9/1 O.T

Chapter 3.3~3.4

Chapter 4에서 Surface 중요

라이브시작하면 강의실에서 수업가능!

3.3 Orientation

2020년 9월 3일 목요일 오후 12:00

If e_1, e_2, e_3 is a frame on R^3 and A is its attitude matrix, then

$e_1 \cdot e_2 \times e_3 = \det A = \pm 1$ by Ex.2.1.4(a) and 6

★ Definition

the frame e_1, e_2, e_3 is positively oriented if this number is +1.(양의 방향-같은 방향)

The frame is negative oriented if it is −1. (음의 방향-방향 반대)

Ex.2.1.4(a) and 6

4. Let $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3), w = (w_1, w_2, w_3)$

Prove that

(a) $u \cdot v \times w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

Pf)

$$u \cdot v \times w = (u_1, u_2, u_3) \cdot \begin{vmatrix} U_1(p) & U_2(p) & U_3(p) \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= (u_1, u_2, u_3) \cdot (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1)$$

$$= (u_1(v_2 w_3 - v_3 w_2), u_2(v_3 w_1 - v_1 w_3), u_3(v_1 w_2 - v_2 w_1))$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

6. If e_1, e_2, e_3 is a frame, show that $e_1 \cdot e_2 \times e_3 = \pm 1$

deduce that any 3×3 orthogonal matrix has determinant ± 1

Pf)

Let $A = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = I_3$

$A\,{}^tA = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} (e_1 \quad e_2 \quad e_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow$ 1= det($A\,{}^tA$)=det(A)det(${}^tA$)=(det(A))^2

Thus

$e_1 \cdot e_2 \times e_3 = \pm 1$ and det(A)=±1

$PO = |e_1 \times e_2||e_3| \cos \theta = 1$

$\theta \to 0$

Positively oriented

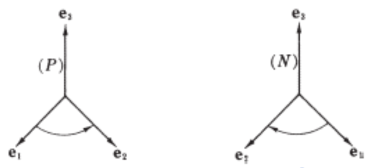


FIG. 3.3

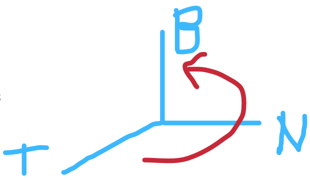
negatively oriented

★ Remark 3.1

(1) The natural frame field U_1, U_2, U_3 at each point of R^3 is positively oriented.

$\because U_1(p) \cdot U_2(p) \times U_3(p) = 1$ for each $p \in R^3$

(2) A frame e_1, e_2, e_3 is positively oriented iff $e_1 \times e_2 = e_3$



$\because \Rightarrow$) $1 = e_1 \cdot e_2 \times e_3 = e_1 \times e_2 \cdot e_3$ by Ex2.1.4(d)

and $|e_1 \times e_2| = |e_1||e_2| \sin \frac{\pi}{2} = 1$.

$\Leftarrow$) $e_1 \cdot e_2 \times e_3 = e_1 \times e_2 \cdot e_3 = e_3 \cdot e_3 = 1$.

Therefore Frenet frames T, N B are positively oriented $\because B = T \times N$

(3)

행렬식의 성질: 한 번은 부호 바뀜

$$e_1 = e_2 \times e_3 = -e_3 \times e_2$$
$$e_2 = e_3 \times e_1 = -e_1 \times e_3$$
$$e_3 = e_1 \times e_2 = -e_2 \times e_1$$

Ex 2.3.4이랑 비슷함

$B = T \times N \Rightarrow 1 = B \cdot T \times N = B \times T \cdot N$ (by def&2.1.ex.4)

Thus

$N = B \times T$

$N = B \times T \Rightarrow 1 = N \cdot B \times T = N \times B \cdot T$

Thus

$T = N \times B$

By alternation rule

$B = -N \times T, T = -B \times N, N = -T \times B$

Ex2.1.4(d)

(d) $u \cdot v \times w = u \times v \cdot w$

Pf)

$$u \cdot v \times w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

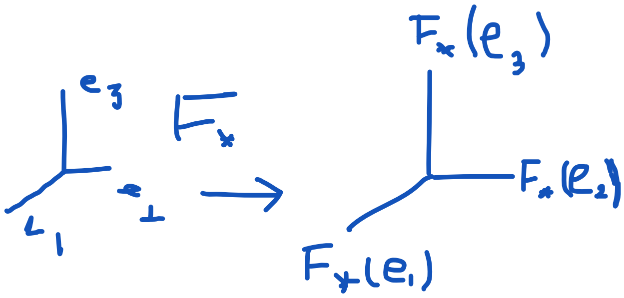
$$= -\begin{vmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \quad (\because 4. (c))$$

$$= \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$= w \cdot u \times v$

$= u \times v \cdot w$

When $F = TC \in \mathcal{E}(3)$, we define $\operatorname{sgn} F = \det C = \pm 1$ C: orthogonal part of the isometry F



★ Lemma 3.2

$F_*(e_1) \cdot F_*(e_2) \times F_*(e_3) = (\operatorname{sgn} F)\ e_1 \cdot e_2 \times e_3$

Pf)

$e_1 = (a_{11}, a_{12}, a_{13})$

$e_2 = (a_{21}, a_{22}, a_{23})$

$e_3 = (a_{31}, a_{32}, a_{33})$

$$F_*(e_1) = C(e_1) = (c_{ij})\begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix} = (c_{11}a_{11} + c_{12}a_{12} + c_{13}a_{13}, c_{21}a_{11} + c_{22}a_{12} + c_{23}a_{13}, c_{31}a_{11} + c_{32}a_{12} + c_{33}a_{13})$$

$$F_*(e_2) = C(e_2) = (c_{ij})\begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \end{pmatrix} = (c_{11}a_{21} + c_{12}a_{22} + c_{13}a_{23}, c_{21}a_{21} + c_{22}a_{22} + c_{23}a_{23}, c_{31}a_{21} + c_{32}a_{22} + c_{33}a_{23})$$

$$F_*(e_3) = C(e_3) = (c_{ij})\begin{pmatrix} a_{31} \\ a_{32} \\ a_{33} \end{pmatrix} = (c_{11}a_{31} + c_{12}a_{32} + c_{13}a_{33}, c_{21}a_{31} + c_{22}a_{32} + c_{23}a_{33}, c_{31}a_{31} + c_{32}a_{32} + c_{33}a_{33})$$

If $e_j = \sum a_{ji} U_i$, then $F_*(e_j) = \sum_{ik} c_{ik} e_{jk} \overline{U}_i$ by thm 3.2.1 $F_*(v_p) = (Cv)_{F(p)}$

Thus the attitude matrix of the frame $F_*(e_1), F_*(e_2), F_*(e_3)$ is $\left(\sum_k c_{ik} a_{jk}\right) = C\ {}^t\!A$

Therefore

$F_*(e_1) \cdot F_*(e_2) \times F_*(e_3) = \det(C\ {}^t\!A)$

$= \det C \cdot \det({}^t\!A)$

$= \det C \cdot \det A$

$= (\operatorname{sgn} F)\ e_1 \cdot e_2 \times e_3$

★ Def 3.3

$F = TC \in \mathcal{E}(3)$ is

orientation-preserving if $\operatorname{sgn} F = \det C = +1$,

orientation-reversing if $\operatorname{sgn} F = \det C = -1$

★ Example 3.4

(1) Translations.

T = T I $\in \mathcal{E}(3)$, where I is the identity $\Rightarrow$ sgn T = det I = 1.

(2) Rotations. as Example.3.1.2

$$C = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{I C} \in \mathcal{E}(3) \Rightarrow \text{sgn C} = \text{det C} = 1.$$

(3) Reflections. $R(p) = (-p_1, p_2, p_3)$ is orthogonal.

★ Lemma 3.5

For a positively oriented frame e_1, e_2, e_3,

$$v \times w = (v_1e_1 + v_2e_2 + v_3e_3) \times (w_1e_1 + w_2e_2 + w_3e_3)$$

$$= (v_2w_3 - v_3w_2)e_1 - \left(v_{1w_3} - v_3w_1\right)e_2 + (v_1w_2 - v_2w_1)e_3$$

$$= \epsilon \begin{vmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \text{ where } \epsilon = e_1 \cdot e_2 \times e_3 = 1$$

★ Thm 3.6

$F_*(v \times w) = (\operatorname{sgn} F)F_*(v) \times F_*(w)$

Pf) Let $v = \sum v_iU_i$ and $w = \sum w_iU_i$ $\Rightarrow v \times w = \begin{vmatrix} U_1 & U_2 & U_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \xrightarrow{F_*} \begin{vmatrix} \end{vmatrix}\underbrace{F_*(U_1)}_{e_1} - \begin{vmatrix} \end{vmatrix}\underbrace{F_*(U_2)}_{e_2} + \begin{vmatrix} \end{vmatrix}\underbrace{F_*(U_3)}_{e_3}$

같음

Let $e_i = F_*(U_i)$

Then $F_*(v) = \sum v_ie_i$ and $F_*(w) = \sum w_ie_i$

By 3.5, $F_*(v) \times F_*(w) = \epsilon \begin{vmatrix} e_1 & e_2 & e_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \epsilon\, F_*(v \times w) = (\operatorname{sgn} F)\, F_*(v \times w)$

$\odot\ e_1 \cdot e_2 \times e_3 = (\operatorname{sgn} F)\ U_1 \cdot U_2 \times U_3 = \operatorname{sgn} F$

3.4 Euclidean Geometry

★

Y is a vector field on α: $I \to R^3$ and $F : R^3 \to R^3$ is a mapping.

$\overline{Y} = F_*(Y)$ is a vector field on $\overline{\alpha} = F(\alpha)$.

i.e. for $t \in I, Y(t) \in T_{\alpha(t)}(R^3)$, and $\overline{Y}(t) = F_*(Y(t)) \in T_{\overline{\alpha}(t)}(R^3)$.

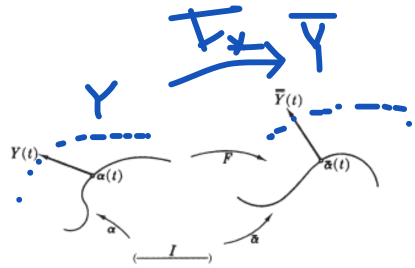


FIG. 3.6

★ Cor4.1

$\overline{Y}' = F_*(Y')$ where $F = TC \in \mathcal{E}(3)$

Thm 2.1 $F_*(v_p) = (Cv)_{F(p)}$

Pf) $Y = \sum y_j U_j \Rightarrow Y' = \sum \frac{dy_j}{dt} U_j$

By thm2.1 $F_*(Y') = \sum c_{ij} \frac{dy_j}{dt} \overline{U}_i$

But $\overline{Y} = F_*(Y) = \sum c_{ij} y_j \overline{U}_i \Rightarrow \overline{Y'} = \sum c_{ij} \frac{dy_j}{dt} \overline{U_i}$

$\therefore \overline{Y}' = F_*(Y')$

★ Note) Isometries preserve acceleration(가속도)

$F = TC \in \mathcal{E}(3) \Rightarrow \overline{\alpha}'' = F_*(\alpha'')$

$\because$) Let $Y = \alpha'$

$\Rightarrow \overline{Y} = F_*(Y) = F_*(\alpha') = \overline{\alpha}'$ by thm I.7.7

$\Rightarrow \overline{\alpha}'' = \overline{Y}' = F_*(Y') = F_*(\alpha'')$ by cor4.1

Thm 1.7.7

$\underbrace{\beta = F(\alpha)}_{= \overline{\alpha}} \Rightarrow \underbrace{\beta'}_{= \overline{\alpha}'} = F_*(\alpha')$

★ Thm4.2

$F = TC \in \mathcal{E}(3)$ and $\operatorname{sgn} F = +1$

Let β be a unit-speed curve in R^3 with $\kappa > 0$, and $\overline{\beta} = F(\beta)$. Then

$\overline{\kappa} = \kappa, \ \overline{T} = F_*(T)$

$\overline{\tau} = (\operatorname{sgn} F)\tau, \ \ \overline{N} = F_*(N)$

$\overline{B} = (\operatorname{sgn} F) F_*(B)$

$\|F_*(\beta')\|^2 = F_*(\beta') \cdot F_*(\beta') = \beta' \cdot \beta' = \|\beta'\|^2$

내적보존(norm)

