, \cdots, f_m)$ is liner transformation.

$$\begin{aligned} F_*(v_p) &= (\frac{d}{dt}(F(p+tv))|_{t=0})_{F(p)} \\ &= (\frac{d}{dt}(F(p))|_{(t=0)})_{F(p)} + (\frac{d}{dt}(tF(v))|_{t=0})_{F(p)} \quad (\because \ F(p+tv) = F(p) + tF(v)\ ) \\ &= F(v)_{F(p)} \end{aligned}$$

6.(a)

 Give an example to demonstrate that one-to-one and onto mapping need not to be a diffeomorphism

 Sol)

$f(x) = x^3$ is one-to-one and onto mapping but inverse function $f^{-1}(x) = (x)^{\frac{1}{3}}$ is not differetiable at x=0

6.(b)

Prove that if a one-to-one and onto mapping $F: R^n \to R^n$ is regular, then it is a diffeomorphism

Pf)

Let one-to-one and onto mapping $F: R^n \to R^n$ be regular

Therefore F:one-to-one and onto $\Rightarrow \exists\ F^{-1}$ and F:regular $\Leftrightarrow$ JF at p has rank=n

Thus F is diffeomorphism

P.50-51 Exercises2-1 (4,5,6)

4. Let $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3), w = (w_1, w_2, w_3)$

Prove that

(a) $u \cdot v \times w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

Pf)

$$u \cdot v \times w = (u_1, u_2, u_3) \cdot \begin{vmatrix} U_1(p) & U_2(p) & U_3(p) \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= (u_1, u_2, u_3) \cdot (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1)$$

$$= (u_1(v_2 w_3 - v_3 w_2), u_2(v_3 w_1 - v_1 w_3), u_3(v_1 w_2 - v_2 w_1))$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

(b) $u \cdot v \times w \neq 0$ iff u, v, w: linear independent

Pf)

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \neq 0 \Leftrightarrow u, v, w: \text{linear independent}$$

Thus

$u \cdot v \times w \neq 0 \Leftrightarrow u, v, w$: linear independent

(c) If any two vectors in $u \cdot v \times w$ are reversed, the product changes sign.

Pf)

By property of determinant, clear

(d) $u \cdot v \times w = u \times v \cdot w$

Pf)

$$u \cdot v \times w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= - \begin{vmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \quad (\because 4.(c))$$

$$= \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= w \cdot u \times v$$

$$= u \times v \cdot w$$

|\matrix(u_1 & u_2 & u_3 @ v_1 & v_2 & v_3 @ w_1 & w_2 & w_3)|

$\times$ \times

$\parallel$ \parallel

5. Prove that $v \times w \neq 0$ iff v, w: linear independent,

and show that $\parallel v \times w \parallel$ is the area of the parallelogram with sides v and w

Pf)

(i)

$v \times w = 0 \Leftrightarrow \parallel v \times w \parallel = \parallel v \parallel \parallel w \parallel sin(\theta) = 0 \Leftrightarrow \theta = 0, \pi$

$\Leftrightarrow v, w$: linear dependence

Thus

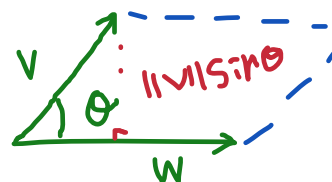$v \times w \neq 0 \Leftrightarrow v, w$: linear independent

(ii)

$\parallel v \times w \parallel^2 = (v_2 w_3 - v_3 w_2)^2 + (v_3 w_1 - v_1 w_3)^2 + (v_1 w_2 - v_2 w_1)^2$

$= (v_1^2 + v_2^2 + v_3^2)(w_1^2 + w_2^2 + w_3^2) - (v_1 w_1 + v_2 w_2 + v_3 w_3)^2$

$= \parallel v \parallel^2 \parallel w \parallel^2 - (v \cdot w)^2$

$= \parallel v \parallel^2 \parallel w \parallel^2 - (\parallel v \parallel \parallel w \parallel cos(\theta))^2$

$= \parallel v \parallel^2 \parallel w \parallel^2 (1 - cos^2(\theta))$

$= \parallel v \parallel^2 \parallel w \parallel^2 sin^2(\theta)$

$\Rightarrow \parallel v \times w \parallel = \parallel v \parallel \parallel w \parallel sin(\theta)$

Thus

$\parallel v \times w \parallel$ is the area of the parallelogram with sides v and w

6. If e_1, e_2, e_3 is a frame, show that $e_1 \cdot e_2 \times e_3 = \pm 1$

deduce that any 3×3 orthogonal matrix has determinant ± 1

Pf)

Let A=$\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}$=$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$= I_3

A$\cdot^t$A=$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$(e_1 \quad e_2 \quad e_3)$=$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow$ 1= det(A$\cdot^t$A)=det(A)det(^{t}A)=(det(A))^2

Thus

$e_1 \cdot e_2 \times e_3$= ± 1 and det(A)=± 1

p.68 Exercises2-3,8

8.(a) $\tilde{\kappa}$

i) $T' = \tilde{\kappa} N \Rightarrow T' \cdot N = \tilde{\kappa}$

ii) $N \cdot T = 0 \Rightarrow N' \cdot T + N \cdot T' = 0$

$\Rightarrow N' \cdot T = -N \cdot T'$

$\Rightarrow N' = -N \cdot T' \cdot T$

$\Rightarrow -\tilde{\kappa} \cdot T \quad (\because \quad 8.(a).(i))$

8.(b)

If $\cos \varphi = x'$, $\sin \varphi = y'$

Then

By def

$T' = \left(-\varphi' \sin \varphi, \varphi' \cos \varphi\right) \& N = (-\sin \varphi, \cos \varphi)$

