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Project 3

MATH345 MATH MODELING

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Spring 2022

Group Report 3

2022.03.21

Abstract

The Great Lakes are the largest freshwater source on Earth, supplying over 30 million people with drinking water.[1] The heavy industrial presence around the Great Lakes has resulted in a serious water pollution problem. In this paper, we address this public health emergency and investigate strategies to reduce the pollution in these lakes to 5% of the current level in 10 years or less. We created a model using differential equations that studies the rate of change of the amount of pollutant in each Great Lake as a function of time. Using data collected by the National Oceanic and the Atmospheric Administration Great Lakes Coastal Forecast System, we were able to determine an effective strategy in reducing the amount of pollution in the Great Lakes. A policy with both a constant pollution limit and pollution removal would lower the pollution to 5% of the current level in 10 years or less, however, exponentially damping the pollution limit with a larger pollution removal would be a more practical solution to this problem.

Keywords: Water pollution, differential equations

Contents

1 Introduction

1.1 Background

Water is integral for life on Earth, and lakes play an important role in providing potable water for many organisms on Earth. However, the current geological era, labeled the Anthropocene, has the highest level of human induced change on the environment, increasing the concentrations of greenhouse gases in the atmosphere, changing the earth's climate, and polluting the earth's waters. Therefore, it is important to limit pollution in waterways in order to preserve the existing potable water, as well as save living organisms in these waterways.

The Great Lakes, which consist of Lake Superior, Lake Huron, Lake Erie, Lake Ontario, and Lake Michigan, are a large interconnected series of freshwater lakes in the United States. These lakes are the largest fresh water source on Earth, supplying over 30 million people with drinking water.[1] The Great Lakes also support a variety of wildlife and ecosystems and play a vital role in the economy. However, the heavy industrial presence around these lakes has resulted in serious water pollution.

Water pollution can be defined as a change in the chemical, physical, or biological health of a waterway due to human activity. This negatively affects the health of the waterway, the health of organisms living around the waterway, and the health of humans. Since the 1960s, there have been many legislative efforts to address the rising pollution levels in the Great Lakes, but it still remains a serious public health issue today. Point source pollution, when pollutants enter the waterway through a specific entry point, is the easiest source of pollution to control and regulate.[1]

This paper will propose a mathematical model that researchers and conservationists can utilize in order to study the impacts single-source pollution can have on a greater system. By focusing on one source on pollution, it is easier to witness the impacts that one source has on all five of the Great Lakes. Then, by focusing on that once source, it is simpler to figure out how to decrease the concentration of pollution in the lakes just by decreasing that amount of pollution coming from that source. Similarly, this model could be utilized to determine the impacts of alternative sources and source placement. Our model enables people to better understand how to determine the indirect impacts pollution can have on greater systems.

1.2 Assumptions

There are several assumptions that must be made in order to form a model:

1. All pollutants originate from Lake Michigan.

   We assume that the source of pollution flows directly into Lake Michigan, and this pollution will then flow into the other lakes.

2. Lake Superior does not contain any pollutants from the source studied in our model.

   Since we assumed that all pollutants originate from Lake Michigan, we must also assume that Lake Superior does not receive pollution from the source being studied. This is due to the direction of flow amongst the Great Lakes.

3. The pollutants are released at a steady rate and with a steady concentration.

   The pollutant from the source flows at a constant speed and concentration, and it does not stop flowing from the source.

4. The pollutant does not decay in the process.

   We assume that the pollutant does not decay within a lake or as it flows from lake to lake, for this would alter the concentration of the pollutant in each lake.

5. The pollutant is distributed evenly within each of the lakes.

   We assume that the pollutant is evenly distributed and not heavily concentrated in certain parts of each of the lakes. This assumption allows our model to be more accurate. This, however, does not assume that each lake has the same concentration of pollution as the other lakes.

6. The pollution flowing upstream is negligible.

   Our model does not study the pollution that flows upstream, as it could alter the concentration of the pollutant. We assume that there is a steady flow in one direction, and that any pollutant that flows back upstream is negligible.

7. The flow rate remains constant.

   Our model takes into account the flow rate. Therefore, we must assume that the rate at which one lake flows into another stays constant in order for our model to be accurate.

8. The volume of each lake remains constant.

   We assume that the amount of water gained or lost as one lake flows into another is negligible in order to assume that total volume of water in each lake remains constant.