Pf) By thm 2.2, $\left|\overline{\beta}'\right| = |F_*(\beta')| = |\beta'| = 1$

$\Rightarrow \overline{T} = \overline{\beta}' = F_*(\beta') = F_*(T), \ \overline{\kappa} = \left|\overline{\beta}''\right| = |F_*(\beta'')| = |\beta''| = \kappa > 0$

Furthermore,

$\overline{N} = \frac{\overline{\beta}''}{\overline{\kappa}} = \frac{F_*(\beta'')}{\kappa} = F_*\left(\frac{\beta''}{\kappa}\right) = F_*(N).$

By thm3.6, $\overline{B} = \overline{T} \times \overline{N} = F_*(T) \times F_*(N) = (sgn\, F)F_*(T \times N) = (sgn\, F)F_*(B)$

But $B' = -\tau N \Rightarrow \tau = -B' \cdot N = B \cdot N'$

thus by thm2.2 $\overline{\tau} = \overline{B} \cdot \overline{N}' = (sgn\, F)F_*(B) \cdot F_*(N') = (sgn\, F)B \cdot N' = (sgn\, F)\tau.$

Thm2.2 내적보존

$F_*(v_p) \cdot F_*(w_p) = v_p \cdot w_p$

Thm 3.6

$F_*(v \times w) = (sgnF)F_{*(v)} \times F_*(w)$

★ Ex4.3

$\beta(s) = \left(cos\, \frac{s}{c}, sin\, \frac{s}{c}, \frac{s}{c}\right)$ from 2.3.3 $a = b = 1, c = \sqrt{2}$

$\Rightarrow \kappa = \tau = \frac{1}{2}$

Let $R(x, y, z) = (x, y, -z)$ be a reflection and $\overline{\beta} = R(\beta)$

$\Rightarrow \overline{\beta}(s) = \left(cos\, \frac{s}{c}, sin\, \frac{s}{c}, -\frac{s}{c}\right)$

But $\operatorname{sgn} R = \det R = -1$

$\Rightarrow \overline{\kappa} = \kappa = \frac{1}{2}$ and $\overline{\tau} = -\tau = -\frac{1}{2}$

Note) $\overline{\beta}$ is the helix ($b = -1$)

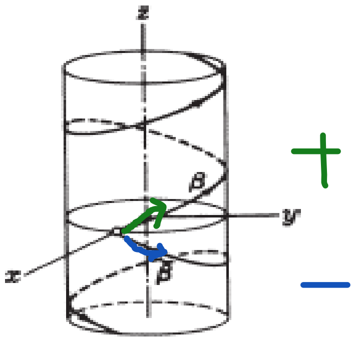


FIG. 3.7

4.1 Sufaces of R^3

위상동형

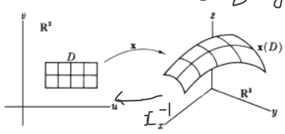


FIG. 4.1

★ Def 1.1

A coordinate patch $x:\ D\ \to R^3$ is a one-to-one regular mapping, where $D \subset R^2$ is open.

A coordinate patch $x:\ D\ \to R^3$ is *proper* if x^{-1}: $x(D) \to D$ is continuous.

Note) D is homeomorphic with $x(D)$

★ Def 1.2

A *surface* in R^3 is a subspace $M < R^3$ such that

$\forall p \in M, \exists$ a proper patch $x : D\ \to M, \exists\ U(p)$ a nbd of p in M, $U(p) \subset x(D)$

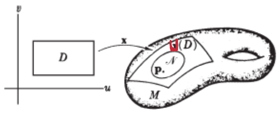


FIG. 4.2

$\boxed{U(p) \subset x(D)}$
$\quad \|$
$B(p,\varepsilon) \cap M$

$\Sigma = S^2 =$ 구 sphere

★ Example)

$\Sigma = \{(p_1,p_2,p_3)\ \in R^3\,|\,|p| = 1\}$ is a surface in R^3

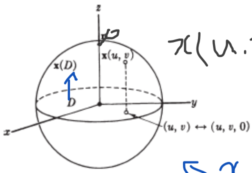
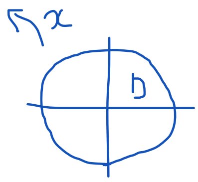


FIG. 4.3

$x(u,v) = (u, v, \sqrt{1-u^2-v^2})$



bd 포함X
D, x 둘다

Case 1) Given $p = (0,0,1) \in \Sigma$ the north pole

Define $x:\ D = \{(u,v)\,|\,|(u,v)| < 1\}\ \to \Sigma$ by $x(u,v)\ =\ \left(u, v, \sqrt{1-u^2-v^2}\right)$

Then $x(0,0)\ =\ p$

(1) x is a 1-1 mapping

(2) x is regular

$\because J(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix}$ where $f(u,v) =\ \sqrt{1-u^2-v^2}$, has *rank* 2 at each point in D

(3) x is proper

$\because x^{-1}$: $x(D) \to D$ is defined by $x^{-1}(p_1,p_2,p_3) =\ (p_1,p_2)$

$U(p) =\ x(D) = \{\ (p_1,p_2,p_3) \in R^3 |\ p_3 > 0\ \} \cap \Sigma$ is open in Σ.

Case 2) Given $p\ =\ (0,0,-1) \in \Sigma$

…

Case 6)(x방향, y방향, z방향 각 2가지씩)

★ Def

If f is a function and $D \subset R^2$ is open,

Then $x:\ D\ \to R^3$ defined by $x(u,v) =\ (u,v,f(u,v))$is a proper patch. "*Monge patches*"

$M\ =\ x(D)$ is a *simple* surface

★ Ex1.3

The surface $M:\ z\ =\ f(x,y)$ where $f:\ R^2\ \to R$ is diff. (f is real -valued function)

$p \in M \Leftrightarrow p_3 =\ f(p_1,p_2)$

Thus $p\ =\ \left(p_1,p_2,f(p_1,p_2)\right) \in M$, the *graph* of f

$x:\ R^2 \to M$ defined by $x(u,v) =\ (u,v,f(u,v))$ (monge patches)

$\Rightarrow\ M$ is a simple surface

$\therefore$ M is a simple surface

★ Thm 1.4

$M: g(x,y,z) = c$ is a surface if $dg \neq 0$ at any point of M

Pf) By Ex.1.5.9

Note that A 1-form $\phi = 0$ at p if $\phi(v_p) = 0$ for all $v_p \in T_p(R^3)$

$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz$

$$dg(v_p) = \frac{\partial g}{\partial x}(p)\,dx(v_p) + \frac{\partial g}{\partial y}(p)\,dy(v_p) + \frac{\partial g}{\partial z}(p)\,dz(v_p)$$

Suppose $p \in M$ and $\left(\frac{\partial g}{\partial z}\right)(p) \neq 0$

By implicit function theorem

$\exists h: D \to V \subset R$ is diff.

where D is a nbd of (p_1,p_2) and $h(D) = V$ is a nbd of p_3

(1) $g(u,v,h(u,v)) = c$ for $(u,v) \in D$

(2) $D \times V = \{(u,v,h(u,v)) \mid (u,v) \in D\}$ is a nbd of p in M

$\Rightarrow x: D \to R^3$ defined by $x(u,v) = (u,v,h(u,v))$

$\Rightarrow x(D) \subset M$

★ Note) The sphere $\sum$□ in R^3 of radius $r > 0$ and center $c = (c_1,c_2,c_3)$

$\Rightarrow \sum$□ $: g(x_1,x_2,x_3) = \sum(x_i - c_i)^2 = r^2$ is a surface

$\because dg = 2\sum(x_i - c_i)\,dx_i \neq 0$ for each $p \in \sum$□

★ Ex 1.5 (A surface of revolution) M

The curve $C: f(x,y) = c$

$p = (p_1,p_2,p_3) \in M \Leftrightarrow \overline{p} = \left(p_1, \sqrt{p_2^2 + p_3^2}, 0\right) \in C$

$\Rightarrow M: g(x,y,z) = f\left(x, \sqrt{y^2+z^2}\right) = c$

$\Rightarrow$ *dg is not zero*

$\because$과제

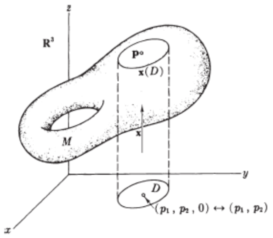


FIG. 4.4

1.5.9

1-form ϕ is zero at a point p provided $\phi(v_p) = 0$ for all tangent vectors at p

A point at which its differential df is zero is called a critical point of the function f

Prove that p is a critical point of f iff $\frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = \frac{\partial f}{\partial z}(p) = 0$

Find all criticla points of $f = (1-x^2)y + (1-y^2)z$

pf)$\Rightarrow$

if p is critical point of f then $df = o$

Thus $0 = df\left(U_i(p)\right) = U_i(p)[f] = \frac{\partial f}{\partial x_i}(p)$

$\therefore)\ \frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = \frac{\partial f}{\partial z}(p) = 0$

$\Leftarrow$

$\forall v_p = v_1U_1(p) + v_2U_2(p) + v_3U_3(p)$,

$df(v_p) = v_1U_1(p)[f] + v_2U_2(p)[f] + v_3U_3(p)[f]$

$= v_1\frac{\partial f}{\partial x}(p) + v_2\frac{\partial f}{\partial y}(p) + v_3\frac{\partial f}{\partial z}(p)$

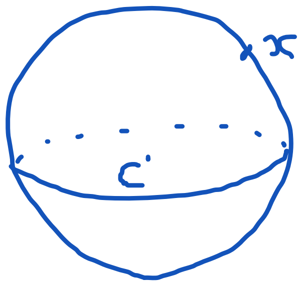
$= 0$

$\therefore)\ df = 0$

$\nabla f = (-2xy, 1-x^2-2yz, 1-y^2) = (0,0,0)$

$\Rightarrow\ y = \pm 1, x = 0, z = \pm\frac{1}{2}$

$\Rightarrow\ (x,y,z) = \pm\left(0,1,\frac{1}{2}\right)$



$x = (x_1, x_2, x_3)$

$p = (p_1, p_2, p_3)$

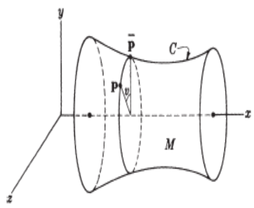


FIG. 4.5

★ Ex 1.6

$D = \{(u,v) | -\pi < u < \pi,\ 0 < v < 1\}$ in R^2

define $x : D \to R^3$ by $x(u,v) = (\sin u, \sin 2u, v)$ is 1-1

x(D) $= M$ is not a surface

$x^{-1}: x(D) \to D$ is not continuous $\Leftrightarrow x$ is not open

$\because\ \ V = \{(u,v) | -\pi/2 < u < \pi/2,\ 0 < v < 1\}$ is a nbd of $\left(0, \frac{1}{2}\right)$ in D

But $x(V)$ is not a nbd of $\left(0,0,\frac{1}{2}\right)$in $x(D)$

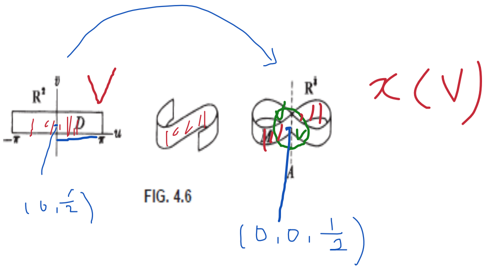
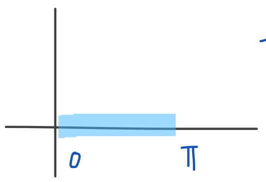
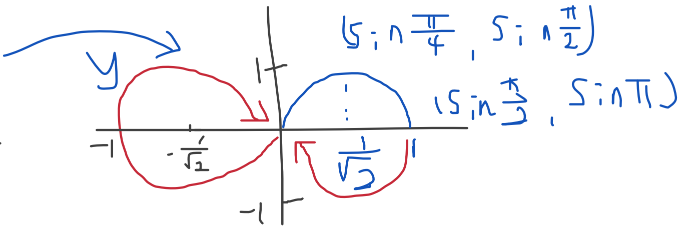
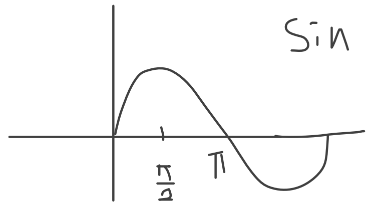


