1.1 Euclidean Space

★ 1.1

R^3={ $p = (p_1, p_2, p_3) \mid p_i \in R^1$} with two operations

$p + q = (p_1 + q_1, p_2 + q_2, p_3 + q_3)$

$ap = (ap_1, ap_2, ap_3)$

is a vector space.

★ 1.2

x : $R^3 \to R^1$ by $x(p) = p_1$,

$x_1 = x \;\; x_2 = y \;\; x_3 = z$

$p = (p_1, p_2, p_3) = \left(x_1(p), x_2(p), x_3(p)\right)$

★ 1.3

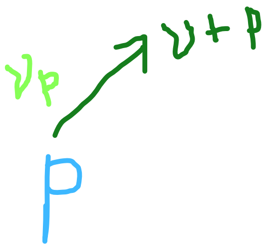
f : $R^3 \to R^1$ is differentiable if

∃ $\frac{\partial f}{\partial x_i}$ and $\frac{\partial f}{\partial x_i}$ is continuous for all i

$(f + g)(p) = f(p) + g(p), (fg)(p) = f(p)g(p)$

1.2 Tangent vectors

2020년 4월 9일 목요일 오후 1:21

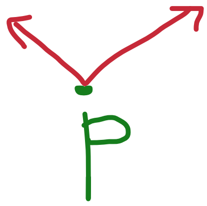


★ 2.1
A tangent vector v_p to R^3
v: vector, p:작용점

★ Tangent bundle $T(R^3) = \cup_{p \in R^3} T_p(R^3)$
tangent space들로 이루어짐

★ 2.2 The tangent space of R^3 at p

$T_p(R^3) = \{ v_p | \ v \in R^3 \}$



★ 2.3
A vector field on R^3 is a function
$V : R^3 \to T(R^3)$by $V(p) \in T_p(R^3)$

★ The set of all vector fields on $R^3$with two operations
$(V + W) = V(p) + W(p)$,
$(fV)(p) = f(p)V(p)$

★ 2.4
The natural frame field on R^3
$U_1(p) = (1,0,0)_p$
Basis ⇒ lineary independent, span

$U_i \in \mathfrak{X}(\mathbb{R}^3)$

★ 2.5 (Spans) $V = \sum v_i U_i$
pf)$\forall p$, $V(p) = \big(v_1(p), v_2(p), v_3(p)\big)_p$

$$
\begin{aligned}
&= v_1(p)(1,0,0)_p + v_2(p)(0,1,0)_p + v_3(p)(0,0,1)_p \\
&= v_1(p)U_1(p) + v_2(p)U_2(p) + v_3(p)U_3(p) \\
&= (v_1U_1)(p) + (v_2U_2)(p) + (v_3U_3)(p) \\
&= (v_1U_1 + v_2U_2 + v_3U_3)(p)
\end{aligned}
$$

$\therefore \mathfrak{X}(R^3) = \langle \{U_i\} \rangle$

★ (Linearly independent)
$\forall p$, $(f_1U_1 + f_2U_2 + f_3U_3)(p) = f_1(p)U_1(p) + f_2(p)U_2(p) + f_3(p)U_3(p)$

$$
\begin{aligned}
&= f_1(p)(1,0,0)_p + f_2(p)(0,1,0)_p + f_3(p)(0,0,1)_p \\
&= \big(f_1(p), f_2(p), f_3(p)\big)_p \\
&= (0,0,0)_p
\end{aligned}
$$

$\Rightarrow f_1(p) = 0, f_2(p) = 0, f_3(p) = 0$
$\Rightarrow f_1 = f_2 = f_3 = 0$

1.3 Directional Derivatives

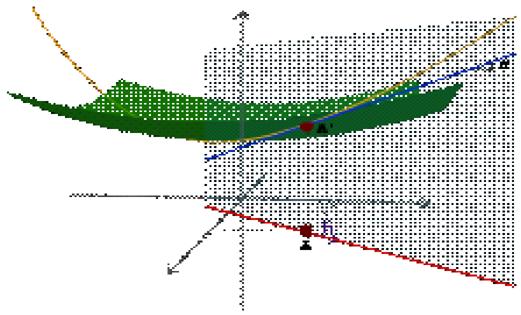
2020년 4월 16일 목요일　　오전 11:00

★ Def 3.1

$f : R^3 \to R$ and $v_p \in T_p(R^3)$

$v_p[f] \ = \ \frac{d}{dt}\left(f(p+tv)\right)\Big|_{t=0}$ is the $derivative\ of\ f\ w.r.t. v_p$

$= \lim_{t\to o} \frac{f(p+tv)-f(p)}{t} = D_v f(p) = Df(p)(v)$ <해기 p.33 2-29>



★ Lemma 3.2

$v_p[f] = \sum v_i \frac{\partial f}{\partial x_i}(p)$

pf) $\alpha(t) \ = \ p+tv \ = \ (p_1 + tv_1, p_2 + tv_2, p_3 + tv_3)$ v_i

$\Rightarrow \ v_p[f] \ = (f \circ \alpha)'(0)$

$= \ f'(\alpha(0)) \cdot \alpha'(0)$

$= \left(\frac{\partial f}{\partial x_1}(p), \ \frac{\partial f}{\partial x_2}(p), \ \frac{\partial f}{\partial x_3}(p)\right)\begin{pmatrix}\alpha_1'(0)\\ \alpha_2'(0)\\ \alpha_3'(0)\end{pmatrix}$

★ thm 3.3 (1)

$(av_p+bw_p)[f] = (av_1 + bw_1, av_2 + bw_2, av_3 + bw_3)_p[f]$

$= \sum \ (av_i + bw_i) \ \frac{\partial f}{\partial x_i} \ (p)$

$= a\sum v_i \frac{\partial f}{\partial x_i}(p) + b\sum w_i \frac{\partial f}{\partial x_i} \ (p)$

$= \ av_p[f] + bw_p[f]$

★ Thm 3.3(2)

$v_p[af + bg] \ = \sum v_i \frac{\partial}{\partial x_i}(af \ + bg)(p)$

$= a\sum v_i \frac{\partial f}{\partial x_i}(p) + b\sum v_i \ \frac{\partial g}{\partial x_i} \ (p)$

$= av_p[f] + \ bv_p[g]$

★ Thm 3.3(3)

$v_p[fg] = \sum v_i (\frac{\partial f}{\partial x_i}(p) \cdot g(p) + f(p) \cdot \frac{\partial g}{\partial x_i}(p))$

$= (\sum v_i \frac{\partial f}{\partial x_i}(p))g(p) + f(p)\left(\sum v_i \frac{\partial g}{\partial x_i}(p)\right)$

$= v_p[f] \cdot g(p) \ + f(p) \cdot v_p[g]$

★ Define $V[f]$: $R^3 \to R^1$ by $(V[f])(p) = \ V(p)[f]$

$\forall p \ \in R^3$

$V[f](p) = \ V(p)[f]$

$= \sum v_i(p) \frac{\partial f}{\partial x_i} \ (p)$

$= \sum \left(v_i \ \frac{\partial f}{\partial x_i}\right)(p)$

$= \left(\sum v_i \frac{\partial f}{\partial x_i}\right)(p)$

$\Rightarrow \ V[f] = \ \left(\sum v_i \frac{\partial f}{\partial x_i}\right)$ $\because)\ U_i[f] = \frac{\partial f}{\partial x_i}$

$\Rightarrow \ V \ = \sum v_i U_i$

★ $\forall p$, $U_1(p)[f] = \frac{d}{dt}\left(f(p_1 + t, p_2, p_3)\right)\Big|_{\{t=0\}} = \frac{\partial f}{\partial x_1}(p)$

$\Rightarrow$ $U_1[f] = \frac{\partial f}{\partial x_1}$

★ Thm 3.4 (1)

pf)

$\forall p \in R^3$

$((fV + gW)[h])(p) = (fV + gW)(p)[h]$

$= (f(p)V(p) + g(p)W(p))[h]$

$= f(p)V(p)[h] + g(p)W(p)[h]$ ($\because$ thm 3.3(1))

$= f(p)(V[h])(p) + g(p)(W[h])(p)$

$= (fV[h])(p) + (gW[h])(p)$

$= (fV[h] + gW[h])(p)$

$\therefore$) $(fV + gW)[h] = fV[h] + gW[h]$

★ Thm3.4 (2)

$\forall a, b \in R$, $\forall p \in R^3$

$V[af + bg](p) = V(p)[af + bg]$

$= aV(p)[f] + bV(p)[g]$ by theorem 3.3 (2)

$= (aV[f] + bV[g])(p)$

$\therefore V[af + bg] = aV[f] + bV[g]$

★ thm3.4 (3) $V[fg] = V[f] \cdot g + f\,V[g]$

$\forall p$

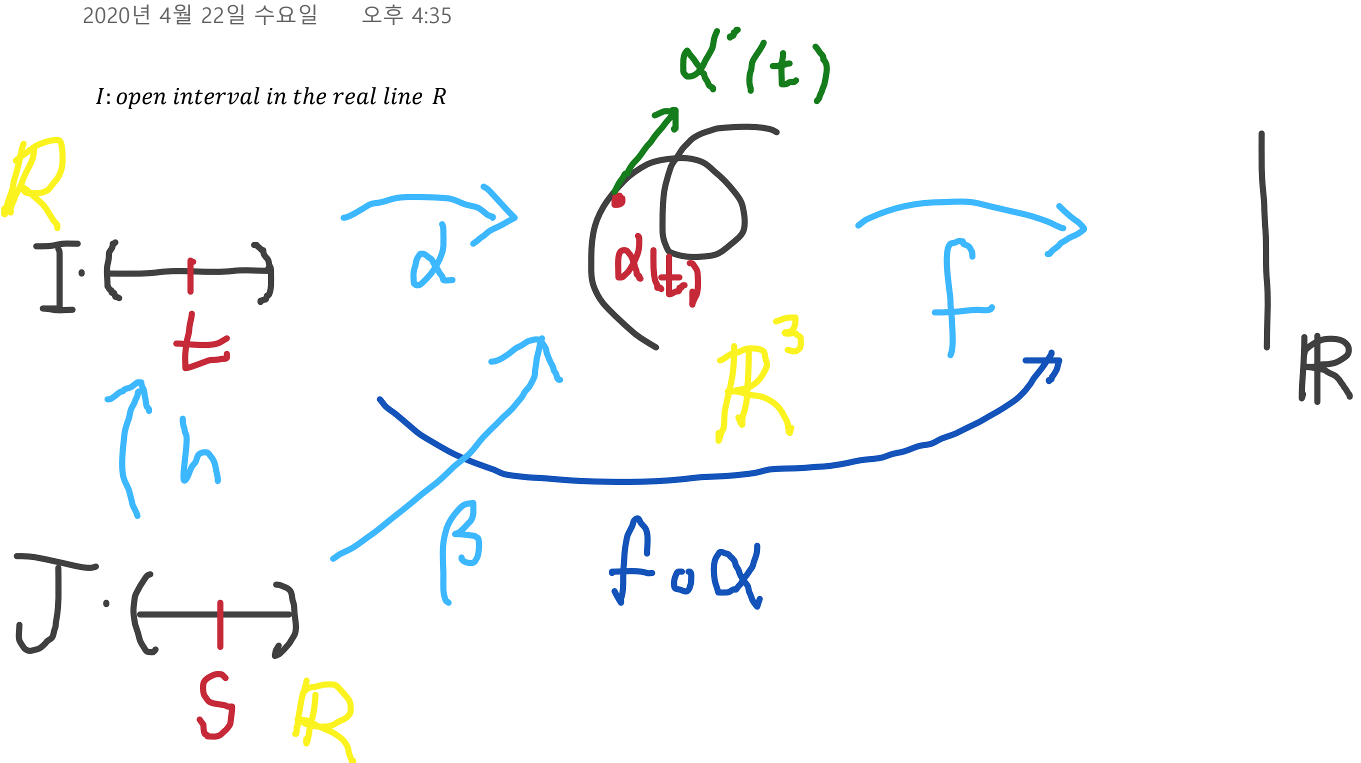
$V(p)[fg] = V(p)[f] \cdot g(p) + f(p) \cdot V(p)[g]$

$= (V[f] \cdot g + f \cdot V[g])(p)$ by 3.3 (3)

1.4 Curves in R^3

2020년 4월 22일 수요일   오후 4:35

I: *open interval in the real line* R



★ Def. 4.1 $\alpha: I \to R^3$ is a *curve* by $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$

Ex. (1) $\alpha(t) = p + tq$ → 직선

(2) *Helix* $\alpha(t) = (a\cos t, a\sin t, bt)$ →

★ Def. 4.3 The *velocity vector* of α at t

$$\alpha'(t) = \left(\frac{d\alpha_1}{dt}(t), \frac{d\alpha_2}{dt}(t), \frac{d\alpha_3}{dt}(t)\right) \in T_{\alpha(t)}(R^3)$$ ($\alpha(t)$의 tangent vector)

$(v_1, v_2, v_3)_p = \Sigma v_i U_i(p)$를 적용하면 $\alpha'(t) = \Sigma \frac{d\alpha_i}{dt}(t) U_i(\alpha(t))$

★ Def. 4.4 A *reparametrization* (재매개화) of α by h is $\beta = \alpha(h): J \to R^3$

★ Lemma 4.5

$\beta'(s) = \left(\frac{dh}{ds}\right)(s)\alpha'(h(s)) = h'(s)\alpha'(h(s))$

Pf) $\beta'(s) = \alpha'(h(s)) \cdot h'(s)$ by Chain Rule

Jacobian

-교수님-

$3\times1 = (3\times1)(1\times1)$

중요!!

pf)

If $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ then $\beta(s) = \alpha(h(s)) = (\alpha_1(h(s)), \alpha_2(h(s)), \alpha_3(h(s)))$ by Chain Rule & definition of velocity

$\beta'(s) = \alpha(h)'(s)$

$= (\alpha_1'(h(s)) \cdot h'(s), \alpha_2'(h(s)) \cdot h'(s), \alpha_3'(h(s)) \cdot h'(s))$

$= \alpha'(h(s)) \cdot h'(s)$

★ Lemma 4.6

$\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t) = (f \circ \alpha)'(t)$

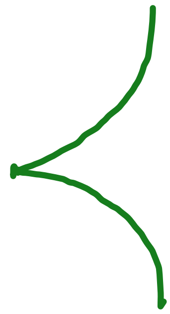
Pf) $\alpha'(t)[f] = \sum \alpha_i'(t)\,\frac{\partial f}{\partial x_i}(\alpha(t)) = f'(\alpha(t)) \cdot \alpha'(t)$ by lemma 3.2 &Chain Rule

Lemma 3.2

$$v_p[f] = \sum v_i \frac{\partial f}{\partial x_i}(p)$$

regular : 모든 속도벡터가 0이 아닌 조건

⇒ 모서리, 첨점이 없음



곡선의 특이점 : cusp

1.5 1−Forms

2020년 4월 22일 수요일 오후 4:36

★ Def 5.1

A $1-form\ \ \phi$ on R^3

$\phi \in \left(T(R^3)\right)^* = \{\phi \mid \phi: T(R^3) \to R$ is linear$\}$

$\forall$ p , $\phi_p \in \left(T_p(R^3)\right)^*$ dual space (tensor space) $\mathcal{T}^{\,1}\left(T_p(R^3)\right)$ $= \Lambda^1\left(T_p\left(\mathbb{R}^3\right)\right)$

0-form

$f\phi = f \wedge \phi$

Two operations

(1) $(\phi + \varphi)(v) = \phi(v) + \varphi(v)$

(2) $(f\phi)(v_p) = f(p)\phi_p(v_p) = f(p)\phi(v_p)$ f는 p에만 의존

★ Define $\phi(V)$: $R^3 \to R$ by $(\phi(V))(p) = \phi\left(V(p)\right)$ for vector fields

$\mathfrak{X}(\mathbb{R}^3) = \langle U_i \rangle$

$T(\mathbb{R}^3)^* = \langle dx_i \rangle$

Then we have

(1) $\phi(fV + gW) = f\phi(V) + g\phi(W)$

∵)

$\left(\phi(fV + gW)\right)(p) = \phi((fV + gW)(p))$

$= \phi\left(f(p)V(p) + g(p)W(p)\right)$

$= \phi\left(f(p)V(p)\right) + \phi\left(g(p)W(p)\right)$ linear

$= f(p)\phi\left(V(p)\right) + g(p)\phi\left(W(p)\right)$ linear

$= (f\phi(V))(p) + (g\phi(W))(p)$

(2) $(f\phi + g\varphi)(V) = f\phi(V) + g\varphi(V)$

∵)

$((f\phi + g\varphi)(V))(p) = (f\phi + g\varphi)\left(V(p)\right)$

$= f(p)\phi\left(V(p)\right) + g(p)\varphi\left(V(p)\right)$ linear& Def5.1

$= (f\phi(V))(p) + (g\varphi(V))(p)$

★ Def 5.2 The $differential$ df of f is the 1-form :

$df(v_p) = v_p[f] = D_v f(p) = Df(p)(v)$

★ Example 5.3 (1)

$dx_i(v_p) = v_p[x_i] = \sum v_j \frac{\partial x_i}{\partial x_j}(p) = \sum v_j \delta_{ij} = v_i$

$\forall v_p$에 대한 dx_i의 값은 벡터의 i번째 값, 작용점 p에는 전혀 의존 안함

(1)에 의해

$\left(dx_i(U_j)\right)(p) = dx_i\left(U_j(p)\right) = \delta_{ij}$ 임을 안다.

