3.3 연습문제

2020년 9월 8일 화요일 오후 1:44

1. Prove

$$\text{sgn}(FG) = \text{sgn}\ F \cdot \text{sgn}\ G = \text{sgn}(GF).$$

Deduce that $\text{sgn}\ F = \text{sgn}\ (F^{-1})$.

pf)

If the orthogonal parts of F and G are A and B,

Then sgn(FG)=det(AB)=det(A)det(B)=det(BA)=sgn(GF) by ex 3.1.2

Thus sgn(FG)=sgn$F$ · sgn$G$=sgn(GF)

also, 1=sgn I=sgn(FF^{-1})=sgn(F)sgn(F^{-1})

By3.1.2

(2.) Given isometries $F = T_a A$ and $G = T_b B$
find translational and orthogonal parts of FG & GF

$(FG)(x) = F(T_b B(x))$
$= T_a A\ (b + Bx)$
$= T_a\ (A(b) + A(B(x)))$
$= a + A(b) + AB(x)$
$= T_c C$ for $c = a + A(b)$, $C = AB$.

$(GF)(x) = b + B(a) + BA(x)$
$= T_d D$ for $d = b + B(a)$, $D = BA$.

2. If H_0 is an orientation-reversing isometry of $\mathbf{R}^3$, show that *every* orientation-reversing isometry has a unique expression H_0F, where F is orientation-preserving.

Pf)

Let G be an orientation-reversing isometry of R^3 and A be the the orthogonal part of G.

$\Rightarrow G = T_a A$ (T_a is translation by a)

$\Rightarrow H_0 = T_b B$ (B is the orthogonal part of G) (by Ex 3.1)

Since H_0 is an isometry of R^3, there exists H_0^{-1} that is also an isometry of R^3.

Thus, $G = H_0(H_0^{-1}G) = H_0F$ where $F = H_0^{-1}G$

Uniqueness) $G = H_0F = H_0F' \Rightarrow F = F'$

3. Let $\mathbf{v} = (3, 1, -1)$ and $\mathbf{w} = (-3, -3, 1)$ be tangent vectors at some point. If C is the orthogonal transformation given in Exercise 4 of Section 1, check the formula

$$C_*(\mathbf{v} \times \mathbf{w}) = (\text{sgn}\ C)C_*(\mathbf{v}) \times C_*(\mathbf{w}).$$

sol)

Let $C = \begin{pmatrix} -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$, $v = (3,1,-1)$, $w = (-3,-3,1)$

Therefore

sgn(C)=det(C)=-1 and $v \times w = (-2,0,-6)$

$C_*(v \times w) = C_*(-2,0,-6)$

$= \sum_{i,j} c_{ij} r_j \overline{U_i}$ where $r = (-2,0,-6)$

$= \left(\frac{10}{3}, \frac{8}{3}, -\frac{14}{3}\right)$

$C_*(v) = C_*(3,1,-1)$

$= \sum_{ij} c_{ij} v_j \overline{U_i}$

$= (-1,3,1)$

$C_*(w) = C_*(-3,-3,1)$

$= \sum_{ij} c_{ij} w_j \overline{U_i}$

$= \left(-\frac{1}{3}, -\frac{11}{3}, -\frac{7}{3}\right)$

Thus $C_*(v) \times C_*(w) = (-1,3,1) \times \left(-\frac{1}{3}, -\frac{11}{3}, -\frac{7}{3}\right)$

$= \left(-\frac{10}{3}, -\frac{8}{3}, \frac{14}{3}\right)$

$\therefore\ C_*(v \times w) = \text{sgn}(C) C_*(v) \times C_*(w)$

$$-\frac{2}{3}\left(\frac{2}{9} + \frac{4}{9}\right) \quad -\frac{4}{9}$$

$$-\frac{2}{3}\left(\frac{4}{9} + \frac{2}{9}\right) - \frac{4}{9}$$

$$-\frac{1}{3}\left(\frac{4}{9} - \frac{1}{9}\right) - \frac{1}{9}$$

$$\begin{vmatrix} U_1 & U_2 & U_3 \\ 3 & 1 & -1 \\ -3 & -3 & 1 \end{vmatrix} = U_1(1-3) - U_2(3-3) + U_3(-9+3)$$

$$= -2U_1 - 6U_3$$

$$\begin{vmatrix} U_1 & U_2 & U_3 \\ -1 & 3 & 1 \\ -\frac{1}{3} & -\frac{11}{3} & -\frac{7}{3} \end{vmatrix} = U_1\left(-\frac{21}{3} + \frac{11}{3}\right) - U_2\left(\frac{7}{3} + \frac{3}{3}\right) + U_3\left(\frac{11}{3} + \frac{3}{3}\right) = \left(-\frac{10}{3}, -\frac{8}{3}, \frac{14}{3}\right)$$

4. A *rotation* is an orthogonal transformation C such that det $C = +1$. Prove that C does, in fact, rotate $\mathbf{R}^3$ around an axis. Explicitly, given a rotation C, show that there exists a number ϑ and points $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ with $\mathbf{e}_i \bullet \mathbf{e}_j = \delta_{ij}$ such that (Fig. 3.5)

$$C(\mathbf{e}_1) = \cos\vartheta\ \mathbf{e}_1 + \sin\vartheta\ \mathbf{e}_2,$$
$$C(\mathbf{e}_2) = -\sin\vartheta\ \mathbf{e}_1 + \cos\vartheta\ \mathbf{e}_2,$$
$$C(\mathbf{e}_3) = \mathbf{e}_3.$$

??

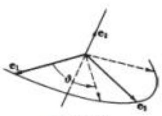


FIG. 3.5

(*Hint:* The fact that the dimension of $\mathbf{R}^3$ is odd means that C has an eigenvalue +1, so there is a point $\mathbf{p} \neq \mathbf{0}$ such that $C(\mathbf{p}) = \mathbf{p}$.)

4. C의 고유치를 λ, 대응하는 고유벡터를 v라 하면

$|\lambda|\,\|\mathbf{v}\| = \|\lambda\mathbf{v}\| = \|C\mathbf{v}\| = \|\mathbf{v}\|$ 이므로 $|\lambda|=1$

C의 고유치 λ, μ에 대응하는 고유벡터를 각각 x, y라 하면

$$\lambda(\mathbf{x}\cdot\mathbf{y}) = (\lambda\mathbf{x})\cdot\mathbf{y} = C\mathbf{x}\cdot\mathbf{y} = \mathbf{x}\cdot C^T\mathbf{y} = \mathbf{x}\cdot C^{-1}\mathbf{y}$$
$$= \mathbf{x}\cdot(\mu^{-1}\mathbf{y}) = \mathbf{x}\cdot(\bar{\mu}\mathbf{y}) = \mu(\mathbf{x}\cdot\mathbf{y})$$

$\lambda \neq \mu$이므로 $\mathbf{x}\cdot\mathbf{y}=0$

C의 특성방정식 $\det(C-\lambda E)=0$는 실계수를 갖는 3차방정식이므로 세 근

$\alpha+\beta i, \alpha-\beta i, \gamma$를 갖는다. $1=|\alpha\pm\beta i|^2=\alpha^2+\beta^2$이므로

$1=\det C=(\alpha+\beta i)(\alpha-\beta i)\gamma=(\alpha^2+\beta^2)\gamma=\gamma$

$\gamma=1$의 고유벡터를 e_3라 하면 $Ce_3=e_3$

$\lambda=\alpha+\beta i$의 고유벡터를 e_1+ie_2라 하면 $\lambda=\alpha-\beta i$의 고유벡터는 e_1-ie_2이다.

$Ce_1 \pm iCe_2 = C(e_1\pm ie_2) = (\alpha\pm\beta i)(e_1\pm ie_2) = \alpha e_1 - \beta e_2 \pm i(\beta e_1+\alpha e_2)$이므로

$Ce_1=\alpha e_1-\beta e_2,\ Ce_2=\beta e_1+\alpha e_2$

$\alpha^2+\beta^2=1$이므로 $\alpha=\cos\theta, \beta=-\sin\theta$인 θ를 택할 수 있다.

$Ce_1=\cos\theta e_1+\sin\theta e_2,\ Ce_2=-\sin\theta e_1+\cos\theta e_2$

$0=(e_1+ie_2)\cdot(e_1-ie_2)=e_1\cdot e_1-e_2\cdot e_2+2ie_1\cdot e_2$이므로

$\|e_1\|=\|e_2\|, e_1\perp e_2$

$0=(e_1+ie_2)\cdot e_3=e_1\cdot e_3+ie_2\cdot e_3$이므로 $e_1\perp e_3, e_2\perp e_3$

$\|e_1\|=\|e_2\|=\|e_3\|=1$이라 가정하여도 일반성을 잃지 않는다.

그러므로 $e_i\cdot e_j=\delta_{ij}$

5. Let a be a point of $\mathbf{R}^3$ such that $\|\mathbf{a}\|=1$. Prove that the formula

$$C(\mathbf{p}) = \mathbf{a}\times\mathbf{p} + (\mathbf{p}\cdot\mathbf{a})\,\mathbf{a}$$

defines an orthogonal transformation. Describe its general effect on $\mathbf{R}^3$.

Sol)

$C(p)\cdot C(q) = (a\times p + (p\cdot a)a)\cdot(a\times q + (q\cdot a)a)$

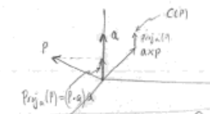
$=(a\times p)\cdot(a\times q)+(q\cdot a)(a\times p\cdot a)+(p\cdot a)(a\cdot a\times q)+(p\cdot a)(q\cdot a)a\cdot a$

$=(a\times p)\cdot(a\times q)+(p\cdot a)(q\cdot a)a\cdot a$

$=(p\cdot q)-(a\cdot q)(p\cdot a)+(p\cdot a)(q\cdot a)$

$=p\cdot q$

∴ C is an orthogonal transformation



• Maybe rotating around the a axis

• $C(a) = a\times a + (a\cdot a)a = a$. Yep, it fixes the a-axis

What follows I'm not completely ☺ about, but it's the best I have now.

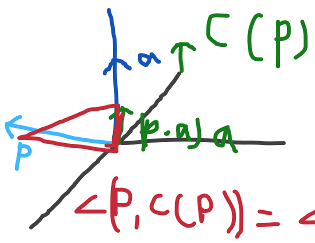
Let's see, what angle? To find $\angle(P, C(P))$ we calculate:

$$\underbrace{P\cdot C(P)}_{\|P\|^2\cos\theta_1} = P\cdot(a\times P+(P\cdot a)a) = \underbrace{(P\cdot a)^2}_{(\|P\|\cos\theta_2)^2}$$

Thus $\cos\theta_1 = \cos^2\theta_2$ where $\theta_1 = \angle(P, C(P))$ and $\theta_2 = \angle(P,a)$. There are two sol^{n}s. ($P\neq a$ is interesting)

1.) $\cos\theta_1 = \cos^2\theta_2 = 1$ $(P=a)$, 2.) $\cos\theta_1 = \cos^2\theta_2 = 0$ $P\neq a$ $\longrightarrow$ $\boxed{\theta = \pi/2}$

C는 a를 축으로 하여 반시계방향으로 $\frac{\pi}{2}$만큼 회전이동시킨 회전변환



$\angle(P, C(P)) = \angle(P, a) = \frac{\pi}{2}$

$C(a)=a$

$$\underbrace{P\cdot C(P)}_{\|P\|^2\cos\theta_1} = P\,(a\times p+(P\cdot a)\,a) = (P\cdot a)^2 \rightarrow (\|P\|\cos\theta_2)^2$$

$\cos\theta_1 = \cos^2\theta_2 = 1$ $\qquad$ $\cos\theta_1 = \cos^2\theta = 0$

$p=a$ $\qquad$ $p\neq a$

6. Prove
(a) The set $O^+(3)$ of all rotations of $\mathbf{R}^3$ is a subgroup of the orthogonal group $O(3)$ (see Ex. 8 of Sec. 3.1).
(b) The set $\mathcal{E}^+(3)$ of all orientation-preserving isometries of $\mathbf{R}^3$ is a subgroup of the Euclidean group $\mathcal{E}(3)$.

a-pf)

By example 3.4.(2)
det(I_3)=1 $\therefore$ $I_3 \in O^+$ ($O^+ \neq \emptyset$)
Let $A, B \in O^+$
$\Rightarrow (AB^{-1})\square^t(AB^{-1})=(A^tB)\square^t(A^tB)$
$=(A^tB)(B^tA)$
$=A^tA$
$=I_3$
$\Rightarrow AB^{-1} \in O^+$
Thus O^+ is a subgroup of O

b-Pf)
det(I_3)=1 $\therefore I_3 \in \mathcal{E}\square^+$ ($\mathcal{E}^+ \neq \emptyset$)
Let F_1,$F_2 \in \mathcal{E}\square^+$, $F_1 = T_aC_1, F_2 = T_bC_2$
$\Rightarrow F_1F_2^{-1} = (T_aC_1)(C_2^{-1}T_{-b})$
$= T_aC\,T_{-b}$ where $C = C_1C_2^{-1}$
$=T_aT_{C(-b)}\,C$ by3.1.1 $CT_a = T_{C(a)}\,C$
$=T_{a+C(b)}C \in \mathcal{E}\square^+$
Thus $\mathcal{E}\square^+$ is a subgroup of $\mathcal{E}$

3.4 연습문제

2020년 9월 15일 화요일 오전 11:00

1. Let $F = TC$ be an isometry of $\mathbf{R}^3$, β a unit speed curve in $\mathbf{R}^3$. Prove
(a) If β is a cylindrical helix, then $F(\beta)$ is a cylindrical helix.
(b) If β has spherical image σ, then $F(\beta)$ has spherical image $C(\sigma)$.

