

COMP3400

Assignment 2 Written

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In addition to this written work there are *four* coding questions. The written work is worth 35 points and the coding questions are worth 60 points totalling 95 points.

Tail Recursion

The *mean* of a collection of observations x_1, x_2, \dots, x_n is given by

$$\bar{x}_n = \frac{1}{n} \sum_{k=1}^n x_k.$$

At first glance it does not seem that \bar{x}_{n+1} is related to \bar{x}_n but notice

$$\bar{x}_{n+1} = \frac{1}{n+1} \sum_{k=1}^{n+1} x_k = \frac{1}{n+1} \left(x_{n+1} + \sum_{k=1}^n x_k \right) = \frac{1}{n+1} (x_{n+1} + n \bar{x}_n)$$

and thereby

$$\bar{x}_{n+1} = \frac{x_{n+1} + n \bar{x}_n}{n+1}.$$

We have shown how to compute \bar{x} *iteratively*. In particular we compute \bar{x}_{n+1} from *only* (n, \bar{x}_n, x_{n+1}) .

Question 1. [2 MARKS]

The definition of *variance* (specifically sample variance) of a collection of observations x_1, x_2, \dots, x_n is

$$S_n^2 = \frac{1}{n} \sum_{k=1}^n (x_k - \bar{x}_n)^2. \quad (1)$$

Take for granted (though it is easy to show) that

$$S_n^2 = \left(\frac{1}{n} \sum_{k=1}^n x_k^2 \right) - \bar{x}_n^2 \quad (2)$$

and use (1) and (2) to derive an iterative definition for variance. That is, produce an equation which computes S_{n+1}^2 from *only* $(n, S_n^2, \bar{x}_n, \bar{x}_{n+1}, x_{n+1})$.

Question 2. [2 MARKS]

Define a *linear recursive*

`lcVariance :: [Float] -> Float`

that computes the variance S_n^2 of a list. Your function must use `lcVariance xs` to compute `lcVariance (x:xs)`. Note, `fromIntegral.length` is compatible with `Float`.

Question 3. [4 MARKS]

Define a *tail recursive* helper function with type:

`trVariance :: Float -> Float -> Float -> [Float] -> Float`

that finds the variance of the list.

Remember your function may *only* be equal to

1. a call to itself with different inputs, or
2. one of the inputs.

Question 4. [1 MARK]

Define

`variance :: [Float] -> Float`

via a single call to `trVariance`.

Question 5. [1 MARK]

Define an iteration invariant for `trVariance` that proves the correctness of `variance`.

Question 6. [5 MARKS]

Prove `trVariance` satisfies your iteration invariant.

Question 7. [1 MARK]

State the bound value for `trVariance`.

Question 8. [2 MARKS]

Prove your bound value is always non-negative and decreasing.

Question 9. [2 MARKS]

Define *two* distinct quick-checks for variance that *both* use lists from `Arbitrary [Float]`. In particular, your quick-checks should be for lists of *arbitrary length*.

Induction

Consider the following definitions for implementing addition on natural numbers.

```
1 data Nat = Zero | Succ Nat deriving Show
2 plus :: Nat -> Nat -> Nat
3 plus m Zero = m
4 plus m (Succ(n)) = Succ (plus m n)
```

Question 10. [7 MARKS]

Using induction prove:

$$\text{plus Zero } n = n$$

for any $n :: \text{Nat}$.

Question 11. [8 MARKS]

Using induction prove that plus is commutative, that is,

$$\text{plus } m \ n = \text{plus } n \ m$$

for any $n, m :: \text{Nat}$.

Hint: Keep m fixed and do induction over n .

Note: There are style marks for this question so be sure to justify each of your lines and include all necessary components of induction.