★ Example 5.3 (2)

$\psi(v_p) = (\sum f_i dx_i)(v_p) = \sum f_i(p) dx_i(v_p) = \sum f_i(p) v_i$

★ Lemma 5.4 $(spans)$

$\phi = \sum f_i dx_i$ where $f_i = \phi(U_i)$

Pf) $\phi(v_p) = \phi\left(\sum v_i U_i(p)\right)$

$= \sum\square\ v_i \phi\left(U_i(p)\right)$

$= \sum\left(\phi(U_i)\right)(p) dx_i(v_p)$

$= \sum\square\ (f_i dx_i)(v_p)$

$\Rightarrow \langle dx_i \rangle = \left(T(\mathbb{R}^3)\right)^*$

Note) dx_i is $linearly\ ind.$

$\sum f_i dx_i = 0 \Rightarrow \forall\ U_j, 0 = (\sum f_i dx_i)(U_j) = f_j$

★ Cor 5.5

$df = \sum\square \frac{\partial f}{\partial x_i} dx_i$

Pf) $df(v_p) = \sum v_i \frac{\partial f}{\partial x_i}(p)$ Lemma 3.2

Note) $d(f + g) = df + dg$ by 3.3(2)

★ Lemma 5.6

$d(fg) = gdf + fdg$

Pf)Using 5.5

$$d(fg) = \sum \frac{\partial(fg)}{\partial x_i} dx_i$$
$$= \sum \left(\frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i}\right) dx_i$$
$$= g\left(\sum \frac{\partial f}{\partial x_i} dx_i\right) + f\left(\sum \frac{\partial g}{\partial x_i} dx_i\right)$$
$$= gdf + fdg$$

★ Lem 5.7

$d(h \circ f) = h'(f)df$

Pf) $(h \circ f)'(p) = h'\left(f(p)\right) \cdot f'(p)$ by C. R.

$$= \left(h'\left(f(p)\right)\right) \cdot \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)(p)$$

$$d(h \circ f) = \sum \frac{\partial (h \circ f)}{\partial x_i} dx_i = \sum h'(f) \frac{\partial f}{\partial x_i} dx_i = h'(f)df$$

★ $x^2 = h \circ x : R^3 \to R^1 \to R^1$ by $p \mapsto x(p) \mapsto \left(x(p)\right)^2$

Thus $d(x^2) = 2x\,dx$

1.6 Differential 1-Forms

2020년 4월 28일 화요일 오후 11:05

★ Note) $dx_i dx_j = dx_i \wedge dx_j = (1+1)!\ \mathrm{Alt}(dx_i \otimes dx_j)$

$dx_i dx_j = -dx_j dx_i$

$\Lambda^1(T(R^3)) = <\{dx_1, dx_2, dx_3\}> \quad \binom{3}{1}$

$\Lambda^2(T(R^3)) = <\{dxdy, dxdz, dydz\}> \binom{3}{2}$

$\Lambda^3(T(R^3)) = <\{dx\, dy\, dz\}> \quad \binom{3}{3}$

$dx_i dx_i = 0$

홀 : 부호 O

짝 : 그대로

★ Lem 6.2

$\phi, \psi \in \Lambda^1(T(R^3))$

$\phi \wedge \psi = (\sum f_i dx_i) \wedge (\sum g_i dx_i) = \sum f_i g_j dx_i dx_j = -\sum g_j f_i dx_j dx_i = -\psi \wedge \phi$

★ Def 6.3

The *exterior derivative* of ϕ

$d\phi = d(\sum f_i dx_i) = \sum df_i \wedge dx_i \in \Lambda^2(T(R^3))$

$d\phi = d\left(\sum_{i<j} f_{ij} dx_i dx_j\right) = \sum df_{ij} \wedge dx_i dx_j$

★ Thm 6.4

(1) $d(fg) = g\,df + f\,dg$

(2) $d(f\phi) = df \wedge \phi + f\,d\phi$

(3) $d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi$

Pf)

(3)Suppose $\phi = f\,dx, \psi = g\,dy$

$d(\phi \wedge \psi) = d(fg\,dxdy) = d(fg) \wedge dxdy$

$= (g\,df + f\,dg) \wedge dxdy$

$= (g(\sum \frac{\partial f}{\partial x_i} dx_i) + f(\sum \frac{\partial g}{\partial x_i} dx_i)) \wedge dxdy$

$= (g\frac{\partial f}{\partial z} dz + f\frac{\partial g}{\partial z} dz) \wedge dxdy$

$= (g\frac{\partial f}{\partial z} + f\frac{\partial g}{\partial z})\, dxdydz$

$d\phi \wedge \psi = \frac{\partial f}{\partial z} dzdx \wedge g dy = g\frac{\partial f}{\partial z} dxdydz$

$\phi \wedge d\psi = f dx \wedge \frac{\partial g}{\partial z} dzdy = -f\frac{\partial g}{\partial z} dxdydz$

$dx \wedge dy$ 있어서

dz 만 ☆

$w \wedge \eta = (-1)^{\boxed{kl}} \eta \wedge w$

$w \in \Lambda^k$

$\eta \in \Lambda^l$

(2)

Suppose $\phi = gdx$

$d(f\phi)=d(fg\,dx)$

$=d(fg)\wedge dx$ ∵) def 6.3

$=(g\,df + f\,dg)\wedge dx$ ∵) lemma 5.6

$=df \wedge g dx + f(dg \wedge dx)$

$=df \wedge \phi + f\,d\phi$

재밌는 이야기

$f \circ f = f^2 = f \cdot f$

합성: self map

곱 :function

$f \quad \mathbb{R}^3 \to \mathbb{R}$

$\Rightarrow$ function

$f: X \to X$

$\Rightarrow$ self map

1.7 mappings

2020년 4월 29일 수요일 오후 8:47

★ Def 7.1 A differentiable function $F : R^n \to R^m$ by
$F(p) = \big(f_1(p), \dots, f_m(p)\big)$ is a *mapping*

★ 3.1 Def
$f : R^3 \to R$ and $v_p \in T_p(R^3)$
$v_p[f] = \frac{d}{dt}\big(f(p+tv)\big)\Big|_{t=0}$ is the *derivative of f w.r.t.* v_p

$= \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t} = D_v f(p) = Df(p)(v)$

★ 7.4 Def. $F_*: T(R^n) \to T(R^m)$ by $F_*(v) = \frac{d}{dt} F(p + tv)\,|_{t=0}$ is the *tangent map* of F

★ 7.5 Pro. $F_*(v) = (v[f_1], \dots, v[f_m])$ $= \sum v[f_i]\, U_i(F(p))$

Pf)

m=3

$\beta(t) = F(p+tv) = \big(f_1(p+tv), f_2(p+tv), f_3(p+tv)\big)$

by Def

$F_*(v) = \beta'(0)$

But

$\frac{d}{dt}\big(f_i(p+tv)\big)\,|_{t=0} = v[f_i]$

Thus

$F_*(v) = (v[f_1], v[f_2], v[f_3])|_{\beta(0)}$ and $\beta(0) = F(p)$

$$\begin{tikzcd} R^n \arrow[r, "F"', "DF(p)\ \ell"] \arrow[d, "\ell.\,\mathrm{iso}"'] & R^m \arrow[d, "\ell\ \mathrm{iso}"] \\ T_p(R^n) \arrow[r, "F_{*p}", "\ell"'] & T_{F(p)}(R^m) \end{tikzcd}$$

J열

$$\left(\frac{\partial f_i}{\partial x_j}(p)\right) = F'(p)$$

★ Cor 7.6 $F_{*_p} : T_p(R^n) \to T_{F(p)}(R^m)$ is linear

Pf)

by thm3.3.(1)

$F_*(av + bw) = aF_*(v) + bF_*(w)$

★ Cor 7.7 $\beta = F(\alpha) \Rightarrow \beta' = F_*(\alpha')$

pf) $\beta' = (\frac{d}{dt} f_1(\alpha), \dots, \frac{d}{dt} f_m(\alpha))$

$= (\alpha'[f_1], \dots, \alpha'[f_m])$ by 4.6

$= F_*(\alpha')$ by 7.5

★ Cor 7.8 $F_*\big(U_j(p)\big) = \sum_i \frac{\partial f}{\partial x_j}(p) U_i\big(F(p)\big)$ *Jacobian matrix* of F at p

★ Def 7.9 F is regular if

(1) F_{*_p} is 1-1

(2) Ker of $F_* = \{0\}$

(3) rank n

$F\ R^n \to R^n\ \det F'(p) \neq 0$

★ Thm 7.10

Inverse Function Theorem

2.1 Dot product

2020년 5월 4일 월요일 오후 7:24

★T(R^n)

● 2학년 때는 R^n 지금은 $T(R^n)$에서

★ Def 1.3 $v_p \cdot w_p = v \cdot w$

★ Def 1.4 $\{e_1, e_2, e_3\} \subset T_p(R^3)$ is a $frame$ at p if $e_i \cdot e_j = \delta_{ij}$

★ Thm 1.5 ($orthonormal\ expansion$) $v = \sum(v \cdot e_i)e_i$

pf) $v = \sum c_i e_i \Rightarrow v \cdot e_j = c_j$

★ Def 1.6 The $attitude\ matrix$ A of the frame $\{e_1, e_2, e_3\} \subset T_p(R^3)$ is

$$A = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = (a_{ij})$$

Note) A is an orthonormal matrix and $|A^t A| = 1 \Rightarrow {}^tA = A^{-1}$

★ Def 1.7 The $cross\ product$ of v and w is

$$v \times w = \begin{vmatrix} U_1(p) & U_2(p) & U_3(p) \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

★ Lem 1.8 $|v \times w|^2 = v \cdot v\, w \cdot w - (v \cdot w)^2$

Pf) Let $v \times w = \sum c_i U_i(p)$

$$\Rightarrow v \cdot (v \times w) = \sum v_i c_i = \begin{vmatrix} v \\ v \\ w \end{vmatrix} = 0 \text{ and } w \cdot (v \times w) = 0$$

$$v \cdot v\, w \cdot w - (v \cdot w)^2 = (\sum v_i^2)(\sum w_j^2) - \left(\sum v_i w_i\right)^2$$
$$= \sum v_i^2 w_j^2 - (\sum v_i^2 w_i^2 + 2 \sum_{(i<j)} v_i w_i v_j w_j)$$
$$= \sum_{i \neq j} v_i^2 w_j^2 - 2 \sum_{i<j} v_i w_i v_j w_j$$

$$|v \times w|^2 = \left(v_2 w_3 - v_3 w_2\right)^2 + \left(v_1 w_3 - v_3 w_1\right)^2 + \left(v_1 w_2 - v_2 w_1\right)^2 \quad Q.E.D.$$

Note) $|v \times w|^2 = |v|^2 |w|^2 - (|v||w|\cos\theta)^2$ by 제2코사인법칙

Thus $|v \times w| = |v||w|\sin\theta$

2.2 curves

2020년 5월 4일 월요일 오후 7:25

The *speed* function v of a curve α is given by $v = |\alpha'|$

The *arc length* of α is $\int_a^b |\alpha'(t)|\, dt$

★ Thm 2.1 α is regular $\Rightarrow$ $\exists$ β (a reparametrization) , $|\beta'| = 1$

Pf) Consider $s(t) = \int_a^t |\alpha'(u)| du$

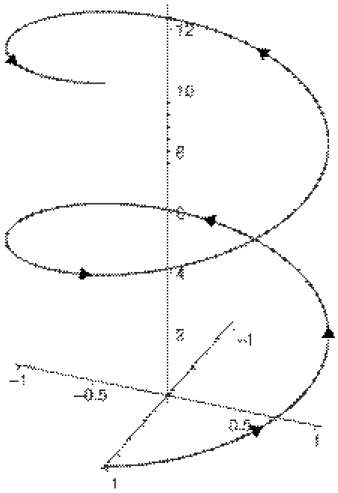
$ds/dt = |\alpha'| > 0$ by regular

$\Rightarrow$ $\exists$ $t = t(s)$ (an inverse ft) , $dt/ds = 1/(ds/dt) > 0$

Let $\beta(s) = \alpha(t(s))$

$\Rightarrow \beta'(s) = (dt\,/\,ds)(s)\ \alpha'(t(s))$ by 4.5

$\Rightarrow |\beta'(s)| = (dt\,/\,ds)(s)\,|\,\alpha'(t(s))\,| = 1$



★ Example) The helix $\alpha(t) = (a\cos t, a\sin t, bt)$

$\Rightarrow \alpha'(t) = (-a\,s\,n\,t, a\ \cos t,\ b)$

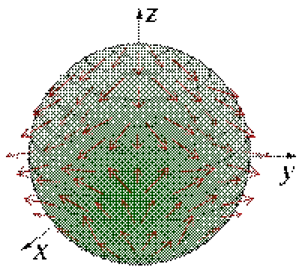
$\Rightarrow |\alpha'(t)| = (a^2 + b^2)^{1/2} = c > 0$

$s(t) = \int_0^t c\,du = ct$ has an inverse ft $t(s) = s/c$

Thus $\beta(s) = \alpha(s/c) = (a\cos \frac{s}{c}, a\ \sin\frac{s}{c}, \frac{bs}{c})$

★ Def 2.2 A vector field Y on α :

$Y : I \to T(R^3)$ by $Y(t) \in T_{\alpha(t)}(R^3)$



A vector field on a sphere

Note) 1) $Y(t) = (y_1(t), y_2(t), y_3(t))_{\alpha(t)} = \sum y_i(t) U_i(\alpha(t))$

2) $Y(t) = (c_1, c_2, c_3)_{\alpha(t)} = \sum c_i U_i(\alpha(t))$ " *parallel* "(평행)

3) $(Y\cdot Z)' = Y'\cdot Z + Y\cdot Z'$

$(Y\cdot Z)'(t) = \frac{d}{dt}(Y\cdot Z)(t)$

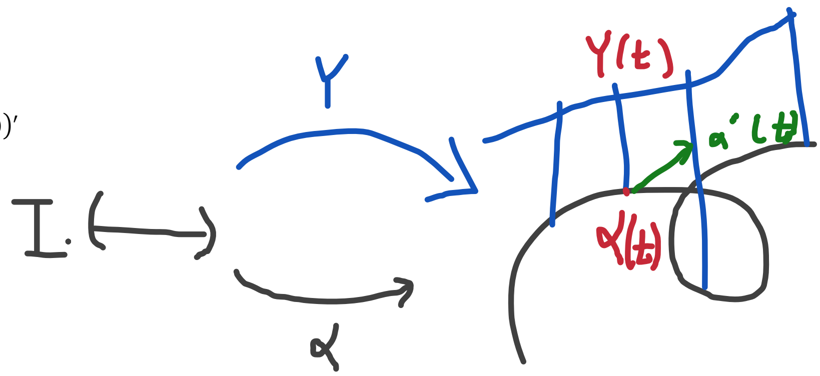
$= \frac{d}{dt}(Y(t)\cdot Z(t))$

$= (y_1(t)z_1(t) + y_2(t)z_2(t) + y_3(t)z_3(t))'$

$= \sum (y_i'(t)z_j(t) + y_i(t)z_j'(t))$

$= (Y'\cdot Z + Y\cdot Z')(t)$

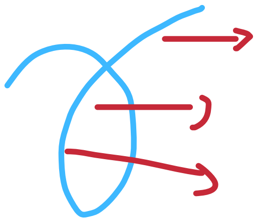
$\therefore$ $(Y\cdot Z)' = Y'\cdot Z + Y\cdot Z'$



★ Lem 2.3 (1) α is constant \iff $\alpha' = 0$

(2) $\alpha(t) = p + tv \iff \alpha'' = 0$

(3) Y is parallel \iff $Y' = 0$



$Y(t) = (\underline{\underline{1}}, 0, 0)_{\alpha(t)}$

x방향

Parallel

평행

2.3 The frenet formulas

2020년 5월 9일 토요일 오후 2:39

★ Note

$\beta : I \to R^3$ is a unit speed curve. $|\beta'(s)| = 1$ (단위속력곡선)

$T = \beta'$ is the unit tangent vector field of β (단위접벡터장)

T' is the curvature vector field of β (곡률벡터장)

$\kappa(s) = |T'(s)| > 0$ is the curvature function of β (곡률함수)

$N = \frac{T'}{\kappa}$ is the principal normal vector field of β (주법벡터장)

$B = T \times N$ is the binormal vector field of β (양법벡터장)