(a)

by thm 4.2

$\bar{\kappa}=\kappa$, $\bar{\tau}=(sgnF)\tau=(detC)\tau=\pm\tau$

$\therefore$ $F(\beta)$ is cylindrical helix

(b)

spherical image : 안 배움

2. Let $Y = (t, 1 - t^2, 1 + t^2)$ be a vector field on the helix

$$\alpha(t) = (\cos t, \sin t, 2t),$$

and let C be the orthogonal transformation

$$C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

Compute $\bar{\alpha} = C(\alpha)$ and $\bar{Y} = C_*(Y)$, and check that

$$C_*(Y') = \bar{Y}', \quad C_*(\alpha'') = \bar{\alpha}'', \quad Y' \bullet \alpha'' = \bar{Y}' \bullet \bar{\alpha}''.$$

$$\bar{\alpha}=C(\alpha)=\begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \cos t \\ \sin t \\ 2t \end{pmatrix}=\left(-cost, \frac{1}{\sqrt{2}}sint\ - t\sqrt{2}, \frac{1}{\sqrt{2}}sint\ + t\sqrt{2}\right)$$

$$\Rightarrow\ \overline{\alpha'}=\left(sint, \frac{1}{\sqrt{2}}cost\ -\ \sqrt{2}, \frac{1}{\sqrt{2}}cost\ +\ \sqrt{2}\right)$$

$$\Rightarrow\ \overline{\alpha''}=\left(cost, -\frac{1}{\sqrt{2}}sint\ , -\frac{1}{\sqrt{2}}sint\ \right)$$

$$\bar{Y}=C_*(Y)=\begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} t \\ 1-t^2 \\ 1+t^2 \end{pmatrix}=\left(-t, -t^2\sqrt{2}, \sqrt{2}\right)$$

$$\Rightarrow \overline{Y'}=\left(-1, -2\sqrt{2}t, 0\right)$$

$$C_*(Y')=\begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 \\ -2t \\ 2t \end{pmatrix}=\ \left(-1, -2\sqrt{2}t, 0\right)$$

$\therefore$ $C_*(Y')=\overline{Y'}$

$$C_*(\alpha'')=\begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} -cost \\ -sint \\ 0 \end{pmatrix}\ =\left(cost, -\frac{1}{\sqrt{2}}sint\ , -\frac{1}{\sqrt{2}}sint\ \right)$$

$\therefore$ $C_*(\alpha'')=\overline{\alpha''}$

$$\begin{aligned}\overline{Y'}\cdot\overline{\alpha''}&=C_*(Y')\cdot C_*(\alpha'')\\ &=\left(-1, -2\sqrt{2}t, 0\right)\cdot\left(cost, -\frac{1}{\sqrt{2}}sint\ , -\frac{1}{\sqrt{2}}sint\ \right)\\ &=(-cost, 2tsint, 0)\end{aligned}$$

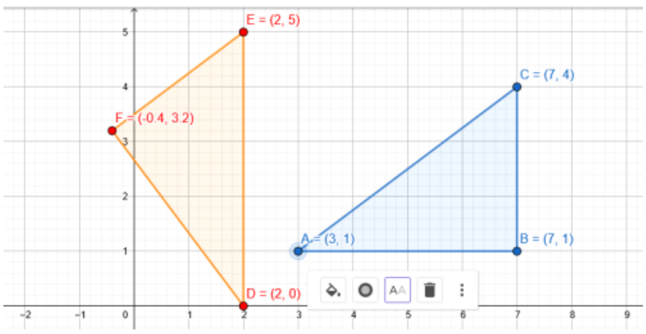
$$\begin{aligned}Y'\cdot\alpha''\ &=(1, -2t, 2t)\cdot(-cost, -sint, 0)\\ &=(-cost, 2tsint, 0)\end{aligned}$$

$\therefore$ $Y'\cdot\alpha''\ =\overline{Y'}\cdot\overline{\alpha''}$

3. Sketch the triangles in $\mathbf{R}^2$ that have vertices

Δ_1: (3, 1), (7, 1), (7, 4), $\quad\Delta_2$: (2, 0), (2, 5), (−2/5, 16/5).

Show that these triangles are congruent by exhibiting an isometry $F = TC$ that carries Δ_1 to Δ_2. (*Hint*: the orthogonal part C is not altered if the triangles are translated.)



If $F = TC$, $T = T_{(p,q)}$, $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then

$$(2,0)=F(3,1)=(p,q)+\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}=(p,q)+(3a+b, 3c+d)$$

$$(2,5)=F(7,4)=(p,q)+\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 7 \\ 4 \end{pmatrix}=(p,q)+(7a+4b, 7c+4d)$$

$$\left(-\frac{2}{5}, \frac{16}{5}\right)=F(7,1)=(p,q)+\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 7 \\ 1 \end{pmatrix}=(p,q)+(7a+b, 7c+d)$$

thus

$C=\begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$& orientation-reversing and $p=3$, $q=-3$

$\therefore$ $\triangle\square_1\equiv\triangle_2$

$\because$

(1)-$(3)\Rightarrow\left(\frac{12}{5}, -\frac{16}{5}\right)=(-4a, -4c)\Rightarrow a=-\frac{3}{5}, c=\frac{4}{5}$

(1)-(2) $\Rightarrow(0, -5)=(-4a-3b, -4c-3d)=\left(\frac{12}{5}-3b, -\frac{16}{5}-3d\right)\Rightarrow b=\frac{4}{5}, d=\frac{3}{5}$

$(2,0)=(p,q)+(-1,3)\Rightarrow p=3, q=-3$

$\equiv$:₩equiv

4. If F: $\mathbf{R}^3 \to \mathbf{R}^3$ is a diffeomorphism such that F_* preserves dot products, show that F is an isometry. (*Hint:* Show that F preserves lengths of curve segments and deduce that F^{-1} does also.)

a: $[0,1] \to R^3$:regular curve by $a(0) = p, a(1) = q$

$L(a)=\int_0^1 \square \parallel a(t) \parallel dt$

$F(a)$:$[0,1] \to R^3$:curve by $F(a(0)) = F(p), F(a(1)) = F(q)$

$(F(a))' = F_*(a')$by tnm1.7.7

F_* preserves norm because it preserves dot products

thus

$\parallel a' \parallel = \parallel F_*(a') \parallel$

$\Rightarrow$ $L(a)=\int_0^1 \square \parallel a' \parallel dt$

$=\int_0^1 \square \parallel F_*(a') \parallel dt$

$=\int_0^1 \square \parallel (F(a))' \parallel dt$

$=L(F(a))$

F is isometry because it preserves length

5. Let F be an isometry of $\mathbf{R}^3$. For each vector field V let $\overline{V}$ be the vector field such that $F_*(V(\mathbf{p})) = \overline{V}(F(\mathbf{p}))$ for all $\mathbf{p}$. Prove that isometries preserve covariant derivatives; that is, show $\overline{\nabla_V \mathrm{W}} = \nabla_{\overline{V}}\overline{\mathrm{W}}$.

$F_*(\nabla_V W)=F_*\left(W(p+tv)'(0)\right)$ by def 2.5.1

$=\overline{W}\left(F(p+tv)'(0)\right)$ by $F_*(W(p)) = \overline{W}(F(p))$

$=\overline{W}\left(TC(p+tv)'(0)\right)$

$=\overline{W}(T(C(p)+tC(v))'(0)$

$=\overline{W}\left(a+C(p)+tC(v)\right)'(0)$

$=\overline{W}\left(F(p)+tC(v)\right)'(0)$

$=\nabla_{F_*(V)}\overline{W}$

Thus

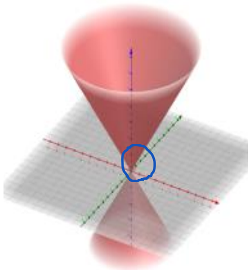
By cor 4.1

$\overline{\nabla_V W}=\nabla_{\overline{V}}\overline{W}$

4.1 연습문제-1

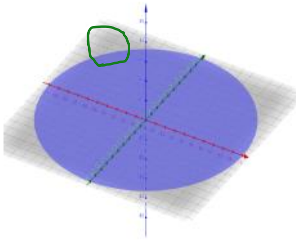
2020년 9월 23일 수요일　　오전 12:24

1-(a)



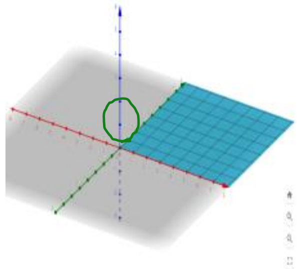
Vertex(꼭짓점)

1-(b)



$x^2 + y^2 = 1$

1-(c)



Z axis

2. A *plane* in $\mathbf{R}^3$ is a surface M: $ax + by + cz = d$, where the numbers a, b, c are necessarily not all zero. Prove that every plane in $\mathbf{R}^3$ may be described by a vector equation as on page 62.

A *plane* in $\mathbf{R}^3$ can be described as the union of all the perpendiculars to a given line at a given point. In vector language then, the *plane through* **p** *orthogonal to* $\mathbf{q} \neq 0$ consists of all points **r** in $\mathbf{R}^3$ such that $(\mathbf{r} - \mathbf{p}) \bullet \mathbf{q} = 0$. By the remark above, we may picture **q** as a tangent vector at **p** as shown in Fig. 2.9.

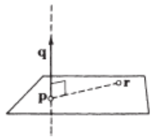


FIG. 2.9

Let $q = (a,b,c), r = (x,y,z), p = (x_0,y_0,z_0)$

$\Rightarrow (r-p)\cdot q = (x-x_0, y-y_0, z-z_0)\cdot(a,b,c)$

$= a(x-x_0) + b(y-y_0) + c(z-z_0)$

$= ax + by + cz - (ax_0 + by_0 + cz_0) = 0$

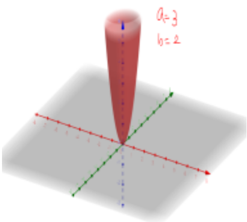
thus

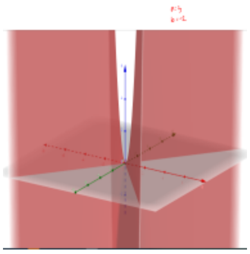
plane in R^3 is $ax + by + cz = d$ where $d = (ax_0 + by_0 + cz_0)$

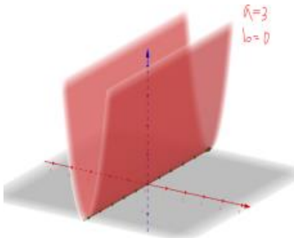
3. Sketch the general shape of the surface M: $z = ax^2 + by^2$ in each of the following cases:

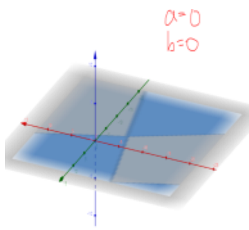
(a) $a > b > 0$.　　(b) $a > 0 > b$.

(c) $a > b = 0$.　　(d) $a = b = 0$.









4. In which of the following cases is the mapping $\mathbf{x}$: $\mathbf{R}^2 \to \mathbf{R}^3$ a patch?

(a) $\mathbf{x}(u,v) = (u, uv, v)$.　　(b) $\mathbf{x}(u,v) = (u^2, u^3, v)$.

(c) $\mathbf{x}(u,v) = (u, u^2, v + v^3)$.　　(d) $\mathbf{x}(u,v) = (\cos 2\pi u, \sin 2\pi u, v)$.

(Recall that $\mathbf{x}$ is one-to-one if and only if $\mathbf{x}(u,v) = \mathbf{x}(u_1,v_1)$ implies $(u,v) = (u_1,v_1)$.)

(a)

Let $x(u,v) = x(u_1,v_1)$

$\Rightarrow (u, uv, v) = (u_1, u_1v_1, v_1)$

$\Rightarrow (u,v) = (u_1,v_1)$

Thus x is 1-1 mapping

$J(x) = \begin{pmatrix} 1 & v & 0 \\ 0 & u & 1 \end{pmatrix}$ has rank2 ⇒ x is a regular

Thus x is patch

(b)

$J(x) = \begin{pmatrix} 2u & 3u^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has rank1 => x is not regular

(c)

Let $x(u,v) = x(u_1,v_1)$

$\Rightarrow (u, u^2, v + v^3) = (u_1, u_1^2, v_1 + v_1^3)$

$\Rightarrow (u,v) = (u_1,v_1)$

thus x is 1-1 mapping

$J(x) = \begin{pmatrix} 1 & 2u & 0 \\ 0 & 0 & 3v^2+1 \end{pmatrix}$ has rank2 ⇒ x is a regular

thus x is patch

(d)

x(0,0)=(1,0,0)=x(1,0)

Thus x is not 1-1 mapping

5. (a) Prove that $M: (x^2+y^2)^2 + 3z^2 = 1$ is a surface.
(b) For which values of c is $M: z(z-2) + xy = c$ a surface?

(a)

Let M: $(x^2+y^2)^2 + 3z^2 = 1$

If $g(x,y,z) = (x^2+y^2)^2 + 3z^2$

Then $dg = 4x(x^2+y^2)dx + 4y(x^2+y^2)dy + 6zdz$

$\Rightarrow dg = 0 \Leftrightarrow x = y = z = 0$

But $(0,0,0) \notin M$

Thus $dg \neq 0$ at all points of M

∴ M is a surface

(b)

Let $M: z(z-2) + xy = c$

If $g(x,y,z) = z(z-2) + xy$

Then $dg = ydx + xdy + 2(z-1)dz$

$\Rightarrow dg = 0 \Leftrightarrow x = y = 0, z = 1$

But $(0,0,1) \notin M$

$\therefore c \neq 1$

6. Determine the intersection $z = 0$ of the *monkey saddle*

$$M: z = f(x,y), \quad f(x,y) = y^3 - 3yx^2,$$

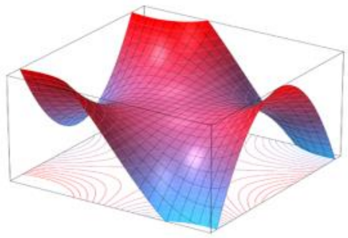
with the xy plane. On which regions of the plane is $f > 0$? $f < 0$? How does this surface get its name? (*Hint:* see Fig. 5.19.)

1) $y = 0 \Leftrightarrow z = 0$

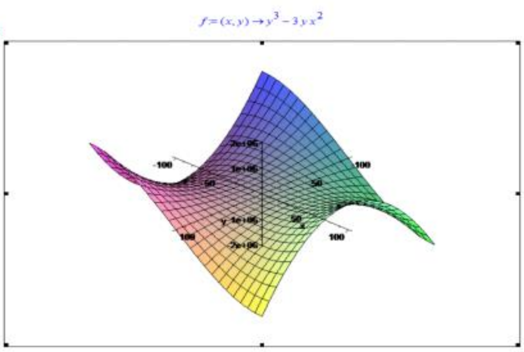
2) $0 = y^3 - 3yx^2$

$\Rightarrow y^2 = 3x^2$

$\Rightarrow y = \pm\sqrt{3}x$



```
> f:=(x,y)->y^3-3*y*x^2;
> plot3d(f(x,y),x=-100..100,y=-100..100);
```



4.1 연습문제-2

2020년 9월 23일 수요일 오전 1:48

7. Let $\mathbf{x}: D \to \mathbf{R}^3$ be a mapping, with

$$\mathbf{x}(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v)).$$

(a) Prove that a point $\mathbf{p} = (p_1, p_2, p_3)$ of $\mathbf{R}^3$ is in the image $\mathbf{x}(D)$ if and only if the equations

$$p_1 = x_1(u, v), \quad p_2 = x_2(u, v), \quad p_3 = x_3(u, v)$$

can be solved for u and v, with (u, v) in D.

(b) If for every point $\mathbf{p}$ in $\mathbf{x}(D)$ these equations have the *unique* solution $u = f_1(p_1, p_2, p_3)$, $v = f_2(p_1, p_2, p_3)$, with (u, v) in D, prove that $\mathbf{x}$ is one-to-one and that $\mathbf{x}^{-1}: \mathbf{x}(D) \to D$ is given by the formula

$$\mathbf{x}^{-1}(\mathbf{p}) = (f_1(\mathbf{p}), f_2(\mathbf{p})).$$

(a)
$p = (p_1, p_2, p_3) \in x(D) \Leftrightarrow \exists (u, v) \in D \ s.t \ x(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v)) = (p_1, p_2, p_3)$

(b)-1
Let $x(u_1, v_1) = x(u_2, v_2) = (p_1, p_2, p_3)$
$u_1 = f_1(p_1, p_2, p_3) = u_2$
$v_1 = f_2(p_1, p_2, p_3) = v_2$
Thus x is 1-1

(b)-2
$x(u, v) = x\left(f_1(p), f_2(p)\right) = p$
$\Rightarrow x^{-1}(p) = (f_1(p), f_2(p))$

8. Let $\mathbf{x}: D \to \mathbf{R}^3$ be the function given by

$$\mathbf{x}(u, v) = (u^2, uv, v^2)$$

on the first quadrant D: $u > 0$, $v > 0$. Show that $\mathbf{x}$ is one-to-one and find a formula for its inverse function $\mathbf{x}^{-1}: \mathbf{x}(D) \to D$. Then prove that $\mathbf{x}$ is a proper patch.