FIG. 4.6

$\exists B(p, \varepsilon) \cap x(D) \subset x(V)$







$y = 2 sinucosu$
$x = sinu$
$\Rightarrow x^2 + \left(\frac{y}{2x}\right)^2 = 1$
$\Rightarrow\ 4x^4 - 4x^2 + y^2 = 0$
$\Rightarrow 4\left(\left(x + \frac{1}{\sqrt{2}}\right)\left(x - \frac{1}{\sqrt{2}}\right)\right)^2 + y^2 = 1$

교수님 과제 설명

Let $h(x,y,z) = \left(x, \sqrt{y^2+z^2}\right)$

$\Rightarrow\ g\ = f \circ h$

$D_1 g(p) = D_1 f(h(p)) \cdot h'(p)$

Note) $p\ \in M\ \Leftrightarrow h(p) \in C$

$\Rightarrow\ g'(p) = \left(D_1 g(p), D_2 g(p), D_3 g(p)\right)$

$(f \circ h)'(p) = f'\left(h(p)\right) \cdot h'(p)$ and $f'\left(h(p)\right) = \left(D_1 f\left(h(p)\right), D_2 f\left(h(p)\right)\right)$

Note) 1.4.6 below, $D_1 f$ and $D_2 f$ are never simultaneously zero at any point of C

$$h'(p) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{p_2}{\sqrt{p_2^2+p_3^2}} & \frac{p_3}{\sqrt{p_2^2+p_3^2}} \end{pmatrix}$$

$\bar{p} = (p_1, 0, 0) \notin C$

Note) If $p_2 = p_3 = 0$ then $(p_1, 0,0) \notin M$

$\Rightarrow\ dg\ =\ D_1 g dx\ +\ D_2 g dy\ +\ D_3 g dx\ \neq 0\ \because$ Ex. 1.5.9

1,7^* Mappings

2020년 10월 1일 목요일 오후 8:29

1.3.1 Definition $f: R^3 \to R$. The (directional) *derivative* of f with respect to v_p

$v_p[f] = \frac{d}{dt}\big(f(p+tv)\big)|_{t=0}$ $\in R$

$= \cancel{(f\circ\alpha)'(0)}$ where $\alpha(t) = p + tv$

$= df(v_p)$ the *differential* df of f (1-form), Def. 1.5.2

$= df(\alpha'(0))$ where $\alpha'(0) = v_p \in T_p(R^n)$

$= D_v f(p) = Df(p)(v)$ by C. Ex. 2-29

1.7.4 Definition $F: R^n \to R^m$ is a mapping. The *tangent map*

$F_{*p}: T_p(R^n) \to T_{F(p)}(R^m)$ by $F_{*p}(\alpha'(0)) = (F\circ\alpha)'(0)$

Note) $F_{*p} = dF_p = DF_p$ and $F_* = dF = DF$ "the *differential* of F"

(1) $n = 3$ and $m = 1$ "the *differential of* F" 1.5.2 Def.

(2) $dF(v) = F_*(v) = (v[f_1], \dots, v[f_m])$ by 1.7.5

$= (df_1(v), \dots, df_m(v))$ ~~by 1.5.2 Def.~~

$= (df_1, \dots, df_m)(v) \Rightarrow dF = d(f_1, \dots, f_m) = \cancel{(df_1, \dots, df_m)}$

(3) $G: R^m \to R^p$ (Chain Rule)

$d(G\circ F)(\alpha'(0)) = ((G\circ F)\circ\alpha)'(0) \overset{\text{Chain Rule}}{=} (G\circ(F\circ\alpha))'(0)$

$= dG((F\circ\alpha)'(0)) = dG\big(dF(\alpha'(0))\big) = (dG\circ dF)(\alpha'(0))$

Thus $d(G\circ F) = dG\circ dF$ Ex.1.7.9.(b)

(4) For $g: R^m \to R$, using (3)

$F_*(v)[g] = dF(v)[g] = dg(dF(v)) = d(g\circ F)(v) = v[g(F)]$ Ex 1.7.7

$df: R^3 \cong T_p(R^3) \to R^1 \cong T_{f(p)}(R^1)$

$\alpha'(0) \longmapsto (f\circ\alpha)'(0)$

$(f\circ\alpha)'(0)$

$DF(p): R^n \xrightarrow{\ell} R^m$

Jacobian $(D_j f_i(p))$ I 7 8

$dF = F_{*p}: T_p(R^n) \xrightarrow{\ell} T_{F(p)}(R^m)$

$\parallel$

R^n

4.2 Patch Computations

★ Def2.1

$x : D \to R^3$ is a patch and $(u_0, v_0) \in D$. Define $patial\ velocity\ vectors$

(1) $x_u(u_0, v_0)$ for $v = v_0$ is the velocity vector at u_0 of the u-parameter curve

(2) $x_v(u_0, v_0)$ for $u = u_0$ is the velocity vector at v_0 of the v-parameter curve

Let $x(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$

Then $x_u = (D_1x_1, D_1x_2, D_1x_3)_x$ and $x_v = (D_2x_1, D_2x_2, D_2x_3)_x$

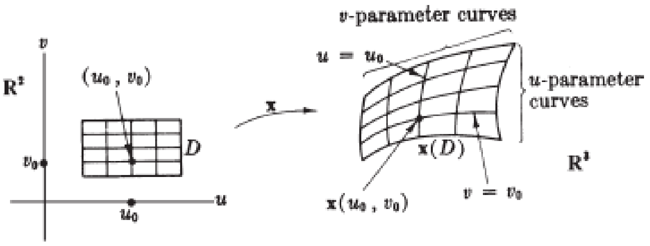


FIG. 4.10

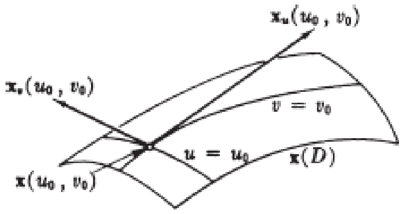


FIG. 4.11

★ Ex2.2

The geographical patch in the sphere Σ□ $= \{ p \in R^3 \mid |p| = 1 \}$

Define D with longitude $-\pi < u < \pi$ and latitude $-\frac{\pi}{2} < v < \frac{\pi}{2}$

Define $x(u,v) = (r \cos v \cos u\,, r \cos v \sin u\,, r \sin v)$

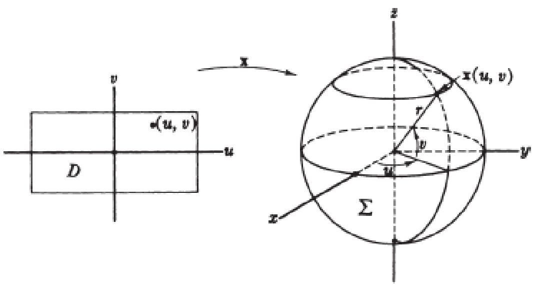


FIG. 4.12

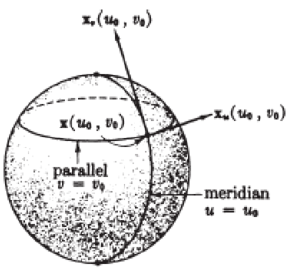


FIG. 4.13

★ Def2.3

A regular mapping $x : D \to R^3$ with $x(D) \subset M$ is a *parametrization* of $x(D)$ in M

Note) Any patch is a 1-1 parametrization

Note) x is regular $\Leftrightarrow x_u \times x_v \neq 0$ for all (u, v)

★ Ex2.4

M is the surface of revolution.(Ex.1.5)

Let $\alpha(u) = \big(g(u), h(u), 0\big)$ be a parametriization of C $(h > 0)$

Define $x(u, v) = \big(g(u), h(u)\cos v\,, h(u)\sin v\big)$is not 1-1

$D = (0,1) \times (-2\pi, 2\pi) \Rightarrow x(D) = M$

$x_u(u, v) = \big(g'(u), \cos v\, h'(u), \sin v h'(u)\big)$

$x_v(u, v) = (0, -h(u)\sin v\,, h(u)\cos v)$

Assume $x_u(u, v) \times x_v(u, v) = \big(h(u)h'(u), -g'(u)h(u)\cos v\,, -g'(u)h(u)\sin v\big) = 0$

$\Rightarrow h'(u) = 0, g'(u)\cos v = 0, g'(u)\sin v = 0$

$\Rightarrow h'(u) = 0$ and $\cos v = \sin v = 0$ (This is a contradiction)

Thus x is a parametrization of M

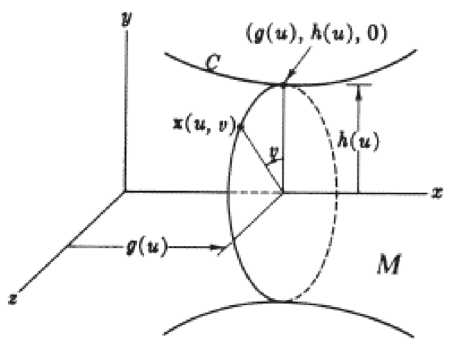


FIG. 4.14

★ Ex2.5

Torus of revolution T

Let $\alpha(u) = (R + r\cos u, r\sin u) = \big(h(u), g(u)\big)$ for C

Define $x: R^2 \to T < R^3$ the $usual\ parametrization$ of T by

$x(u,v) = \big(h(u)\cos v, h(u)\sin v, g(u)\big)$

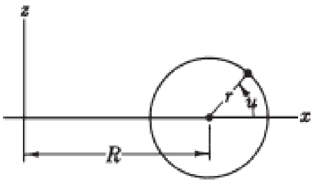


FIG. 4.15

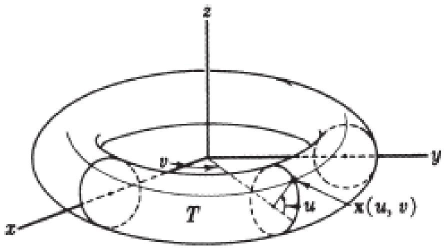


FIG. 4.16

4.3_1 Differentiable Functions

2020년 10월 13일 화요일 오후 6:18

★ Definition

(1) $f: M \to R$ is a function and $x: D \to M$ is a coordinate patch

$f \circ x: D \to R$ is a *coordinate expression* for f

f is *differentiable* if $\forall f \circ x$ are differentiable

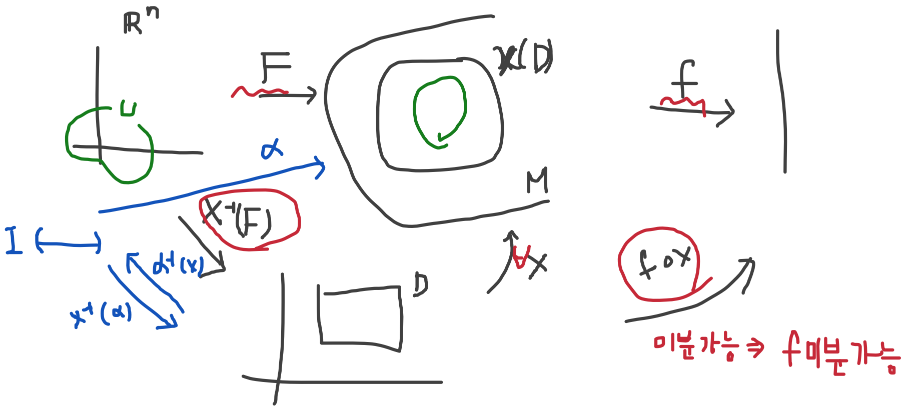
(2) $F: R^n \to M$ is a function and $x: D \to M$ is a coordinate patch

$x^{-1}(F): R^n \to D \subset R^2$ is a *coordinate expression* for F

Note) $x(D)$ is open in M, $U = F^{-1}(x(D))$ is open in R^n !

$x^{-1}(F): U \subset R^n \to D \subset R^2$

F is *differentiable* if $\forall x^{-1}(F)$ are differentiable



★ Lemma3.1

If $\alpha: I \to M$ is differentiable with $\alpha(I) \subset x(D)$, then $\exists |$ $(a_1, a_2), \alpha = x(a_1, a_2)$

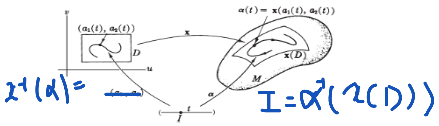
Pf) Note $I = \alpha^{-1}(x(D))$

$x^{-1}(\alpha): I \to D \subset R^2$ is differentiable

Let $x^{-1}(\alpha) = (a_1, a_2) \Rightarrow \alpha = x(a_1, a_2)$

(Uniqueness) $\alpha = x(b_1, b_2)$

$\Rightarrow (a_1, a_2) = x^{-1}(\alpha) = x^{-1}(x(b_1, b_2)) = (b_1, b_2)$



These functions a_1, a_2 are called the *coordinate functions* of the curve α with respect to the patch x.