$\kappa = \frac{1}{r}$

$0\ T \cdot (T \times N) = 0$, N도 수직

$|T'(s)| > 0$

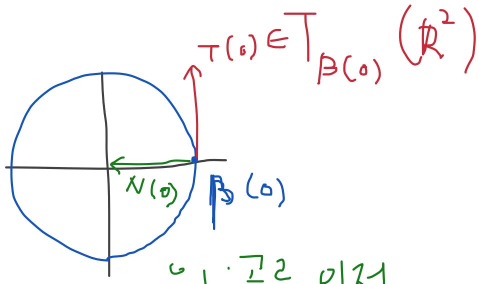
0포함X

Ex. $\beta(s) = (\cos s, \sin s)$

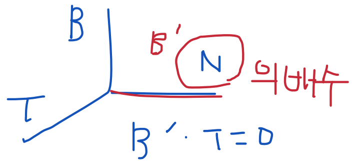
$T(s) = (-\sin s, \cos s)$

$T'(s) = (-\cos s, -\sin s) = N(s)$

$\because \kappa(s) = |(-\cos s, -\sin s)| = 1$

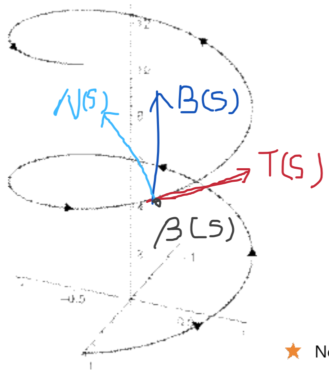


원: 곡률 일정



$B' \cdot T = 0$

$B' \cdot B = 0$



★ Lem 3.1

T, N, B is the *Frenet frame field* (프레네 틀장) on β

pf) $|N| = \frac{1}{\kappa}|T'| = 1$

$0 = (T \cdot T)' = 2T' \cdot T \Rightarrow T \cdot N = 0$

Then $|B| = |T||N| \sin\left(\frac{\pi}{2}\right) = 1$ ♡

δ_{ij} $\|N\| = 1$

$\|T\| = 1$

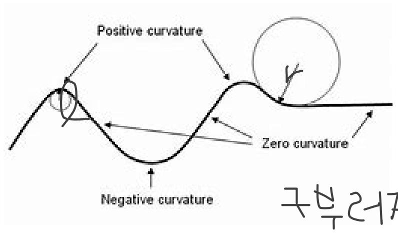
$\|B\| = 1$

★ Note

$B \cdot B = 1 \Rightarrow B' \cdot B = 0$
$B \cdot T = 0 \Rightarrow B' \cdot T = -B \cdot T' = -B \cdot \kappa N = 0$

Define the *torsion* function τ of β by $B' = -\tau N$ (비틀림)

곡률↑



$k = \frac{1}{r}$

구부러진 정도

★ Thm3.2

$N' = -\kappa T + \tau B$

pf) By orthonormal expansion (thm1.5)

$N' = N' \cdot T\,T + N' \cdot N\,N + N' \cdot B\,B$

$N' \cdot T = -N \cdot T' = -\kappa$

$N' \cdot N = 0$

$N' \cdot B = -N \cdot B' = -N \cdot (-\tau N) = \tau$ ♡

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} & \kappa & \\ -\kappa & & +\tau \\ & -\tau & \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

정의, clear

★ 3.3 Ex. The unit speed helix

$\beta(s) = (a \cos\frac{s}{c}, a \sin\frac{s}{c}, \frac{bs}{c})$ where $c = (a^2 + b^2)^{1/2}$

$T(s) = \beta'(s) = (-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c})$

$T'(s) = (-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0)$

Thus

$\kappa(s) = |T'(s)| = \frac{a}{c^2} > 0$

$N(s) = (-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0)$

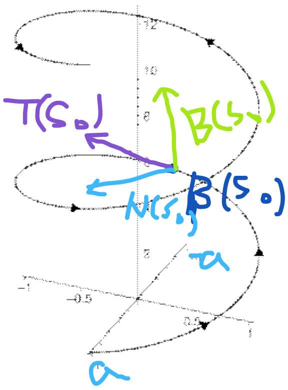
$B(s) = T(s) \times N(s) = (\frac{b}{c}\sin\frac{s}{c}, -\frac{b}{c}\cos\frac{s}{c}, \frac{a}{c})$

$B'(s) = (\frac{b}{c^2}\cos\frac{s}{c}, \frac{b}{c^2}\sin\frac{s}{c}, 0)$

$B'(s) = -\tau(s)N(s) \Rightarrow \tau(s) = \frac{b}{c^2}$

s_0

$\beta(c\pi) = (a\cos\pi, a\sin\pi, b\pi)$
$= (-a, 0, b\pi)$



★ Cor. 3.5

β is a plane curve(평면곡선) $\Leftrightarrow \tau = 0$

pf) $\Rightarrow$) $\exists p \; and \; q, (\beta(s) - p) \cdot q = 0$

$\Rightarrow \beta'(s) \cdot q = \beta''(s) \cdot q = 0$

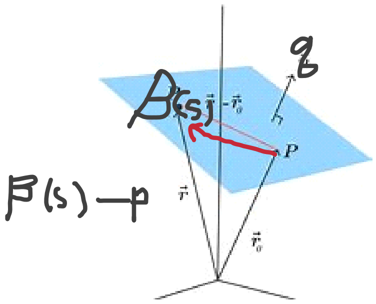
$\Rightarrow B = \pm q/|q| \quad \Rightarrow 0 = B' = -\tau N \Rightarrow \tau = 0$

$\Leftarrow$) $\tau = 0 \Rightarrow B' = 0 \Rightarrow B$ is parallel

Let $f(s) = (\beta(s) - \beta(0)) \cdot B$

Then $f' = \beta' \cdot B = 0$ and $f(0) = 0$ Thus $f = 0$

$(\beta(s) - \beta(0)) \cdot B = 0$



★ Lemma 3.6

κ is constant and $\tau = 0 \Rightarrow \beta(s) \in S^1\left(c, \frac{1}{\kappa}\right)$

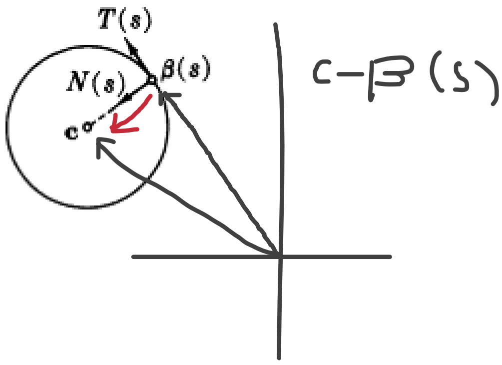
pf) By 3.5 β is a plane curve

Let $\gamma = \beta + \left(\frac{1}{\kappa}\right) N$

then $\gamma' = \beta' + \frac{1}{\kappa} N' = T + \frac{1}{\kappa}(-\kappa T) = 0$

Hence γ is constant $c \in R^3$

But $d(c, \beta(s)) = |\frac{1}{\kappa} N(s)| = \frac{1}{\kappa}$



2.5 Covariant Derivatives

★ Def 5.1 The *covariant derivative* of a v.f. W with respect to v is

$$\nabla_v W = W(p+tv)'(0) \in T_p(R^3)$$

★ Lem 5.2 $W = \sum w_i U_i$ => $\nabla_v W = \sum v[w_i]\, U_i(p)$

pf) $W(p+tv) = \sum w_i\,(p+tv)\; U_i(p+tv)$

then $\nabla_v W = W(p+tv)'(0) = \sum v[w_i] U_i(p)$ by 1.3.1

★ Thm 5.3 (4) $v[Y \cdot Z] = \nabla_v Y \cdot Z(p) + Y(p) \cdot \nabla_v Z$

pf) $v[Y \cdot Z] = v[\sum y_i z_i] = \sum v[y_i] z_i(p) + \sum y_i(p) v[z_i]$ by 1.3.3

By 5.2 we have done

5.3(1)

$\nabla_{av+bw} Y = \sum (av+bw)[y_i] U_i(p)$

$= a\sum v[y_i] U_i(p) + b\sum w[y_i] U_i(p)$ (by1.3.3(1))

$= a\nabla_v Y + b\nabla_w Y$

5.4(1)

$(\nabla_{fV+gW} Y)(p) = \nabla_{(fV+gW)(p)} Y$

$= \sum (fV+gW)(p)[y_i] U_i(p)$

$= \sum (fV[y_i](p) + gW[y_i])(p) U_i(p)$

$= (f\sum V[y_i] U_i)(p) + (g\sum W[y_i] U_i)(p)$

$= (f\nabla_V Y)(p) + (g\nabla_W Y)(p)$

∴) $\nabla_{fV+gW} Y = f\nabla_V Y + g\nabla_W Y$

5.3(2)

$\nabla_v(aY+bZ) = \sum (v[ay_i + bz_i]) U_i(p)$

$= \sum (av[y_i] + bv[z_i]) U_i(p)$ by 1.3.3(2)

$= a\sum v[y_i] U_i(p) + b\sum v[z_i] U_i(p)$

$= a\nabla_v Y + b\nabla_v Z$

5.3(3)

$\nabla_v(fY) = \sum v[fy_i]U_i(p)$

$= \sum (v[f] \cdot y_i(p) + f(p) \cdot v[y_i])U_i(p)$ by 1.3.3(3)

$= v[f] \sum y_i(p)U_i(p) + f(p) \sum v[y_i]U_i(p)$

$= v[f]Y(p) + f(p)\, \nabla_v Y$

5.4(4)

$(V[Y \cdot Z])(p) = (V[\sum y_i z_i])(p)$

$= V(p)[\sum y_i z_i]$

$= \sum(V[y_i])(p) z_i(p) + \sum y_i(p)(V[z_i])(p)$ by 1.3.3(2), (3)

$= (\sum V[y_i] z_i)(p) + (\sum y_i V[z_i])(p)$

∴) $V[Y \cdot Z] = \nabla_V Y \cdot Z + Y \cdot \nabla_V Z$

★ Def. Define a v.f. $(\nabla_V W)(p) = \nabla_{V(p)} W \in T_p(R^3)$

★ Note) $(\nabla_V W)(p) = \nabla_{V(p)} W$

$= \sum V(p)[w_i]\, U_i(p) = \sum (V[w_i]\, U_i)(p) = (\sum V[w_i]\, U_i)(p)$

Thus $\nabla_V W = \sum V[w_i]\, U_i$

★ Ex) $V = (y - x)U_1 + xyU_3$ and $W = x^2 U_1 + yzU_3$

$V[x^2] = (y-x)U_1[x^2] = 2x(y-x) + 0$

$V[yz] = 0 + xy^2$

Thus $\nabla_V W = V[x^2]U_1 + V[yz]U_3 = 2x(y-x)U_1 + xy^2U_3$

2.6 Frame Fields

2020년 5월 24일 일요일 오전 9:39

★ Def 6.1 vector field E_1, E_2, E_3 on R^3 is a $frame\ fileld$ on R^3 if $E_i \cdot E_j = \delta_{ij}$

Note)

U_1, U_2, U_3 is the natural frame field on R^3

★ 6.2 Ex. (1) The $cylindrical\ f.f.$(기둥틀장) on R^3

Let r, θ, z be the usual cylindrical coordinate functions on R^3

$E_1 = \cos\theta U_1 + \sin\theta U_2$ by r ↑

$E_2 = -\sin\theta U_1 + \cos\theta U_2$ by θ ↑

$E_3 = U_3$ by z ↑

Clearly $E_i \cdot E_j = \delta_{ij}$

(2) The $spherical$ f.f. (구면틀장)on R^3

Let ρ, θ, φ be the spherical coordinate functions on R^3

$F_1 = \cos\varphi E_1 + \sin\varphi E_3$ by ρ ↑

$F_2 = E_2$ by θ ↑

$F_3 = -\sin\varphi E_1 + \cos\varphi E_3$ by φ ↑

★ Lem 6.3 (1) $V = \sum f_i E_i$ where $f_i = V \cdot E_i$

(2) $V \cdot W = \sum f_i g_i$ and $|V|^2 = \sum f_i^2$

2.7 Connection Forms

2020년 5월 24일 일요일 오전 10:18

★

$\nabla_v E_1 = \sum c_{1j} E_j(p)$

$\nabla_v E_2 = \sum c_{2j} E_j(p)$

$\nabla_v E_3 = \sum c_{3j} E_j(p)$ where $c_{ij} = \nabla_v E_i \cdot E_j\,(p)$

Note) c_{ij} depends on v

Define $\omega_{ij}: T(R^3) \to R$ by $\omega_{ij}\,(v) = \nabla_v E_i \cdot E_j\,(p)$

★ Lem 7.1 ω_{ij} is a $connection$ 1-form and $\omega_{ji} = -\omega_{ij}$ (Alternating)

pf) $\omega_{ij}(av + bw) = \nabla_{av+bw} E_i \cdot E_j\,(p)$

$= a\nabla_v E_i \cdot E_j\,(p) + b\nabla_w E_i \cdot E_j\,(p)$ by 5.3(2)

By 5.3(4) $0 = v[\delta_{ij}] = v[E_i \cdot E_j] = \nabla_v E_i \cdot E_j\,(p) + \nabla_v E_j \cdot E_i\,(p)$

$\therefore\ \omega_{ji} = -\omega_{ij}$

★ Thm 7.2 $\nabla_V E_i = \sum \omega_{ij}\,(V) E_j$

pf) Note $(\nabla_V E_i)(p) = \nabla_{V(p)} E_i$ and $(\omega_{ij}\,(V) E_j)(p) = \omega_{ij}\left(V(p)\right) E_j(p)$.

By orthonormal expansion

★ Define $A = \left(a_{ij}\right)$ is the $attitude\ matrix$ of the frame feild E_1, E_2, E_3

Note) $A(p) = \left(a_{ij}(p)\right)$ is the attitude matrix of the frame $E_1(p), E_2(p), E_3(p)$ (def. 1.6)

Define The $differential$ of A is $dA = \left(da_{ij}\right)$

★ Thm 7.3 $\omega = dA\ {}^tA$ or $\omega_{ij} = \sum_k a_{jk}\, da_{ik}$

pf) $\omega_{ij}(v) = \nabla_v E_i \cdot E_j(p)$ and $E_j = \sum a_{jk} U_k$

By 5.2 $\nabla_v E_i = \sum_k v[a_{ik}] U_k$

=> $\omega_{ij}(v) = \sum_k v[a_{ik}]\ a_{jk}(p)$

$= \sum_k da_{ik}(v)\ a_{jk}(p)$

$= \left(\sum_k a_{jk}\ da_{ik}\right)(v)$

Ex. For the cylindrical f.f.

$$A = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus $\omega = dA\ {}^tA = \begin{pmatrix} -sin\,\theta d\theta & cos\,\theta d\theta & 0 \\ -\cos\theta d\theta & -sin\,\theta d\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} cos\,\theta & -sin\,\theta & 0 \\ sin\,\theta & cos\,\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & d\theta & 0 \\ -d\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$_(*)

Furthermore by (*)

$\nabla_V E_1 = d\theta(V)E_2 = V[\theta]E_2$

$\nabla_V E_2 = -d\theta(V)E_1 = -V[\theta]E_1$

$\nabla_V E_3 = 0$

2.8 the structural equations

2020년 6월 3일 수요일 오후 4:34

★ Def 8.1

The $dual\ 1-forms\ \ \theta_1,\ \theta_2, \theta_3$ of the frame field E_1, E_2, E_3 are $\theta_i(v) = v \cdot E_i(p)$

$(\theta_i(E_j))(p)$

$= \theta_i(E_j(p))$

★ Note)

$\theta_i(E_j) = \delta_{ij}$ "Dual" $\theta_i \leftrightarrow E_i$

$dx_i(v) = v_i = v \cdot U_i(p) \Rightarrow dx_i = \theta_i \qquad dx_i \leftrightarrow U_i$

$V = \sum (V \cdot E_i) E_i = \sum \theta_i(V) E_i$

★ Lem 8.2 $\theta_i \leftrightarrow E_i$

$$\phi = \sum \phi(E_i)\theta_i$$

$\phi(fV) = f\phi(V)$ — f

pf) $(\sum \phi(E_i)\theta_i)(V) = \sum \phi(E_i)\theta_i(V)$ by p.24

$= \sum \theta_i(V)\, \phi(E_i)$

$= \phi(\sum \theta_i(V)\, E_i) = \phi(V)$

★ $\theta_i = \sum a_{ij} dx_j$

★ Note) $E_i = \sum a_{ij} U_j$ where $A = (a_{ij})$ is the attitude matrix of E_i

$\Rightarrow \theta_i = \sum a_{ij} dx_j$

A 2

$\because\ dx_i \leftrightarrow U_i$

$\Rightarrow \theta_i = \sum \theta_i(U_j) dx_j$

But $\theta_i(U_j) = U_j \cdot E_i = U_j \cdot (\sum a_{ik} U_k) = a_{ij}$

★ Thm 8.3 (Cartan structural equations) $\theta_i \leftrightarrow E_i$

$w = dA\,{}^tA$

$\hookrightarrow {}^t w = {}^t(dA\,{}^tA)$

$= A\,{}^t dA$

Let $\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}$ and $d\xi = \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$ we have $\theta = A\, d\xi$

(1) $d\theta_i = \sum \omega_{ij} \wedge \theta_j$ i.e. $d\theta = \omega\theta$

pf) $d\theta = d(A\, d\xi) = dA \cdot d\xi = dA\,{}^tA \cdot A\, d\xi = \omega\theta$

(2) $d\omega_{ij} = \sum \omega_{ik} \wedge \omega_{kj}$ i.e. $d\omega = \omega\omega$

pf) Note that $d(df\, g) = d(g\, df) = dg \wedge df = -df \wedge dg$

$d\omega = d(dA\,{}^tA) = -dA \cdot d({}^tA) = -dA\,{}^tA \cdot A^t\,(dA) = -\omega\,{}^t\omega = \omega\omega$

★ 8.4 Ex. On spherical frame field

$F_1 = \cos\varphi E_1 + \sin\varphi E_3 = \cos\varphi(\cos\theta U_1 + \sin\theta U_2) + \sin\varphi U_3$
$F_2 = E_2 = -\sin\theta U_1 + \cos\theta U_2$
$F_3 = -\sin\varphi E_1 + \cos\varphi E_3 = -\sin\varphi(\cos\theta U_1 + \sin\theta U_2) + \cos\varphi U_3$

$$A = (a_{ij}) = \begin{pmatrix} \cos\varphi\cos\theta & \cos\varphi\sin\theta & \sin\varphi \\ -\sin\theta & \cos\theta & 0 \\ -\sin\varphi\cos\theta & -\sin\varphi\sin\theta & \cos\varphi \end{pmatrix}$$