$T' = \tilde{\kappa} N = (-\tilde{\kappa} \sin \varphi, \tilde{\kappa} \cos \varphi)$

$\therefore \tilde{\kappa} = \varphi'$

8.(c)

i)

$\beta(t) = (r\cos \frac{t}{r}, r\sin \frac{t}{r})$

$\Rightarrow T = (-\sin \Box \frac{t}{r}, \cos \Box \frac{t}{r}) \& N = (-\cos \Box \frac{t}{r}, -\sin \Box \frac{t}{r})$

$\Rightarrow T' = (-\frac{1}{r}\cos \Box \frac{t}{r}, -\frac{1}{r}\sin \Box \frac{t}{r})$

$\therefore \tilde{\kappa} = \frac{1}{r}$

ii)

$\beta(t) = (r\cos(-\frac{t}{r}), r\sin(-\frac{t}{r}))$

$\Rightarrow T = (\sin \Box \left(-\frac{t}{r}\right), -\cos \Box \left(-\frac{t}{r}\right)) \& N = (\cos \Box \left(-\frac{t}{r}\right), \sin \Box \left(-\frac{t}{r}\right))$

$\Rightarrow T' = (-\frac{1}{r}\cos(-\frac{t}{r}), -\frac{1}{r}\sin(-\frac{t}{r}))$

$\therefore \tilde{\kappa} = -\frac{1}{r}$

8.(d)

If $\beta = (x, y)$ then $T = \left(x', y'\right) \& N = \left(-y', x'\right)$

$\Rightarrow T' = \tilde{\kappa} N$

$\Rightarrow \tilde{\kappa} = -\frac{x''}{y'} = \frac{y''}{x'}$

So

$T' = \left(x'', y''\right) \Rightarrow \|T'\| = \sqrt{(x'')^2 + \left(y''\right)^2}$

$= \sqrt{\left(-y' \tilde{\kappa}\right)^2 + (x' \tilde{\kappa})^2}$

$= \sqrt{(\tilde{\kappa})^2}$

$= |\tilde{\kappa}|$

p.84 Exercises2-5,(1,3,4,5)

1. Consider the tangent vector $\mathbf{v} = (1, -1, 2)$ at the point $\mathbf{p} = (1, 3, -1)$. Compute $\nabla_\mathbf{v} W$ directly from the definition, where
    (a) $W = x^2 U_1 + y U_2$.　　　　　　　　(b) $W = x U_1 + x^2 U_2 - z^2 U_3$.

2. Let $V = -y U_1 + x U_3$ and $W = \cos x U_1 + \sin x U_2$. Express the following covariant derivatives in terms of U_1, U_2, U_3:
    (a) $\nabla_V W$.　　　　　　　　　　　(b) $\nabla_V V$.
    (c) $\nabla_V(z^2 W)$.　　　　　　　　　(d) $\nabla_W(V)$.
    (e) $\nabla_V(\nabla_V W)$.　　　　　　　　(f) $\nabla_V(x V - z W)$.

3. If W is a vector field with constant length $\|W\|$, prove that for any vector field V, the covariant derivative $\nabla_V W$ is everywhere orthogonal to W.

4. Let X be the special vector field $\sum x_i U_i$, where x_1, x_2, x_3 are the natural coordinate functions of $\mathbf{R}^3$. Prove that $\nabla_V X = V$ for every vector field V.

5. Let W be a vector field defined on a region containing a regular curve α. Then $t \to W(\alpha(t))$ is a vector field on α called the *restriction* of W to α and denoted by W_α.
    (a) Prove that $\nabla_{\alpha'(t)} W = (W_\alpha)'(t)$.
    (b) Deduce that the straight line in Definition 5.1 may be replaced by *any* curve with initial velocity $\mathbf{v}$. Thus the derivative Y' of a vector field Y on a curve α is (almost) $\nabla_{\alpha'} Y$.

1.(a)

$\because$ 3.2 Lemma　$v_p[f] = \sum v_i \frac{\partial f}{\partial x_i}(p)$

$\nabla_v W = \sum v[w_i] U_i(p)$

$\quad = v[x^2] U_1(p) + v[y] U_2(p) + 0$

$\quad = 2 U_1(p) - U_2(p)$

1.(b)

$\nabla_v W = \sum v[w_i] U_i(p)$

$\quad = v[x] U_1(p) + v[x^2] U_2(p) + v[-z^2] U_3(p)$

$\quad = U_1(p) + 2 U_2(p) + 4 U_3(p)$

3.

Let $W = \sum w_i U_i$ & $\|W\| = c$

$\Rightarrow W \cdot W = c^2$

$\Rightarrow 2 W' \cdot W = 0 \Rightarrow 0 = W' \cdot W = \sum w_i \frac{\partial w_i}{\partial x_i} = \sum w_i U_i[w_i]$

So $0 = \sum w_i U_i[w_i] = \sum w_i \frac{\partial w_i}{\partial x_i} = \sum V[w_i] w_i = \left(\sum V[w_i] U_i \right) \cdot \left(\sum w_i U_i \right) = (\nabla_V W) \cdot W$

Thus $\nabla_V W \perp W$

4.