2 Methods

The primary goal, that the pollution level reducing to 5% of the current level in 10 years or less will be tested in the mathematical model we constructed. First, we constructed some assumptions to model the real world case. Then, we brainstormed on what variables would affect the strategy of reducing the pollution of the Great Lake system. We used compartment modeling with differential equations to model the pollution level of the Great Lake system. Also, we determined that the input pollution would enter the lake in a constant rate, instead of a fluctuating rate or all at once. We used the data from a study done from Rochester Institute of Technology [4], American Association for the Advancement of Science [5], and The Great Lakes Commission [2] to incorporate in our numerical example.

Data

The data was collected from the Rochester Institute of Technology website, from years 2009 to 2014 and it was originally collected by the National Oceanic and Atmospheric Administration Great Lakes Coastal Forecast System. [4] This dataset includes the plastic pollution data specifically on the plastic debris. Moreover, the data from the American Association for the Advancement of Science includes the flow rate data. The data on the volume of the lakes of 2020 were collected from the Great Lakes Commission website [2].

3 Our Model

3.1 Variables and Model

Our model finds the pollution concentration in each of the five main Great Lakes. Then, we will be able to determine the total concentration of pollutant in the lakes, and figure out how to limit this pollutant in the future.

First, this is a diagram of the flow of the five main Great Lakes.

Figure 1: Lake Flow Diagram[3]

Next, we will define the necessary variables.

- $u_0(t)$ = the amount of pollutant created each year in the source of pollution, which is a function of t (kg per year).
- $u_i(t)$ = amount of pollutant in each lake, $\forall i = 1, 2, 3, 4, 5$, which are functions of t (kg per year).
- r_i = rate of amount of pollutant removed in each lake each year, $\forall i = 1, 2, 3, 4, 5$ (% per year).
- v_i = volume of each of the lakes, $\forall i = 1, 2, 3, 4, 5$ (km^3).

- F_{jk} = the flow rate, where the lake flows from j into k (km^3 per year).
- t = time (year).

Also, our variable u is a function of time, t.
Our Model:

$$\begin{aligned}
\frac{du_1}{dt} &= u_0(t) - \frac{u_1(t)}{v_1}F_{13} - r_1u_1 \\
\frac{du_2}{dt} &= -\frac{u_2(t)}{v_2}F_{23} - r_2u_2 \\
\frac{du_3}{dt} &= \frac{u_1(t)}{v_1}F_{13} + \frac{u_2(t)}{v_2}F_{23} - \frac{u_3(t)}{v_3}F_{34} - r_3u_3 \\
\frac{du_4}{dt} &= \frac{u_3(t)}{v_3}F_{34} - \frac{u_4(t)}{v_4}F_{45} - r_4u_4 \\
\frac{du_5}{dt} &= \frac{u_4(t)}{v_4}F_{45} - \frac{u_5(t)}{v_5}F_5 - r_5u_5
\end{aligned}$$

Then, in order to determine the total concentration, we consider the maximum of pollutant concentration for each lake:

$$u = \max_{1 \leq i \leq 5} \frac{u_i}{v_i}.$$

Based on decreasing of u we test whether a policy achieve the goal of reducing the pollution to 5% of current case.

3.2 Model explained

The goal of our model is to determine the amount of pollution in five of the Great Lakes. We assume that there is one main source of pollution, and that the other sources are negligible. Therefore, we assume that we know the amount of pollution that is being put into the first lakes, and we also know the how the lakes flow into one another and interact.

The differential equations study the rate of change of kilograms of pollution in the lake over time. First, the u_0 value is important to understand. This is the amount of pollutants put into the lake per year. This is significant as it will have a direct impact on the total amount of pollutant in all of the lakes, and it will help determine the policy for minimizing pollution in the Great Lakes. Next, the amount of pollution in lake 1 is determined. Lake 1 is Lake Michigan, as this lake has a large amount of pollution, and this lake flows into three of the four other lakes. Lake Michigan does not flow into Lake Superior, so we assume that Lake Superior has no added amount of pollution from this source of pollution that originates in Lake Michigan. Lake Michigan does flow into the other three lakes, so it is important to study the impacts the pollution into Lake Michigan have on those lakes. Also, there is an original amount of pollution in Lake Superior, so that is important and impacts the amount of pollution in the other lakes as that is able to flow into the other lakes.

