Ch1. Preliminaries: mixtures and Markov chains

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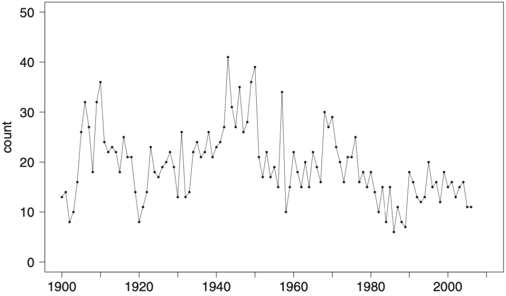
Introduction

- **Hidden Markov models (HMMs)** are models in which the distribution that generates an observation depends on the state of an unobserved Markov process.
- HMMs provide flexible general-purpose models for time series.
- This chapter is about
  1. brief and informal introduction of **HMMs**.
  2. **finite mixture distribution** that is marginal distribution of HMM.
  3. **Markov chains** which provide the underlying 'parameter process' of HMM .

Example: the series of annual counts of earthquakes for 1900-2006

13	14	8	10	16	26	32	27	18	32	36	24	22	23	22	18	25	21	21	14
8	11	14	23	18	17	19	20	22	19	13	26	13	14	22	24	21	22	26	21
23	24	27	41	31	27	35	26	28	36	39	21	17	22	17	19	15	34	10	15
22	18	15	20	15	22	19	16	30	27	29	23	20	16	21	21	25	16	18	15
18	14	10	15	8	15	6	11	8	7	18	16	13	12	13	20	15	16	12	18
15	16	13	15	16	11	11													

Table: Number of major earthquakes in the world, 1900–2006



Example: the series of annual counts of earthquakes for 1900-2006

- For this series, the application of standard models such as ARMA models would be not appropriate.
  $\rightarrow$ such models are based on the normal distribution.

- The usual model for unbounded counts is the Poisson distribution
  $\rightarrow$ but the series displays overdispersion and strong positive serial dependence.

- **HMMs** can accommodate both overdispersion and serial dependence.

Introduction of Hidden Markov model

- Attractive features of HMMs
  - their general mathematical tractability.
  - the likelihood is relatively straightforward to compute.
- Introduce the **basic HMM**: is univariate and based on a homogenous Markov chain and has neither trand nor seasonal variation.
- Ignore information that may be available on covariates

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Definition and properties

- Consider again the series of earthquake counts.
- A standard model for unbounded counts is the Poisson distribution, with its probability function $p(x) = e^{-\lambda}\lambda^x/x!$.
- However, for the earthquakes series the sample variance, $s^2 \approx 52$ is much larger than the sample mean, $\bar{x} \approx 19$, which indicates strong overdispersion.

Definition and properties

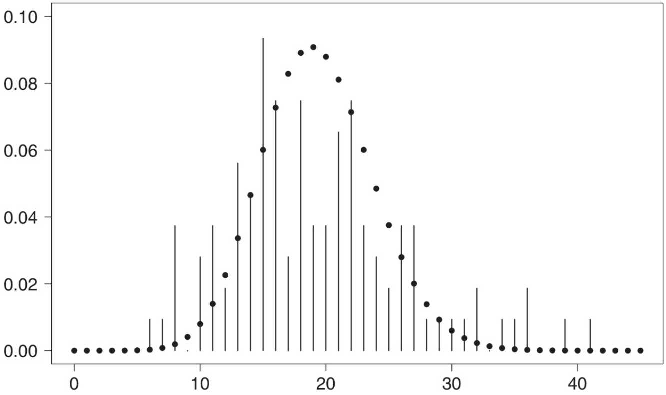

Figure: Major earthquakes, 1900-2006: bar plot of relative frequencies of counts, and fitted Poisson distribution.

Definition and properties

- One method of dealing with overdispersed observations is to use a mixture model.
- **Mixture model** is a probabilistic model for representing the presence of subpopulations within an overall population
- the population may consist of unobserved groups, each having a distinct distribution for the observed variable.
- That provides a principled approach to modeling such complex data.

Example: the series of annual counts of earthquakes for 1900-2006

- Suppose that each count in the earthquakes series is generated by one of two Poisson distributions, with means λ_1 and λ_2.
- The choice of mean is determined by some other random mechanism which we call the **parameter process**.
- Suppose also that λ_1 is selected with probability δ_1 and λ_2 with probability $\delta_2 = 1 - \delta_1$.
- If the parameter process is a series of independent random variables, the term 'independent mixture'.

Definition and properties

- In general, an independent mixture distribution consists of a finite number m, of component distributions and a 'mixing distribution' which selects from these components.