Let $X = x_1 U_1 + x_2 U_2 + x_3 U_3$ & $V = v_1 U_1 + v_2 U_2 + v_3 U_3$

$\Rightarrow \nabla_V X = \sum V[x_i]U_i = \sum v_i U_i = V$

5.(a)

Let $W = \sum w_i U_i$

$\nabla_{\alpha'(t)} W = \sum \alpha'(t)[w_i]U_i(\alpha(t)) = \sum \frac{d}{dt}(w_i(\alpha))(t)U_i(\alpha(t)) = \sum \frac{d}{dt}(w_i(\alpha) U_i(\alpha))(t) = (W_\alpha)'(t)$

$\because$ Lemma 4.6 $\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t)$

5.(b)

Let $\alpha'(0) = v, \alpha(0) = p$ & $\alpha: curve$

By 5.(a)

$\nabla_v W = (W_\alpha)'(0)$

$\qquad = \nabla_{\alpha'(0)} W$

Thus

Derivative W' of a verctor field W on a curve α is $\nabla_{\alpha'} W$

1. If V and W are vector fields on $\mathbf{R}^3$ that are linearly independent at each point, show that

$$E_1 = \frac{V}{|V|}, \quad E_2 = \frac{\tilde{W}}{|\tilde{W}|}, \quad E_3 = E_1 \times E_2$$

is a frame field, where $\tilde{W} = W - (W \cdot E_1)E_1$.

Pf)

$E_1 = \dfrac{V}{||V||} \Rightarrow ||E_1|| = 1$

$E_2 = \dfrac{\tilde{W}}{||\tilde{W}||} \Rightarrow ||E_2|| = 1$

$E_3 = E_1 \times E_2 \Rightarrow ||E_3|| = 1 \because \text{Lem 2.1.8}$

Lem 2.1.8 $\quad |v \times w|^2 = v \cdot v\, w \cdot w - (v \cdot w)^2$

$E_1 \cdot E_2 = \dfrac{V}{||V||} \cdot \dfrac{\tilde{W}}{||\tilde{W}||} = \dfrac{1}{||V||\,||W||}\left(V \cdot W - \dfrac{W \cdot V}{||V||}\dfrac{V \cdot V}{||V||}\right) = \dfrac{1}{||V||\,||W||}(V \cdot W - W \cdot V) = 0$

$E_1 \cdot E_3 = E_1 \cdot E_1 \times E_2 = 0$

$E_2 \cdot E_3 = E_2 \cdot E_1 \times E_2 = 0$

$\therefore E_1, E_2, E_3 :$ frame filed

3. Find a frame field E_1, E_2, E_3 such that

$$E_1 = \cos x\, U_1 + \sin x \cos z\, U_2 + \sin x \sin z\, U_3.$$

Pf)

$E_1 = \cos x U_1 + \sin x(\cos z U_2 + \sin z U_3) \Rightarrow ||E_1|| = 1$

Let $E_2 = -\sin z U_2 + \cos z U_3$

$\Rightarrow ||E_2|| = 1$

$E_1 \cdot E_2 = -\sin x \sin z \cos z + \sin x \sin z \cos z = 0$

$E_3 = E_1 \times E_2 \Rightarrow ||E_3|| = 1$

$E_1 \cdot E_3 = E_1 \cdot E_1 \times E_2 = 0$

$E_2 \cdot E_3 = E_2 \cdot E_1 \times E_2 = 0$

1. For any function f, show that the vector fields

$$E_1 = (\sin f\, U_1 + U_2 - \cos f\, U_3)/\sqrt{2},$$
$$E_2 = (\sin f\, U_1 - U_2 - \cos f\, U_3)/\sqrt{2},$$
$$E_3 = \cos f\, U_1 + \sin f\, U_3$$

form a frame field, and find its connection forms.

2. Find the connection forms of the natural frame field U_1, U_2, U_3.

3. For any function f, show that

$$A = \begin{pmatrix} \cos^2 f & \cos f \sin f & \sin f \\ \sin f \cos f & \sin^2 f & -\cos f \\ -\sin f & \cos f & 0 \end{pmatrix}$$

is the attitude matrix of a frame field, and compute its connection forms.

4. Prove that the connection forms of the spherical frame field are

$$\omega_{12} = \cos\varphi\, d\vartheta, \quad \omega_{13} = d\varphi, \quad \omega_{23} = \sin\varphi\, d\vartheta.$$

5. If E_1, E_2, E_3 is a frame field and $W = \sum f_i E_i$, prove the *covariant derivative formula*:

$$\nabla_v W = \sum_j \left\{ V[f_j] + \sum_i f_i \omega_{ij}(V) \right\} E_j.$$

6. Let E_1, E_2, E_3 be the cylindrical frame field. If V is a vector field such that $V[r] = r$ and $V[\vartheta] = 1$, compute $\nabla_V (r \cos\vartheta E_1 + r \sin\vartheta E_3)$.

1.

$$E_1 = \frac{(\sin f U_1 + U_2 - \cos f U_3)}{\sqrt{2}} \Rightarrow |E_1| = 1$$

$$E_2 = \frac{(\sin f U_1 - U_2 - \cos f U_3)}{\sqrt{2}} \Rightarrow |E_2| = 1$$

$$E_3 = \cos f U_1 + \sin f U_3 \Rightarrow |E_3| = 1$$

$$E_1 \cdot E_2 = \frac{\sin^2 f}{2} - \frac{1}{2} + \frac{\cos^2 f}{2} = 0$$

$$E_1 \cdot E_3 = \frac{\sin f \cos f}{\sqrt{2}} - \frac{\sin f \cos f}{\sqrt{2}} = 0$$

$$E_2 \cdot E_3 = \frac{\sin f \cos f}{\sqrt{2}} - \frac{\sin f \cos f}{\sqrt{2}} = 0$$