The source pollutant, the flow rate from the source, the volume of lake 1, and the flow rate out of lake 1 are all important factors that determine the amount of pollution in lake 1. Once this is determined, this amount will help to find the amount of pollutant in the remaining lakes.

The flow rates in and out of the lakes are significant because they help to determine how much of that pollutant will stay in the lake and how much will be able to travel into another lake. The higher the flow rate, the more likely the pollutant is to travel further into the next lakes. Moreover, it is important to note that each of the lakes has an inflow and an outflow rate, so the lakes have constant motion that will constantly alter the amounts of pollution in the lakes. This stable inflow and outflow ensures that the lake remains in some form of equilibrium.

The amounts of pollutant in each of the lakes are extremely important in determining the amount of pollution in other lakes. Knowing the amount of pollutant in lake 1 is absolutely critical for determining the amount of pollution in lake 3 because lake 1 flows directly into lake 3. Then, it is clear that this amount of pollution will have a direct impact on the pollution in other lakes. Therefore, each of the differential equations are directly dependent on one another. Once the pollution amount in lake 1 is known, the pollution amount in lake 3 can be found. This is found from the knowledge of the amounts of pollution of lake 1 and lake 2, as well as knowing the volumes of these lakes and the flow rates. Then, same procedures can be performed for lakes 4 and 5.

The variable r is the removal factor. When multiplied by the amount of pollutant, it represents the amount of pollutant removed from each of the lakes. This is important in order to minimize the amount of pollution in the Great Lakes because it may be necessary to not only limit what is going into the lakes, but also taking some of the existing pollution out of the lakes.

3.3 Solution

With parameters and initial condition, we use Heun's method to solve the differential equations system and compare different combinations of pollution limit and pollutant removal actions to see how to lower the pollution level to 5% of current level with practical method. See **Appendix 5** for codes used in this section.

3.3.1 Keep Current Pollution

Based on the data, if the amount of pollutants damped into the lake keeps current level and no policies would be applied, the pollutant concentrations would explosively grow as shown by **Figure 2**, which indicates the necessity of take measures to control pollution.

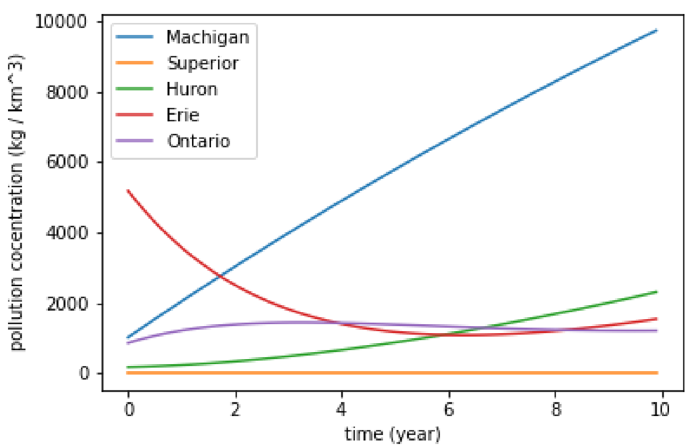

Figure 2: Pollution Concentration without Pollution Control

3.3.2 Constant Pollution Limit with Pollution Removal

One way to control pollutant is to limit the pollutant dumped into the Michigan Lake (as we assumed, the main pollutant source) by constant and take physical and chemical measures to remove the pollutant in each lake dynamically. Based on the data, if the amount of pollutants damped into the lake is limited to below 10^5 kg per year, which is $\frac{1}{50}$ of current polluting level, and in each lake 20% of pollutant is dynamically removed, after 10 years the pollution level would be lowered to 5% of current level with practical method, with pollution concentrations shown by **Figure 3**.