★ Thm 3.2

Mapping $F: R^n \to R^3$ with $F(R^n) \subset M$ is differentiable i.e.

$\forall x$, $x^{-1}(F): R^n \to D \subset R^2$ is differentiable (by *John McCleary*)

Pf) Note $V = F^{-1}(x(D))$ is open in R^n

Let $h = x^{-1}(F): V \to D$

Let $x(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v))$ and $x(u_0, v_0) = p \in M, F(z) = p$

Suppose $D_1x_1 \cdot D_2x_2 - D_2x_1 \cdot D_1x_2 \neq 0$ at $(u_0, v_0) = h(z) = x^{-1}(p)$ by regular

Define $G: D \times R \to R^3$ by

$G(u, v, t) = x(u, v) + (0,0,t) = (x_1(u, v), x_2(u, v), x_3(u, v) + t)$

$\Rightarrow$ G is differentiable and $G(u, v, 0) = x(u, v)$

The Jacobian of G is $J(G) = \begin{pmatrix} D_1x_1 & D_2x_1 & 0 \\ D_1x_2 & D_2x_2 & 0 \\ D_1x_3 & D_2x_3 & 1 \end{pmatrix}$

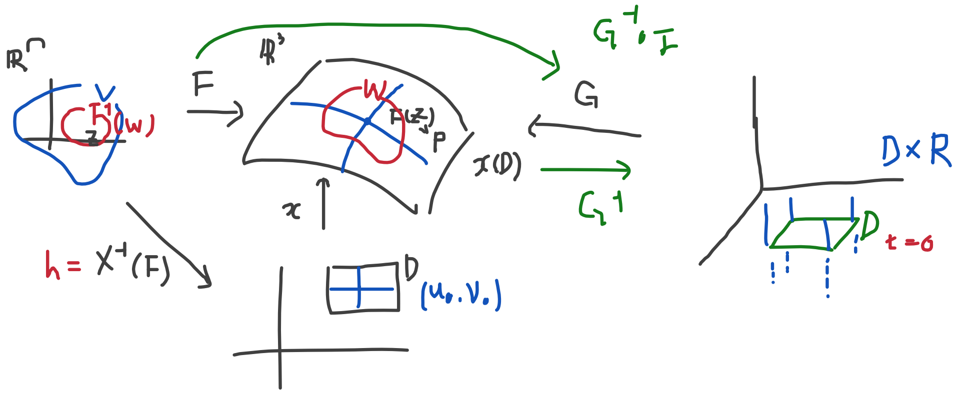
$\Rightarrow \det J(G)(u_0, v_{0,0}) \neq 0$

By I.F.T. $\exists$ W (a nbd of $G(u_0, v_{0,0}) = x(u_0, v_0)$), G^{-1} is differentiable

$\Rightarrow (h(z), 0) = (u_0, v_0, 0) = (G^{-1} \circ F)(z)$

$\because$ $G(u_0, v_{0,0}) = x(u_0, v_0) = p = F(z)$

Thus h is differentiable



$$J = \begin{pmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \\ \frac{\partial x_3}{\partial u} & \frac{\partial x_3}{\partial v} \end{pmatrix}$$

rank 2

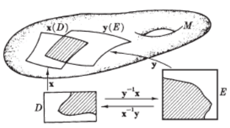
★ cor3.3

if x and y are patches in $M \subset R^3$ whose images overlap,

then the composite functions $x^{-1}y$ and $y^{-1}x$ are mappings defined on open sets of R^2

pf)

과제



★ Cor3.4

If x and y are overlapping patches in M,

Then $\exists |$ differentiable functions $\bar{u}, \bar{v}$ such that $y(u, v) = x(\bar{u}(u, v), \bar{v}(u, v))$ for all (u, v) in the domain of $x^{-1}y$

$x^{-1}(y) : (u,v) \longmapsto (\bar{u}(u,v), \bar{v}(u,v))$

$\Rightarrow y(u,v) = x(\bar{u}(\ ), \bar{v}(\ ))$

$\Rightarrow y = x(\bar{u}, \bar{v})$

if W= $Im\, x \cap Im\, y$ then $x^{-1} \circ y : y^{-1}(W) \to x^{-1}(W)$ is differentiable function by cor 3.3

$\exists \bar{u}, \bar{v}$ such that $x^{-1} \circ y(u,v) = (\bar{u}(u,v), \bar{v}(u,v))$

$\therefore y(u,v) = x(\bar{u}(u,v), \bar{v}(u,v))$

If $y(u,v) = x(\bar{u}_1(u,v), \overline{v_1}(u,v)) = x(\bar{u}_2(u,v), \bar{v}_2(u,v))$

Then $x^{-1} \circ y(u,v) = (\bar{u}_1(u,v), \overline{v_1}(u,v)) = (\bar{u}_2(u,v), \bar{v}_2(u,v))$

Thus $\bar{u}_1 = \bar{u}_2, \bar{v}_1 = \bar{v}_2$

4.3_2 Tangent vectors

2020년 10월 13일 화요일 오후 6:19

★ Def 3.5

v_p is *tangent* to M at p if $\exists\ \alpha$ (a curve in M), $v_p = \alpha'(0)$

$T_p(M) = \{v_p | v_p$ is tangent to M at $p\}$ the *tangent plane* of M at p

$T_p(M) = \langle \{x_u, x_v\} \rangle < T_p(\mathbb{R}^3)$ ↱2 ↱3

← span

$x_u \times x_v \neq 0$

independent

★ Lem3.6

$x(u_0, v_0) = p \in M$

$v_p \in T_p(M) \Leftrightarrow \exists c_i.\ v = c_1 x_u(u_0, v_0) + c_2\, x_v(u_0, v_0)$

Pf) Note the parameter curves $x_u, x_v \in T_p(M)$

$\Rightarrow$) $\exists \alpha,\ \alpha'(0) = v$ and $\alpha(0) = p$

$\underset{3.1}{\Rightarrow} \alpha = x(a_1, a_2)$

$\Rightarrow \alpha' = x_u(a_1, a_2)\frac{da_1}{dt} + x_v(a_1, a_2)\frac{da_2}{dt}$

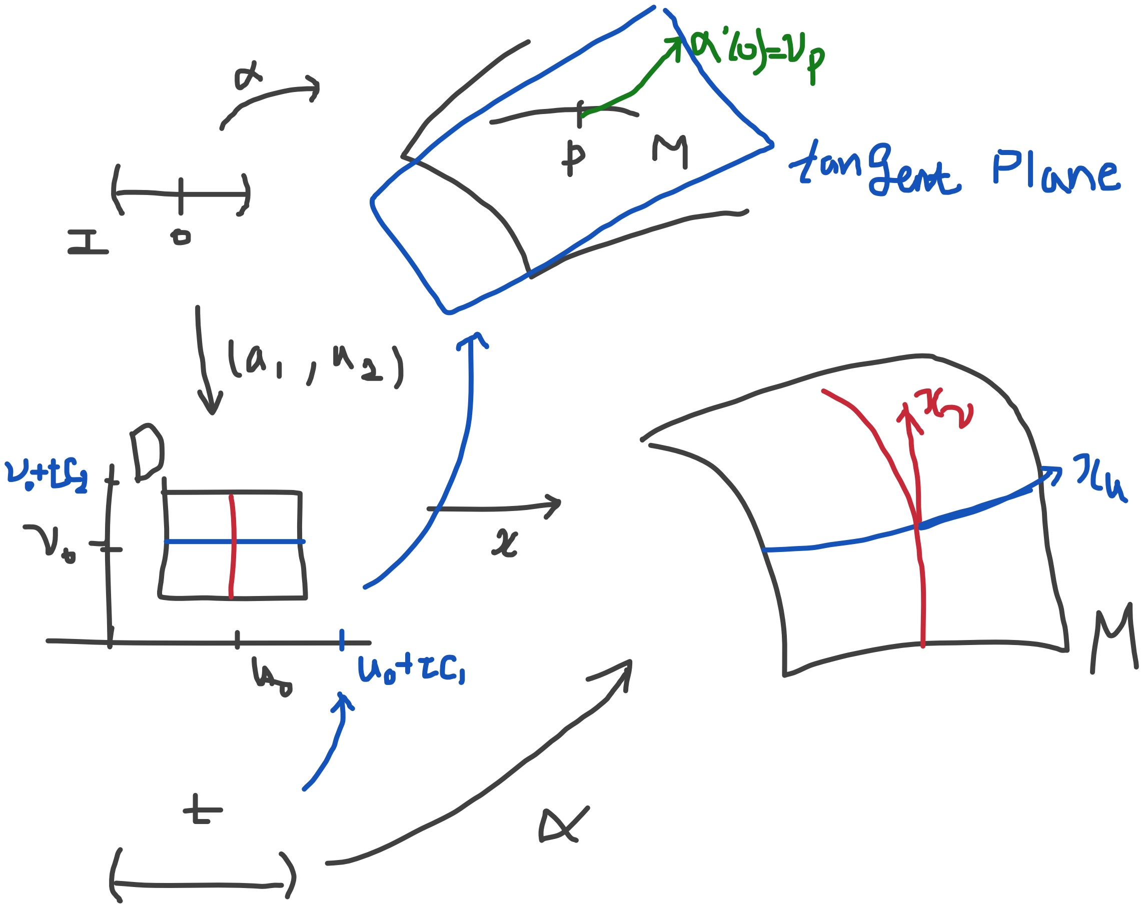
$\Rightarrow v = \alpha'(0) = \frac{da_1}{dt}(0)\, x_u(u_0, v_0) + \frac{da_2}{dt}(0)\, x_v(u_0, v_0)$ by x is inj.

$\Leftarrow$) $v = c_1 x_u(u_0, v_0) + c_2\, x_v(u_0, v_0)$

$\Rightarrow \alpha'(t) = c_1 x_u + c_2 x_v$

Let $\alpha(t) = x(u_0 + tc_1, v_0 + tc_2)$

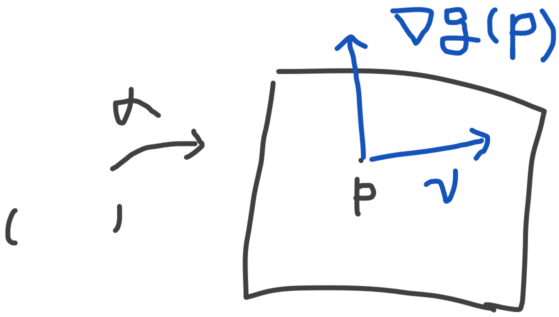
Then $\alpha'(0) = v \in T_p(M)$



★ Def3.7

A $Euclidean\ vector\ field\ Z$ on M is a function

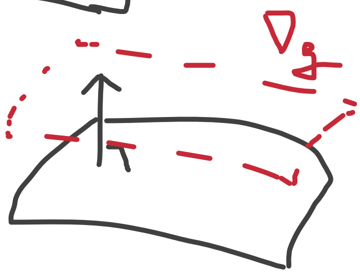
$Z: M \to T(M)$ by $Z(p) \in T_p(M)$



★ Lem 3.8

$M:\ g = c$

$\nabla g = \sum (D_i g) U_i$ is a nonvanishing normal v.f. on M



Pf) $(\nabla g)(p) \cdot v = 0$ for all $v \in T_p(M)$!!