- In the case of two components, the mixture distribution depends on two probability or density functions:

component	1	2
probability or density function	$p_1(x)$	$p_2(x)$

- To specify the component, one needs a discrete random variable C which performs the mixing:

$$C = \begin{cases} 1 & \textit{with } \delta_1 \\ 2 & \textit{with probability } \delta_2 = 1 - \delta_2 \end{cases}$$

Definition and properties

- Let $\delta_1, ..., \delta_m$ denote the probabilities assigned to the different components, and let $p_1, ..., p_m$ denote their probability or density functions.
- Let X denote the random variable which has the mixture distribution.
- In discrete case, the probability function of X is given by

$$\begin{aligned} p(x) &= \sum_{i=1}^{m} Pr(X = x|C = i)Pr(C = i) \\ &= \sum_{i=1}^{m} \delta_i p_i(x) \end{aligned}$$

Definition and properties

- The expectation of the mixture can be given in terms of the expectations of the component distributions.
- Letting Y_i denote the random variable with probability function p_i,

$$E(X) = \sum_{i=1}^{m} Pr(C = i)E(X|C = i) = \sum_{i=1}^{m} \delta_i E(Y_i)$$

- The same result holds for a mixture of continuous distributions.

Parameter estimation

- The estimation of the parameters of a mixture distribution is often performed by maximum likelihood (ML).
- The likelihood of a mixture model with m components is given by,

$$L(\theta_1, ..., \theta_m, \delta_1, ..., \delta_m | x_1, ..., x_m) = \prod_{j=1}^{n} \sum_{i=m}^{m} \delta_i p_i(x_j, \theta_i) \quad (1)$$

- $\theta_1, ..., \theta_m$: the parameter vectors of the component distributions
  $\delta_1, ..., \delta_m$: the mixing parameters, totalling 1
  $x_1, ..., x_m$: n observations

Parameter estimation

- In the case of component distributions each specified by one parameter, $2m - 1$ independent parameters have to be estimated.
- Except perhaps in special cases, analytic maximization of such a likelihood is not possible.

Parameter estimation

- Suppose that $m = 2$ and the two components are Poisson-distributed with mean λ_1 and λ_2.
- Let δ_1 and $\delta_2 = 1 - \delta_1$ be the mixing parameters.
- The mixture distribution p is then given by

$$p(x) = \delta_1 \frac{\lambda_1^x e^{-\lambda_1}}{x!} + \delta_2 \frac{\lambda_2^x e^{-\lambda_2}}{x!}$$

- Since $\delta_2 = 1 - \delta_1$, there are only three parameters to be estimated: $\lambda_1, \lambda_2, \delta_1$
- The likelihood is

$$L(\lambda_1, \lambda_2, \delta_1 | x_1, ..., x_m) = \prod_{i=1}^{n} (\delta_1 \frac{\lambda_1^{x_i} e^{-\lambda_1}}{x_i!} + (1 - \delta_1) \frac{\lambda_2^{x_i} i e^{-\lambda_2}}{x_i!})$$

Parameter estimation

- The analytic maximization of L with respect to $\lambda_1, \lambda_2, \delta_1$ would be awkward, as *L* is the product of *n* factors, each of which is a sum.
- Taking the logarithm and then differentiating does not greatly simplify matters either.
- Therefore parameter estimation is more conveniently carried out by **direct** numerical maximization of the likelihood or **using EM algorithm**.

Unbounded likelihood in mixtures

- The likelihood of mixtures of continuous distributions is unbounded.
- For instance, in the case of a mixture of normal distributions, the likelihood becomes **arbitrarily large** if one sets a component mean equal to one of the observations and the corresponding variance to tend to zero.
- If the likelihood is thus unbounded, the ML estimates simply 'do not exist'.

Unbounded likelihood in mixtures

- Thus, replace each density value in a likelihood by the probability of the interval corresponding to the recorded value.
- In the context of independent mixtures one replaces the expression (1) for the likelihood by the discrete likelihood

$$L = \prod_{j=1}^{n} \sum_{i=m}^{m} \delta_i \int_{a_j}^{b_j} p_i(x, \theta_i) dx \tag{2}$$

where the interval (a_j, b_j) consists of those values which, if observed, would be recorded as x_j .

Examples of fitted mixture models: Poisson distribution

Model	i	δ_i	λ_i	$-logL$	Mean	Variance
$m = 1$	1	1.000	19.364	391.9189	19.364	19.364
$m = 2$	1	0.676	15.777	**360.3690**	19.364	46.182
	2	0.324	26.840			
$m = 3$	1	0.278	12.736	356.8489	19.364	51.170
	2	0.593	19.785			
	3	0.130	31.629			
$m = 4$	1	0.093	10.584	356.7337	19.364	51.638
	2	0.354	15.528			
	3	0.437	20.969			
	4	0.116	32.079			
observation					19.364	51.573

Table: Poisson independent mixture models fitted to the earthquakes series

Examples of fitted mixture models: Poisson distribution

- There is a very clear improvement in likelihood resulting from the addition of a second component.
- There is very little improvement from addition of a fourth apparently insufficient to justify the additional two parameter.
- It is clear that the mixtures fit the observations much better than does a single Poisson distribution, and visually the three- and four-state models seem adequate.

