p.135~136 Ex4.1.5 (9/22)

1.5 Ex. (A surface of revolution) M

The curve $C : f(x, y) = c$

$p = (p_1, p_2, p_3) \in M \Leftrightarrow \overline{p} = \left(p_1, \sqrt{p_2^2 + p_3^2}, 0\right) \in C$

$\Rightarrow M : g(x, y, z) = f\left(x, \sqrt{y^2 + z^2}\right) = c$

$\Rightarrow dg\ is\ not\ zero$

$\because$

$$
\begin{aligned}
dg &= \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz \\
&= \frac{\partial}{\partial x}\left(f\left(x, \sqrt{y^2 + z^2}\right)\right)dx + \frac{\partial}{\partial y}\left(f\left(x, \sqrt{y^2 + z^2}\right)\right)dy + \frac{\partial}{\partial z}\left(f\left(x, \sqrt{y^2 + z^2}\right)\right)dz \\
&= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\frac{\partial}{\partial y}\left(\sqrt{y^2 + z^2}\right) + \frac{\partial f}{\partial y}\frac{\partial}{\partial z}\left(\sqrt{y^2 + z^2}\right) \\
&= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\left(\frac{ydy}{\sqrt{y^2 + z^2}} + \frac{zdz}{\sqrt{y^2 + z^2}}\right)
\end{aligned}
$$

$\neq 0$

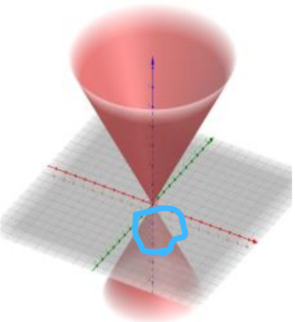
$\because$ $f = c$ is curve $\Rightarrow \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \neq 0$ and $y > 0 \Rightarrow \sqrt{y^2 + z^2} > 0$

p.137 Exercises4-1,(1-4)

2020년 9월 24일 목요일 오전 11:25

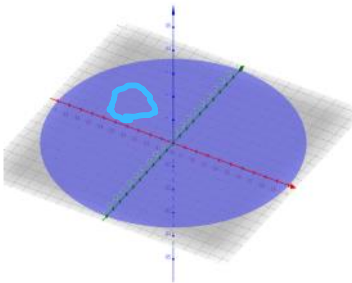
1. None of the following subsets M of $\mathbf{R}^3$ are surfaces. At which points $\mathbf{p}$ is it impossible to find a proper patch in M that will cover a neighborhood of $\mathbf{p}$ in M? (Sketch M—formal proofs not required.)
(a) Cone M: $z^2 = x^2 + y^2$
(b) Closed disk M: $x^2 + y^2 \leqq 1$, $z = 0$.
(c) Folded plane M: $xy = 0$, $x \geqq 0$, $y \geqq 0$.

1-(a)



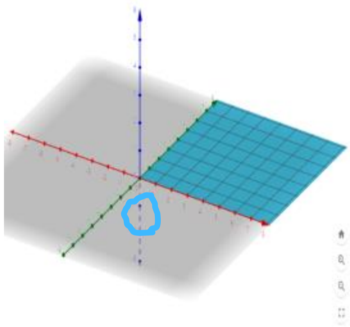
Vertex

1-(b)



$x^2 + y^2 = 1$

1-(c)



Z axis

2. A *plane* in $\mathbf{R}^3$ is a surface M: $ax + by + cz = d$, where the numbers a, b, c are necessarily not all zero. Prove that every plane in $\mathbf{R}^3$ may be described by a vector equation as on page 62.

p.62

A *plane* in $\mathbf{R}^3$ can be described as the union of all the perpendiculars to a given line at a given point. In vector language then, the *plane through* $\mathbf{p}$ *orthogonal to* $\mathbf{q} \neq 0$ consists of all points $\mathbf{r}$ in $\mathbf{R}^3$ such that $(\mathbf{r} - \mathbf{p}) \cdot \mathbf{q} = 0$. By the remark above, we may picture $\mathbf{q}$ as a tangent vector at $\mathbf{p}$ as shown in Fig. 2.9.

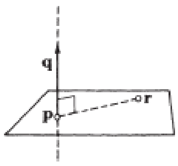


FIG. 2.9

Let $q = (a, b, c), r = (x, y, z), p = (x_0, y_0, z_0)$

$\Rightarrow (r - p) \cdot q = (x - x_0, y - y_0, z - z_0) \cdot (a, b, c)$

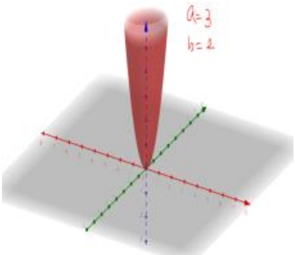
$= a(x - x_0) + b(y - y_0) + c(z - z_0)$

$= ax + by + cz - (ax_0 + by_0 + cz_0) = 0$

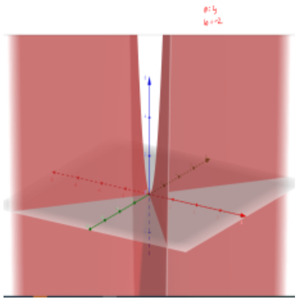
thus

plane in R^3 is $ax + by + cz = d$ where $d = (ax_0 + by_0 + cz_0)$

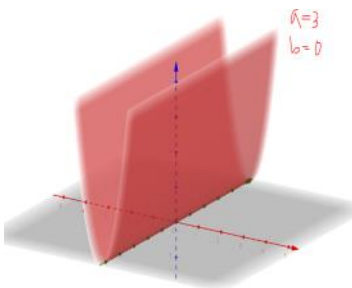
3. Sketch the general shape of the surface M: $z = ax^2 + by^2$ in each of the following cases:
(a) $a > b > 0$. (b) $a > 0 > b$.
(c) $a > b = 0$. (d) $a = b = 0$.



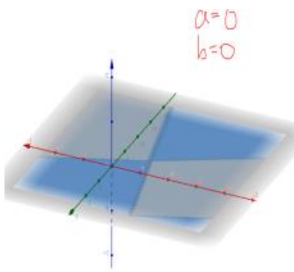
a=3, b=2



a=3,b=-2



a=3,b=0



a=b=0

4. In which of the following cases is the mapping $\mathbf{x}: \mathbf{R}^2 \to \mathbf{R}^3$ a patch?

(a) $\mathbf{x}(u, v) = (u, uv, v)$. (b) $\mathbf{x}(u, v) = (u^2, u^3, v)$.