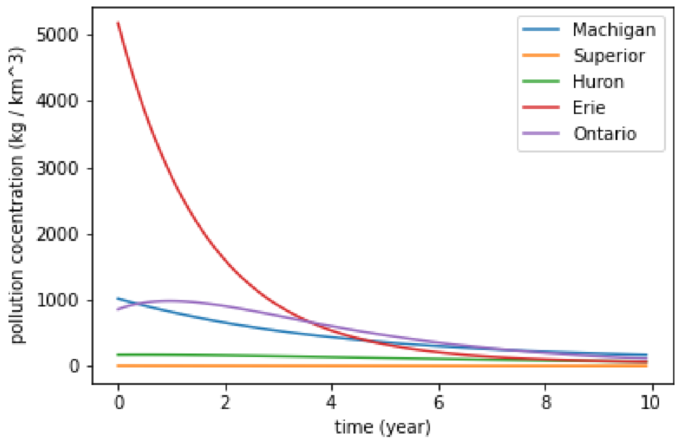

Figure 3: Constant Limitation and 20% Pollutant Removal

3.3.3 Exponentially Damping Pollution Limit with Pollution Removal

Though method in previous session achieves our goal, the $\frac{1}{50}$ limitation and 20% pollutant removal are not practical as policies. Therefore, we attempted to limit the pollutant dumped into the Michigan Lake by exponentially damping limitation $5 \times 10^6 \times e^{-3t}$ kg per year to enable industries to adapt to the change of policies. In fact with this more progressive limitation, we can achieve our 10-year goal by removing 15% of pollutant in each lake dynamically, instead of 20%. That is to say if we keep the 20% removal rate we could achieve the 5% pollution goal in less than 10 years. Pollution concentrations under this exponentially damping limitation and with 15% removal rate are shown by **Figure 4**.

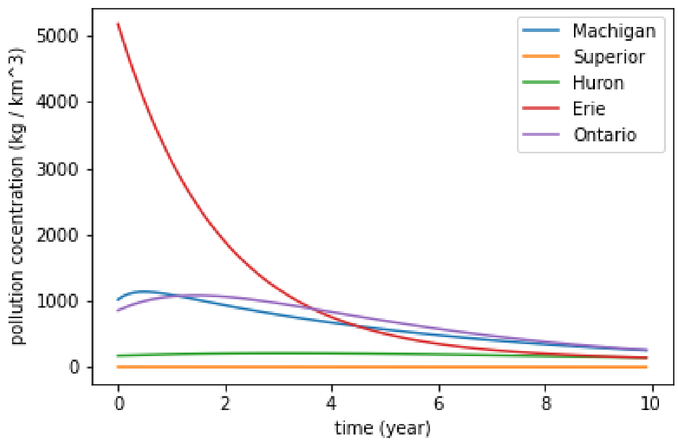

Figure 4: Exponentially Damping Limitation and 15% Pollutant Removal

4 Discussion

4.1 Strengths and Weaknesses

Even though many assumptions were made in order to form a more comprehensive mathematical model, it still gives a strong understanding of the pollution level in the Great Lakes System. First, our model is able to determine how a single-source of pollution can impact all of the connected lakes. This is significant as it shows how the system is highly integrated, and therefore, even the smallest amount of pollution in one lake has vast impacts on the other lakes. Next, another strength of our model is that one can use it for multiple different sources. Even though we focus on the single-source of pollution beginning in Lake Michigan, it can easily be extended to other lakes and other sources of pollution. The user would need to ensure that the flow of the lakes is still maintained in the model. Another significant strength of our model is that our model considers the different flow rates of the different lakes. This leads to a more accurate representation of a real lake system, and ensures that the different flow rates can account for differences in amount of pollutants. This also means that this model can be implemented for other lake systems because these lake systems may have different features that can be inputted into our model. Moreover, our model is able to find the different amounts of pollutants in each of the lakes separately. This enables the policy maker to figure out which of the lakes are more likely to be most polluted, furthering their ability to make more specific policies.

However, while our model is successful in multiple ways, it also does experience some weaknesses. First, our model is only able to consider one source of pollution at a time. This is a weakness because in reality, there may be numerous different sources of pollution, including sources of constant flows of pollution as well as one-off dumping into the lake. Another potential weakness is that we assume that the source of the pollution has a constant flow. This may not always be the case because there may be sources of pollution that are less consistent. Also, the source of pollution may have less pollutant at different points of time, and our model does not take this into consideration. Another weakness that we encountered is that our data only focused on plastic debris. While our model is capable of incorporating all types of pollutant, the data still lacks information that would be critical in order to better come up with a policy and conclusion. Moreover, this plastic pollution data is from a long time ago, so it is critical to try find better data in order to better understand the current amount of pollution in the lakes. It is challenging to find accurate information about the current amount of pollution since this information is safeguarded because people often don't want to release information about the amount of pollution in their lakes and ecological systems.