$\alpha'(0) = v$

Choose $\alpha, \alpha'(0) = v$ and $\alpha(0) = p$

$\Rightarrow g(\alpha) = g(\alpha_1, \alpha_2, \alpha_3) = c$

$\Rightarrow \sum D_i g(\alpha) \frac{d\alpha_i}{dt} = 0$

$\Rightarrow \sum D_i g(p) \frac{d\alpha_i}{dt}(0) = \sum D_i g(p) v_i = (\nabla g)(p) \cdot v = 0$

4.4 Differential Form on a Surface

2020년 10월 28일 수요일 오전 9:08

★ Def 4.1

A 2-$form$ on a surface on M

$\eta : T_p(M) \times T_p(M) \to R$ is an alternating 2-tensor .i.e. $\eta \in \Lambda^2\left(T_p(M)\right)$

Note) $\eta(v,v) = 0$

★ Lem 4.2

$\eta \in \Lambda^2\left(T_p(M)\right)$ and $v,w \in T_p(M)$ are independent $\Rightarrow$ $\eta(av + bw, cv + dw) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \eta(v,w)$

Pf) $\eta(av + bw, cv + dw)$

$= ac\,\eta(v,v) + ad\,\eta(v,w) + bc\eta(w,v) + bd\eta(w,w)$

$= ad\,\eta(v,w) - bc\eta(v,w)$

Def-2학년 해석기하학

$T \in \mathcal{T}^k(V)$ and $S \in \mathcal{T}^\ell(V)$

(1) $T \otimes S\,(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+\ell}) = T(v_1, \dots, v_k) \cdot S(v_{k+1}, \dots, v_{k+\ell})$

(2) $Alt(T)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}\sigma \cdot T(v_{\sigma(1)}, \dots, v_{\sigma(k)})$

$\omega \in \Lambda^k(V)$and $\eta \in \Lambda^l(V)$

(3) The wedge product $\omega \wedge \eta = \frac{(k+\ell)!}{k!\ell!} Alt\,(\omega \otimes \eta)$

(4) $\omega \wedge \eta = (-1)^{k\ell}\,\eta \wedge \omega$

★ Def 4.3

For $\phi, \psi \in \mathcal{T}^1\left(T_p(M)\right) = \Lambda^1\left(T_p(M)\right)$ are 1-forms

$(\phi \wedge \psi)(v,w) = \left(\frac{(1+1)!}{1!1!} Alt(\phi \otimes \psi)\right)(v,w)$

$= (\phi \otimes \psi)(v,w)$ for σ = identity

$-(\phi \otimes \psi)(w,v)$ for $\sigma = (1,2)$

$= \phi(v)\psi(w) - \phi(w)\psi(v)$ for all $v,w \in T_p(M)$

Note) $\phi \wedge \phi = 0$ by (4)

★ Def 4.4

ϕ is a 1-form on M. The $exterior\ derivative$ $d\phi$ of ϕ is a 2-form by

$d\phi(x_u, x_v) = \frac{\partial}{\partial u}\left(\phi(x_v)\right) - \frac{\partial}{\partial v}\left(\phi(x_u)\right)$ for any patch x in M

Note) $T_p(M) = < \{x_u, x_v\} >$ by 4.3.6

★

$T_p(M) = < \{x_u, x_v\} > = < \left\{x_{u_1}, x_{u_2}\right\} >$ by (4.3.6)

The dual basis $\varphi_i\left(x_{u_j}\right) = \delta_{ij} \Rightarrow \phi = \sum f_i \varphi_i$ is a 1-form on M

Note) The exterior derivative of ϕ is $d\phi = \sum df_i \wedge \varphi_i$ by (1.6.3)

$(d\phi)\left(x_{u_1}, x_{u_2}\right) = \left(\sum df_i \wedge \varphi_i\right)\left(x_{u_1}, x_{u_2}\right)$

$= \sum(df_i \wedge \varphi_i)\left(x_{u_1}, x_{u_2}\right)$

$= \sum\left[(df_i)\left(x_{u_1}\right)\varphi_i\left(x_{u_2}\right) - (df_i)\left(x_{u_2}\right)\varphi_i\left(x_{u_1}\right)\right]$ by (4.4.3)

$= (df_2)\left(x_{u_1}\right) - (df_1)\left(x_{u_2}\right)$

$= x_{u_1}[f_2] - x_{u_2}[f_1]$ by (1.5.2)

$= \sum \frac{\partial x_i}{\partial u_1} D_i f_2 - \sum \frac{\partial x_i}{\partial u_2} D_i f_1$ by (1.3.2)

$= \frac{\partial(f_2 \circ x)}{\partial u_1} - \frac{\partial(f_1 \circ x)}{\partial u_2}$

$= \frac{\partial\left(\phi\left(x_{u_2}\right)\right)}{\partial u_1} - \frac{\partial\left(\phi\left(x_{u_1}\right)\right)}{\partial u_2}$

$\because \phi\left(x_{u_2}\right) = \sum(f_i\varphi_i)\left(x_{u_2}\right)$

$= \sum f_i(x)\varphi_i\left(x_{u_2}\right)$ by $(f\phi)(v_p) = f(p)\phi(v_p)$ and $x_{u_2} = \left(\frac{\partial x_1}{\partial x_{u_2}}, \frac{\partial x_2}{\partial x_{u_2}}, \frac{\partial x_3}{\partial x_{u_2}}\right)_x$

$= f_2(x)$

★ Lem 4.5

$d_x\phi = d_y\phi$ on $x(D) \cap y(E)$

Pf) $\forall v_1, v_2,\ \ d_x\phi(v_1, v_2) = d_y\phi(v_1, v_2)$

We show that $d_x\phi(y_u, y_v) = d_y\phi(y_u, y_v)$!! by (4.4.2)

By (4.3.4) $y = x(\bar{u}, \bar{v})$

$\Rightarrow y_u = \frac{\partial \bar{u}}{\partial u} x_u + \frac{\partial \bar{v}}{\partial u} x_v$ and $y_v = \frac{\partial \bar{u}}{\partial v} x_u + \frac{\partial \bar{v}}{\partial v} x_v$ (1)

where x_u and x_v are evaluated on $(\bar{u}, \bar{v})$

By 4.4.2

(2) $(d_x\phi)(y_u, y_v) = J \cdot (d_x\phi)(x_u, x_v) = J \cdot \left\{ \frac{\partial}{\partial \bar{u}}\left(\phi(x_v)\right) - \frac{\partial}{\partial \bar{v}}\left(\phi(x_u)\right) \right\}$, $J = \begin{vmatrix} \frac{\partial \bar{u}}{\partial u} & \frac{\partial \bar{v}}{\partial u} \\ \frac{\partial \bar{u}}{\partial v} & \frac{\partial \bar{v}}{\partial v} \end{vmatrix}$

(3) $d_y\phi(y_u, y_v) = \frac{\partial}{\partial u}\left(\phi(y_v)\right) - \frac{\partial}{\partial v}\left(\phi(y_u)\right)$

By (1) $\phi(y_v) = \frac{\partial \bar{u}}{\partial v}\phi(x_u) + \frac{\partial \bar{v}}{\partial v}\phi(x_v)$

$\frac{\partial}{\partial u}\left(\phi(y_v)\right) = \left(\frac{\partial}{\partial u}\right)\left(\frac{\partial \bar{u}}{\partial v}\right)\left(\phi(x_u)\right) + \frac{\partial \bar{u}}{\partial v}\frac{\partial}{\partial u}\left(\phi(x_u)\right) + \left(\frac{\partial}{\partial u}\right)\left(\frac{\partial \bar{v}}{\partial v}\right)\left(\phi(x_v)\right) + \frac{\partial \bar{v}}{\partial v}\frac{\partial}{\partial u}\left(\phi(x_v)\right)$

$= \left(\frac{\partial}{\partial u}\right)\left(\frac{\partial \bar{u}}{\partial v}\right)\left(\phi(x_u)\right) + \frac{\partial \bar{u}}{\partial v}\left(\frac{\partial}{\partial \bar{u}}\left(\phi(x_u)\right)\frac{\partial \bar{u}}{\partial u} + \frac{\partial}{\partial \bar{v}}\left(\phi(x_u)\right)\frac{\partial \bar{v}}{\partial u}\right)$

$+\ \cdots$ Report

Note)

(1) $d : \Lambda^1\left(T_p(M)\right) \to \Lambda^2\left(T_p(M)\right)$

(2) $\Lambda^3\left(T_p(M)\right) = 0 \ \because\ T_p(M) = <\{x_u, x_v\}>$ has dimension 2

Note) (1) $\phi = \psi$ is a 1-form on $x(D) \Leftrightarrow \phi(x_u) = \psi(x_u)$ and $\phi(x_v) = \psi(x_v)$

(2) $\mu = \nu$ is a 2-form on $x(D) \Leftrightarrow \mu(x_u, x_v) = \nu(x_u, x_v)$ by (4.4.2)

★ Thm 4.6

$d(df) = 0$

Pf) Let $\psi = df \Rightarrow (d\psi)(x_u, x_v) = 0$!! by (4.4.2)

$\psi(x_u) = df(x_u) = x_u[f] = \sum \frac{\partial x_i}{\partial u} D_i f = \frac{\partial}{\partial u}(fx)$ and $\psi(x_v) = \frac{\partial}{\partial v}(fx)$

$\Rightarrow d\psi(x_u, x_v) = \frac{\partial}{\partial u}(\psi x_v) - \frac{\partial}{\partial v}(\psi x_u) = 0$

★ Def 4.8

ϕ is *closed* if $d\phi = 0$, and ϕ is *exact* if $\exists \xi, d\xi = \phi$

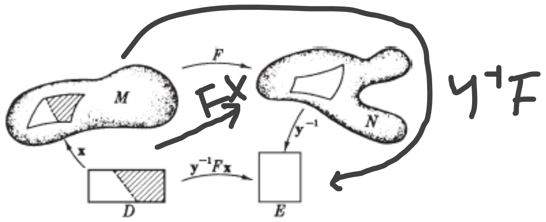
4.5 Mappings of Surfaces

2020년 11월 8일 일요일 오후 9:36

★ Def5.1

A *mapping* of surfaces $F : M \to N$ is differentiable if for each patch x in M and y in N, $y^{-1}Fx$ is differentiable.

$= y^{-1}(Fx) = (y^{-1}F)x$



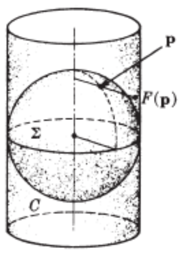
★ Example 5.2

(1) Define $F : \Sigma \to C$ by

$x(u,v) = (\cos v \cos u, \cos v \sin u, \sin v)$ and $y(u,v) = (\cos u, \sin u, v)$

$\Rightarrow F(x(u,v)) = (\cos u, sinu, \sin v) = y(u, \sin v)$

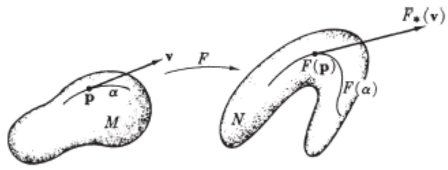
$\Rightarrow (y^{-1}Fx)(u,v) = (u, \sin v)$



★ Def5.3

Let $F : M \to N$ be a mapping of surfaces.

The *tangent map* $F_{*p} : T_p(M) \to T_{F(p)}(N)$ is defined by $F_{*p}(v) = F_{*p}(\alpha'(0)) = (F \circ \alpha)'(0)$



1.7.4 Definition $F : R^n \to R^m$ is a mapping. The *tangent map*

$F_{*p} : T_p(R^n) \to T_{F(p)}(R^m)$ by $F_{*p}(\alpha'(0)) = (F \circ \alpha)'(0)$

Note) $F_{*p} = dF_p = DF_p$ and $F_* = dF = DF$ "the *differential* of F"

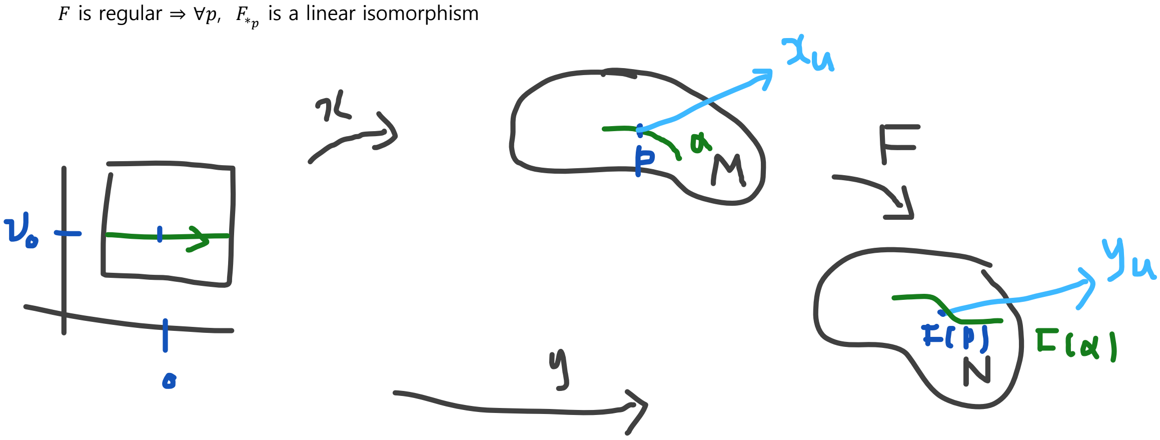
Note)

(1) $y = F(x) \Rightarrow F_*(x_u) = F_*(\alpha'(0)) = (F \circ \alpha)'(0) = y_u$

where $\alpha(u) = x(u, v_0) \Rightarrow (F \circ \alpha)(u) = (F(x))(u, v_0) = y(u, v_0)$