$x_1 = \rho\cos\varphi\cos\theta,\quad x_2 = \rho\cos\varphi\sin\theta,\quad x_3 = \rho\sin\varphi$

$dx_1 = \cos\varphi\cos\theta d\rho + \rho\cos\theta(-\sin\varphi)\, d\varphi + \rho\cos\varphi(-\sin\theta)\, d\theta$ by 5.5
$dx_2 = \cos\varphi\sin\theta d\rho + \rho\sin\theta(-\sin\varphi)\, d\varphi + \rho\cos\varphi(\cos\theta)\, d\theta$

$dx_3 = \sin\varphi\ d\rho + \rho\cos\varphi d\varphi$

$\theta_1 = (\cos\varphi\cos\theta)dx_1 + (\cos\varphi\sin\theta)dx_2 + (\sin\varphi)dx_3$ by 5.2^*

$= \cos^2\varphi\ \cos^2\theta\ d\rho + \rho(-\sin\varphi)\cos\varphi\ \cos^2\theta\ d\varphi + \rho\ \cos^2\varphi(-\sin\theta)\cos\theta d\theta$

$+ \cos^2\varphi\sin^2\theta d\rho + \rho(-\sin\varphi)\cos\varphi sin^2\theta\ d\varphi + \rho\ \cos^2\varphi\sin\theta(\cos\theta)\ d\theta$

$+\sin^2\varphi\ d\rho$ **+** $\rho\ sin\ \varphi\ \cos\varphi\ d\ \varphi$

= $(\cos^2\varphi\cos^2\theta + \cos^2\varphi\sin^2\theta + \sin^2\varphi)d\rho$

$+ (\rho(-\sin\varphi)\cos\varphi\cos^2\theta + \rho(-\sin\varphi)\cos\varphi\sin^2\theta + \rho\ sin\ \varphi\ \cos\varphi)d\varphi$

$= d\rho$

From 7.3 $\omega = dA^tA$, $\omega_{12} = d(\cos\varphi\cos\theta)(-\sin\theta) + d(\cos\varphi\ \sin\theta)(\cos\theta)$

$= (-\sin\theta)\ (\cos\theta(-\sin\varphi)\ d\varphi + \cos\varphi(-\sin\theta)\ d\theta)$

$+ (\cos\theta)\ (\sin\theta(-\sin\varphi)\ d\varphi + \cos\varphi(\cos\theta)\ d\theta)$

$= \cos\varphi\ d\theta$

From 8.3(1) $d\theta_3 = \sum\omega_{3j}\wedge\theta_j$

$= \omega_{31}\wedge\theta_1 + \omega_{32}\wedge\theta_2 = -d\varphi\wedge d\rho + (-\sin\varphi d\theta)\wedge(\rho\cos\varphi d\theta) = d\rho\wedge d\varphi$

From 8.3(2) $d\omega_{12} = \sum\ \omega_{1k}\wedge\omega_{k2}$

$= \omega_{13}\wedge\omega_{32} = d\varphi\wedge(-\sin\varphi\ d\theta) = -\ \sin\varphi\ d\varphi\wedge d\theta$

Note) $d\omega_{12} = d(\cos\varphi d\theta) = d(\cos\varphi)\wedge d\theta = -\ \sin\varphi\ d\varphi\wedge d\theta$ by def 1.6.3

Note에 더 쉬움

연습문제 2 : $d\theta$, $d\omega$

3.1 Isometries in R^3

2020년 6월 4일 목요일 오후 5:59

★ Def 1.1 An *isometry* of R^3 is a mapping $F: R^3 \to R^3$ s.t. $d\big(F(p), F(q)\big) = d(p,q)$

Denote) $F \in \mathcal{E}(3)$

★ 1.2 Ex. (1) Translations $T_a(p) = p + a$ Note) T_a is not linear

$d(T_a(p), T_a(q)) = d(p+a, q+a)$
$= \|(p+a) - (q+a)\|$
$= \|p - q\|$
$= d(p,q)$

(2) Rotations around a coordinate axis

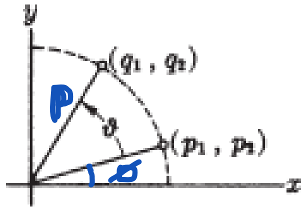


FIG. 3.1

$q_1 = \rho\cos(\phi + \theta) = p_1\cos\theta - p_2\sin\theta$
$q_2 = \rho\sin(\phi + \theta) = p_1\sin\theta + p_2\cos\theta$

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

Thus $C(p) = (p_1\cos\theta - p_2\sin\theta, p_1\sin\theta + p_2\cos\theta, p_3)$

Note) C is linear

★ Lem 1.3 $F, G \in \mathcal{E}(3) \Rightarrow G\,F \in \mathcal{E}(3)$

★ Lem 1.4

(1) $T_aT_b = T_bT_a$

(2) $(T_a)^{-1} = T_{-a}$

(3) $\forall p, q, \exists! \; T, T(p) = q$

pf) (Existence) $T_{q-p}(p) = q$

(Uniqueness) $T_a(p) = q \Rightarrow p + a = q \Rightarrow a = q - p$

Note) $T(p) = p$ for some $p \Rightarrow T = I$ by (3)

Def. A linear transformation $C: R^3 \to R^3$ is *orthogonal*

if it preserves the dot product.

Denote) $C \in \mathcal{O}(3)$

★ Lem 1.5 $C \in \mathcal{E}(3)$

pf) $|C(p)|^2 = C(p)\cdot C(p) = p \cdot p = |p|^2$

Thus $d\big(C(p), C(q)\big) = \big|C(p) - C(q)\big| = |C(p-q)| = |p - q| = d(p,q)$

★ Lem 1.6

$F \in \mathcal{E}(3)$ and $F(0) = 0 \Rightarrow F \in \mathcal{O}(3)$

pf) (1) it preserves the dot product !

$|F(p)| = d\left(0, F(p)\right) = d\left(F(0), F(p)\right) = d(0,p) = |p|$ for all p

Then $\left(F(p) - F(q)\right) \cdot \left(F(p) - F(q)\right) = |F(p) - F(q)|^2 = |p-q|^2 = (p-q)\cdot(p-q)$

Thus $|F(p)|^2 - 2F(p)\cdot F(q) + |F(q)|^2 = |p|^2 - 2p\cdot q + |q|^2$
$\Rightarrow F(p)\cdot F(q) = p \cdot q$

(2) (Linear)

Let $u_1 = (1,0,0), u_2 = (0,1,0)$ and $u_3 = (0,0,1)$

Then $F(u_i)\cdot F(u_j) = \delta_{ij}$ by (1)

By orthonormal expansion, $F(p) = \sum F(p)\cdot F(u_i) F(u_i) = \sum \square\, p_i F(u_i)$

Thus $F(ap + bq) = \sum (ap_i + bq_i) F(u_i) = aF(p) + bF(q)$

★ Thm 1.7

$F \in \mathcal{E}(3) \Rightarrow \exists | T$ and $C \in \mathcal{E}(3)$, $F = TC$

pf) $(Existence)$ Let $T = T_{F(0)}$

Then $(T^{-1}F)(0) = F(0) - F(0) = 0$, thus $T^{-1}F \in \mathcal{O}(3)$ by 1.6

Take $C = T^{-1}F$ We have $F = TC$

$(Uniqueness)$ Let $TC = F = \overline{TC}$

Then $C = T^{-1}\,\overline{TC}$

Thus $0 = C(0) = T^{-1}\overline{TC}(0) = T^{-1}\overline{T}(0)$ by linear

By 1.4(3) $T^{-1}\,\overline{T} = I$, we have $T = \overline{T}$

$TC = \overline{T}\,\overline{C} = T\,\overline{C}$
$\Rightarrow C = \overline{C}$

$q = F(p) = T_a C(p)$

$= a + C(p)$

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} C_{ij} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

3.2 the tangent map of an isometry

2020년 6월 11일 목요일 오전 9:43

★ Thm 2.1

F = $TC \in \mathcal{E}(3) \Rightarrow F_*(v_p) = (Cv)_{F(p)}$

pf)

$F(p+tv) = T\big(C(p) + tC(v)\big) = a + C(p) + tC(v) = F(p) + tC(v)$

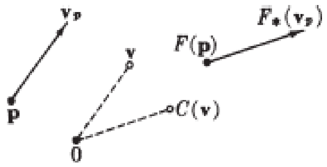


FIG. 3.2

★ Cor 2.2

$F_*(v) \cdot F_*(w) = C(v) \cdot C(w) = v \cdot w$

★ Thm2.3

∃| $F \in \mathcal{E}(3)$, $F_*(e_i) = f_i$

pf) Let A be the attitude matrix of e_i and let B be the attitude matrix of f_i

$A(e_i) = u_i$ and $B(f_i) = u_i$

$\Rightarrow C = {}^tBA$, $C(e_i) = f_i$

Take $F = T_{q-C(p)}C$

$\Rightarrow F(p) = q$ and $F_*(e_i) = \left(C_{e_i}\right)_{F(p)} = \left(f_i\right)_q = f_i$

1.5.1-4연습문제

2020년 4월 23일 목요일 오후 12:29

★ 1.5.1

Let $v=(1,2,-3)$ and $p=(0,-2,1)$ .Evaluate the following 1-forms on the tangent vector v_p

(a)

$(y^2dx)(v_p)=y^2(p)dx(v_p)=4\cdot 1=4$

(b)

$(zdy-ydz)(v_p)=z(p)dy(v_p)-y(p)dz(v_p)=1\cdot 2-(-2\cdot -3)=2-6=-4$

(c)

$((z^2-1)dx-dy+x^2dz)(v_p)=(z^2-1)(p)dx(v_p)-dy(v_p)+x^2(p)dz(v_p)=0\cdot 1-2+0\cdot(-3)=-2$

★ 1.5.2

If $\phi=\sum f_i dx_i$ and $V=\sum v_i U_i$, show that the 1-form ϕ evaluated on the vector field V is the function $\phi(V)=\sum f_i v_i$.

Pf)

$$\begin{aligned}\phi(V) &= \phi(\Sigma v_i U_i)\\ &= \Sigma\left(\phi(v_i U_i)\right)\\ &= \Sigma\left(v_i\phi(U_i)\right)\\ &= \Sigma(v_i f_i) \quad \text{by Lemma 5.4}\end{aligned}$$

★ 1.5.3

Evaluate the 1-form $\phi=x^2dx-y^2dz$ on the vector fields

$V=xU_1+yU_2+zU_3$

$W=xy(U_1-U_3)+yz(U_1-U_2)$, and $\left(\frac{1}{x}\right)V+\left(\frac{1}{y}\right)W$

1) $\phi(V)=\phi(xU_1+yU_2+zU_3)=(x^2dx-y^2dz)(xU_1+yU_2+zU_3)=x^3-y^2z$

2) $\phi\left(xy(U_1-U_3)+yz(U_1-U_2)\right)=(x^2dx-y^2dz)\left(xy(U_1-U_3)+yz(U_1-U_2)\right)=x^3y+xy^3+x^2yz=xy(x^2+y^2+xz)$

3) $\frac{1}{x}V+\frac{1}{y}W=U_1+\frac{y}{x}U_2+\frac{z}{x}U_3+xU_1-xU_3+zU_1-zU_2=(1+x+z)U_1+\left(\frac{y}{x}-z\right)U_2+\left(\frac{z}{x}-x\right)U_3$

$\phi\left(\frac{1}{x}V+\frac{1}{y}W\right)=x^2(1+x+z)-y^2\left(\frac{z}{x}-x\right)$

★ 1.5.4

Express the following differentials in terms of df:

(a)

$d(f^5)=5\cdot(f^4)df$

(b)

$d\left(\sqrt{f}\right)=\frac{1}{2\sqrt{f}}df$

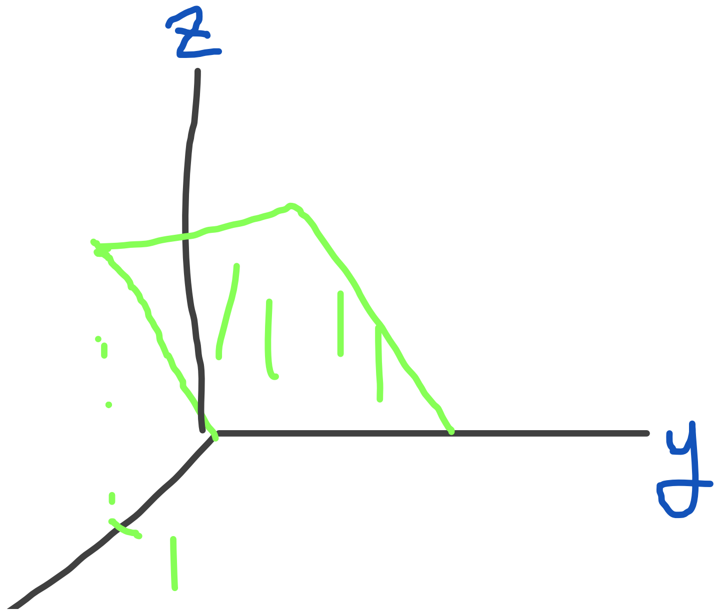
(c)

$$\begin{aligned}d(\log(1+f^2)) &= \frac{1}{1+f^2}d(1+f^2)\\ &= \frac{1}{1+f^2}(d(1)+2fdf)\\ &= \frac{2f}{1+f^2}df\end{aligned}$$

$O_n \mathbb{R}^2$

$\pi_1 = x$

$\mathbb{R}^2 \to \mathbb{R}$

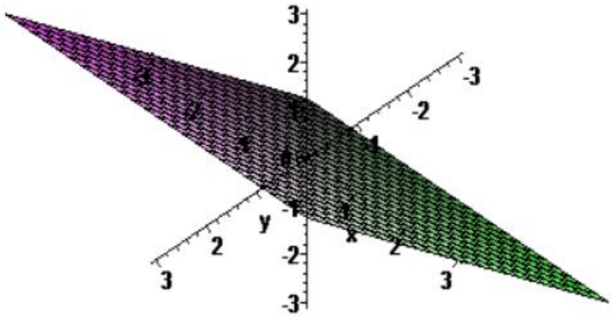


π ..

```
f:=(x,y)->x;
```

$$f := (x, y) \to x$$

```
plot3d(f(x,y),x=-3..3,y=-3..3);
```



1.5.5-11 연습문제

2020년 4월 24일 금요일 오후 3:46

5.

(a) $d\left((x^2+y^2+z^2)^{\frac{1}{2}}\right)=\frac{1}{2\sqrt{x^2+y^2+z^2}}\,d(x^2+y^2+z^2)=\frac{2x}{2\sqrt{x^2+y^2+z^2}}dx+\frac{2y}{2\sqrt{x^2+y^2+z^2}}dy+\frac{2z}{2\sqrt{x^2+y^2+z^2}}dz=\frac{1}{\sqrt{x^2+y^2+z^2}}(xdx+ydy+zdz)$

(b) $d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)=\left(\frac{1}{1+\left(\frac{y}{x}\right)^2}\right)\cdot d\left(\frac{y}{x}\right)=\left(\frac{x^2}{x^2+y^2}\right)\cdot\left(-\frac{y}{x^2}\right)dx+\left(\frac{x^2}{x^2+y^2}\right)\cdot\left(\frac{1}{x}\right)dy=-\frac{y}{x^2+y^2}dx+\frac{x}{x^2+y^2}dy$

6. in each case compute the differential of f and find the directional derivative $v_p[f]$, for v_p as in exercise 1

$v=(1,2,-3), p=(0,-2,1)$

$(a) f=xy^2-yz^2$

$\Rightarrow\ df=y^2dx+(2xy-z^2)dy-2yzdz$

$\Rightarrow\ df[v_p]=1\cdot(-2)^2+2\cdot(-1)+(-3)\cdot(-2\cdot-2)=4-2-12=-10$ $\quad\because) df[v_p]=v_p[f]=\sum v_i\frac{\partial f}{\partial x_i}(p)$

$(b) f=xe^{(yz)}$

$\Rightarrow\ df=e^{yz}dx+xz\cdot e^{yz}dy+xy\cdot e^{yz}dz$

$\Rightarrow\ df[v_p]=1\cdot e^{-2}+2\cdot 0+(-3)\cdot 0=e^{-2}$

$(c)\ f=\sin(xy)\cos(xz)$

$\Rightarrow df=(y\cos(xy)\cdot\cos(xz)-z\sin(xy)\sin(xz))dx+(x\cos(xy)\cdot\cos(xz))dy+(-x\sin(xz)\cdot\sin(xy))dz$

$\Rightarrow df[v_p]=(-2\cdot\cos(0)\cdot\cos(0)-1\cdot\sin(0)\cdot\sin(0))\cdot 1+0+0=-2$

7.