(c) $\mathbf{x}(u, v) = (u, u^2, v + v^3)$. (d) $\mathbf{x}(u, v) = (\cos 2\pi u, \sin 2\pi u, v)$.

(Recall that $\mathbf{x}$ is one-to-one if and only if $\mathbf{x}(u, v) = \mathbf{x}(u_1, v_1)$ implies $(u, v) = (u_1, v_1)$.)

(a)

Let $x(u,v) = x(u_1, v_1)$

$\Rightarrow (u, uv, v) = (u_1, u_1v_1, v_1)$

$\Rightarrow (u,v) = (u_1, v_1)$

Thus x is 1-1 mapping

$J(x) = \begin{pmatrix} 1 & v & 0 \\ 0 & u & 1 \end{pmatrix}$ has rank2 ⇒ x is a regular

Thus x is patch

(b)

$J(x) = \begin{pmatrix} 2u & 3u^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has rank1 => x is not regular

(c)

Let $x(u,v) = x(u_1, v_1)$

$\Rightarrow (u, u^2, v + v^3) = (u_1, u_1^2, v_1 + v_1^3)$

$\Rightarrow (u,v) = (u_1, v_1)$

thus x is 1-1 mapping

$J(x) = \begin{pmatrix} 1 & 2u & 0 \\ 0 & 0 & 3v^2 + 1 \end{pmatrix}$has rank2 ⇒ x is a regular mapping

thus x is patch

(d)

x(0,0)=(1,0,0)=x(1,0)

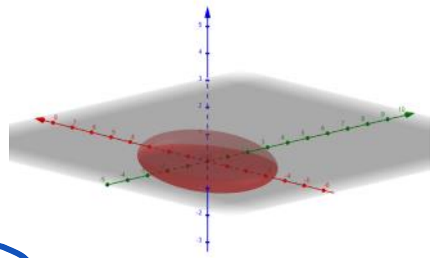
Thus x is not 1-1 mapping

p.148 Exercises4-2,9

2020년 10월 6일 화요일 오후 3:19

9. In each case, (i) show that M is a surface, and sketch its general shape when $a = 3$, $b = 2$, $c = 1$; (ii) show that $\mathbf{x}$ is a parametrization in M and describe what part of M it covers.

(a) *Ellipsoid* $M: \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$,
$\mathbf{x}(u, v) = (a\cos u \cos v, b\cos u \sin v, c \sin u)$ on $D: -\pi/2 < u < \pi/2$.



i)

Let $g(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

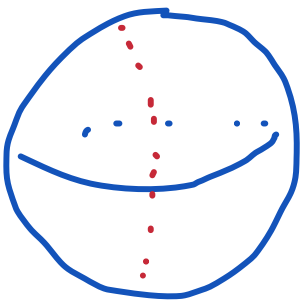
$(F\phi)(V_p)$
$= f(p)\phi(V_p)$

$\Rightarrow dg = \frac{2x}{a^2}dx + \frac{2y}{b^2}dy + \frac{2z}{c^2}dz$

$dg(v_0) = \frac{2x}{a^2}(0)dx(v_0) + \frac{2y}{b^2}(0)dy(v_0) + \frac{2z}{c^2}(0)dx(v_0)$

$= 0 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3$ for all $v_0 \in T_0(R^3)$

$= 0 + 0 + 0 = 0$



but $0 = p = (0,0,0) \notin M$

thus $dg \neq 0$ at any point of M

$\therefore M$ is surface

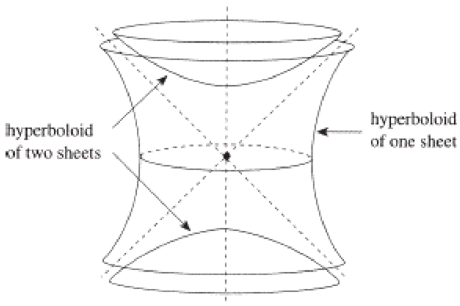
ii) 2.2 Geographical patch

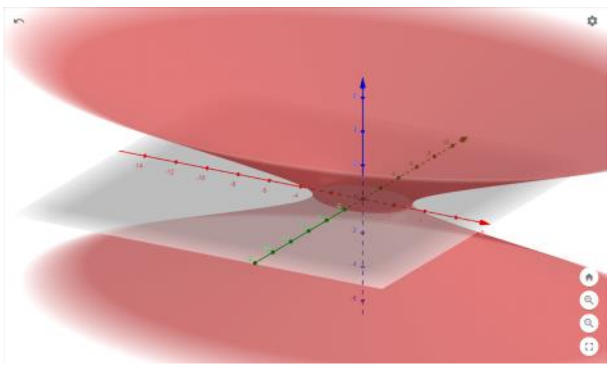
$x_u = (-asinucosv, -bsinusinv, ccosu)$
$x_v = (-acosusinv, bcosucosv, 0)$

$x_u \times x_v = \begin{vmatrix} U_1 & U_2 & U_3 \\ -asinucosv & -bsinusinv & ccosu \\ -acosusinv & bcosucosv & 0 \end{vmatrix} = -bccos^2ucosvU_1 + accos^2 u\, sinvU_2 - absinucosuU_3 \neq 0$ where $D: -\frac{\pi}{2} < u < \frac{\pi}{2}$

$\therefore x$ is parametrization of $x(D)$ in M

(b) *Hyperboloid of one sheet* (Fig. 4.21).
$M: \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, $\mathbf{x}(u, v) = (a\cosh u\cos v, b\cosh u \sin v, c\sinh u)$
on $\mathbf{R}^2$.





i)

Let $g(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

$\Rightarrow dg = \frac{2x}{a^2}dx + \frac{2y}{b^2}dy - \frac{2z}{c^2}dz$

$dg(v_0) = \frac{2x}{a^2}(0)dx(v_0) + \frac{2y}{b^2}(0)dy(v_0) - \frac{2z}{c^2}(0)dx(v_0)$

$= 0 \cdot v_1 + 0 \cdot v_2 - 0 \cdot v_3$ for all $v_0 \in T_0(R^3)$

$= 0 + 0 - 0 = 0$

but $0 = p = (0,0,0) \notin M$

thus $dg \neq 0$ at any point of M

$\therefore M$ is surface

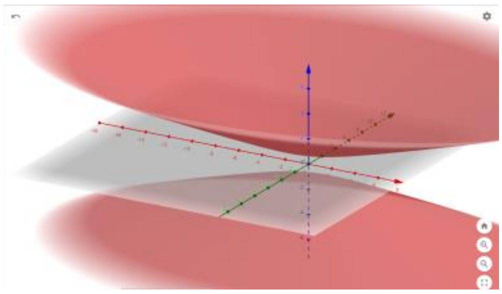
ii)