(2) $F : M \to N$ is *regular* if $\forall p$, F_{*p} is 1-1

F is regular $\Rightarrow \forall p$, F_{*p} is a linear isomorphism



★ Thm 5.4

${F_*}_p : T_p(M) \to T_{F(p)}(N)$ is a linear isomorphism at $p \in M$

$\Rightarrow \exists\ \mathcal{U}$ a nbd of p and $\mathcal{V}$ of $F(p)$, $F| : \mathcal{U} \to \mathcal{V}$ is a diffeomorphism by (1.7.10)

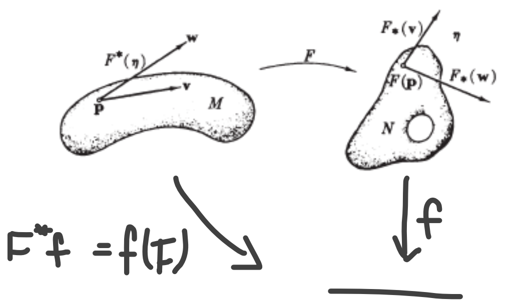
★ Def5.6 $(Pullback)$

Let $F : M \to N$ be a mapping of surfaces

The tangent map ${F_*}_p : T_p(M) \to T_{F(p)}(N)$

(1) ϕ is 1-form on N, define $F^* : \Lambda^1\big(T_p(N)\big) \to \Lambda^1\big(T_p(M)\big)$ by $(F^*\phi)(v) = \phi(F_* v)$ for $v \in T_p(M)$ "The fullback of ϕ by F"

(2) η is a 2-form on N, define the 2-form $F^*\eta$ on M by $(F^*\eta)(v,w) = \eta(F_*v, F_*w)$ for $v, w \in T_p(M)$



Note) A function f (a 0-form) on N, $F^*f = f(F)$ is the 0-form on M

★ Thm5.7

(1) $F^*(\xi + \eta) = F^*\xi + F^*\eta$

(2) $F^*(\xi \wedge \eta) = F^*\xi \wedge F^*\eta$

(3) $F^*(d\xi) = d(F^*\xi)$

(1)

Pf) ξ, η are 1-form and v is a tangent vector in M

$\Rightarrow F^*(\xi + \eta)(v) = (\xi + \eta)(F_*v) = \xi(F_*v) + \eta(F_*v) = (F^*\xi)(v) + (F^*\eta)(v)$

(2)

Pf) ξ, η are 1-form and v, w are tangent vectors in M

$\Rightarrow F^*(\xi \wedge \eta)(v,w) = (\xi \wedge \eta)(F_*v, F_*w)$

$= \xi(F_*v)\eta(F_*w) - \xi(F_*w)\eta(F_*v)$ by (4.4.3)

$= (F^*\xi)(v)(F^*\eta)(w) - (F^*\xi)(w)(F^*\eta)(v)$

$= (F^*\xi \wedge F^*\eta)(v,w)$

(3)

Pf) ξ is a 1-form on N

$d(F^*\xi)(x_u, x_v) = \frac{\partial}{\partial u}\big((F^*\xi)(x_v)\big) - \frac{\partial}{\partial v}\big((F^*\xi)(x_u)\big)$ by (4.4.4)

$= \frac{\partial}{\partial u}\Big(\xi\big(F_*(x_v)\big)\Big) - \frac{\partial}{\partial v}\Big(\xi\big(F_*(x_u)\big)\Big)$

$= \frac{\partial}{\partial u}\big(\xi(y_v)\big) - \frac{\partial}{\partial v}\big(\xi(y_u)\big)$ by p.163

$= d\xi(y_u, y_v)$ (by ex4.4.6)

$= d\xi(F_*x_u, F_*x_v)$

$= \big(F^*(d\xi)\big)(x_u, x_v)$

4.6 Integration of Forms

2020년 11월 19일 목요일 오전 11:38

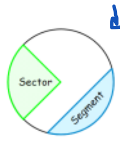
dimension 2

Two meanings:

- The smallest part of a circle made when it is cut by a line,

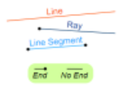
or

- Part of a line or curve.



활꼴

See: Segment (circle)



선분

Let $\alpha: [a, b] \to M$ be a curve segment in M

R이니까 dt 1개

$\alpha_*: T_t([a,b]) \to T_{\alpha(t)}(M)$ is a tangent map by 1.7.4

The pullback α^*: $\Lambda^1(T_{\alpha(t)}(M)) \to \Lambda^1(T_t([a,b]))$ by $\alpha^*\phi = f\,dt = (\alpha^*\phi)(U_1)\,dt$

and $(\alpha^*\phi)(U_1(t)) = \phi\left(\alpha_*(U_1(t))\right) = \phi(\alpha'(t))$ by 1.7.8

linear map

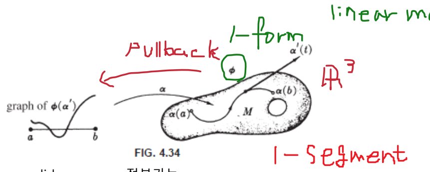


FIG. 4.34

surface → euclidean space : 적분가능

★ Def 1.7.4

$F_*: T(R^n) \to T(R^m)$ by $F_*(v) = \frac{d}{dt}F(p + tv)\,|_{t=0}$ is the *tangent map* of F

★ Cor 1.7.8

$F_*\left(U_j(p)\right) = \sum_i \frac{\partial f}{\partial x_j}(p) U_i\left(F(p)\right)$

$\alpha_*(U_1(t)) = \alpha'(t)$

$= (\alpha_1'(t), \alpha_2'(t), \alpha_3'(t))_{\alpha(t)}$

★ Def 6.1

$\phi \in \Lambda^1\ (T_{\alpha(t)}(M))$ and let $\alpha: [a,b] \to M$ be a curve segment.

Then the *integral* of ϕ *over* α is

기억 ϕ

$\int_\alpha \phi = \int_{[a,b]} \alpha^*\phi = \int_a^b \phi(\alpha'(t))dt$ *" Line integral "*

→ R 1-form

$\phi = df$

★ Thm 6.2 $f: M \to R$ and $\alpha: [a,b] \to M$ from $p = \alpha(a)$ to $q = \alpha(b)$.

Then $\int_\alpha df = f(q) - f(p)$

Pf)

$\int_\alpha df = \int_a^b df(\alpha'(t))dt$

But $df(\alpha'(t)) = \alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t)$ by 1.4.6

Thus $\int_\alpha df = \int_a^b \frac{d(f(\alpha))}{dt}(t)dt = f(\alpha(b)) - f(\alpha(a)) = f(q) - f(p)$

★ Lemma 4.6

$\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t) = (f \circ \alpha)'(t)$

$R = \{(u,v) \in R^2 \mid a \le u \le b, c \le v \le d\}$ and a *2-segment* is a differential map $x: R \to M$

$x_*: T_{(u,v)}(R) \to T_{x(u,v)}(M)$ is the tangent map

$$J(x) = \begin{pmatrix} D_1x_1 & D_2x_1 \\ D_1x_2 & D_2x_2 \\ D_1x_3 & D_2x_3 \end{pmatrix}$$

$= (x_u\ x_v)$

The pullback x^*: $\Lambda^2(T_{x(u,v)}(M)) \to \Lambda^2(T_{(u,v)}(R))$

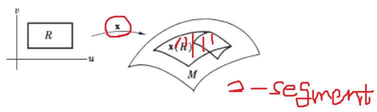
η is a 2-form on $M \Rightarrow x^*\eta = h\,du\,dv$ by Ex. 4.7

and $h = (x^*\eta)(U_1, U_2) = \eta(x_*U_1, x_*U_2) = \eta(x_u, x_v)$ by 1.7.8

★ Def 6.3

η is a 2-form on M and $x: R \to M$ is a 2-segment in M.

The *integral* of η *over* x is $\iint_x \eta = \iint_R x^*\eta = \int_a^b \int_c^d \eta(x_u, x_v)\,du\,dv$

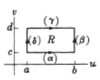


★ Def 6.4

$x: R \to M$ is a 2-segment in M. The *edge curves* of x are the curve segments

$\alpha(u) = x(u,c), \beta(v) = x(b,v), \gamma(u) = (u,d), \delta(v) = x(a,v)$.

The *boundary* ∂x of x is $\partial x = \alpha + \beta - \gamma - \delta$



Then $\int_{\partial x} \phi = \int_\alpha \phi + \int_\beta \phi - \int_\gamma \phi - \int_\delta \phi$

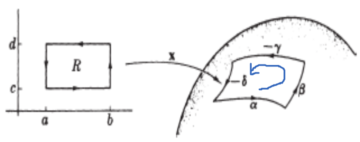


FIG. 4.38

★ Thm 6.5($Stokes' theorem$)

ϕ is a 1-form on M and $x: R \to M$ is a 2-segment $\iint_x d\phi = \int_{\partial x} \phi$

Pf) $\iint_R d\phi(x_u, x_v)\ dudv = \iint_R (\frac{\partial}{\partial u}(\phi x_v) - \frac{\partial}{\partial v}(\phi x_u))\ dudv$

$= \iint_R \frac{\partial g}{\partial u}\ dudv\ - \iint_R \frac{\partial f}{\partial v}\ dudv$ where $f = \phi(x_u)$ and $g = \phi(x_v)$

$\iint_R \frac{\partial g}{\partial u} dudv = \int_c^d I(v)\ dv$ where $I(v) = \int_a^b \frac{\partial g}{\partial u}(u, v)\, du = g(b, v) - g(a, v)$

$\Rightarrow \iint_R \frac{\partial g}{\partial u}\ dudv = \int_c^d g(b, v)\ dv - \int_c^d g(a, v)\ dv$ (2)

But $g(b, v) = \phi(x_v(b, v)) = \phi(\beta'(v))$ $\because \beta(v) = x(b, v)$

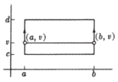


FIG. 4.39

$\Rightarrow \int_c^d g(b, v)\ dv = \int_c^d \phi(\beta'(v))\ dv = \int_\beta \phi$

$\Rightarrow \iint_R \frac{\partial g}{\partial u} dudv = \int_\beta \phi\ - \int_\delta \phi$ (3)

On the other hand, by $Fubini's\ theorem$


▶ 푸비니 정리

정리
- f가 직사각형 $R=\{(x,y)|a \le x \le b, c \le y \le d\}$ 상에서 연속이면 다음 식이 성립한다.
$\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$

$\iint_R \frac{\partial f}{\partial v}\ dudv = \iint_R \frac{\partial f}{\partial v}\ dvdu = \int_a^b I(u)\ du$ where $I(u) = \int_c^d \frac{\partial f}{\partial v}(u, v)\, dv = f(u, d) - f(u, c)$

$\Rightarrow \iint_R \frac{\partial f}{\partial v}\ dudv = \int_a^b f(u, d)\ du - \int_a^b f(u, c)\ du$ (2')

But $f(u, d) = \phi(x_u(u, d)) = \phi(\gamma'(u))$ $\because \gamma(u) = x(u, d)$

$\Rightarrow \int_a^b f(u, d)\ du = \int_a^b \phi(\gamma'(u))\ du = \int_\gamma \phi$

$\Rightarrow \iint_R \frac{\partial f}{\partial v} dudv = \int_\gamma \phi\ - \int_\alpha \phi$ (3')

Thus $\iint_R d\phi(x_u, x_v)\ dudv = (\int_\beta \phi\ - \int_\delta \phi\ ) - (\int_\gamma \phi\ - \int_\alpha \phi\ ) = \int_{\partial x} \phi$

★ Lem 6.6

Let $\alpha(h)$: $[a, b] \to [c, d] \to M$ be a reparametrization and ϕ be a 1-form on M

(1) If h is orientation-preserving, then $\int_{\alpha(h)} \phi\ = \int_\alpha \phi$

(2) If h is orientation-reversing, then $\int_{\alpha(h)} \phi = -\int_\alpha \phi$

Pf) (1) By 1.4.5 $(\alpha(h))'(t) = \left(\frac{dh}{dt}\right)(t)\alpha'(h(t))$

$\int_{\alpha(h)} \phi = \int_a^b \phi\left((\alpha(h))'(t)\right)\ dt = \int_a^b \phi\left(\alpha'(h(t))\right) \left(\frac{dh}{dt}\right)(t)\ dt$

Let $u = h(t)$

$\int_{\alpha(h)} \phi = \int_c^d \phi(\alpha'(u))\, du = \int_\alpha \phi$

Note) Given $\beta : [t_0, t_1] \to M$.