(1)
$x(u, v) = x(u_1, v_1)$
$\Rightarrow (u^2, uv, v^2) = (u_1^2, u_1 v_1, v_1^2)$
$\Rightarrow u = u_1, v = v_1 \quad \because u, u_1, v, v_1 > 0$
Thus x is 1-1

(2)
$x_u(u, v) = (2u, v, 0), x_v(u, v) = (0, u, 2v)$
$\Rightarrow \ x_u \times x_v = (2v^2, -4uv, 2u^2) \neq 0$
$\because \| x_u \times x_v \| = 4u^4 + 16u^2v^2 + 4v^4 > 0$
Thus x is regular

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ 2u & v & 0 \\ 0 & u & 2v \end{vmatrix} = 2v^2 u_1 - 4uv u_2 + 2u^2 u_3$$

(3)
$x^{-1}(x, y, z) = (\sqrt{x}, \sqrt{z})$:continuous

$\Rightarrow$ x is proper patch

9. Let $\mathbf{x}: \mathbf{R}^2 \to \mathbf{R}^3$ be the mapping

$$\mathbf{x}(u, v) = (u + v, u - v, uv).$$

Show that $\mathbf{x}$ is a proper patch and that the image of $\mathbf{x}$ is the entire surface M: $z = (x^2 - y^2)/4$.

1) x is a 1-1 mapping
2) x is a regular
$\because$
$J(x) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ v & u \end{pmatrix}$ has $rank$ 2
3) $x^{-1}(x, y, z) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right), z = \frac{x^2-y^2}{4}$: continuous
$\Rightarrow$ x is proper patch

If $x(u, v) = (x, y, z)$ then $x = u + v, y = u - v, z = uv$
$\Rightarrow x^2 = u^2 + 2uv + v^2, y^2 = u^2 - 2uv + v^2$
$\Rightarrow \frac{x^2-y^2}{4} = uv = z \ , -\infty < x, y < \infty$
Thus the image of x is $z = \frac{x^2-y^2}{4}$
Let $g(x, y, z) = \frac{x^2-y^2}{4} - z$
$\Rightarrow dg = \frac{x}{2}dx - \frac{y}{2}dy - dz \neq 0$
Thus M: $z = \frac{x^2-y^2}{4}$ is surface (by thm 1.4)

10. If $F: \mathbf{R}^3 \to \mathbf{R}^3$ is a diffeomorphism and M is a surface in $\mathbf{R}^3$, prove that the image $F(M)$ is also a surface in $\mathbf{R}^3$. (*Hint:* If $\mathbf{x}$ is a patch in M, then the composite function $F(\mathbf{x})$ is regular, since $F(\mathbf{x})_* = F_* \mathbf{x}_*$ by Ex. 9 of Sec. 1.7.)

$\forall q \in F(M), \exists\, p \in M$ $s.t$ $q = F(p)$

Since M is surface , $\exists\, x: D \to M$ $s.t$ $x(D)$ is a nbd of p

If $y = F(x): D \to F(M)$,

Then y(D)=F(x(D)) is a nbd of q and $y_* = F_*(x_*)$ by ex.1.7.9

F_*:isomorphism and x_*:1-1$\Rightarrow$ y_*:1-1

Thus y is patch

Therefore F(M) is surface

11. Prove this special case of Exercise 10: If F is a diffeomorphism of $\mathbf{R}^3$, then the image of the surface M: $g = c$ is $\overline{M}$: $\overline{g} = c$, where $\overline{g} = g(F^{-1})$ and $\overline{M}$ is a surface. (*Hint:* If $dg(\mathbf{v}) \neq 0$ at $\mathbf{p}$ in M, show by using Ex. 7 of Sec. 1.7 that $d\overline{g}(F_*\mathbf{v}) \neq 0$ at $F(\mathbf{p})$.)

Pf)

Let $q \in F(M)$ tnen $p = F^{-1}(q) \in M$

since M is surface, $\exists\, v \in T_p(M)$ such that $dg(v) \neq 0$

thus $F_*(v) \in T_{F(p)}(\overline{M})$

and $d\bar{g}(F_*(v))$=$F_*(v)[\bar{g}]$=$v[\bar{g} \circ F]$=$v[g \circ F^{-1} \circ F]$=$v[g]$=$dg(v) \neq 0$

$\therefore$ $\overline{M}$ is surface

$d\bar{g}(F_*(v))$=$dg\, dF^{-1}(dF(v))$=$dg(v) \neq 0$

12. Let C be a Curve in the xy plane that is symmetric about the x axis. Assume C crosses the x axis and always does so orthogonally. Explain why there can be only one or two crossings. Thus C is either an arc or is closed (Fig. 4.9). Revolving C about the x axis gives a surface M, called an *augmented surface of revolution*. Explain how to define patches in M at the crossing points.

$f(x,(-y)^2) = f(x,y^2)$이므로 C는 $x-$축 대칭이다.

C가 $x-$축 대칭이고 연결집합이므로 $x-$축과 적어도 한 점에서 만난다.

C가 $x-$축 위의 세 점 $(p,0),(q,0),(r,0), p<q<r$에서 만난다고 가정하면

C가 $x-$축 대칭이므로 C는 점 $(q,0)$에서 self intersect한다.

이는 C가 곡선이라는 가정에 모순이다.

따라서 C는 $x-$축과 많아야 두 점에서 만난다.

$M: f(x,y^2+z^2)=c$라 하자.

$df = \frac{\partial f}{\partial x}dx + 2y\frac{\partial f}{\partial y}dy + 2z\frac{\partial f}{\partial y}dz$이고

$$(\frac{\partial f}{\partial x})^2 + (2y\frac{\partial f}{\partial x})^2 + (2z\frac{\partial f}{\partial y})^2 = (\frac{\partial f}{\partial x})^2 + 4(y^2+z^2)(\frac{\partial f}{\partial y})^2$$
$$\geq (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 > 0$$

이므로 M은 곡면이다.

4.2 연습문제-1

2020년 10월 6일 화요일 오후 8:57

1. Find a parametrization of the entire surface obtained by revolving:
(a) C: $y = \cosh x$ around the x axis (catenoid).
(b) C: $(x-2)^2 + y^2 = 1$ around the y axis (torus).
(c) C: $z = x^2$ around the z axis (paraboloid).

(a)

By p.143 $x(u,v) = \left(g(u), h(u)cosv, h(u)sinv\right)$

$x(u,v) = (u, coshu\ cosv, coshu\ sinv)$where $-\infty < u < \infty, 0 \le v < 2\pi$

(b)

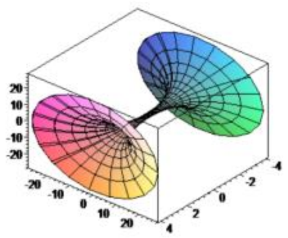
Let $x = 2 + cosu$ then $y = sinu$

$x(u,v) = (\ (2+cosu)sinv, sinu, (2+cosu)cosv\ )$where $0 \le\ u, v\ < 2\pi$

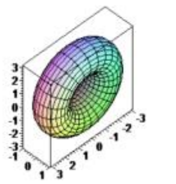
(c)

$x(u,v) = \left(u, v, u^2+v^2\right)$

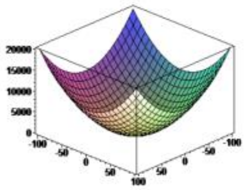
```
> x:=(u,v)->[u,cosh(u)*cos(v),cosh(u)*sin(v)];
```
$x := (u,v) \rightarrow [u, \cosh(u)\cos(v), \cosh(u)\sin(v)]$
```
> plot3d(x(u,v),u=-4..4,v=-4..4);
```



```
x:=(u,v)->[(2+cos(u))*sin(v),sin(u),(2+cos(u))*cos(v)];
```
$x := (u,v) \rightarrow [(2+\cos(u))\sin(v), \sin(u), (2+\cos(u))\cos(v)]$
```
plot3d(x(u,v),u=0..2*Pi,v=0..2*Pi);
```



```
> x:=(u,v)->[u,v,u^2+v^2];
```
$x := (u,v) \rightarrow [u, v, u^2+v^2]$
```
> plot3d(x(u,v),u=-100..100,v=-100..100);
```



2. Partial velocities $\mathbf{x}_u$ and $\mathbf{x}_v$ are defined for an arbitrary mapping $\mathbf{x}: D \to \mathbf{R}^3$, so we can consider the real-valued functions

$$E = \mathbf{x}_u \cdot \mathbf{x}_u, \quad F = \mathbf{x}_u \cdot \mathbf{x}_v, \quad G = \mathbf{x}_v \cdot \mathbf{x}_v$$

on D. Prove

$$\|\mathbf{x}_u \times \mathbf{x}_v\|^2 = EG - F^2.$$

Deduce that $\mathbf{x}$ is a regular mapping if and only if $EG - F^2$ is never zero. (This is often the easiest way to check regularity. We will see, beginning in the next chapter, that the functions E, F, G are fundamental to the geometry of surfaces.)

Let $E = x_u \cdot x_u$, $F = x_u \cdot x_v$, $G = x_v \cdot x_v$

By lemma2.1.8 $|v \times w|^2 = \ v \cdot v\, w \cdot w\ -\ (v \cdot w)^2$

$|x_u \times x_v|^2$=$x_u \cdot x_u x_v \cdot x_v - \left(x_u \cdot x_v\right)^2 = EG - F^2$

and

$regular \Leftrightarrow\ \ x_u \times x_v \neq 0 \Rightarrow |x_u \times x_v|^2 \neq 0$

Thus

regular $\Leftrightarrow EG - F^2 \neq 0$

3. A *generalized cone* is a ruled surface with a parametrization of the form

$$\mathbf{x}(u,v) = \mathbf{p} + v\delta(u).$$

Thus all rulings pass through the vertex $\mathbf{p}$ (Fig. 4.17). Show that $\mathbf{x}$ is regular if and only if v and $\delta \times \delta'$ are never zero. (Thus the vertex is never part of the cone. Unless the term *generalized* is used, we assume that δ is a closed curve and require either $v > 0$ or $v < 0$.)

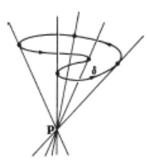


FIG. 4.17

Let $x(u,v) = p + v\delta(u)$ and $\delta(u) = \left(x_1(u), x_2(u), x_3(u)\right)$ then $x(u,v) = p + v\left(x_1(u), x_2(u), x_3(u)\right)$

$x_u = \left(vx_1'(u), vx_2'(u), vx_3'(u)\right)$=$v\delta'(u)$ and $x_v = \left(x_1(u), x_2(u), x_3(u)\right)$=$\delta(u)$

Thus

$x_u \times x_v =\ v\delta'(u) \times \delta(u)$

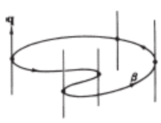
$regular\ \ \Leftrightarrow x_u \times x_v \neq 0$

$\therefore regular\ \ \Leftrightarrow \delta' \times \delta \neq 0, v \neq 0$

4. A *generalized cylinder* is a ruled surface for which the rulings are all Euclidean parallel (Fig. 4.18). Thus there is always a parametrization of the form

$$\mathbf{x}(u,v) = \beta(u) + v\mathbf{q} \quad (\mathbf{q} \in \mathbf{R}^3).$$

Prove: (a) Regularity of $\mathbf{x}$ is equivalent to $\beta' \times \mathbf{q}$ never zero.
(b) If C: $f(x,y) = a$ is a Curve in the plane, show that in $\mathbf{R}^3$ the same equation defines a surface $\bar{C}$. If $t \to (x(t), y(t))$ is a parametrization of C, find a parametrization of $\bar{C}$ that shows it is a generalized cylinder.
Generalized cylinders are a rather broad category—including Euclidean



Prove: (a) Regularity of **x** is equivalent to $\beta' \times \mathbf{q}$ never zero.

(b) If C: $f(x, y) = a$ is a Curve in the plane, show that in $\mathbf{R}^3$ the same equation defines a surface $\bar{C}$. If $t \to (x(t), y(t))$ is a parametrization of C, find a parametrization of $\bar{C}$ that shows it is a generalized cylinder.

Generalized cylinders are a rather broad category—including Euclidean planes when β is a straight line—so unless the term *generalized* is used, we assume that cylinders are over *closed* curves β.

FIG. 4.18

(a)

$x(u,v) = \beta(u) + vq \Rightarrow x_u = \beta'(u), x_v = q$

by ex3

$x_u \times x_v = \beta'(u) \times q$

Thus x is a regular $\Leftrightarrow \beta'(u) \times q \neq 0$

(b)

??

5. A line L is attached orthogonally to an axis A (Fig. 4.19). If L moves steadily along A, rotating at constant speed, then L sweeps out a *helicoid* H. When A is the z axis, H is the image of the mapping $\mathbf{x}$: $\mathbf{R}^2 \to \mathbf{R}^3$ such that

$$\mathbf{x}(u, v) = (u \cos v, u \sin v, bv) \quad (b \neq 0).$$

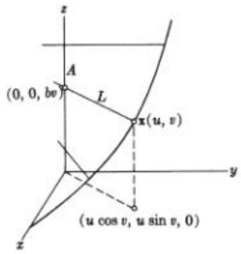


FIG. 4.19

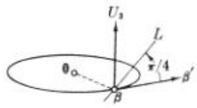


FIG. 4.20

(a) Prove that **x** is a patch.
(b) Describe its parameter curves.
(c) Express the helicoid in the implicit form $g = c$.
(d) (*Computer graphics.*) Plot one full turn ($0 \leqq v \leqq 2\pi$) of a helicoid with $b = 1/2$. Restrict the rulings to $-1 \leqq u \leqq 1$.

(a)

Let $x(u,v) = x(u_1, v_1)$

$\Rightarrow (ucosv, usinv, bv) = (u_1cosv_1, u_1sinv_1, bv_1)$

$\Rightarrow (u,v) = (u_1, v_1)$

Thus x is 1-1 mapping

$J(x) = \begin{pmatrix} cosv & sinv & 0 \\ -usinv & ucosv & b \end{pmatrix}$ has rank2 $\Rightarrow$ x is a regular

or by ex2 $|x_u \times x_v|^2 = EG - F^2 = b^2 + u^2 \neq 0 \Rightarrow$ x is a regular

Thus x is patch

(b):

u-parameter curve: $(ucosv_0, usinv_0, bv_0) \Rightarrow straight\ line$

v-parameter curve: $(u_0cosv, u_0sinv, bv) \Rightarrow helix$

(c)

Let $(ucosv, usinv, bv) = (x, y, z)$ then $xsin\left(\frac{z}{b}\right) = ycos\left(\frac{z}{b}\right)$

Thus

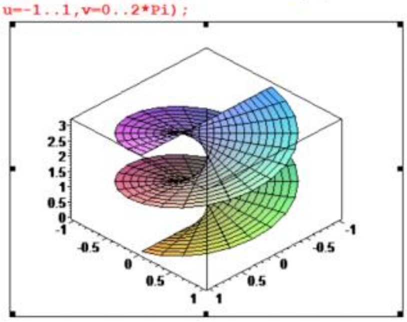
H: $xsin\left(\frac{z}{b}\right) - ycos\left(\frac{z}{b}\right) = 0$

(d)

```
> x:=(u,v)->[u*cos(v),u*sin(v),1/2*v];
```

$$x := (u, v) \to \left[u\cos(v), u\sin(v), \frac{1}{2}v\right]$$

```
> plot3d(x(u,v),u=-1..1,v=0..2*Pi);
```



4.2 연습문제-2

2020년 10월 6일 화요일 오후 9:22

6. (a) Show that the *saddle surface* M: $z = xy$ is doubly ruled: Find two ruled parametrizations with different rulings.