(a) $\phi(v_p)=v_1-v_3$

$\phi(av_p+bw_p)=(av_1+bw_1)-(av_3+bw_3)=a\phi(v_p)+b\phi(w_p)$

Thus ϕ is a 1-form

$\Rightarrow \phi(U_1)=1=f_1$

$\Rightarrow \phi=dx-dz$

(b) $\phi(2v_p)=p_1-p_3\neq 2(p_1-p_3)=2\phi(v_p)$

$\therefore$ ϕ is not 1-form

(c) $\phi(v_p)=v_1p_3+v_2p_1$

$\phi(av_p+bw_p)=(av_1+bw_1)p_3+(av_2+bw_2)p_1=a(v_1p_3+v_2p_1)+b(w_1p_3+w_2p_1)=a\phi(v_p)+b\phi(w_p)$

Thus ϕ is a 1-form

$\Rightarrow \phi(U_1)=p_3=z$

$\Rightarrow \phi=xdx+xdy$

(d) $\phi(v_p)=v_p[f]=v_p[x^2+y^2]$

$v_p[x^2+y^2]=\frac{v_1(\partial f)}{\partial x}(p)+\frac{v_2(\partial f)}{\partial y}(p)+\frac{v_3(\partial f)}{\partial z}(p)=v_1\cdot(2p_1)+v_2\cdot(2p_2)=2p_1v_1+2p_2v_2$

$\therefore \phi=2xdx+2ydy$

(e) $\phi(v_p)=0$

Clear

$\phi=0$

(f) $\phi(2v_p)=(p_1)^2\neq 2(p_1)^2=2\phi(v_p)$

$\therefore$ ϕ is not 1-form

8.

$d(fg)=gdf+fdg$

$$\begin{aligned} d(fg)&=\sum\frac{\partial fg}{\partial x_i}dx_i\\ &=\sum\left(\frac{\partial f}{\partial x_i}g+f\frac{\partial g}{\partial x_i}\right)dx_i\\ &=g\sum\frac{\partial f}{\partial x_i}dx_i+f\sum\frac{\partial g}{\partial x_i}dx_i\\ &=gdf+fdg\end{aligned}$$

9. 1-form ϕ is zero at a point p provided $\phi(v_p)=0$ for all tangent vectors at p

A point at which its differential df is zero is called a critical point of the function f

Prove that p is a critical point of f iff $\frac{\partial f}{\partial x}(p)=\frac{\partial f}{\partial y}(p)=\frac{\partial f}{\partial z}(p)=0$

Find all criticla points of $f = (1-x^2)y + (1-y^2)z$

pf)⇒

if p is critical point of f then $df = o$

Thus $0 = df\left(U_i(p)\right) = U_i(p)[f] = \frac{\partial f}{\partial x_i}(p)$

∴) $\frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = \frac{\partial f}{\partial z}(p) = 0$

⇐

$\forall v_p = v_1 U_1(p) + v_2 U_2(p) + v_3 U_3(p)$,

$df(v_p) = v_1 U_1(p)[f] + v_2 U_2(p)[f] + v_3 U_3(p)[f]$

$\quad = v_1 \frac{\partial f}{\partial x}(p) + v_2 \frac{\partial f}{\partial y}(p) + v_3 \frac{\partial f}{\partial z}(p)$

$\quad = 0$

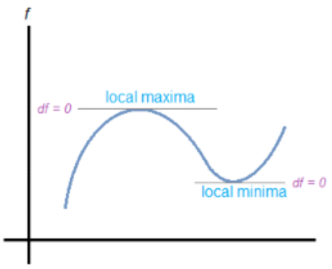
∴) $df = 0$

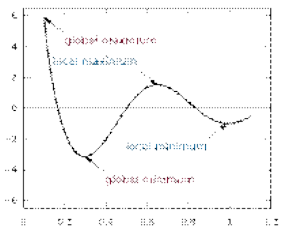
$\nabla f = \left(-2xy, 1-x^2-2yz, 1-y^2\right) = (0,0,0)$

$\Rightarrow \quad y = \pm 1, x = 0, z = \pm\frac{1}{2}$

$\Rightarrow \ (x,y,z) = \pm\left(0,1,\frac{1}{2}\right)$

10.





11.(a)

$f(x) \approx f(a) + Df(a)(x-a)$

$f(v+p) - f(p) \approx Df(p)(v) = df(v_p)$

11.(b)

It is sometimes asserted that df is the linear approximation(선형근사) of Δf.

Compute exact and approximate values of $f(0.9,1.6,1.2) - f(1,1.5,1)$, where $f = \frac{x^2 y}{z}$

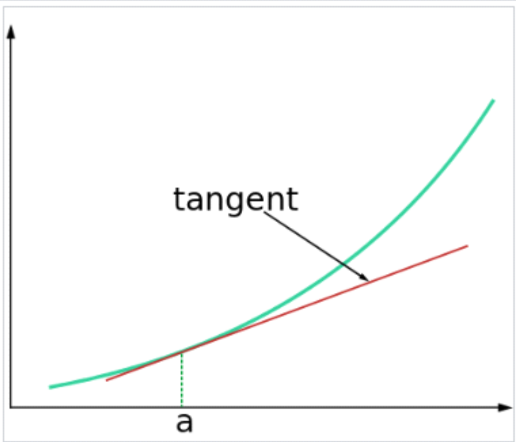
sol)

$f(v+p) - f(p)$=$f(0.9,1.6,1.2) - f(1,1.5,1)$=$1.08 - 1.5 = -0.42$ ∵(a)

df=$\frac{2xy}{z}dx$+$\frac{x^2}{z}dy$-$\frac{x^2 y}{z}dz$

$df(v_p)$=$(-0.1, 0.1, 0.2)_p\left[\frac{x^2 y}{z}\right]$= $-0.3 + 0.1 - 0.3 = -0.5$ ∵lemma3.2

∵ $v = (-0.1, 0.1, 0.2)$,$p = (1, 1.5, 1)$



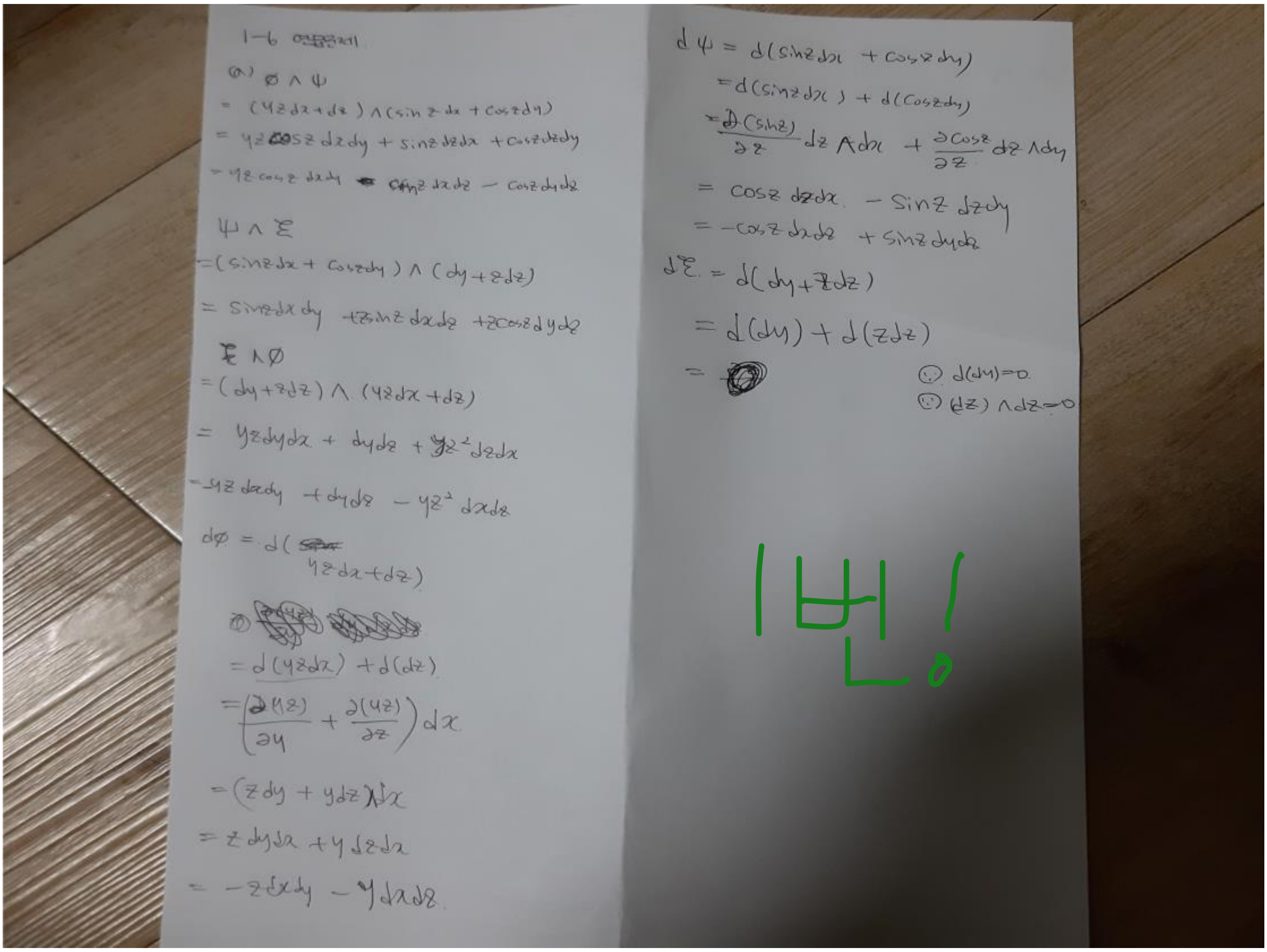
$(a, f(a))$에서의 접선

어떤 함수를 선형 함수, 즉 일차 함수로 근사하는 것을 말한다. 아이디어는 그림과 같이 어떤 점 근처를 확대하면 확대할수록 (미분 가능한) 함수의 그래프와 그 점에서의 접선은 비슷해 진다는 사실로부터 온다.

https://ko.wikipedia.org/wiki/%EC%84%A0%ED%98%95_%EA%B7%BC%EC%82%AC

1.6.1,5-9연습문제

2020년 4월 29일 수요일 오후 9:21



1-6 연습문제

(a) $\phi \wedge \psi$

$= (yz\,dx + dz) \wedge (\sin z\,dx + \cos z\,dy)$

$= yz\cos z\,dx\,dy + \sin z\,dz\,dx + \cos z\,dz\,dy$

$= yz\cos z\,dx\,dy - \sin z\,dx\,dz - \cos z\,dy\,dz$

$\psi \wedge \xi$

$= (\sin z\,dx + \cos z\,dy) \wedge (dy + z\,dz)$

$= \sin z\,dx\,dy + z\sin z\,dx\,dz + z\cos z\,dy\,dz$

$\xi \wedge \phi$

$= (dy + z\,dz) \wedge (yz\,dx + dz)$

$= yz\,dy\,dx + dy\,dz + yz^2\,dz\,dx$

$= -yz\,dx\,dy + dy\,dz - yz^2\,dx\,dz$

$d\phi = d(yz\,dx + dz)$

$= d(yz\,dx) + d(dz)$

$= \left(\frac{\partial(yz)}{\partial y} + \frac{\partial(yz)}{\partial z}\right)dx$

$= (z\,dy + y\,dz) \wedge dx$

$= z\,dy\,dx + y\,dz\,dx$

$= -z\,dx\,dy - y\,dx\,dz$

$d\psi = d(\sin z\,dx + \cos z\,dy)$

$= d(\sin z\,dx) + d(\cos z\,dy)$

$= \frac{\partial(\sin z)}{\partial z}dz \wedge dx + \frac{\partial \cos z}{\partial z}dz \wedge dy$

$= \cos z\,dz\,dx - \sin z\,dz\,dy$

$= -\cos z\,dx\,dz + \sin z\,dy\,dz$

$d\xi = d(dy + z\,dz)$

$= d(dy) + d(z\,dz)$

$= 0$

∵ $d(dy) = 0$

∵ $(dz) \wedge dz = 0$

1번!

5.

$\phi_1 \wedge \phi_2 = (f_{11}dx_1 + f_{12}dx_2 + f_{13}dx_3) \wedge (f_{21}dx_1 + f_{22}dx_2 + f_{23}dx_3)$

$= f_{11}f_{22}dx_1dx_2 + f_{11}f_{23}dx_1dx_3 + f_{12}f_{21}dx_2dx_1 + f_{12}f_{23}dx_2dx_3 + f_{13}f_{21}dx_3dx_1 + f_{13}f_{22}dx_3dx_2$

$\Rightarrow \phi_1 \wedge \phi_2 \wedge \phi_3$

$= (f_{11}dx_1 + f_{12}dx_2 + f_{13}dx_3) \wedge (f_{21}dx_1 + f_{22}dx_2 + f_{23}dx_3) \wedge (f_{31}dx_1 + f_{32}dx_2 + f_{33}dx_3)$

$= f_{11}\begin{vmatrix} f_{22}, & f_{33} \\ f_{23}, & f_{32} \end{vmatrix} dx_1dx_2dx_3 - f_{12}\begin{vmatrix} f_{21}, & f_{33} \\ f_{23}, & f_{31} \end{vmatrix} dx_1dx_2dx_3 + f_{13}\begin{vmatrix} f_{21}, & f_{32} \\ f_{22}, & f_{31} \end{vmatrix} dx_1dx_2dx_3$

$= \begin{vmatrix} f_{11}, & f_{12}, & f_{13} \\ f_{21}, & f_{22}, & f_{23} \\ f_{31}, & f_{32}, & f_{33} \end{vmatrix} dx_1dx_2dx_3$

Thus

$$\phi_1 \wedge \phi_2 \wedge \phi_3 = \begin{vmatrix} f_{(11)} & f_{(12)} & f_{(13)} \\ f_{(21)} & f_{(22)} & f_{(23)} \\ f_{(31)} & f_{(32)} & f_{(33)} \end{vmatrix} dxdydz$$

6.

$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial z}dz = \cos(\theta)\,dr - r\sin(\theta)d\theta$

$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta + \frac{\partial y}{\partial z}dz = \sin(\theta)\,dr\ +\ r\cos(\theta)\,d\theta$

$dz = dz$

Therefore

$dxdydz = dx\ \wedge dy\ \wedge dz$

$=(\cos(\theta)\,dr - r\sin(\theta)d\theta) \wedge (\sin(\theta)\,dr\ +\ r\cos(\theta)\,d\theta) \wedge dz$

$=(r\cos^2(\theta)\,drd\theta - r\sin^2(\theta)\,d\theta dr)\wedge dz$

$=(r\left(\cos^2(\theta) + \sin^2(\theta)\right)drd\theta)\wedge dz$

$=r\ drd\theta dz$

7.

$\phi =\ f_1dx_1 +\ f_2dx_2 +\ f_3dx_3$

$\Rightarrow\ \ d\phi=(\ \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\ )dx_1dx_2\ +\ (\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}\ )\ dx_1dx_3\ +\ (\ \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\ )\ dx_2dx_3$

$\Rightarrow d\left(d\phi\right)=(\frac{\partial^2 f_2}{\partial x_3\partial x_1} - \frac{\partial^2 f_1}{\partial x_3\partial x_2})dx_3dx_1dx_2 +\ (\frac{\partial^2 f_3}{\partial x_2\partial x_1} - \frac{\partial^2 f_1}{\partial x_2\partial x_3})dx_2dx_1dx_3 + (\frac{\partial^2 f_3}{\partial x_1\partial x_2} - \frac{\partial^2 f_2}{\partial x_1\partial x_3})dx_1dx_2dx_3$

$=(\frac{\partial^2 f_2}{\partial x_3\partial x_1} - \frac{\partial^2 f_1}{\partial x_3\partial x_2} + \frac{\partial^2 f_1}{\partial x_2\partial x_3} - \frac{\partial^2 f_3}{\partial x_2\partial x_1} + \frac{\partial^2 f_3}{\partial x_1\partial x_2} - \frac{\partial^2 f_2}{\partial x_1\partial x_3})dx_1dx_2dx_3$

$=0$

8.

(a) $df\ = \sum \frac{\partial f}{\partial x_i}\ dx_i$

$\leftrightarrow \sum \frac{\partial f}{\partial x_i}\ U_i$ by (1)

$=\ grad\ f$

(b) $d\phi\ =\ (\ \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\ )dx_1dx_2\ +\ (\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}\ )\ dx_1dx_3\ +\ (\ \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\ )\ dx_2dx_3$

$=\ (\ \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\ )\ dx_2dx_3 + (\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1})\ dx_3dx_1 + (\ \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\ )dx_1dx_2$

$\leftrightarrow \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right)U_1 +\ \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right)U_2 +\ \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)U_3$ by(2)

$=\ curl\ V$

(c) $\eta=f_1dydz\ +\ f_2dzdx\ +\ f_3dxdy$

$d\eta=\frac{\partial f_1}{\partial x}dxdydz\ +\ \frac{\partial f_2}{\partial y}dxdydz\ +\ \frac{\partial f_3}{\partial z}dxdydz$

$div\ V = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} +\ \frac{\partial f_3}{\partial z}$

$\therefore\ \ d\eta \leftrightarrow (divV)dxdydz$

9.

$df = \sum \frac{\partial f}{\partial x_i}dx_i$, $dg = \sum \frac{\partial g}{\partial x_i}dx_i$

$df\ \wedge dg = \sum \frac{\partial f}{\partial x_i}dx_i \wedge \sum \frac{\partial g}{\partial x_i}dx_i$

$=(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy) \wedge (\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy)$

$=\ \frac{\partial f}{\partial x}\frac{\partial g}{\partial y}\ dxdy\ + \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\ dydx$

$=(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x})dxdy$

Thus

$$df \wedge dg = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} \mathrm{dxdy}$$

$$\begin{aligned} dg \wedge df &= (\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy) \wedge (\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy) \\ &= \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}\ dxdy + \frac{\partial g}{\partial y}\frac{\partial f}{\partial x}\ dydx \\ &= (\frac{\partial g}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial x})dxdy \\ &= (-\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial x})dxdy \\ &= -(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x})dxdy \\ &= -df \wedge dg \end{aligned}$$

1.7 1-3연습문제

2020년 5월 1일 금요일   오후 5:19

1.