Define an orientation-reversing reparametrization of β

$(-\beta) : [t_1, t_0] \to M$ by $(-\beta)(t) = \beta(t_0 + t_1 - t)$

$\Rightarrow \int_{-\beta} \phi = -\int_\beta \phi$

4.6^* Example

2020년 11월 21일 토요일   오후 6:26

Consider a vector field V on M as a forse field, and a curve $\alpha : [a,b] \to M$ as a description of a moving particle, with $\alpha(t)$ in position at time t.

What is the total amount of work W done by the force on the particle as it move from $p = \alpha(a)$ to $q = \alpha(b)$?

For small Δt, the subsegment of α from $\alpha(t)$ to $\alpha(t+\Delta t)$ is approximated by the straight line segment $\Delta t\, \alpha'(t)$

$\because \frac{\alpha(t+\Delta t) - \alpha(t)}{\Delta t} \to \alpha'(t)$

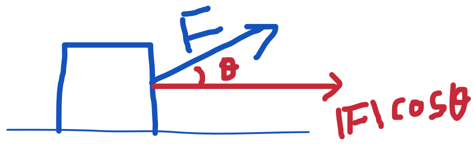
Work is done on the particle only by the *component* of *force* tangent to α, $V(\alpha) \cdot T = V(\alpha) \cdot \frac{\alpha'}{|\alpha'|} = |V(\alpha)|\cos\theta$,
where θ is the angle between $V((\alpha(t))$ and $\alpha'(t)$

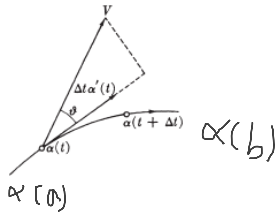
$|\alpha(t+\Delta t) - \alpha(t)| \to |\alpha'(t)|\, \Delta t$

Thus the work done by the force during time Δt is approximately the force times the distance $|\alpha'(t)|\Delta t$

힘 × 거리 = W

$\lim_{\Delta t \to 0} \sum \left(V(\alpha(t)) \cdot \frac{\alpha'(t)}{|\alpha'(t)|} \right) |\alpha'(t)| \Delta t \;\Rightarrow\; W = \int_a^b V(\alpha(t)) \cdot \alpha'(t)\, dt$





Define the 1-fom ϕ by $\phi(w_p) = w \cdot V(p)$ dual to V

$\Rightarrow \phi(\alpha'(t)) = \alpha'(t) \cdot V(\alpha(t))$

$= \int_a^b \phi(\alpha'(t))\, dt$

$\Rightarrow W = \int_\alpha \phi$

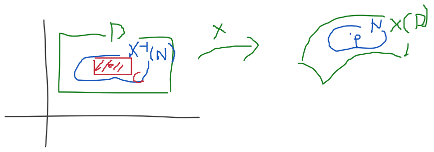
https://en.wikipedia.org/wiki/Force_field_(physics)

물체에 작용하는 힘이 물체의 위치 및 속도 등을 통해 임의적으로 결정될 때, 넓은 뜻으로 힘의 공간을 말한다. 즉 만유인력이나 정전기력과 같은 경우, 단위 질량이나 단위 전하를 가진 입자에 작용하는 힘은 공간의 위치로 인하여 정해진다. 하전된 입자가 전자기장에 의해 받는 힘은 이와 같이 간단한 성질을 가지지는 못하지만 공간 좌표 및 시간의 함수인 전자기장의 세기와 입자의 속도로써 정해진다. 이 외에 핵력 역시 중간 자기장에 의한 역장으로 볼 수 있다.

4.7 Topological Properties of surfaces

2020년 11월 26일 목요일 오후 3:11

C = R(xy)



★ Def 7.1

M is connected if $\forall\ p, q \in M$, $\exists\ \alpha$ (a curve segment) in M from p to q

path

Ex. 9(a) M is locally path connected

$\because$ *Given* $p \in M$ $\Rightarrow \exists\ x$ *(a proper patch),* $x(D)$ *is a path connected nbd of* p

Thus M is connected $\underset{T.25.5}{\Longrightarrow}$ M is path connected

topology

Theorem 25.5. *If X is a topological space, each path component of X lies in a component of X. If X is locally path connected, then the components and the path components of X are the same.*

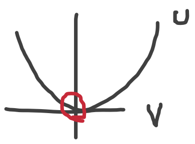
topology

Definition. A space X is said to be ***locally connected at x*** if for every neighborhood U of x, there is a connected neighborhood V of x contained in U. If X is locally connected at each of its points, it is said simply to be ***locally connected***. Similarly, a space X is said to be ***locally path connected at x*** if for every neighborhood U of x, there is a path-connected neighborhood V of x contained in U. If X is locally path connected at each of its points, then it is said to be ***locally path connected***.

separation

topology

Definition. Let X be a topological space. A ***separation*** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be ***connected*** if there does not exist a separation of X.



connected

Definition. Given points x and y of the space X, a ***path*** in X from x to y is a continuous map $f : [a, b] \to X$ of some closed interval in the real line into X, such that $f(a) = x$ and $f(b) = y$. A space X is said to be ***path connected*** if every pair of points of X can be joined by a path in X.

★ Lemma 7.2

M is cpt $\Leftrightarrow \exists\ x_i$ (a 2-segment), $M = \cup_{i=1}^{n} Im(x_i)$

Pf) $\Rightarrow$) Given $p \in M$

$\exists x$ (a proper patch), $\exists N(p)$ (a nbd of p in M), $N(p) \subset x(D)$and $x(u, v) = p$

$(u, v) \in x^{-1}\left(N(p)\right) \subset D$

$\Rightarrow \exists\ (u - \epsilon, u + \epsilon) \times (v - \epsilon, v + \epsilon) \subset x^{-1}\left(N(p)\right)$

Let $C = \left[u - \frac{\epsilon}{2}, u + \frac{\epsilon}{2}\right] \times \left[v - \frac{\epsilon}{2}, v + \frac{\epsilon}{2}\right]$

$\Rightarrow x|_C : C \to M$ is a 2-segment with $x(Int\ C)$ is open in M and $p \in x(Int\ C) \subset x(C) = Im(x)$

$\Rightarrow \{\ x_i(IntC_i) \mid x_i|_{C_i}$ is a 2-segment in M $\}$ is an open covering of M

$\Leftarrow$) Since $Im(x)$ is cpt for any 2-segment $x : C \to M$

$\Rightarrow$ $M = \cup_{i=1}^{n} Im(x_i)$ is cpt

★ Lemma 7.3

If $C \subset M$ is cpt and $f : C \to R$ is cts, then f takes a maximum

$\subset M$

$* \ f(x) : C \xrightarrow{x} c(M) \xrightarrow{f} \mathbb{R}$

Theorem 27.4 (Extreme value theorem). *Let $f : X \to Y$ be continuous, where Y is an ordered set in the order topology. If X is compact, then there exist points c and d in X such that $f(c) \le f(x) \le f(d)$ for every $x \in X$.*

The extreme value theorem of calculus is the special case of this theorem that occurs when we take X to be a closed interval in $\mathbb{R}$ and Y to be $\mathbb{R}$.

★ Def 7.4 M is orientable if $\exists\mu$ (a diff. 2-form on M), $\mu \neq 0$ at every point $p \in M$

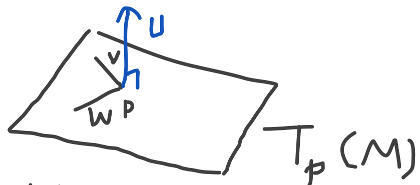
★ Note) R^2 is orientable $\because$ $dudv(U_1, U_2) = 1$ in EX.4.4.(2) $du(U_1) = 1$ and $du(U_2) = 0$

δ_{ij}

$$dudv\left(U_1(p), U_2(p)\right) \underset{4.3}{=} du\left(U_1(p)\right) dv\left(\overset{=1}{U_2(p)}\right) + du\left(U_2(p)\right) dv\left(\overset{=0}{U_1(p)}\right) = 1$$

$\sigma = (1, 2)$

★ Pro 7.5 M is orientable $\Leftrightarrow \exists\ U$ (a unit normal vector field on M)



Pf) $\Leftarrow$) If $v, w \in T_p(M)$ are linearly independent,

define $\mu(v,w) = U(p)\cdot v \times w \ \neq 0 \quad \because$ they are linearly independent.

Note) v_i are l.ind. $\Rightarrow v_1 \cdot v_2 \times v_3 = \det(v_1, v_2, v_3) \neq 0$

$\because$ otherwise $\det\square = 0$, a controdiction to the volume element of R^3

$v' \times w' = \begin{vmatrix} a & b \\ c & d \end{vmatrix} v \times w$

$\mu(v', w') = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \mu(v,w)$

성질 비슷

2-form

외적

$\Rightarrow$) Suppose μ is a nonvanishing 2-form on M

If $v, w \in T_p(M)$ are linearly independent, then $\mu(v,w) \neq 0$

Define a vector field on M, $\quad Z(p) = \frac{v \times w}{\mu(v,w)} \neq 0$

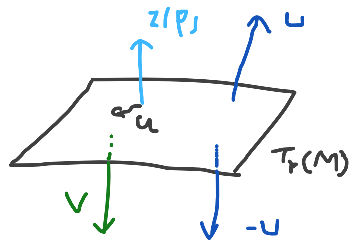
$\Rightarrow Z(p)$ is independent of the choice of v, w

$\because$ By 4.2, if $v' = av + bw$ and $w' = cv + dw$, then

$\frac{v' \times w'}{\mu(v',w')} = \frac{v \times w}{\mu(v,w)}$

And $Z(p)$ is normal to $M \quad \because Z(p)\cdot u = 0$ for every $u \in T_p(M)$

Take $U = \frac{Z}{|Z|}$

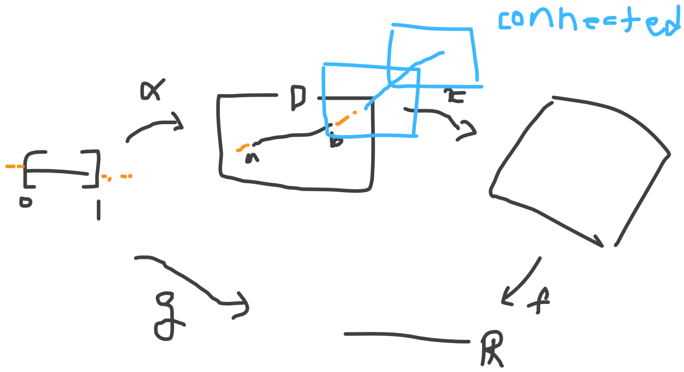


★ Let V be a unit normal on M $\qquad V\cdot U = |V||U|\cos\theta$, $\theta = 0$ or π

$\Rightarrow \forall p,\ V(p)\cdot U(p) = \pm 1 \Rightarrow V = \pm U$ by Ex.4

Ex.4

(a) Given $p \in M$, $\exists$ a proper patch $x: D \to M$



$\Rightarrow df(x_u) = D_1(f(x))$ and $df(x_v) = D_2(f(x))$ where $f(x)$ is a coordinate expession and Ex. 4.3(4)

Let $h = f(x)$, then $h'(u,v) = (D_1 h(u,v), D_2 h(u,v)) = 0$ for every $(u,v) \in D$,

Given $a, b \in D$ and $\alpha(t) = (1-t)a + tb \in D$ by convex

Define $g: [0,1] \to R$ by $g(t) = h(\alpha(t))$

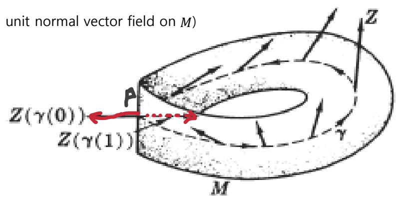
$\Rightarrow g'(t) = h'(\alpha(t))\cdot \alpha'(t) = 0$ for all t

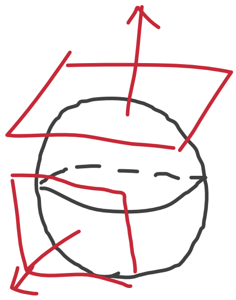
$\Rightarrow g$ is constant in $[0,1]$, thus $g(0) = g(1)$ by mean value therem and $g(0) = \lim_{t\to 0^+}\square\ g(t)$

$\Rightarrow f(x(a)) = h(a) = g(0) = f(x(b)) \qquad$ Note) $a \neq b \Rightarrow x(a) \neq x(b)$