$x_u = (asinhucosv, bsinhusinv, ccoshu)$
$x_v = (-acoshusinv, bcoshucosv, 0)$

$x_u \times x_v = \begin{vmatrix} U_1 & U_2 & U_3 \\ asinhucosv & bsinhusinv & ccoshu \\ -acoshusinv & bcoshucosv & 0 \end{vmatrix} = -bccosh^2ucosvU_1 - accosh^2\,u\,sinvU_2 + absinhucoshuU_3 \neq 0$ where $D = R^2$

$\therefore$ x is parametrization in M

(c) *Hyperboloid of two sheets* (Fig. 4.21).
$M: \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$, $\mathbf{x}(u, v) = (a \sinh u \cos v, b \sinh u \sin v, c \cosh u)$
on D: $u \neq 0$.



i)

Let $g(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

$\Rightarrow dg = \frac{2x}{a^2}dx + \frac{2y}{b^2}dy - \frac{2z}{c^2}dz$

$dg(v_0) = \frac{2x}{a^2}(0)dx(v_0) + \frac{2y}{b^2}(0)dy(v_0) - \frac{2z}{c^2}(0)dx(v_0)$

$= 0 \cdot v_1 + 0 \cdot v_2 - 0 \cdot v_3$ for all $v_0 \in T_0(R^3)$

$= 0 + 0 - 0 = 0$

but $0 = p = (0,0,0) \notin M$

thus $dg \neq 0$ at any point of M

$\therefore M$ is surface

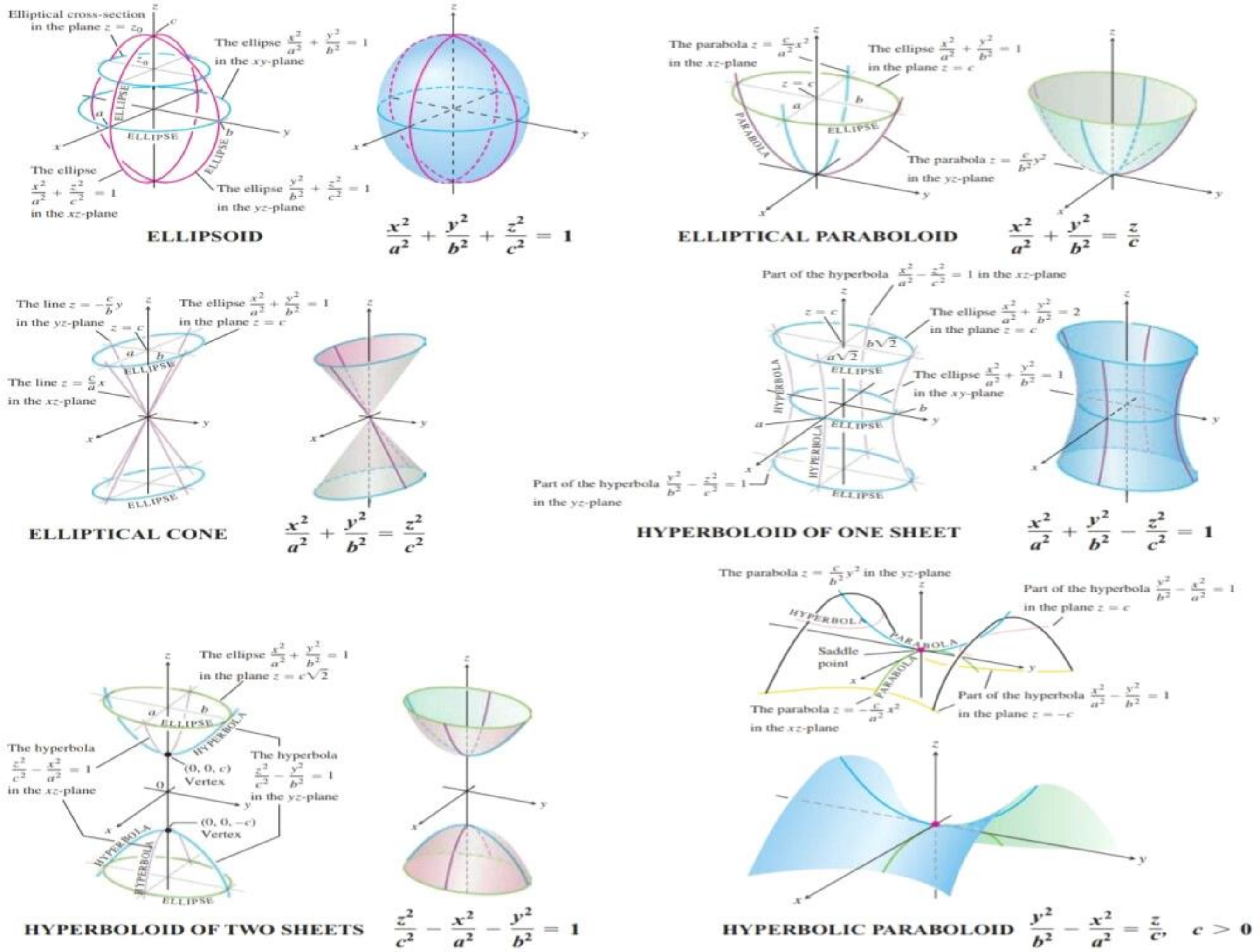
ii)

$x_u = (acoshucosv, bcoshusinv, csinhu)$

$x_v = (-asinhusinv, bsinhucosv, 0)$

$x_u \times x_v = \begin{vmatrix} U_1 & U_2 & U_3 \\ acoshucosv & bcoshusinv & csinhu \\ -asinhusinv & bsinhucosv & 0 \end{vmatrix} = -bcsinh^2ucosvU_1 - acsinh^2\,u\,sinvU_2 + absinhucoshuU_3 \neq 0$ where $D: u \neq 0$