(a)

$u^2 - v^2 = 0\,, 2uv = 0$

$\Rightarrow\ \ u = v = 0$

$\therefore)\ \ p = (0{,}0)$

(b)

$u^2 - v^2 = 8, 2uv = 6$

$=> uv = 3,\ |u| > |v|$

$\therefore)\ \ p = (3{,}1),\ p = (-3, -1)$

(c)

$u^2 - v^2 =\ u,\ 2uv = v$

$\Rightarrow\ u = 0\ or\ 1\,, v = 0$

$\therefore)\ p = (0{,}0), p = (1{,}0)$

2.

(a)

$v = 1$

$F(u, 1) = (u^2 - 1, 2u\,)$

$\alpha(u) = (u^2 - 1\,, 2u) = (x, y)$

$u = \frac{y}{2} \Rightarrow \left(\frac{y^2}{4} - 1, y\right)$

2.

(b)

$u = 1$

$F(1, v) = (1 - v^2, 2v)$

$\beta(v) = (1 - v^2, 2v) = (x, y)$

$v = \frac{y}{2} \Rightarrow \left(1 - \frac{y^2}{4}, y\right)$

2.

(c)

(i) $v = 0, 0 \le u \le 1$

$F(u, 0) = (u^2, 0) = (x, y) \Leftrightarrow y = 0, 0 \le x \le 1$

(ii) $u = 0, 0 \le v \le 1$

$F(0, v) = (-v^2, 0) = (x, y) \Leftrightarrow y = 0, -1 \le x \le 0$

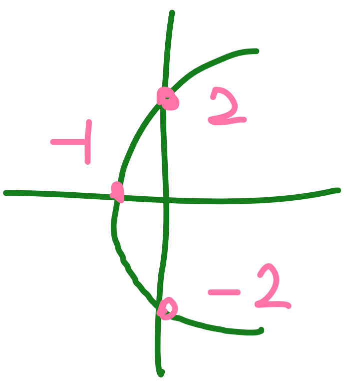
(iii) $v = 1, 0 \le u \le 1$

$F(u, 1) = (u^2 - 1\,, 2u) = (x, y) \Leftrightarrow x = \frac{y^2}{4} - 1, 0 \le y \le 2$

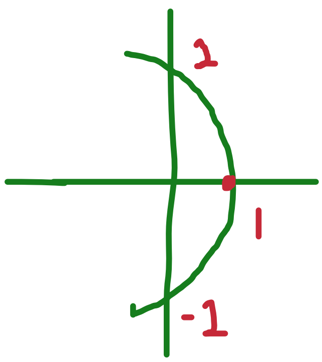
(iv) $u = 1, 0 \le v \le 1$

$F(1, v) = (1 - v^2, 2v) = (x, y) \Leftrightarrow x = 1 - \frac{y^2}{4}, 0 \le y \le 2$

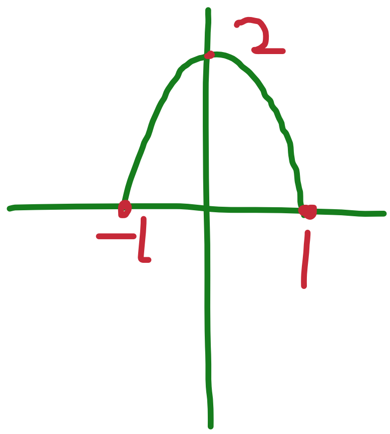
(a)



(b)



(c)



3.

$F_*(v) = \frac{d}{dt} F(p\ +\ tv)\ |_{t=0}$

$= \frac{d}{dt}\left((p_1 +\ tv_1)^2 -\ (p_2 +\ tv_2)^2, 2(p_1 + tv_1)(p_2 + tv_2)\right)|_{t=0}$

$= \left(2v_1(p_1 +\ tv_1) - 2v_2(p_2 +\ tv_2), 2v_2(p_1 + tv_1) + 2v_1(p_2 + tv_2)\right)|_{(t=0)}$

$= 2(p_1 v_1 - \ p_2 v_2, p_1 v_2 +\ p_2 v_1)$

2.1 7-10 연습문제

2020년 5월 7일 목요일 오후 2:01

7. If $\mathbf{u}$ is a unit vector, then the *component* of $\mathbf{v}$ in the $\mathbf{u}$ direction is

$$(\mathbf{v} \cdot \mathbf{u})\mathbf{u} = \|\mathbf{v}\| \cos\vartheta\, \mathbf{u}.$$

Show that $\mathbf{v}$ has a unique expression $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ and $\mathbf{v}_1$ is the component of $\mathbf{v}$ in the $\mathbf{u}$ direction.

8. Prove: The volume of the parallelepiped with sides $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is $\pm\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$ (Fig. 2.5). (*Hint:* Use the indicated unit vector $\mathbf{e} = \mathbf{v} \times \mathbf{w}/\|\mathbf{v} \times \mathbf{w}\|$.)

9. Prove, using ε-neighborhoods, that each of the following subsets of $\mathbf{R}^3$ is open:
(a) All points $\mathbf{p}$ such that $\|\mathbf{p}\| < 1$.
(b) All $\mathbf{p}$ such that $p_3 > 0$. (*Hint:* $|p_i - q_i| \leqq d(\mathbf{p}, \mathbf{q})$.)

10. In each case, let S be the set of all points $\mathbf{p}$ that satisfy the given condition. Describe S, and decide whether it is *open*.
(a) $p_1{}^2 + p_2{}^2 + p_3{}^2 = 1$.
(b) $p_3 \neq 0$.
(c) $p_1 = p_2 \neq p_3$.
(d) $p_1{}^2 + p_2{}^2 < 9$.

7.

$v_1 = (v \cdot u)u, v_2 = v - v_1$

(i)

$v_1 \cdot v_2 = ((v \cdot u)u) \cdot (v - (v \cdot u)u)$

$= (v \cdot u)^2 - (v \cdot u)^2|u|^2$

$= 0$ ($\because$ u:unit vector)

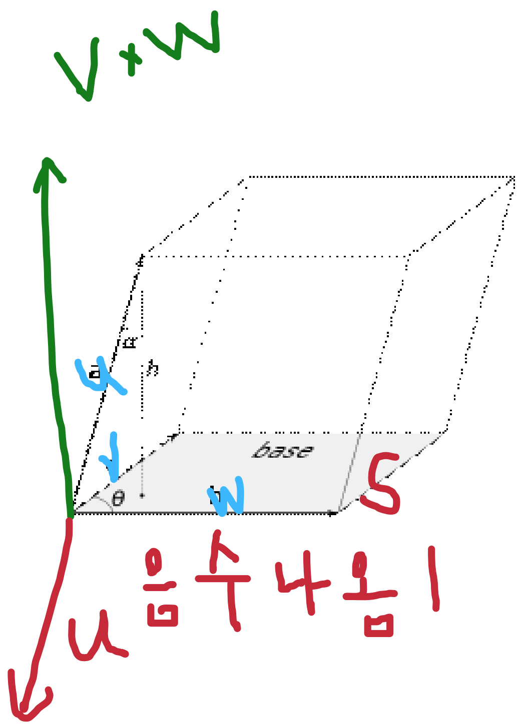
(ii)

Let $v = v_1 + v_2 = v_1 + w_2$

$v_1 \cdot v_2 = 0 = v_1 \cdot w_2$ ($\because$ (i))

Thus

$v_2 = w_2$



8.

Let $\| v \times w \| = \| v \| \| w \| \sin(\theta) = s$

$\Rightarrow \| v \times w \|$is the area of the parallelogram with sides v and w

$u \cdot v \times w = |u||v \times w| \cos(\alpha)$

$= sh$ ($\because$ $h = |u|\cos(\alpha)$)

Thus

$|u \cdot v \times w|$ is the volume of the parallelepied withe sides u, v, w

9.(a)

Sol)

Let $p \in B(0,1)$, $\varepsilon = 1 - \| p \|$

$\Rightarrow \exists B(p,\varepsilon)$ $s.t$ $p \in B(p,\varepsilon) \subset B(0,1)$

$\therefore$ open

9.(a)

Sol)

Let $p \in B(0,1)$, $\varepsilon = 1 - \| p \|$

$\Rightarrow \exists B(p,\varepsilon)\ s.t\ p \in B(p,\varepsilon) \subset B(0,1)$

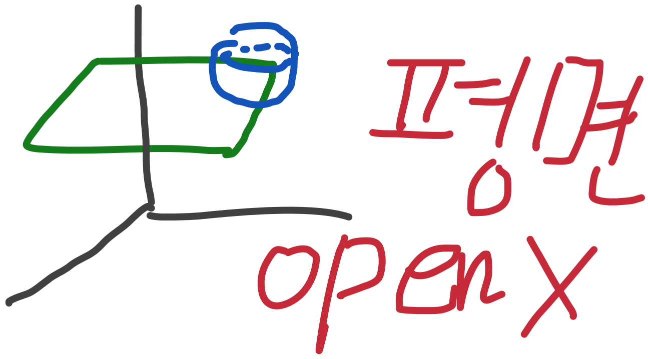
∴ open

9.(b)

Let $S = \{p : p_3 > 0\}$, $\epsilon = p_3$

$\forall p \in S$, $\exists B(p,\varepsilon)\ s.t\ p \in B(p,\varepsilon) \subset S$

∴ open



10.(a)

$S = \{p : p_1^2 + p_2^2 + p_3^2 = 1\}$

$p = (1,0,0) \in S$

$\Rightarrow \forall \epsilon > 0$, $\nexists\ B(p,\epsilon)$, $B(p,\epsilon) \subset S$

∴ S is not open

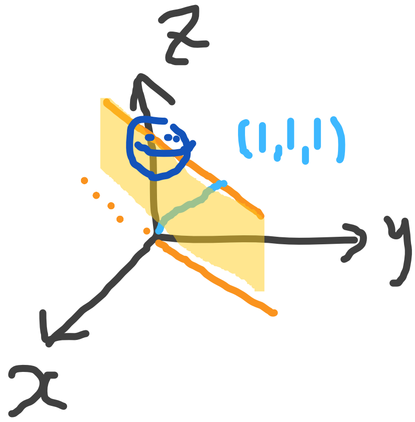
10.(b)

$S = \{p : p_3 \neq 0\}$

$\forall p \in S, \epsilon = |p_3|$

$\exists B(p,\varepsilon)\ s.t\ p \in B(p,\varepsilon) \subset S$

∴ S is open



10.(c)

$p = (0,0,1) \in S$

$\Rightarrow \forall \epsilon > 0$, $\nexists\ B(p,\epsilon)$, $B(p,\epsilon) \subset S$

∴ S is not open

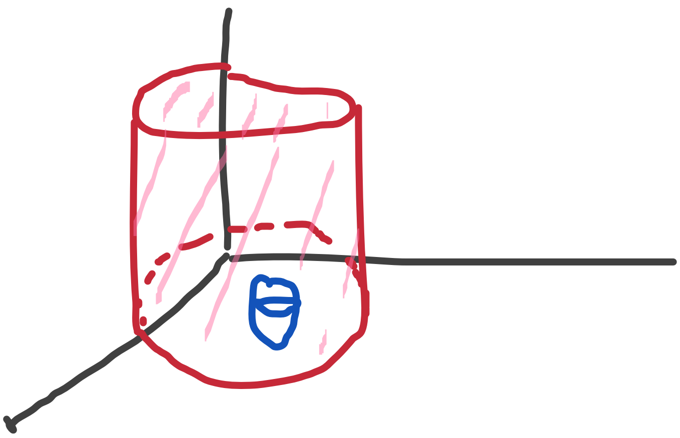
10.(d)

$S = \{p : p_1^2 + p_2^2 < 9\}$

$\forall p \in S$, $\epsilon = 3 - \sqrt{p_1^2 + p_2^2} > 0$

$\exists B(p,\varepsilon)\ s.t\ p \in B(p,\varepsilon) \subset S$

∴ S is open



2.2 1-5연습문제

2020년 5월 12일 화요일 오전 1:06

1.

For the curve $\alpha(t)=\left(2t, t^2, \frac{t^3}{3}\right)$

(a) find the velocity, speed, and acceleration for arbitrary t, and at t = 1;

(b) find the arc length function s = s(t) (based at t = 0), and determine

the arc length of a from t = -1 to t = +1.

$\alpha(t)=\left(2t, t^2, \frac{t^3}{3}\right)$

$\Rightarrow \alpha'(t)=\left(2, 2t, t^2\right)$, $\alpha(1) = (2,2,1)$

$\Rightarrow \| \alpha'(t) \| = \sqrt{4+4t^2+t^4} = 2+t^2$, $\| \alpha'(1) \| = 3$

$\Rightarrow \alpha''(t)=(0,2,2t)$, $\alpha''(1) = (0,2,2)$

$\Rightarrow s(t)=\int_0^t \| \alpha'(u) \| \, du = \int_0^t \left(t^2+2\right)du = \left[\frac{u^3}{3} + 2u\right]_0^t = \frac{t^3}{3} + 2t$

⇒t:-1..1

$s = \int_{-1}^{1}\left(t^2+2\right)dt = \left[\frac{t^3}{3} + 2t\right]_{-1}^{1} = \frac{1}{3} + 2 + \frac{1}{3} + 2 = \frac{14}{3}$

2.

Curve has constant speed iff its acceleration is everywhere orthogonal to its velocity

Pf)

⇒

Assume $\alpha: I \to R^3$ by $\alpha(t) = \left(\alpha_1(t), \alpha_2(t), \alpha_3(t)\right)$

Let $|\alpha'(t)| = K,\ K \in R$

$\Leftrightarrow K^2 = \left(\alpha_1'(t)\right)^2 + \left(\alpha_2'(t)\right)^2 + \left(\alpha_3'(t)\right)^2$

$\Leftrightarrow\ 0 = 2(\alpha_1'(t)\alpha_1''(t) + \alpha_2'(t)\alpha_2''(t) + \alpha_3'(t)\alpha_3''(t))$

$\Leftrightarrow\ 0 = \alpha'_1(t)\alpha_1''(t) + \alpha_2'(t)\alpha_2''(t) + \alpha_3'(t)\alpha_3''(t)$

$\Leftrightarrow\ 0 = (\alpha_1'(t), \alpha_2'(t), \alpha_3'(t)) \cdot (\alpha_1''(t), \alpha_2''(t), \alpha_3''(t))$

Thus

Curve has constant speed iff its acceleration of α is everywhere orthogonal to its velocity

3.보강시간

$\alpha(t)$=$(co\,sh(t), si\,nh(t), t)$

$\Rightarrow\ \alpha'(t)$=$(si\,nh(t), co\,sh(t), 1)$

$\Rightarrow\ = \sqrt{\sinh^2(t)+\cosh^2(t)+1}\ = \sqrt{2}\cosh(t) \qquad (\because\ \cosh^2(t)-\sinh^2(t)=1)$

$\Rightarrow\ s(t)= \int_0^t \sqrt{2}\cosh(u)\,du = \sqrt{2}\sinh(t)$

$\Rightarrow\ t = \sinh^{-1} \frac{s}{\sqrt{2}}$

$\therefore\ \beta(s) = (\cosh(\sinh^{-1}\frac{s}{\sqrt{2}}), \frac{s}{\sqrt{2}}, \sinh^{-1}\frac{s}{\sqrt{2}})$

4.보강시간

$\alpha(1)=(2,1,0)$and $\alpha(2)=\ (4,4,log2)$.

$$l=\int_1^2 \|\alpha'(t)\|\,dt = \int_1^2 \sqrt{2^2+(2t)^2+\left(\frac{1}{t}\right)^2}\,dt = \int_1^2 \left(2t+\frac{1}{t}\right)dt = \left[t^2+\ logt\right]_1^2 = (4+\log 2)-(1+0)=3+\log 2$$

5.

Suppose that β_1 and β_2 are unit-speed reparametrizations of the same curve α.

Show that there is a number s_0 such that $\beta_2(s)\ =\ \beta_1(s+s_0)$ for all s.

What is the geometric significance of s_0?

Pf)

let $\beta_1(s')=\alpha(t(s'))$, $\beta_2(s)=\alpha(t(s))$, $s'=s'(s)$

$\Rightarrow \beta_2(s)=\beta_1(s'(s))$

$\Rightarrow \frac{d\beta_2}{ds}=\frac{ds'}{ds}\frac{d\beta_1}{ds'}$

$\Rightarrow\ \ \|\frac{d\beta_2}{ds}\| = \|\frac{ds'}{ds}\frac{d\beta_1}{ds'}\| = \left|\frac{ds'}{ds}\right|\ \|\frac{d\beta_1}{ds'}\|$

But

since $\|\frac{d\beta_2}{ds}\|=\|\frac{d\beta_1}{ds'}\|=1$, $\left|\frac{ds'}{ds}\right|=1$

Thus

$$\frac{ds'}{ds}=\pm 1$$

If β_1,β_2 have the same direction as α, then $\frac{ds'}{ds}=1$

Therefore $s'=s+s_0$ for some s_0

$\therefore\ \ \beta_2(s)=\beta_1(s+\ s_0)$

s_0=$\int_{t_1}^{t_2} \|\ \alpha(u)\ \|\ du$

2.2 6-11 연습문제

2020년 5월 8일 금요일 오후 12:20

6.