$\therefore E_1, E_2, E_3$:frame filed

$$A = \begin{pmatrix} \frac{\sin f}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{\cos f}{\sqrt{2}} \\ \frac{\sin f}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{\cos f}{\sqrt{2}} \\ \cos f & 0 & \sin f \end{pmatrix}$$

$$\therefore \omega = dA^t A = \begin{pmatrix} \frac{\cos f}{\sqrt{2}} df & 0 & \frac{\sin f}{\sqrt{2}} df \\ \frac{\cos f}{\sqrt{2}} df & 0 & \frac{\sin f}{\sqrt{2}} df \\ -\sin f df & 0 & \cos f df \end{pmatrix} \begin{pmatrix} \frac{\sin f}{\sqrt{2}} & \frac{\sin f}{\sqrt{2}} & \cos f \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{\cos f}{\sqrt{2}} & -\frac{\cos f}{\sqrt{2}} & \sin f \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} df \\ 0 & 0 & \frac{1}{\sqrt{2}} df \\ -\frac{1}{\sqrt{2}} df & -\frac{1}{\sqrt{2}} df & 0 \end{pmatrix}$$

2.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \omega = 0$$

3.

$$A = \begin{pmatrix} \cos^2 f & \cos f \sin f & \sin f \\ \sin f \cos f & \sin^2 f & -\cos f \\ -\sin f & \cos f & 0 \end{pmatrix}$$

$$\omega = dA^t A$$

$$= \begin{pmatrix} -2\sin f \cos f\, df & \cos^2 f - \sin^2 f\, df & \cos f df \\ \cos^2 f - \sin^2 f\, df & 2\cos f \sin f\, df & \sin f df \\ -\cos f\, df & -\sin f df & 0 \end{pmatrix} \begin{pmatrix} \cos^2 f & \sin f \cos f & -\sin f \\ \cos f \sin f & \sin^2 f & \cos f \\ \sin f & -\cos f & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -df & \cos f df \\ df & 0 & \sin f df \\ -\cos f\, df & -\sin f df & 0 \end{pmatrix}$$

$\because (\cos^2 f - \sin^2 f) = 2\cos^2 f - 1 = 1 - 2\sin^2 f$

$(\cos^2 f - \sin^2 f)(\cos f \sin f) = 2\sin f \cos^3 f - \sin f \cos f$

$\cos^3 f - \sin^2 f \cos f = \cos f - 2\sin^2 f \cos f$

$E_1 = \cos^2 f\, U_1 + \cos f \sin f U_2 + \sin f U_3 \Rightarrow |E_1| = 1 \quad \because \cos^2 f \sin^2 f + \sin^2 f = 1 - \cos^4 f$

$E_2 = \sin f \cos f U_1 + \sin^2 f U_2 - \cos f U_3 \Rightarrow |E_2| = 1$

$E_3 = -\sin f U_1 + \cos f U_2 \Rightarrow |E_3| = 1$

$E_1 \cdot E_2 = \sin f \cos^3 f + \sin^3 f \cos f - \sin f \cos f = \sin f \cos f (\cos^2 f + \sin^2 f - 1) = 0$

$E_1 \cdot E_3 = -\sin f \cos^2 f + \cos^2 f \sin f = 0$

$E_2 \cdot E_3 = -\sin^2 f \cos f + \sin^2 f \cos f = 0$

$\therefore E_1, E_2, E_3$:frame filed

4.

$$A = \begin{pmatrix} \cos\theta\cos\varphi & \sin\theta\cos\varphi & \sin\varphi \\ -\sin\theta & \cos\theta & 0 \\ -\cos\theta\sin\varphi & -\sin\theta\sin\varphi & \cos\varphi \end{pmatrix}$$

$$\omega = dA^t A$$

$$= \begin{pmatrix} -\sin\theta\cos\varphi d\theta - \cos\theta\sin\varphi d\varphi & \cos\theta\cos\varphi d\theta - \sin\theta\sin\varphi d\varphi & \cos\varphi d\varphi \\ -\cos\theta d\theta & -\sin\theta d\theta & 0 \\ \sin\theta\sin\varphi d\theta - \cos\theta\cos\varphi d\varphi & -\cos\theta\sin\varphi d\theta - \sin\theta\cos\varphi d\varphi & -\sin\varphi d\varphi \end{pmatrix} \begin{pmatrix} \cos\theta\cos\varphi & -\sin\theta & -\cos\theta\sin\varphi \\ \sin\theta\cos\varphi & \cos\theta & -\sin\theta\sin\varphi \\ \sin\varphi & 0 & \cos\varphi \end{pmatrix}$$

$$= \begin{pmatrix} 0 & cos\varphi d\theta & d\varphi \\ -cos\varphi d\theta & 0 & sin\varphi d\theta \\ -d\varphi & -sin\varphi d\theta & 0 \end{pmatrix}$$

$\because \ (-sin\theta cos\varphi d\theta - cos\theta sin\varphi d\varphi)cos\theta cos\varphi = -sin\theta \ cos\theta cos^2\varphi \ d\theta - cos^2\theta sin\varphi cos\varphi \ d\varphi$

$(cos\theta cos\varphi d\theta - sin\theta sin\varphi d\varphi)(sin\theta cos\varphi) = sin\theta \ cos\theta cos^2\varphi \ d\theta - sin^2\theta sin\varphi cos\varphi \ d\varphi$

$(-sin\theta cos\varphi d\theta - cos\theta sin\varphi d\varphi)(-sin\theta) = sin^2\theta \ cos\varphi d\theta + sin\theta cos\theta sin\varphi d\varphi$

$(cos\theta cos\varphi d\theta - sin\theta sin\varphi d\varphi)(cos\theta) = cos^2\theta cos\varphi \ d\theta - sin\theta cos\theta sin\varphi d\varphi$

5.

$$\nabla_V W = \nabla_V \left(\sum_i f_i E_i \right)$$

$= \sum_i \nabla_V (f_i E_i)$

$= \sum_i (V[f_i]E_i + f_i \nabla_V E_i)$ by cor 2.5.4.(3)

$= \sum_i V[F_i]E_i + \sum_i (f_i \sum_j \omega_{ij}(V)E_j)$ by thm 2.7.2

$= \sum_j \{V[f_i] + \sum_i f_i \omega_{ij}(V)\}E_j$

6.