$\therefore$ x is parametrization in M


Elliptical cross-section in the plane $z = z_0$
The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the xy-plane
The ellipse $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$ in the xz-plane
The ellipse $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ in the yz-plane
ELLIPSOID
$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
The parabola $z = \frac{c}{a^2}x^2$ in the xz-plane
The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the plane $z = c$
The parabola $z = \frac{c}{b^2}y^2$ in the yz-plane
ELLIPTICAL PARABOLOID
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$
The line $z = -\frac{c}{b}y$ in the yz-plane
The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the plane $z = c$
The line $z = \frac{c}{a}x$ in the xz-plane
ELLIPTICAL CONE
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$
Part of the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ in the xz-plane
The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$ in the plane $z = c$
The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the xy-plane
Part of the hyperbola $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ in the yz-plane
HYPERBOLOID OF ONE SHEET
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the plane $z = c\sqrt{2}$
The hyperbola $\frac{z^2}{c^2} - \frac{x^2}{a^2} = 1$ in the xz-plane
$(0, 0, c)$ Vertex
The hyperbola $\frac{z^2}{c^2} - \frac{y^2}{b^2} = 1$ in the yz-plane
$(0, 0, -c)$ Vertex
HYPERBOLOID OF TWO SHEETS
$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
The parabola $z = \frac{c}{b^2}y^2$ in the yz-plane
Part of the hyperbola $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ in the plane $z = c$
Saddle point
The parabola $z = -\frac{c}{a^2}x^2$ in the xz-plane
Part of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in the plane $z = -c$
HYPERBOLIC PARABOLOID
$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, \quad c > 0$

p.151 corollary 3.3

2020년 10월 13일 화요일 오후 2:41

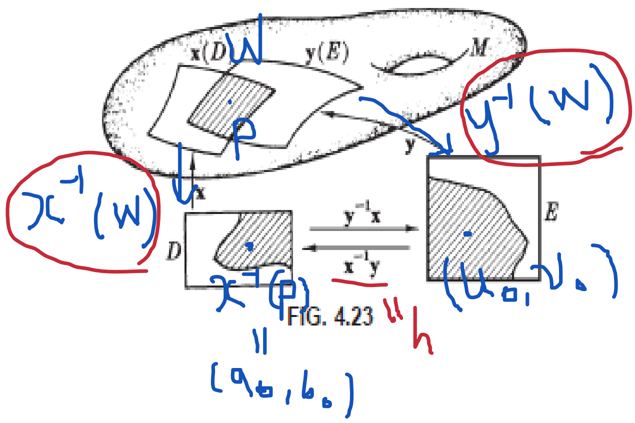
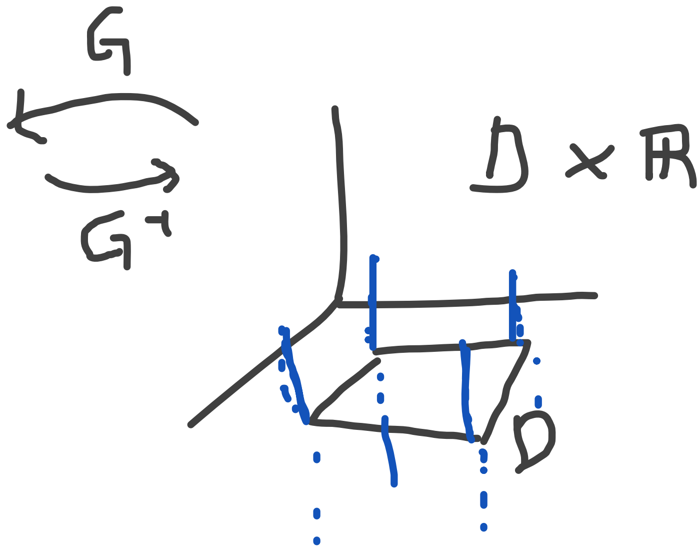


FIG. 4.23



Suppose M is a surface and $x : D \to M$ and $y : E \to M$ are coordinate patch with $W = x(D) \cap y(E) \subset M$

$h = x^{-1}(y) : y^{-1}(W) \to x^{-1}(W)$ is 1-1& conti

Let $(u_0, v_0) \in y^{-1}(W)$ and $y(u_0, v_0) = p \in W$,$y(z) = p$

Let $x(a,b) = (x_1(a,b), x_2(a,b), x_3(a,b))$

Suppose $D_1x_1 \cdot D_2x_2 - D_2x_1 \cdot D_1x_2 \neq 0$ at $(a_0, b_0) = h(u_0, v_0) = x^{-1}(p)$ by regular

Define $G : D \times R \to R^3$ by

$G(u,v,t) = x(u,v) + (0,0,t) = (x_1(u,v), x_2(u,v), x_3(u,v) + t)$

$\Rightarrow$ G is differentiable and $G(u,v,0) = x(u,v)$

The Jacobian of G is $J(G) = \begin{pmatrix} D_1x_1 & D_2x_1 & 0 \\ D_1x_2 & D_2x_2 & 0 \\ D_1x_3 & D_2x_3 & 1 \end{pmatrix}$

$\Rightarrow \det J(G)\,(a_0, b_0, 0) \neq 0$

By I.F.T. ∃ V (a nbd of $G(a_0, b_0, 0) = y(u_0, v_0)$), G^{-1} is differentiable

$\Rightarrow (h(u_0, v_0), 0) = (a_0, b_0, 0) = G^{-1} \circ y(u_0, v_0)$

$\because\ G(a_0, b_0, 0) = y(u_0, v_0) = p = y(z)$

Thus h is differentiable

p.156 Exercises4-3,(1-3)

2020년 10월 15일 목요일 오전 11:37

1. Let $\mathbf{x}$ be the geographical patch in the sphere Σ (Ex. 2.2). Find the coordinate expression $f(\mathbf{x})$ for the following functions on Σ:
(a) $f(\mathbf{p}) = p_1^2 + p_2^2$. (b) $f(\mathbf{p}) = (p_1 - p_2)^2 + p_3^2$.

$x(u,v) = (rcosvcosu, rcosvsinu, rsinv)$

a)

$f(p) = p_1^2 + p_2^2$

$\Rightarrow f(x) = (rcosvcosu)^2 + (rcosvsinu)^2 = r^2\cos^2 v$

b)

$f(p) = (p_1 - p_2)^2 + p_3^2$

$\Rightarrow f(x) = (rcosvcosu - rcovsinu)^2 + r^2\sin^2 v$

$= (rcosv(cosu - sinu))^2 + r^2\sin^2 v$

$= r^2\cos^2 v\,(cosu - sinu)^2 + r^2\sin^2 v$

$= r^2\cos^2 v\,(\cos^2 u - 2cosvsinv + \sin^2 u) + r^2\sin^2 v$

$= r^2\cos^2 v\,(1 - 2cosusinu) + r^2\sin^2 v$

$= r^2 - 2r^2\cos^2 v\, cosusinu$

3. (a) Prove Corollary 3.4.
(b) Derive the chain rule

$$\mathbf{y}_u = \frac{\partial \bar{u}}{\partial u}\mathbf{x}_u + \frac{\partial \bar{v}}{\partial u}\mathbf{x}_v, \quad \mathbf{y}_v = \frac{\partial \bar{u}}{\partial v}\mathbf{x}_u + \frac{\partial \bar{v}}{\partial v}\mathbf{x}_v,$$

where $\mathbf{x}_u$ and $\mathbf{x}_v$ are evaluated on $(\bar{u}, \bar{v})$.
(c) Deduce that $\mathbf{y}_u \times \mathbf{y}_v = J\,\mathbf{x}_u \times \mathbf{x}_v$, where J is the Jacobian of the mapping $\mathbf{x}^{-1}\mathbf{y} = (\bar{u}, \bar{v})\colon D \to \mathbf{R}^2$.