(b) (*Computer graphics.*) Plot a representative portion of M, using a patch for which the parameter curves are rulings.

7. Let β be a unit-speed parametrization of the unit circle in the xy plane. Construct a ruled surface as follows: Move a line L along β in such a way that L is always orthogonal to the radius of the circle and makes constant angle $\pi/4$ with β' (Fig. 4.20).

(a) Derive this parametrization of the resulting ruled surface M:

$$\mathbf{x}(u, v) = \beta(u) + v(\beta'(u) + U_3).$$

(b) Express $\mathbf{x}$ explicitly in terms of v and coordinate functions for β.

(c) Deduce that M is given implicitly by the equation

$$x^2 + y^2 - z^2 = 1.$$

(d) Show that if the angle $\pi/4$ above is changed to $-\pi/4$, the same surface M results. Thus M is doubly ruled.

(e) Sketch this surface M showing the two rulings through each of the points (1, 0, 0) and (2, 1, 2).

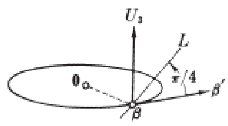


FIG. 4.20

(a)

Let $\beta(u) = (cosu, sinu, 0)$ then by FIG4.20 $x(u,v) = \beta(u) + v(\beta'(u) + u_3)$

(b)

$x(u,v) = (cosu\ , sinu, 0) + v(\ (-sinu, cosu, 0) + (0,0,1)) = (cosu - vsinu, sinu + vcosu, v)$

(c)

Let $(cosu - vsinu, sinu + vcosu, v) = (x, y, z)$ then $x^2 + y^2 - z^2 = 1$

(d)

$x(u,v) = \beta(u) + v(\beta'(u) - u_3) = (cosu - vsinu, sinu + vcosu, -v)$ at $\angle L\beta\beta' = -\frac{\pi}{4}$

(e)

$x(u,v) = (1\ , 0, 0) \Rightarrow u = 0, v = 0$

Thus

Ruling is $\beta(u) + v(\beta'(u) + u_3) = (1,0,0) + v(0,1,1)$

$x(u,v) = (2\ , 1, 2) \Rightarrow cosu = \dfrac{4}{5}, v = 2\ (\because cosu - 2sinu = 2, sinu + 2cosu = 1)$

Thus

Ruling is $\beta(u) + v(\beta'(u) + u_3) = \left(\frac{4}{5}, \frac{3}{5}, 0\right) + v\left(-\frac{3}{5}, \frac{4}{5}, 1\right)$

8. Let M be the surface of revolution gotten by revolving the curve $t \to (g(t), h(t), 0)$ about the x axis ($h > 0$). Show that:

(a) If g' is never zero, then M has a parametrization of the form

$$\mathbf{x}(u, v) = (u, f(u)\cos v, f(u)\sin v).$$

(b) If h' is never zero, then M has a parametrization of the form

$$\mathbf{x}(u, v) = (f(u), u\cos v, u\sin v).$$

A *quadric surface* is a surface M: $g = 0$ in $\mathbf{R}^3$ such that g contains at most quadratic terms in x_1, x_2, x_3, that is,

$$g = \sum_{i,j} a_{ij}x_ix_j + \sum_i b_ix_i + c.$$

Trivial cases excepted, every quadric surface is congruent to one of the five types described in the next two exercises. (Use of computers is optional in these exercises.)

10. Sketch the following surfaces (graphs of functions) for $a = 2$, $b = 1$:

(a) *Elliptic paraboloid.* M: $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$. Show that

$$\mathbf{x}(u, v) = (au\cos v, bu\sin v, u^2), \quad u > 0,$$

is a parametrization that omits only one point of M.

(b) *Hyperbolic paraboloid.* M: $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$.

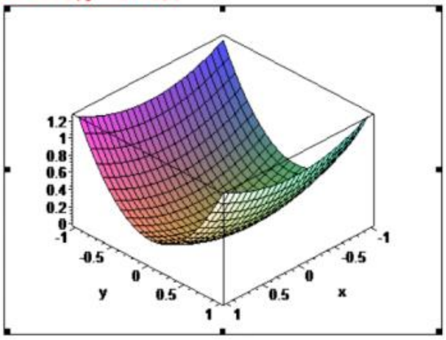
Show that M is covered by the single patch

$$\mathbf{x}(u, v) = (a(u + v), b(u - v), 4uv) \text{ on } \mathbf{R}^2.$$

```
> M:=(x,y)->x^2/4+y^2;
```

$$M := (x, y) \to \frac{1}{4}x^2 + y^2$$

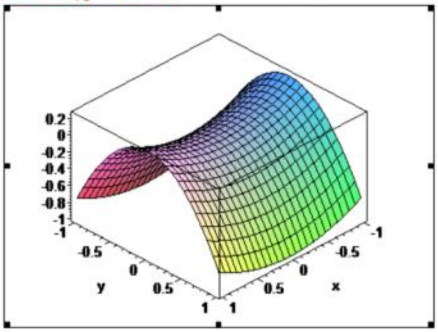
```
> plot3d(M(x,y),x=-1..1,y=-1..1);
```




```
> M:=(x,y)->x^2/4-y^2;
```

$$M := (x, y) \to \frac{1}{4}x^2 - y^2$$

```
> plot3d(M(x,y),x=-1..1,y=-1..1);
```



11. *Doubly ruled quadrics.*

(a) Show that the hyperbolic paraboloid M in the preceding exercise is doubly ruled.

(b) (*Computer graphics.*) For $a = 2$, $b = 1$ use the patch in (b) of Exercise 10 to plot a portion of M. (Keep the same scale on all axes; the parameter curves will be the rulings.)

(c) Find two different ruled parametrizations of the hyperboloid of one sheet by using the scheme in the special case, Exercise 7.

(d) (*Computer graphics.*) Plot a portion of each of these parametrizations, taking $a = 1.5$, $b = 1$, $c = 2$.

4.3연습문제

2020년 10월 21일 수요일 오전 1:10

4. Let $\mathbf{x}$ be a patch in M.
(a) If $\mathbf{x}_*$ is the tangent map of $\mathbf{x}$ (Sec. 7 of Ch. 1), show that

$$\mathbf{x}_*(U_1) = \mathbf{x}_u, \quad \mathbf{x}_*(U_2) = \mathbf{x}_v,$$

where U_1, U_2 is the natural frame field on $\mathbf{R}^2$.
(b) If f is a differentiable function on M, prove

$$\mathbf{x}_u[f] = \frac{\partial}{\partial u}(f(\mathbf{x})), \quad \mathbf{x}_v[f] = \frac{\partial}{\partial v}(f(\mathbf{x})).$$

(a)

$$x_*\left(U_1(p)\right) = \frac{d}{dt}x\left(p + tU_1(p)\right)\Big|_{t=0}$$
$$= \frac{d}{dt}x(p_1 + t\,, p_2)\Big|_{t=0}$$
$$= x_u(p_1 + t\,, p_2)\Big|_{t=0}$$
$$= x_u(p_1\,, p_2)$$
$$= x_u(p)$$

$$x_*\left(U_2(p)\right) = \frac{d}{dt}x\left(p + tU_2(p)\right)\Big|_{t=0}$$
$$= \frac{d}{dt}x(p_1\,, p_2 + t)\Big|_{t=0}$$
$$= x_v(p_1\,, p_2 + t)\Big|_{t=0}$$
$$= x_v(p_1\,, p_2)$$
$$= x_v(p)$$

(b)

$$x_u[f] = x_*(U_1)[f] = U_1[f(x)] = \frac{\partial}{\partial u}f(x)$$
$$x_v[f] = x_*(U_2)[f] = U_2[f(x)] = \frac{\partial}{\partial v}f(x)$$

5. Prove that:
(a) $\mathbf{v} = (v_1, v_2, v_3)$ is tangent to $M: z = f(x, y)$ at a point $\mathbf{p}$ of M if and only if

$$v_3 = \frac{\partial f}{\partial x}(p_1, p_2)v_1 + \frac{\partial f}{\partial y}(p_1, p_2)v_2.$$

(b) If $\mathbf{x}$ is a patch in an arbitrary surface M, then $\mathbf{v}$ is tangent to M at $\mathbf{x}(u, v)$ if and only if

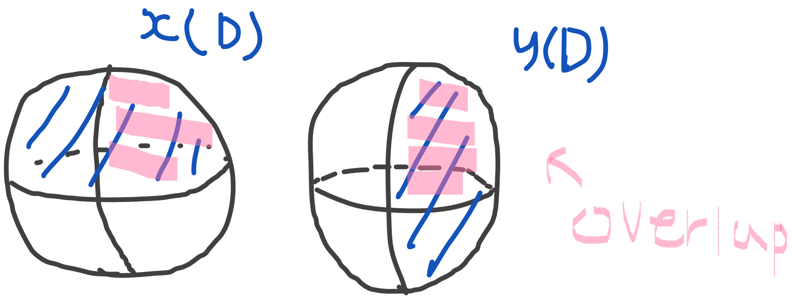
$$\mathbf{v} \bullet \mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) = 0.$$

(a)
Given $M: g = z - f(x,y) = 0,\ with\ (\nabla g) = (-f_x, -f_y, 1)$
$v = (v_1, v_2, v_3)$ is tangent to M at p iff $v \cdot (\nabla g)(p) = 0$
$\Leftrightarrow v \cdot (\nabla g) = (v_1, v_2, v_3) \cdot (-f_x, -f_y, 1) = -v_1 f_x - v_2 f_y + v_3 = 0$
$\Leftrightarrow v_3 = \frac{\partial f}{\partial x}(p_1, p_2)v_1 + \frac{\partial f}{\partial y}(p_1, p_2)v_2$
(b)
Since $x_u\,(u,v)$ and $x_v(u,v)$ is tangent to M
$x_u\,(u,v) \times x_v(u,v)$ is orthogonal to the tangent plane of M.
Thus v is tangent to M at p iff $v \cdot x_u\,(u,v) \times x_v(u,v) = 0$

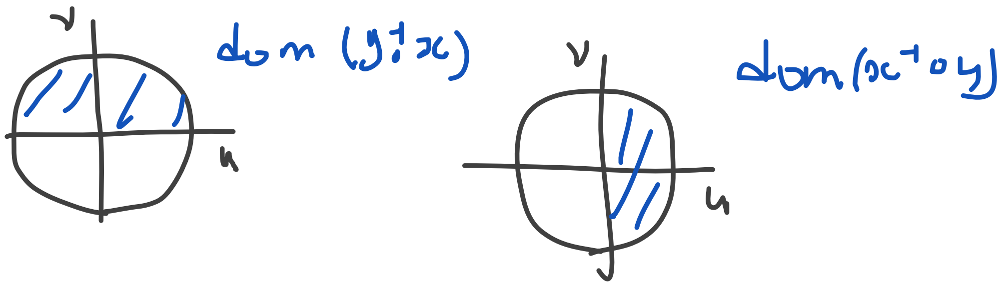
6. Let $\mathbf{x}$ and $\mathbf{y}$ be the patches in the unit sphere Σ that are defined on the unit disk D: $u^2 + v^2 < 1$ by

$$\mathbf{x}(u, v) = (u, v, f(u, v)), \quad \mathbf{y}(u, v) = (v, f(u, v), u),$$

where $f = \sqrt{1 - u^2 - v^2}$.
(a) On a sketch of Σ indicate the images $\mathbf{x}(D)$ and $\mathbf{y}(D)$, and the region on which they overlap.
(b) At which points of D is $\mathbf{y}^{-1}\mathbf{x}$ defined? Find a formula for this function.
(c) At which points of D is $\mathbf{x}^{-1}\mathbf{y}$ defined? Find a formula for this function.



(a)
$x(D) = x^2 + y^2 + z^2 = 1\,, z > 0$
$y(D) = x^2 + y^2 + z^2 = 1, y > 0$
(b)
$y^{-1} \circ x\,(u,v) = \left(\sqrt{1 - u^2 - v^2}, u\right)$
$x^{-1} \circ\ y\,(u,v) = \left(v, \sqrt{1 - u^2 - v^2}\right)$



7. Find a nonvanishing normal vector field on $M: z = xy$ and two tangent vector fields that are linearly independent at each point.

If $g = z - xy$ then $(\nabla g) = (-y, -x, 1)$ is normal vector field.
$V = (f_1, f_2, f_3)$ is tangent vector field of M iff $V \cdot (\nabla g) = 0$
$\Leftrightarrow V \cdot (\nabla g) = (f_1, f_2, f_3) \cdot (-y, -x, 1) = -yf_1 - xf_2 + f_3 = 0$
$\therefore V_1 = (x, 0, z), V_2 = (0, y, z)$

8. Let C be the circular cone parametrized by

$$\mathbf{x}(u, v) = v(\cos u, \sin u, 1).$$

If α is the curve $\mathbf{x}(\sqrt{2}t, e^t)$:
(a) Express α' in terms of $\mathbf{x}_u$ and $\mathbf{x}_v$.
(b) Show that at each point of α, the velocity α' bisects the angle between $\mathbf{x}_u$ and $\mathbf{x}_v$. (*Hint:* Verify that

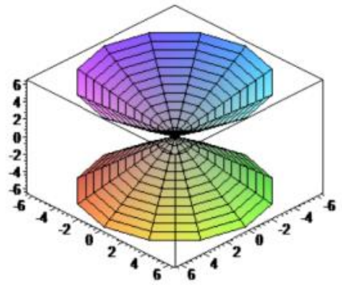
$$\alpha' \bullet \left(\frac{\mathbf{x}_u}{\|\mathbf{x}_u\|}\right) = \alpha' \bullet \left(\frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}\right),$$

where $\mathbf{x}_u$ and $\mathbf{x}_v$ are evaluated on $(\sqrt{2}t, e^t)$.)
(c) Make a sketch of the cone C showing the curve α.

```
> x:=(u,v)->[v*cos(u),v*sin(u),v];
```

$$x := (u, v) \rightarrow [v\cos(u), v\sin(u), v]$$

```
> plot3d(x(u,v),u=-2*Pi..2*Pi,v=-2*Pi..2*Pi);
```



$x_u = (-vsinu, vcosu, 0)$,$x_v = (cosu, sinu, 1)$

(a)

$\alpha' = \sqrt{2}x_u + e^t x_v$

(b)

$x_u \cdot x_u = v^2$,$x_u \cdot x_v = 0, x_v \cdot x_v = 2, u = \sqrt{2}, v = e^t$

$$\alpha' \cdot \left(\frac{x_u}{\| x_u \|}\right) = \sqrt{2}e^t$$

$$\alpha' \cdot \left(\frac{x_v}{\| x_v \|}\right) = \sqrt{2}e^t$$

$$\Rightarrow \ \alpha' \cdot \left(\frac{x_u}{\|x_u\|}\right) = \alpha' \cdot \left(\frac{x_v}{\|x_v\|}\right)$$

Thus α' bisects the angle betweenx_u and x_v.