$\nabla_V (rcos\theta E_1 + rsin\theta E_3) = \nabla_V (rcos\theta E_1) + \nabla (rsin\theta E_3)$ by cor 2.5.4.(2)

$\qquad\qquad\qquad = V[rcos\theta]E_1 + rcos\theta \nabla_V E_1 + V[rsin\theta]E_3 + rsin\theta \nabla_V E_3$ by cor 2.5.4.(3)

$\qquad\qquad\qquad = -rsin\theta d\theta(V)E_1 + rcos\theta E_2 + rcos\theta d\theta(V)E_3$ by p.92 & thm 2.7.2

$\qquad\qquad\qquad = -rsin\theta E_1 + rcos\theta E_2 + rcos\theta E_3$ by $d\theta(V) = 1$

$\therefore$
p.92

$$A = \begin{pmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus $\omega = dA \ {}^t A = \begin{pmatrix} -sin\theta d\theta & cos\theta d\theta & 0 \\ -cos\theta d\theta & -sin\theta d\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & d\theta & 0 \\ -d\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

7.2 Thm. $\nabla_V E_i = \sum \omega_{ij} (V)E_j$

2.check all the structural equations of the spherical frame field.

by Ex.2.2-7.4

$$\omega = \begin{pmatrix} 0 & cos\varphi d\theta & d\varphi \\ -cos\varphi d\theta & 0 & sin\varphi d\theta \\ -d\varphi & -sin\varphi d\theta & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} cos\theta cos\varphi & sin\theta cos\varphi & sin\varphi \\ -sin\theta & cos\theta & 0 \\ -cos\theta sin\varphi & -sin\theta sin\varphi & cos\varphi \end{pmatrix}$$

by ex2.8.4

$dx_1 = \cos\varphi\cos\theta d\rho + \rho\cos\theta(-\sin\varphi)d\varphi + \rho\cos\varphi(-\sin\theta)d\theta$ by 1.5.5

$dx_2 = \cos\varphi\sin\theta d\rho + \rho\sin\theta(-\sin\varphi)d\varphi + \rho\cos\varphi(\cos\theta)d\theta$

$dx_3 = \sin\varphi\,d\rho + \rho\cos\varphi d\varphi$

by $\theta_i = \sum a_{ij}dx_j$

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} cos\theta cos\varphi & sin\theta cos\varphi & sin\varphi \\ -sin\theta & cos\theta & 0 \\ -cos\theta sin\varphi & -sin\theta sin\varphi & cos\varphi \end{pmatrix} \begin{pmatrix} \cos\varphi\cos\theta d\rho + \rho\cos\theta(-\sin\varphi)d\varphi + \rho\cos\varphi(-\sin\theta)d\theta \\ \cos\varphi\sin\theta d\rho + \rho\sin\theta(-\sin\varphi)d\varphi + \rho\cos\varphi(\cos\theta)d\theta \\ \sin\varphi\,d\rho + \rho\cos\varphi d\varphi \end{pmatrix}$$

$$= \begin{pmatrix} d\rho \\ \rho cos\varphi d\theta \\ \rho d\varphi \end{pmatrix}$$

From 2.8.3(1) structural equation

$d\theta_1 = \sum \omega_{1j} \wedge \theta_j$
$= \omega_{12} \wedge \theta_2 + \omega_{13} \wedge \theta_3$
$= cos\varphi d\theta \wedge \ \rho cos\varphi d\theta + d\varphi \wedge \rho d\varphi$
$= 0$

(1) the *first structural equations*:

$$d\theta_i = \sum_j \omega_{ij} \wedge \theta_j \quad (1 \leqq i \leqq 3);$$

$d\theta_2 = \sum \omega_{2j} \wedge \theta_j$
$= \omega_{21} \wedge \theta_1 + \omega_{23} \wedge \theta_3$
$= -cos\varphi d\theta \wedge d\rho + sin\varphi d\theta \wedge \rho d\varphi$
$= cos\varphi d\rho d\theta - \rho sin\varphi d\varphi d\theta$

$d\theta_3 = \sum \omega_{3j} \wedge \theta_j$
$= \omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2$
$= -d\varphi \wedge d\rho + (-sin\varphi d\theta) \wedge (\rho cos\varphi d\theta)$
$= d\rho \wedge d\varphi$
$= d\rho d\varphi$

$\therefore \quad d\theta = \begin{pmatrix} 0 \\ cos\varphi d\rho d\theta - \rho sin\varphi d\varphi d\theta \\ d\rho d\varphi \end{pmatrix}$