(a)

If W= $Im\, x \cap Im\, y$ then $x^{-1} \circ y : y^{-1}(W) \to x^{-1}(W)$ is differentiable function by cor 3.3

$\exists \bar{u}, \bar{v}$ such that $x^{-1} \circ y(u,v) = (\bar{u}(u,v), \bar{v}(u,v))$

$\therefore y(u,v) = x(\bar{u}(u,v), \bar{v}(u,v))$

(uniqueness)

If $y(u,v) = x(\bar{u}_1(u,v), \overline{v_1}(u,v)) = x(\bar{u}_2(u,v), \bar{v}_2(u,v))$

Then $x^{-1} \circ y(u,v) = (\bar{u}_1(u,v), \overline{v_1}(u,v)) = (\bar{u}_2(u,v), \bar{v}_2(u,v))$

Thus $\bar{u}_1 = \bar{u}_2, \bar{v}_1 = \bar{v}_2$

(b)

$y(u,v) = x(\bar{u}(u,v), \bar{v}(u,v))$

$\Rightarrow y_u = \frac{\partial}{\partial u}x(\bar{u},\bar{v}) = \frac{\partial \bar{u}}{\partial u}\cdot\frac{\partial x}{\partial \bar{u}}(\bar{u},\bar{v}) + \frac{\partial \bar{v}}{\partial u}\cdot\frac{\partial x}{\partial \bar{v}}(\bar{u},\bar{v}) = \frac{\partial \bar{u}}{\partial u}x_u + \frac{\partial \bar{v}}{\partial u}x_v$

$\Rightarrow y_v = \frac{\partial}{\partial v}x(\bar{u},\bar{v}) = \frac{\partial \bar{u}}{\partial v}\cdot\frac{\partial x}{\partial \bar{u}}(\bar{u},\bar{v}) + \frac{\partial \bar{v}}{\partial v}\cdot\frac{\partial x}{\partial \bar{v}}(\bar{u},\bar{v}) = \frac{\partial \bar{u}}{\partial v}x_u + \frac{\partial \bar{v}}{\partial v}x_v$

(c)

$y_u \times y_v = (\frac{\partial \bar{u}}{\partial u}x_u + \frac{\partial \bar{v}}{\partial u}x_v) \times (\frac{\partial \bar{u}}{\partial v}x_u + \frac{\partial \bar{v}}{\partial v}x_v) = Jx_u \times x_v$ where J is jacobian of $x^{-1}\ y = (\bar{u}, \bar{v}) : D \to R^2$

2. Let $\mathbf{x}$ be the usual parametrization of the torus (Ex. 2.5).
(a) Find the Euclidean coordinates $\alpha_1, \alpha_2, \alpha_3$ of the curve $\alpha(t) = \mathbf{x}(t, t)$.
(b) Show that α is periodic, and find its period $p > 0$, the smallest number such that $\alpha(t + p) = \alpha(t)$ for all t.

Usual parametrization of torus : $x(u,v) = ((R + rcosu)cosv, (R + rcosu)sinv, rsinu)$

(a)

$\alpha(t) = x(t,t) = ((R + rcost)cost, (R + rcost)sint, rsint)$

$\alpha_1 = (R + rcost)cost$

$\alpha_2 = (R + rcost)sint$

$\alpha_3 = rsint$

(b)

$p = 2\pi$

p.161 lemma4.4.5, p.164 Exercises4-4,(1-3)

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Lemma 4.5

$d_x\phi = d_y\phi$ on $x(D) \cap y(E)$

Pf) $\forall v_1, v_2,\ d_x\phi(v_1, v_2) = d_y\phi(v_1, v_2)$

We show that $d_x\phi(y_u, y_v) = d_y\phi(y_u, y_v)$!! by (4.4.2)

By (4.3.4) $y = x(\bar{u}, \bar{v})$

$\Rightarrow y_u = \frac{\partial \bar{u}}{\partial u} x_u + \frac{\partial \bar{v}}{\partial u} x_v$ and $y_v = \frac{\partial \bar{u}}{\partial v} x_u + \frac{\partial \bar{v}}{\partial v} x_v$ (1)

where x_u and x_v are evaluated on $(\bar{u}, \bar{v})$

By 4.4.2

(2) $(d_x\phi)(y_u, y_v) = J \cdot (d_x\phi)(x_u, x_v) = J \cdot \left\{\frac{\partial}{\partial \bar{u}}\left(\phi(x_v)\right) - \frac{\partial}{\partial \bar{v}}\left(\phi(x_u)\right)\right\}$, $J = \begin{vmatrix} \frac{\partial \bar{u}}{\partial u} & \frac{\partial \bar{v}}{\partial u} \\ \frac{\partial \bar{u}}{\partial v} & \frac{\partial \bar{v}}{\partial v} \end{vmatrix}$

(3) $d_y\phi(y_u, y_v) = \frac{\partial}{\partial u}\left(\phi(y_v)\right) - \frac{\partial}{\partial v}\left(\phi(y_u)\right)$

By (1) $\phi(y_v) = \frac{\partial \bar{u}}{\partial v}\phi(x_u) + \frac{\partial \bar{v}}{\partial v}\phi(x_v)$, $\phi(y_u) = \frac{\partial \bar{u}}{\partial u}\phi(x_u) + \frac{\partial \bar{v}}{\partial u}\phi(x_v)$