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Definition

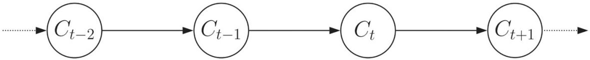
- Our treatment is restricted to those few aspects of discrete time Markov chains that we need.
- A sequence of discrete random variables $\{C_t : t \in \mathbb{N}\}$ is said to be a (discrete-time) **Markov chain** (MC) if, for all $t \in \mathbb{N}$, it satisfies the **Markov property**

$$Pr(C_{t+1}|C_t, ..., C_1) = Pr(C_{t+1}|C_t)$$

- To conditioning only on the most recent value C_t.

Definition

- The random variables $\{C_t\}$ are displayed in the following directed graph in which the future are dependent **only through the present**.



- Define **transition probabilities**:

$$\gamma_{ij}(t) = Pr(C_{s+t} = j | C_s = i)$$

- If these probabilities do not depend on s, the Markov chain is called **homogeneous**.
- We shall assume that the Markov chain under discussion is homogeneous.

Definition

- The matrix $\Gamma(t)$ is defined as the matrix with (i, j) element $\gamma_{ij}(t)$.
- An important property of all finite state-space homogeneous Markov chains is that they satisfy the **Chapman-Kolmogorov equations**: $\Gamma(t + u) = \Gamma(t)\Gamma(u)$.
- For all $t \in \mathbb{N}, \quad \Gamma(t) = \Gamma(1)^t$
- The matrix of t-step transition probabilities is the tth power of $\Gamma(1)$, the matrix of one-step transition probabilities.

Definition

- The matrix $\Gamma(1)$ which will be abbreviated as Γ, is a square matrix of probabilities with row sums equal to 1:

$$\Gamma = \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1m} \\ \vdots & \ddots & \vdots \\ \gamma_{m1} & \cdots & a_{mm} \end{pmatrix}$$

  where m denotes the number of states of the Markov chain.

- The row sums are equal to 1 can be written as $\Gamma\mathbf{1}' = \mathbf{1}'$

- We shall refer to Γ as the (one-step) **transition probability matrix** (t.p.m.).

Definition

- The unconditional probabilities $Pr(C_t = j)$ of a Markov chain being in a given state at a given time t are often of interest.

- Denote these by the row vector

$$u(t) = \ (Pr(C_t = 1), \cdots, Pr(C_t = m)), \ t \in \mathbb{N}$$

- We refer to $u(1)$ as the initial distribution of the Markov chain.

- The distribution at time $t+1$ from that at t we postmultiply by the transition probability matrix Γ:

$$u(t+1) = u(t)\Gamma. \tag{3}$$

Stationary distributions

- A Markov chain with transition probability matrixΓ is said to have **stationary distribution** δ if $\delta\Gamma = \delta$ and $\delta\mathbf{1}' = 1$.
- Since $u(t+1) = u(t)\Gamma$, a Markov chain started from its stationary distribution will continue to have that distribution at all subsequent time points.
- An irreducible (homogeneous, discrete-time, finite state-space) Markov chain has a unique, strictly positive, stationary distribution.
- Always assume aperiodicity and irreducibility of stationary Markov chains.

Autocorrelation function

- To compare the autocorrelation function (ACF) of a hidden Markov model with that of its underlying Markov chain $\{C_t\}$, on the states $1, 2, ..., m$.
- Assume that these states are quantitative and not merely categorical, and stationary and irreducible.
- Then, for all non-negative integers k,

$$Cov(C_t, C_{t+k}) = \delta V \Gamma^k v' - (\delta v')^2 \tag{4}$$

Autocorrelation function

- If Γ is diagonalizable, and its eigenvalues (other than 1) are denoted by $\omega_2, \omega_3, \cdots, \omega_m$, then Γ can be written as

$$\Gamma = U \Omega U^{-1}$$

where Ω is $diag(1, \omega_2, \omega_3, \cdots, \omega_m)$ and the columns of U are corresponding right eigenvectors of Γ.

- Then, for all non-negative integers k,

$$\begin{aligned} Cov(C_t, C_{t+k}) &= \delta V U \Omega^k U^{-1} v' - (\delta v')^2 \\ &= a \Omega^k b' - a_1 b_1 \\ &= \sum_{i=2}^{m} a_i b_i \omega_i^k, \end{aligned}$$

where $a = \delta V U$ and $b' = U^{-1} v'$.