$\because$
$\theta_1 = cos\theta cos\varphi(\cos\varphi\cos\theta d\rho + \rho\cos\theta(-\sin\varphi)d\varphi + \rho\cos\varphi(-\sin\theta)d\theta) + sin\theta cos\varphi(\cos\varphi\sin\theta d\rho + \rho\sin\theta(-\sin\varphi)d\varphi + \rho\cos\varphi(\cos\theta)d\theta) + sin\varphi(\sin\varphi\,d\rho + \rho\cos\varphi d\varphi)$
$= (\cos^2\theta\cos^2\varphi\,d\rho - \rho\cos^2\theta\cos\varphi\sin\varphi\,d\varphi - \rho\cos^2\theta\cos\theta\sin\theta\,d\theta) + (\sin^2\theta\cos^2\varphi\,d\rho - \rho\sin^2\theta\cos\varphi\sin\varphi\,d\varphi + \rho\cos^2\varphi\cos\theta\sin\theta\,d\theta) + (\sin^2\varphi\,d\rho + \rho\cos\varphi\sin\varphi d\varphi)$
$= \cos^2\theta\cos^2\varphi\,d\rho + \sin^2\theta\cos^2\varphi\,d\rho - \rho\cos^2\theta\cos\varphi\sin\varphi\,d\varphi - \rho\sin^2\theta\cos\varphi\sin\varphi\,d\varphi - \rho\cos^2\theta\cos\theta\sin\theta\,d\theta + \rho\cos^2\varphi\cos\theta\sin\theta\,d\theta + \sin^2\varphi\,d\rho + \rho\cos\varphi\sin\varphi d\varphi$
$= \cos^2\varphi\,d\rho - \rho cos\varphi\sin\varphi\,d\varphi + \sin^2\varphi\,d\rho + \rho\cos\varphi\sin\varphi d\varphi$
$= \cos^2\varphi\,d\rho + \sin^2\varphi\,d\rho - \rho cos\varphi\sin\varphi\,d\varphi + \rho\cos\varphi\sin\varphi d\varphi$
$= d\rho$

$\theta_2 = -sin\theta(\cos\varphi\cos\theta d\rho + \rho\cos\theta(-\sin\varphi)d\varphi + \rho\cos\varphi(-\sin\theta)d\theta) + cos\theta(\cos\varphi\sin\theta d\rho + \rho\sin\theta(-\sin\varphi)d\varphi + \rho\cos\varphi(\cos\theta)d\theta)$
$= (-\cos\varphi sin\theta\cos\theta d\rho + \rho sin\theta cos\theta sin\varphi d\varphi + \rho cos\varphi\sin^2\theta d\theta) + (\cos\varphi sin\theta\cos\theta d\rho - \rho sin\theta cos\theta sin\varphi d\varphi + \rho cos\varphi\cos^2\theta d\theta)$
$= -\cos\varphi sin\theta\cos\theta d\rho + \cos\varphi sin\theta\cos\theta d\rho + \rho sin\theta cos\theta sin\varphi d\varphi - \rho sin\theta cos\theta sin\varphi d\varphi + \rho cos\varphi\sin^2\theta d\theta + \rho cos\varphi\cos^2\theta d\theta$
$= \rho cos\varphi d\theta$

θ_3
$= -cos\theta sin\varphi(\cos\varphi\cos\theta d\rho + \rho\cos\theta(-\sin\varphi)d\varphi + \rho\cos\varphi(-\sin\theta)d\theta) - sin\theta sin\varphi(\cos\varphi\sin\theta d\rho + \rho\sin\theta(-\sin\varphi)d\varphi + \rho\cos\varphi(\cos\theta)d\theta) + cos\varphi(\sin\varphi\,d\rho + \rho\cos\varphi d\varphi)$
$= (-\cos^2\theta\,cos\varphi sin\varphi\,d\rho + \rho\cos^2\theta\sin^2\varphi\,d\varphi + \rho cos\varphi sin\varphi cos\theta\sin\theta\,d\theta) + (-\sin^2\theta\,cos\varphi sin\varphi\,d\rho + \rho\sin^2\theta\sin^2\varphi\,d\varphi - \rho cos\varphi sin\varphi cos\theta\sin\theta\,d\theta) + (cos\varphi sin\varphi d\rho + \rho\cos^2\varphi d\varphi)$
$= -\cos^2\theta\,cos\varphi sin\varphi\,d\rho - \sin^2\theta\,cos\varphi sin\varphi\,d\rho + \rho\cos^2\theta\sin^2\varphi\,d\varphi + \rho\sin^2\theta\sin^2\varphi\,d\varphi + \rho cos\varphi sin\varphi cos\theta\sin\theta\,d\theta - \rho cos\varphi sin\varphi cos\theta\sin\theta\,d\theta + cos\varphi sin\varphi d\rho + \rho\cos^2\varphi d\varphi$
$= -\cos\varphi sin\varphi\,d\rho + \rho\sin^2\varphi\,d\varphi + cos\varphi sin\varphi d\rho + \rho\cos^2\varphi d\varphi$
$= -\cos\varphi sin\varphi\,d\rho + cos\varphi sin\varphi d\rho + \rho\sin^2\varphi\,d\varphi + \rho\cos^2\varphi d\varphi$
$= \rho d\varphi$

2.8.3(2) structural equation

By def 1.6.3

$d\omega_{12} = d(\cos\varphi d\theta) = d(\cos\varphi) \wedge d\theta = -\sin\varphi\,d\varphi \wedge d\theta = -d\omega_{21}$
$d\omega_{13} = d(d\varphi) = 0 = -d\omega_{31}$
$d\omega_{23} = d(\sin\varphi d\theta) = d(\sin\varphi) \wedge d\theta = \cos\varphi d\varphi \wedge d\theta = -d\omega_{32}$

p.106–107Exercises3–1,(7,8,9)

7. Prove that the set $\mathscr{E}(3)$ of all isometries of $\mathbf{R}^3$ forms a group—with composition of functions as the operation. $\mathscr{E}(3)$ is called the *Euclidean group* of order 3.

A subset H of a group G is a *subgroup* of G provided (1) if g_1 and g_2 are in H, then so is g_1g_2, (2) is g is in H, so is g^{-1}, and hence (3) the identity element e of G is in H. A subgroup H of G is automatically a group.