9. If $\mathbf{z}$ is a nonzero vector normal to M at $\mathbf{p}$, let $\bar{T}_p(M)$ be the Euclidean plane through $\mathbf{p}$ orthogonal to $\mathbf{z}$. Prove:
(a) If each tangent vector $\mathbf{v}_p$ to M at $\mathbf{p}$ is replaced by its *tip* $\mathbf{p} + \mathbf{v}$, then $T_p(M)$ becomes $\bar{T}_p(M)$. Thus $\bar{T}_p(M)$ gives a concrete representation of $T_p(M)$ in $\mathbf{R}^3$. It is called the *Euclidean tangent plane to M at p.*
(b) If $\mathbf{x}$ is a patch in M, then $\bar{T}_{\mathbf{x}(u,v)}(M)$ consists of all points $\mathbf{r}$ in $\mathbf{R}^3$ such that $(\mathbf{r} - \mathbf{x}(u, v)) \bullet \mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v) = 0$.
(c) If M is given implicitly by $g = c$, then $\bar{T}_p(M)$ consists of all points $\mathbf{r}$ in $\mathbf{R}^3$ such that $(\mathbf{r} - \mathbf{p}) \bullet (\nabla g)(\mathbf{p}) = 0$.

(a)

$\bar{T}_{p(M)}$=$\{r|(r-p) \cdot z = 0\}$

Hence v_p is in $T_{p(M)}(v \cdot z = 0)$ iff $p + v$ is in $\bar{T}_{p(M)}$

(b)

$x_u(u,v) \times x_v(u,v)$is normal vector of M

By (a)

$\bar{T}_{p(M)}$=$\{r|(r - x(u,v)) \cdot x_u(u,v) \times x_v(u,v) = 0\}$

(c)

$(\nabla g)(p)$is normal vector of M

By (a)

$\bar{T}_{p(M)}$=$\{r|(r-p) \cdot (\nabla g)(p) = 0\}$

10. In each case below find an equation of the form $ax + by + cz = d$ for the plane $\bar{T}_p(M)$.
(a) $\mathbf{p} = (0, 0, 0)$ and M is the sphere

$$x^2 + y^2 + (z-1)^2 = 1.$$

(b) $\mathbf{p} = (1, -2, 3)$ and M is the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{18} = 1.$$

(c) $\mathbf{p} = \mathbf{x}(2, \pi/4)$, where M is the helicoid parametrized by

$$\mathbf{x}(u, v) = (u\cos v, u\sin v, 2v).$$

(a)

Let $f(x,y,z) = x^2 + y^2 + (z-1)^2$

$\nabla f\,(0,0,0)$=$(0,0,-2)$ ($\because \nabla f = (2x, 2y, 2(z-1))$)

Thus $0(x-0) + 0(y-0) - 2(z-0) = 0 \Rightarrow z = 0$

(b)

Let $f(x,y,z) = \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{18} = 1$

$\nabla f\,(1,-2,3)$=$\left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{3}\right)$ ($\because \nabla f = \left(\frac{x}{2}, \frac{y}{8}, \frac{z}{9}\right)$)

Thus $\frac{1}{2}(x-1) - \frac{1}{4}(y+2) + 2(z-3) = 0 \Rightarrow 6x - 3y + 4z = 24$

(c)

Let $x(u,v) = (ucosv, usinv, 2v)$

$x_u\left(2, \frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), x_v\left(2, \frac{\pi}{4}\right) = \left(-\sqrt{2}, \sqrt{2}, 2\right)$ $(\because x_u = (cosv, sinv, 0), x_v = (-usinv, ucosv, 2))$

$x_u \times x_v = \left(\sqrt{2}, -\sqrt{2}, 2\right)$

Thus $\sqrt{2}\left(x - \sqrt{2}\right) - \sqrt{2}\left(y - \sqrt{2}\right) + 2\left(z - \frac{\pi}{2}\right) = 0 \Rightarrow x - y + \sqrt{2}z = \frac{\pi}{2}$

11. (*Continuation of Ex. 2.*) With $\mathbf{x}$ the usual parametrization of the torus of revolution T, consider the curve $\alpha: \mathbf{R} \to T$ such that $\alpha(t) = \mathbf{x}(at, bt)$.
(a) If a/b is a rational number, show that α is a simple closed curve in T, that is, periodic with no self-crossings.
(b) If a/b is irrational, show α is one-to-one. Such a curve is called a *winding line* on the torus. It is *dense* in T in the sense that given any $\varepsilon > 0$, α comes within distance ε of every point of T.
(c) (*Computer graphics.*) For reference, plot the torus T with $R = 3$, $r = 1$ (see Ex. 2.5). Then plot the following curves in T:
(i) $\alpha(t) = \mathbf{x}(3t, 5t)$ on intervals $0 \leq t \leq b$, for $b = \pi, 2\pi$, and larger values. Estimate the period of α, in this case the smallest number $T > 0$ such that $\alpha(T) = \alpha(0)$.
(ii) $\alpha(t) = \mathbf{x}(\pi t, 5t)$ on intervals $0 \leq t \leq b$, for increasing values of b. (Keep the curve reasonably smooth.)

$\alpha(t) = x(at, bt) = ((R + r\cos at)\cos bt, (R + r\cos at)\sin bt, r\sin at)$

(a)

Let $\frac{a}{b} = \frac{m}{n}$, $m, n \in Z$

if $\triangle t = \frac{m}{a}2\pi = \frac{n}{b}2\pi$, then $\alpha(t + \triangle t) = \alpha(t)$

hence α is periodic

If $\alpha(s) = \alpha(t)$, then

$((R + r\cos as)\cos bs, (R + r\cos as)\sin bs, r\sin as) = ((R + r\cos at)\cos bt, (R + r\cos at)\sin bt, r\sin at)$

Thus

$\sin as = \sin at, \cos as = \cos at, \sin bs = \sin bt, \cos bs = \cos bt$

$\Rightarrow as = at + 2k\pi, k \in Z, bs = bt + 2l\pi, l \in Z$

Therefore $\frac{a}{b} = \frac{k}{l} = \frac{m}{n}$, $\exists\, c \in Z$ s.t $k = cm, l = cn$

$\Rightarrow s = t + \frac{k}{a}2\pi = t + \frac{cm}{a}2\pi = t + c\triangle t$

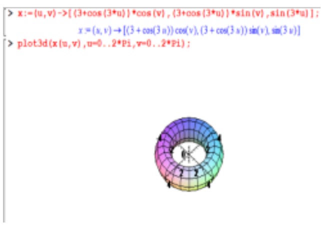
$\therefore \alpha$ is simple closed curve

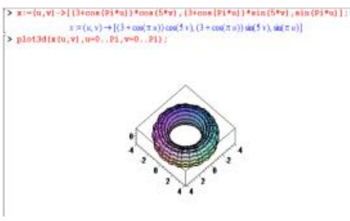
(b)

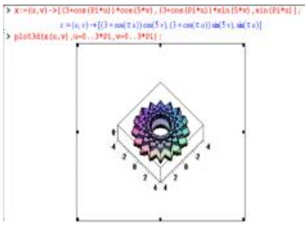
Let $\alpha(s) = \alpha(t), s \neq t$

By (a) $\frac{a}{b} = \frac{k}{l}$ ↯ ($\because \frac{a}{b}$ is irrational)

Thus s=t, $\alpha: 1-1$







12. A Euclidean vector field $Z = \sum z_i U_i$ on M is *differentiable* provided its coordinate functions z_1, z_2, z_3 (on M) are differentiable. If V is a tangent vector field on M, show that
(a) For every patch $\mathbf{x}: D \to M$, V can be written as

$$V(\mathbf{x}(u, v)) = f(u, v)\mathbf{x}_u(u, v) + g(u, v)\mathbf{x}_v(u, v).$$

(b) V is differentiable if and only if the functions f and g (on D) are differentiable.

The following exercises deal with *open sets* in a surface M in $\mathbf{R}^3$, that is, sets $\mathscr{U}$ in M that contain a neighborhood in M of each of their points.

(a)

$T_{x(u,v)}(M) = \{a x_u(u,v) + b x_v(u,v) : a, b \in R\}$ & $(u,v) \in D \Rightarrow V(x(u,v)) \in T_{x(u,v)}(M)$

$\Rightarrow \exists\, f(u,v), g(u,v) \in R$ s.t $V(x(u,v)) = f(u,v)x_u(u,v) + g(u,v)x_v(u,v)$

(b)

clear

13. Prove that if $\mathbf{y}: E \to M$ is a proper patch, then $\mathbf{y}$ carries open sets in E to open sets in M. Deduce that if $\mathbf{x}: D \to M$ is an arbitrary patch, then the image $\mathbf{x}(D)$ is an open set in M. (*Hint:* To prove the latter assertion, use Cor. 3.3.)

$y: E \to M$ is proper patch $\Rightarrow y^{-1}$: conti

Hence $y^{-1}(E)$: open set $\Rightarrow y(E) = (y^{-1})^{-1}(E)$: open set

let $x: D \to M$ be arbitrary patch, and hold the proper patch $y_i: E_i \to M$, which is $D \subset \cup_i E_i$

$x(D) = \cup_i\, y_i\left(y^{-1} \circ y_i(E_i)\right)$ and $x^{-1} \circ y_i(E_i)$: open set $\Rightarrow y^{-1}\left(x^{-1} \circ y_i(E_i)\right)$:open set

Thus $x(D)$ is open set

14. Prove that *every* patch $\mathbf{x}: D \to M$ in a surface M in $\mathbf{R}^3$ is proper. (*Hint:* Use Ex. 13. Note that $(\mathbf{x}^{-1}\mathbf{y})\mathbf{y}^{-1}$ is continuous and agrees with $\mathbf{x}^{-1}$ on an open set in $\mathbf{x}(D)$.)

$x^{-1} = (x^{-1}y)y^{-1}$ is continuous on open set $y^{-1}x(D) \cap E$

$\therefore$ x is proper patch

15. If $\mathcal{U}$ is a subset of a surface M in $\mathbf{R}^3$, prove that $\mathcal{U}$ is itself a surface in $\mathbf{R}^3$ if and only if $\mathcal{U}$ is an open set of M.

$\Leftarrow$

$\forall p \in \mathcal{U} \subset$ M and M is surface , so $\exists$ x: D $\rightarrow$ M(proper patch) s.t image contains n.b.d of p in M

If D is limited to $x^{-1}(M \cap \mathcal{U})$, it becomes patch of $\mathcal{U}$

Thus $\mathcal{U}$ is surface

$\Rightarrow$

Let p be point of $\mathcal{U}$

$\mathcal{U}$ is image of surface and $\exists$x: D $\rightarrow$ $\mathcal{U}$(proper patch) s.t contain O_1(n.b.d of p in $\mathcal{U}$)

Also p $\in$ M,

M is image of surface and $\exists$y: E $\rightarrow$ M(proper patch) s.t contain O_2(n.b.d of p in M)

$x^{-1}(O_1) \subset y^{-1}(O_2)$ and $y^{-1}(O_2)$,$x^{-1}(O_1)$: open set of R^2

$O_1 = x\left(x^{-1}(O_1)\right), O_2 = y(y^{-1}(O_2))$:open set

Thus $\mathcal{U}$ is open set of M

4.4연습문제

2020년 10월 29일 목요일 오전 1:12

1. Prove the Leibnizian formulas

$$d(fg) = g\,df + f\,dg, \quad d(f\phi) = df \wedge \phi + f\,d\phi,$$

where f and g are functions on M and ϕ is a 1-form.
(*Hint:* By definition, $(f\phi)(\mathbf{v}_p) = f(\mathbf{p})\phi(\mathbf{v}_p)$; hence $f\phi$ evaluated on $\mathbf{x}_u$ is $f(\mathbf{x})\phi(\mathbf{x}_u)$.)

(1) $d(fg)(x_u)=x_u\,[fg]$
$=\sum \frac{\partial x_i}{\partial u} D_i(fg)(x)$
$=\sum \frac{\partial x_i}{\partial u}(gD_i f + fD_i g)(x)$
$=g(x)\sum \frac{\partial x_i}{\partial u}(D_i f)(x) + f(x)\sum \frac{\partial x_i}{\partial u}(D_i g)(x)$
$=g(x)df(x_u) + f(x)dg(x_u)$
$=(gdf + fdg)(x_u)$
(2) $d(fg)(x_v) = (gdf + fdg)(x_v)$
$\therefore d(fg) = gdf + fdg$

$d(f\phi)(x_u,x_v)=\frac{\partial}{\partial u}(f\phi)(x_v) - \frac{\partial}{\partial v}(f\phi)(x_u)$
$=\frac{\partial}{\partial u}\left(f(x)\phi(x_v)\right) - \frac{\partial}{\partial v}\left(f(x)\phi(x_u)\right)$
$=\frac{\partial f(x)}{\partial u}\phi(x_v) + f(x)\frac{\partial}{\partial u}\phi(x_v) - \frac{\partial f(x)}{\partial v}\phi(x_u) - f(x)\frac{\partial}{\partial v}\phi(x_u)$
$=\frac{\partial f(x)}{\partial u}\phi(x_v) - \frac{\partial f(x)}{\partial v}\phi(x_u) + f(x)\left(\frac{\partial}{\partial u}\phi(x_v) - \frac{\partial}{\partial v}\phi(x_u)\right)$
$=(df \wedge \phi + fd\phi)(x_u,x_v)$
$\therefore d(f\phi) = (df \wedge \phi + fd\phi)$

2. (a) Prove formulas (1) and (2) in Example 4.7 using the remark preceding Example 4.7. (*Hint:* Show $(du_1du_2)(U_1, U_2) = 1$.)
(b) Derive the remaining formulas using the properties of d and the wedge product.

If f is a function, ϕ a 1-form, and η a 2-form, then

(1) $\phi = f_1du_1 + f_2du_2$, where $f_i = \phi(U_i)$.
(2) $\eta = g\,du_1du_2$, where $g = \eta(U_1, U_2)$.
(3) for $\psi = g_1du_1 + g_2du_2$ and ϕ as above,
$$\phi \wedge \psi = (f_1g_2 - f_2g_1)du_1\,du_2.$$
(4) $df = \dfrac{\partial f}{\partial u_1}du_1 + \dfrac{\partial f}{\partial u_2}du_2.$
(5) $d\phi = \left(\dfrac{\partial f_2}{\partial u_1} - \dfrac{\partial f_1}{\partial u_2}\right)du_1\,du_2$ (ϕ as above).