$$\begin{aligned}\frac{\partial}{\partial u}\left(\phi(y_v)\right) &= \left(\frac{\partial}{\partial u}\right)\left(\frac{\partial \bar{u}}{\partial v}\right)\left(\phi(x_u)\right) + \frac{\partial \bar{u}}{\partial v}\frac{\partial}{\partial u}\left(\phi(x_u)\right) + \left(\frac{\partial}{\partial u}\right)\left(\frac{\partial \bar{v}}{\partial v}\right)\left(\phi(x_v)\right) + \frac{\partial \bar{v}}{\partial v}\frac{\partial}{\partial u}\left(\phi(x_v)\right) \\ &= \left(\frac{\partial}{\partial u}\right)\left(\frac{\partial \bar{u}}{\partial v}\right)\left(\phi(x_u)\right) + \frac{\partial \bar{u}}{\partial v}\left(\frac{\partial}{\partial \bar{u}}\left(\phi(x_u)\right)\frac{\partial \bar{u}}{\partial u} + \frac{\partial}{\partial \bar{v}}\left(\phi(x_u)\right)\frac{\partial \bar{v}}{\partial u}\right) + \left(\frac{\partial}{\partial u}\right)\left(\frac{\partial \bar{v}}{\partial v}\right)\left(\phi(x_v)\right) + \frac{\partial \bar{v}}{\partial v}\left(\frac{\partial}{\partial \bar{u}}\left(\phi(x_v)\right)\frac{\partial \bar{u}}{\partial u} + \frac{\partial}{\partial \bar{v}}\left(\phi(x_v)\right)\frac{\partial \bar{v}}{\partial u}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial v}\left(\phi(y_u)\right) &= \left(\frac{\partial}{\partial v}\right)\left(\frac{\partial \bar{u}}{\partial u}\right)\left(\phi(x_u)\right) + \frac{\partial \bar{u}}{\partial u}\frac{\partial}{\partial v}\left(\phi(x_u)\right) + \left(\frac{\partial}{\partial v}\right)\left(\frac{\partial \bar{v}}{\partial u}\right)\left(\phi(x_v)\right) + \frac{\partial \bar{v}}{\partial u}\frac{\partial}{\partial v}\left(\phi(x_v)\right) \\ &= \left(\frac{\partial}{\partial v}\right)\left(\frac{\partial \bar{u}}{\partial u}\right)\left(\phi(x_u)\right) + \frac{\partial \bar{u}}{\partial u}\left(\frac{\partial}{\partial \bar{v}}\left(\phi(x_u)\right)\frac{\partial \bar{v}}{\partial v} + \frac{\partial}{\partial \bar{u}}\left(\phi(x_u)\right)\frac{\partial \bar{u}}{\partial v}\right) + \left(\frac{\partial}{\partial v}\right)\left(\frac{\partial \bar{v}}{\partial u}\right)\left(\phi(x_v)\right) + \frac{\partial \bar{v}}{\partial u}\left(\frac{\partial}{\partial \bar{u}}\left(\phi(x_v)\right)\frac{\partial \bar{u}}{\partial v} + \frac{\partial}{\partial \bar{v}}\left(\phi(x_v)\right)\frac{\partial \bar{v}}{\partial v}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial u}\left(\phi(y_v)\right) - \frac{\partial}{\partial v}\left(\phi(y_u)\right) &= \frac{\partial \bar{u}}{\partial v}\left(\frac{\partial}{\partial \bar{v}}\left(\phi(x_u)\right)\frac{\partial \bar{v}}{\partial u}\right) + \frac{\partial \bar{v}}{\partial v}\left(\frac{\partial}{\partial \bar{u}}\left(\phi(x_v)\right)\frac{\partial \bar{u}}{\partial u}\right) - \frac{\partial \bar{u}}{\partial u}\left(\frac{\partial}{\partial \bar{v}}\left(\phi(x_u)\right)\frac{\partial \bar{v}}{\partial v}\right) - \frac{\partial \bar{v}}{\partial u}\left(\frac{\partial}{\partial \bar{u}}\left(\phi(x_v)\right)\frac{\partial \bar{u}}{\partial v}\right) \\ &= \frac{\partial \bar{u}}{\partial u}\frac{\partial \bar{v}}{\partial v}\left(\frac{\partial}{\partial \bar{u}}\left(\phi(x_v)\right) - \frac{\partial}{\partial \bar{v}}\left(\phi(x_u)\right)\right) - \frac{\partial \bar{u}}{\partial v}\frac{\partial \bar{v}}{\partial u}\left(\frac{\partial}{\partial \bar{u}}\left(\phi(x_v)\right) - \frac{\partial}{\partial \bar{v}}\left(\phi(x_u)\right)\right) \\ &= \left(\frac{\partial \bar{u}}{\partial u}\frac{\partial \bar{v}}{\partial v} - \frac{\partial \bar{u}}{\partial v}\frac{\partial \bar{v}}{\partial u}\right)\left(\frac{\partial}{\partial \bar{u}}\left(\phi(x_v)\right) - \frac{\partial}{\partial \bar{v}}\left(\phi(x_u)\right)\right) \\ &= J \cdot \left\{\frac{\partial}{\partial \bar{u}}\left(\phi(x_v)\right) - \frac{\partial}{\partial \bar{v}}\left(\phi(x_u)\right)\right\}, \ J = \begin{vmatrix} \frac{\partial \bar{u}}{\partial u} & \frac{\partial \bar{v}}{\partial u} \\ \frac{\partial \bar{u}}{\partial v} & \frac{\partial \bar{v}}{\partial v} \end{vmatrix}\end{aligned}$$

1. Prove the Leibnizian formulas

$$d(fg) = g\,df + f\,dg, \quad d(f\phi) = df \wedge \phi + f\,d\phi,$$

where f and g are functions on M and ϕ is a 1-form.
(*Hint:* By definition, $(f\phi)(\mathbf{v}_p) = f(\mathbf{p})\phi(\mathbf{v}_p)$; hence $f\phi$ evaluated on $\mathbf{x}_u$ is $f(\mathbf{x})\phi(\mathbf{x}_u)$.)

(1) $d(fg)(x_u) = x_u[fg]$

$$\begin{aligned} &= \sum \frac{\partial x_i}{\partial u} D_i(fg)(x) \\ &= \sum \frac{\partial x_i}{\partial u}(gD_if + fD_ig)(x) \\ &= g(x)\sum \frac{\partial x_i}{\partial u}(D_if)(x) + f(x)\sum \frac{\partial x_i}{\partial u}(D_ig)(x) \\ &= g(x)df(x_u) + f(x)dg(x_u) \\ &= (gdf + fdg)(x_u)\end{aligned}$$

(2) $d(fg)(x_v) = (gdf + fdg)(x_v)$

$\therefore d(fg) = gdf + fdg$

$$\begin{aligned} d(f\phi)(x_u, x_v) &= \frac{\partial}{\partial u}(f\phi)(x_v) - \frac{\partial}{\partial v}(f\phi)(x_u) \\ &= \frac{\partial}{\partial u}\left(f(x)\phi(x_v)\right) - \frac{\partial}{\partial v}\left(f(x)\phi(x_u)\right) \\ &= \frac{\partial f(x)}{\partial u}\phi(x_v) + f(x)\frac{\partial}{\partial u}\phi(x_v) - \frac{\partial f(x)}{\partial v}\phi(x_u) - f(x)\frac{\partial}{\partial v}\phi(x_u) \\ &= \frac{\partial f(x)}{\partial u}\phi(x_v) - \frac{\partial f(x)}{\partial v}\phi(x_u) + f(x)\left(\frac{\partial}{\partial u}\phi(x_v) - \frac{\partial}{\partial v}\phi(x_u)\right) \\ &= (df \wedge \phi + fd\phi)(x_u, x_v)\end{aligned}$$

$\therefore d(f\phi) = (df \wedge \phi + fd\phi)$

2. (a) Prove formulas (1) and (2) in Example 4.7 using the remark preceding Example 4.7. (*Hint:* Show $(du_1du_2)(U_1, U_2) = 1$.)
(b) Derive the remaining formulas using the properties of d and the wedge product.