Let Y be a vector field on the helix $\alpha(t) = (\cos(t), \sin(t), t)$.

In each of the following cases, express Y in the form $\sum y_i U_i$:

(a) $Y(t)$ is the vector from $\alpha(t)$ to the origin of R^3.

Sol)

$Y(t)$=(0,0,0) - $\alpha(t)$=$-(\cos(t), \sin(t), t)_{\alpha(t)}$

$\Rightarrow Y = -\cos(t)\, U_1 - \sin(t)\, U_2 - tU_3$

(b) $Y(t) = \alpha'(t)$ - $\alpha''(t)$.

Sol)

$Y(t)=(-\sin(t), \cos(t), 1)_{\alpha(t)} - (-\cos(t), -\sin(t), 0)_{\alpha(t)}$

$\Rightarrow Y=(\cos(t) - \sin(t))U_1 + (\cos(t) + \sin(t))U_2 + U_3$

(c) $Y(t)$ has unit length and is orthogonal to both $\alpha'(t)$ and $\alpha''(t)$.

Sol)

Let $Y(t)=(y_1, y_2, y_3)_{\alpha(t)}$

$y_1^2+y_2^2+y_3^2=1$,

$Y(t) \cdot \alpha'(t) = -y_1 \sin(t) + y_2 \cos(t) + y_3 = 0$,

$Y(t) \cdot \alpha''(t) = -y_1 \cos(t) - y_2 \sin(t) = 0$

$\Rightarrow Y(t)=(-\frac{1}{\sqrt{2}}\sin(t), \frac{1}{\sqrt{2}}\cos(t), -\frac{1}{\sqrt{2}})_{\alpha(t)}$ or $(\frac{1}{\sqrt{2}}\sin(t), -\frac{1}{\sqrt{2}}\cos(t), \frac{1}{\sqrt{2}})_{\alpha(t)}$

$\Rightarrow Y=-\frac{1}{\sqrt{2}}\sin(t)\, U_1 + \frac{1}{\sqrt{2}}\cos(t)\, U_2 - \frac{1}{\sqrt{2}}U_3$ or $\frac{1}{\sqrt{2}}\sin(t)\, U_1 - \frac{1}{\sqrt{2}}\cos(t)\, U_2 + \frac{1}{\sqrt{2}}U_3$

(d) $Y(t)$ is the vector from $\alpha(t)$ to $\alpha(t+\pi)$.

Sol)

$Y(t) = \alpha(t+\pi) - \alpha(t)$

$\quad = (-\cos(t), -\sin(t), t+\pi)_{\alpha(t)} - (\cos(t), \sin(t), t)_{\alpha(t)}$

$\Rightarrow Y = -2\cos(t)\, U_1 - 2\sin(t)\, U_2 + \pi U_3$

7.

₩int_a^b ₩parallel ₩alpha'(h(t)) ₩cdot h'(t) ₩parallel dt

A reparametrization $\alpha(h)$: $[c,d] \to R^3$of a curve segment α: $[a,b] \to R^3$ is monotone provided either

(i) $h' \geq 0, h(c) = a, h(d) = b$ **or** (ii) $h' \leq 0, h(c) = b, h(d) = a$.

Prove that monotone reparametrization does not change arc length.

Pf)

Prove that i) $h' \geq 0, h(c) = a, h(d) = b$

Let $\beta(t)=\alpha(h(t))$,$a \leq t \leq b$

$\int_a^b \| \beta'(t) \| dt = \int_a^b \| \alpha'(h(t)) \cdot h'(t) \| dt$

$\quad = \int_a^b \| \alpha'(h(t)) \| \cdot h'(t)\, dt$

$\quad = \int_c^d \| \alpha'(s) \| ds \quad (s = h(t))$

Thus monotone reparametrization does not change arc length.

8.

Let Y be a vector field on a curve α. If $\alpha(h)$ is a reparametrization of α,

show that the reparametrization $Y(h)$ is a vector field on $\alpha(h)$, and prove the chain rule $Y(h)' = h'Y'(h)$.

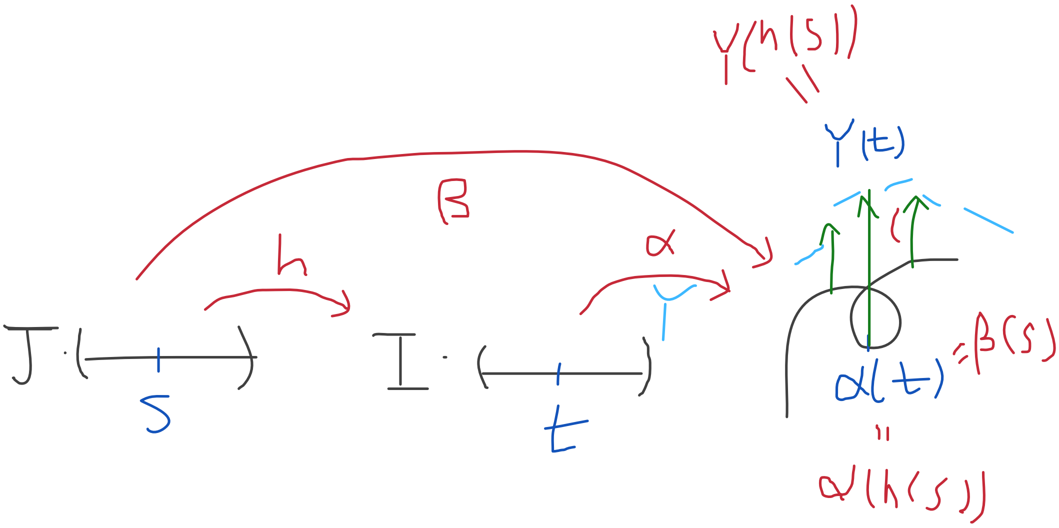
pf)

$Y(h) = (y_1(h), y_2(h), y_3(h))_{\alpha(h)}$

$\Rightarrow Y(h)' = (y_1(h)h', y_2(h)h', y_3(h)h')_{\alpha(h)}$

$=h'\left(y_1(h), y_{2(h)}, y_3(h)\right)_{\alpha(h)}$

$=h'Y'(h)$



9.

(Numerical integration.)

The curve segments

$\alpha(t)=(\sin(t), t^2\cos(t), \sin(2t))$, $\beta(t)=(t^2\sin(t), t^2, t^2(1+\cos(t)))$

defined on $0 \le t \le \pi$, run from the origin 0 to $(0, \pi^2, 0)$. Which is shorter?

Pf)

$L(\alpha)=\int_0^{\pi} \| \alpha'(t) \| \, dt$

$\quad = \int_0^{\pi} \| (\cos(t), 2t\cos(t) - t^2\sin(t), 2\cos(2t)) \| \, dt$

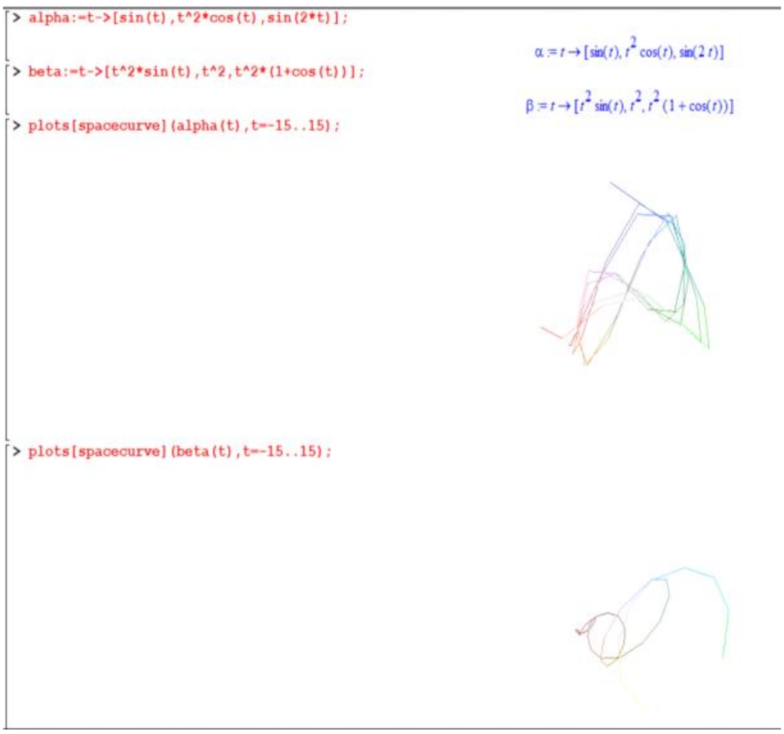
$\quad \simeq\ 12.9153$

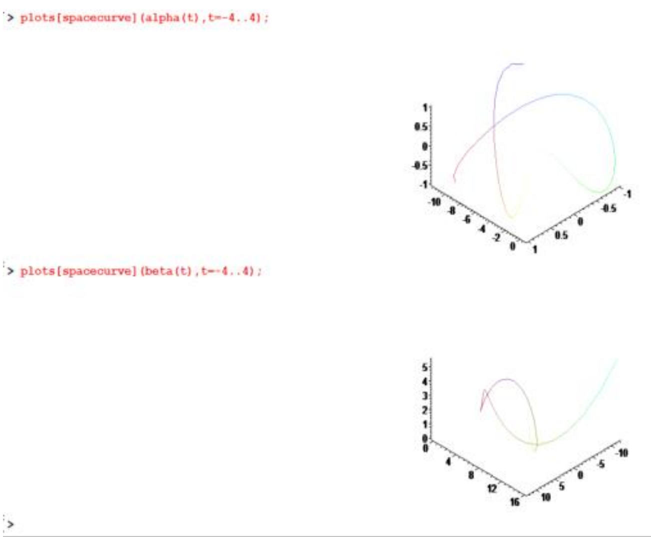
$L(\beta)=\int_0^{\pi} \| \beta'(t) \| \, dt$

$\quad = \int_0^{\pi} \| (2t\cos(t) + t^2\cos(t), 2t, 2t(1+\cos(t)) - t^2\sin(t)) \| \, dt$

$\quad \simeq 14.461$

$\therefore\ L(\beta) > L(\alpha)$





```
> alpha:=t->[sin(t),t^2*cos(t),sin(2*t)];
```

$$\alpha := t \to [\sin(t), t^2\cos(t), \sin(2t)]$$

```
> dalpha:=diff(alpha(t),t);
```

$$dalpha = [\cos(t), 2t\cos(t) - t^2\sin(t), 2\cos(2t)]$$

```
> int(sqrt((cos(t))^2+(2*t*cos(t)-t^2*sin(t))^2+(2*cos(2*t))^2),t=0..Pi);
```

$$\int_0^{\pi} \sqrt{\cos(t)^2 + (2t\cos(t) - t^2\sin(t))^2 + 4\cos(2t)^2}\, dt$$

```
> evalf(%);
```

$$12.91534010$$

```
> beta:=t->[t^2*sin(t),t^2,t^2*(1+cos(t))];
```

$$\beta := t \to [t^2\sin(t), t^2, t^2(1+\cos(t))]$$

```
> dbeta:=diff(beta(t),t);
```

$$dbeta = [2t\sin(t) + t^2\cos(t), 2t, 2t(1+\cos(t)) - t^2\sin(t)]$$

```
> int(sqrt((t*(2*sin(t)+t*cos(t)))^2+4*t^2+(t*(2+2*cos(t)-t*sin(t)))^2),t=0..Pi);
```

$$\int_0^{\pi} \sqrt{t^2(2\sin(t) + t\cos(t))^2 + 4t^2 + t^2(2 + 2\cos(t) - t\sin(t))^2}\, dt$$

```
> evalf(%);
```

$$14.46098839$$

10.보강시간

$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ and $\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t))$

$\left(\frac{d\alpha_1}{dt}(t), \frac{d\alpha_2}{dt}(t), \frac{d\alpha_3}{dt}(t)\right) = \alpha'(t) = \beta'(t) = \left(\frac{d\beta_1}{dt}(t), \frac{d\beta_2}{dt}(t), \frac{d\beta_3}{dt}(t)\right)$ by parallel

$\Rightarrow \dfrac{d\alpha_i}{dt}(t) = \dfrac{d\beta_i}{dt}(t) for\ i = 1,2,3\ and\ \forall t \in I$

$\Rightarrow \frac{d(\beta_i - \alpha_i)}{dt}(t) = 0\ for\ i = 1,2,3\ and\ \forall t \in I$

Then $\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t))$

$= (\alpha_1(t) + p_1, \alpha_2(t) + p_2, \alpha_3(t) + p_3)$

$= (\alpha_1(t), \alpha_2(t), \alpha_3(t)) + (p_1, p_2, p_3)$

$= \alpha(t) + P\ \ [where\ P = (p_1, p_2, p_3)]$

$\therefore \beta(t) = \alpha(t) + P\ \ \forall t \in I$ i.e. α and β are parallel

11.

Prove that a straight line is the shortest distance between two points in R3.

Use the following scheme; let α: $[a, b] \to R3$ be an arbitrary curve segment from $p = \alpha(a)$ to $q = \alpha(b)$.

Let $u = \frac{q-p}{\| q-p \|}$

(a) If σ is a straight line segment from p to q, say

$\sigma(t)$=$(1-t)p + tq$

show that $L(\sigma) = d(p, q)$.

Pf)

$\sigma(t)$=$(1-t)p + tq$=$p + t(q-p)$,$0 \le t \le 1$

$L(\sigma(t))$=$\int_0^1 \| \sigma'(t) \|\, dt$

$=\int_0^1 \| q-p \|\, dt$

$=\| q-p \|$

$=d(q-p)$

(b) From $\| \alpha' \| \ge \alpha' \cdot u$, deduce $L(\alpha) \ge d(p, q)$, where $L(\alpha)$ is the length of α and d is Euclidean distance.

Pf)

$\alpha' \cdot u = \| \alpha' \| \| u \| \cos(\theta)$

$= \| \alpha' \| \cos(\theta)$ ($\because \| u \| = 1$)

$\le \| \alpha' \|$

$L(\alpha) = \int_a^b \| \alpha'(t) \| \, dt$

$\ge \int_a^b \alpha'(t) \cdot u \, dt$

$= [\alpha(t) \cdot u]_a^b$

$= \alpha(b) \cdot u - \alpha(a) \cdot u$

$= (p - q) \cdot u$

$= \| p - q \|$

$= d(p, q)$

(c) Furthermore, show that if $L(\alpha) = d(p,q)$, then (but for parametrization) α is a straight line segment. (Hint: write $\alpha' = (\alpha' \cdot u)u + Y$, where $Y \cdot u = 0$.

Pf)

If $\alpha' = (\alpha' \cdot u)u + Y$, where $Y \cdot u = 0$.

Then $L(\alpha) = \int_a^b \| \alpha'(t) \| \, dt$

$= \int_a^b \sqrt{(\alpha'(t) \cdot u)^2 + \| Y(t) \|^2} \, dt$

By (11.(b)) $d(p,q) = \int_a^b \alpha'(t) \cdot u \, dt$

$= \int_a^b \sqrt{(\alpha'(t) \cdot u)^2} \, dt$

Thus

$L(\alpha) = d(p,q) \Rightarrow \| Y(c) \| = 0 \Rightarrow Y = 0$

Therefore

$\alpha'(t) = (\alpha'(t) \cdot u)u$

Thus

α is a straight line segment

2.3 1-4 연습문제

2020년 5월 13일 수요일 오후 4:31

1.

$\beta(s)=(\frac{4}{5}\cos(s), 1-\sin(s), -\frac{3}{5}\cos(s))$

$T(s)=\beta'(s)=(-\frac{4}{5}\sin(s), -\cos(s), \frac{3}{5}\sin(s)) \Rightarrow \| \beta'(s) \|=1$

$T'(s)=\beta''(s)=(-\frac{4}{5}\cos(s), \sin(s), \frac{3}{5}\cos(s))=N(s)$ by $\kappa(s)=\| T'(s) \|=1$

$B(s)=T(s)\times N(s) = \left(-\frac{3}{5}, 0, -\frac{4}{5}\right)$

$B'(s)=(0,0,0) \Rightarrow \tau = 0$

Thus

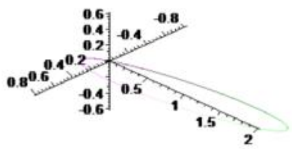
β is plane curve by cor3.5

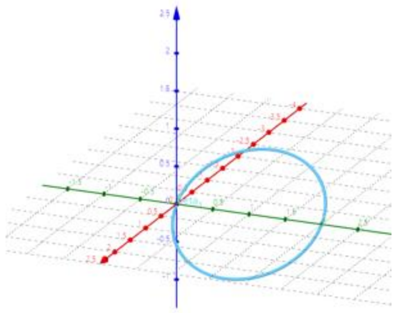
So

$\beta(s)+\frac{1}{\kappa}N(s)=(0,1,0) \Rightarrow \| \beta(s)-(0,1,0) \|=1$ by lemma 3.6

∴ center of $\beta(s)$: (0,1,0), radius of $\beta(s)$: 1

$\beta := s \rightarrow \left[\frac{4}{5}\cos(s), 1-\sin(s), \frac{3}{5}\cos(s)\right]$





2.