(a)

(1)

$\phi(v_p) = \phi\left(\sum v_i U_i(p)\right)$
$= \sum v_i \phi\left(U_i(p)\right)$
$= \sum\left(\phi(U_i)\right)(p)(du_i)(v_p)$
$= \sum(f_i du_i)(v_p)$
$\therefore\ \phi = f_1du_1 + f_2du_2$

(2)

v_p and w_p are linearly independent
$\eta(v_p,w_p)=\eta\left(v_1U_1(p) + v_2U_2(p), w_1U_1(p) + w_2U_2(p)\right)$
$=\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\eta\left(U_1(p),U_2(p)\right)$
$=\eta\left(U_1(p),U_2(p)\right)(v_1w_2 - w_1v_2)$
$=\left(\eta(U_1,U_2)\right)(p)\left(du_1(v_p)du_2(w_p) - du_1(w_p)du_2(v_p)\right)$
$=g(p)(du_1 \wedge du_2)(v_p,w_p)$
$=(gdu_1du_2)(v_p,w_p)$
$\therefore\ \eta = gdu_1du_2$

(2)
From $(\phi(V))(p) = \phi\left(V(p)\right)$
$((du_1du_2)(U_1,U_2))(p) = (du_1du_2)\left(U_1(p),U_2(p)\right)$
$= du_1\left(U_1(p)\right)du_2\left(U_2(p)\right)$
$-\ du_1\left(U_2(p)\right)du_2\left(U_1(p)\right)$ by (4.4.3)
$= 1 \cdot 1 - 0 \cdot 0 = 1$ for all p
Thus,
$g\,du_1du_2(p)$
$= \eta(((U_1,U_2)du_1du_2)(p))$
$= \eta(p)$

$gdu_1du_2(U_1,U_2) = g = \eta(U_1,U_2)$

$\Rightarrow gdu_1du_2 = \eta$

(b)

(3)

$\phi \wedge \psi = (f_1du_1 + f_2du_2) \wedge (g_1du_1 + g_2du_2)$
$= f_1g_2du_1du_2 + f_2g_1du_2du_1$
$= f_1g_2du_1du_2 - f_2g_1du_1du_2$

(4)

$df(v_p) = \sum v_i\frac{\partial f}{\partial u_i}(p) = \sum\left(\frac{\partial f}{\partial u_i}du_i\right)(v_p)$
$\therefore\ df = \frac{\partial f}{\partial u_1}du_1 + \frac{\partial f}{\partial u_2}du_2$

(5)

$d\phi = d\left(\sum f_i du_i\right) = \sum df_i \wedge du_i = \frac{\partial f_2}{\partial u_1}du_1du_2 + \frac{\partial f_1}{\partial u_2}du_2du_1 = \frac{\partial f_2}{\partial u_1}du_1du_2 - \frac{\partial f_1}{\partial u_2}du_1du_2$

3. If f is a real-valued function on a surface, and g is a function on the real line, show that

$$\mathbf{v}_p[g(f)] = g'(f)\mathbf{v}_p[f].$$

Deduce that

$$d(g(f)) = g'(f)df.$$

Lemma 4.6
$\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t) = (f \circ \alpha)'(t)$

Let α be a curve with initial velocity v at p

$v_p[g(f)]=(gf\alpha)'(0)$
$=g'(f(\alpha))(0)(f\alpha)'(0)$
$=g'\big(f(p)\big)v_p[f] \ \because lemma1.4.6$

$d(g(f))(v_p)=v_p\ [g(f)]$
$=g'\big(f(p)\big)v_p[f]$
$=g'\big(f(p)\big)df(v_p)$
$=(g'(f)df)(v_p)$

$\therefore\ d\big(g(f)\big)=(g'(f)df)$

4. If f, g, and h are functions on a surface M, and ϕ is a 1-form, prove:
(a) $d(fgh) = ghdf + fhdg + fgdh$,
(b) $d(\phi f) = fd\phi - \phi \wedge df$. $(\phi f = f\phi)$.
(c) $(df \wedge dg)(\mathbf{v}, \mathbf{w}) = \mathbf{v}[f]\mathbf{w}[g] - \mathbf{v}[g]\mathbf{w}[f]$.

(a)

$d(fgh)(x_u)=x_u\,[fgh]$
$=\sum\frac{\partial x_i}{\partial u}D_i(fgh)(x)$
$=\sum\frac{\partial x_i}{\partial u}(ghD_if+fhD_ig+fgD_ih)(x)$
$=gh(x)\sum\frac{\partial x_i}{\partial u}(D_if)(x)+fh(x)\sum\frac{\partial x_i}{\partial u}(D_ig)(x)+fg(x)\sum\frac{\partial x_i}{\partial u}(D_ih)(x)$
$=gh(x)df(x_u)+fh(x)dg(x_u)+fg(x)dh(x_u)$
$=(ghdf+fhdg+fgdh)(x_u)$

$d(fgh)(x_v)=\ (ghdf\ +\ fhdg+fgdh)(x_v)$

$\therefore\ d(fgh)=(ghdf\ +\ fhdg+fgdh)$

(b)

Suppose $\phi=gdx$

$d(f\phi)=d(fg\,dx)$
$=d(fg)\wedge dx$
$=(g\,df\ +\ f\,dg)\wedge dx$
$=df\ \wedge gdx\ +\ f(dg\ \wedge dx)$
$=df\ \wedge\phi+\ f\ d\phi$
$=fd\phi\ -\phi\wedge\ df$

(c)

$(df\wedge dg)(v,w)=(df)(v)(dg)(w)-df(w)dg(v)$
$=\mathrm{v}[\mathrm{f}]\mathrm{w}[\mathrm{g}]-\mathrm{w}[\mathrm{f}]\mathrm{v}[\mathrm{g}]$

5. Suppose that M is covered by open sets $\mathcal{U}_1, \ldots, \mathcal{U}_k$, and on each $\mathcal{U}_i$ there is defined a function f_i such that $f_i - f_j$ is constant on the overlap of $\mathcal{U}_i$ and $\mathcal{U}_j$. Show that there is a 1-form ϕ on M such that $\phi = df_i$ on each $\mathcal{U}_i$. Generalize to the case of 1-forms ϕ_i such that $\phi_i - \phi_j$ is closed.

Let $\phi=df_i$ on $\mathcal{U}_\mathrm{i}$
On the overlap of $\mathcal{U}_\mathrm{i}$ and $\mathcal{U}_\mathrm{j}$, $df_i-df_j=d(f_i\ -\ f_j)=0$
$\Rightarrow\ \phi$: well defined
Thus ϕ is 1-form defined on M

7. If $\mathbf{x}: D \to M$ is a patch in M, let $\tilde{u}$ and $\tilde{v}$ be the coordinate functions of $\mathbf{x}^{-1}$, so $\mathbf{x}^{-1}(\mathbf{p}) = (\tilde{u}(\mathbf{p}), \tilde{v}(\mathbf{p}))$ for all $\mathbf{p}$ in $\mathbf{x}(D)$. Show that
(a) $\tilde{u}$ and $\tilde{v}$ are differentiable functions on $\mathbf{x}(D)$ such that:

$$\tilde{u}(\mathbf{x}(u, v)) = u, \quad \tilde{v}(\mathbf{x}(u, v)) = v.$$

These functions constitute the *coordinate system* associated with $\mathbf{x}$.

(b) $d\tilde{u}(\mathbf{x}_u) = 1, \quad d\tilde{u}(\mathbf{x}_v) = 0,$
$d\tilde{v}(\mathbf{x}_u) = 0, \quad d\tilde{v}(\mathbf{x}_v) = 1.$

(c) If ϕ is a 1-form and η is a 2-form, then

$$\phi = f\ d\tilde{u} + g\ d\tilde{v}, \quad \text{where } f(\mathbf{x}) = \phi(\mathbf{x}_u), g(\mathbf{x}) = \phi(\mathbf{x}_v);$$
$$\eta = h\ d\tilde{u}\ d\tilde{v}, \quad \text{where } h(\mathbf{x}) = \eta(\mathbf{x}_u, \mathbf{x}_v).$$

(*Hint:* for (b) use Ex. 4(b) of Sec. 3.)

(a)

$(u,v)=x^{-1}\circ x(u,v)\ =x^{-1}\big(\tilde{u}(\mathrm{x}(\mathrm{u},\mathrm{v})),\tilde{v}(\mathrm{x}(\mathrm{u},\mathrm{v}))\big)$
$\therefore u=\tilde{u}\big(\mathrm{x}(\mathrm{u},\mathrm{v})\big),\mathrm{v}=\tilde{v}\big(\mathrm{x}(\mathrm{u},\mathrm{v})\big)$

(b)

$d\tilde{u}(x_u)=\mathrm{x_u}[\tilde{u}]=\frac{\partial(\tilde{u}(x))}{\partial u}=\frac{\partial u}{\partial u}=1$

$d\tilde{u}(x_v)=\mathrm{x_v}[\tilde{u}]=\frac{\partial(\tilde{u}(x))}{\partial v}=\frac{\partial u}{\partial v}=0$

$d\tilde{v}(x_u)=\mathrm{x_u}[\tilde{v}]=\frac{\partial(\tilde{v}(x))}{\partial u}=\frac{\partial v}{\partial u}=0$

$d\tilde{v}(x_v)=\mathrm{x_v}[\tilde{v}]=\frac{\partial(\tilde{v}(x))}{\partial v}=\frac{\partial v}{\partial v}=1$

(c)

Let $\phi=fd\tilde{u}+gd\tilde{v}$

$\phi(x_u)=(fd\tilde{u}+gd\tilde{v})(x_u)$

$=f(x)d\tilde{u}(x_u)+g(x)d\tilde{v}(x_u)$

$=f(x)$

$\phi(x_v)=(fd\tilde{u}+gd\tilde{v})(x_v)$

$=f(x)d\tilde{u}(x_v)+g(x)d\tilde{v}(x_v)$

$=g(x)$

$\therefore f(x)=\phi(x_u),\ g(x)=\phi(x_v)$

Let $\eta=hd\tilde{u}d\tilde{v}$

$\eta(x_u,x_v)=(hd\tilde{u}d\tilde{v})(x_u,x_v)$

$=h(x)\{d\tilde{u}(x_u)d\tilde{v}(x_v)-d\tilde{u}(x_v)d\tilde{v}(x_u)\}$

$=h(x)$

$\therefore h(x)=\eta(x_u,x_v)$

Ex 4.4.6

2020년 11월 12일 목요일 오후 12:23

Exercise 4.4.6 $y : E \subset R^2 \to M \subset R^3$ is a mapping

$\underset{4.3.2}{\Longrightarrow}$ the coordinate expression $x^{-1}(y) : y^{-1}(x(D)) \to D$ is differentiable for each overlaping patch x in M

Note) $y^{-1}(x(D))$ is open in R^2

Let $x^{-1}(y) = (\bar{u}, \bar{v})$, i.e., $y = x(\bar{u}, \bar{v})$

$\Rightarrow y_u = \frac{\partial \bar{u}}{\partial u} x_u + \frac{\partial \bar{v}}{\partial u} x_v$ and $y_v = \frac{\partial \bar{u}}{\partial v} x_u + \frac{\partial \bar{v}}{\partial v} x_v$

4.2

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$$\underset{4.4.2}{\Longrightarrow} d\phi(y_u, y_v) = J \cdot d\phi(x_u, x_v)$$
$$= J \cdot [\, \frac{\partial}{\partial \bar{u}}\big(\phi(x_v)\big) - \frac{\partial}{\partial \bar{v}}\big(\phi(x_u)\big) \,]$$
$$= \frac{\partial}{\partial u}\big(\phi(y_v)\big) - \frac{\partial}{\partial v}\big(\phi(y_u)\big) \quad (*)$$

$$(*)\ \frac{\partial}{\partial u}\big(\phi(y_v)\big) = \frac{\partial}{\partial u}[\, \frac{\partial \bar{u}}{\partial v}\big(\phi(x_u)\big) + \frac{\partial \bar{v}}{\partial v}\big(\phi(x_v)\big) \,]$$

$$= \frac{\partial}{\partial u}\big(\phi(x_u)\big)\frac{\partial \bar{u}}{\partial v} + \frac{\partial}{\partial u}\frac{\partial \bar{u}}{\partial v}\big(\phi(x_u)\big) + \frac{\partial}{\partial u}\big(\phi(x_v)\big)\frac{\partial \bar{v}}{\partial v} + \frac{\partial}{\partial u}\frac{\partial \bar{v}}{\partial v}\big(\phi(x_v)\big)$$

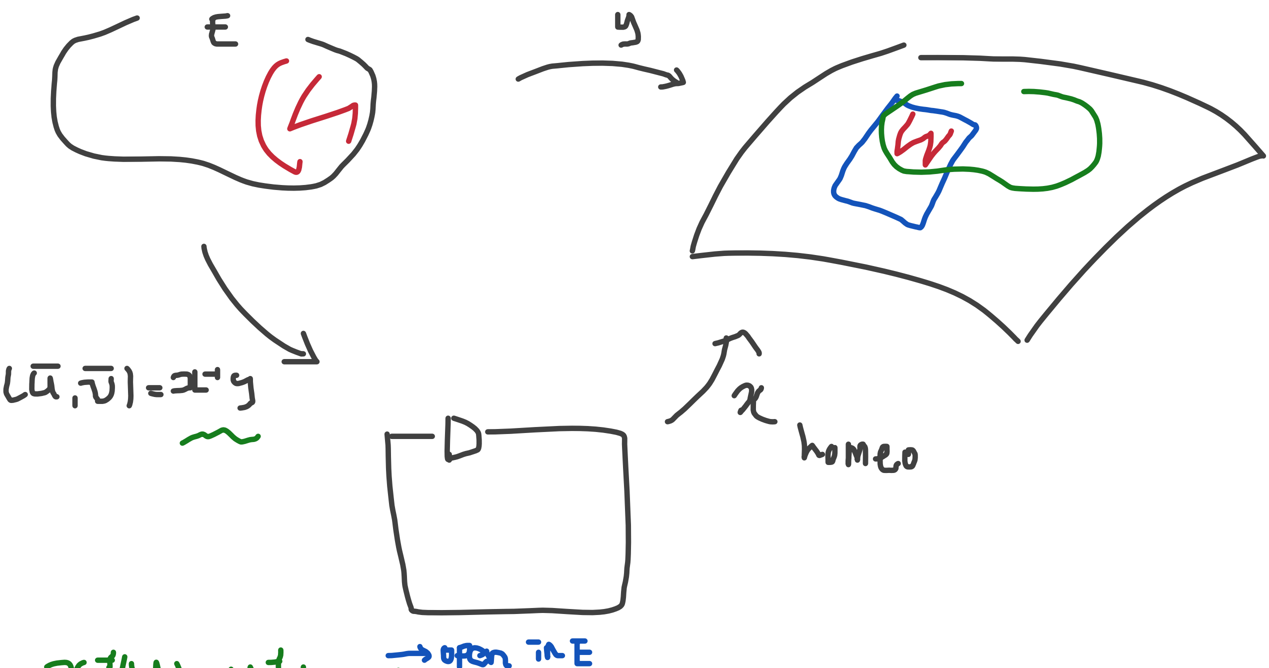
$$= [\, \frac{\partial}{\partial \bar{u}}\big(\phi(x_u)\big)\frac{\partial \bar{u}}{\partial u} + \frac{\partial}{\partial \bar{v}}\big(\phi(x_u)\big)\frac{\partial \bar{v}}{\partial u} \,]\frac{\partial \bar{u}}{\partial v} + \frac{\partial}{\partial u}\frac{\partial \bar{u}}{\partial v}\big(\phi(x_u)\big) + [\, \frac{\partial}{\partial \bar{u}}\big(\phi(x_v)\big)\frac{\partial \bar{u}}{\partial u} + \frac{\partial}{\partial \bar{v}}\big(\phi(x_v)\big)\frac{\partial \bar{v}}{\partial u} \,]\frac{\partial \bar{v}}{\partial v} + \frac{\partial}{\partial u}\frac{\partial \bar{v}}{\partial v}\big(\phi(x_v)\big)$$

$$\frac{\partial}{\partial v}\big(\phi(y_u)\big) = \frac{\partial}{\partial v}[\, \frac{\partial \bar{u}}{\partial u}\big(\phi(x_u)\big) + \frac{\partial \bar{v}}{\partial u}\big(\phi(x_v)\big) \,]$$

$$= \frac{\partial}{\partial v}\big(\phi(x_u)\big)\frac{\partial \bar{u}}{\partial u} + \frac{\partial}{\partial v}\frac{\partial \bar{u}}{\partial u}\big(\phi(x_u)\big) + \frac{\partial}{\partial v}\big(\phi(x_v)\big)\frac{\partial \bar{v}}{\partial u} + \frac{\partial}{\partial v}\frac{\partial \bar{v}}{\partial u}\big(\phi(x_v)\big)$$

$$= [\, \frac{\partial}{\partial \bar{u}}\big(\phi(x_u)\big)\frac{\partial \bar{u}}{\partial v} + \frac{\partial}{\partial \bar{v}}\big(\phi(x_u)\big)\frac{\partial \bar{v}}{\partial v} \,]\frac{\partial \bar{u}}{\partial u} + \frac{\partial}{\partial v}\frac{\partial \bar{u}}{\partial u}\big(\phi(x_u)\big) + [\, \frac{\partial}{\partial \bar{u}}\big(\phi(x_v)\big)\frac{\partial \bar{u}}{\partial v} + \frac{\partial}{\partial \bar{v}}\big(\phi(x_v)\big)\frac{\partial \bar{v}}{\partial v} \,]\frac{\partial \bar{v}}{\partial u} + \frac{\partial}{\partial v}\frac{\partial \bar{v}}{\partial u}\big(\phi(x_v)\big)$$

$$\Rightarrow \frac{\partial}{\partial u}\big(\phi(y_v)\big) - \frac{\partial}{\partial v}\big(\phi(y_u)\big) = J \cdot [\, \frac{\partial}{\partial \bar{u}}\big(\phi(x_v)\big) - \frac{\partial}{\partial \bar{v}}\big(\phi(x_u)\big) \,],\ J = \begin{vmatrix} \frac{\partial \bar{u}}{\partial u} & \frac{\partial \bar{v}}{\partial u} \\ \frac{\partial \bar{u}}{\partial v} & \frac{\partial \bar{v}}{\partial v} \end{vmatrix}$$