$\Rightarrow f$ is constant in $x(D)$ $\qquad \xRightarrow{\text{connected}} f$ is constant in M

$\nexists\ U$ (a unit normal vector field on M)





$\exists\ U$ (a unit normal vector field on M)

해석기하학 4장 p.82~84(2018년에 안 배운 부분)

2020년 12월 1일 화요일 오후 4:32

$\Lambda^k(V)$ ($V = \mathbb{R}^n$) 의 차원 $\binom{n}{k}$

0보다 크다

4,1-12

2020년 11월 13일 금요일 오전 9:53

4.6 $\omega(w_1,\dots,w_n) = \overset{\pm 1}{\det(a_{ij})} \cdot \omega(v_1,\dots,v_n)$

Let $\omega \neq 0 \in \Lambda^n(V)$ for $\dim(V) = n$

Let $\mathcal{B} = \{\, v_1,\dots,v_n \mid v_1,\dots,v_n$ is a basis for $V \,\}$

Define an equivalence relation on $\mathcal{B}$ by 동치

$v_1,\dots,v_n \sim w_1,\dots,w_n \Leftrightarrow [\; w_i = \sum a_{ij} v_j \Rightarrow \det(a_{ij}) > 0 \;]$

$\Leftrightarrow [\; \omega(w_1,\dots,w_n) > 0 \Rightarrow \omega(v_1,\dots,v_n) > 0 \;]$ by 4-6

$[e_1, e_2, e_3] = [e_2, e_3, e_1]$

$-[e_2, e_1, e_3]$

Denote the equivalence class $[v_1,\dots,v_n]$ of $v_1,\dots,v_n$ " an *orientation* for V"

$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -|\,\cdot\,| = -1$

Thus $\mathcal{B}/\sim \; = \{\, [v_1,\dots,v_n],\, -[v_1,\dots,v_n] \,\}$ 2가지

Denote) $[e_1,\dots,e_n]$ is the *usual orientation* for $\mathbb{R}^n$ 표준

Note) $\omega \in \Lambda^n(\mathbb{R}^n)$ with $\omega(e_1,\dots,e_n) = 1$

$\Rightarrow \forall\, w_i = \sum a_{ij} e_j$, $\omega(w_1,\dots,w_n) = \det(a_{ij}) = \det(w_1,\dots,w_n)$ by 4-6

$\Rightarrow \omega = \det$. $\quad \det \in \Lambda^n(\mathbb{R}^n)$

$T_p(M)$ · dim 2

$\Lambda^2(T_p(M))$ · dim 1

$M(p)$ $\quad dudv$

$\det \in \Lambda^n(\mathbb{R}^n)$

volume element

Suppose an inner product T for V is given

If $v_1,\dots,v_n$ and $w_1,\dots,w_n$ are orthonormal w.r.t. T, and $A = (a_{ij})$ for $w_i = \sum a_{ij} v_j$, then

$\delta_{ij} = T(w_i, w_j) = \sum_{k,\ell=1}^{n} a_{ik} a_{j\ell}\, T(v_k, v_\ell) = \sum_{k=1}^{n} a_{ik} a_{jk}$

$\Rightarrow I = (\delta_{ij}) = A \cdot A^T \Rightarrow \det A = \pm 1$

If $\omega \in \Lambda^n(V)$ with $\omega(v_1,\dots,v_n) = \pm 1$, then $\omega(w_1,\dots,w_n) = \pm 1$ by 4-6

f^*T

Claim) $\mu = [v_1,\dots,v_n]$ is an orientation for V and $v_1,\dots,v_n$ is orthonormal w.r.t. T

$\Rightarrow \exists!\; \omega \in \Lambda^n(V)$ with $\omega(v_1,\dots,v_n) = 1$ $\quad \exists!\; w \in \Lambda^n(V)$, $w(v_1,\dots,v_n) = 1$

$\because$ ~~(Existence)~~ Define a linear transformation $f : V \to \mathbb{R}^n$ by $f(v_i) = e_i$

$\Rightarrow f^* : \Lambda^n(\mathbb{R}^n) \to \Lambda^n(V)$ is the pullback

$\Rightarrow (f^*\det)(v_1,\dots,v_n) = \det(f(v_1),\dots,f(v_n)) = \det(e_1,\dots,e_n) = 1$

(*Uniqueness*) If $\omega' \in \Lambda^n(V)$, $\omega'(v_1,\dots,v_n) = 1$ and $w_i = \sum a_{ij} v_j$, then by 4-6

$\omega'(w_1,\dots,w_n) = \det(a_{ij}) \cdot \omega'(v_1,\dots,v_n) = \det(a_{ij}) \cdot \omega(v_1,\dots,v_n) = \omega(w_1,\dots,w_n)$ $\quad \det(e_1,\dots,e_n) = 1$

$\Rightarrow \omega' = \omega$

Definition $\omega = f^*\det$ is called the *volume element* of V determined by T and μ

Note) (1) $\det$ is the volume element of $\mathbb{R}^n$ determind by $\langle , \rangle$ and $[e_1,\dots,e_n]$

(2) $|\det(v_1,\dots,v_n)|$ is the volume of the parallelepiped spanned by the line segments from 0 to each of $v_1,\dots,v_n$

https://simple.wikipedia.org/wiki/Parallelepiped

$\Lambda^k(V) \subset \mathcal{T}^k(V)$

$T \in \mathcal{T}^k(V)$

$$\mathrm{Alt}(T)(v_1,\dots,v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}\sigma \cdot T(v_{\sigma(1)},\dots,v_{\sigma(k)}),$$

4-3 Theorem

(1) *If* $T \in \mathfrak{J}^k(V)$, *then* $Alt(T) \in \Lambda^k(V)$.
(2) *If* $\omega \in \Lambda^k(V)$, *then* $Alt(\omega) = \omega$.
(3) *If* $T \in \mathfrak{J}^k(V)$, *then* $Alt(Alt(T)) = Alt(T)$.

4-4 Theorem

(1) *If* $S \in \mathfrak{J}^k(V)$ and $T \in \mathfrak{J}^l(V)$ *and* $Alt(S) = 0$, *then*

$$Alt(S \otimes T) = Alt(T \otimes S) = 0.$$

(2) $$Alt(Alt(\omega \otimes \eta) \otimes \theta) = Alt(\omega \otimes \eta \otimes \theta) = Alt(\omega \otimes Alt(\eta \otimes \theta)).$$

(3) *If* $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^l(V)$, *and* $\theta \in \Lambda^m(V)$, *then*

$\omega \wedge \eta \in \Lambda^{k+l}(V)$ by

$$\omega \wedge \eta = \frac{(k+l)!}{k!\,l!} \mathrm{Alt}(\omega \otimes \eta).$$

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{(k+l+m)!}{k!\,l!\,m!} Alt(\omega \otimes \eta \otimes \theta).$$

4-6 Theorem. *Let* $v_1, \ldots, v_n$ *be a basis for* V, *and let* $\omega \in \Lambda^n(V)$. If $w_i = \sum_{j=1}^n a_{ij} v_j$ *are* n *vectors in* V, *then*

$$\omega(w_1, \ldots, w_n) = \underbrace{\det(a_{ij})}_{\pm 1} \cdot \omega(v_1, \ldots, v_n).$$

Proof. Define $\eta \in \mathfrak{J}^n(\mathbf{R}^n)$ by

$$\eta((a_{11}, \ldots, a_{1n}), \ldots, (a_{n1}, \ldots, a_{nn})) = \omega(\underbrace{\Sigma a_{1j} v_j}_{w_1}, \ldots, \underbrace{\Sigma a_{nj} v_j}_{w_n}).$$

$\nearrow$ dim 1

Clearly $\eta \in \Lambda^n(\mathbf{R}^n)$ so $\eta = \lambda \cdot \det$ for some $\lambda \in \mathbf{R}$ and $\lambda = \eta(e_1, \ldots, e_n) = \omega(v_1, \ldots, v_n)$. ∎

Theorem 4-6 shows that a non-zero $\omega \in \Lambda^n(V)$ splits the bases of V into two disjoint groups, those with $\omega(v_1, \ldots, v_n) > 0$ and those for which $\omega(v_1, \ldots, v_n) < 0$; if $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are two bases and $A = (a_{ij})$ is defined by $w_i = \Sigma a_{ij}v_j$, then $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are in the same group if and only if $\det A > 0$. This criterion is independent of ω and can always be used to divide the bases of V into two disjoint groups. Either of these two groups is called an **orientation** for V. The orientation to which a basis $v_1, \ldots, v_n$ belongs is denoted $[v_1, \ldots, v_n]$ and the other orientation is denoted $-[v_1, \ldots, v_n]$. In $\mathbf{R}^n$ we define the **usual orientation** as $[e_1, \ldots, e_n]$.

The fact that $\dim \Lambda^n(\mathbf{R}^n) = 1$ is probably not new to you, since det is often defined as the unique element $\omega \in \Lambda^n(\mathbf{R}^n)$ such that $\omega(e_1, \ldots, e_n) = 1$. For a general vector space V there is no extra criterion of this sort to distinguish a particular $\omega \in \Lambda^n(V)$. Suppose, however, that an inner product T for V is given. If $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ are two bases which are orthonormal with respect to T, and the matrix $A = (a_{ij})$ is defined by $w_i = \Sigma_{j=1}^n a_{ij}v_j$, then

$$\delta_{ij} = T(w_i, w_j) = \sum_{k,l=1}^{n} a_{ik}a_{jl}T(v_k, v_l)$$
$$= \sum_{k=1}^{n} a_{ik}a_{jk}.$$

In other words, if A^{T} denotes the transpose of the matrix A, then we have $A \cdot A^{\mathrm{T}} = I$, so $\det A = \pm 1$. It follows from Theorem 4-6 that if $\omega \in \Lambda^n(V)$ satisfies $\omega(v_1, \ldots, v_n) = \pm 1$, then $\omega(w_1, \ldots, w_n) = \pm 1$. If an orientation μ for V has also been given, it follows that there is a unique $\omega \in \Lambda^n(V)$ such that $\omega(v_1, \ldots, v_n) = 1$ whenever $v_1, \ldots, v_n$ is an orthonormal basis such that $[v_1, \ldots, v_n] = \mu$. This unique ω is called the **volume element** of V, determined by the inner product T and orientation μ. Note that det is the volume element of $\mathbf{R}^n$ determined by the usual inner product and usual orientation, and that $|\det(v_1, \ldots, v_n)|$ is the volume of the parallelipiped spanned by the line segments from 0 to each of $v_1, \ldots, v_n$.

표준

평행육면체 부피

1.1.6 비기

6. If r, ϑ, z are the cylindrical coordinate functions on $\mathbf{R}^3$, then $x = r\cos\vartheta$, $y = r\sin\vartheta$, $z = z$. Compute the *volume element $dx\,dy\,dz$ of $\mathbf{R}^3$* in cylindrical coordinates. (That is, express $dx\,dy\,dz$ in terms of the functions r, ϑ, z, and their differentials.)

$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial z} dz = \cos(\theta)\, dr - r \sin(\theta) d\theta$
$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial z} dz = \sin(\theta)\, dr + r\cos(\theta)\, d\theta$
$dz = dz$

Therefore
$dxdydz = dx \wedge dy \wedge dz$
$= (\cos(\theta)\, dr - r\sin(\theta) d\theta) \wedge (\sin(\theta)\, dr + r\cos(\theta)\, d\theta) \wedge dz$
$= (r\cos^2(\theta)\, dr d\theta - r\sin^2(\theta)\, d\theta dr) \wedge dz$
$= (r(\cos^2(\theta) + \sin^2(\theta)) dr d\theta) \wedge dz$
$= r\, dr d\theta dz$

비기 2.1.8

8. Prove: The volume of the parallelepiped with sides $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is $\pm \mathbf{u} \bullet \mathbf{v} \times \mathbf{w}$ (Fig. 2.5). (*Hint:* Use the indicated unit vector $\mathbf{e} = \mathbf{v} \times \mathbf{w} / \| \mathbf{v} \times \mathbf{w} \|$.)

Let $\| v \times w \| = \| v \| \| w \| \sin(\theta) = s$
$\Rightarrow \| v \times w \|$ is the area of the parallelogram with sides v and w
$u \cdot v \times w = |u||v \times w| \cos(\alpha)$
$= sh \quad (\because h = |u|\cos(\alpha))$
Thus
$|u \cdot v \times w|$ is the volume of the parallelepied withe sides u, v, w