★ 1.3 Lem. $F, G \in \mathcal{E}(3) \Rightarrow G\,F \in \mathcal{E}(3)$

Pf)

Let G be set of all isometries of R^3

By lemma 3.1.3

$\forall g_1, g_2 \in G \Rightarrow g_1 g_2 \in G$

$\forall p, q \in R^3$, $id: R^3 \to R^3$ by $id(p) = p \ and \ id(q) = q$

Therefore

$d(p,q) = d\left(id(p), id(q)\right)$

Thus

id is isometry of R^3 and $id \in G$

$\forall g \in G, \exists g^{-1} \in G$ s.t $g \circ g^{-1} = g^{-1} \circ g = e$

$\therefore$ G forms a group with composition of function as the operation

8. Prove that the set $\mathscr{T}(3)$ of all translations of $\mathbf{R}^3$ and the set $O(3)$ of all orthogonal transformations of $\mathbf{R}^3$ are each subgroups of the Euclidean group $\mathscr{E}(3)$. $O(3)$ is called the *orthogonal group* of order 3. Which isometries of $\mathbf{R}^3$ are in both these subgroups?

It is easy to check that the results of this section, though stated for $\mathbf{R}^3$, remain valid for Euclidean spaces $\mathbf{R}^n$ of any dimension.

pf)

Let T_a, T_b be transformations of R^3

(1) $\{(T_a)(T_b)\}(p) = (T_a)(p + b)$
$= (p + b) + a$
$= p + (a + b)$
$= (T_{a+b})(p).$

Thus

$(T_a)(T_b) = T_{a+b}$ is translation.

(2) $(T_a)(p) = p + a = q$
$\Rightarrow (T_a)^{-1}(q) = p = q + (-a) = T_{-a}(q)$

Thus

$(T_a)^{-1} = T_{-a}$ is translation.

(3)$id(p) = p$ is translation. clear

$(id = T_0)$

$\therefore \mathcal{T}(3)$ is subgroup of $\mathcal{E}(3)$

Let C_1, C_2 be orthogonal transformations of R^3

(1) $(C_1 C_2)(p) \cdot (C_1 C_2)(q) = C_2(p) \cdot C_2(q) = p \cdot q$

Thus

$C_1 C_2$ is orthogonal transformation.

(2)

$$p \cdot q = (C_1 C_1^{-1})(p) \cdot (C_1 C_1^{-1})(q)$$
$$= C_1 \left(C_1^{-1}(p) \right) \cdot C_1 \left(C_1^{-1}(q) \right)$$
$$= C_1^{-1}(p) \cdot C_1^{-1}(q)$$

Thus

C_1^{-1} is orthogonal transformation

(3) $id(p) = p$ is orthogonal transformation. clear

∴ $O(3)$ is subgroup of $\mathcal{E}(3)$

Let $p = a(\neq 0), q = 0$

Then

$$p \cdot q = a \cdot 0 = 0$$
$$T_a(p) \cdot T_a(q) = T_a(a) \cdot T_a(0) = (a + a) \cdot (a + 0) = 2a \cdot a = 2\,|\square|a|\square|\square^2$$
$$\Rightarrow 2\,|\square|a|\square|\square^2 \neq 0$$

Therefore a=0

Thus *id is only isometry of R^3 which are in both of the subgroups T (3) and 0 (3)*

9. (a) Give an explicit description of an arbitrary 2×2 orthogonal matrix C. (*Hint:* Use an angle and a sign.)
    (b) Give a formula for an arbitrary isometry F of $\mathbf{R} = \mathbf{R}^1$.

R^2

(a)
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be arbitary 2×2 orthogonal matrix
then
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix}$$
$$\Rightarrow a^2 + b^2 = 1,\ ac + bd = 0, c^2 + d^2 = 1$$
$$\Rightarrow \begin{cases} a = \cos\theta \\ b = \sin\theta \\ c = \sin\theta \text{ or } -\sin\theta \\ d = -\cos\theta \text{ or } \cos\theta,\ \theta \in [0,2\pi) \end{cases}$$
thus
$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \text{or} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix},\ \theta \in [0,2\pi)$$

� $\left(C_{11} \right)_{1\times 1} \left(C_{11} \right)_{1\times 1} = 1$

$\Rightarrow C_{11} = \pm 1$

(b)
Let F be isometry of R
Then
By thm 3.1.7
$$F = T_a C$$
$$\Rightarrow F(p) = T_a C(p) = a + C(p) = a \pm p$$

$C(p) = \pm p$

★ 1.7 Thm. $F \in \mathcal{E}(3) \Rightarrow \exists\,|\,T$ and $C \in \mathcal{E}(3),\ F = T\,C$

P.110 Exercises3.2 ,4

q와 수직이고 p 를 지나는 평면
⇒C(q)와 수직이고 F(p)를 지나는 평면

4. (a) Prove that an isometry $F = TC$ carries the plane through **p** orthogonal to $\mathbf{q} \neq 0$ to the plane through $F(\mathbf{p})$ orthogonal to $C(\mathbf{q})$.

(b) If P is the plane through $(1/2, -1, 0)$ orthogonal to $(0, 1, 0)$ find an isometry $F = TC$ such that $F(P)$ is the plane through $(1, -2, 1)$ orthogonal to $(1, 0, -1)$.