$\beta(s) = \left(\frac{(1+s)^{\frac{3}{2}}}{3}, \frac{(1-s)^{\frac{3}{2}}}{3}, \frac{s}{\sqrt{2}}\right)$

$T(s)=\beta'(s)=\left(\frac{(1+s)^{\frac{1}{2}}}{2}, -\frac{(1-s)^{\frac{1}{2}}}{2}, \frac{1}{\sqrt{2}}\right) \Rightarrow \| \beta'(s) \|=1$

$T'(s)=\beta''(s)=\left(\frac{(1+s)^{-\frac{1}{2}}}{4}, \frac{(1-s)^{-\frac{1}{2}}}{4}, 0\right) \Rightarrow \kappa(s)=\| T'(s) \|=\sqrt{\frac{1}{16}\left(\frac{1}{1+s}+\frac{1}{1-s}\right)}=\frac{1}{2\sqrt{2}(1+s)^{\frac{1}{2}}(1-s)^{\frac{1}{2}}}$

$N(s)=\frac{T'(s)}{\kappa(s)}=\left(\frac{(1-s)^{\frac{1}{2}}}{\sqrt{2}}, \frac{(1+s)^{\frac{1}{2}}}{\sqrt{2}}, 0\right)$

$B(s)=T(s)\times N(s) = \left(-\frac{(1+s)^{\frac{1}{2}}}{2}, \frac{(1-s)^{\frac{1}{2}}}{2}, \frac{1}{\sqrt{2}}\right)$

$-\tau N(s)=B'(s)=\left(-\frac{(1+s)^{-\frac{1}{2}}}{4}, \frac{(1-s)^{-\frac{1}{2}}}{4}, 0\right) \Rightarrow \tau=\frac{1}{2\sqrt{2}(1+s)^{\frac{1}{2}}(1-s)^{\frac{1}{2}}}$

3.

$T' = \kappa N$

∵)

$T'(s)=(-\frac{a}{c^2}\cos\left(\frac{s}{c}\right), -\frac{a}{c^2}\sin\left(\frac{s}{c}\right), 0)$

$\kappa(s)N(s)=\frac{a}{c^2}(-cos\left(\frac{s}{c}\right)-sin\left(\frac{s}{c}\right), 0)$

$N' = -\kappa T + \tau B$

∵)

$N'(s)=(\frac{1}{c}\sin\left(\frac{s}{c}\right), -\frac{1}{c}\cos\left(\frac{s}{c}\right), 0)$

$-\kappa(s)T(s)=-\frac{a}{c^2}\left(-\frac{a}{c}\sin\left(\frac{s}{c}\right), \frac{a}{c}\cos\left(\frac{s}{c}\right), \frac{b}{c}\right)$

$\tau(s)B(s)=\frac{b}{c^2}(\frac{b}{c}sin\left(\frac{s}{c}\right)-\frac{b}{c}cos\left(\frac{s}{c}\right), \frac{a}{c})$

$\Rightarrow -\kappa(s)T(s)+\tau(s)B(s)=(\frac{1}{c}\sin\left(\frac{s}{c}\right), -\frac{1}{c}\cos\left(\frac{s}{c}\right), 0)$

$B' = -\tau N$

∵)

$\beta'(s)=(\frac{b}{c^2}\cos\left(\frac{s}{c}\right), \frac{b}{c^2}\sin\left(\frac{s}{0}\right), 0)$

$-\tau(s)N(s) = -\frac{b}{c^2}(-cos\left(\frac{s}{c}\right), -sin\left(\frac{s}{c}\right), 0)$

3.3 Ex. The unit speed helix

$\beta(s) = (a\cos\frac{s}{c}, a\sin\frac{s}{c}, \frac{bs}{c})$ where $c = (a^2+b^2)^{1/2}$

$T(s) = \beta'(s) = (-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c})$

$T'(s) = (-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0)$

Thus

$\kappa(s) = |T'(s)| = \frac{a}{c^2} > 0$

$N(s) = (-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0)$

$B(s) = T(s)\times N(s) = (\frac{b}{c}\,sin\,\frac{s}{c}, -\frac{b}{c}\,cos\,\frac{s}{c}, \frac{a}{c}$

$B'(s) = (\frac{b}{c^2}\,cos\,\frac{s}{c}, \frac{b}{c^2}\,sin\,\frac{s}{c}, 0)$

$B'(s) = -\tau(s)N(s) \Rightarrow \tau(s) = \frac{b}{c^2}$

4.

$B = T\times N \Rightarrow 1 = B\cdot T\times N = B\times T\cdot N$ (by def&2.1.ex.4)

Thus

$N = B\times T$

$N = B\times T \Rightarrow 1 = N\cdot B\times T = N\times B\cdot T$

Thus

$T = N\times B$

By alternation rule

$B = -N\times T$, $T = -B\times N$, $N = -T\times B$

2.3 5-6-(i) 연습문제

2020년 5월 14일 목요일 오전 11:50

$\overline{\mathrm{N}}$ $\overline{\mathrm{B}}$ $\overline{\tau}$ $\widehat{\beta}$ $\overline{\mathrm{T}}$ $\tilde{\kappa}\gamma$

$\overline{\mathrm{N}}\overline{\mathrm{B}}\overline{\tau}\widehat{\beta}\overline{\mathrm{T}}\tilde{\kappa}\gamma$

\bot

5.
$A = \tau T + \kappa B$
$T' = A \times T$
$\because$
$A \times T = (\tau T + \kappa B) \times T = \tau T \times T + \kappa B \times T = \kappa B \times T = \kappa N$ ($\because$ $N = B \times T$) $= T'$

$N' = A \times N$
$\because$
$A \times N = (\tau T + \kappa B) \times N = \tau T \times N + \kappa B \times N = \tau B - \kappa T$ ($\because$ $B = T \times N, -T = B \times N$) $= N'$

$B' = A \times B$
$\because$
$A \times B = (\tau T + \kappa B) \times B = \tau T \times B + \kappa B \times B = \tau T \times B = -\tau N$ ($\because$ $-N = T \times B$) $= B'$

6.
$\gamma(s) = c + r\cos\frac{s}{r}e_1 + r\sin\frac{s}{r}e_2$
i)
if
$\gamma_1(s) = c_1 + r_1\cos\frac{s}{r_1}e_1 + r_1\sin\frac{s}{r_1}e_2$,$\gamma_2(s) = c_2 + r_2\cos\frac{s}{r_2}e_1' + r_2\sin\frac{s}{r_2}e_2{}'$
then
$\gamma_1'(s) = -\sin\frac{s}{r_1}e_1 + \cos\frac{s}{r_1}e_2$,$\gamma'_2(s) = -\sin\frac{s}{r_2}e_1{}' + \cos\frac{s}{r_2}e_2{}'$
$\gamma_1''(s) = -\frac{1}{r_1}\cos\frac{s}{r_1}e_1 - \frac{1}{r_1}\sin\frac{s}{r_1}e_2$,$\gamma''_2(s) = -\frac{1}{r_2}\cos\frac{s}{r_2}e_1' - \frac{1}{r_2}\sin\frac{s}{r_2}e_2'$
$\Rightarrow$ $\beta(0) = \gamma_1(0) = \gamma_2(0)$ $\Rightarrow$ $c_1 + r_1e_1 = c_2 + r_2e_1'$

$\Rightarrow$ $\beta'(0) = \gamma'_1(0) = \gamma'_2(0)$ $\Rightarrow$ $e_2 = e_2'$

$\Rightarrow$ $\beta''(0) = \gamma''_1(0) = \gamma''_2(0)$ $\Rightarrow$ $-\frac{1}{r_1}e_1 = -\frac{1}{r_2}e_1'$
Thus
$\exists\gamma$

2.5 2연습문제

2020년 5월 22일 금요일 오후 10:48

EX. 2.5(2)

2번 - (1)

$V = -yU_1 + xU_3, \quad W = \cos x\, U_1 + \sin x\, U_2$

(a) $\nabla_V W$

$\nabla_V W = \sum V[w_i] U_i$

$= V[\cos x] U_1 + V[\sin x] U_2 + 0$

$= y \sin x\, U_1 - y \cos x\, U_2$

$-yU_1[\cos x] = -y \cdot -\sin x$

$-yU_1[\sin x] = -y \cdot \cos x$

(b) $\nabla_V V$

$\nabla_V V = \sum V[v_i] U_i$

$= V[-y] U_1 + 0 + V[x] U_3$

$= 0 + 0 - yU_3$

$xU_3[-y] = 0$

$-yU_1[x] = -y$

(c) $\nabla_V (z^2 W)$

$\nabla_V (z^2 W) = \sum V[z^2 w_i] U_i$

$= \sum (V[z^2] w_i + z^2 V[w_i]) U_i$

$= V[z^2] W + z^2 V[\cos x] U_1 + z^2 V[\sin x] U_2 + 0$

$= 2xz \cos x\, U_1 + 2xz \sin x\, U_2 + z^2 y \sin x\, U_1 - z^2 y \cos x\, U_2$

$= (yz^2 \sin x + 2xz \cos x) U_1 + (2xz \sin x - yz^2 \cos x) U_2$

$V[z^2] = xU_3[z^2]$

$= 2xz$

(d) $\nabla_W V$

$\nabla_W V = \sum W[v_i] U_i$

$= W[-y] U_1 + 0 + W[x] U_3$

$= -\sin x \, U_1 + \cos x \, U_3$

(∵) $W[-y] = \sin x \, U_2[-y]$

$= -\sin x$

$W[x] = \cos x \, U_1[x] = \cos x$

(e) $\nabla_V (\nabla_V W)$

(a) $\nabla_V W = y \sin x \, U_1 - y \cos x \, U_2 = E$

$\nabla_V (\nabla_V W) = \nabla_V E = \sum V[e_i] U_i$

$= V[y \sin x] U_1 + V[-y \cos x] U_2 + 0$

$= -y^2 \cos x \, U_1 - y^2 \sin x \, U_2$

(∵) $V[y \sin x] = V[y] \cdot \sin x + y \cdot V[\sin x] = 0 + y \cdot -y \cos x = -y^2 \cos x$

$V[-y \cos x] = V[-y] \cdot \cos x - y V[\cos x] = 0 - y \cdot -y \cdot -\sin x = -y^2 \sin x$

(f) $\nabla_V (xV - zW)$

$\nabla_V xV - \nabla_V zW = V[x] \cdot V + x \nabla_V V - V[z] \cdot W - z \nabla_V W$

(∵) $V[x] = -y$, $V[z] = x$

$= +y^2 U_1 - xy U_3 - xy U_3 - x \cos x \, U_1 - x \sin x \, U_2 - zy \sin x \, U_1 + zy \cos x \, U_2$

$= (y^2 - x \cos x - yz \sin x) U_1 - (x \sin x - yz \cos x) U_2 - 2xy U_3$

2번-(2)

공변도함수 그림그리기

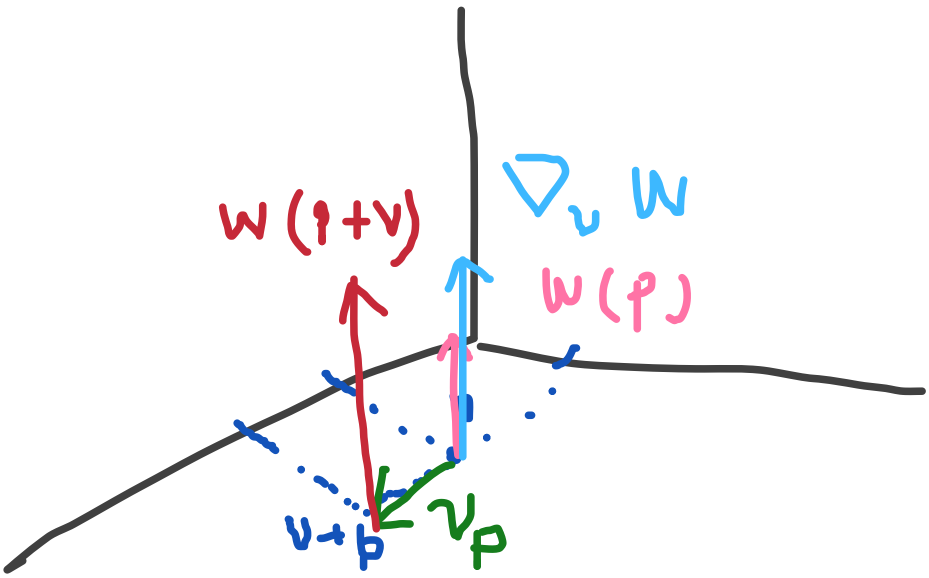
2020년 6월 4일 목요일　　오후 5:53

$W = xU_3 \;\; p = (1,1,0)\;,\; v = (1,0\,,0)$

$p + tv = (1 + t, 1, 0)$

$W(p + tv) = (1 + t)U_3$

$\nabla_v W = U_3$



2.8 3-4연습문제

2020년 6월 3일 수요일 오후 8:18

$$A = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3.(a)

$dx=\cos\theta dr\text{-}r sin\theta d\theta$

$dy=\sin\theta\, dr+r cos\theta d\theta$

$dz=dz$

$\Rightarrow \theta_1 = \cos\theta\ (\cos\theta dr\text{-}r sin\theta d\theta) + \sin\theta\ (\sin\theta\, dr+r cos\theta d\theta)$

$=\cos^2\theta dr\text{-}r sin\theta \cos\theta\, d\theta+\sin^2\theta dr\ +r sin\theta \cos\theta\, d\theta$

$=dr$

$\theta_2=-\sin\theta(\cos\theta dr\text{-}r sin\theta d\theta)+\cos\theta(\sin\theta\, dr+r cos\theta d\theta)$

$=-\cos\theta \sin\theta dr+r sin^2\theta d\theta\ +\cos\theta \sin\theta dr+r cos^2\theta d\theta$

$=r d\theta$

$\theta_3=dz$

3.(b)

$E_1[r]=dr\left(E_1\right)=\theta_1\left(E_1\right) = \mathrm{E}_1 \cdot \mathrm{E}_1 = 1$

$E_2[\theta]=d\theta\left(E_2\right)=\frac{1}{r}\theta_2(E_2)=\frac{1}{r}$

$E_3[z]=dz\left(E_3\right)=\theta_3\left(E_3\right)=1$

$E_1[\theta]=d\theta\left(E_1\right)=\frac{1}{r}\theta_2\left(E_1\right)=0$

$E_1[z]=dz\left(E_1\right)=\theta_3\left(E_1\right)=0$

$E_2[r]=dr\left(E_2\right)=\theta_1\left(E_2\right)=0$

$E_2[z]=dz\left(E_2\right)=\theta_3\left(E_2\right)=0$

$E_3[r]=dr\left(E_3\right)=\theta_1\left(E_3\right)=0$

$E_3[\theta]=d\theta\left(E_3\right)=\frac{1}{r}\theta_2\left(E_3\right)=0$

$\because\ \theta_i\left(E_i\right) = \delta_{ij}$

3.(c)

$E_1\left[f\right]= df\left(E_1\right)=(\frac{\partial f}{\partial r}dr+\frac{\partial f}{\partial \theta}d\theta+\frac{\partial f}{\partial z}dz)(E_1)=\frac{\partial f}{\partial r}dr(E_1)=\frac{\partial f}{\partial r}$

$E_2\left[f\right]=df\left(E_2\right)=(\frac{\partial f}{\partial r}dr+\frac{\partial f}{\partial \theta}d\theta+\frac{\partial f}{\partial z}dz)(E_2)=\frac{\partial f}{\partial \theta}d\theta(E_2)=\frac{1}{r}\frac{\partial f}{\partial \theta}$

$E_3\left[f\right]=df\left(E_3\right)=(\frac{\partial f}{\partial r}dr+\frac{\partial f}{\partial \theta}d\theta+\frac{\partial f}{\partial z}dz)(E_3)=\frac{\partial f}{\partial z}dz(E_3)=\frac{\partial f}{\partial z}$

$\because$ 3.(b)

4.(a)

$$A=\begin{pmatrix} cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \ \&\ x = r\cos\psi,\ y = r\sin\psi, z = z$$

$$\Rightarrow d\xi = \begin{pmatrix} \cos\psi dr \text{ -}rsin\psi d\psi \\ \sin\psi\, dr + rcos\psi d\psi \\ dz \end{pmatrix}$$

$$\Rightarrow \theta = A\cdot d\xi = \begin{pmatrix} cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\psi dr \text{ -}rsin\psi d\psi \\ \sin\psi\, dr + rcos\psi d\psi \\ dz \end{pmatrix} = \begin{pmatrix} dr \\ rd\psi \\ dz \end{pmatrix}$$

$$\Rightarrow \omega = dA\,{}^{t}A=\begin{pmatrix} -sin\psi d\psi & cos\,\psi\, d\psi & 0 \\ -\,cos\psi\, d\psi & -sin\,\psi d\,\psi & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & d\psi & 0 \\ -d\psi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \text{ connection form: } \omega = \begin{pmatrix} 0 & d\psi & 0 \\ -d\psi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ dual 1-form:}\theta = \begin{pmatrix} dr \\ rd\psi \\ dz \end{pmatrix}$$

4.(b)

First Structural equation

$d\theta\ =\ \omega\theta$

$\because$

$$d\theta=\begin{pmatrix} 0 \\ drd\psi \\ 0 \end{pmatrix}$$

$$\omega\theta =\begin{pmatrix} 0 & d\psi & 0 \\ -d\psi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} dr \\ rd\psi \\ dz \end{pmatrix}$$

$$=\begin{pmatrix} 0 \\ drd\psi \\ 0 \end{pmatrix}$$

Second structural equation

$d\omega= \omega\omega$

$\because$

$$d\omega =\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\omega\omega =\begin{pmatrix} 0 & d\psi & 0 \\ -d\psi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & d\psi & 0 \\ -d\psi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$