If f is a function, ϕ a 1-form, and η a 2-form, then
(1) $\phi = f_1du_1 + f_2du_2$, where $f_i = \phi(U_i)$.
(2) $\eta = gdu_1du_2$, where $g = \eta(U_1, U_2)$.
(3) for $\psi = g_1du_1 + g_2du_2$ and ϕ as above,
$$\phi \wedge \psi = (f_1g_2 - f_2g_1)du_1\, du_2.$$
(4) $df = \dfrac{\partial f}{\partial u_1}du_1 + \dfrac{\partial f}{\partial u_2}du_2.$
(5) $d\phi = \left(\dfrac{\partial f_2}{\partial u_1} - \dfrac{\partial f_1}{\partial u_2}\right)du_1\, du_2$ (ϕ as above).

(a)
(1)
$$\begin{aligned}\phi(v_p) &= \phi\left(\sum v_i U_i(p)\right)\\ &= \sum v_i \phi\left(U_i(p)\right)\\ &= \sum\left(\phi(U_i)\right)(p)(du_i)(v_p)\\ &= \sum(f_i du_i)(v_p)\end{aligned}$$
$\therefore\ \phi = f_1du_1 + f_2du_2$
(2)
v_p and w_p are linearly independent
$$\begin{aligned}\eta(v_p, w_p) &= \eta\left(v_1U_1(p) + v_2U_2(p), w_1U_1(p) + w_2U_2(p)\right)\\ &= \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2\end{vmatrix}\eta\left(U_1(p), U_2(p)\right)\\ &= \eta\left(U_1(p), U_2(p)\right)(v_1w_2 - w_1v_2)\\ &= \left(\eta(U_1, U_2)\right)(p)\left(du_1(v_p)du_2(w_p) - du_1(w_p)du_2(v_p)\right)\\ &= g(p)(du_1 \wedge du_2)(v_p, w_p)\\ &= (gdu_1du_2)(v_p, w_p)\end{aligned}$$
$\therefore\ \eta = gdu_1du_2$
(b)
(3)
$$\begin{aligned}\phi \wedge \psi &= (f_1du_1 + f_2du_2) \wedge (g_1du_1 + g_2du_2)\\ &= \mathrm{f_1g_2du_1du_2 + f_2g_1du_2du_1}\\ &= \mathrm{f_1g_2du_1du_2 - f_2g_1du_1du_2}\\ &= \mathrm{(f_1g_2 - f_2g_1)du_1du_2}\end{aligned}$$
(4)
$$df(v_p) = \sum v_i \frac{\partial f}{\partial u_i}(p) = \sum\left(\frac{\partial f}{\partial u_i}du_i\right)(v_p)$$
$\therefore\ df = \frac{\partial f}{\partial u_1}du_1 + \frac{\partial f}{\partial u_2}du_2$
(5)
$$d\phi = d\left(\sum f_i du_i\right) = \sum df_i \wedge du_i = \frac{\partial f_2}{\partial u_1}du_1du_2 + \frac{\partial f_1}{\partial u_2}du_2du_1 = \frac{\partial f_2}{\partial u_1}du_1du_2 - \frac{\partial f_1}{\partial u_2}du_1du_2$$

3. If f is a real-valued function on a surface, and g is a function on the real line, show that
$$\mathbf{v}_p[g(f)] = g'(f)\mathbf{v}_p[f].$$
Deduce that
$$d(g(f)) = g'(f)df.$$

- Lemma 4.6
$$\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t) = (f \circ \alpha)'(t)$$

Let α be a curve with initial velocity v at p
$$\begin{aligned}v_p[g(f)] &= (gf\alpha)'(0)\\ &= g'(f(\alpha))(0)(f\alpha)'(0)\\ &= g'\left(f(p)\right)v_p[f] \ \because lemma 1.4.6\end{aligned}$$
$$\begin{aligned}d(g(f))(v_p) &= v_p\,[g(f)]\\ &= g'\left(f(p)\right)v_p[f]\\ &= g'\left(f(p)\right)df(v_p)\\ &= (g'(f)df)(v_p)\end{aligned}$$
$\therefore\ d\left(g(f)\right) = (g'(f)df)$

p.165 Exercises4-4-6,p.172Exercises4-5,(1,3)

2020년 11월 10일 화요일 오후 2:58

6. Let $\mathbf{y}: E \to M$ be an arbitrary mapping of an open set of $\mathbf{R}^2$ into a surface M. If ϕ is a 1-form on M, show that the formula

$$d\phi(\mathbf{y}_u, \mathbf{y}_v) = \frac{\partial}{\partial u}(\phi(\mathbf{y}_v)) - \frac{\partial}{\partial v}(\phi(\mathbf{y}_u))$$

is still valid even when $\mathbf{y}$ is not regular or one-to-one.
(*Hint:* In the proof of Lem. 4.5, check that equation (3) is still valid in this case.)

??? 죄송합니다...

♣

1. Let M and N be surfaces in $\mathbf{R}^3$. If $F: \mathbf{R}^3 \to \mathbf{R}^3$ is a mapping such that the image $F(M)$ of M is contained in N, then the restriction of F to M is a function $F|M: M \to N$. Prove that $F|M$ is a mapping of surfaces. (*Hint:* Use Thm. 3.2.)

3.2 Theorem Let M be a surface in $\mathbf{R}^3$. If $F: \mathbf{R}^n \to \mathbf{R}^3$ is a (differentiable) mapping whose image lies in M, then considered as a function $F: \mathbf{R}^n \to M$ into M, F is differentiable (as defined above).