Pf)
a)

Let A=$\{x: (x - p) \cdot q = 0 \}$ and $B=\{y: (y - F(p)) \cdot C(q) = 0\}$

$$x \in A \Rightarrow (F(x) - F(p)) \cdot C(q) = \{(a + C(x)) - (a + C(p))\} \cdot C(q)$$
$$= (C(x) - C(p)) \cdot C(q)$$
$$= C(x - p) \cdot C(q) \quad \text{by C : linear}$$
$$= (x - p) \cdot q$$
$$= 0$$

$\Rightarrow F(x) \in B$

Pf)
b)

By ex. 4.(a)

$p = \left(\frac{1}{2}, -1, 0\right),\ q = (0,1,0) \neq 0,\ F(p) = (1, -2, 1),\ C(q) = \frac{1}{\sqrt{2}}(1, 0, -1)$

$$\Rightarrow A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$C = {}^{t}BA = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$C(p) = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2\sqrt{2}} \\ -1 \\ 0 \end{pmatrix}$$

$a = F(p) - C(p) = (1, -2, 1) - \left(\frac{\sqrt{2}}{4}, -1, 0\right) = \left(\frac{4-\sqrt{2}}{4}, -1, 1\right)$

T_a: translation part , C: orthogonal part

3.

$\beta(s) = (a \cos \frac{s}{c}, a \sin \frac{s}{c}, \frac{bs}{c})$ where $c = (a^2 + b^2)^{\frac{1}{2}}$

$T(s) = \beta'(s) = (-\frac{a}{c} \sin \frac{s}{c}, \frac{a}{c} \cos \frac{s}{c}, \frac{b}{c})$

$T'(s) = (-\frac{a}{c^2} \cos \frac{s}{c}, -\frac{a}{c^2} \sin \frac{s}{c}, 0)$

$\kappa(s) = |T'(s)| = \frac{a}{c^2} > 0$

Since

$\mathrm{T}'(s) = \kappa(s)\mathrm{N}(s)$

$N(s) = (-\cos \frac{s}{c}, -\sin \frac{s}{c}, 0)$

$B(s) = T(s) \times N(s) = (\frac{b}{c} \sin \frac{s}{c}, -\frac{b}{c} \cos \frac{s}{c}, \frac{a}{c})$

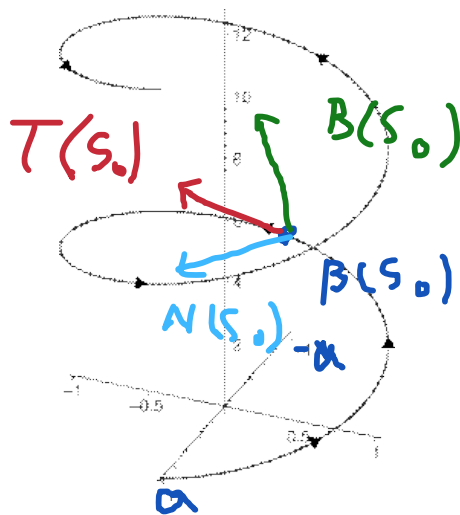$B'(s) = (\frac{b}{c^2} \cos \frac{s}{c}, \frac{b}{c^2} \sin \frac{s}{c}, 0)$

$B'(s) = -\tau(s)N(s) \Rightarrow \tau(s) = \frac{b}{c^2}$

$\therefore$

$B(s) = T(s) \times N(s) = \begin{vmatrix} U_1(\beta(s)) & U_2(\beta(s)) & U_3(\beta(s)) \\ -\frac{a}{c} \sin \frac{s}{c} & \frac{a}{c} \cos \frac{s}{c} & \frac{b}{c} \\ -\cos \frac{s}{c} & -\sin \frac{s}{c} & 0 \end{vmatrix}$
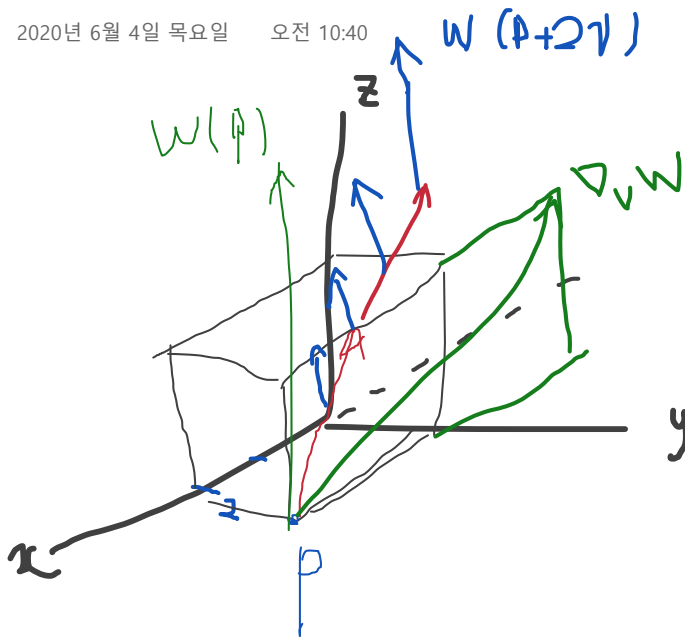
$= (\frac{b}{c} \sin \frac{s}{c})U_1(\beta(s)) - (\frac{b}{c} \cos \frac{s}{c}) U_2(\beta(s)) + \frac{a}{c}U_3(\beta(s))$

$= (\frac{b}{c} \sin \frac{s}{c}, -\frac{b}{c} \cos \frac{s}{c}, \frac{a}{c})$

$\beta(c\pi) = (a\cos \pi, a\sin \pi, b\pi)$

$= (-a, 0, b\pi)$

z

$W(P)$

$W(P+2V)$

$\nabla_V W$

y

x

P

$W = x^2 U_1 + yz U_3$

$\Rightarrow W(2,1,0) = (4,0,0)$

$$\begin{matrix} 2 & 0 \\ -\frac{1}{2} & 0 \\ & \frac{3}{2} \end{matrix} \quad 1$$

$W(P+V) = (1, 0, 2)$

$W(P+\frac{1}{2}V) = (\frac{9}{4}, 0, 1)$

$\nabla_V W = (-4, 0, 2)_P$

$W(P+2V) = (0, 0, 4)$