$x^{-1}(y)$ $\underbrace{y^{-1}(x(D))}_{\text{open in } \mathbb{R}^2} \xrightarrow{} D$ → open in E

open in $\mathbb{R}^2$ ($\because$ E is open in $\mathbb{R}^2$)

4.5연습문제

2020년 11월 5일 목요일　　오후 10:00

4. Use the preceding exercise to construct a mapping of the helicoid H (Ex. 2.5) onto the torus T (Ex. 2.5) such that the rulings of H are carried to the meridians of T.

Let $x(u,v) = (ucosv, usinv, bv), y(u,v) = \big((R + rcosu)cosv, (R + rcosu)sinv, rsinu\big), b \neq 0$

$F: H \to T$ by $F\big(x(u,v)\big) = y(u,v)$

5. If Σ is the sphere $\|\mathbf{p}\| = r$, the mapping $A: \Sigma \to \Sigma$ such that $A(\mathbf{p}) = -\mathbf{p}$ is called the *antipodal map* of Σ. Prove that A is a diffeomorphism and that $A_*(\mathbf{v}_p) = (-\mathbf{v})_{-p}$.

$\sum$ is surface in R^3

By Exercise 1

A is differentiable.

since $AA = I \Rightarrow A^{-1} = A$, so A is diffeomorphism. ($\because A^{-1}$: differetiable)

Let $\alpha(0) = p, \alpha'(0) = v$

$A_*(v_p) = A_*(\alpha'(0)_p) = (A\alpha)'(0)_{A(p)} = A'(\alpha(0))\alpha'(0)_{-p} = -\alpha'(0)_{-p} = -v_{-p}$

6. A regular mapping $F: M \to N$ of surfaces is often called a *local diffeomorphism*. For such a mapping F, prove that, in fact, every point $\mathbf{p}$ of M has a neighborhood $\mathscr{U}$ such that $F|\mathscr{U}$ is a diffeomorphism of $\mathscr{U}$ onto a neighborhood of $F(\mathbf{p})$ in N.

By inverse function theorem & thm 5.4

$\forall\, p \in M, {F_*}_p: T_p(M) \to T_{F(p)}(N)$ is isomorphism

And $\exists\, \mathcal{U}$ a nbd of p & $F(\mathcal{U})$ a nbd of $F(p)$, $F|_{\mathcal{U}}: \mathcal{U} \to F(\mathcal{U})$ is diffeomorphism

7. If $\mathbf{x}: D \to M$ is a parametrization, prove that the restriction of $\mathbf{x}$ to a sufficiently small neighborhood of a point (u_0, v_0) in D is a patch in M. (Thus any parametrization can be cut into patches.)

$x: D \to M$: regular

By inverse function theorem

x is a patch in M of a neighborhood of $\forall (u_0, v_0) \in D$

8. Let $F: M \to N$ be a mapping. If $\mathbf{x}$ is a patch in M, then as in the text, let $\mathbf{y} = F(\mathbf{x})$. (Although $\mathbf{y}$ maps into N, it is not necessarily a patch.) For a curve

$$\alpha(t) = \mathbf{x}(a_1(t), a_2(t))$$

in M, show that the image curve $\bar{\alpha} = F(\alpha)$ in N, has velocity

$$\bar{\alpha}' = \frac{da_1}{dt}\mathbf{y}_u(a_1, a_2) + \frac{da_2}{dt}\mathbf{y}_v(a_1, a_2).$$

$\bar{\alpha} = F(\alpha) = F\big(x(a_1, a_2)\big) = y(a_1, a_2)$

By chain rule

$\overline{\alpha'} = \frac{da_1}{dt} y_u(a_1, a_2) + \frac{da_2}{dt} y_v(a_1, a_2)$

9. Prove: (a) The invariance property needed to justify the definition (5.3) of the tangent map.
(b) Tangent maps $F_*: T_p(M) \to T_{F(p)}(N)$ are linear transformations.

(a)

Let $\alpha_1(0) = \alpha_2(0),\ \alpha_1'(0) = \alpha_2'(0) = v$

$\alpha_1(t) = x\big(a_1(t), a_2(t)\big), \alpha_2(t) = x(b_1(t).b_2(t))$

By Exercise 8

$\overline{\alpha_1'} = \frac{da_1}{dt} y_u(a_1, a_2) + \frac{da_2}{dt} y_v(a_1, a_2), \overline{\alpha'}_2 = \frac{db_1}{dt} y_u(b_1, b_2) + \frac{db_2}{dt} y_v(b_1, b_2)$

$\Rightarrow \overline{\alpha_1'}(0) =, \overline{\alpha'}_2(0)$

Thus $F_*(v)$ is well defined no matter how you choose the curve with the initial velocity v.

(b)

Let

$\alpha_1 = x(a_1, a_2), c_1 = \frac{da_1}{dt}, d_1 = \frac{da_2}{dt}$

$\alpha_2 = x(b_1, b_2), c_2 = \frac{db_1}{dt}, d_2 = \frac{db_2}{dt}$

If $v_1 = c_1 x_u + d_1 x_v$, $v_2 = c_2 x_u + d_2 x_v$

Then

$\forall s, k \in R$, $F_*(sv_1 + kv_2) = F_*\big((sc_1 + kc_2)x_u + (sd_1 + kd_2)x_v\big)$

$= (sc_1 + kc_2)y_u + (sd_1 + kd_2)y_v$

$= s(c_1 y_u + d_1 y_v) + k(c_2 y_u + d_2 y_v)$

$= sF_*(v_1) + kF_*(v_2)$

Thus F_* is linear transformation

5. A line L is attached orthogonally to an axis A (Fig. 4.19). If L moves steadily along A, rotating at constant speed, then L sweeps out a *helicoid* H. When A is the z axis, H is the image of the mapping $\mathbf{x}: \mathbf{R}^2 \to \mathbf{R}^3$ such that

$$\mathbf{x}(u, v) = (u \cos v, u \sin v, bv) \quad (b \neq 0).$$

2.5 Example *Torus of revolution T.* This is the surface of revolution obtained when the profile curve C is a circle. Suppose that C is the circle in the xz plane with radius $r > 0$ and center $(R, 0, 0)$. We shall rotate about the z axis; hence we must require $R > r$ to keep C from meeting the axis of revolution. A natural parametrization (Fig. 4.15) for C is

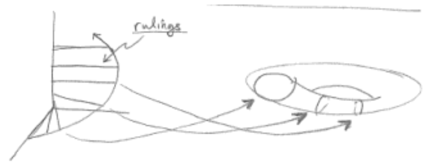
$$\alpha(u) = (R + r\cos u, r\sin u).$$

Thus by the remarks above we must have $g(u) = r\sin u$ (distance *along* the z axis) and $h(u) = R + r\cos u$ (distance *from* the z axis). The general argument in Example 2.4—with coordinate axes permuted—then yields the parametrization

$$\begin{aligned}\mathbf{x}(u, v) &= (h(u)\cos v, h(u)\sin v, g(u)) \\ &= ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u).\end{aligned}$$

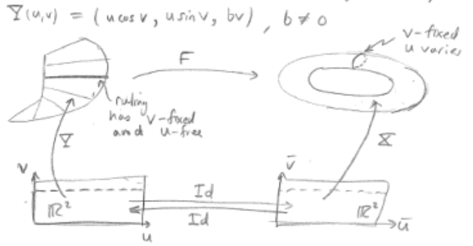
We call $\mathbf{x}$ the *usual parametrization* of the torus (Fig. 4.16). Its domain is the whole plane $\mathbf{R}^2$, and it is periodic in both u and v:

$$\mathbf{x}(u + 2\pi, v) = \mathbf{x}(u, v + 2\pi) = \mathbf{x}(u, v) \quad \text{for all } (u, v).$$



$T: \Sigma(\bar{u}, \bar{v}) = \big((R + r\cos\bar{u})\cos\bar{v}, (R + r\cos\bar{u})\sin\bar{v}, r\sin(\bar{u})\big), r, R \neq 0$

$H: \Upsilon(u, v) = (u\cos v, u\sin v, bv), b \neq 0$



$F(x,y,z) = \Sigma \circ \Upsilon^{-1}(x, y, z)$

$= \Sigma\left(\sqrt{x^2 + y^2}, \frac{1}{b}z\right)$, $\bar{u} = \sqrt{x^2+y^2}$, $\bar{v} = z/b$

$= \boxed{\left((R + r\cos\sqrt{x^2+y^2})\cos(z/b), (R + r\cos\sqrt{x^2+y^2})\sin(z/b), r\sin\sqrt{x^2+y^2}\right)}$

for $(x, y, z) \in H$

We can check my inverse for Υ　those are $x \neq y$ why?

$\Upsilon(\Upsilon^{-1}(x,y,z)) = \Upsilon(\sqrt{x^2+y^2}, z/b) = \left(\sqrt{x^2+y^2}\cos(\frac{z}{b}), \sqrt{x^2+y^2}\sin(\frac{z}{b}), z\right)$

4.6연습문제

2020년 11월 19일 목요일 오후 12:34

4. The 1-form

$$\psi = \frac{u\,dv - v\,du}{u^2+v^2}$$

is well-defined on the plane $\mathbf{R}^2$ with the origin $\mathbf{0}$ removed. Show:
(a) ψ is closed but not exact on $\mathbf{R}^2 - \mathbf{0}$. (*Hint:* Integrate around the unit circle and use Ex. 3.)
(b) The restriction of ψ to, say, the right half-plane $u > 0$ is exact.

$(u^2+v^2)-u\cdot 2u$

$(u^2+v^2)-v\cdot 2v$

(a)

$$\psi = \frac{u}{u^2+v^2}dv - \frac{v}{u^2+v^2}du$$

$$\Rightarrow d\psi = \frac{\partial}{\partial u}\left(\frac{u}{u^2+v^2}\right)dudv - \frac{\partial}{\partial v}\left(\frac{v}{u^2+v^2}\right)dvdu$$

$$= \left(\frac{-u^2+v^2}{(u^2+v^2)^2} + \frac{u^2-v^2}{(u^2+v^2)^2}\right)dudv$$

$$=0$$

Let $\alpha(t) = (\cos t, \sin t), 0 \le t \le 2\pi$

$\Rightarrow \alpha(0) = \alpha(2\pi)$

$\Rightarrow$ if ψ is exact, then $\int_\alpha \psi = 0$

But $\int_\alpha \psi = \int_0^{2\pi} dt = 2\pi$

$\because \phi(\alpha'(t))dt = \cos t^2 dt + \sin t^2 dt = dt$

$\therefore$ ψ is not exact

(b)

$$f = \tan^{-1}\frac{v}{u} \Rightarrow df = \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv = \psi$$

$$\because \frac{\partial f}{\partial u}du = -\frac{\frac{v}{u^2}}{1+\frac{v^2}{u^2}}du = -\frac{v}{u^2+v^2}du,\ \frac{\partial f}{\partial v}dv = \frac{\frac{1}{u}}{1+\frac{v^2}{u^2}}dv = \frac{u}{u^2+v^2}dv2$$

$\therefore$ ψ is exact

5. (a) Show that every curve α in $\mathbf{R}^2$ that does not pass through the origin has an (orientation-preserving) reparametrization in the polar form

$$\alpha(t) = (r(t)\cos\vartheta(t), r(t)\sin\vartheta(t)).$$

(*Hint:* Use Ex. 12 of Sec. 2.1.)
If the curve α: $[a, b] \to \mathbf{R}^2 - \mathbf{0}$ is closed, prove:
(b) $\mathrm{wind}(\alpha) = \dfrac{\vartheta(b) - \vartheta(a)}{2\pi}$ is an integer.

This integer, called the *winding number of α about* $\mathbf{0}$, represents the total algebraic number of times α has gone around the origin in the counterclockwise direction. (Note that $\mathrm{wind}(\alpha) = \mathrm{wind}(\alpha/\|\alpha\|)$.)
(c) If ψ is the 1-form in Exercise 4, then $\mathrm{wind}(\alpha) = \dfrac{1}{2\pi}\displaystyle\int_\alpha \psi$.
(d) If $\alpha = (f, g)$, then

$$\mathrm{wind}(\alpha) = \frac{1}{2\pi}\int_a^b \frac{fg' - gf'}{f^2+g^2}dt = \frac{1}{2\pi}\int_a^b \frac{\det(\alpha(t), \alpha'(t))}{\alpha(t)\bullet\alpha(t)}dt.$$

(The determinant is of the 2×2 matrix whose rows are $\alpha(t)$ and $\alpha'(t)$.)

12. *Angle functions.* Let f and g be differentiable real-valued functions on an interval I. Suppose that $f^2 + g^2 = 1$ and that ϑ_0 is a number such that $f(0) = \cos\vartheta_0$, $g(0) = \sin\vartheta_0$. If ϑ is the function such that

$$\vartheta(t) = \vartheta_0 + \int_0^t (fg' - gf')du,$$

prove that

$$f = \cos\vartheta, \quad g = \sin\vartheta.$$

Hint: We want $(f - \cos\vartheta)^2 + (g - \sin\vartheta)^2 = 0$, so show that its derivative is zero.

The point of this exercise is that ϑ is a differentiable function, unambiguously defined on the whole interval I.