3.3 Corollary If $\mathbf{x}$ and $\mathbf{y}$ are patches in a surface M in $\mathbf{R}^3$ whose images overlap, then the composite functions $\mathbf{x}^{-1}\mathbf{y}$ and $\mathbf{y}^{-1}\mathbf{x}$ are (differentiable) mappings defined on open sets of $\mathbf{R}^2$.

Pf

If $x: D \to M$ is a patch, then $\underset{G}{F(x)}: D \to N$ is a differentiable mapping by thm 3.2

~~$y^{-1}Fx$ is differentiable for any patch y in N~~ $y^{-1}(G)$

∴ $F|M$ is a mapping of surfaces.(∵ def 5.1)

♣

3. Let M be a *simple surface*, that is, one that is the image of a single proper patch $\mathbf{x}: D \to \mathbf{R}^3$. If $\mathbf{y}: D \to N$ is any mapping into a surface N, show that the function $F: M \to N$ such that

$$F(\mathbf{x}(u, v)) = \mathbf{y}(u, v) \quad \text{for all } (u, v) \text{ in } D$$

is a mapping of surfaces. (*Hint:* Write $F = \mathbf{y}\mathbf{x}^{-1}$, and use Cor. 3.3.)

$x(D) \subset R^3$
$\shortparallel$ open

$Fx = y$ E
$F = yx^{-1}$ $R^3 \to D \to N$
(전체X)

강력

$\bar{y}^{-1}y$: thm3.2
$x^{-1}\bar{x}$: cor 3.3⇒ thm 3.2

Pf

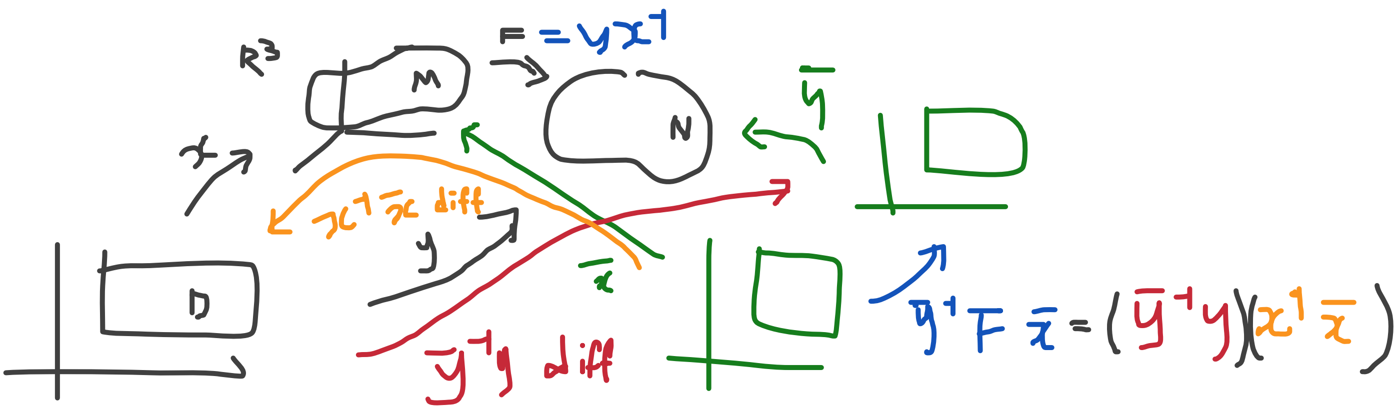
Let $\bar{x}$ be patch in M and $\bar{y}$ be patch in N

Let $F = yx^{-1}$

then $\bar{y}^{-1}F\,\bar{x} = (\bar{y}^{-1}y)(x^{-1}\bar{x})$ is differentiable (∵ $\bar{y}^{-1}y$, $x^{-1}\bar{x}$: differentiable(∵ thm 3.2) and composition of differentiable functions ⇒ differentiable)

∴ F is a mapping of surfaces.(∵ def 5.1)

♣



p.180~181 Exercises4-6,(1-3)

2020년 11월 19일 목요일 오후 8:32

1. If α is a curve in $\mathbf{R}^2$ and ϕ is a 1-form, prove this computational rule for finding $\phi(\alpha')dt$: Substitute $u = \alpha_1$ and $v = \alpha_2$ into the coordinate expression $\phi = f(u, v)\, du + g(u, v)\, dv$.

$\phi(\alpha')dt = \phi(\alpha_1', \alpha_2')dt$

$= \phi(\alpha_1' U_1 + \alpha_2' U_2)dt$

$= \left(\alpha_1' \phi(U_1) + \alpha_2' \phi(U_2)\right) dt$

$= \left(\alpha_1' f(\alpha_1, \alpha_2) + \alpha_2' g(\alpha_1, \alpha_2)\right) dt$

$= f(\alpha_1, \alpha_2)d\alpha_1 + g(\alpha_1, \alpha_2)d\alpha_2$

$= f(u, v)du + g(u, v)dv$,$u = \alpha_1$, $v = \alpha_2$ ♣

2. Let $\alpha: [-1, 1] \to \mathbf{R}^2$ be the curve segment given by $\alpha(t) = (t, t^2)$.

(a) If $\phi = v^2\, du + 2uv\, dv$, compute $\int_\alpha \phi$.

(b) Find a function f such that $df = \phi$ and check Theorem 6.2 in this case.

(a)

By exercise 1

$\phi(\alpha')dt = t^4 dt + 2t \cdot 2t^3 dt$

$= 5t^4 dt$

$\int_{-1}^{1} 5t^4 dt = [t^5]_{-1}^{1} = 1 - (-1) = 2$

(b)

$f = uv^2 \Rightarrow df = v^2 du + 2uv\, dv$

Thus

$\int_\alpha \phi = \int_\alpha df = f(\alpha(1)) - f(\alpha(-1)) = 2$ ♣

3. Let ϕ be a 1-form on a surface M. Show:

(a) If ϕ is closed, then $\int_{\partial \mathbf{x}} \phi = 0$ for every 2-segment $\mathbf{x}$ in M.

(b) If ϕ is exact, then more generally,

$$\int_\alpha \phi = \sum_i \int_{\alpha_i} \phi = 0$$

★ Def 4.8

ϕ is *closed* if $d\phi = 0$, and ϕ is *exact* if $\exists \xi, d\xi = \phi$

(a)

$\int_{\partial x} \phi = \int \int_x d\phi = \int \int_x 0 = 0$ by def 4.8

(b)

$\sum_i \int_{\alpha_i} \phi = \sum_i \int_{\alpha_i} df$

$= \sum_i (f(q) - f(p))$ ($\because$ thm 6.2)

$= 0$ ($\because$ f(q) = f(p)) ♣