4.2 Potential Future Research

Our mathematical model is strong in many ways and has some weaknesses. It is important to consider some potential future extensions and modifications to our model. First, it would be incredibly beneficial to include the initial pollution amounts of each of the lakes in the model. This could be added to the amount of pollutant function in order to find the total amount of pollutant in each of the lakes. Our model, at present, is only able to determine the amount of added pollutant concentration in the lake from the source pollution. However, it is important to also note the existing pollution in the lakes from years of pollution occurring. Next, another future extension for our model would be to incorporate run-off and air pollution factors of pollution. Run-off from agriculture and air pollution caused by human industries have a direct and indirect impact on waterway pollution. Therefore, it is incredibly important to include these pollutant factors in the future. It is also incredibly challenging because it is difficult to measure air pollution and there is no source for this pollution. Also, run-off happens inconsistently and is not a constant flow, so this leads to another challenge in incorporating these sources of pollution. A third potential future extension to our model is to consider multiple sources of pollution. Our model focuses on one source and studies the impacts that source has on the other connected lakes. By incorporating multiple sources of pollution, our model would have a more accurate representation of the pollution added into the lakes over time. Finally, in the future, we would strive to take into account for different sources of pollutants rather than just focusing on plastic pollution. This would lead to a more thorough understanding of the pollution taking place in the lakes, and it would enable us to make stronger conclusions regarding a policy.

5 Appendix

```
import numpy as np
from matplotlib import pyplot as plt

# initialize problem
# Flow rate / volume
F1V1 = 5013 * 3600 * 24 * 365 / 10**9 / 4918
F2V2 = 2067 * 3600 * 24 * 365 / 10**9 / 12100
F3V3 = 5100 * 3600 * 24 * 365 / 10**9 / 3538
F4V4 = 5551 * 3600 * 24 * 365 / 10**9 / 483
F5V5 = 6627 * 3600 * 24 * 365 / 10**9 / 1639

A = np.array([[-F1V1, 0, 0, 0, 0], [0, -F2V2, 0, 0, 0], \
    [F1V1, F2V2, -F3V3, 0, 0], [0, 0, F3V3, -F4V4, 0], [0, 0, 0, F4V4, -F5V5]])

# initial condition
u0 = np.array([5*10**6, 32000, 6*10**5, 2.5*10**6, 1.4*10**6])
# initial pollutant amount
v = np.array([4918, 12100, 3538, 483, 1639]) # lake volumes
t0 = 0
tf = 10 # 10 years
n = 5 # 5 lakes
```

```
dt = .1 # time step

# huen method
def heun(dt, A, b):
    count = int(round((tf-t0) / dt))
    t = t0
    u = np.zeros([count, n])
    u[0,:] = u0.transpose()
    # Heun method
    for i in range(0, count - 1):
        t += dt
        ui = u[i, :]
        u[i + 1, :] = ui + dt / 2 * ((2 + dt) * A.dot(ui) + (1 + dt) * b[i, :] + b
    return u

def check(u):
    c = u/v
    plt.plot(np.arange(0, 10, .1), c)
    plt.legend(['Machigan', 'Superior', 'Huron', 'Erie', 'Ontario'])
    plt.xlabel('time (year)')
    plt.ylabel('pollution cocentration (kg / km^3)')
    plt.savefig('pic.png')
    print('Initial Pollutant Concentration')
    print(c[0, :])
    print('Final Pollutant Concentration')
    print(c[-1, :])
    print('Achieve 5%:')
    print(max(c[-1, :]) <= max(0.05 * c[0, :]))

# case 1
# constant u_0 and keep current pollutant
count = int(round((tf-t0) / dt))
b = np.zeros([count, n])
b[:, 0] = 5 * 10 ** 6
u = heun(dt, A=A, b=b)
check(u)

# case 2
```

```
# constant u_0 with limitation and remove pollutants dynamically
count = int(round((tf-t0) / dt))
b = np.zeros([count, n])
b[:, 0] = 10 ** 5
A1 = A - .2 * np.eye(5)
u = heun(dt, A=A1, b=b)
check(u)

# case 3
# exponentially damping u_0 and remove pollutants dynamically
count = int(round((tf-t0) / dt))
A1 = A - .125 * np.eye(5)
b = np.zeros([count, n])
t = 0
for i in range(0, count):
    t += dt
    b[i, 0] = 5 * 10**6 * np.exp(-3*t)
u = heun(dt, A=A1, b=b)
check(u)
```

References

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