(a)

Let $r(t) = \|\alpha(t)\|$, $f = U_1 \cdot \frac{\alpha}{\|\alpha\|}$, $g = U_2 \cdot \frac{\alpha}{\|\alpha\|}$

$\Rightarrow f^2 + g^2 = 1$

by exercise 2.1.12

$\exists\,\theta$ such that $f = \cos\theta$ & $g = \sin\theta$

thus

$$\alpha(t) = (\|\alpha(t)\|f(t), \|\alpha(t)\|g(t))$$
$$=(\|\alpha(t)\|\cos\theta(t), \|\alpha(t)\|\sin\theta(t))$$
$$=(r(t)\cos\theta(t), r(t)\sin\theta(t))$$

(b)

$\theta(a)$ and $\theta(b)$ measure the same the angle

$\Rightarrow$ $\theta(a)$ and $\theta(b)$ differ by some integer multiple of 2π

Thus $wind(\alpha)$ is integer

(c)

$$wind(\alpha) = wind\left(\frac{\alpha}{\|\alpha\|}\right) \Rightarrow r(t) = 1$$

$$\alpha(t) = (\cos\theta(t), \sin\theta(t)), \psi = \frac{u}{u^2+v^2}dv - \frac{v}{u^2+v^2}du$$

$$\Rightarrow \frac{1}{2\pi}\int_\alpha \psi = \frac{1}{2\pi}\int_a^b \psi(\alpha'(t))dt$$
$$=\frac{1}{2\pi}\int_a^b \psi(-\sin\theta(t)\cdot\theta'(t), \cos\theta(t)\cdot\theta'(t))dt$$
$$=\frac{1}{2\pi}\int_a^b \theta'(t)dt$$
$$=\frac{1}{2\pi}(\theta(b) - \theta(a))$$
$$= wind(\alpha)$$

(d)

by exercise 2.1.12

$$\theta(t) = \theta_0 + \int_0^t (fg' - gf')dt$$

$$\Rightarrow wind(\alpha) = \frac{1}{2\pi}(\theta(b) - \theta(a)) = \frac{1}{2\pi}\int_a^b \frac{fg'-gf'}{f^2+g^2}dt$$

$$\frac{\det(\alpha, \alpha')}{\alpha\cdot\alpha} = \frac{\begin{vmatrix} f & g \\ f' & g' \end{vmatrix}}{f^2+g^2} = \frac{fg' - gf'}{f^2+g^2}$$

Thus

$$wind(\alpha) = \frac{1}{2\pi}\int_a^b \frac{fg'-gf'}{f^2+g^2}dt = \frac{1}{2\pi}\int_a^b \frac{\det(\alpha,\alpha')}{\alpha\cdot\alpha}dt$$

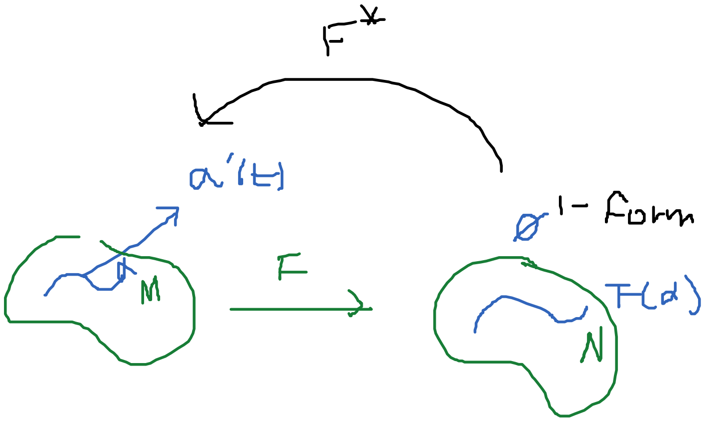
7. Let $F\colon M \to N$ be a mapping. Prove:
(a) If α is a curve segment in M, and ϕ is a 1-form on N, then $\int_\alpha F^*\phi = \int_{F(\alpha)} \phi$.
(b) If $\mathbf{x}$ is 2-segment in M, and η is a 2-form on N, then $\iint_{\mathbf{x}} F^*\eta = \iint_{F(\mathbf{x})} \eta$.

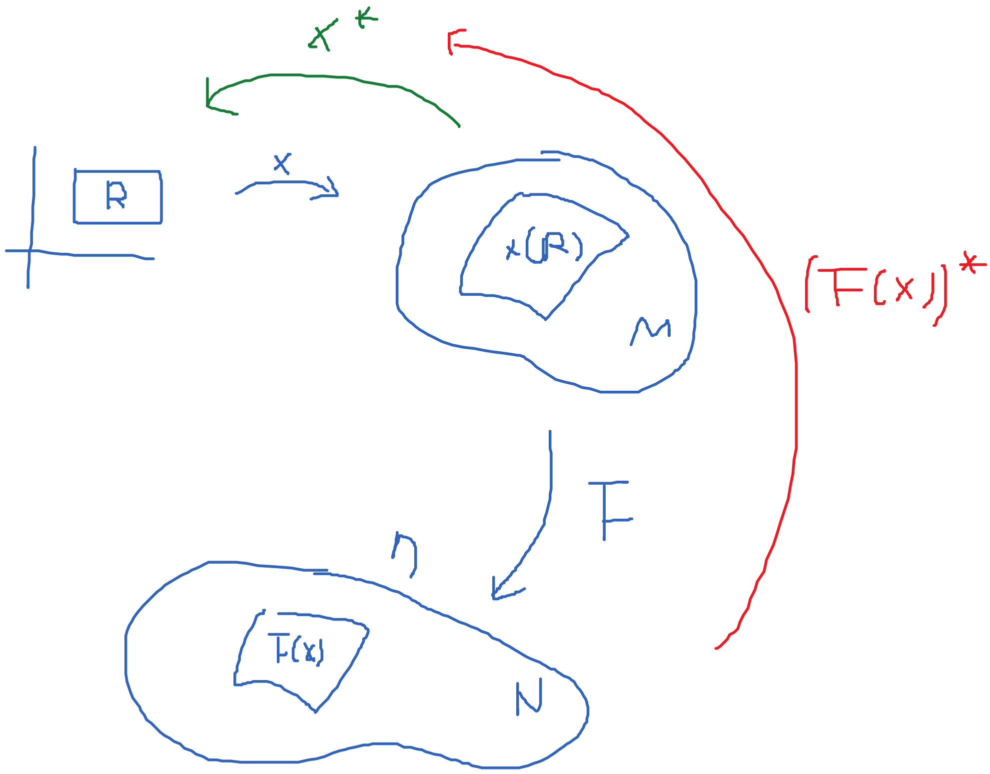
(a)

$$\int_\alpha F^*\phi = \int_a^b (F^*\phi)(\alpha')dt$$
$$=\int_a^b \phi(F_*\alpha')dt$$
$$=\int_a^b \phi((F\circ\alpha)')dt$$
$$=\int_{F(\alpha)} \phi$$

(b)

$$\iint_x F^*\eta = \iint_R x^*F^*\eta = \iint_R (F(x))^*\eta = \iint_{F(x)} \eta$$





8. Let $\mathbf{x}$ be a patch in a surface M. For a curve segment

$$\alpha(t) = \mathbf{x}(a_1(t), a_2(t)), \quad a \leqq t \leqq b,$$

in $\mathbf{x}(R)$, show that

$$\int_\alpha \phi = \int_a^b \left(\phi(\mathbf{x}_u)\frac{da_1}{dt} + \phi(\mathbf{x}_v)\frac{da_2}{dt} \right) dt,$$

where $\mathbf{x}_u$ and $\mathbf{x}_v$ are evaluated on (a_1, a_2). (This generalizes Ex. 1, which is recovered by using the identity patch $\mathbf{x}(u, v) = (u, v)$ in $\mathbf{R}^2$.)

$$\alpha'(t) = x_u \frac{da_1}{dt} + x_v \frac{da_2}{dt}$$
$$\Rightarrow \int_\alpha \phi = \int_a^b \phi(\alpha'(t))dt$$
$$=\int_a^b \phi\left(x_u \frac{da_1}{dt} + x_v \frac{da_2}{dt}\right)dt$$
$$=\int_a^b (\phi(x_u)\frac{da_1}{dt} + \phi(x_v)\frac{da_2}{dt})dt$$

9. Let $\mathbf{x}$ be the usual parametrization of the torus T (Ex. 2.5). For integers m and n, let α be the closed curve

$$\alpha(t) = \mathbf{x}(mt, nt) \quad (0 \leq t \leq 2\pi).$$

Find:

(a) $\int_\alpha \xi$, where ξ is the 1-form such that $\xi(\mathbf{x}_u) = 1$ and $\xi(\mathbf{x}_v) = 0$.

(b) $\int_\alpha \eta$, where η is the 1-form such that $\eta(\mathbf{x}_u) = 0$ and $\eta(\mathbf{x}_v) = 1$.

For an arbitrary closed curve γ in T, $\int_\gamma \xi/(2\pi)$ is an integer that counts the total (algebraic) number of times γ goes around the torus in the general direction of the parallels, and $\int_\gamma \eta/(2\pi)$ gives a similar count for the meridians. (This suggests the informal notation $\xi = d\vartheta$, $\eta = d\varphi$, but see Ex. 7 of Sec. 7.)

(a)

Usual parametrization of torus : $x(u,v) = \big((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u\big)$

$\alpha(t) = x(mt, nt), 0 \leq t \leq 2\pi$

$\Rightarrow \alpha' = m x_u + n x_v$

$$\int_\alpha \xi = \int_0^{2\pi} \xi(\alpha')dt$$

$= \int_0^{2\pi} \xi(m x_u + n x_v)dt$

$= \int_0^{2\pi} m\xi(x_u)dt + \int_0^{2\pi} n\xi(x_v)dt$

$= \int_0^{2\pi} m dt$

$= 2\pi m$

(b)

$$\int_\alpha \eta = \int_0^{2\pi} \eta(\alpha')dt$$

$= \int_0^{2\pi} \eta(m x_u + n x_v)dt$

$= \int_0^{2\pi} m\eta(x_u)dt + \int_0^{2\pi} n\eta(x_v)dt$

$= \int_0^{2\pi} n dt$

$= 2\pi n$

10. Let $\mathbf{x}: R \to M$ be a 2-segment defined on the unit square R: $0 \leq u$, $v \leq 1$. If ϕ is the 1-form on M such that

$$\phi(\mathbf{x}_u) = u + v \quad \text{and} \quad \phi(\mathbf{x}_v) = uv,$$

compute $\iint_{\mathbf{x}} d\phi$ and $\int_{\partial\mathbf{x}} \phi$ separately, and check the results by Stokes' theorem. (*Hint:* $\mathbf{x}^* d\phi = d(\mathbf{x}^*\phi)$.)

$R: 0 \leq u, v \leq 1$

$\phi(x_u) = u + v, \phi(x_v) = uv$

$d\phi(x_u, x_v) = \frac{\partial}{\partial u}\big(\phi(x_v)\big) - \frac{\partial}{\partial v}\big(\phi(x_u)\big)$

$= v - 1$

$$\iint_x d\phi = \iint_R x^* d\phi$$

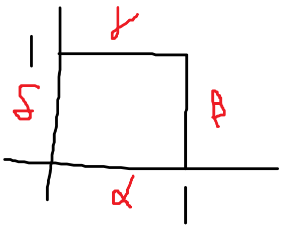
$= \int_0^1 \int_0^1 d\phi(x_u, x_v)dudv$

$= \int_0^1 \int_0^1 \left(\frac{\partial}{\partial u}(\phi x_v) - \frac{\partial}{\partial v}(\phi x_u)\right)dudv$

$= \int_0^1 (v-1)dv$

$= \left[\frac{1}{2}v^2 - v\right]_0^1$

$= -\frac{1}{2}$



α $\phi(x_u)(u,0) = u$

γ $\phi(x_u)(u,1) = u+1$

δ $\phi(x_v)(0,v) = 0$

β $\phi(x_v)(1,v) = v$

$\int_{\partial x} \phi = \int_\alpha \phi + \int_\beta \phi - \int_\gamma \phi - \int_\delta \phi$

$= \int_0^1 u\, du + \int_0^1 v\, dv - \int_0^1 (u+1)du - \int_0^1 0 dv$

$= \frac{1}{2} + \frac{1}{2} - \left(\frac{3}{2}\right)$

$= -\frac{1}{2}$

11. Same as Exercise 10, except that $\boldsymbol{R}$: $0 \leqq u \leqq \pi/2$, $0 \leqq v \leqq \pi$, and $\phi(\mathbf{x}_u) = u\cos v$, $\phi(\mathbf{x}_v) = v\sin u$.

The following exercise is a 2-dimensional analogue of Lemma 6.6. However, with future applications in mind, we generalize 2-segments $\mathbf{x}$: $R \to M$ by replacing the rectangle R by any compact region $\mathscr{R}$ in $\mathbf{R}^2$ whose boundary consists of smooth curve segments. (Compactness ensures that integrals over $\mathscr{R}$ will be finite.)

$R: 0 \le u \le \frac{\pi}{2}, 0 \le v \le \pi$

$\phi(x_u) = u\cos v, \phi(x_v) = v\sin u$

$d\phi(x_u, x_v) = \frac{\partial}{\partial u}\left(\phi(x_v)\right) - \frac{\partial}{\partial v}\left(\phi(x_u)\right)$

$= \frac{\partial}{\partial u}(v\sin u) - \frac{\partial}{\partial v}(u\cos v)$

thus

$\iint_x d\phi = \iint_R x^* d\phi$

$= \int_0^\pi \int_0^{\frac{\pi}{2}} d\phi(x_u, x_v)\,dudv$

$= \int_0^\pi \int_0^{\frac{\pi}{2}} (v\cos u + u\sin v)\,dudv$

$= \int_0^\pi \int_0^{\frac{\pi}{2}} v\cos u\ dudv + \int_0^\pi \int_0^{\frac{\pi}{2}} u\sin v\ dudv$

$= \int_0^\pi \int_0^{\frac{\pi}{2}} v\cos u\ dudv + \int_0^\pi \int_0^{\frac{\pi}{2}} u\sin v\ dudv$

$= \int_0^\pi v[\sin u]_0^{\frac{\pi}{2}} dv + \int_0^\pi \sin v \left[\frac{1}{2}u^2\right]_0^{\frac{\pi}{2}} dv$

$= \int_0^\pi v\ dv + \int_0^\pi \frac{\pi^2}{8} \sin v dv$

$= \frac{\pi^2}{2} - \frac{\pi^2}{8}[\cos v]_0^\pi$

$= \frac{\pi^2}{2} + \frac{\pi^2}{4}$

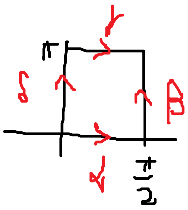
$= \frac{3\pi^2}{4}$

$\int_{\partial x} \phi = \int_\alpha \phi + \int_\beta \phi - \int_\gamma \phi - \int_\delta \phi$

$= \int_0^{\frac{\pi}{2}} u\ du + \int_0^\pi v\ dv - \int_0^{\frac{\pi}{2}} -(u)du - \int_0^\pi 0dv$

$= \frac{\pi^2}{8} + \frac{\pi^2}{2} + \frac{\pi^2}{8}$

$= \frac{3}{4}\pi^2$



$\alpha: \phi(x_u)(u, 0) = u$

$\gamma: \phi(x_u)(u, \pi) = -u$

$\delta: \phi(x_v)(0, v) = 0$

$\beta: \phi(x_v)(\frac{\pi}{2}, v) = v$